Algorithm Design Strategies II

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Overview

- Counting basic operations – Recap
- Deterministic vs Non-Deterministic Algorithms
- Problem Types and Design Strategies
- Algorithm Efficiency and Complexity Analysis
- Brute-Force
- Divide-and-Conquer
- Decrease-and-Conquer
- Example: computing powers

Running Time vs. Operations Count

- Running time is not (very) useful for comparing algorithms
  - Speed of particular computers
  - Chosen computer language
  - Quality of programming implementation
  - Compiler optimizations

- Evaluate efficiency in an independent way
  - Count the "basic operations" !!
    - Contribute the most to overall running time

Formal Analysis – Pencil and paper

- Understand algorithm behavior
  - Count arithmetic operations / comparisons
  - Find a closed formula !!
  - Identify best, worst and average case situations, if that is the case
- Iterative algorithms
  - Loops : how many iterations ?
  - Set a sum for the basic operation counts
- Recursive algorithms
  - How many recursive calls ?
  - Establish and solve appropriate recurrences

Return value? – Number of iterations?

```
int f1(int n) {
   int i,r=0;
   for(i = 1; i <= n; i++)
      r += i;
   return r;
}
```

```
int f2(int n) {
   int i,j,r=0;
   for(i = 1; i <= n; i++)
      for(j = 1; j <= n; j++)
         r += 1;
   return r;
}
```

```
int f3(int n) {
   int i,j,r=0;
   for(i = 1; i <= n; i++)
      for(j = i; j <= n; j++)
         r += 1;
   return r;
}
```

```
int f4(int n) {
   int i,j,r=0;
   for(i = 1; i <= n; i++)
      for(j = 1; j <= i; j++)
         r += j;
   return r;
}
```

Closed formulas? – Comput. tests?

- f1(n) = n (n + 1) / 2 n_iters1(n) = n
- f2(n) = n^2 n_iters2(n) = f2(n)
- f3(n) = n (n + 1) / 2 n_iters3(n) = f3(n)
- f4(n) = n (n + 1) (n + 2) / 6
- n_iters4(n) = n (n + 1) / 2
- Use WolframAlpha to get / check results !

Return value? – Number of calls?

```
unsigned int
r1(unsigned int n) {
   if(n == 0) return 0;
   return 1 + r1(n – 1);
}
```

```
unsigned int
r2(unsigned int n) {
   if(n == 0) return 0;
   if(n == 1) return 1;
   return n + r2(n – 2);
}
```

```
unsigned int
r3(unsigned int n) {
   if(n == 0) return 0;
   return 1 + 2 * r3(n – 1);
}
```

```
unsigned int
r4(unsigned int n) {
   if(n == 0) return 0;
   return 1 + r4(n – 1) + r4(n – 1);
}
```

Closed formulas? – Comput. tests?

- $r1(n) = n$ $\quad\quad$ $n\_calls1(n) = r1(n)$

- $r2(n) = n\,(n + 2) / 4$, if n is even
- $r2(n) = 1 + (n - 1)\,(n + 3) / 4$, if n is odd

- $n\_calls2(n) = \mathit{floor}(n / 2)$

- Use WolframAlpha to get / check results !

Closed formulas? – Comput. tests?

- $r3(n) = 2^n - 1$ $\quad\quad\quad$ $n\_calls3(n) = n\_calls1(n)$
- $r4(n) = r3(n) = 2^n - 1$
- $n\_calls4(n) = 2 \times (2^n - 1) = 2 \times r4(n)$

- r3 and r4 compute the same result
- BUT, r4 will take much more time…
  - How far can you go with your computer?

Algorithms

- Algorithm
  - Sequence of non-ambiguous instructions
  - Finite amount of time

- Input to an algorithm
  - An instance of the problem the algorithm solves

- How to classify / group algorithms?
  - Type of problems solved
  - Design techniques
  - Deterministic vs non-deterministic

Deterministic Algorithms

- A deterministic algorithm
  - Returns the same answer no matter how many times it is called on the same data.
  - Always takes the same steps to complete the task when applied to the same data.
- The most familiar kind of algorithm !
- There is a more formal definition in terms of state machines…

Non-Deterministic Algorithms

- A non-deterministic algorithm
  - Can exhibit different behavior, for the same input data, on different runs.
  - As opposed to a deterministic algorithm !
- Often used to obtain approximate solutions to given problem instances
  - When it is too costly to find exact solutions using a deterministic algorithm

Non-Deterministic Algorithms

- How to behave differently from run to run ?

