Computers and Mathematics with Applications 73 (2017) 374–384

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# An improved non-traditional finite element formulation for solving three-dimensional elliptic interface problems

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ARTICLE INFO

*Article history:*
Received 8 April 2016
Received in revised form 22 August 2016
Accepted 24 November 2016
Available online 15 December 2016

*Keywords:*
Non-traditional finite element method
Elliptic equation
Three-Dimensional interface problems
Jump condition
Matrix coefficient

ABSTRACT

Solving elliptic equations with interfaces has wide applications in engineering and science. The real world problems are mostly in three dimensions, while an efficient and accurate solver is a challenge. Some existing methods that work well in two dimensions are too complicated to be generalized to three dimensions. Although traditional finite element method using body-fitted grid is well-established, the expensive cost of mesh generation is an issue. In this paper, an efficient non-traditional finite element method with non-body-fitted grids is proposed to solve elliptic interface problems. The special cases when the interface cuts though grid points are handled carefully, rather than perturbing the cutting point away to apply the method for general case. Both Dirichlet and Neumann boundary conditions are considered. Numerical experiments show that this method is approximately second order accurate in the $L^\infty$ norm and $L^2$ norm for piecewise smooth solutions. The large sparse matrix for our linear system also has nice structure and properties.


## 1. Introduction

In recent years, there has been tremendous interest in developing efficient numerical methods for interface problems. In numerical mathematics, ordinary differential equations or partial differential equations are often used to model interface problems involving materials with different properties, such as conductivities, densities or diffusions. Elliptic problem with internal interfaces is the basic form of this kind of problem, the governing equations of which has discontinuous coefficients at interfaces and sometimes singular source term exists. Due to the poor global smoothness of the solution and the irregularity of the interfaces, the solution for the differential equation model is usually not smooth and even discontinuous. In this way, standard finite element or finite difference method are not suitable for interface problems, hence designing highly efficient methods for these problems are desired. To solve this problem, a large number of numerical methods are designed for interface problems, which can be classified into three main categories, including finite difference methods, finite element methods and finite volume methods.

The numerical model of interface problem was first proposed by Peskin [1] in 1977 to study blood flow through heart valves. The immersed boundary method (IBM) [2] is a mathematical formulation and numerical approach to such problems. This method uses a numerical approximation of the delta-function, which smears out the solution on a thin finite band around the interface gama. In the following years, Peskin focused on the improvement of the immersed boundary method

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http://dx.doi.org/10.1016/j.camwa.2016.11.035
0898-1221/

and several adaptive version of the method are introduced [3–6], actual second order convergence rates are obtained by considering the interaction of a viscous incompressible flow and an anisotropic incompressible viscoelastic shell. In 1994, LeVeque and Li [7–9] proposed the immersed interface method (IIM) to solve elliptic equations with discontinuous coefficients and singular source term. This method is based on the finite difference method under the Cartesian grid. Standard finite difference or finite element method is employed away from the interface, while the grid points or elements near the interface are amended by the interface condition. This method is successfully applied to incompressible Stokes equation and Navier–Stokes equations with singular source term. Liu, Fedkiw and Kang proposed the ghost fluid method (GFM) to solve the elliptic equations with interfaces in 2000 [10,11], which is widely used in solving incompressible two-phase or multiphase flow problem, interface tracking problem and irregular domain problems with Dirichlet boundary condition. Wei et al. pays attention to interface problems with geometry singularity since 2007 and developed the second order accurate method called matched interface and boundary method (MIB) [12–14]. This method is successfully applied to biomathematics research on molecular level.

So far, there are mainly two kinds of effective finite element methods, one is fitted interface finite element method, in which meshes are generated along the interface. The other is immersed interface finite element method, in which the mesh generation does not rely on the interface, while the construction of finite element space relies on the jump condition of the interface. As for fitted mesh method, Chen and Zou [15] considered the finite element methods for solving second order elliptic and parabolic interface problems. The fitted methods are computationally costly especially for problems with moving interfaces because repeated grid generation is called for. Examples of immersed interface method are immersed finite element method [16–23], adaptive immersed interface method [24], extended finite element method [25–27]. The penalty finite element method [28,29] modifies the bilinear form near the interface by penalizing the jump of the solution value (with no general flux jump) across the interface. Recently a few unfitted mesh methods based on discontinuous Galerkin method using well known interior penalty technique to deal with jump and flux conditions for elliptic interface problem [30–32].

Also, there has been a large body of work from the finite volume perspective for developing high order methods for elliptic equations in complex domains, such as [33,34] for two-dimensional problems and [35] for three-dimensional problems. In [36], the immersed finite volume element method (IFVE) is developed by combining the finite volume element method and the immersed finite element method.

Non-traditional finite element method has been developed for solving interface problems involving several important types of partial differential equations, such as elliptic equations [37–40], elliptic interface problems with triple junction points [41] and elasticity interface problems [42,43]. The main idea is that the test function and trial function have different basis and the resulting linear system is non-symmetric but positive definite. Although the three dimensional elliptic interface problem was discussed in [39] using this type of formulation, whenever the interface hit grid points, it is perturbed away.

