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A SYSTEMATIC CONSTRUCTION OF FINITE ELEMENT COMMUTING EXACT SEQUENCES*

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Abstract. We present a systematic construction of finite element exact sequences with a commuting diagram property for the de Rham complex in one-, two-, and three-space dimensions. We apply the construction in two-space dimensions to rediscover two families of exact sequences for triangles and three for squares, and to uncover one new family of exact sequence for squares and two new families of exact sequences for general polygonal elements. We apply the construction in three-space dimensions to rediscover two families of exact sequences for tetrahedra, three for cubes, and one for prisms, and to uncover four new families of exact sequences for pyramids, three for prisms, and one for cubes.

Key words. finite elements, commuting diagrams, exact sequences, polyhedral elements

AMS subject classifications. 65N30, 65M60, 35L65

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1. Introduction.

In this paper, we provide a systematic construction of finite element exact sequences with a commuting diagram property, which we call *finite element commuting exact sequences*, for the de Rham complex in one-, two-, and three-dimensional domains. The construction of these finite element commuting exact sequences for the three-dimensional case relies on the construction of finite element commuting exact sequences for the two-dimensional case, which in turn relies on the construction of finite element commuting exact sequences in one-space dimension. Moreover, for each dimension, the construction is carried out on a single polytopic element K in such a way that proper continuity properties hold which guarantee the commutativity of the diagram when K is replaced by the triangulation of the domain Ω, $\Omega_h := \{K\}$.

To be able to discuss our results, we need to recall the definition of a finite element commuting sequence; we follow [4]. The de Rham complex on a domain $\Omega \in \mathbb{R}^3$ is the sequence of spaces and mappings

$$0 \longrightarrow C^\infty(\Omega) \xrightarrow{\nabla} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\nabla\times} C^\infty(\Omega;\mathbb{R}^3) \xrightarrow{\nabla\cdot} C^\infty(\Omega) \longrightarrow 0,$$

and the L^2-de Rham complex is

$$0 \longrightarrow H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} H(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \longrightarrow 0.$$

By a finite element commuting exact sequence on a polyhedral domain $\Omega \subset \mathbb{R}^3$, we mean a (finite-dimensional) discrete de Rham subcomplex on the triangulation Ω_h of Ω

$$0 \longrightarrow H^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla} E^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(\Omega_h) \longrightarrow 0$$

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of the L^2-de Rham complex such that its extended complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla} E^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(\Omega_h) \longrightarrow 0$$

is exact (image of each map in the above sequence is the kernel of the next one), and it is equipped with a set of canonical commuting projection operators (that can induce a set of degrees of freedom) with the de Rham complex; i.e., the following diagram commutes

$$\begin{array}{ccccccccc}
0 \longrightarrow C^\infty(\Omega) & \xrightarrow{\nabla} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\nabla\times} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\nabla\cdot} & C^\infty(\Omega) \longrightarrow 0 \\
\downarrow \Pi_H & & \downarrow \Pi_E & & \downarrow \Pi_V & & \downarrow \Pi_W \\
0 \longrightarrow H^{3\mathrm{d}}(\Omega_h) & \xrightarrow{\nabla} & E^{3\mathrm{d}}(\Omega_h) & \xrightarrow{\nabla\times} & V^{3\mathrm{d}}(\Omega_h) & \xrightarrow{\nabla\cdot} & W^{3\mathrm{d}}(\Omega_h) \longrightarrow 0.
\end{array}$$

Here Π_H, Π_E, Π_V, Π_W are the canonical commuting projection operators. Finite element commuting exact sequences on a single polyhedral element and in one- and two-space dimensions are defined in a similar way.

For simplicity, and whenever there is no possibility of confusion, we abbreviate *finite element commuting exact sequence* by *commuting exact sequence*, keeping in mind that all their projection operators have finite-dimensional images.

A brief historical overview. Let us place our contribution into historical perspective. The importance of finite element commuting exact sequences for the devising of stable finite element methods has been amply discussed in [4, 6] and the references therein. Our work is part of the ongoing effort of constructing new commuting exact sequences. Next, we briefly review such effort and show that it can be thought of as the confluence of the work on *minimal compatible finite element systems* by Christiansen and Gillette [11] and our work on *M-decompositions* for mixed methods for second-order diffusion [16, 14, 15].

Most of the previous work on the construction of these sequences in three-space dimension, especially those using piecewise polynomials of arbitrary degree, focuses on the explicit construction of shape-functions on one of four particular *reference* polyhedra, namely, the reference tetrahedron, hexahedron (cube), prism, and pyramid. See [23, 24] for sequences on the reference tetrahedron and reference hexahedron, [3, 17] on the reference hexahedron, [28] on the reference prism, and [25, 26] on the reference pyramid. All of these spaces in these sequences are spanned by polynomial shape functions, except those in [25, 26] which also contain rational shape functions.

Currently, there are two ways of constructing commuting exact sequences on general polyhedral meshes. The first is provided by the *virtual element methods* (VEM); see the 2013 paper [7] and the 2015 paper [8]. This approach defines the basis functions of the local spaces on each polyhedral element in terms of solutions to certain partial differential equations. The explicit form of these basis functions, which usually are not computable, is *not* needed by the methods, but a set of *unisolvent* degrees of freedom, which can be used to *exactly* compute integrals related to the polynomial parts of the basis functions, needs be constructed. In our construction, we avoid basis functions defined as solutions to partial differential equations; we provide basis functions ready for implementation.

The second way is provided by the so-called *finite element system* (FES) developed in 2011 in [12, sect. 5]; see also the recent papers [13, 11]. Therein, the notion of a *compatible* FES was introduced, see [12, Def. 5.12], which was then proven to be equivalent to the existence of a commuting diagram, see [12, Prop. 5.44]. The construction of a FES with a commuting diagram was thus reduced to the construction of a compatible FES. In [12, Ex. 5.29], the authors obtained compatible FESs on a general (n-dimensional) polytope mesh via *element agglomeration* from (available) compatible FESs defined on a refined (simplicial) mesh. However, the resulting compatible FES on the polytope mesh either provides identical spaces on the refined (simplicial) mesh or requires finding a subsystem of *locally harmonic forms*, which is not a trivial undertaking.

In [11, Cor. 3.2], a criterion for finding compatible FESs, containing certain prescribed functions, with the *smallest* possible dimension was presented, thus giving rise to the concept of a *minimal compatible FES* (mcFES). It was also shown how to verify the minimality of the compatible FES by a simple dimension count equation in [11, Cor. 3.2]. The minimality was proven for three finite element systems, namely, the *trimmed polynomial differential forms* [4] on a simplicial mesh (in three-dimensions this is the exact sequence due to Nédélec [23]), the serendipity elements [3] on a cubic mesh, and the TNT elements [17] on a cubic mesh.

The present work can be considered as a way to actually construct mcFESs in one-, two-, and three-space dimensions. Indeed, we develop a systematic construction of commuting exact sequences, that is, compatible FES [12], on a general polytope in one-, two-, and three-space dimensions. The construction provides already known commuting exact sequences as well as a variety of new ones. Next, we give a rough idea of this construction. As we are going to see, the construction can be thought of as an extension to the setting of exact sequences for the de Rham complex of the technique of M-decompositions (for mixed methods for steady-state diffusion problems) introduced in [16, 14, 15].

The main steps of the construction. To describe our construction, we only need to do it in a local, elementwise manner. Indeed, since a *commuting exact sequence* is nothing but a *compatible FES* by [12, Prop. 5.44], we do not need to explicitly construct the commuting projection operators, as pointed out in [12]. As a consequence, we only need to work inductively in the space dimension and restrict ourselves to a single element K.

We proceed in three steps as follows.

Step 1. Let K be a given polyhedron. We start with a set of commuting exact sequences on the faces of K,

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \{\mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : \ \mathrm{f} \in \mathcal{F}(K)\},$$

where $\mathcal{F}(K)$ is the set of faces of K. Each of the commuting exact sequences

$$\mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : H^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla} E^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(\mathrm{f})$$

was *previously* obtained by applying the systematic construction in the two dimensional case. Moreover, we require a *compatibility* condition on the edges, namely, that if the faces f_1 and f_2 share the edge e, the trace on the edge e of $H^{2\mathrm{d}}(\mathrm{f}_1)$ must coincide with that of $H^{2\mathrm{d}}(\mathrm{f}_2)$, and that the tangential trace on the edge e of $E^{2\mathrm{d}}(\mathrm{f}_1)$ must coincide with that of $E^{2\mathrm{d}}(\mathrm{f}_2)$. We note that this compatibility condition can be easily incorporated in the construction of commuting exact sequences in two-space dimensions.

Step 2. We consider a given candidate, for the commuting exact sequence we are seeking, of the form

$$\mathrm{S}_g^{3\mathrm{d}}(K) : H_g^{3\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W_g^{3\mathrm{d}}(K).$$

Here the subindex g stands for "given." We require $\mathrm{S}_g^{3\mathrm{d}}(K)$ to be exact; to have, for each face f of K, the trace of $H_g^{3\mathrm{d}}(K)$ on the face f in the space $H^{2\mathrm{d}}(\mathrm{f})$, the tangential trace of $E_g^{3\mathrm{d}}(K)$ on the face f in the space $E^{2\mathrm{d}}(\mathrm{f})$, and the normal trace of $V_g^{3\mathrm{d}}(K)$ on the face f in the space $W^{2\mathrm{d}}(\mathrm{f})$; and to have the constant functions in the space $W_g^{3\mathrm{d}}(K)$. We note that this sequence is usually *not* a commuting exact sequence.

Step 3. We *characterize* the spaces $\delta H^{3\mathrm{d}}(K)$ and $\delta E^{3\mathrm{d}}(K)$ as the spaces with smallest dimension such that

$$\begin{array}{ccccccc} H_g^{3\mathrm{d}}(K) & \xrightarrow{\nabla} & E_g^{3\mathrm{d}}(K) & \xrightarrow{\nabla\times} & V_g^{3\mathrm{d}}(K) & \xrightarrow{\nabla\cdot} & W_g^{3\mathrm{d}}(K) \\ \oplus\delta H^{3\mathrm{d}}(K) & & \oplus\nabla\,\delta H^{3\mathrm{d}}(K) & & \oplus\nabla\times\delta E^{3\mathrm{d}}(K) & & \\ & & \oplus\delta E^{3\mathrm{d}}(K) & & & & \end{array}$$

is a commuting exact sequence for which, for each face f of K, the trace of $H_g^{3\mathrm{d}}(K)\oplus\delta H^{3\mathrm{d}}(K)$ on the face f constitutes the space $H^{2\mathrm{d}}(\mathrm{f})$, the tangential trace of $E_g^{3\mathrm{d}}(K)\oplus\nabla\,\delta H^{3\mathrm{d}}(K)\oplus\delta E^{3\mathrm{d}}(K)$ on the face f constitutes the space $E^{2\mathrm{d}}(\mathrm{f})$, and the normal trace of $V_g^{3\mathrm{d}}(K)\oplus\nabla\times\delta E^{3\mathrm{d}}(K)$ on the face f constitutes the space $W^{2\mathrm{d}}(\mathrm{f})$.

This completes the rough description of the systematic construction. Note that the spaces $\delta H^{3\mathrm{d}}(K)$ and $\delta E^{3\mathrm{d}}(K)$ are *not* necessarily unique even though their dimension is uniquely determined. Note also that the construction of the space $\delta E^{3\mathrm{d}}(K)$, which is the *most difficult* part of the general construction, is essentially a particular case of the construction of M-decompositions for mixed methods, see [16, 14, 15]. Thus, as previously pointed out, the present construction can be considered to be an extension of the M-decomposition approach to the setting of exact sequences for the de Rham complex.

Recovering old and uncovering new sequences. Let us describe the commuting exact sequences in two- and three-space dimensions we obtain by applying the systematic construction just sketched.

The two-dimensional case. In two dimensions, we apply the systematic construction to obtain *eight* families of commuting exact sequences: two on the reference triangle, four on the reference square, and two on a general polygon.

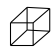
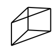
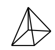
The two sequences on the reference triangle and two of the four sequences on the reference square correspond to the Raviart–Thomas [27] and Brezzi–Douglas–Marini [9] spaces. The third sequence on the reference square is the TNT sequence proposed by Cockburn and Qiu [17]. The last sequence on the reference square is a small modification of the sequence corresponding to the Brezzi–Douglas–Marini space.

The highlight of our construction in two-space dimensions is the discovery of two new families of commuting exact sequences on general polygonal elements with spaces including polynomials of arbitrary degree.

The three-dimensional case. We also apply the construction to each of the above-mentioned four reference polyhedra to discover and rediscover concrete examples of commuting exact sequences. For a general polyhedron, our construction of commuting exact sequences (for spaces including high-degree polynomials) is significantly more difficult than that of the already mentioned cases. It will be carried out elsewhere.

