Introduction to Randomized Algorithms I

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Overview

- Deterministic vs Non-Deterministic Algorithms
- Randomized Algorithms
- Randomness as a Source of Efficiency – Example
- Simulation of Random Events
- Examples of Statistical Experiments
- Examples of Simple Games

Algorithms

- Algorithm
  - Sequence of non-ambiguous instructions
  - Finite amount of time

- Input to an algorithm
  - An instance of the problem the algorithm solves

- How to classify / group algorithms?
  - Type of problems solved
  - Design techniques
  - Deterministic vs non-deterministic

Deterministic Algorithms

- A deterministic algorithm
  - Returns the same answer no matter how many times it is called on the same data.
  - Always takes the same steps to complete the task when applied to the same data.

- The most familiar kind of algorithm !

- There is a more formal definition in terms of state machines…

Non-Deterministic Algorithms

- A non-deterministic algorithm
  - Can exhibit different behavior, for the same input data, on different runs.
  - As opposed to a deterministic algorithm !

- Often used to obtain approximate solutions to given problem instances
  - When it is too costly to find exact solutions using a deterministic algorithm

Non-Deterministic Algorithms

- How to behave differently from run to run ?

- Factors of non-deterministic behavior
  - External state other than the input data
    - User input / timer values / random values
  - Timing-sensitive operation on multiple processor machines
  - Hardware errors might force state to change in unexpected ways

Randomized Algorithms

- Use a degree of randomness as part of an algorithm's logic

- Algorithm behavior can be guided by random bits as an auxiliary input
  - Take decisions by tossing coins !

- Aiming at good performance on average !

Randomized Algorithms

- What is the effect of randomness?
- Algorithm running time and / or algorithm output are random variables
  - Determined by the random bits / by the coin tossing results

Randomness as a source of efficiency

- Computers C_1 and C_2 at separate locations
  - Connected via a network
- Initial copies of the same DB: DB_1 and DB_2
- BUT, contents evolve over time !
- DB changes have been done simultaneously
- Do DB_1 and DB_2 contain the same data ?

Deterministic approach

- DBs of size n bits (e.g., $n = 10^{16}$)
- Is the data on both computers the same ?
  - Yes / No – Decision Problem

- What is the number of bits that have to be exchaged, between C_1 and C_2, to solve the problem ?
- At least n bits !!
  - Send the entire DB, without communication errors

Randomized approach

- Contents of DB_1 are a string X of n bits
- Contents of DB_2 are a string Y of n bits

- C_1 makes a uniform random choice of a prime number p from $[2, n^2]$
- Computes s = Number(X) mod p
  - String X is the binary rep. of natural Number(X)
- And sends (s, p) to C_2

Randomized approach

- C_2 reads (s, p)
- Computes r = Number(Y) mod p
- If $s \neq r$, then C_2 outputs "$X \neq Y$"
- If $s = r$, then C_2 outputs "$X = Y$"
- Reliable answers ?

Randomized approach

- Size of the message (s, p) ?
- At most,

$$4 \times \text{ceil}(\log_2 n) \text{ bits}$$

- Given that $s \leq p < n^2$
- $n = 10^{16}$ implies a message of, at most, 256 bits

Randomized approach

- Reliability of the final answer ?

- If $X = Y$, then the answer is always correct !!
- If $X \neq Y$, then the answer might be wrong !!

- For $X \neq Y$, the output might be "$X = Y$", if the chosen prime was a "bad" prime for (X, Y)
  - Number(X) mod p = Number(Y) mod p, with $X \neq Y$

Randomized approach

- Choose p from {2, 3, 5, 7, 11, 13, 17, 19, 23}
- $p = 7$

- $X = 01111 \quad \rightarrow \quad \text{Number}(X) = 15$
- $Y = 10110 \quad \rightarrow \quad \text{Number}(Y) = 22$

- $\text{Number}(X) \bmod p = \text{Number}(Y) \bmod p$
- BUT, $X \neq Y$

Randomized approach

- Error probability ?
- At most, $(\ln n^2) / n$, which presents no real risk…
- For n = 10^{16} the error probability is, at most,

  0.36892×10^{-14}

Randomized approach

- If we want to be safer, we can use 10 rand. chosen primes
  - 10 independent repetitions
  - Message will be 10 times larger !

- Error probability ?
  - Are all 10 primes "bad" primes ?

- For n = 10^{16} the error probability is smaller than 0.4717×10^{-141}

Random Number Generators

- The source of randomness is usually a random number generator
  - Repeated calls return a stream of numbers
  - That appear to be randomly chosen
  - From some range / interval
- In reality, they are pseudo-random numbers !
  - Generated by particular recurrence relations
  - It is possible to calculate each value from a sequence of preceeding values !!