In this paper, we propose an improved numerical method for solving the three-dimensional elliptic interface problem with Dirichlet Boundary Condition and Neumann Boundary Condition. We discussed all the possible ways the interface cuts the grid. In other words, the interface is not perturbed away if it hits grid points. Matrix coefficients $\beta^+$ and $\beta^-$ can be handled. Due to the weak formulation, the case with second derivative blowing up at a point can be handled as well. Extensive numerical experiments demonstrate that our new method is effective for all kinds of possible problems and can achieve second order accuracy in the $L^\infty$ norm and $L^2$ norm.

## 2. Equations and weak formulation

Consider an open bounded domain $\Omega \subset R^3$. Let $\Gamma$ be an interface of co-dimension 1, which divides $\Omega$ into disjoint open subdomains, $\Omega^-$ and $\Omega^+$, hence $\Omega = \Omega^- \bigcup \Omega^+ \bigcup \Gamma$. Assume that the boundary $\partial\Omega$ and the boundary of each subdomain $\partial\Omega^\pm$ are Lipschitz continuous as submanifolds. Since $\partial\Omega^\pm$ are Lipschitz continuous, so is $\Gamma$. A unit normal vector of $\Gamma$ can be defined a.e. on $\Gamma$.

We seek solutions of the variable coefficient elliptic equation away from the interface $\Gamma$ given by

$$-\bigtriangledown \cdot (\beta(\mathbf{x}) \bigtriangledown u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus \Gamma \tag{1}$$

in which $\mathbf{x} = (x_1, x_2, x_3)$ denotes the spatial variables and $\bigtriangledown$ is the gradient operator. The coefficient $\beta(\mathbf{x})$ is assumed to be a $3 \times 3$ matrix that is uniformly elliptic on each disjoint subdomain, $\Omega^-$ and $\Omega^+$, and its components are continuously differentiable on each disjoint subdomain, but they may be discontinuous across the interface $\Gamma$. The right-hand side $f(\mathbf{x})$ is assumed to lie in $L^2(\Omega)$.

Given functions $a$ and $b$ along the interface $\Gamma$, we prescribe the jump conditions

$$\begin{cases} [u]_\Gamma(\mathbf{x}) \equiv u^+(\mathbf{x}) - u^-(x) = a(\mathbf{x}) \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_\Gamma(\mathbf{x}) \equiv \mathbf{n} \cdot (\beta^+(\mathbf{x}) \bigtriangledown u^+(\mathbf{x})) - \mathbf{n} \cdot (\beta^-(\mathbf{x}) \bigtriangledown u^-(\mathbf{x})) = b(\mathbf{x}). \end{cases} \tag{2}$$

The "$\pm$" superscripts refer to limits taken from within the subdomains $\Omega^\pm$.

Finally, the boundary conditions are given by

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega, \tag{3}$$

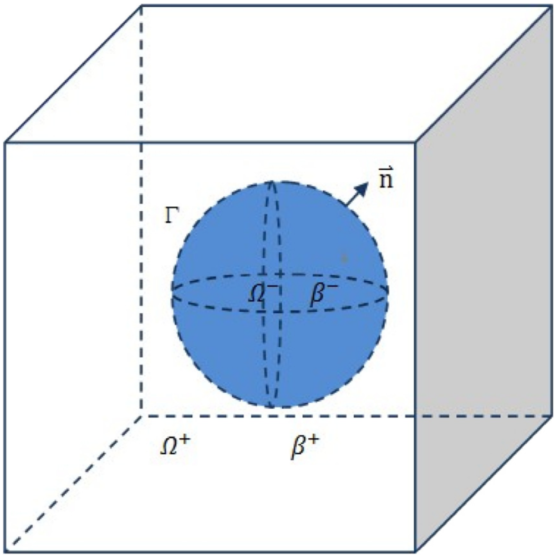



**Fig. 1.** Setup of the problem.

or

$$\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{4}$$

The setup of the problem is illustrated in Fig. 1.

The jump conditions are enforced strongly in the local system. Also, the flux jump condition appears in the weak formulation.

We introduce the weak solution by the standard procedure of multiplying Eq. (1) by a test function $\psi$ and integrating by parts: for the problem with Dirichlet Boundary Condition,

$$\int_{\Omega^+} \beta \bigtriangledown u \cdot \bigtriangledown\psi + \int_{\Omega^-} \beta \bigtriangledown u \cdot \bigtriangledown\psi = \int_{\Omega} f\psi - \int_{\Gamma} b\psi, \tag{5}$$

for the problem with Neumann Boundary Condition,

$$\int_{\Omega^+} \beta \bigtriangledown u \cdot \bigtriangledown\psi + \int_{\Omega^-} \beta \bigtriangledown u \cdot \bigtriangledown\psi + \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}}\psi = \int_{\Omega} f\psi - \int_{\Gamma} b\psi, \tag{6}$$

where $\psi$ is in $H_0^1$ for Eq. (5) and $H^1$ for Eq. (6). Note that although the test function $\psi$ is the same as in the standard finite element method, the function $u$ is not a linear combination of such basis functions. Instead, the jump conditions are enforced strongly. A detailed discussion is provided in [39].