We obtain *fourteen* families of commuting exact sequences: two on the reference tetrahedron, four on the reference cube, four on the reference prism, and four on the reference pyramid. In Table 1, we indicate if these are known or new commuting

Table 1
Commuting exact sequences obtained as the application of the systematic construction. The index k is associated to a polynomial degree. The symbol ✓ indicates that the sequence is new. Sequences in the same row are such that the spaces of traces are the same on faces of the same shape.

sequence				
$S_{1,k}$	[24]	[3]	✓	✓
$S_{2,k}$	[23]	✓[22]	✓	✓
$S_{3,k}$	-	[17]	✓	✓
$S_{4,k}$	-	[23]	[20]	✓

exact sequences. The known results obtained in [23, 24, 3, 17, 28] on a tetrahedron, cube, and prism fit nicely within our construction. On a reference prism, there is an additional family of commuting exact sequence, namely, the one proposed in [28]; it uses the $H(\text{div})$ and L^2-spaces obtained in [10]. The sequence introduced in [20] is a slight modification of this one. On a reference pyramid, there are two additional families of exact sequences in [25, 26]; our spaces are significantly smaller. Finally, let us note that the *trimmed serendipity finite element differential forms* $S_r^-\Lambda^k$ recently introduced and studied in [22] are closely related to our second sequence $S_{2,k}$ on the reference cube. The resulting spaces are the same. Therein, a nice set of unisolvent degrees of freedom is also introduced for $S_r^-\Lambda^k$.

Our fourteen commuting exact sequences can be gathered into *four* groups of sequences each of which is displayed in a row in Table 1. These *four* groups of sequences are such that the H^1-, $H(\text{curl})$-, and $H(\text{div})$-trace spaces on similar faces are the *same*. As a consequence, each of the *four* groups of sequences can be patched into a *hybrid polyhedral mesh* $\Omega_h = \{K\}$, where the elements K are suitably defined by affine mappings of the four reference polyhedral elements. This way of regrouping the sequences is motivated by the work proposed in [20].

Organization of the material. The rest of the paper is organized as follows. In section 2, we present our main results on the systematic construction of commuting exact sequences in one-, two-, and three-space dimensions. Then in section 3, we apply the systematic construction to explicitly obtain commuting exact sequences on the reference interval in one-space dimension; on the reference triangle, reference square, and on a general polygon in two-space dimensions; and on the above-mentioned four reference polyhedra in three-space dimensions. Section 4 is devoted to the proofs of results in section 2, and section 5 is devoted to the proofs results in section 3. We end in section 6 with some concluding remarks.

2. A systematic construction of commuting exact sequences.

In this section, we introduce the notation used throughout the paper. We then define the concept of a *compatible exact sequence* in one-, two-, and three-space dimensions, which, in differential form language, is nothing but the *compatible* FES introduced in [12, Def. 5.12]. Let us recall that the theory of FESs introduced in [12, Prop. 5.44] establishes the equivalence of a compatible exact sequence and a sequence admitting a commuting diagram. As a consequence, the construction of a sequence admitting a commuting diagram is reduced to the construction of a compatible exact sequence. We use this powerful result and devote ourselves to developing a systematic construction of compatible exact sequences in one-, two-, and three-space dimensions.

In section 3, this approach is applied to obtain many compatible exact sequences with explicitly defined shape functions for elements of various shapes.

2.1. Notation. Let us introduce the notation we use in the rest of the paper.

Geometry. We denote by $K \subset \mathbb{R}^d$ a segment if $d = 1$, a polygon if $d = 2$, and a polyhedron if $d = 3$. We denote its boundary by ∂K, the set of its vertices by $\mathcal{V}(K)$, the set of its edges (for $d = 2, 3$) by $\mathcal{E}(K)$, and the set of its faces (for $d = 3$) by $\mathcal{F}(K)$.

Trace operators. For a scalar-valued function v on K with sufficient regularity, we denote by $\mathrm{tr}_H^{\mathrm{v}} v := v|_{\mathrm{v}}$ the trace of v on a vertex $\mathrm{v} \in \mathcal{V}(K)$, by $\mathrm{tr}_H^{\mathrm{e}} v := v|_{\mathrm{e}}$ the trace of v on an edge $\mathrm{e} \in \mathcal{E}(K)$, by $\mathrm{tr}_H^{\mathrm{f}} v := v|_{\mathrm{f}}$ the trace of v on a face $\mathrm{f} \in \mathcal{F}(K)$, and by $\mathrm{tr}_H v := v|_{\partial K}$ the trace of v on the whole boundary ∂K.

If $d \geq 2$, for a d-dimensional vector-valued function v with sufficient regularity, we denote by $\mathrm{tr}_E^{\mathrm{e}} v := (v \cdot t_{\mathrm{e}})|_{\mathrm{e}}$, where t_{e} is the unit vector in the direction of the edge e, the tangential trace of v on an edge $\mathrm{e} \in \mathcal{E}(K)$. We denote by $\mathrm{tr}_E^{\mathrm{f}} v := (n_{\mathrm{f}} \times (v \times n_{\mathrm{f}}))|_{\mathrm{f}}$, where n_{f} is the unit outward normal to the face f, the tangential trace of v on a face $\mathrm{f} \in \mathcal{F}(K)$. Finally, we denote by

$$\mathrm{tr}_E v := \begin{cases} (v \cdot t_{\partial K})|_{\partial K} & \text{if } d = 2, \\ (n_{\partial K} \times (v \times n_{\partial K}))|_{\partial K} & \text{if } d = 3, \end{cases}$$

the tangential trace of v on the whole boundary ∂K. Here, $t_{\partial K}$ on the edge e is nothing but t_{e}. Similarly, $n_{\partial K}$ on the face f is nothing but n_{f}.

Finally, if $d = 3$, for a d-dimensional vector-valued function v with sufficient regularity, we denote by $\mathrm{tr}_V^{\mathrm{f}} v := (v \cdot n_{\mathrm{f}})|_{\mathrm{f}}$ the normal trace of v on a face $\mathrm{f} \in \mathcal{F}(K)$, and by $\mathrm{tr}_V v := (v \cdot n_{\partial K})|_{\partial K}$ the normal trace of v on the whole boundary ∂K.

Differential operators. The gradient operator (for $d = 1, 2, 3$) is denoted by "∇", the curl operator (for $d = 2, 3$) is denoted by "$\nabla\times$" (when $d = 2$, $\nabla\times(v_1, v_2) = -\partial_y v_1 + \partial_x v_2$), and the divergence operator (for $d = 3$) is denoted by "$\nabla\cdot$".

Sequences and bubble spaces. Here we give the definitions of an exact sequence and its sequence of traces. To emphasize that functions in a given finite-dimensional space are defined on a domain of $\mathbb{R}^d$, we use the superscript dd.

Definition 2.1 (exact sequences). *The sequences*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{3\mathrm{d}}(K) \xrightarrow{\nabla} E^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(K) \longrightarrow 0 \quad \textit{for } K \subset \mathbb{R}^3,$$
$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{2\mathrm{d}}(K) \xrightarrow{\nabla} E^{2\mathrm{d}}(K) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(K) \longrightarrow 0 \quad \textit{for } K \subset \mathbb{R}^2,$$
$$0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{1\mathrm{d}}(K) \xrightarrow{\nabla} W^{1\mathrm{d}}(K) \longrightarrow 0 \quad \textit{for } K \subset \mathbb{R}^1,$$

are said to be exact *if the image of each map is the kernel of the next one.*

Definition 2.2 (sequence of traces). *Let $K \subset \mathbb{R}^3$ be a polyhedron. For any sequence*

$$\mathrm{S}(K): \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{3\mathrm{d}}(K) \xrightarrow{\nabla} E^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(K) \longrightarrow 0,$$

and for any face $\mathrm{f} \in \mathcal{F}(K)$*, we define its sequence of traces on* f *as*

$$\mathrm{tr}^{\mathrm{f}}(\mathrm{S(K)}): \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} \mathrm{tr}_H^{\mathrm{f}}(H^{3\mathrm{d}}(K)) \xrightarrow{\nabla} \mathrm{tr}_E^{\mathrm{f}}(E^{3\mathrm{d}}(K)) \xrightarrow{\nabla\times} \mathrm{tr}_V^{\mathrm{f}}(V^{3\mathrm{d}}(K)) \longrightarrow 0,$$

and for any edge $\mathrm{e} \in \mathcal{E}(K)$*, we define its sequence of traces on* e *by*

$$\mathrm{tr}^{\mathrm{e}}\,(\mathrm{S(K)}): \quad 0 \longrightarrow \mathbb{R} \xrightarrow{i} \mathrm{tr}^{\mathrm{e}}_H(H^{3\mathrm{d}}(K)) \xrightarrow{\nabla} \mathrm{tr}^{\mathrm{e}}_E(E^{3\mathrm{d}}(K)) \longrightarrow 0.$$

The sequence of traces on an edge for a two-dimensional sequence is defined in the same way.

Definition 2.3 (Bubble spaces). *Let* $K \subset \mathbb{R}^3$ *be a polyhedron. For any sequence*

$$\mathrm{S}(K): \quad 0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{3\mathrm{d}}(K) \xrightarrow{\nabla} E^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(K) \longrightarrow 0,$$

we define the related H^1*-,* $H(\mathrm{curl})$*-,* $H(\mathrm{div})$*-, and* L^2-bubble *spaces as*

$$\begin{aligned}
\overset{\circ}{H}{}^{3\mathrm{d}}(K) &:= \{v \in H^{3\mathrm{d}}(K):\ \mathrm{tr}_H v = 0\},\\
\overset{\circ}{E}{}^{3\mathrm{d}}(K) &:= \{v \in E^{3\mathrm{d}}(K):\ \mathrm{tr}_E v = 0\},\\
\overset{\circ}{V}{}^{3\mathrm{d}}(K) &:= \{v \in V^{3\mathrm{d}}(K):\ \mathrm{tr}_V v = 0\},\\
\overset{\circ}{W}{}^{3\mathrm{d}}(K) &:= \left\{v \in W^{3\mathrm{d}}(K):\ \int_K v = 0\right\},
\end{aligned}$$

respectively. Similar, obvious definitions for bubble spaces hold for the two- and one-dimensional cases.

From now on, we remove the first two and last terms in the definition of the exact sequence to simplify the notation. For example, we simply write $H^{1\mathrm{d}}(\Omega) \xrightarrow{\nabla} W^{1\mathrm{d}}(\Omega)$, instead of writing $0 \longrightarrow \mathbb{R} \xrightarrow{\subset} H^{1\mathrm{d}}(\Omega) \xrightarrow{\nabla} W^{1\mathrm{d}}(\Omega) \longrightarrow 0$.

Finally, let us emphasize that *all* the spaces in the sequences considered below have *finite* dimension.

Polynomial spaces. We denote the polynomial space of degree at most p with argument $(x, y, z) \in \mathbb{R}^3$ by

$$\mathcal{P}_p(x, y, z) := \mathrm{span}\{x^i y^j z^k:\ i, j, k \geq 0,\ i + j + k \leq p\},$$

and we denote the *homogeneous* polynomial space of total degree p by

$$\widetilde{\mathcal{P}}_p(x, y, z) := \mathrm{span}\{x^i y^j z^k:\ i, j, k \geq 0,\ i + j + k = p\}.$$

We denote the tensor-product polynomial space of degree at most p by

$$\mathcal{Q}_p(x, y, z) := \mathcal{P}_p(x) \otimes \mathcal{P}_p(y) \otimes \mathcal{P}_p(z) = \mathrm{span}\{x^i y^j z^k:\ 0 \leq i, j, k \leq p\}.$$

We also denote the polynomial space of degree at most p in the (x, y) variable and of degree at most p in the z variable by

$$\mathcal{P}_{p|p}(x, y, z) := \mathcal{P}_p(x, y) \otimes \mathcal{P}_p(z).$$

Similar definitions hold in the two- and one-dimensional cases.

Given an element $K \subset \mathbb{R}^d$, we denote $\mathcal{P}_p(K)$ to be the space of polynomials with degree at most p defined on K and proceed similarly for $\widetilde{\mathcal{P}}_p(K)$, $\mathcal{Q}_p(K)$, $\mathcal{P}_{p|p}(K)$. We denote by $\boldsymbol{\mathcal{P}}_p(K)$, respectively $\widetilde{\boldsymbol{\mathcal{P}}}_p(K)$, $\boldsymbol{\mathcal{Q}}_p(K)$, and $\boldsymbol{\mathcal{P}}_{p|p}(K)$, the vector-valued functions whose components lie in $\mathcal{P}_p(K)$, respectively $\widetilde{\mathcal{P}}_p(K)$, $\mathcal{Q}_p(K)$, and $\mathcal{P}_{p|p}(K)$.