Random Number Generators

- Check the story of Daniel Corriveau at
  - http://www.americancasinoguide.com/gambling-stories/costly-casino-mistakes-the-keno-mix-up.html
- What happened ?

Python – The `random` Module

- For integers
  - `randint(…)`
  - `randrange(…)`

- For sequences
  - `choice(…)`
  - `sample(…)`
  - …

Python – The `random` Module

- For generating real-valued distributions
  - `random()` `# next random float in [0,1)`
  - `uniform(…)`
  - `gauss(…)`
  - …

Python – The `random` Module

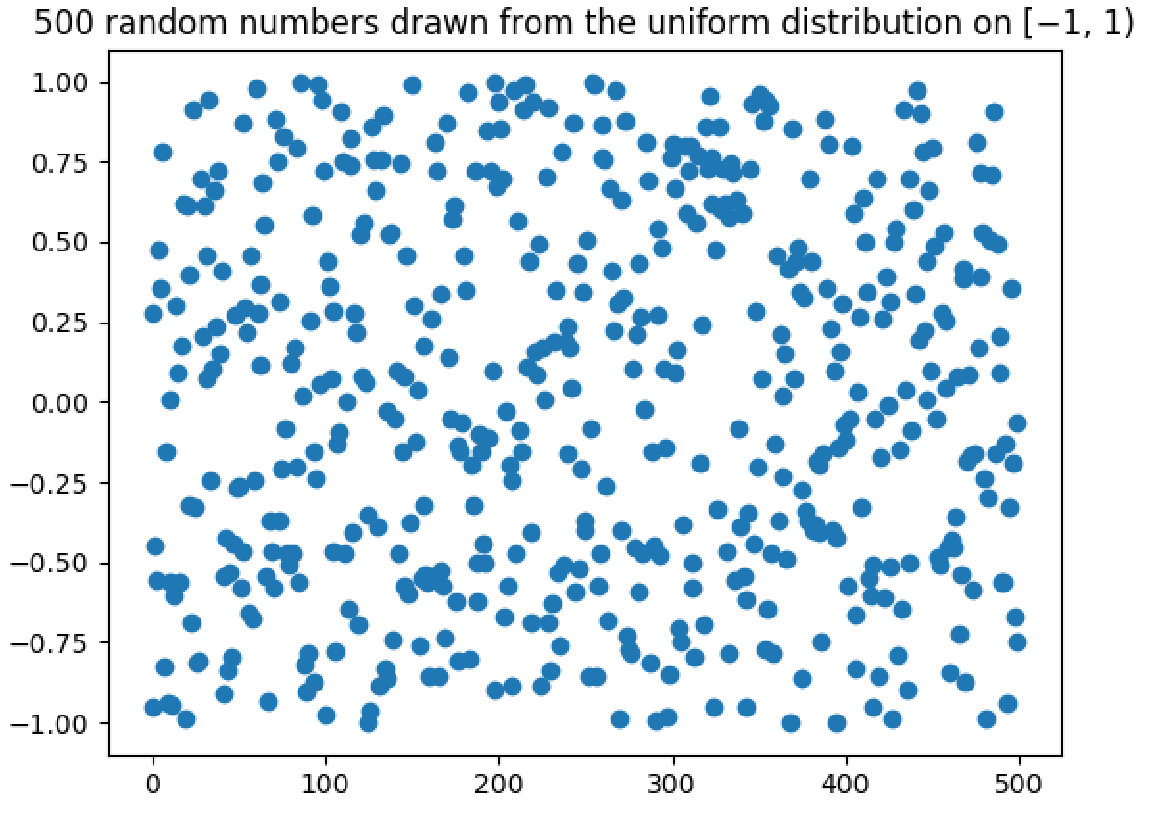
- Reproducibility
  - It might be useful to reproduce the sequences given by a pseudo random number generator
- Re-using of seed values
  - Same sequence should be reproducible from run to run, as long as multiple threads are not running
  - `seed(…)`