- Factors of non-deterministic behavior
  - External state other than the input data
    - User input / timer values / random values
  - Timing-sensitive operations on multiple processor machines
  - Hardware errors might force state to change in unexpected ways

Problem Types

- Searching
- Sorting
- String Processing
- Graph / Network problems
- Combinatorial problems
- Bioinformatics
- …

- Examples of algorithms ?

Searching

- Which items?
  - Numbers, strings, records (key?), etc.

- Possible representations?
  - Arrays, lists, trees, etc.

- Ordered vs. non-ordered items
- Dynamically changing set?

- Sequential vs. binary search
- Others?

Sorting

- Which items?
  - Numbers, strings, records (key?), etc.

- Possible representations?
  - Arrays, lists, trees, etc.

- Use an indexing array?

- Which ordering? Repeated items?
- Stable? In-place?

- How many algorithms do you know?
- Which ones are the "most efficient"? When?

String Processing

- Text strings, bit strings, gene sequences, etc.

- String matching?
- Longest-common substring?
- String-edit distance?
- Other problems / algorithms?

Graph / Network Problems

- Modeling the real-world!
- Dense vs. sparse graphs / networks
- Representations
  - Adjacency matrices vs. lists
  - Forward-star and reverse-star forms
- Depth vs. breadth traversals
- Shortest path? K-shortest paths?
- Minimum spanning tree?
- Traveling salesman !
- Other problems?

Combinatorial Problems

- Find a permutation, combination or subset !!

- What are the constraints?
- Are we optimizing some property?
  - Max value, min cost, etc.

- The most difficult problems in computing !!
- No (known?) polynomial algorithms for some problems !!

- Instance size vs. execution time
  - Exhaustive search?
- Optimal solutions vs. approximations

- Examples
  - N-Queens / Knapsack / Traveling salesman

Bioinformatics

- Applications in molecular biology

- Dealing with sequences (DNA or proteins)
  - Storing
  - Mapping and analyzing
  - Aligning

Algorithm Design Techniques

- Design techniques / strategies / paradigms

- General approaches to problem solving

- Apply to
  - Various problem types
  - Different application areas

Algorithm Design Techniques

- Brute-Force
- Divide-and-Conquer
- Decrease-and-Conquer
- Transform-and-Conquer
- Dynamic Programming
- Greedy Algorithms

- Examples of algorithms ?

- What about problems / instances that cannot be solved within a reasonable amount of time ?

Brute-Force

- Direct approaches
  - Selection sort
  - Sequential search
  - …

- Exhaustive search
  - Problem instances of small (?!) size
  - Traveling salesman
  - Knapsack
  - …

Divide-and-Conquer

- Recursive decomposition into “smaller” prob. instances
- Solve them all !

- Sorting
  - Mergesort
  - Quicksort

- Multiplication
  - Multiplying large integers
  - Strassen matrix multiplication

- …

Decrease-and-Conquer

- Successive decomposition into a “smaller” problem instance

- How small is it?
  - Decrease-by-one
  - Decrease by a constant factor
  - Variable-size decrease

- Examples
  - Binary search
  - Interpolation search
  - Fake-coin problem

Transform-and-Conquer

- Solve a different problem and get the desired result
  - Problem reduction
- Sometimes, perform some kind of pre-processing on the data
- Examples
  - Searching on ordered and balanced trees
    - AVL and 2-3 trees
  - Heapsort

Dynamic Programming

- Decomposition into overlapping (smaller !) sub-problems
  - Avoid solving them all !!
  - Proceed bottom-up
  - Store results and use them !!

- Simple examples
  - Computing Fibonacci numbers
  - Computing binomial coefficients
  - …

- Other
  - Graphs: Warshall alg.; Floyd alg; etc.
  - Knapsack

Greedy Algorithms

- Construct a solution through a sequence of steps
  - Expand a partially constructed solution

- The choice made at each step is
  - Feasible : satisfies constraints
  - Locally optimal : best choice at each step
  - Irrevocable

- Examples
  - Coin-changing problem
  - Graphs
    - Dijkstra’s shortest-path algorithm
    - Prim’s minimum-spanning tree algorithm
    - Kruskal’s minimum-spanning tree algorithm

Limitations of Algorithmic Power

- How to cope?

