Introduction to Randomized Algorithms III

Joaquim Madeira

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Overview

- Randomized Algorithms
- Random Sampling
- Fermat's Primality Test
- Las Vegas Algorithms
- Monte Carlo Algorithms
- Randomized Algs. for Optimization Problems

Randomized Algorithms

- Use a degree of randomness as part of an algorithm's logic
- Algorithm behavior can be guided by random bits as an auxiliary input
  - Take decisions by tossing coins !
- Aiming at good performance on average !

Randomized Algorithms

- What is the effect of randomness?
- Algorithm running time and / or algorithm output are random variables
  - Determined by the random bits / by the coin tossing results

Randomized Algorithms

- Monte Carlo methods
  - Rely on repeated random sampling to achieve numerical results
  - Often used in physical and mathematical problems

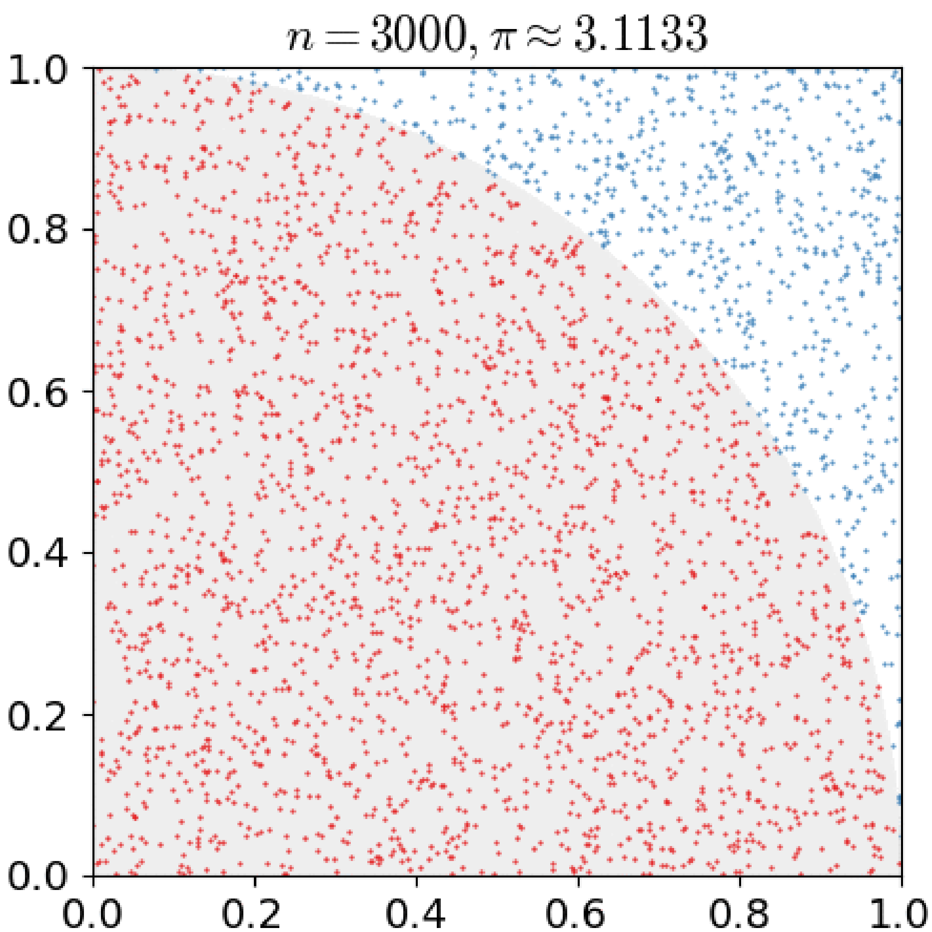
Monte Carlo Methods

- Define the domain of possible values
- Generate random values inside the domain
  - Probability distribution ?
- Process the generated values
  - Deterministic computation
- Compute the desired result
  - Approximation !!

What are we computing ?

```
def monte_carlo_approximation( num_points ):

    domain_area = 4

    inside_counter = 0

    for i in range( num_points ):

        x = random.uniform( -1, 1 )

        y = random.uniform( -1, 1 )

        if x * x + y * y <= 1.0:

            inside_counter += 1

    return ( inside_counter / num_points ) * domain_area
```

Another method



[Wikipedia]

Randomized Algorithms

- Las Vegas algorithms
  - Use the random bits to reduce expected running time or memory usage
  - BUT always terminate with a correct result, in a bounded amount of time

Example – Las Vegas Algorithm

- Array of n >= 2 elements, half are 'a', the other half 'b'
- Find an 'a' in the array

```
repeat
    Randomly select an array element;
until 'a' is found;
```

- Algorithm succeeds with probability 1
- Expected running time over many calls is O(1)

Probabilistic Algorithms

- Randomized algorithms that:
- Have a chance of producing an incorrect result
  - Monte Carlo algorithms
- Have a chance of failing to produce a result
  - Signaling a failure
  - Failing to terminate (!?!)