Whenever there is no possible confusion, we write $\mathcal{P}_p$ instead of $\mathcal{P}_p(K)$.

2.2. Compatible exact sequences. Here we introduce the concept of *compatible exact sequences* in one-, two-, and three-space dimensions, which is just a reformulation, in our notation, of the *compatible* FES in differential form language introduced in [12, Def. 5.12].

The main result of a compatible FES in [12, Prop. 5.44], see also [13, Prop. 2.8], provides the equivalence of a compatible FES with a FES admitting a commuting diagram. This powerful result reduces the search for a commuting diagram to that of a compatible FES (or compatible exact sequence in our notation). In it, the *harmonic interpolator*, a generalization of the projection-based interpolation operator proposed in [18, 19], was used to obtain the commuting diagram; we reformulate these harmonic interpolators in one-, two-, and three-space dimensions using our notation in the Appendix A.

Let us recall the notion of a compatible FES. We quote verbatim from Definition 2.3 in [13]:

> Consider now the following two conditions on an element system E on a cellular complex $\mathcal{T}$:
>
> - *Extensions.* For each $T \in \mathcal{T}$ and $k \in \mathbb{N}$, the restriction operator (pullback to the boundary) $E^k(T) \longrightarrow E^k(\partial T)$ is onto. The kernel of this map is denoted $E_0^k(T)$.
> - *Local exactness.* The following sequence is exact for each $T \in \mathcal{T}$:
>
> $$0 \longrightarrow \mathbb{R} \longrightarrow E^0(T) \xrightarrow{d} E^1(T) \xrightarrow{d} \cdots \xrightarrow{d} E^{\dim T}(T) \longrightarrow 0.$$
>
>   The second arrow sends an element of $\mathbb{R}$ to the constant function on T taking this value.
>
> We will say that an element system *admits extensions* if the first condition holds, is *locally exact* if the second condition holds and is *compatible* if both hold.

For a comprehensive theory of FESs, a precise definition of a *cellular complex* and an *element system*, we refer to [12, sect. 5] and [13]. See also [11] where the concept of a minimal compatible FES was introduced.

Now, we are ready to reformulate the notion of a compatible FES in one- to three-space dimensions in our notation.

Definition 2.4 (One-dimensional compatible exact sequence). *Let K be a segment. Consider the finite-dimensional exact sequence*

$$\mathrm{S}^{1\mathrm{d}}(K): \quad H^{1\mathrm{d}}(K) \xrightarrow{\nabla} W^{1\mathrm{d}}(K).$$

We say that the sequence $\mathrm{S}^{1\mathrm{d}}(K)$ *is a* compatible exact sequence *if*

(i) $\dim \mathrm{tr}_H H^{1\mathrm{d}}(K) = \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 = 2.$

Definition 2.5 (Two-dimensional compatible exact sequence). *Let K be a polygon. Consider the finite-dimensional exact sequence*

$$\mathrm{S}^{2\mathrm{d}}(K): \quad H^{2\mathrm{d}}(K) \xrightarrow{\nabla} E^{2\mathrm{d}}(K) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(K),$$

and, for every edge $\mathrm{e} \in \mathcal{E}(K)$*, its sequence of traces*

$$\mathrm{tr}^{\mathrm{e}}(\mathrm{S}^{2\mathrm{d}}(K)): \quad H^{1\mathrm{d}}(\mathrm{e}) \xrightarrow{\nabla} W^{1\mathrm{d}}(\mathrm{e}),$$

where $H^{1\mathrm{d}}(\mathrm{e}) := \mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}(K)$ *and* $W^{1\mathrm{d}}(\mathrm{e}) := \mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(K)$. *We say that the sequence* $\mathrm{S}^{2\mathrm{d}}(K)$ *is a* compatible exact sequence *if*

(i) *For each edge* $\mathrm{e} \in \mathcal{E}(K)$, *the sequence* $\mathrm{tr}^{\mathrm{e}}(\mathrm{S}^{2\mathrm{d}}(K))$ *is a (one-dimensional) compatible exact sequence,*

(ii) $$\begin{cases} \dim \mathrm{tr}_H(H^{2\mathrm{d}}(K)) = \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 + \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}),\\ \dim \mathrm{tr}_E(E^{2\mathrm{d}}(K)) = \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim W^{1\mathrm{d}}(\mathrm{e}). \end{cases}$$

Definition 2.6 (Three-dimensional compatible exact sequence). *Let* K *be a polyhedron. Consider the exact sequence*

$$\mathrm{S}^{3\mathrm{d}}(K): \quad H^{3\mathrm{d}}(K) \xrightarrow{\nabla} E^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(K)$$

and its sequences of traces for all faces $\mathrm{f} \in \mathcal{F}(K)$

$$\mathrm{tr}^{\mathrm{f}}(\mathrm{S}^{3\mathrm{d}}(K)): \quad H^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla} E^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(\mathrm{f})$$

and all edges $\mathrm{e} \in \mathcal{E}(K)$

$$\mathrm{tr}^{\mathrm{e}}(\mathrm{S}^{3\mathrm{d}}(K)): \quad H^{1\mathrm{d}}(\mathrm{e}) \xrightarrow{\nabla} W^{1\mathrm{d}}(\mathrm{e}),$$

where $H^{2\mathrm{d}}(\mathrm{f}) \times E^{2\mathrm{d}}(\mathrm{f}) \times W^{2\mathrm{d}}(\mathrm{f}) := \mathrm{tr}_H^{\mathrm{f}} H^{3\mathrm{d}}(K) \times \mathrm{tr}_E^{\mathrm{f}} E^{3\mathrm{d}}(K) \times \mathrm{tr}_V^{\mathrm{f}} V^{3\mathrm{d}}(K)$, *and* $H^{1\mathrm{d}}(\mathrm{e}) \times W^{1\mathrm{d}}(\mathrm{e}) := \mathrm{tr}_H^{\mathrm{e}} H^{3\mathrm{d}}(K) \times \mathrm{tr}_E^{\mathrm{e}} E^{3\mathrm{d}}(K)$.

We say that the sequence $\mathrm{S}^{3\mathrm{d}}(K)$ *is a* compatible exact sequence *if*

(i) *For each face* f, *the sequence* $\mathrm{tr}^{\mathrm{f}}(\mathrm{S}^{3\mathrm{d}}(K))$ *is a (two-dimensional) compatible exact sequence.*

(ii) $$\begin{cases} \dim \mathrm{tr}_H H^{3\mathrm{d}}(K) = \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 + \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}) + \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim \overset{\circ}{H}{}^{2\mathrm{d}}(\mathrm{f}),\\ \dim \mathrm{tr}_E E^{3\mathrm{d}}(K) = \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim W^{1\mathrm{d}}(\mathrm{e}) + \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim \overset{\circ}{E}{}^{2\mathrm{d}}(\mathrm{f}),\\ \dim \mathrm{tr}_V V^{3\mathrm{d}}(K) = \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim W^{2\mathrm{d}}(\mathrm{f}). \end{cases}$$

Note that our definition is inductive on the space dimension. This is a slight difference from the original, more compact definition of a compatible FES quoted above. We find such inductive definition useful to express our main results in the next subsection.

The following result establishes the equivalence of compatible exact sequence and compatible FES.

Proposition 2.7. *Let* $K \in \mathbb{R}^d$ $(d = 1, 2, 3)$ *be a* d*-dimensional polytope. Then, with a change of scalar/vector fields to differential forms, a compatible exact sequence on* K *defined above is the corresponding compatible FES on* $\triangleleft(K)$, *the set that contains the element* K *and its vertices, edges (if* $d \geq 2$*), and faces (if* $d = 3$*).*

Proof. The result is proven by induction on the space dimension. The result in one dimension is trivial. We skip the proof for the two-dimensional case and focus on the proof of the three-dimensional case.

Compatible exact sequence $\Longrightarrow$ compatible FES: The dimension count equalities ensure the corresponding restriction operators (traces) from K to ∂K are onto, by induction hypothesis the restriction operators from a face f of K to $\partial\mathrm{f}$ are onto, and the restriction operators from an edge e of K to $\partial\mathrm{e}$ are onto; hence the FES admits an extension. The exactness of the sequence $\mathrm{S}^{3\mathrm{d}}(K)$ along with the compatibility and exactness of the trace sequences $\mathrm{tr}^{\mathrm{f}}(\mathrm{S}^{3\mathrm{d}}(K))$ ensure local exactness.

Compatible FES $\Longrightarrow$ compatible exact sequence: By induction hypothesis, the trace sequences are (two-dimensional) compatible exact sequences. The dimension count equalities are direct consequences of onto of the restriction operators from K to ∂K. This completes the proof. ∎

The following result is a direct consequence of [13, Prop. 2.8].

Proposition 2.8. *Let $K \in \mathbb{R}^d$ $(d = 1, 2, 3)$ be a d-dimensional polytope. The following statements are equivalent:*

- *$S(K)$ is a compatible exact sequence on K.*
- *$S(K)$ admits a commuting diagram.*

Note that, in the proof of [13, Prop. 2.8], the so-called *harmonic interpolator* is used to obtain the commuting diagram. We give a reformulation of this concept in our notation in Appendix A. These harmonic interpolators naturally induce a set of degrees of freedom for the corresponding spaces; see [13, Prop. 2.8]. The choice of degrees of freedom is certainly not unique and one needs to pick them so as to strike a balance between mathematical simplicity and computational efficiency. In this paper, we do not intend to answer the question of which set of degrees of freedom (or basis functions) shall be used. We refer the reader to [1, 28, 5, 20] for examples of degrees of freedom (or basis functions) for spaces involving polynomials of high degree.

2.3. The construction of compatible exact sequences. Here, we give our main results on the systematic construction of compatible exact sequences. Again, for each space dimension, we have a specific construction. The corresponding proofs are given in section 4. The resulting compatible exact sequences are all minimal sequences containing a prescribed exact sequence in the sense of [11, Cor. 3.2].

The one-dimensional case. In one-space dimension, the construction of compatible exact sequences is fairly simple.

Theorem 2.9. *Let K be a segment. Let any given exact sequence*

$$\mathrm{S}_g^{1\mathrm{d}}(K): \quad H_g^{1\mathrm{d}}(K) \xrightarrow{\nabla} W_g^{1\mathrm{d}}(K)$$

be such that $\mathcal{P}_0(K) \subset W_g^{1\mathrm{d}}(K)$. Then, it is compatible.

The two-dimensional case. In two-space dimensions, the systematic construction of compatible exact sequence is more involved than in the one-dimensional case, not only because of the geometry but also because we seek a compatible exact sequence with certain given traces on each of the edges of the element. Those traces are compatible exact sequences (found while dealing with the one-dimensional case) which we gather in the set $\mathrm{S}^{1\mathrm{d}}(\partial K)$. The sequence with which we begin the construction must then be what we call $\mathrm{S}^{1\mathrm{d}}(\partial K)$-*admissible*. We define this term next.

Definition 2.10 ($\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible exact sequence). *Let*

$$\mathrm{S}^{1\mathrm{d}}(\partial K) := \{\mathrm{S}^{1\mathrm{d}}(\mathrm{e}): \quad H^{1\mathrm{d}}(\mathrm{e}) \xrightarrow{\nabla} W^{1\mathrm{d}}(\mathrm{e}) \quad \forall e \in \mathcal{E}(K)\}$$

be a set of one-dimensional exact sequences. Then, we say that a given two-dimensional exact sequence

$$H_g^{2\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{2\mathrm{d}}(K) \xrightarrow{\nabla\times} W_g^{2\mathrm{d}}(K)$$

is $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible if

(i) $\mathrm{tr}^{\mathrm{e}}_H\left(H^{2\mathrm{d}}_g(K)\right) \times \mathrm{tr}^{\mathrm{e}}_E\left(E^{2\mathrm{d}}_g(K)\right) \subset H^{1\mathrm{d}}(\mathrm{e}) \times W^{1\mathrm{d}}(\mathrm{e}) \quad \forall\ e \in \mathcal{E}(K)$.
(ii) $\mathcal{P}_0(K) \subset W^{2\mathrm{d}}_g(K)$.

The next theorem is the main result of the two-dimensional case. It shows how to construct a compatible exact sequence by suitably enriching a given $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible exact sequence. The compatible exact sequence we seek is such that its traces on the edges *coincide* with the exact sequences of the set $\mathrm{S}^{1\mathrm{d}}(\partial K)$.