- Backtracking
  - N-Queens problem
  - …

- Branch-and-Bound
  - Assignment problem
  - Knapsack problem
  - TSP
  - …

- Approximation algorithms for NP-hard problems
  - Knapsack problem
  - TSP
  - ...

Fundamental Data Structures

- Algorithms operate on data !

- How to organize and store related data items?
  - Data structures (DS)

- Which operations should be provided?
  - Abstract data types (ADT) or classes (in OO languages)

- How to choose?
  - Identify the most common operations on the data
  - Identify the needs of particular algorithms

- Different algorithms for the same problem often require different data structures
  - Efficiency !!

Fundamental Data Structures

- Arrays
  - 1D, 2D, …

- Linked Lists
  - Single pointer vs. two pointers per node
  - List of lists
  - …

- Trees
  - Binary tree
  - Quaternary tree
  - …

Common Abstract Data Types

- Stack
- Queue
- Priority Queue
- Ordered List
- Binary Search Tree
- …
- Graph / Network
- …

Algorithm Efficiency

- Analyze algorithm efficiency
  - Running time ?
  - Memory space ?

- Time
  - How fast does an algorithm run?

- Space
  - Does an algorithm require additional memory?

Efficiency Analysis

- How fast does an algorithm run ?
  - Most algorithms run longer on larger inputs !
- How to relate running time to input size ?
- How to rank / compare algorithms ?
  - If there is more than one available…
- How to estimate running time for larger problem instances ?

Running Time vs. Operations Count

- Running time is not (very) useful for comparing algorithms
  - Speed of particular computers
  - Chosen computer language
  - Quality of programming implementation
  - Compiler optimizations

- Evaluate efficiency in an independent way
  - Count the "basic operations" !!
    - Contribute the most to overall running time

Input Size

- Relate operations count / running time to input size !!
  - Number of array / matrix / list elements
  - …

- Relate size metric to the main operations of an algorithm
  - Working with individual chars vs. with words
  - Number of bits in binary rep., when checking if n is prime
  - …

Formal Analysis – Pencil and paper

- Understand algorithm behavior
  - Count arithmetic operations / comparisons
  - Find a closed formula !!
  - Identify best, worst and average case situations, if that is the case
- Iterative algorithms
  - Loops : how many iterations ?
  - Set a sum for the basic operation counts
- Recursive algorithms
  - How many recursive calls ?
  - Establish and solve appropriate recurrences

Worst, Best and Average Cases

- Running time depends on input size
- BUT, for some algorithms, it might also depend on particular data configurations !!

- Sequential search on a n-element array
  - Non-ordered array ?
  - Ordered array ?
  - Increasing vs. decreasing order
  - Probability of a successful search ?

Worst, Best and Average Cases

- Worst case : W(n)
  - Input(s) of size n for which an algorithm runs longest
  - Upper bound for operations count
- Best case : B(n)
  - Input(s) of size n for which an algorithm runs fastest
  - Lower bound for operations count
  - Not very useful…
- Average case : A(n)
  - Behavior for "typical" or "random" inputs
  - Establish assumptions about possible inputs of size n
  - For some algorithms, much better than worst case !!

Growth Rate

- Identify algorithm efficiency for large input sizes
- How fast does the running time (i.e., number of operations) of an algorithm grow, when input size becomes (much) larger ?
- What happens when the input size
  - doubles ?
  - increases ten-fold ?
  - …
- How to represent such growth rate?

Orders of Growth

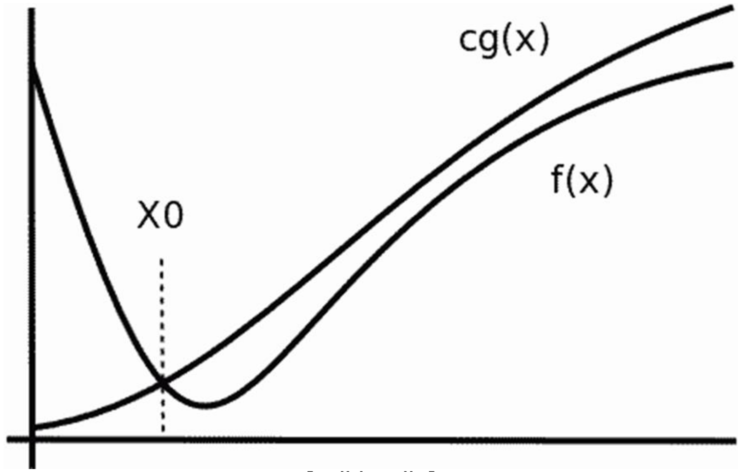
- Approximate values for some common functions