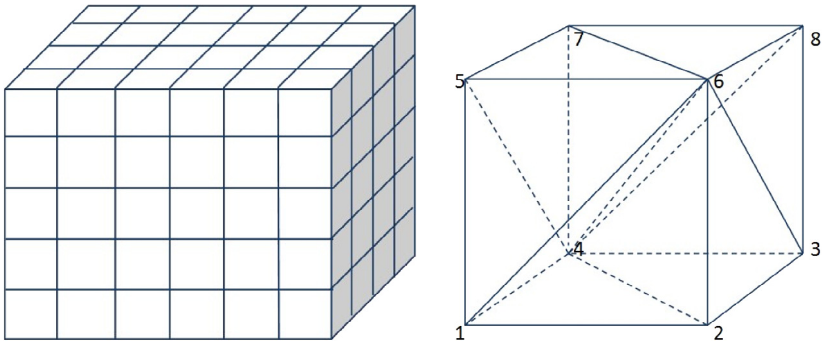
## 3. Numerical method

For ease of discussion in this section, and for accuracy test in the next section, we assume that $\beta$ and $f$ are smooth on $\Omega^+$ and $\Omega^-$, but they may be discontinuous across the interface $\Gamma$. However $\partial\Omega$, $\partial\Omega^-$ and $\partial\Omega^+$ are kept to be Lipschitz continuous. We assume that there is a Lipschitz continuous and piecewise smooth level-set function $\phi$ on $\Omega$, where $\Gamma = \{\phi = 0\}$, $\Omega^- = \{\phi < 0\}$ and $\Omega^+ = \{\phi > 0\}$. A unit vector $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$ can be obtained on $\overline{\Omega}$, which is a unit normal vector of $\Gamma$ at $\Gamma$ pointing from $\Omega^-$ to $\Omega^+$.

In this paper, we restrict ourselves to a cube cell domain $\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max}) \times (z_{\min}, z_{\max})$ in 3D space, and $\beta$ is a $3 \times 3$ matrix that is uniformly elliptic in each subdomain. Given positive integers $I$, $J$ and $K$, set $\Delta x = (x_{\max} - x_{\min})/I$, $\Delta y = (y_{\max} - y_{\min})/J$ and $\Delta z = (z_{\max} - z_{\min})/K$. We define a uniform Cartesian grid $(x_i, y_j, z_k) = (x_{\min} + i\Delta x, y_{\min} + j\Delta y, z_{\min} + k\Delta z)$ for $i = 0, \ldots, I$, $j = 0, \ldots, J$ and $k = 0, \ldots, K$. Each $(x_i, y_j, z_k)$ is called a grid point. For the case $i = 0, I$ or $j = 0, J$ or $k = 0, K$, a grid point is called a boundary point, otherwise it is called an interior point. The grid size is defined as $h = \max(\Delta x, \Delta y, \Delta z) > 0$.

Two sets of grid functions are needed and they are denoted by

$$H^{1,h} = \{\omega^h = (\omega_{i,j,k}) : 0 \le i \le I, 0 \le j \le J, 0 \le k \le K\}$$




(a) Cube cells. (b) Tetrahedralization.

**Fig. 2.** Grids.

and

$$H_0^{1,h} = \{\omega^h = (\omega_{i,j,k}) \in H^{1,h} : \omega_{i,j,k} = 0 \texttt{ if } i = 0, I \texttt{ or } j = 0, J \texttt{ or } k = 0, K\}.$$

Cut each cube cell region $[x_i, x_{i+1}] \times [y_j, y_{j+1}] \times [z_k, z_{k+1}]$ into six tetrahedron regions, collecting all those tetrahedron regions, a uniform tetrahedralization $T^h : \bigcup_{L \in T^h} L$ is obtained, see Fig. 2.

If $\phi(x_i, y_j, z_k) \le 0$, we count the grid point $(x_i, y_j, z_k)$ as in $\overline{\Omega^-}$; otherwise we count it as in $\Omega^+$. We call an edge (an edge of a tetrahedron in the tetrahedralization) an interface edge if two of its ends (vertices of tetrahedrons in the tetrahedralization) belong to different subdomains; otherwise we call it a regular edge.

We call a cell $L$ an interface cell if its vertices are on the interface or belong to different subdomains. In the interface cell, we write $L = L^+ \bigcup L^-$. $L^+$ and $L^-$ are separated by a plane segment, denoted by $\Gamma_L^h$. There are seven kinds of plane segments, see Fig. 3. The vertices of the plane segment $\Gamma_L^h$ are located on the interface $\Gamma$ and their locations can be calculated from the linear interpolations of the discrete level-set functions $\phi^h = \phi(x_i, y_j, z_k)$. The vertices of $L^+$ are located in $\Omega^+ \bigcup \Gamma$ and the vertices of $L^-$ are located in $\Omega^- \bigcup \Gamma$. $L^+$ and $L^-$ are approximations of regions $L \bigcap \Omega^+$ and $L \bigcap \Omega^-$, respectively. We call a cell $L$ a regular cell if all its vertices belong to the same subdomain, either $\Omega^+$ or $\Omega^-$. For a regular cell, we also write $L = L^+ \bigcup L^-$, where $L^- = \{\}$ (empty set) if all vertices of $L$ are in $\Omega^+$, and $L^+ = \{\}$ if all vertices of $L$ are in $\Omega^-$. Clearly $\Gamma_L^h = \{\}$ in a regular cell. We use $|L^+|$ and $|L^-|$ to represent the areas of $L^+$ and $L^-$, respectively.