Python – The `secrets` Module

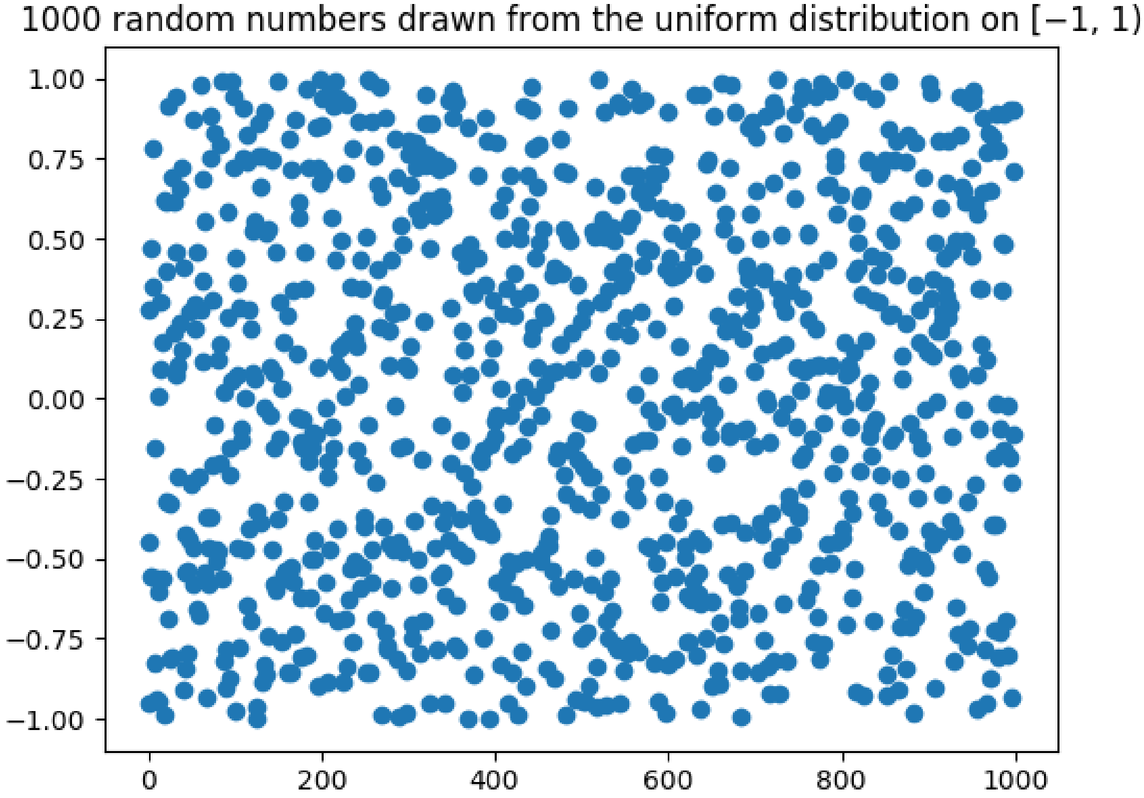
- The pseudo-random generators of the `random` module should not be used for security purposes !

- For security or cryptographic uses, use the `secrets` module instead !
  - Generation of secure random numbers

