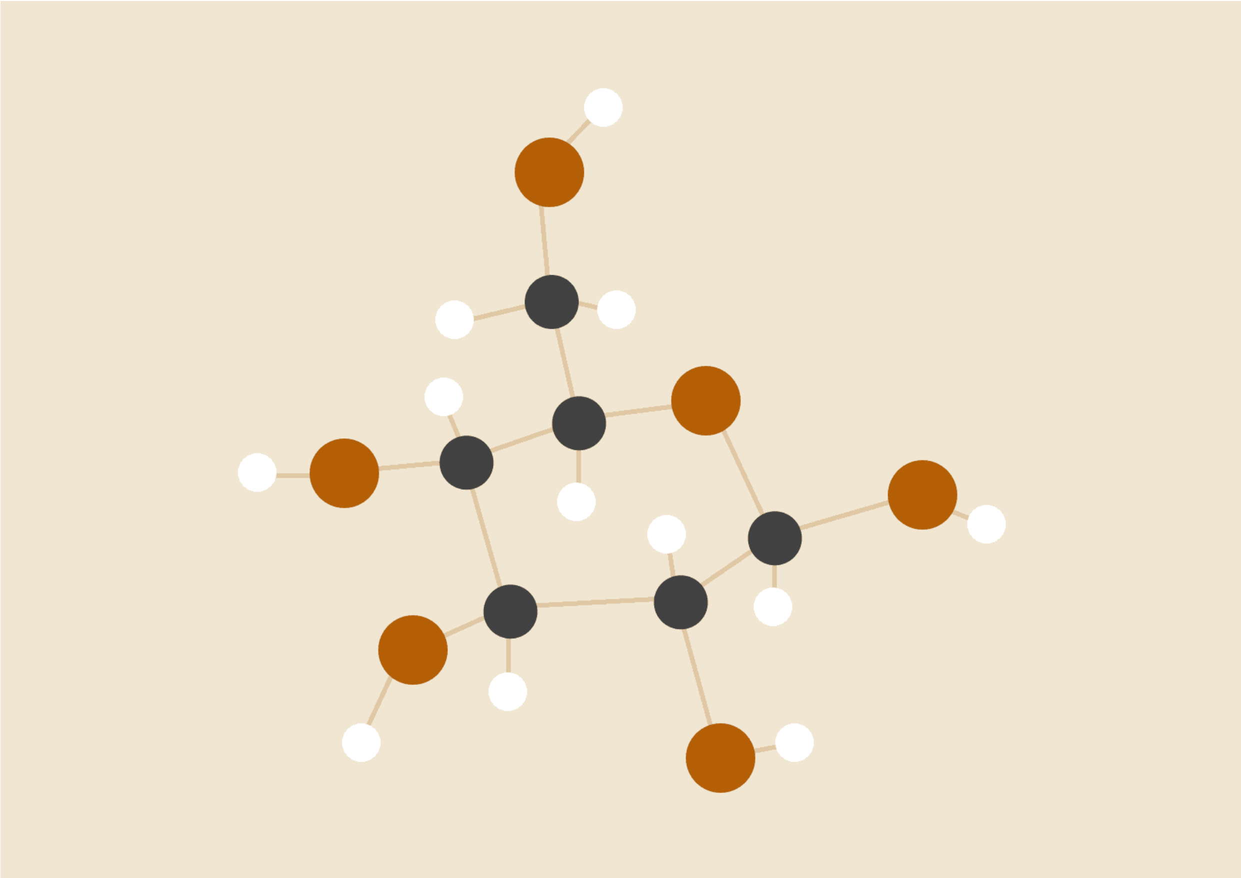
SOLVING HARMONIC OSCILLATOR

Using Suzuki-Trotter Decomposition to Simulate Time Evolution



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FROM THE PAPER

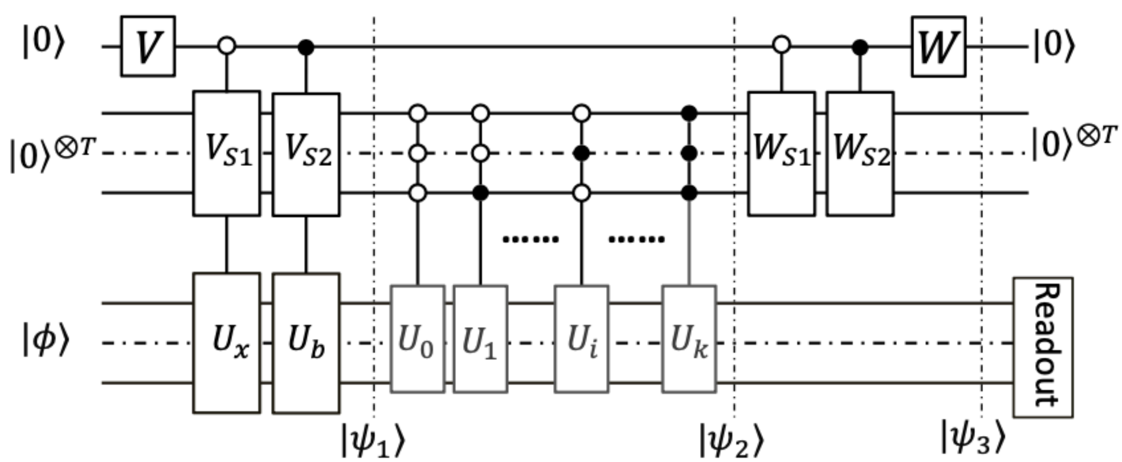
The paper given shows the algorithm of solving a linear differential equation in the general form:

$\frac{dx(t)}{dt} = Mx(t) + b$

The solution to this involves the matrix exponential e^{Mt}. In the paper, this is approximated by using a Taylor Series expansion. This algorithm constructs a circuit that simulates the power of matrix M.

The vector can be represented using a series of quantum states(qubits), and the matrix M can be represented by a unitary operator, which gives us insight on building a time evolution operator.

DEEP DIVE INTO THE CIRCUIT



The initial state $x(0)$ and b are stored in the first register qubit, a series of ancilla qubits are stored in the second register, lastly work qubit is represented by $|\phi>$. The first controlled operation U_x and U_b entangle the first register qubit with the first ancilla qubit, this controls whether the work qubit evolve according to $x(0)$ or b. The Taylor coefficients of $x(0)$ are stored in matrix V_{S1} and the coefficients of b are stored in matrix

V_{S2}. After operation U_x and U_b, the zeroth order terms of the Taylor Series are introduced. The subsequent controlled operation entangles the first qubit with the following ancilla qubits one by one, introducing higher order Taylor terms one by one. At the end the entire Taylor series is represented with a global superposition(Eq.7)