Two finite element isomorphism mappings are needed from coefficient vector to finite element functions. The first one is $T^h : H^{1,h} \to H_0^1(\Omega)$. For any $\psi^h \in H_0^{1,h}$, $T^h(\psi^h)$ is a standard continuous piecewise linear function, which is a linear function in every tetrahedron cell and $T^h(\psi^h)$ matches $\psi^h$ on grid points. Clearly such a function set, denoted by $H_0^{1,h}$, is a finite dimensional subspace of $H_0^1(\Omega)$. The second operator $U^h$ is constructed as follows. For any $u^h \in H^{1,h}$ with $u^h = g^h$ at boundary points, $U^h(u^h)$ is a piecewise linear function and matches $u^h$ on grid points. It is a linear function in each regular cell, just like the first operator $U^h(u^h) = T^h(u^h)$ in a regular cell. In each interface cell, it consists of two pieces of linear functions, one is on $L^+$ and the other is on $L^-$. The location of its discontinuity in the interface cell is the plane segment $\Gamma_L^h$. Note that two end points of the plane segment are located on the interface $\Gamma$, and hence the interface condition $[u] = a$ could be and is enforced exactly at the vertices of $\Gamma_L^h$. In each interface cell, the interface condition $[\beta \nabla u \cdot \mathbf{n}] = b$ is enforced with the value $b$ at the barycenter of $\Gamma_L^h$.

We shall construct an approximate solution to the interface problem taking into account the jump conditions. Note that the Neumann jump condition $[(\beta \nabla u) \cdot \mathbf{n}] = b$ along the interface $\Gamma$ is already absorbed into the weak formulation, hence, we only need to take care of the Dirichlet jump condition $[u] = a$ along the interface $\Gamma$. We shall seek an approximate solution which is piecewise linear on both subdomains $\Omega^-$ and $\Omega^+$, but discontinuous along the interface $\Gamma$. Clearly, in Cases 1–3, when a vertex $(x_i, y_i, z_i)$ of the interface cell $L$ is on the interface, we need to get two values $u^+(x_i, y_i, z_i)$ and $u^-(x_i, y_i, z_i)$ defined at one point. To this end, we introduce a globally piecewise linear approximation $u^h(x, y, z)$ defined below:

if $\phi(x_2, y_2, z_2) > 0$,

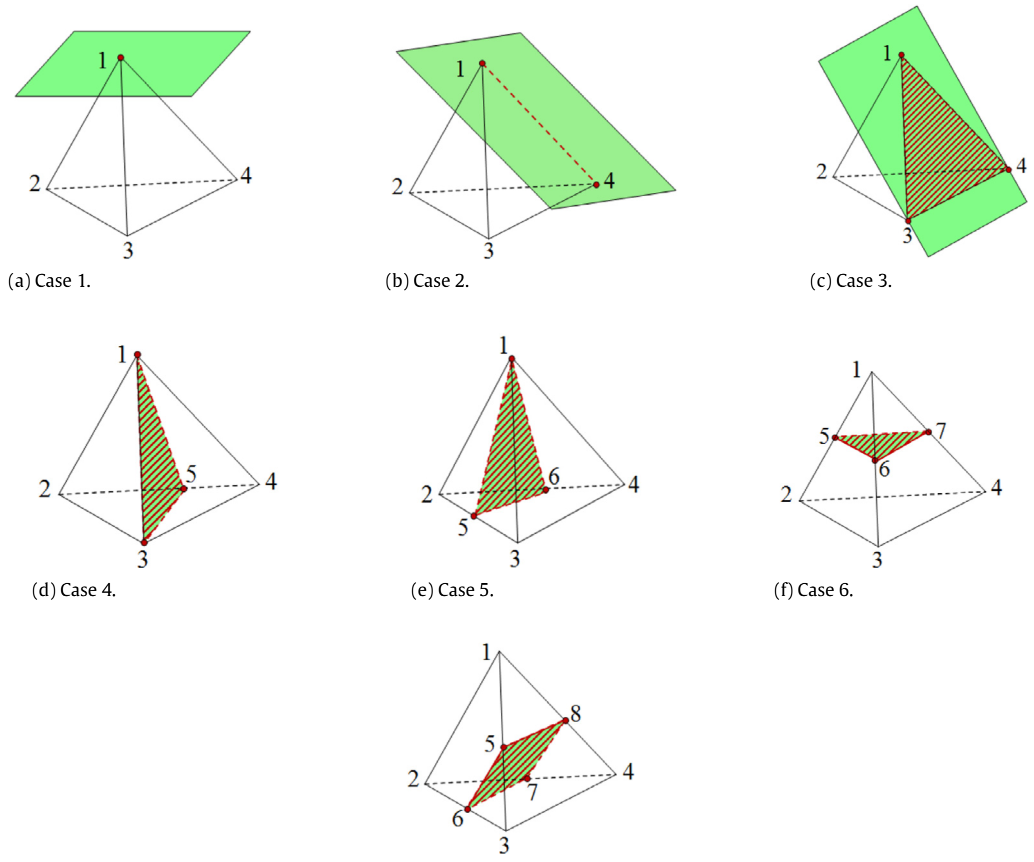
$$u^h(x_i, y_i, z_i) = u^+(x_i, y_i, z_i), \quad i = 1, 2, 3, 4$$

and if $\phi(x_2, y_2, z_2) < 0$,

$$u^h(x_i, y_i, z_i) = \begin{cases} u^+(x_i, y_i, z_i), & \text{if } \phi(x_i, y_i, z_i) = 0, \\ u^-(x_i, y_i, z_i), & \text{if } \phi(x_i, y_i, z_i) < 0. \end{cases} \quad i = 1, 2, 3, 4.$$

In **Case 1–2**, we have the following equation

$$\int_L \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) = \int_L f T^h(\psi^h). \tag{7}$$



(a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4. (e) Case 5. (f) Case 6. (g) Case 7.

