Linear Block Codes

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Binary Block Codes

Binary Block Code

Let $\mathbb{F}_2$ be the set $\{0, 1\}$.

Definition

An (n, k) binary block code is a subset of $\mathbb{F}_2^n$ containing 2^k elements

Example

$n = 3$, $k = 1$, $C = \{000, 111\}$

Example

$n \geq 2$, C = Set of vectors of even Hamming weight in $\mathbb{F}_2^n$,
$k = n - 1$
$n = 3$, $k = 2$, $C = \{000, 011, 101, 110\}$
This code is called the single parity check code

Encoding Binary Block Codes

The encoder maps k-bit information blocks to codewords.

Definition

An encoder for an (n, k) binary block code C is an injective function from $\mathbb{F}_2^k$ to C

Example (3-Repetition Code)

$0 \rightarrow 000, 1 \rightarrow 111$
or
$1 \rightarrow 000, 0 \rightarrow 111$

Decoding Binary Block Codes

The decoder maps n-bit received blocks to codewords

Definition

A decoder for an (n, k) binary block code is a function from $\mathbb{F}_2^n$ to C

Example (3-Repetition Code)

$n = 3$, $C = \{000, 111\}$

$$
\begin{array}{ll}
000 \to 000 & 111 \to 111 \\
001 \to 000 & 110 \to 111 \\
010 \to 000 & 101 \to 111 \\
100 \to 000 & 011 \to 111
\end{array}
$$

Since encoding is injective, information bits can be recovered as $000 \to 0, 111 \to 1$

Optimal Decoder for Binary Block Codes

- Optimality criterion: Maximum probability of correct decision
- Let $\mathbf{x} \in C$ be the transmitted codeword
- Let $\mathbf{y} \in \mathbb{F}_2^n$ be the received vector
- Maximum a posteriori (MAP) decoder is optimal

$$\hat{\mathbf{x}}_{MAP} = \text{argmax}_{\mathbf{x} \in C} \Pr(\mathbf{x}|\mathbf{y})$$

- If all codewords are equally likely to be transmitted, then maximum likelihood (ML) decoder is optimal

$$\hat{\mathbf{x}}_{ML} = \text{argmax}_{\mathbf{x} \in C} \Pr(\mathbf{y}|\mathbf{x})$$

- Over a BSC with $p < \frac{1}{2}$, the minimum distance decoder is optimal if the codewords are equally likely

$$\hat{\mathbf{x}} = \text{argmin}_{\mathbf{x} \in C} d(\mathbf{x}, \mathbf{y})$$

Error Correction Capability of Binary Block Codes

Definition

The minimum distance of a block code C is defined as

$$d_{min} = \min_{\mathbf{x},\mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} d(\mathbf{x}, \mathbf{y})$$

Example (3-Repetition Code)

$C = \{000, 111\}$, $d_{min} = 3$

Example (Single Parity Check Code)

C = Set of vectors of even weight in $\mathbb{F}_2^n$, $d_{min} = 2$

Theorem

For a binary block code with minimum distance d_{min}, the minimum distance decoder can correct upto $\lfloor \frac{d_{min}-1}{2} \rfloor$ errors.

Complexity of Encoding and Decoding

Encoder

- Map from $\mathbb{F}_2^k$ to C
- Worst case storage requirement = $O(n2^k)$

Decoder

- Map from $\mathbb{F}_2^n$ to C
- $\hat{\mathbf{x}}_{ML} = \operatorname{argmax}_{\mathbf{x} \in C} \Pr(\mathbf{y}|\mathbf{x})$
- Worst case storage requirement = $O(n2^k)$
- Time complexity = $O(n2^k)$

Need more structure to reduce complexity

Binary Linear Block Codes

Vector Spaces over $\mathbb{F}_2$

- Define the following operations on $\mathbb{F}_2$
- Addition $+$
  - $0 + 0 = 0$
  - $0 + 1 = 1$
  - $1 + 0 = 1$
  - $1 + 1 = 0$
- Multiplication $\times$
  - $0 \times 0 = 0$
  - $0 \times 1 = 0$
  - $1 \times 0 = 0$
  - $1 \times 1 = 1$
- $\mathbb{F}_2$ is also represented as GF(2)

Fact

The set $\mathbb{F}_2^n$ is a vector space over $\mathbb{F}_2$

Binary Linear Block Code

Definition

An (n, k) binary linear block code is a k-dimensional subspace of $\mathbb{F}_2^n$

Theorem

Let S be a nonempty subset of $\mathbb{F}_2^n$. Then S is a subspace of $\mathbb{F}_2^n$ if $\mathbf{u} + \mathbf{v} \in S$ for any two $\mathbf{u}$ and $\mathbf{v}$ in S.