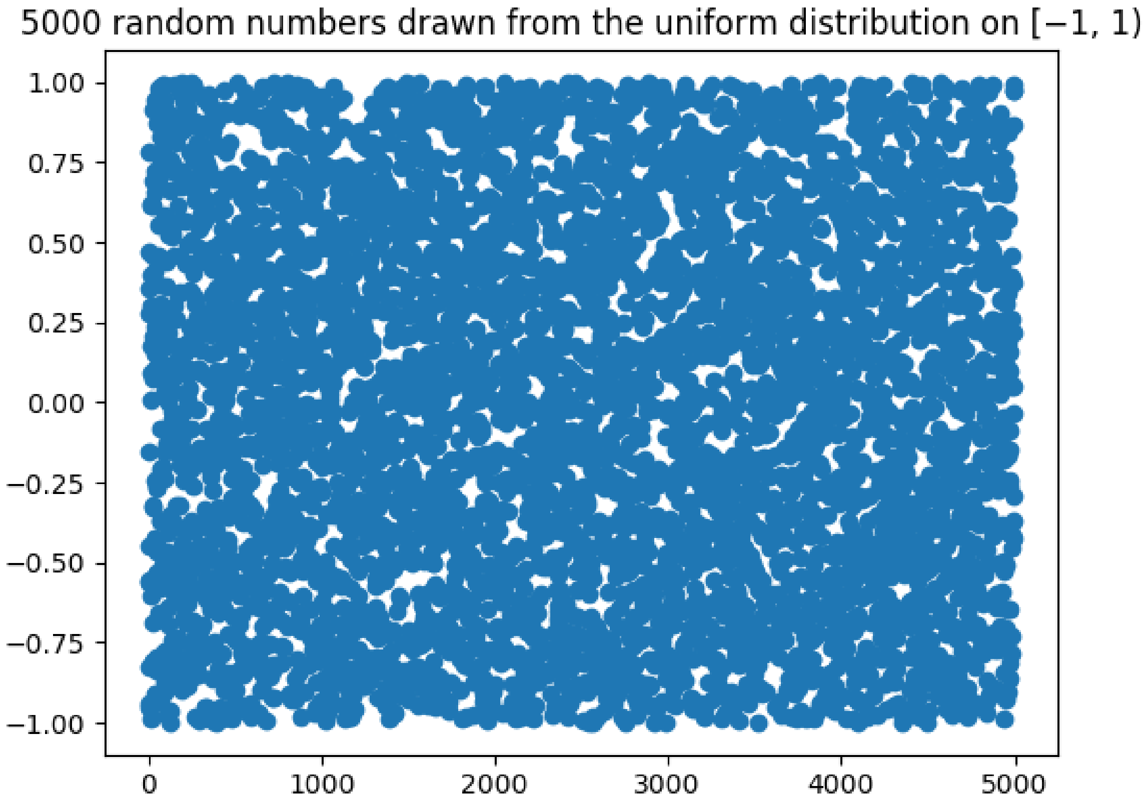
Uniform Distribution



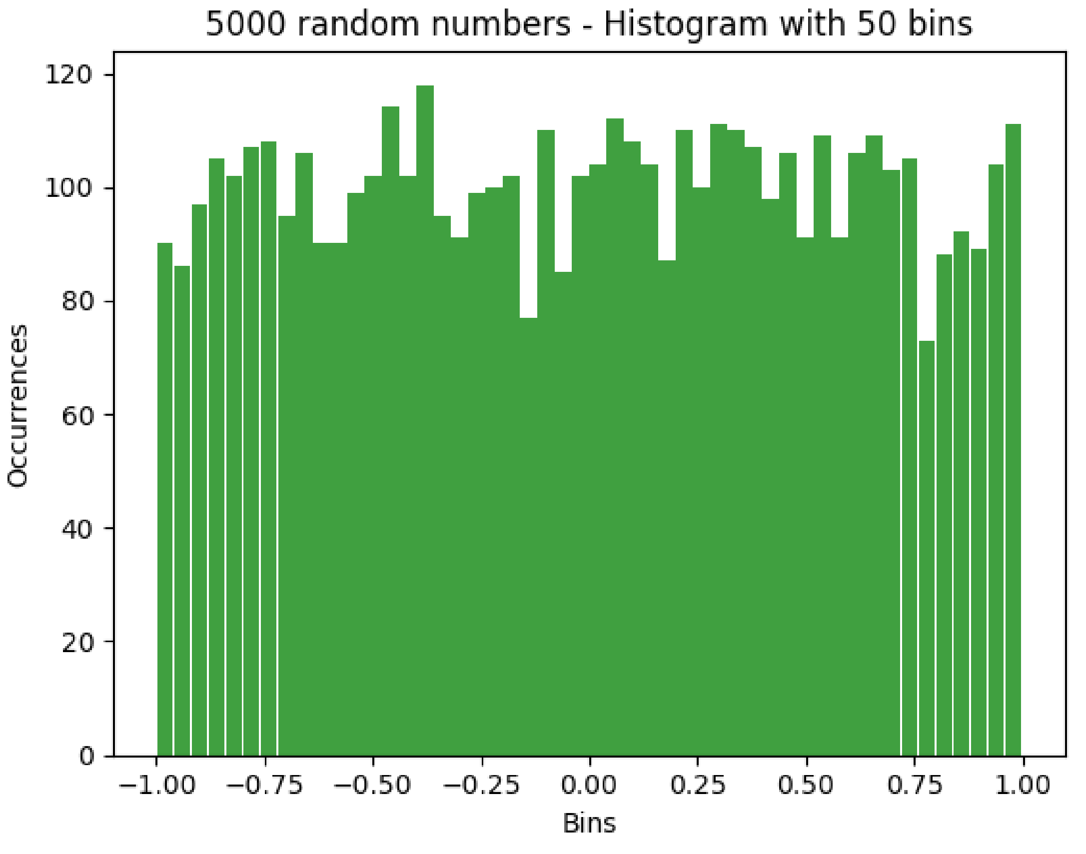
Uniform Distribution