THEOREM 2.11. *Let K be a polygon, let*

$$\mathrm{S}^{1\mathrm{d}}(\partial K) = \{\mathrm{S}^{1\mathrm{d}}(\mathrm{e}) : \quad H^{1\mathrm{d}}(\mathrm{e}) \xrightarrow{\nabla} W^{1\mathrm{d}}(\mathrm{e}) \quad \forall e \in \mathcal{E}(K)\}$$

be a set of compatible exact sequences*, and let*

$$H^{2\mathrm{d}}_g(K) \xrightarrow{\nabla} E^{2\mathrm{d}}_g(K) \xrightarrow{\nabla\times} W^{2\mathrm{d}}_g(K)$$

be a given $\mathrm{S}^{1\mathrm{d}}(\partial K)$*-admissible exact sequence. Let the space $\delta H^{2\mathrm{d}}_g(K) \subset H^1(K)$ satisfy the following properties:*
(i) $\mathrm{tr}^{\mathrm{e}}_H \delta H^{2\mathrm{d}}_g(K) \subset H^{1\mathrm{d}}(\mathrm{e})$ *for all edges* $\mathrm{e} \in \mathcal{E}(K)$.
(ii) $\delta H^{2\mathrm{d}}_g(K) \cap H^{2\mathrm{d}}_g(K) = \{0\}$.
(iii) $\{v \in H^{2\mathrm{d}}_g(K) \oplus \delta H^{2\mathrm{d}}_g(K) : \ \mathrm{tr}_H v = 0\} = \overset{\circ}{H}{}^{2\mathrm{d}}_g(K)$.
(iv) $\dim \delta H^{2\mathrm{d}}_g(K) = \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 + \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim \overset{\circ}{H}{}^{1\mathrm{d}}(e) + \dim \overset{\circ}{H}{}^{2\mathrm{d}}_g(K) - \dim H^{2\mathrm{d}}_g(K)$.

Then, the following sequence

$$\mathrm{S}^{2\mathrm{d}}(K): \quad H^{2\mathrm{d}}_g(K) \oplus \delta H^{2\mathrm{d}}_g(K) \quad \xrightarrow{\nabla} \quad E^{2\mathrm{d}}_g(K) \oplus \nabla\, \delta H^{2\mathrm{d}}_g(K) \quad \xrightarrow{\nabla\times} \quad W^{2\mathrm{d}}_g(K)$$

is a compatible exact sequence. Moreover, it is also a minimal compatible exact sequence containing the exact sequence

$$H^{2\mathrm{d}}_g(K) \xrightarrow{\nabla} E^{2\mathrm{d}}_g(K) \xrightarrow{\nabla\times} W^{2\mathrm{d}}_g(K).$$

Let us relate this result with the theory of M-decompositions developed in [16]. That theory has to do with the right-most part of the commuting diagram. Since the operator $\nabla\times$ was replaced by the divergence operator $\nabla\cdot$ and since

$$\nabla \times (v_1, v_2) = -\partial_{x_2} v_1 + \partial_{x_1} v_2 = \nabla \cdot (v_2, -v_1) = \nabla \cdot (v_1, v_2)^{\mathrm{rot}},$$

we have that $\nabla \times E^{2\mathrm{d}}(K) = \nabla \cdot E^{2\mathrm{d}}_{\mathrm{rot}}(K)$, with the obvious notation. Moreover, it was shown in [16, Prop. 5.1] that $\left(E^{2\mathrm{d}}_g(K) \oplus \nabla\, \delta H^{2\mathrm{d}}_g(K)\right)^{\mathrm{rot}} \times W^{2\mathrm{d}}_g(K)$ is the *smallest* space containing $\left(E^{2\mathrm{d}}_g(K)\right)^{\mathrm{rot}} \times W^{2\mathrm{d}}_g(K)$ which admits an $M(\partial K)$-decomposition with the trace space

$$M(\partial K) := \{\mu \in L^2(\partial K) : \ \ \mu|_{\mathrm{e}} \in W^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K)\}.$$

The three-dimensional case. Now, we present our most involved result, the general construction in three-space dimensions. As we did in the two-dimensional case, we have to provide a set of compatible exact sequences on the faces of the element K, $\mathrm{S}^{2\mathrm{d}}(\partial K)$; they were previously obtained when dealing with the two-dimensional case. Then, we have to start from certain exact sequences we call $\mathrm{S}^{2\mathrm{d}}(\partial K)$-*admissible* which we define next.

DEFINITION 2.12 ($\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible exact sequence). *Let*

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \{\mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : \quad H^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla} E^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(\mathrm{f}),\ \mathrm{f} \in \mathcal{F}(K)\}$$

be a set of two-dimensional exact sequences. We say that a given three-dimensional exact sequence

$$H_g^{3\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W_g^{3\mathrm{d}}(K)$$

is $\mathrm{S}^{2\mathrm{d}}(\partial K)$*-admissible if*

(i) $\mathrm{tr}_H^{\mathrm{f}} H^{3\mathrm{d}}(K) \times \mathrm{tr}_E^{\mathrm{f}} E_g^{3\mathrm{d}}(K) \times \mathrm{tr}_V^{\mathrm{f}} V_g^{3\mathrm{d}}(K) \subset H^{2\mathrm{d}}(\mathrm{f}) \times E^{2\mathrm{d}}(\mathrm{f}) \times W^{2\mathrm{d}}(\mathrm{f})\ \forall \mathrm{f} \in \mathcal{F}(K)$.
(ii) $\mathcal{P}_0(K) \subset W_g^{3\mathrm{d}}(K)$.

The following theorem is our main result. It shows how to enrich an $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible exact sequence to get a commuting exact sequence. Note also that the traces on the faces of the sequence we seek must *coincide* with the sequence of traces in the set $\mathrm{S}^{2\mathrm{d}}(\partial K)$.

THEOREM 2.13. *Let K be a polyhedron and let*

$$\mathrm{S}^{2\mathrm{d}}(\partial K) = \{\mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : \quad H^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla} E^{2\mathrm{d}}(\mathrm{f}) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(\mathrm{f}),\ \mathrm{f} \in \mathcal{F}(K)\}$$

be a set of compatible exact sequences *satisfying the following* compatibility *condition*

$$\mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}(\mathrm{f}_1) \times \mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(\mathrm{f}_1) = \mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}(\mathrm{f}_2) \times \mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(\mathrm{f}_2) =: H^{1\mathrm{d}}(\mathrm{e}) \times W^{1\mathrm{d}}(\mathrm{e}),$$

for any faces $\mathrm{f}_1, \mathrm{f}_2 \in \mathcal{F}(K)$ sharing an edge $\mathrm{e} := \mathcal{E}(\mathrm{f}_1) \cap \mathcal{E}(\mathrm{f}_2)$. Let

$$H_g^{3\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W_g^{3\mathrm{d}}(K)$$

be a given $\mathrm{S}^{2\mathrm{d}}(\partial K)$*-admissible exact sequence.*

Let the spaces $\delta H_g^{3\mathrm{d}}(K) \times \delta E_g^{3\mathrm{d}}(K) \subset H^1(K) \times H(\mathrm{curl}, K)$ satisfy the following properties:

Properties of $\delta H_g^{3\mathrm{d}}(K)$

(i) $\mathrm{tr}_H^{\mathrm{f}} \delta H_g^{3\mathrm{d}}(K) \subset H^{2\mathrm{d}}(\mathrm{f})$ *for all faces* $\mathrm{f} \in \mathcal{F}(K)$.
(ii) $\delta H_g^{3\mathrm{d}}(K) \cap H_g^{3\mathrm{d}}(K) = \{0\}$.
(iii) $\{v \in H_g^{3\mathrm{d}}(K) \oplus \delta H_g^{3\mathrm{d}}(K) :\ \mathrm{tr}_H v = 0\} = \overset{\circ}{H}{}_g^{3\mathrm{d}}(K)$.
(iv) $$\dim \delta H_g^{3\mathrm{d}}(K) = \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 + \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}) + \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim \overset{\circ}{H}{}^{2\mathrm{d}}(\mathrm{f}) + \dim \overset{\circ}{H}{}_g^{3\mathrm{d}}(K) - \dim H_g^{3\mathrm{d}}(K).$$

Properties of $\delta E_g^{3\mathrm{d}}(K)$

(i) $\mathrm{tr}_E^{\mathrm{f}} \delta E_g^{3\mathrm{d}}(K) \subset E^{2\mathrm{d}}(\mathrm{f})$ *for all faces* $\mathrm{f} \in \mathcal{F}(K)$.
(ii) $\nabla\times \delta E_g^{3\mathrm{d}}(K) \cap V_g^{3\mathrm{d}}(K) = \{0\}$.
(iii) $\{v \in V_g^{3\mathrm{d}}(K) \oplus \nabla\times \delta E_g^{3\mathrm{d}}(K) :\ \mathrm{tr}_V v = 0,\ \nabla\cdot v = 0\} = \{v \in \overset{\circ}{V}{}_g^{3\mathrm{d}}(K) :\ \nabla\cdot v = 0\}$.
(iv) $$\dim \delta E_g^{3\mathrm{d}}(K) = \dim \nabla\times \delta E_g^{3\mathrm{d}}(K) = \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim W^{2\mathrm{d}}(\mathrm{f}) + \dim \overset{\circ}{W}{}_g^{3\mathrm{d}}(K) + \dim\{v \in \overset{\circ}{V}{}_g^{3\mathrm{d}}(K) :\ \nabla\cdot v = 0\} - \dim V_g^{3\mathrm{d}}(K).$$

Then, the sequence

$$\mathrm{S}^{3\mathrm{d}}(K): \begin{array}{c} H_g^{3\mathrm{d}}(K) \\ \oplus\, \delta H_g^{3\mathrm{d}}(K) \end{array} \xrightarrow{\nabla} \begin{array}{c} E_g^{3\mathrm{d}}(K) \\ \oplus\, \nabla\, \delta H_g^{3\mathrm{d}}(K) \oplus \delta E_g^{3\mathrm{d}}(K) \end{array} \xrightarrow{\nabla\times} \begin{array}{c} V_g^{3\mathrm{d}}(K) \\ \oplus\, \nabla\times\, \delta E_g^{3\mathrm{d}}(K) \end{array} \xrightarrow{\nabla\cdot} W_g^{3\mathrm{d}}(K)$$

is a commuting exact sequence. Moreover, it is a minimal commuting exact sequence containing the exact sequence

$$H_g^{3\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W_g^{3\mathrm{d}}(K).$$

Note that, unlike the two-dimensional case, we are requiring the sequences of the set $\mathrm{S}^{2\mathrm{d}}(\partial K)$ to satisfy a compatibility condition on each of the edges of the polyhedron. Such a compatibility condition was automatically satisfied, and hence was not required, in the two-dimensional case.

Let us relate this result with the theory of M-decompositions introduced in [16]. As for the two-dimensional case, the part of this theory concerned with mixed methods is associated with the right-most side of the diagram. Indeed, in [16, Prop. 5.1] it is shown that the space $\big(V_g^{3\mathrm{d}}(K) \oplus \nabla\times\, \delta E_g^{3\mathrm{d}}(K)\big) \times W_g^{3\mathrm{d}}(K)$ is the *smallest* one containing $V_g^{3\mathrm{d}}(K) \times W_g^{3\mathrm{d}}(K)$ and admitting an $M(\partial K)$-decomposition with the trace space

$$M(\partial K) := \{\mu \in L^2(\partial K) : \ \ \mu|_{\mathrm{f}} \in W^{2\mathrm{d}}(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K)\}.$$

3. Applications.

Now, we apply our main results on the systematic construction of compatible exact sequences in section 2 to explicitly construct them on various element shapes in one-, two-, and three-space dimensions.

To be consistent with the notation used in the Introduction, we refer to the *compatible exact sequences* as *commuting exact sequences*, bearing in mind that they are exactly the same object.

3.1. The one-dimensional case.

In one-space dimension, we consider the element $K \subset \mathbb{R}^1$ to be the reference interval $\{x : \ 0 < x < 1\}$. The most useful commuting exact sequence on K is given below.

THEOREM 3.1. *Let $K \subset \mathbb{R}^1$ be the reference interval with coordinate x. Then, the following sequence on K is a commuting exact sequence for $k \geq 0$:*

$$\mathcal{P}_{k+1}(x) \xrightarrow{\nabla} \mathcal{P}_k(x).$$

3.2. The two-dimensional case.

In two-space dimensions, we proceed as follows. We first consider the element K to be either the reference triangle $\{(x, y) : \ x > 0, y > 0, x + y < 1\}$ or the reference square $\{(x, y) : \ 0 < x < 1, 0 < y < 1\}$.

We present two commuting exact sequences on the reference triangle and four on the reference square whose spaces only consist of polynomials. All the sequences, except the second one on the reference square, are well-known in the literature. Then, we consider the case in which K is a general polygon, and present two *new* commuting exact sequences which contain nonpolynomial functions; this is based on the results in [14] on the construction of M-decompositions in two dimensions. All these commuting exact sequences are the *smallest* ones, as stated in Theorem 2.11, that contain a certain given exact sequence and have a certain prescribed sequence of traces on each edge.