Example (3-Repetition Code)

$C = \{000, 111\} \neq \phi$

$000 + 000 = 000$, $000 + 111 = 111$, $111 + 111 = 000$

Example (Single Parity Check Code)

C = Set of vectors of even weight in $\mathbb{F}_2^n$

$\mathrm{wt}(\mathbf{u} + \mathbf{v}) = \mathrm{wt}(\mathbf{u}) + \mathrm{wt}(\mathbf{v}) - 2\,\mathrm{wt}(\mathbf{u} \cap \mathbf{v})$

Encoding Binary Linear Block Codes

Definition

A generator matrix for a k-dimensional binary linear block code C is a $k \times n$ matrix $\mathbf{G}$ whose rows form a basis for C.

Linear Block Code Encoder

Let $\mathbf{u}$ be a $1 \times k$ binary vector of information bits. The corresponding codeword is

$$\mathbf{v} = \mathbf{u}\mathbf{G}$$

Example (3-Repetition Code)

$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Encoding Binary Linear Block Codes

Example (Single Parity Check Code)

$n = 3$, $k = 2$, $C = \{000, 011, 101, 110\}$

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Encoding Complexity of Binary Linear Block Codes

- Need to store $\mathbf{G}$
- Storage requirement = $O(nk) \ll O(n2^k)$
- Time complexity = $O(nk)$
- Complexity can be reduced further by imposing more structure in addition to linearity
- Decoding complexity? What is the optimal decoder?

Decoding Binary Linear Block Codes

- Codewords are equally likely $\Rightarrow$ ML decoder is optimal

$$\hat{\mathbf{x}}_{ML} = \text{argmax}_{\mathbf{x} \in C} \Pr(\mathbf{y}|\mathbf{x})$$

- Equally likely codewords and channel is BSC $\Rightarrow$ Minimum distance decoder is optimal

$$\hat{\mathbf{x}}_{ML} = \text{argmin}_{\mathbf{x} \in C} d(\mathbf{x}, \mathbf{y})$$

- To exploit linear structure to reduce decoding complexity, we need to study the dual code

Inner Product of Vectors in $\mathbb{F}_2^n$

Definition

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ belong to $\mathbb{F}_2^n$. The inner product of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i$$

$\mathbf{u} \cdot \mathbf{v} = 0 \Rightarrow \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Examples

- $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0$
- $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 1 \cdot 0 + 1 \cdot 1 + 0 \cdot 1 = 1$
- $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$
- $\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 0$
  Nonzero vectors can be self-orthogonal

Dual Code of a Linear Block Code

Definition

Let C be an (n, k) binary linear block code. Let $C^\perp$ be the set of vectors in $\mathbb{F}_2^n$ which are orthogonal to all the codewords in C.

$$C^\perp = \left\{ \mathbf{u} \in \mathbb{F}_2^n \middle| \ \mathbf{u} \cdot \mathbf{v} = 0 \ \text{ for all } \ \mathbf{v} \in C \right\}$$

$C^\perp$ is a linear block code and is called the dual code of C.

Example (3-Repetition Code)

$C = \{000, 111\}$, $C^\perp = ?$

$$\begin{array}{ll} 000 \cdot 111 = 0 & 111 \cdot 111 = 1 \\ 001 \cdot 111 = 1 & 110 \cdot 111 = 0 \\ 010 \cdot 111 = 1 & 101 \cdot 111 = 0 \\ 100 \cdot 111 = 1 & 011 \cdot 111 = 0 \end{array}$$

$C^\perp = \{000, 011, 101, 110\}$ = Single Parity Check Code

Dimension of the Dual Code

Example (3-Repetition Code and SPC Code)

$C = \{000, 111\}$, $\dim C = 1$
$C^{\perp} = \{000, 011, 101, 110\}$, $\dim C^{\perp} = 2$
$\dim C + \dim C^{\perp} = 1 + 2 = 3$

Theorem

$\dim C + \dim C^{\perp} = n$

Corollary

C is an (n, k) binary linear block code $\Rightarrow$ $C^{\perp}$ is an $(n, n-k)$ binary linear block code

Parity Check Matrix of a Code

Definition

Let C be an (n, k) binary linear block code and let $C^{\perp}$ be its dual code. A generator matrix $\mathbf{H}$ for $C^{\perp}$ is called a parity check matrix for C.