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10	3.3×10^1	10^2	10^3	10^3	3.6×10^6
10^2	6.6	10^2	6.6×10^2	10^4	10^6	1.3×10^{30}	9.3×10^{157}
10^3	10	10^3	10^4	10^6	10^9	?	?
10^4	13	10^4	1.3×10^5	10^8	10^{12}	?	?
10^5	17	10^5	1.7×10^6	10^{10}	10^{15}	?	?
10^6	20	10^6	2.0×10^7	10^{12}	10^{18}	?	?

Asymptotic Notations

- Order of growth of operations count indicates efficiency
- How to compare / rank algorithms for the same problem?
  - Compare their orders of growth !!
- Useful notations: $O(n)$, $\Omega(n)$, $\Theta(n)$

Big-Oh Notation

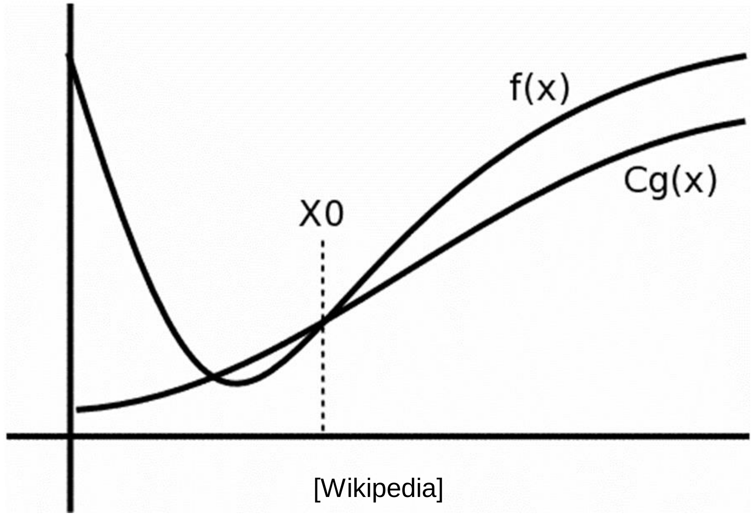


[Wikipedia]

- Asymptotic upper bound

- O(g(n)) : set of all functions with smaller or same order of growth as g(n)
  - $t(n) \leq c\ g(n)$, for all $n \geq n_0$, positive constant c
  - t(n), g(n) : non-negative functions on the set of natural numbers

Big-Omega Notation

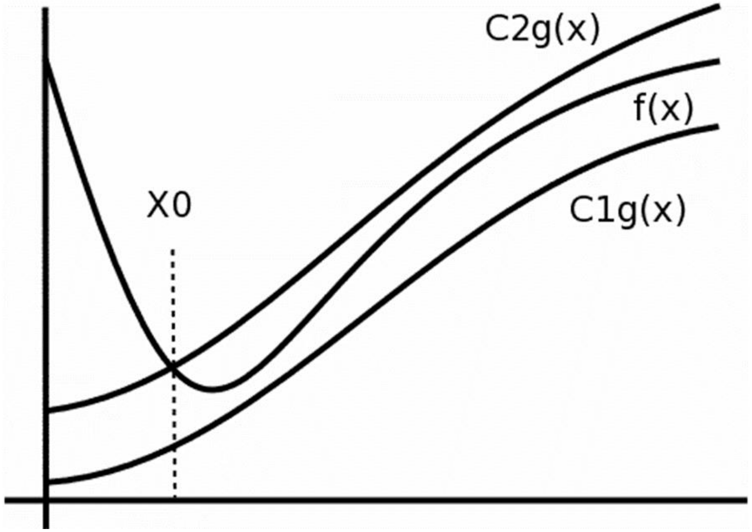


- Asymptotic lower bound

- Ω(g(n)) : set of all functions with larger or same order of growth as g(n)