Example – Monte Carlo Algorithm

- Array of $n >= 2$ elements, half are 'a', the other half 'b'
- Find an 'a' in the array

```
i = 0;
repeat
      Randomly select an array element;
      i = i + 1;
until i == k or 'a' is found;
```

Example – Monte Carlo Algorithm

- Run time is fixed !
- If an ‘a’ is found, the algorithm succeeds; else, it fails !
  - Compare with the Las Vegas version

- What is the probability of having found an ‘a’ after k iterations ?
  - Is it “large” or “small” ?

- Expected running time over many calls is O(1)

Choosing a Random Sample

- How to choose a random sample of size K from a set of size N ?
  - Random sampling a “population”

- Possible goal – Identify common features
- Why sampling ? – Population is too large !

- Every subset of size K < N must be given equal chance of being chosen !
  - How many such subsets ?

Choosing a Random Sample

```
in_sample[i] = false, for i=1 to N;
while( count < K )
      r = rand_int( 1, N );
      if( not in_sample[r] )
            in_sample[r] = true;
            count++;
Read the selected samples from file;
```

Choosing a Random Sample

- Do we have a valid random sample ?
  - Higher probability for some subsets ?
- Yes, the selection process is unbiased !
- What is the probability of each subset of K elements being chosen ?

Choosing a Random Sample

- Problem : we might need a VERY LARGE boolean array !
- K << N , usually
- Use an integer array of size K to keep the sorted set of selected indices !
- How to modify the previous algorithm ?

Choosing a Random Sample

- ISSUE : we are assuming that the entire set of size N is known in advance

- What if that is not the case ?
  - Need a on-line algorithm !

- Reservoir Sampling algorithms
  - N is unknown
  - N is TOO LARGE

Reservoir Sampling – Algorithm R

```
// Jeffrey Vitter, 1985

// 1 - Fill the reservoir array

Read K objects into array reservoir[1..K];

num_objs_read = K;
```

Reservoir Sampling – Algorithm R

```
// 2 - Sampling and Replacement
while( not end of input )
     obj = Get next object;
     r = rand_int( 1, num_objs_read );
     if( r <= K )
          reservoir[r] = obj;
     num_objs_read++;
```

Random Sampling – Tasks

- Implement the three previous random sampling algorithms
- Analyze their behaviour for different test cases
  - How many random numbers are generated by the first algorithms ?
  - How many reservoir replacements are done in the last algorithm ?

Monte Carlo Algorithms – Recap

- Guaranteed to be fast !
- Might not find the correct solution !
- BUT, the probability of finding a correct answer can be computed and controlled
  - Incorrect answer with negligible probability !
  - That is what makes them useful !

Decision Problems

- Decision problems
  - Answer is yes / no – true / false

- Some decision problems are "difficult"
  - No known polynomial algorithm, at the moment
  - Alternatives to "brute-force" for large instances ?

- Monte Carlo algorithms are useful here !
  - Fast execution
  - Negligible error probability

Monte Carlo Algorithms

- For decision problems, Monte Carlo algorithms can be:

- Yes-biased
  - A yes answer is always correct !
  - A no answer might be correct, with some probab.
- No-biased
  - A no answer is always correct !
  - A yes answer might be correct, with some probab.

Primality Testing

- Given a positive integer p
- Is it a prime ?
  - Yes / No
- Important decision problem ?
  - Applications ?
- Naïve deterministic algorithm ?

Naïve Primality Testing - Task

- Implement a first, naïve brute-force primality testing algorithm
- Improve your previous algorithm in order to avoid using unnecessary divisors !!
  - Even divisors ?
  - Stopping criterion ?
- What is the largest prime you can find in a few seconds ?

Fermat’s Primality Test

```
boolean fermat_test( P ) // P > 3
    a = rand_int( 2, P - 2 );
    if( power( a, P - 1 ) % P != 1 )
        return false; // Composite !!
    return true; // Meaning ?
```

Does it always work ?

- Result for $p = 15$ and $a = 2$?
  - Composite !
  - 2 is a Fermat-witness for 15
- Result for $p = 341$ and $a = 3$?
  - Composite ! $p = 11 \times 31$
  - 3 is a Fermat-witness for 341
- Result for $p = 341$ and $a = 2$?
  - 2 is a Fermat-liar for 341 !!

Iterated Fermat's Primality Test

```
boolean is_prime( P, K ) // P > 3
    for( i = 0; i < K; i++ )
        // Repeating Fermat's test
        a = rand_int( 2, P - 2 );
        if( power( a, P - 1 ) % P != 1 )
            return false; // Composite !
    return true; // PROBABLY prime !
```

Fermat's Primality Testing

- Fermat's Little Theorem (1640)
  - If the integer number p is prime, then for every integer a, 1 <= a < p

$$a^{p-1} \bmod p = 1$$

- How accurate is a true answer ?
- How much confidence in a true answer ?
  - Fermat liars vs Fermat witnesses
  - What is the proportion of Fermat witnesses ?