We end by obtaining commuting exact sequences on two-dimensional polygonal meshes.

Triangle.

THEOREM 3.2. *Let K be the reference triangle with coordinates (x, y). Then, the following two sequences on K are commuting exact sequences for $k \geq 0$,*

$$\begin{array}{llcccl} \mathrm{S}^{\triangle}_{1,\mathrm{k}}(K): & \mathcal{P}_{k+2}(x,y) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_{k+1}(x,y) & \xrightarrow{\nabla\times} & \mathcal{P}_k(x,y), \\ \mathrm{S}^{\triangle}_{2,\mathrm{k}}(K): & \mathcal{P}_{k+1}(x,y) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_{k}(x,y) \oplus \boldsymbol{x} \times \widetilde{\mathcal{P}}_k(x,y) & \xrightarrow{\nabla\times} & \mathcal{P}_k(x,y). \end{array}$$

Here $\boldsymbol{x} \times p = (y\,p, -x\,p)^t$ for a scalar function p.

Moreover, the sequence of traces for $\mathrm{S}^{\triangle}_{1,\mathrm{k}}$ on an edge $\mathrm{e} \in \mathcal{E}(K)$ is

$$\mathrm{tr}^{\mathrm{e}}\left(\mathrm{S}^{\triangle}_{1,\mathrm{k}}(K)\right) : \mathcal{P}_{k+2}(\mathrm{e}) \longrightarrow \mathcal{P}_{k+1}(\mathrm{e}),$$

and that for $\mathrm{S}^{\triangle}_{2,\mathrm{k}}$ is

$$\mathrm{tr}^{\mathrm{e}}\left(\mathrm{S}^{\triangle}_{2,\mathrm{k}}(K)\right) : \mathcal{P}_{k+1}(\mathrm{e}) \longrightarrow \mathcal{P}_{k}(\mathrm{e}).$$

These two sequences are well-known. Indeed, the first sequence $\mathrm{S}^{\triangle}_{1,\mathrm{k}}$ is mainly due to Brezzi, Douglas, and Marini [9] since its $H(\mathrm{curl})$-space is a ninety-degree rotation, that is, the mapping (v_1, v_2) to $(-v_2, v_1)$, of the $H(\mathrm{div})$-space, usually called the BDM space of degree $k+1$. Its H^1 and L^2 spaces are the Lagrange polynomial spaces of degree $k+2$ and discontinuous polynomial space of degree k. The second sequence $\mathrm{S}^{\triangle}_{2,\mathrm{k}}$ is mainly due to Raviart and Thomas [27] since its $H(\mathrm{curl})$-space is a ninety-degree rotation of the $H(\mathrm{div})$-space, usually called the RT space of degree k. Its H^1- and L^2-spaces are the Lagrange polynomial spaces of degree $k+1$ and discontinuous polynomial space of degree k.

Square.

THEOREM 3.3. *Let K be the reference square with coordinates (x, y). Then, the following four sequences are commuting exact sequences for $k \geq 0$:*

$$\begin{array}{llcccl} \mathrm{S}^{\square}_{1,\mathrm{k}}(K): & \begin{array}{c}\mathcal{P}_{k+2}(x,y)\\ \oplus\delta H^{2,I}_{k+2}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+1}(x,y)\\ \oplus\nabla\,\delta H^{2,I}_{k+2}\end{array} & \xrightarrow{\nabla\times} & \mathcal{P}_k(x,y), \\ \mathrm{S}^{\square}_{2,\mathrm{k}}(K): & \begin{array}{c}\mathcal{P}_{k+1}(x,y)\\ \oplus\delta H^{2,I}_{k+1}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k}(x,y) \oplus \boldsymbol{x} \times \widetilde{\mathcal{P}}_k(x,y)\\ \oplus\nabla\,\delta H^{2,I}_{k+1}\end{array} & \xrightarrow{\nabla\times} & \mathcal{P}_k(x,y), \\ \hline \mathrm{S}^{\square}_{3,\mathrm{k}}(K): & \begin{array}{c}\mathcal{Q}_{k}(x,y) \oplus \{x^{k+1}, y^{k+1}\}\\ \oplus\delta H^{2,I}_{k+1}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{Q}}_{k}(x,y) \oplus \boldsymbol{x} \times \{x^k y^k\}\\ \oplus\nabla\,\delta H^{2,I}_{k+1}\end{array} & \xrightarrow{\nabla\times} & \mathcal{Q}_k(x,y), \\ \mathrm{S}^{\square}_{4,\mathrm{k}}(K): & \mathcal{Q}_{k+1}(x,y) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{Q}}_{k}(x,y) \oplus \begin{pmatrix} y^{k+1}\mathcal{P}_k(x) \\ x^{k+1}\mathcal{P}_k(y) \end{pmatrix} & \xrightarrow{\nabla\times} & \mathcal{Q}_k(x,y). \end{array}$$

Here the additional space $\delta H^{2,I}_k$, for $k \geq 1$, takes the following form:

$$\delta H^{2,I}_k := \mathrm{span}\{x\,y^k, y\,x^k\}.$$

Moreover, the sequence of traces for $\mathrm{S}^{\square}_{1,\mathrm{k}}$ on an edge $\mathrm{e} \in \mathcal{E}(K)$ is

$$\mathrm{tr}^{\mathrm{e}}\left(\mathrm{S}^{\square}_{1,\mathrm{k}}(K)\right) : \mathcal{P}_{k+2}(\mathrm{e}) \longrightarrow \mathcal{P}_{k+1}(\mathrm{e}),$$

and that for $\mathrm{S}^{\square}_{\mathrm{i},\mathrm{k}}$ with $i \in \{2, 3, 4\}$ is

$$\mathrm{tr}^{\mathrm{e}}\left(\mathrm{S}^{\square}_{\mathrm{i},\mathrm{k}}(K)\right) : \mathcal{P}_{k+1}(\mathrm{e}) \longrightarrow \mathcal{P}_{k}(\mathrm{e}).$$

Note that for $k = 0$, the last three sequences are exactly the same. Here the second sequence is new and the other three are well-known. The first sequence $\mathrm{S}^{\square}_{1,\mathrm{k}}$ is mainly due to Brezzi, Douglas, and Marini [9] since its $H(\mathrm{curl})$-space is a ninety-degree rotation of the $H(\mathrm{div})$-space, usually called the BDM space, obtained in [9], of degree $k+1$ on the square. Its H^1- and L^2-spaces are the serendipity polynomial spaces of degree $k+2$ and discontinuous polynomial space of degree k. The second sequence $\mathrm{S}_{2,\mathrm{k}}^{\square}$ is a new one resulting in a new family of $H(\mathrm{curl})$-spaces. The third sequence is the TNT sequence [17] on the square. The foruth is mainly due to Raviart and Thomas [27] since its $H(\mathrm{curl})$-space is a ninety-degree rotation of the $H(\mathrm{div})$-space, usually called the RT space, obtained in [27], of degree k on the square. Its H^1- and L^2-spaces are the tensor-product Lagrange polynomial space of degree $k+1$ and discontinuous tensor-product polynomial space of degree k.

As already mentioned in the Introduction, we note that our second sequence shares the same spaces as the one in the recent manuscript [22] where the authors carefully study the spaces in the differential form language (for any space dimension). Therein, a nice set of unisolvent degrees of freedom is also introduced. This remark carries over to our second sequence on the reference cube in Theorem 3.6 below. Moreover, the last part of the commuting diagram of the first and second sequences were also obtained in [2].

Polygon. The explicit construction of (high-order) commuting exact sequences on a general polygon K is not known. Here we fill this gap by presenting two families of commuting exact sequences by applying Theorem 2.11. To do so, we take advantage of the recent results on constructing M-decompositions in [14] to deal with the rightmost part of the diagram.

To state our result, we need to introduce some notation. Let $\{\mathrm{v}_i\}_{i=1}^{ne}$ be the set of vertices of the polygonal element K which we take to be counter-clockwise ordered. Let $\{\mathrm{e}_i\}_{i=1}^{ne}$ be the set of edges of K where the edge e_i connects the vertices v_i and v_{i+1}. Here the subindexes are integers module ne. We also define, for $1 \le i \le ne$, λ_i to be the linear function that vanishes on edge e_i and reaches maximum value 1 in the closure of the element K. To each vertex v_i, $i = 1, \ldots, ne$, we associate a function ξ_i satisfying the following conditions:

(L.1) $\xi_i \in H^1(K)$,
(L.2) $\xi_i|_{\mathrm{e}_j} \in \mathcal{P}_1(\mathrm{e}_j),\ j = 1, \ldots, ne$,
(L.3) $\xi_i(\mathrm{v}_j) = \delta_{i,j},\ j = 1, \ldots, ne$,

where $\delta_{i,j}$ is the Kronecker delta. Note that conditions (L.2) and (L.3) together ensure that the trace of ξ_i on the edges is only non-zero at e_i and e_{i+1}, where they are linear. These functions, with examples given in [14], are not polynomials if the polygon K is not a triangle or a parallelogram. Note that when the polygon K is star shaped with respect to an interior node v_o, we can obtain ξ_i by first subdividing the polygon into triangles, for example, by connecting v_o with all the vertices of the polygon, and then setting ξ_i to be a (piecewise) linear function on each subdivided triangle taking value 1 at the vertex v_i and 0 at all the other vertices and the node v_o.

Now, we are ready to state the result.

THEOREM 3.4. *Let K be a polygon of ne edges with coordinates (x, y) such that none of its edges lie on the same line. Then, the following two sequences on K are commuting exact sequences for $k \ge 0$,*

$$
\begin{array}{llcccc}
\mathrm{S}_{1,\mathrm{k}}^{\mathrm{poly}}(K): & \begin{array}{c}\mathcal{P}_{k+2}(x,y)\\ \oplus\delta H^{2,II}_{k+2}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+1}(x,y)\\ \oplus\nabla\,\delta H^{2,II}_{k+2}\end{array} & \xrightarrow{\nabla\times} & \mathcal{P}_k(x,y),\\
\mathrm{S}_{2,\mathrm{k}}^{\mathrm{poly}}(K): & \begin{array}{c}\mathcal{P}_{k+1}(x,y)\\ \oplus\delta H^{2,II}_{k+1}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k(x,y)\oplus\boldsymbol{x}\times\widetilde{\mathcal{P}}_k(x,y)\\ \oplus\nabla\,\delta H^{2,II}_{k+1}\end{array} & \xrightarrow{\nabla\times} & \mathcal{P}_k(x,y).
\end{array}
$$

Here the additional space $\delta H_k^{2,II}$*, for* $k \geq 1$*, takes the following form:*

$$\delta H_k^{2,II} = \oplus_{i=3}^{ne} \Psi_{i,k},$$

where

$$\Psi_{i,k} = \begin{cases} \mathrm{span}\{\xi_{i+1}\lambda_{i+1}^a : \ \max\{k+3-i, 0\} \leq a \leq k-1\} & \text{if } 3 \leq i \leq ne-1, \\ \mathrm{span}\{\xi_{i+1}\lambda_{i+1}^a : \ \max\{k+4-i, 1\} \leq a \leq k-1\} & \text{if } i = ne, \end{cases}$$

and the functions $\{\xi_i\}_{i=1}^{ne}$ *are assumed to satisfy conditions (L).*

Moreover, the sequence of traces for $\mathrm{S}_{1,\mathrm{k}}^{\mathrm{poly}}$ *on an edge* $\mathrm{e} \in \mathcal{E}(K)$ *is*

$$\mathrm{tr}^{\mathrm{e}}\left(\mathrm{S}_{1,\mathrm{k}}^{\triangle}(K)\right) : \mathcal{P}_{k+2}(\mathrm{e}) \longrightarrow \mathcal{P}_{k+1}(\mathrm{e}),$$

and that for $\mathrm{S}_{2,\mathrm{k}}^{\mathrm{poly}}$ *is*

$$\mathrm{tr}^{\mathrm{e}}\left(\mathrm{S}_{2,\mathrm{k}}^{\triangle}(K)\right) : \mathcal{P}_{k+1}(\mathrm{e}) \longrightarrow \mathcal{P}_{k}(\mathrm{e}).$$

Note that we can also deal with the case when the polygon K has edges lying on the same line; of course, the resulting additional space would be different. For example, in the case when K is a triangle with a hanging node, see the results in [14, sect. 4.2].