APPLY TO HO

The given Harmonic Oscillator:

$\frac{d^2y}{dt^2} + \omega^2 y = 0,\ \omega = 1$

Boundary condition: $y(0) = 1,\ \frac{dy}{dt}|_{t=0} = 1.$

This equation can be written as:

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

The system evolve based on e^{-iHt}, the Hamiltonian can be expressed as:

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

SUZUKI-TROTTER DECOMPOSITION

Suzuki-Trotter Decomposition can be written as:

$U(t) \approx (e^{-i\widehat{V}dt}e^{-i\widehat{T}dt})^n,\ n = time\ steps,\ dt = \frac{t}{n}.$

Potential energy is corresponding to position basis, we try to represent this by Pauli-Z matrices:

$V = \frac{\omega}{2}(Z_1 + Z_2)$

Kinetic energy is corresponding to momentum basis, we try to represent this by Pauli-X matrices:

$T = \frac{1}{2}(X_1 + X_2)$

BURNING QUESTIONS

This challenge provides a great opportunity for us to dive deep into multiple algorithms and try to understand them from first principle. Although the more we explore, the more questions we encounter:

1. Do we need to apply QFT for this problem?
2. The algorithm shown in the paper used Taylor expansion, is Trotterization a better way to do approximation
3. If operator T and V do commute with each other, does trotterization provide an exact solution?
4. What does “a qubit evolve in time” really means?