Fermat's Primality Testing

- Theorem
  - If the integer number p is not a prime, then at most half of the integers a, 1 <= a < p, satisfy the equation in Fermat's Little Theorem.

- Consequence ?
  - 1 test – error probability at most 50%
  - 2 tests – error probability at most 25%
  - …
  - 10 tests – error probability is negligible (!?!)

Remarks

- 1 and (p – 1) are trivial Fermat-liars
  - Do not use them !
- Exponentiation and integer division are "expensive" operations
- There are some "stubborn" composite numbers
  - Carmichael numbers : 561, 1105, 1729, 2465, …

Carmichael Numbers

- 561, 1105, 1729, 2465, 2821, 6601, …
- Odd composite number n which satisfies

$$b^{n-1} \equiv 1 \pmod{n}$$

  - For all integers b which are relatively prime to n
- What happens for 561 for a few values of b ?
- Consequence for Fermat's primality test ?

Primality Testing – Better alternatives

- Solovay-Strassen, 1977
  - First Monte Carlo algorithm for primality testing
- Miller-Rabin, 1980
- Baillie-PSW, 1980
- Also, PRIMES is in P – 2002
  - But, Monte Carlo primality testing mostly used !

Primality Testing

- Implement Fermat's primality test
- Generate some random positive integers and check if any of them is a prime
  - Check your results using the OEIS
- Use the Fermat's primality test to list the first Mersenne primes
  - It is not the fastest way !! But, it is OK for us…
  - Check your results using the OEIS

Las Vegas Algorithms – Recap

- Guaranteed to give the correct answer !
- BUT their execution time is probabilistic !
  - Although expected to be fast in general
- The probability of an efficient execution time can be computed and controlled
  - Very long execution times with low probability !
  - That is what makes them useful !

Randomized Search

```
boolean las_vegas_search( a[], N, Target )
    for( i = 0; i < N; i++ )
        // Ensure no repeated indices
        test = rand_int( 0, N - 1 );
        if( a[test] == Target )
            return true;
    return false;
```

Randomized Search

- On average, faster than linear search
  - Whenever the array contains multiple occurrences of the target

- Compare the performance of linear search and randomized search
  - Generate random arrays
  - Select random targets
  - Count the number of array comparisons !

Randomized Quicksort

- How can we transform Quicksort into a randomized algorithm ?

- The probability of Worst Case behaviour will be much smaller !

- Standard vs randomized Quicksort
  - Compare their performance
  - Generate random arrays of large sizes
  - Number of array comparisons and exchanges ?

Randomized Algs. for Opt. Problems

- Compute an approximate solution for optimization problems

- Execute k runs of a randomized algorithm

- Final result ?
- The best of the k solutions computed
  - Regarding the optimization goal
  - I.e., the objective function

Probability of (in)success

- Probability of computing no optimal solution, in a given run of the algorithm

$$\left(1-\frac{1}{n}\right)$$

- After n runs we have

$$\left(1-\frac{1}{n}\right)^n < \frac{1}{e}$$

- The probability of having obtained an optimal solution is at least

$$1-\frac{1}{e}$$

Nearly-optimal solutions

- We are usually happy with a feasible solution that does not differ much from an optimal solution
- It is an approximation strategy !!
- Move from exponential or factorial time complexity to polynomial time complexity !

Approximation Accuracy – Min Prob

- Minimize function f()
- Approximate solution : s_a
- Exact solution : s^*

- Relative error : $re(s_a) = (f(s_a) – f(s^*)) / f(s^*)$

- Accuracy ratio : $r(s_a) = f(s_a) / f(s^*)$

- Performance ratio : R_A
  - The lowest upper bound of possible $r(s_a)$ values
  - Should be as close to 1 as possible
  - Indicates the quality of the approximation algorithm

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- Performance ratio : R_A
  - The largest lower bound of possible $r(s_a)$ values
  - Should be as close to 1 as possible

Main Goals

- Improve the accuracy ratio
  - I.e., the "quality" of the approximate solution
- Produce feasible solutions whose cost / value is not very far from the optimal cost / value
- With high probability !
- And without taking too much time !!

Tasks – TSP

- Develop a randomized algorithm for the TSP

- For some test instances
- Compute optimal solutions using exhaustive search
- Compute approximate solutions after 100, 1000, 10000, … iterations
- Evaluate the accuracy of the obtained approximate solutions

References

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  - Chapter 6
- J. Hromkovic, *Design and Analysis of Randomized Algorithms*, Springer, 2005
  - Chapter 2
- M. Dietzfelbinger, *Primality Testing in Polynomial Time*, Springer, 2004
  - Chapter 5