  - t(n) ≥ c g(n), for all n ≥ n_0, positive constant c

Big-Theta Notation



[Wikipedia]

- Asymptotic tight bound

- $\Theta(g(n))$: set of all functions with the same order of growth as g(n)

  - $c_1\, g(n) \leq t(n) \leq c_2\, g(n)$, for all $n \geq n_0$, positive constants c_1 , c_2

  - t(n) in $O(g(n))$ and t(n) in $\Omega(g(n))$

Asymptotic Notation

- Hide unimportant details about how fast a function grows
  - Forget constants and lower-order terms

- $T_1(n) = 2\ n^2 + 3000\ n + 5$
- $T_2(n) = 10\ n^2 + 100\ n - 23$
- For large values of n, $T_2(n)$ grows faster than $T_1(n)$
- BUT, both grow quadratically : $\Theta(n^2)$

Asymptotic Notation – Example

- $T(n) = 10\ n^2 + 100\ n - 23$

$T(n) = O(n^2)$ $T(n) = O(n^3)$ $T(n) \neq O(n)$

$T(n) = \Omega(n^2)$ $T(n) \neq \Omega(n^3)$ $T(n) = \Omega(n)$

$T(n) = \Theta(n^2)$ $T(n) \neq \Theta(n^3)$ $T(n) \neq \Theta(n)$

Efficiency Classes

- O(1) : constant
  - Which algorithms?
- O(log n) : logarithmic
  - E.g., decrease-and-conquer
- O(n) : linear
  - Processing all elements of an array, list, etc.
- O(n log n) : n-log-n
  - E.g., divide-and-conquer

Efficiency Classes

- $O(n^k)$: polynomial (quadratic, cubic, etc.)
  - k nested loops
- $O(2^n)$: exponential
  - Generating all subsets of an n-element set
- $O(n!)$: factorial
  - Generating all permutations of an n-element set

Empirical Analysis

- Run the algorithm on a sample of test inputs
  - Input data should represent all possible cases
  - Input data should encompass large (set) sizes
  - Pseudo-random data
- Record and analyze – Tables
  - operation counts
  - running times (?)
- Identify best, worst and average case behavior
  - If that is the case…
- Identify complexity classes

Example – Table of operations count

n	1	2	4	8	16	32	64	128	256
$M(n)$	1	3	10	36	136	528	2080	8256	32896

- M(n) : the number of operations carried out
- Complexity order ?
- Closed formula for the number of operations ?

Another table of operations count

n	1	2	3	4	5	6	7	8	9	10
$M(n)$	1	3	7	15	31	63	127	255	511	1023

- M(n) : the number of operations carried out
- Complexity order ?
- Closed formula for the number of operations ?

Empirical Analysis

- Problems
  - Inadequate sample input data
    - Size? Configurations?
  - Dependence of running times
- Advantages
  - Avoid difficult formal analysis
  - Allow predicting the running time for different input data sets
    - Interpolation and extrapolation (?)
- BUT, some problems / instances cannot be solved quickly enough…

Brute-Force

- The (most) straightforward approach to solving a problem

- Directly based on
  - The problem statement
  - The definitions involved

- Strengths
  - Simplicity
  - Applicable to different kinds of problems

- Weaknesses
  - (Very!) Low efficiency in some cases
  - Useful only for instances of (relatively) small size !!

Brute-Force

- Where to apply?

- Numerical problems, searching, sorting, etc.
  - Acceptable efficiency
  - Can be used for large problem instances

- Combinatorial problems
  - Exhaustive search
  - Set of candidate solutions grows very fast
  - Used only for reduced size instances

Brute-Force

- How many examples do you know?

- Add n numbers
- Direct matrix multiplication
- Sequential search
- Selection sort
- Bubble sort
- …

Brute-Force – Tasks

- Compute b^n, with $n \geq 0$, using

  $b^n = b \times b \times \ldots \times b$ $\qquad$ $b^n = b \times b^{n-1}$

- Base cases for the recursion ?
- Number of multiplications ?
  - Formal + Empirical analysis
- Any gains from the recursive approach ?