**Fig. 3.** Interface cell.

In **Case 3**, if $\phi(x_2, y_2, z_2) > 0$, we have

$$\int_{L^+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) = \int_{L^+} f T^h(\psi^h) \tag{8}$$

and if $\phi(x_2, y_2, z_2) < 0$, we have

$$\int_{L^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) = \int_{L^-} f T^h(\psi^h) - \int_{\Gamma_L^h} b T^h(\psi^h). \tag{9}$$

In **Case 4**, the local system can be constructed as follows using the same way:

$$\begin{cases} [u]_5 = a_5, \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_6 = b_6 \end{cases}$$

where point 6 is the barycenter of points 1, 3, 5.

The least squares method is used to solve this system and get the values of $u_5^{\pm}$, they are denoted by the linear combinations of $u_1$, $u_2$, $u_3$ and $u_4$. Then we have the following equation

$$\int_{L^+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) + \int_{L^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) = \int_{L^+} f T^h(\psi^h) + \int_{L^-} f T^h(\psi^h) - \int_{\Gamma_L^h} b T^h(\psi^h). \tag{10}$$

In **Case 5**, the local system defined on the interface cell $L$ can be constructed as

$$\begin{cases} [u]_5 = a_5, \\ [u]_6 = a_6, \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_7 = b_7 \end{cases}$$

where point 7 is the barycenter of points 1, 5, 6.

The least squares method is used to solve this system and get the values of $u_5^\pm, u_6^\pm$, they are denoted by the linear combinations of $u_1, u_2, u_3$ and $u_4$. Then we have Eq. (10) defined on this cell as well.

In **Case 6**, notice that points 2, 3, 5, 6 are coplanar, the value of point 6 can be denoted as a linear combination of the values of points $2, 3, 5 : u_6 = c_1u_2 + c_2u_3 + c_3u_5$, so are the values of points 3, 4, 6, 7 and 2, 4, 5, 6. Then a local system defined on the interface cell $L$ can be constructed as

$$\begin{cases} [u]_5 = a_5, \\ [u]_6 = a_6, \\ [u]_7 = a_7, \\ c_1u_2 + c_2u_3 + c_3u_5 = u_6 \\ d_1u_3 + d_2u_4 + d_3u_6 = u_7 \\ e_1u_2 + e_2u_4 + e_3u_5 = u_7 \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_9 = b_9 \end{cases}$$

where point 9 is the barycenter of points 5, 6, 7.

The least squares method is used to solve this system and get the values of $u_5^\pm, u_6^\pm, u_7^\pm$, they are denoted by the linear combinations of $u_1, u_2, u_3$ and $u_4$. Then we have Eq. (10) defined on this cell as well.

In **Case 7**, the local system can be constructed as follows using the same way:

$$\begin{cases} [u]_5 = a_5, \\ [u]_6 = a_6, \\ [u]_7 = a_7, \\ [u]_8 = a_8, \\ c_1u_1 + c_2u_2 + c_3u_5 = u_6 \\ d_1u_1 + d_2u_2 + d_3u_7 = u_8 \\ e_1u_3 + e_2u_4 + e_3u_6 = u_7 \\ f_1u_3 + f_2u_4 + f_3u_5 = u_8 \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_9 = b_9 \\ [(\beta \bigtriangledown u) \cdot \mathbf{n}]_{10} = b_{10} \end{cases}$$

where point 9 is the barycenter of points 5, 6, 7 and point 10 is the barycenter of points 5, 7, 8. Note that the points 5, 6, 7, 8 are not necessarily in the same plane. That is why it is broken into triangles 5, 6, 7 and 5, 7, 8.

We use least squares method to solve this system and get the values of $u_5^\pm, u_6^\pm, u_7^\pm$ and $u_8^\pm$, they are denoted by the linear combinations of $u_1, u_2, u_3$ and $u_4$. Then we have Eq. (10) defined on this cell as well.

The local systems above could produce a way to generate all the nonzero elements of the large sparse linear system to be solved in the end. The main idea is to write the solution on the interface cut points in terms of linear combinations of the unknowns on the uniform Cartesian grid.