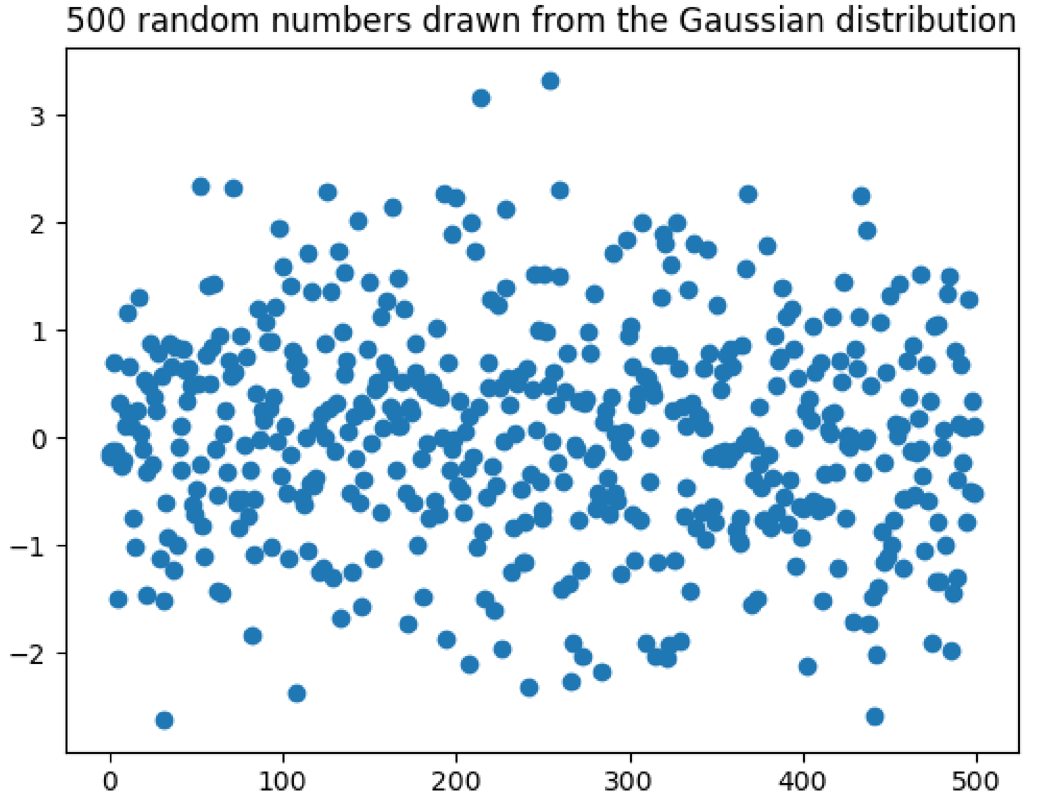
Uniform Distribution



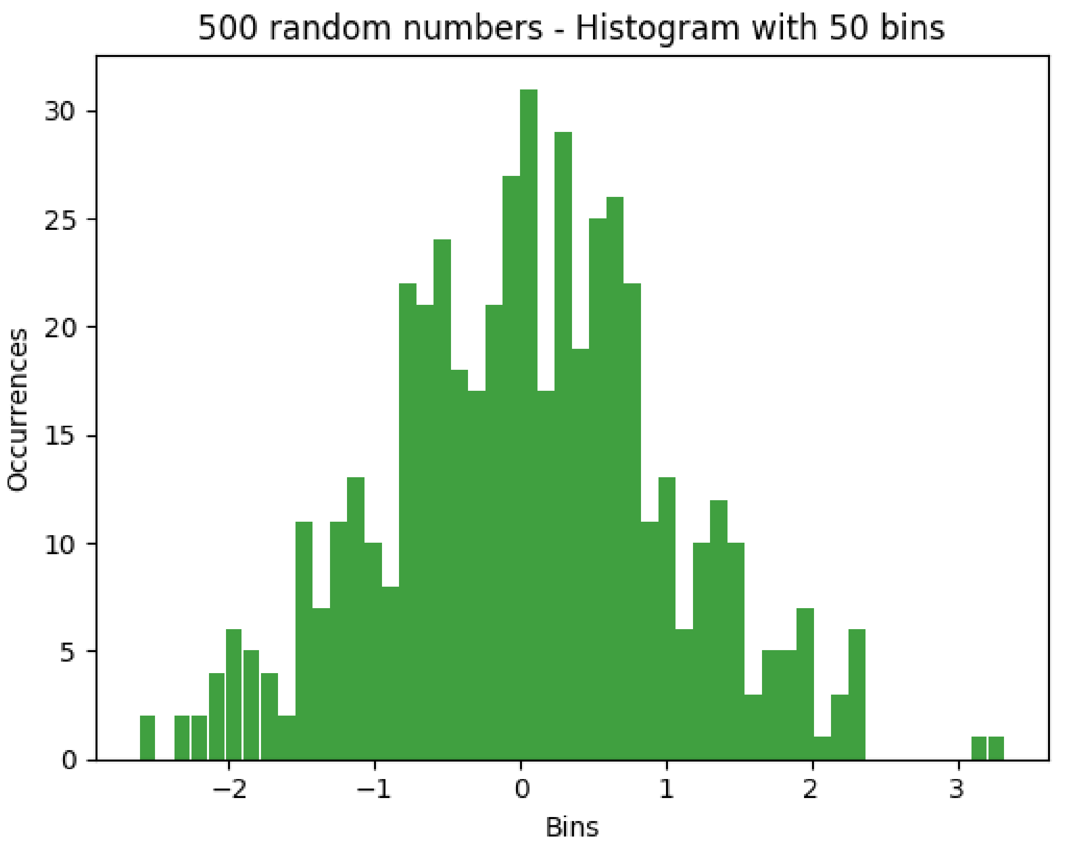
Uniform Distribution