Divide-And-Conquer

- The best known algorithm design technique

- General framework
  - Divide a problem instance into (two or more) similar, smaller instances
  - The smaller instances are solved recursively
  - Solutions for smaller instances are combined to get the solution of the original problem, if needed

Divide-And-Conquer

- In each subdivision step, the smaller instances should have approx. the same size !
  - This might not happen, for some particular instances

- All smaller problem instances have to be solved !!
  - Usually two new smaller instances, at each step

- When do we stop the subdivision process ?
  - Base cases ? Just one or more ?
  - Smaller instances might be solved by another algorithm

Divide-And-Conquer

- This recursive strategy can be implemented
  - Using recursive functions / procedures (obvious solution !)
  - Iteratively, using a stack, queue, etc.
    - Choose which sub-problem to solve next !!

- Problems ?
  - Recursion is slow !
    - Identify all possible base cases
    - Solve small instances using other algorithms
  - Not the best approach for simple problems !
    - E.g., adding N numbers
  - Sub-problems might overlap !
    - Reuse previous results / solutions !

Divide-And-Conquer – Tasks

- Compute b^n, with $n \geq 0$, using

$$b^n = b^{n \text{ div } 2} \times b^{(n+1) \text{ div } 2}$$

- Base cases ?
- Always use two recursive calls !!
- Number of multiplications ?
  - Formal + Empirical analysis

Decrease-And-Conquer

- Exploit the relationship between
  - A solution to a given problem instance
  - A solution to a smaller instance of the same problem

- General framework (Top-Down)
  - Identify ONE similar and smaller problem instance
  - The smaller instance is solved recursively
  - Solutions for smaller instances are processed to get the solution of the original problem, if needed

- Compare with Divide-and-Conquer !!

Decrease-And-Conquer – Tasks

- Compute b^n, with $n \geq 0$, using

$$b^n = b^{n \text{ div } 2} \times b^{n \text{ div } 2} \text{ , if } n \text{ is even}$$

$$b^n = b \times b^{(n-1) \text{ div } 2} \times b^{(n-1) \text{ div } 2} \text{ , if } n \text{ is odd}$$

- Base cases ?
- Use just ONE recursive call !!
- Number of multiplications ?
  - Formal + Empirical analysis

Decrease-And-Conquer – Extra-Task

- Compute b^n, with $n \geq 0$, using

$$b^n = b^{n \text{ div } 2} \times b^{n \text{ div } 2} \text{ , if } n \text{ is even}$$

$$b^n = b \times b^{(n-1) \text{ div } 2} \times b^{(n-1) \text{ div } 2} \text{ , if } n \text{ is odd}$$

- Develop an iterative version !!
- It should have the same behavior as the recursive version
  - Same algorithm, but a different implementation

a^b – Brute-Force – Iterative algorithm

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a \times \ldots \times a$$

- Number of multiplications ?
  - Formal + Empirical analysis

a^b – Brute-Force – Iterative algorithm

```python
def powerIterV1( a, b ) :

    """ Computing a**b using a loop """

    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"

    assert (a != 0) or (b != 0), "Cannot compute 0**0 !"

    res = 1

    for i in range( 1, b + 1 ) :

        res *= a

    return res
```

- Number of multiplications ?

a^b – Brute-Force – Iterative algorithm

n	2**n	#Mults
0	1	0
1	2	1
2	4	2
3	8	3
4	16	4
5	32	5
6	64	6
7	128	7
8	256	8
9	512	9
10	1024	10
11	2048	11

a^b – Brute-Force – Recursive alg.

- Compute a^b, with $b \geq 0$, using

$$a^b = a \times a^{b-1}, \text{ with } a^0 = 1$$

- Number of multiplications ?
  - Formal + Empirical analysis

- Any gains ?

a^b – Brute-Force – Recursive alg.

```
def powerRecV1( a, b ) :
    """ Computing a**b recursively --- Direct algorithm """
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    assert (a != 0) or (b != 0), "Cannot compute 0**0 !"
    if b == 0 :
        return 1
    return a * powerRecV1( a, b - 1 )
```

- Number of multiplications ?

a^b – Brute-Force – Recursive alg.

```
  n     2**n    #Mults
----------------------
   0        1     0
   1        2     1
   2        4     2
   3        8     3
   4       16     4
   5       32     5
   6       64     6
   7      128     7
   8      256     8
   9      512     9
  10     1024    10
  11     2048    11
```

a^b – Divide-And-Conquer

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{(b+1) \text{ div } 2}$$

- Base cases ?
- Always use two recursive calls !!