Note also that in the notation used in [14], we have that $\delta V_{\mathrm{fillM}} := \nabla\times\,\delta H_{k+1}^{2,II}$ is the *filling* space that guarantees that the pair $V \times W := \boldsymbol{\mathcal{P}}_k \oplus \delta V_{\mathrm{fillM}} \times \mathcal{P}_k$ admits an M-decomposition for the trace space

$$M(\partial K) := \{\mu \in L^2(\partial K) : \mu|_{\mathrm{e}} \in \mathcal{P}_k(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K)\}.$$

When K is a convex quadrilateral, a construction of $H(\mathrm{div})$-conforming spaces of the form $\boldsymbol{\mathcal{P}}_k(K) \oplus \nabla\times\,\delta H(K)$ was also presented in [2]. While the filling space $\nabla\times\,\delta H(K)$ shares the same dimension (1 if $k = 0$ and 2 if $k \geq 1$) as those in [14] on a convex quadrilateral, they were not constructed directly on the quadrilateral element K as was done in [14] but by two (one if $k = 0$) Piola-mapped divergence-free polynomial functions from the reference square to K; the resulting functions are rational functions. Moreover, $H(\mathrm{div})$-conforming finite element shape functions were proposed [2, sect. 5].

Let us now briefly comment on the special cases when K is a triangle or a parallelogram. In these cases, the element K can be considered as a physical element obtained from an affine mapping of the reference triangle or the reference square. We can easily obtain *mapped* commuting exact sequences on the physical element K from those in Theorem 3.2 on the reference triangle, and those in Theorem 3.3 on the reference square via proper linear mapping functions. To simplify the notation, we still denote the mapped sequences on a physical triangle K as $\mathrm{S}_{1,\mathrm{k}}^{\triangle}(K)$ and $\mathrm{S}_{2,\mathrm{k}}^{\triangle}(K)$, and those on the physical parallelogram as $\mathrm{S}_{\mathrm{i},\mathrm{k}}^{\square}(K)$ for $i \in \{1, 2, 3, 4\}$. These mapped sequences have an advantage over those in Theorem 3.4, namely, that numerical integration only needs to be done on the reference elements.

Whole mesh. Using the sequences in Theorem 3.2 and Theorem 3.3, we can readily obtain *mapped* commuting exact sequences on a hybrid mesh, $\Omega_h := \{K\}$, of a polygonal domain Ω, where each physical element K is an affine mapping of the reference triangle or the reference square. We obtain two families of *mapped* sequences with the following form,

$$\mathrm{S}_{2\mathrm{d}}(\Omega_h) : \quad H^{2\mathrm{d}}(\Omega_h) \xrightarrow{\nabla} E^{2\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\times} W^{2\mathrm{d}}(\Omega_h),$$

where $H^{2\mathrm{d}}(\Omega_h) \times E^{2\mathrm{d}}(\Omega_h) \times W^{2\mathrm{d}}(\Omega_h) \subset H^1(\Omega) \times H(\mathrm{curl}, \Omega) \times L^2(\Omega)$. The restriction on an element K of the first family of sequences is $\mathrm{S}^{\triangle}_{1,\mathrm{k}}(K)$ if K is a triangle, and is $\mathrm{S}^{\square}_{1,\mathrm{k}}(K)$ if K is a parallelogram. The restriction on an element K of the second family of sequences is $\mathrm{S}^{\triangle}_{2,\mathrm{k}}(K)$ if K is a triangle, and is any of $\mathrm{S}^{\square}_{\mathrm{i},\mathrm{k}}(K)$ for $i \in \{2, 3, 4\}$ if K is a parallelogram. The reason we are able to do so is due to the trace compatibility of the sequences $\mathrm{S}^{\triangle}_{1,\mathrm{k}}(K)$ and $\mathrm{S}^{\square}_{1,\mathrm{k}}(K)$, and the trace compatibility of the sequences $\mathrm{S}^{\triangle}_{2,\mathrm{k}}(K)$ and $\mathrm{S}^{\square}_{\mathrm{i},\mathrm{k}}$ for $i \in \{2, 3, 4\}$ in Theorem 3.2 and Theorem 3.3.

On the other hand, using the sequences in Theorem 3.4, we can readily obtain two families of *non-mapped* commuting exact sequences on a more general polygonal mesh, $\Omega_h := \{K\}$, of a polygonal domain Ω, where each physical element K is a polygon, and the restriction on an element K of the first family of sequences is $\mathrm{S}^{\mathrm{poly}}_{1,\mathrm{k}}(K)$, and that for the second is $\mathrm{S}^{\mathrm{poly}}_{2,\mathrm{k}}(K)$.

3.3. The three-dimensional case. In three-space dimensions, we first consider the element $K \subset \mathbb{R}^3$ to be any of the following four reference polyhedra:

	tetrahedron:	$\{(x, y, z) : 0 < x,\ 0 < y,\ 0 < z,\ x + y + z < 1\}$,
	hexahedron:	$\{(x, y, z) : 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\}$,
	prism:	$\{(x, y, z) : 0 < x,\ 0 < y,\ 0 < z < 1,\ x + y < 1\}$,
	pyramid:	$\{(x, y, z) : 0 < x,\ 0 < y,\ 0 < z,\ x + z < 1,\ y + z < 1\}$.

We present two commuting exact sequences on the reference tetrahedron, four on the reference hexahedron, four on the reference prism, and four on the reference pyramid. All these commuting exact sequences are the *smallest* ones, as described in Theorem 2.13, which contain a certain given exact sequence and have a certain prescribed sequence of traces on each face. We then obtain commuting exact sequences on polyhedral meshes made of mapped tetrahedra, hexahedra, prisms, and pyramids.

Tetrahedron.

THEOREM 3.5. *Let K be the reference tetrahedron with coordinates (x, y, z). Then, the following two sequences on K are exact for $k \geq 0$:*

$$\begin{array}{llcccccc}
\mathrm{S}^{\mathrm{tet}}_{1,\mathrm{k}}: & \mathcal{P}_{k+3} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_{k+2} & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{P}}_{k+1} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k, \\
\mathrm{S}^{\mathrm{tet}}_{2,\mathrm{k}}: & \mathcal{P}_{k+1} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x} \times \widetilde{\boldsymbol{\mathcal{P}}}_k & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x}\, \widetilde{\mathcal{P}}_k & \xrightarrow{\nabla\times} & \mathcal{P}_k.
\end{array}$$

Here $\boldsymbol{x} = (x, y, z)^t$ and $\mathcal{P}_k$ denotes $\mathcal{P}_k(x, y, z)$.

Moreover, the sequence of traces on a face f *for $\mathrm{S}^{\mathrm{tet}}_{1,\mathrm{k}}$ is the sequence $\mathrm{S}^{\triangle}_{1,\mathrm{k+1}}(\mathrm{f})$, and that for $\mathrm{S}^{\mathrm{tet}}_{2,\mathrm{k}}$ is the sequence $\mathrm{S}^{\triangle}_{2,\mathrm{k}}(\mathrm{f})$.*

Both of these sequences are well-known. The first sequence $\mathrm{S}^{\mathrm{tet}}_{1,\mathrm{k}}$ is mainly due to Nédélec [24] since its $H(\mathrm{curl})$- and $H(\mathrm{div})$-spaces are Nédélec's edge and face spaces of second kind of degree $k + 2$ and $k + 1$, respectively. Its H^1- and L^2- spaces are the Lagrange polynomial space of degree $k + 3$, and the discontinuous polynomial space of degree k. The second sequence $\mathrm{S}^{\mathrm{tet}}_{2,\mathrm{k}}$ is mainly due to Nédélec [23] since its $H(\mathrm{curl})$- and $H(\mathrm{div})$-spaces are Nédélec's edge and face spaces of first kind of degree k, respectively. Its H^1- and L^2-spaces are the Lagrange polynomial space of degree $k + 1$, and the discontinuous polynomial space of degree k, respectively.

Hexahedron.

THEOREM 3.6. *Let K be the reference hexahedron with coordinates (x, y, z). Then, the following four sequences are exact for $k \geq 0$:*

$$
\begin{array}{lcccccccc}
\mathrm{S}^{\boxplus}_{1,\mathrm{k}}: & \begin{array}{c}\mathcal{P}_{k+3}\\ \oplus\delta H^{3,I}_{k+3}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+2}\\ \oplus\nabla\,\delta H^{3,I}_{k+3}\\ \oplus\delta E^{3,I}_{k+2}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+1}\\ \oplus\nabla\times\,\delta E^{3,I}_{k+2}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k, \\
\mathrm{S}^{\boxplus}_{2,\mathrm{k}}: & \begin{array}{c}\mathcal{P}_{k+1}\\ \oplus\delta H^{3,I}_{k+1}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k}\oplus\boldsymbol{x}\times\widetilde{\boldsymbol{\mathcal{P}}}_k\\ \oplus\nabla\,\delta H^{3,I}_{k+1}\\ \oplus\delta E^{3,I}_{k+1}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k}\oplus\boldsymbol{x}\,\widetilde{\mathcal{P}}_k\\ \oplus\nabla\times\,\delta E^{3,I}_{k+1}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k, \\
\hline
\mathrm{S}^{\boxplus}_{3,\mathrm{k}}:: & \begin{array}{c}\mathcal{Q}_k\oplus\left\{\begin{array}{c}x^{k+1},\\ y^{k+1},\\ z^{k+1}\end{array}\right\}\\ \oplus\delta H^{3,II}_{k+1}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{Q}}_k\oplus\boldsymbol{x}\times\left\{\begin{array}{c}y^kz^k\,\nabla x,\\ z^kx^k\,\nabla y,\\ x^ky^k\,\nabla z\end{array}\right\}\\ \oplus\nabla\,\delta H^{3,II}_{k+1}\\ \oplus\delta E^{3,II}_{k+1}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{Q}}_k\oplus\boldsymbol{x}\,\{x^ky^kz^k\}\\ \oplus\nabla\times\,\delta E^{3,II}_{k+1}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{Q}_k, \\
\mathrm{S}^{\boxplus}_{4,\mathrm{k}}: & \mathcal{Q}_{k+1} & \xrightarrow{\nabla} & \begin{pmatrix}\mathcal{P}_{k,k+1,k+1}\\ \mathcal{P}_{k+1,k,k+1}\\ \mathcal{P}_{k+1,k+1,k}\end{pmatrix} & \xrightarrow{\nabla\times} & \begin{pmatrix}\mathcal{P}_{k+1,k,k}\\ \mathcal{P}_{k,k+1,k}\\ \mathcal{P}_{k,k,k+1}\end{pmatrix} & \xrightarrow{\nabla\cdot} & \mathcal{Q}_k.
\end{array}
$$

Here the additional spaces $\delta H^{3,I}_k, \delta H^{3,II}_k$ and $\delta E^{3,I}_k, \delta E^{3,II}_k$ for $k \geq 1$ take the following forms:

$$
\delta H^{3,I}_k := \operatorname{span}\left\{\begin{array}{c} x\,y\,z^k,\ y\,z\,x^k,\ z\,x\,y^k, \\ x\,\widetilde{\mathcal{P}}_k(y,z),\ y\,\widetilde{\mathcal{P}}_k(z,x),\ z\,\widetilde{\mathcal{P}}_k(x,y) \end{array}\right\},
$$

$$
\delta H^{3,II}_k := \operatorname{span}\left\{\begin{array}{c} x\,y\,z^k,\ y\,z\,x^k,\ z\,x\,y^k, \\ x\,y^k,\ y\,z^k,\ z\,x^k, \\ x\,z^k,\ y\,x^k,\ z\,y^k \end{array}\right\},
$$

$$
\delta E^{3,I}_k := \operatorname{span}\left\{\begin{array}{c} x\,\widetilde{\mathcal{P}}_{k-1}(y,z)(y\,\nabla z - z\,\nabla y), \\ y\,\widetilde{\mathcal{P}}_{k-1}(z,x)(z\,\nabla x - x\,\nabla z), \\ z\,\widetilde{\mathcal{P}}_{k-1}(x,y)(x\,\nabla y - y\,\nabla x) \end{array}\right\},
$$

$$
\delta E^{3,II}_k := \operatorname{span}\left\{\begin{array}{c} x(y^k\,\nabla z - z^k\,\nabla y),\ y(z^k\,\nabla x - x^k\,\nabla z), \\ z(x^k\,\nabla y - y^k\,\nabla x),\ x\,y^{k-1}\,z^{k-1}(y\,\nabla z - z\,\nabla y), \\ y\,z^{k-1}\,x^{k-1}(z\,\nabla x - x\,\nabla z),\ z\,x^{k-1}\,y^{k-1}(x\,\nabla y - y\,\nabla x) \end{array}\right\}.
$$

Moreover, the sequence of traces on a square face f *for $\mathrm{S}^{\boxplus}_{1,\mathrm{k}}$ is the sequence $\mathrm{S}^{\square}_{1,\mathrm{k+1}}(\mathrm{f})$, and that for $\mathrm{S}^{\boxplus}_{\mathrm{i,k}}$ is the sequence $\mathrm{S}^{\square}_{\mathrm{i,k}}(\mathrm{f})$ for $i \in \{2, 3, 4\}$.*

Note that for $k = 0$, the last three sequences are exactly the same. Here the second sequence is new; the other three are known. The first sequence $\mathrm{S}^{\boxplus}_{1,\mathrm{k}}$ is the serendipity sequence of Arnold and Awanou [3]. The second sequence is a slight variation of the first one. It displays a new family of $H(\mathrm{curl})$- and $H(\mathrm{div})$-spaces. The third sequence $\mathrm{S}^{\boxplus}_{3,\mathrm{k}}$ is the TNT sequence of Cockburn and Qiu [17] (with a slight variation in the space representation). The last sequence is mainly due to Nédélec [23] since its $H(\mathrm{curl})$- and $H(\mathrm{div})$-spaces are Nédélec's edge and face spaces of first kind of degree k on the cube, respectively. Its H^1- and L^2-spaces are the tensor-product Lagrange polynomial space of degree $k+1$ and the discontinuous tensor-product polynomial space of degree k respectively.