Example (3-Repetition Code)

$C = \{000, 111\}$
$C^{\perp} = \{000, 011, 101, 110\}$

A generator matrix of $C^{\perp}$ is $\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\mathbf{H}$ is a parity check matrix of C.

Parity Check Matrix Completely Describes a Code

Theorem

Let C be a linear block code with parity check matrix **H**. *Then*

$$\mathbf{v} \in C \iff \mathbf{v} \cdot \mathbf{H}^T = \mathbf{0}$$

Example (3-Repetition Code)

$C = \{000, 111\}$, $\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Forward direction: $\mathbf{v} \in C \Rightarrow \mathbf{v} \cdot \mathbf{H}^T = \mathbf{0}$

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Parity Check Matrix Completely Describes a Code

Theorem

Let C be a linear block code with parity check matrix $\mathbf{H}$. Then

$$\mathbf{v} \in C \iff \mathbf{v} \cdot \mathbf{H}^T = \mathbf{0}$$

Example (3-Repetition Code)

$C = \{000, 111\}$, $\mathbf{H} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Reverse direction: $\mathbf{v} \in C \Leftarrow \mathbf{v} \cdot \mathbf{H}^T = \mathbf{0}$

$$\mathbf{v} \cdot \mathbf{H}^T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} v_1 + v_3 & v_2 + v_3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{v} \cdot \mathbf{H}^T = \mathbf{0} \quad &\Rightarrow \quad v_1 + v_3 = 0, v_2 + v_3 = 0 \\ &\Rightarrow \quad v_1 = v_3, v_2 = v_3 \Rightarrow v_1 = v_2 = v_3 \end{aligned}$$

Decoding Binary Linear Block Codes

- Let a codeword $\mathbf{x}$ be sent through a BSC to get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{x} + \mathbf{e}$$

  where $\mathbf{e}$ is the error vector
- The probability of observing $\mathbf{y}$ given $\mathbf{x}$ was transmitted is given by

$$\begin{aligned} \Pr(\mathbf{y}|\mathbf{x}) &= p^{d(\mathbf{x},\mathbf{y})}(1-p)^{n-d(\mathbf{x},\mathbf{y})} \\ &= p^{\mathrm{wt}(\mathbf{e})}(1-p)^{n-\mathrm{wt}(\mathbf{e})} \\ &= (1-p)^n \left(\frac{p}{1-p}\right)^{\mathrm{wt}(\mathbf{e})} \end{aligned}$$

- If $p < \frac{1}{2}$, lower weight error vectors are more likely

Decoding Binary Linear Block Codes

- Optimal decoder is given by

$$\begin{aligned}\hat{\mathbf{x}}_{ML} &= \text{argmin}_{\mathbf{x}\in C} d(\mathbf{x},\mathbf{y}) \\ &= \mathbf{y} + \hat{\mathbf{e}}_{ML}\end{aligned}$$

  where $\hat{\mathbf{e}}_{ML}$ = Most likely error vector such that $\mathbf{y} + \mathbf{e} \in C$.
- $\mathbf{y} + \mathbf{e} \in C \iff (\mathbf{y}+\mathbf{e}) \cdot \mathbf{H}^T = \mathbf{0} \iff \mathbf{e} \cdot \mathbf{H}^T = \mathbf{y} \cdot \mathbf{H}^T$
- If $\mathbf{s} = \mathbf{y} \cdot \mathbf{H}^T$, the most likely error vector is

$$\hat{\mathbf{e}}_{ML} = \underset{\mathbf{e}\in\mathbb{F}_2^n, \mathbf{e}\cdot\mathbf{H}^T=\mathbf{s}}{\text{argmin}} \; \text{wt}(\mathbf{e})$$

- Time complexity = $O\left(p(n)2^k\right)$ where p is a polynomial
- For each $\mathbf{s}$, the $\hat{\mathbf{e}}_{ML}$ can be precomputed and stored
- $\mathbf{s}$ is $1 \times n-k$ binary vector $\Rightarrow$ Storage required is $O(n2^{n-k})$

Summary

Complexity Comparison

General Block Codes

- Encoding = $O(n2^k)$
- Decoding = $O(n2^k)$

Linear Block Codes

- Encoding = $O(nk)$
- Decoding = $O(p(n)2^{\min(k,n-k)})$

Observations

- Linear structure in codes reduces encoding complexity
- Decoding complexity is still exponential
- Need for codes with low complexity decoders

Questions? Takeaways?