- Number of multiplications ?
  - Formal + Empirical analysis
- Is it better than the direct algorithm ?

a^b – Divide-And-Conquer

```
def powerRecV3( a, b ) :
    """ Computing a**b recursively --- Blind Div & Conq strategy"""
    # TWO base cases are needed !!
    # Otherwise, we would not stop when b == 1 !!
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    if b == 0 :
        return 1
    if b == 1 :
        return a
    return powerRecV3( a, b // 2 ) * powerRecV3( a, (b + 1) // 2 )
```

- Number of multiplications ?

Formal analysis

$$M(n) = M(n \text{ div } 2) + M((n+1) \text{ div } 2) + 1$$

- Easier to solve if n is a power of 2
  - $n = 2^k$, $k = \log_2 n$

$$M(n) = M(n / 2) + M(n / 2) + 1$$
$$= 2\,M(n / 2) + 1 = \ldots$$

- Closed formula ? Complexity order ?

a^b – Divide-And-Conquer

n	2**n	#Mults
0	1	0
1	2	0
2	4	1
3	8	2
4	16	3
5	32	4
6	64	5
7	128	6
8	256	7
9	512	8
10	1024	9
11	2048	10

a^b – Decrease-And-Conquer

- Compute a^b, with $b \geq 0$, using

$$a^b = a^{b \text{ div } 2} \times a^{b \text{ div } 2} \text{, if } b \text{ is even}$$

$$a^b = a \times a^{(b-1) \text{ div } 2} \times a^{(b-1) \text{ div } 2} \text{, if } b \text{ is odd}$$

- Base cases ?
- Use just ONE recursive call !!

- Number of multiplications ?
  - Formal + Empirical analysis

a^b – Decrease-And-Conquer

```
def powerRecV6( a, b ) :
    """ Computing a**b recursively --- Smart Dec & Conq strategy"""
    assert (type( b ) == int) and (b >= 0), "Wrong exponent!"
    if b == 0 :
        return 1
    p = powerRecV6( a, b // 2 )
    if (b % 2) == 0 :
        return p * p
    return a * p * p
```

- Number of multiplications ?

Formal analysis

$$M(n) = M(n \text{ div } 2) + 1 \text{ , if n is even}$$

$$M(n) = M((n-1) \text{ div } 2) + 2 \text{ , if n is odd}$$

- Check some examples with pencil and paper
  - Do you understand what is happening?
  - Best vs. worst cases ?
- Closed formula ? Complexity order ?
- Is it better than the previous algorithms ?

a^b – Decrease-And-Conquer

n	2**n	#Mults
0	1	0
1	2	2
2	4	3
3	8	4
4	16	4
5	32	5
6	64	5
7	128	6
8	256	5
9	512	6
10	1024	6
11	2048	7
12	4096	6
13	8192	7
14	16384	7
15	32768	8
16	65536	6
17	131072	7

Decrease-And-Conquer – Extra-Task

- Compute b^n, with $n \geq 0$, using

$$b^n = b^{n \text{ div } 2} \times b^{n \text{ div } 2} \text{ , if } n \text{ is even}$$

$$b^n = b \times b^{(n-1) \text{ div } 2} \times b^{(n-1) \text{ div } 2} \text{ , if } n \text{ is odd}$$

- Develop an iterative version !!

- It should have the same behavior as the recursive version
  - Same algorithm, but a different implementation

New Task – Counting

- Given an array with non-negative integer values

- Count the number of even-valued elements

- Implement the 3 strategies:
  - Brute-Force / Div & C / Dec & C

- Formal + Empirical analysis : Comparisons

New Task – Sequential Search

- Given an array with non-negative integer values
- Use the iterative Sequential Search algorithm to look for a given value
- Formal + Empirical analysis : Comparisons
- Best / Worst / Average Cases ?

References

- A. Levitin, *Introduction to the Design and Analysis of Algorithms,* 3rd Ed., Pearson, 2012
  - Chapter 1 + Chapter 2
- D. Vrajitoru and W. Knight, *Practical Analysis of Algorithms*, Springer, 2014
  - Chapter 1 + Chapter 3 + Chapter 5
- T. H. Cormen et al., *Introduction to Algorithms*, 3rd Ed., MIT Press, 2009
  - Chapter 1 + Chapter 2 + Chapter 3