Prism.

THEOREM 3.7. *Let K be the reference prism with coordinates (x, y, z). Then, the following four sequences are exact for $k \geq 0$:*

$$
\begin{array}{lcccccccc}
\mathrm{S}_{1,\mathrm{k}}^{\boxdot}: & \begin{array}{c}\mathcal{P}_{k+3}\\ \oplus\delta H_{k+3}^{3,III}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+2}\\ \oplus\nabla\,\delta H_{k+3}^{3,III}\\ \oplus\delta E_{k+2}^{3,III}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+1}\\ \oplus\nabla\times\delta E_{k+2}^{3,III}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k,\\
\mathrm{S}_{2,\mathrm{k}}^{\boxdot}: & \begin{array}{c}\mathcal{P}_{k+1}\\ \oplus\delta H_{k+1}^{3,III}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\times\widetilde{\boldsymbol{\mathcal{P}}}_k\\ \oplus\nabla\,\delta H_{k+1}^{3,III}\\ \oplus\delta E_{k+1}^{3,III}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\widetilde{\mathcal{P}}_k\\ \oplus\nabla\times\delta E_{k+1}^{3,III}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k,\\
\hline
\mathrm{S}_{3,\mathrm{k}}^{\boxdot}: & \begin{array}{c}\mathcal{P}_{k|k}\\ \oplus\widetilde{\mathcal{P}}_{k+1}(x,y)\oplus\{z^{k+1}\}\\ \oplus\delta H_{k+1}^{3,III}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k|k}\\ \oplus\begin{pmatrix}\begin{pmatrix}y\\-x\end{pmatrix}\widetilde{\mathcal{P}}_k(x,y)\\ \widetilde{\mathcal{P}}_{k+1}(x,y)\,z^k\end{pmatrix}\\ \oplus\nabla\,\delta H_{k+1}^{3,III}\\ \oplus\delta E_{k+1}^{3,III}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k|k}\\ \oplus\begin{pmatrix}0\\0\\ \widetilde{\mathcal{P}}_k(x,y)\,z^{k+1}\end{pmatrix}\\ \oplus\nabla\times\delta E_{k+1}^{3,III}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_{k|k},\\
\mathrm{S}_{4,\mathrm{k}}^{\boxdot}: & \mathcal{P}_{k+1|k+1} & \xrightarrow{\nabla} & \begin{pmatrix}\boldsymbol{N}_k(x,y)\otimes\mathcal{P}_{k+1}(z)\\ \mathcal{P}_{k+1|k}\end{pmatrix} & \xrightarrow{\nabla\times} & \begin{pmatrix}\boldsymbol{RT}_k(x,y)\otimes\mathcal{P}_k(z)\\ \mathcal{P}_{k|k+1}\end{pmatrix} & \xrightarrow{\nabla\cdot} & \mathcal{P}_{k|k}.
\end{array}
$$

Here, we have

$$
\boldsymbol{N}_k(x,y) := \boldsymbol{\mathcal{P}}_k(x,y) \oplus \begin{pmatrix}-y\\x\end{pmatrix}\widetilde{\mathcal{P}}_k(x,y), \quad \boldsymbol{RT}_k(x,y) := \boldsymbol{\mathcal{P}}_k(x,y) \oplus \begin{pmatrix}x\\y\end{pmatrix}\widetilde{\mathcal{P}}_k(x,y),
$$

and the additional spaces $\delta H_k^{3,III}$ and $\delta E_k^{3,III}$ for $k \geq 1$ take the following forms:

$$
\begin{aligned}
\delta H_k^{3,III} &:= \operatorname{span}\left\{\; z^k\,\widetilde{\mathcal{P}}_1(x,y),\ z\,\widetilde{\mathcal{P}}_k(x,y)\;\right\},\\
\delta E_k^{3,III} &:= \operatorname{span}\left\{\; z^k(x\,\nabla y - y\,\nabla x),\ z\,\widetilde{\mathcal{P}}_{k-1}(x,y)(x\,\nabla y - y\,\nabla x)\;\right\}.
\end{aligned}
$$

Moreover, the sequence of traces for these four sequences are

$$
\mathrm{tr}^{\mathrm{f}}\left(\mathrm{S}_{1,\mathrm{k}}^{\boxdot}\right) = \begin{cases}\mathrm{S}_{1,\mathrm{k+1}}^{\triangle}(\mathrm{f}) & \textit{triangle } \mathrm{f},\\ \mathrm{S}_{1,\mathrm{k+1}}^{\square}(\mathrm{f}) & \textit{square } \mathrm{f},\end{cases} \qquad \mathrm{tr}^{\mathrm{f}}\left(\mathrm{S}_{2,\mathrm{k}}^{\boxdot}\right) = \begin{cases}\mathrm{S}_{2,\mathrm{k}}^{\triangle}(\mathrm{f}) & \textit{triangle } \mathrm{f},\\ \mathrm{S}_{2,\mathrm{k}}^{\square}(\mathrm{f}) & \textit{square } \mathrm{f},\end{cases}
$$

$$
\mathrm{tr}^{\mathrm{f}}\left(\mathrm{S}_{3,\mathrm{k}}^{\boxdot}\right) = \begin{cases}\mathrm{S}_{2,\mathrm{k}}^{\triangle}(\mathrm{f}) & \textit{triangle } \mathrm{f},\\ \widetilde{\mathrm{S}_{3,\mathrm{k}}^{\square}}(\mathrm{f}) & \textit{square } \mathrm{f},\end{cases} \qquad \mathrm{tr}^{\mathrm{f}}\left(\mathrm{S}_{1,\mathrm{k}}^{\boxdot}\right) = \begin{cases}\mathrm{S}_{2,\mathrm{k}}^{\triangle}(\mathrm{f}) & \textit{triangle } \mathrm{f},\\ \mathrm{S}_{4,\mathrm{k}}^{\square}(\mathrm{f}) & \textit{square } \mathrm{f},\end{cases}
$$

where the sequence $\widetilde{\mathrm{S}_{3,\mathrm{k}}^{\square}}(\mathrm{f})$ on the square face with coordinates (ξ, z) is a slight modification of the sequence $\mathrm{S}_{3,\mathrm{k}}^{\square}(\mathrm{f})$, with the same H^1- and L^2-spaces but a different $H(\mathrm{curl})$-space, namely, $\mathbf{Q}_k(\xi, z) \oplus \xi^{k+1} z^k\,\nabla z \oplus \nabla\{\xi\, z^{k+1}, z\,\xi^{k+1}\}$.

Note that for $k = 0$, the sequences $\mathrm{S}_{2,\mathrm{k}}^{\boxdot}$ and $\mathrm{S}_{4,\mathrm{k}}^{\boxdot}$ are exactly the same; they are slightly different from the sequence $\mathrm{S}_{3,\mathrm{k}}^{\boxdot}$. Note also that the $H(\mathrm{curl})$-space for $\widetilde{\mathrm{S}_{3,\mathrm{k}}^{\square}}(\mathrm{f})$ is not invariant under the coordinate permutation $(\xi, z) \to (z, \xi)$, where (ξ, z) are the coordinates on f. We did not find a prismatic sequence with trace space on a square face f exactly equal to $\mathrm{S}_{3,\mathrm{k}}^{\square}(\mathrm{f})$. This brings about a small complication for constructing the $H(\mathrm{curl})$-conforming finite element spaces on a hybrid mesh; see the discussion on commuting exact sequences on the whole polyhedral mesh below.

Finally, note that while the fourth sequence is known, see [20], the other three are new. These four sequences can be considered as the extensions of the related sequences on the reference cube to the reference prism.

Pyramid.

THEOREM 3.8. *Let K be the reference pyramid with coordinates (x, y, z). Then, the following four sequences are exact for $k \geq 0$:*

$$
\begin{array}{lccccccc}
\mathrm{S}_{1,k}^{\blacktriangle}: & \begin{array}{c}\mathcal{P}_{k+3}\\ \oplus\delta H_{k+3}^{3,IV}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+2}\\ \oplus\nabla\,\delta H_{k+3}^{3,IV}\\ \oplus\delta E_{k+2}^{3,IV}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_{k+1}\\ \oplus\nabla\times\delta E_{k+2}^{3,IV}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k,\\
\mathrm{S}_{2,k}^{\blacktriangle}: & \begin{array}{c}\mathcal{P}_{k+1}\\ \oplus\delta H_{k+1}^{3,IV}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\times\widetilde{\boldsymbol{\mathcal{P}}}_k\\ \oplus\nabla\,\delta H_{k+1}^{3,IV}\\ \oplus\delta E_{k+1}^{3,IV}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\,\widetilde{\mathcal{P}}_k\\ \oplus\nabla\times\delta E_{k+1}^{3,IV}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k,
\end{array}
$$

$$
\begin{array}{lccccccc}
\mathrm{S}_{3,k}^{\blacktriangle}: & \begin{array}{c}\mathcal{P}_{k+1}\\ \oplus\delta H_{k+1}^{3,V}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\times\widetilde{\boldsymbol{\mathcal{P}}}_k\\ \oplus\nabla\,\delta H_{k+1}^{3,V}\\ \oplus\delta E_{k+1}^{3,V}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\,\widetilde{\mathcal{P}}_k\\ \oplus\nabla\times\delta E_{k+1}^{3,V}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k,\\
\mathrm{S}_{4,k}^{\blacktriangle}: & \begin{array}{c}\mathcal{P}_{k+1}\\ \oplus\delta H_{k+1}^{3,VI}\end{array} & \xrightarrow{\nabla} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\times\widetilde{\boldsymbol{\mathcal{P}}}_k\\ \oplus\nabla\,\delta H_{k+1}^{3,VI}\\ \oplus\delta E_{k+1}^{3,V}\end{array} & \xrightarrow{\nabla\times} & \begin{array}{c}\boldsymbol{\mathcal{P}}_k\oplus\boldsymbol{x}\,\widetilde{\mathcal{P}}_k\\ \oplus\nabla\times\delta E_{k+1}^{3,V}\end{array} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k,
\end{array}
$$

Here the additional spaces $\delta H_k^{3,IV}, \delta H_k^{3,V}, \delta H_k^{3,VI}$ and $\delta E_k^{3,IV}, \delta E_k^{3,V}, \delta E_k^{3,VI}$ for $k \geq 1$ takes the following forms:

$$
\begin{aligned}
\delta H_k^{3,IV} &:= \mathrm{span}\left\{ \tfrac{x\,y}{1-z} z^{k-1}, \tfrac{x\,y\,z}{1-z}\widetilde{\mathcal{P}}_{k-2}(x,z), \tfrac{x\,y\,z}{1-z}\widetilde{\mathcal{P}}_{k-2}(y,z) \right\},\\
\delta H_k^{3,V} &:= \delta H_k^{3,IV} \oplus \mathrm{span}\left\{ \tfrac{x^\alpha\,y^\beta}{(1-z)^{\min\{\alpha,\beta\}}} : \begin{array}{c}\alpha=1,\beta=k \ \textit{or}\ \alpha=k,\beta=1\\ \textit{or}\ \alpha\leq k-1,\ \beta\leq k-1,\ k+1\leq\alpha+\beta\end{array} \right\},\\
\delta H_k^{3,VI} &:= \delta H_k^{3,IV} \oplus \mathrm{span}\left\{ \tfrac{x^\alpha\,y^\beta}{(1-z)^{\min\{\alpha,\beta\}}} : \quad \alpha\leq k,\ \beta\leq k,\ k+1\leq\alpha+\beta \right\},\\
\delta E_k^{3,IV} &:= \mathrm{span}\left\{ \tfrac{x\,y^k}{1-z}\nabla z, \tfrac{y\,x^k}{1-z}\nabla z, \tfrac{x\,y\,z}{1-z}\nabla x \right\},\\
\delta E_k^{3,V} &:= \delta E_k^{3,IV} \oplus \mathrm{span}\left\{ \tfrac{x^\alpha\,y^{\beta+1}}{(1-z)^{\min\{\alpha,\beta\}}}\nabla x : \quad \alpha\leq k,\ \beta\leq k,\ k+1\leq\alpha+\beta \right\}.
\end{aligned}
$$

Moreover, the sequence of traces for these four sequences are

$$
\mathrm{tr}^{\mathrm{f}}\left(\mathrm{S}_{1,k}^{\blacktriangle}\right) = \begin{cases} \mathrm{S}_{1,k+1}^{\triangle}(\mathrm{f}) & \textit{if } \mathrm{f} \textit{ is a triangle},\\ \mathrm{S}_{1,k+1}^{\square}(\mathrm{f}) & \textit{if } \mathrm{f} \textit{ is the base square}, \end{cases}
$$

$$
\mathrm{tr}^{\mathrm{f}}\left(\mathrm{S}_{i,k}^{\blacktriangle}\right) = \begin{cases} \mathrm{S}_{2,k}^{\triangle}(\mathrm{f}) & \textit{if } \mathrm{f} \textit{ is a triangle},\\ \mathrm{S}_{i,k}^{\square}(\mathrm{f}) & \textit{if } \mathrm{f} \textit{ is the base square}, \end{cases} \qquad \textit{for } i \in \{2,3,4\}.
$$

These sequences can be considered as the extensions, from the reference cube to the reference pyramid, of the related sequences. Note that unlike all the previous sequences, the function spaces here include rational functions with polynomial traces.