Gaussian Distribution



Gaussian Distribution



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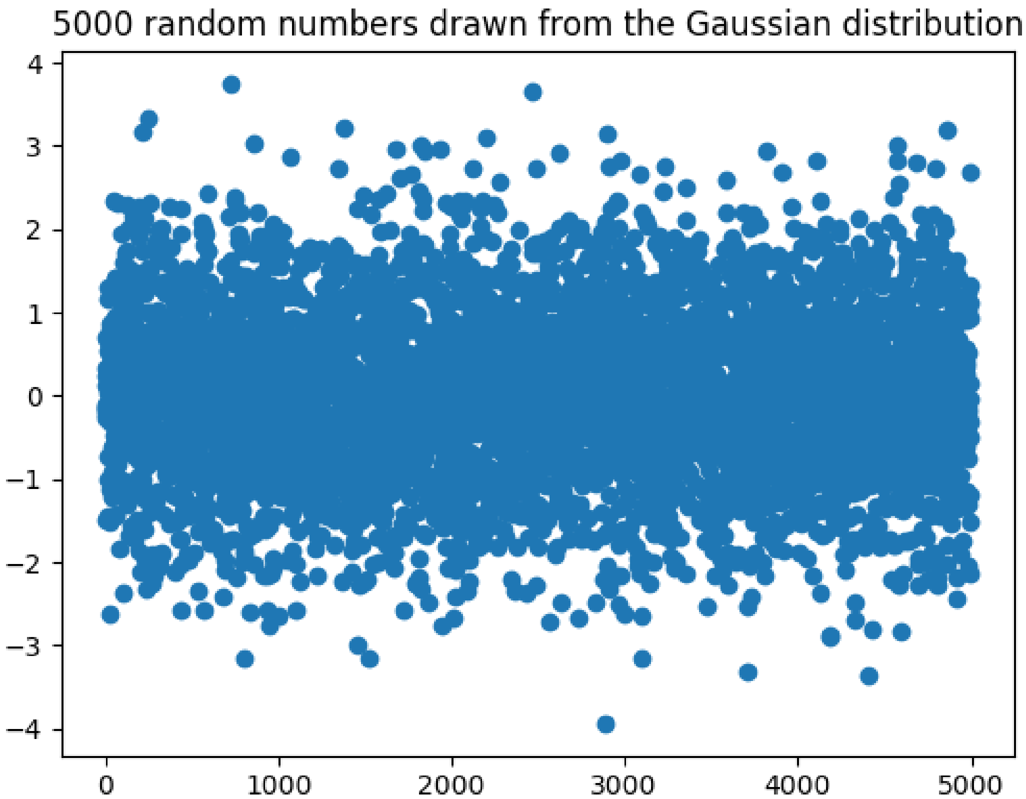