We propose the following method:

**Method 3.1.** Find a discrete function $u^h \in H^{1,h}$, $\psi^h \in H_0^{1,h}$ when it is a Dirichlet Boundary Condition problem and $\psi^h \in H^{1,h}$ when it is a Neumann Boundary Condition problem, we have

$$\sum_{L\in T^h} \left( \int_{L^+} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) + \int_{L^-} \beta \nabla U^h(u^h) \cdot \nabla T^h(\psi^h) \right)$$
$$= \sum_{L\in T^h} \left( \int_{L^+} fT^h(\psi^h) + \int_{L^-} fT^h(\psi^h) - \int_{\Gamma_L^h} bT^h(\psi^h) \right). \tag{11}$$

**Remark.** The traditional finite element method has beautiful convergence proof and error estimates. However, solving interface problems using body-fitted grid needs costly grid generation, especially in three dimensions. Although the theoretical work on our non-traditional finite element method is an open problem, our method has the clear advantage of easy coding and less computational cost, which makes our method practical solving real problems.

## 4. Numerical experiments

In all numerical experiments below, the level-set function $\phi$, the coefficients $\beta^\pm$ and the solutions

$$u = \begin{cases} u^+, \ \mathtt{in}\ \Omega^+, \\ u^-, \ \mathtt{in}\ \Omega^-, \end{cases}$$

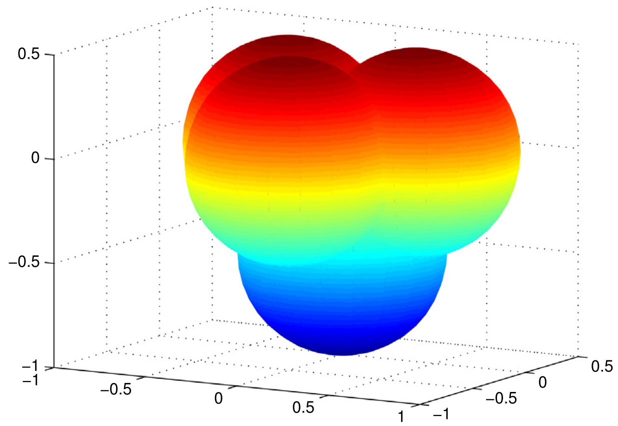


**Fig. 4.** Interface of Example 1: Four balls.

are given. Hence

$$
\begin{aligned}
&f = -\nabla \cdot (\beta \nabla u),\\
&a = u^+ - u^-,\\
&b = (\beta^+ \nabla u^+) \cdot \mathbf{n} - (\beta^- \nabla u^-) \cdot \mathbf{n}
\end{aligned}
$$

on the whole domain $\Omega = (-1, 1) \times (-1, 1) \times (-1, 1)$. Since the solutions are given, $g$ is obtained if it is a Dirichlet Boundary Condition problem.

The following examples show the validity of our method for elliptic problems with Dirichlet and Neumann boundary conditions. Examples 1–2 are problems with Dirichlet Boundary Conditions, Example 3 is a problem with Neumann Boundary Conditions, and Example 4 is an elliptic problem with both types of boundary conditions.

For Neumann problem, the solution could differ by a constant. We subtract the average of the solution so that both the numerical and true solutions have zero average on each side of the interface, so that we could compare.

**Example 1.** The coefficients $\beta^\pm$ and the solutions $u^\pm$ are given as follows:

$$
\begin{aligned}
&\beta^+(x, y, z) = \begin{pmatrix} 4\sin^2(x) + 6 & \sin(x+y)z & xy \\ \sin(x+y)z & 2z^2 + \cos^2(x^2) + 3 & 0.5\sin(xy) \\ xy & 0.5\sin(xy) & \cos^2(xy+z) + 5 \end{pmatrix},\\
&\beta^-(x, y, z) = \begin{pmatrix} xz + \cos(x+y) + 3 & x & 0.2\sin(y-x) \\ x & z^2 + 5 & yz \\ 0.2\sin(y-x) & yz & \sin^2(z) + 2 \end{pmatrix},\\
&u^+(x, y, z) = \exp(x^2 + y^2 + z^2),\\
&u^-(x, y, z) = \log(x^2 + y^2 + z^2 + 3).
\end{aligned}
$$

We consider two kinds of interfaces in this example:

(I) When the level-set function $\phi$ is given as:

$$
\begin{aligned}
&\phi 1(x, y, z) = x^2 + y^2 + (z + 0.5)^2 - 0.25,\\
&\phi 2(x, y, z) = (x - 0.4)^2 + y^2 + z^2 - 0.25,\\
&\phi 3(x, y, z) = (x + 0.3)^2 + y^2 + z^2 - 0.25,\\
&\phi 4(x, y, z) = x^2 + (y + 0.5)^2 + z^2 - 0.25,\\
&\phi(x, y, z) = \min(\phi 1, \min(\min(\phi 2, \phi 3), \phi 4)).
\end{aligned}
$$

The interface is shown in Fig. 4. The error of this problem with Dirichlet Boundary Condition on different grids is shown in Table 1.

**Table 1**
Numerical results of Example 1: Four balls.

| $n_x \times n_y \times n_z$ | $\|u - u^h\|_\infty$ | Order | $\|u - u^h\|_2$ | Order |
|---|---|---|---|---|
| $6 \times 6 \times 6$ | 0.30500 | | 0.16488 | |
| $12 \times 12 \times 12$ | 0.08327 | 1.8724 | 0.03693 | 2.1585 |
| $24 \times 24 \times 24$ | 0.02214 | 1.9109 | 0.00873 | 2.0808 |
| $48 \times 48 \times 48$ | 0.00566 | 1.9690 | 0.00211 | 2.0478 |
| $96 \times 96 \times 96$ | 0.00149 | 1.9283 | 0.00052 | 2.0243 |

**Table 2**
Numerical results of Example 1: Five balls.

| $n_x \times n_y \times n_z$ | $\|u - u^h\|_\infty$ | Order | $\|u - u^h\|_2$ | Order |
|---|---|---|---|---|
| $6 \times 6 \times 6$ | 0.30697 | | 0.14850 | |
| $12 \times 12 \times 12$ | 0.08395 | 1.8706 | 0.03484 | 2.0917 |
| $24 \times 24 \times 24$ | 0.02224 | 1.9161 | 0.00829 | 2.0719 |
| $48 \times 48 \times 48$ | 0.00568 | 1.9705 | 0.00201 | 2.0433 |
| $96 \times 96 \times 96$ | 0.00142 | 1.9943 | 0.00049 | 2.0222 |

(II) When the level-set function $\phi$ is given as:

$$\begin{aligned}
\phi 1(x, y, z) &= x^2 + y^2 + (z - 0.7)^2 - 0.07,\\
\phi 2(x, y, z) &= x^2 + y^2 + (z + 0.7)^2 - 0.08,\\
\phi 3(x, y, z) &= (x - 0.3)^2 + y^2 + z^2 - 0.15,\\
\phi 4(x, y, z) &= (x + 0.2)^2 + y^2 + z^2 - 0.09,\\
\phi 5(x, y, z) &= x^2 + (y + 0.1)^2 + z^2 - 0.06,\\
\phi(x, y, z) &= \min(\min(\phi 1, \min(\min(\phi 2, \phi 3), \phi 4)), \phi 5).
\end{aligned}$$

The interface is shown in Fig. 5. The error of this problem with Dirichlet Boundary Condition on different grids is shown in Table 2.

**Example 2.** In this example, the geometry and the level-set function $\phi$ are very complicated, it can be given by:

$$\begin{aligned}
\phi 1(x, y, z) &= x^2 + y^2 - 0.02,\\
\phi 2(x, y, z) &= x^2 + y^2 - 0.1,\\
\phi 3(x, y, z) &= x^2 + y^2 - 0.3,\\
\phi 4(x, y, z) &= x^2 + y^2 - 0.6,\\
\phi 5(x, y, z) &= x^2 + y^2 - 0.9,\\
\phi 6(x, y, z) &= y^2 + z^2 - 0.9,\\
m1(x, y, z) &= -\min(\phi 2, -\phi 3),\\
m2(x, y, z) &= -\min(\phi 4, -\phi 5),\\
m3(x, y, z) &= \min(\phi 1, \min(m1, m2)),\\
\phi(x, y, z) &= \max(m3, \phi 6).
\end{aligned}$$

The coefficients $\beta^{\pm}$ and the solution $u^{\pm}$ are given as follows:

$$\begin{aligned}
\beta^+(x, y, z) &= \begin{pmatrix} 4x^2 + 6 & \sin(x + y) & xy \\ \sin(x + y) & 2z^2 + 3 & 0.5\sin(x) \\ xy & 0.5\sin(x) & \cos^2(xy + z) + 5 \end{pmatrix},\\
\beta^-(x, y, z) &= \begin{pmatrix} \cos^2(x + y) + 3 & z & 0.2\sin(z - x) \\ z & z^2 + 5 & y \\ 0.2\sin(z - x) & y & \sin^2(z) + 2 \end{pmatrix},\\
u^+(x, y, z) &= 5 - \sin(x^3) + 3y^2 + z,\\
u^-(x, y, z) &= -\cos(x) + 3y + z^3.
\end{aligned}$$

Table 3 shows the error of the problem with Dirichlet Boundary Condition on different grids.

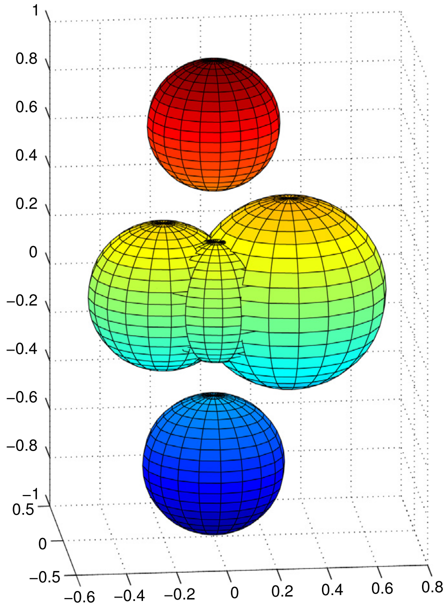



**Fig. 5.** Interface of Example 1: Five balls.

**Example 3.** The level-set function $\phi$, the coefficients $\beta^{\pm}$ and the solution $u^{\pm}$ are given as follows:

$$\phi(x,y,z) = x^2 + y^2 + (z-1)^2 - 0.25,$$

$$\beta^+(x,y,z) = \begin{pmatrix} 4x^2+6 & \sin(x+y) & x \\ \sin(x+y) & 2z^2+3 & 0.5\sin(x) \\ x & 0.5\sin(x) & \cos(z)+5 \end{pmatrix},$$

$$\beta^-(x,y,z) = \begin{pmatrix} \cos(x)+3 & z & 0.2\sin(z-x) \\ z & z^2+5 & y \\ 0.2\sin(z-x) & y & \sin^2(z)+2 \end{pmatrix},$$

$$u^+(x,y,z) = 5 - \sin(x^2) + 3y^2 + z,$$

$$u^-(x,y,z) = -\cos(x) + 3y + z^2.$$

Table 4 shows the error of the problem with Neumann Boundary Condition on different grids.