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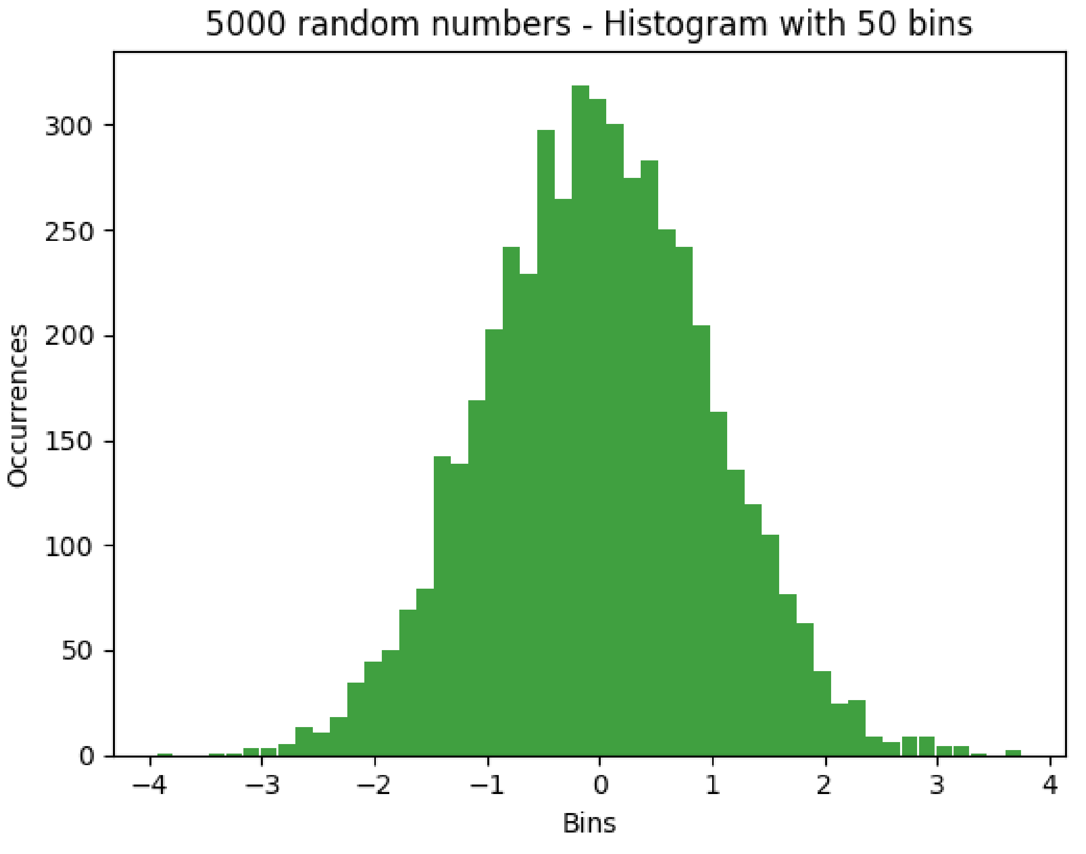
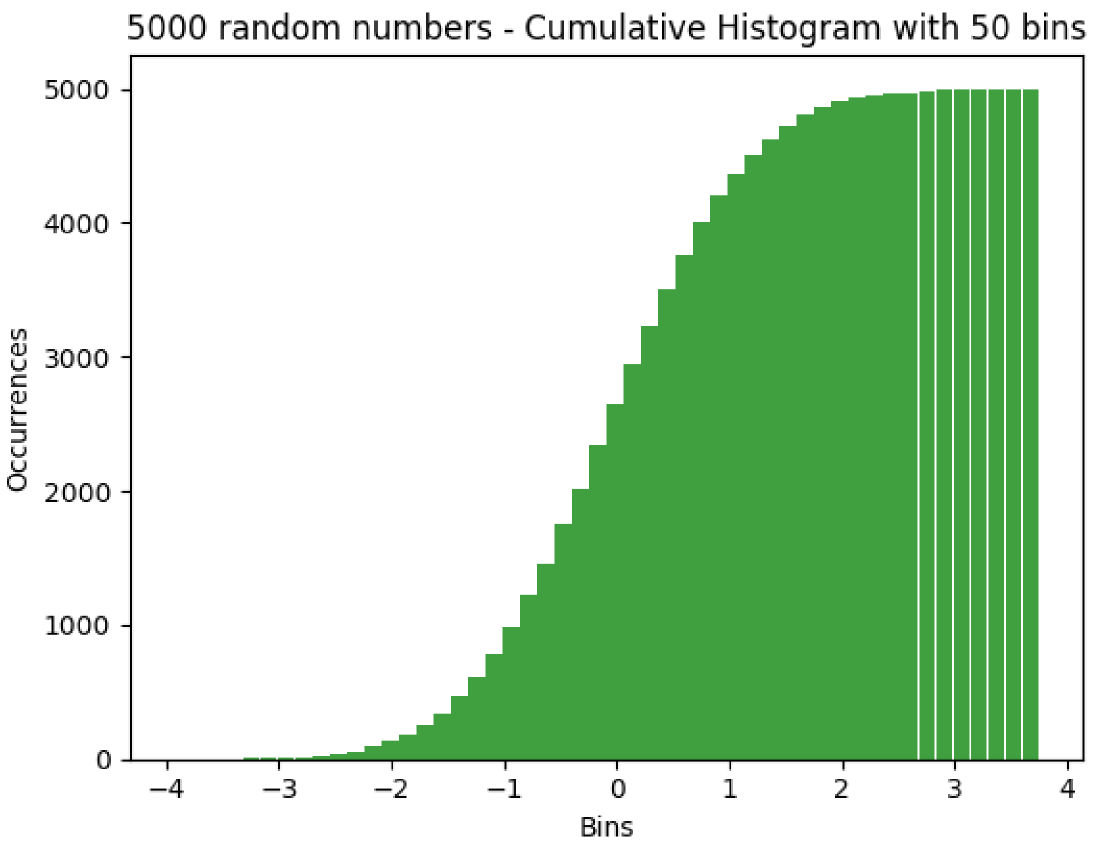