**Example 4.** The level-set function $\phi$, the coefficients $\beta^{\pm}$ and the solution $u^{\pm}$ are given as follows:

$$\phi(x,y,z) = (x-0.4)^2 + y^2 + z^2 - 0.16,$$

$$\beta^+(x,y,z) = \begin{pmatrix} 4x^2+6 & \sin(x+y) & xy \\ \sin(x+y) & 2z^2+3 & 0.5\sin(x) \\ xy & 0.5\sin(x) & \cos^2(xy+z)+2 \end{pmatrix},$$

$$\beta^-(x,y,z) = \begin{pmatrix} \cos^2(x+y)+3 & z & 0.2\sin(z-x) \\ z & z^2+5 & y \\ 0.2\sin(z-x) & y & \sin^2(z)+2 \end{pmatrix},$$

$$u^+(x,y,z) = (x^2+y^2+z^2)^{5/6},$$

$$u^-(x,y,z) = \sin(x+y).$$

Note that the true solution $u^+$ has its second derivative blow up at the origin. Our method can handle such problem. Tables 5 and 6 show the error of the problem with Dirichlet and Neumann Boundary Condition on different grids.

## 5. Conclusion

We developed an improved non-traditional finite element method for matrix-valued coefficient elliptic equations with sharp-edged interfaces in three dimensions. Compared with the previous work in [39], the novelty is that all possible ways

**Table 3**
Numerical results of Example 2.

| $n_x \times n_y \times n_z$ | $\|u - u^h\|_\infty$ | Order |
|---|---|---|
| $6 \times 6 \times 6$ | 0.02797 | |
| $12 \times 12 \times 12$ | 0.00837 | 1.7404 |
| $24 \times 24 \times 24$ | 0.00230 | 1.8657 |
| $48 \times 48 \times 48$ | 0.00063 | 1.8702 |
| $96 \times 96 \times 96$ | 0.00018 | 1.8164 |

**Table 4**
Numerical results of Example 3.

| $n_x \times n_y \times n_z$ | $\|u - u^h\|_\infty$ | Order | $\|u - u^h\|_2$ | Order |
|---|---|---|---|---|
| $8 \times 8 \times 8$ | 0.11018 | | 0.02025 | |
| $16 \times 16 \times 16$ | 0.03765 | 1.5492 | 0.00514 | 1.9778 |
| $32 \times 32 \times 32$ | 0.01170 | 1.6861 | 0.00091 | 2.4962 |
| $64 \times 64 \times 64$ | 0.00363 | 1.6904 | 0.00020 | 2.2024 |

**Table 5**
Numerical results of Example 4: Dirichlet boundary condition.

| $n_x \times n_y \times n_z$ | $\|u - u^h\|_\infty$ | Order |
|---|---|---|
| $6 \times 6 \times 6$ | 0.02251 | |
| $12 \times 12 \times 12$ | 0.00736 | 1.6125 |
| $24 \times 24 \times 24$ | 0.00230 | 1.6780 |
| $48 \times 48 \times 48$ | 0.00071 | 1.6961 |
| $96 \times 96 \times 96$ | 0.00022 | 1.7011 |

**Table 6**
Numerical results of Example 4: Neumann boundary condition.

| $n_x \times n_y \times n_z$ | $\|u - u^h\|_\infty$ | Order |
|---|---|---|
| $8 \times 8 \times 8$ | 0.05955 | |
| $16 \times 16 \times 16$ | 0.01929 | 1.6266 |
| $32 \times 32 \times 32$ | 0.00519 | 1.8936 |
| $64 \times 64 \times 64$ | 0.00151 | 1.7789 |

the interface cuts though grids are considered. If the interface touches a grid point, instead of perturbing it away to apply the general case, we handle it carefully as in Fig. 3. Compared with standard finite element method using body-fitted grid, our non-body-fitted grid has the potential advantage that no mesh generation is needed when dealing with moving interface problems. Mesh generation for standard finite element method is costly in three dimensions. Further, even for elliptic interface problem itself, the number of unknowns in the large linear system is less than that of the body-fitted finite element method since uniform Cartesian grid is used without any added nodes, and therefore our method is faster. Also, both Dirichlet and Neumann boundary conditions are considered. Our extensive numerical experiments indicate that the method is about second-order accurate in the $L^\infty$ norm and $L^2$ norm, and it is stable even for interface problems with very complicated geometry.

## Acknowledgments

L. Shi's research is supported by Science Foundations of China University of Political Science and Law (Nos. 1000-10816106 and 1000-10816330). S. Hou's research is supported by NSF grant DMS-1317994 and Dr. Walter Koss Professorship. L. Wang's research is supported by Science Foundations of China University of Petroleum-Beijing (Nos. 2462015BJB05, 2462015YQ0604 and 2462015QZDX02).

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