Gaussian Distribution



Simulation of Random Events

- Model random events, such that simulated outcomes closely match real-world outcomes

- Analyze simulated outcomes to gain insight !

- Why approximate the real-world ?
  - No precise mathematical description…
  - OR
  - Less time / effort / cost than other approaches

Simulations have to be useful

- How to mirror the real-world ?
- 1st – Prepare the experiment !
- Identify the possible outcomes
- Link each outcome to one (or more) random number(s)
- Choose a source of random numbers

Simulations have to be useful

- 2nd – Run the experiment loop !

- Choose one (or more) random number(s)
- Record the simulated outcome

- 3rd – Analyze the data and report results
  - Histogram
  - …

Applications

- Simulation of real-world systems for which the input is random in some way
  - Queueing in check-out lines
  - …
- Simulation of statistical experiments
  - Tossing balanced / biased coins
  - Throwing fair / unfair dice
  - …

A Coin Experiment – V1

- Toss a balanced coin n times
  - n >= 1 – parameter of the experiment
  - n independent replications of the simplest exp.
- Record the total score of the experiment
  - 1 for heads or 0 for tails
- What do you expect ?

A Coin Experiment – V2

- The coin is now biased !!
- It turns up heads only 45% of the time
- Again, toss the biased coin n times
- And record the total score
- Now, what do you expect ?

Tasks – Simulations

- Simulate both coin experiments
- For n = 1, 3, 5, and 7
- Run the simulations 10, 100 and 1000 times
- Observe the outcomes
  - Histograms

A Die Experiment – V1

- Throw a standard 6-sided die n times
- Record the total score of the experiment
- What do you expect ?

A Die Experiment – V2

- The die is now an unfair die !!
- For which an ace is twice as likely to turn up as any other face
- Again throw the unfair die n times
- And record the total score
- What do you expect ?

Tasks – Simulations

- Simulate both die experiments
- For n = 1, 3, 5, and 7
- Run the simulations 10, 100 and 1000 times
- Observe the outcomes
  - Histograms

Another experiment with coins

- Toss two balanced coins n times !!
- Record the total score of the experiment
  - 1 for heads or 0 for tails
- What do you expect ?

Another experiment with dice

- Throw a pair of fair dice n times !!
- Record the sum of the faces that turn up
- What do you expect ?

Tasks – Simulations

- Simulate both experiments
- For n = 1, 3, 5, and 7
- Run the simulations 10, 100 and 1000 times
- Observe the outcomes
  - Histograms

A Die-Coin Experiment

- A standard die is thrown and then a coin is tossed the number of times shown on the die
  - Compound experiment
  - Second, dependent stage
- Record the total coin score

- Randomization of the first coin experiment !

A Coin-Dice Experiment

- A coin is tossed
- If the coin lands heads, a red die is throw
- If the coin lands tails, a green die is thrown
  - Again, a compound experiment
- Record the die color and score

Tasks – Simulations

- Simulate both experiments
- Run the simulations 10, 100 and 1000 times
- Observe the outcomes
  - Histograms

Extra Tasks

- Simulate experiments using k-sided dice

[Wikipedia]

Task – A Simple Game

- You pay 1 euro to roll two dice
  - Red + Green
- You win 2 euros, if there are more eyes on red than on the green die
- Should you play this game ?
- Run a few simulations and decide !!

Task – A Simple Game

- You roll two dice and, beforehand, guess the sum of the eyes: n eyes
- If the guess turns out to be right, you earn n euros; otherwise, you pay 1 euro
- Should you play this game ?
- Run a few simulations and decide !!

References

- D. Vrajitoru and W. Knight, *Practical Analysis of Algorithms*, Springer, 2014
  - Chapter 6
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  - Chapter 8