We remark that similar *serendipity-type* pyramidal H^1-conforming space as that in $\mathrm{S}_{1,k}^{\blacktriangle}$ (with the same space dimension) was recently introduced in [21]. The serendipity space in [21], containing $\mathcal{P}_k(K)$, was obtained by mapping certain rational functions from the *infinite pyramid* to the reference pyramid, extending similar results in [25, 26] with a significant dimension reduction. A set of degrees of freedom, similar as those in [3] on cubes, was also identified. We believe the H^1-conforming space in [21] (obtained from the infinite pyramid) and that in $\mathrm{S}^{\text{pyr}}_{1,\mathrm{k}}$ (obtained directly from the reference pyramid) should be closely related since both enrich the same amount of rational functions (whose precise definition seems to be different) to $\mathcal{P}_k(K)$ to achieve conformity.

On the other hand, let us point out that our fourth sequence $\mathrm{S}^{\text{pyr}}_{4,\mathrm{k}}$ is significantly smaller than the related pyramidal sequence presented in [20], which was originally obtained in [25].

Whole mesh. Using the sequences in Theorem 3.5 to Theorem 3.8, we can readily obtain *mapped* commuting exact sequences on a hybrid mesh, $\Omega_h := \{K\}$, of a polyhedral domain Ω, where each physical element K is an affine mapping of any of the four reference polyhedra. We obtain *four* families of *mapped* sequences with the following form,

$$\mathrm{S}_{3\mathrm{d}}(\Omega_h): \quad H^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla} E^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\times} V^{3\mathrm{d}}(\Omega_h) \xrightarrow{\nabla\cdot} W^{3\mathrm{d}}(\Omega_h),$$

where $H^{3\mathrm{d}}(\Omega_h) \times E^{3\mathrm{d}}(\Omega_h) \times V^{3\mathrm{d}}(\Omega_h) \times W^{3\mathrm{d}}(\Omega_h) \subset H^1(\Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}, \Omega) \times L^2(\Omega)$.

The restriction on an element K of the first family of sequences is $\mathrm{S}^{\text{tet}}_{1,\mathrm{k}}(K)$ if K is a tetrahedron, $\mathrm{S}^{\text{cube}}_{1,\mathrm{k}}(K)$ if K is a parallelepiped, $\mathrm{S}^{\text{prism}}_{1,\mathrm{k}}(K)$ if K is a parallel prism, and $\mathrm{S}^{\text{pyr}}_{1,\mathrm{k}}(K)$ if K is a pyramid with a parallelogram base.

The restriction on an element K of the second family of sequences is $\mathrm{S}^{\text{tet}}_{2,\mathrm{k}}(K)$ if K is a tetrahedron, $\mathrm{S}^{\text{cube}}_{2,\mathrm{k}}(K)$ if K is a parallelepiped, $\mathrm{S}^{\text{prism}}_{2,\mathrm{k}}(K)$ if K is a parallel prism, and $\mathrm{S}^{\text{pyr}}_{2,\mathrm{k}}(K)$ if K is a pyramid with a parallelogram base.

The restriction on an element K of the third family of sequences, defined on a hybrid mesh without prisms, is $\mathrm{S}^{\text{tet}}_{2,\mathrm{k}}(K)$ if K is a tetrahedron, $\mathrm{S}^{\text{cube}}_{3,\mathrm{k}}(K)$ if K is a parallelepiped, and $\mathrm{S}^{\text{pyr}}_{3,\mathrm{k}}(K)$ if K is a pyramid with a parallelogram base.

The restriction on an element K of the fourth family of sequences is $\mathrm{S}^{\text{tet}}_{2,\mathrm{k}}(K)$ if K is a tetrahedron, $\mathrm{S}^{\text{cube}}_{4,\mathrm{k}}(K)$ if K is a parallelepiped, $\mathrm{S}^{\text{prism}}_{4,\mathrm{k}}(K)$ if K is a parallel prism, and $\mathrm{S}^{\text{pyr}}_{4,\mathrm{k}}(K)$ if K is a pyramid with a parallelogram base. Note that this family of sequences is a modification of the one considered in [20] on a hybrid mesh with a smaller pyramidal sequence.

The reason we are able to do so is due to the trace compatibilities in Theorem 3.5 to Theorem 3.8. In particular, we mention that we exclude prisms in the hybrid mesh of the third family of sequences mainly due to the incompatibility of the $H(\mathrm{curl})$ trace of $\mathrm{S}^{\text{prism}}_{3,\mathrm{k}}$ and that of $\mathrm{S}^{\text{cube}}_{3,\mathrm{k}}$, which differ by a single function.

This concludes the application of the systematic construction.

4. Proofs of the main results in section 2.

4.1. Proof of Theorem 2.9.

Proof. Suppose the segment K has coordinate x. By the exactness of the sequence $\mathrm{S}^{1\mathrm{d}}_g(K)$, we have $\mathbb{R} = \mathrm{Ker}_\nabla H^{1\mathrm{d}}_g(K)$ (and hence $1 \in H^{1\mathrm{d}}_g(K)$), and $\mathcal{P}_0(K) \subset W^{1\mathrm{d}}_g(K) = \nabla H^{1\mathrm{d}}(K)$ (and hence $x \in H^{1\mathrm{d}}_g(K)$). This implies that $\mathcal{P}_1(K) \subset H^{1\mathrm{d}}_g(K)$, and hence $\dim \mathrm{tr}_H(H^{1\mathrm{d}}_g(K)) = \dim \mathrm{tr}_H(\mathcal{P}_1(K)) = 2$. So $\mathrm{S}^{1\mathrm{d}}_g(K)$ is a compatible exact sequence. $\square$

4.2. Proof of Theorem 2.11. To simplify the notation, we set $H^{2\mathrm{d}}(K) := H^{2\mathrm{d}}_g(K) \oplus \delta H^{2\mathrm{d}}_g(K)$, $E^{2\mathrm{d}}(K) := E^{2\mathrm{d}}_g(K) \oplus \nabla\, \delta H^{2\mathrm{d}}_g(K)$, and $W^{2\mathrm{d}}(K) := W^{2\mathrm{d}}_g(K)$.

The exactness of the sequence $\mathrm{S}^{2\mathrm{d}}(K)$ follows directly from the exactness of the given sequence

$$H_g^{2\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{2\mathrm{d}}(K) \xrightarrow{\nabla\times} W_g^{2\mathrm{d}}(K).$$

It is easy to show that for any exact sequence $H^{2\mathrm{d}}(K) \to E^{2\mathrm{d}}(K) \to W^{2\mathrm{d}}(K)$, we have

$$\dim H^{2\mathrm{d}}(K) - \dim E^{2\mathrm{d}}(K) + \dim W^{2\mathrm{d}}(K) = 1, \tag{4.1a}$$

$$\nabla \overset{\circ}{H}{}^{2\mathrm{d}} = \{v \in \overset{\circ}{E}{}^{2\mathrm{d}} : \ \nabla\times v = 0\}, \tag{4.1b}$$

$$\nabla\times \overset{\circ}{E}{}^{2\mathrm{d}}(K) \subset \overset{\circ}{W}{}^{2\mathrm{d}}(K). \tag{4.1c}$$

In view of Definition 2.5 on a compatible exact sequence, we need to prove that the two dimension count identities of property (ii) are satisfied, and that the sequence of traces on each edge is a compatible exact sequence.

We have

$$\begin{aligned} \dim \mathrm{tr}_H H^{2\mathrm{d}}(K) &= \dim H^{2\mathrm{d}}(K) - \dim \overset{\circ}{H}{}^{2\mathrm{d}}(K) \\ &= \dim H_g^{2\mathrm{d}}(K) + \dim \delta H_g^{2\mathrm{d}}(K) - \dim \overset{\circ}{H}{}^{2\mathrm{d}}(K) \\ &= \dim H_g^{2\mathrm{d}}(K) + \dim \delta H_g^{2\mathrm{d}}(K) - \dim \overset{\circ}{H}{}_g^{2\mathrm{d}}(K) \\ &= \sum_{\mathrm{e}\in\mathcal{E}(K)} (\dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}) + 1), \end{aligned} \tag{4.2}$$

where the second equality is due to property (ii) of $\delta H_g^{2\mathrm{d}}(K)$, the third one is due to property (iii), and the last one is due to property (iv). Now, by property (i) of Definition 2.10 of an $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible exact sequence, and by property (i) of $\delta H_g^{2\mathrm{d}}(K)$, we have

$$\mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}(K) \subset H^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K).$$

On the other hand, since

$$\mathrm{tr}_H H^{2\mathrm{d}}(K)\big|_{\mathrm{e}} \subset \mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}(K) \quad \forall \mathrm{e} \in \mathcal{E}(K),$$

we have

$$\dim \mathrm{tr}_H H^{2\mathrm{d}}(K) \le \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim \overline{\overset{\circ}{\mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}}}(K) + \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 \le \sum_{\mathrm{e}\in\mathcal{E}(K)} (\dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}) + 1).$$

By (4.2), we have the above inequalities are indeed equalities, and

$$\mathrm{tr}_H^{\mathrm{e}} H^{2\mathrm{d}}(K) = H^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K). \tag{4.3}$$

This proves the first dimension count identity of property (ii) for a compatible exact sequence.

Next, let us prove the second dimension count identity. We have that

$$\begin{aligned} \dim \mathrm{tr}_E E^{2\mathrm{d}}(K) &= \dim E^{2\mathrm{d}}(K) - \dim \overset{\circ}{E}{}^{2\mathrm{d}}(K) \\ &= \dim H^{2\mathrm{d}}(K) + \dim W^{2\mathrm{d}}(K) - 1 \\ &\quad - \dim \nabla\times \overset{\circ}{E}{}^{2\mathrm{d}}(K) - \dim\{v \in \overset{\circ}{E}{}^{2\mathrm{d}}(K) : \ \nabla\times v = 0\} \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&= \dim H^{2\mathrm{d}}(K) + \dim \overset{\circ}{W}{}^{2\mathrm{d}}(K) - \dim \nabla\times \overset{\circ}{E}{}^{2\mathrm{d}}(K) - \dim \overset{\circ}{H}{}^{2\mathrm{d}}(K)\\
&= \sum_{\mathrm{e}\in\mathcal{E}(K)} (\dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}) + 1) + \dim \overset{\circ}{W}{}^{2\mathrm{d}}(K) - \dim \nabla\times \overset{\circ}{E}{}^{2\mathrm{d}}(K)\\
&\geq \sum_{\mathrm{e}\in\mathcal{E}(K)} (\dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}) + 1)\\
&= \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim W^{1\mathrm{d}}(\mathrm{e}),
\end{aligned}
$$

where the second equality is due to (4.1a) of an exact sequence, the third equality is due to (4.1b) and property (ii) of Definition 2.10, the fourth equality is due to (4.2), the fifth inequality is due to (4.1c), and the last equality is due to the exactness of the sequences $\mathrm{S}^{1\mathrm{d}}(\mathrm{e})$. Now, by property (i) of Definition 2.10 for an $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible exact sequence, we have

$$\mathrm{tr}_E^{\mathrm{e}} E_g^{2\mathrm{d}}(K) \subset W^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K),$$

and by property (i) of $\delta H_g^{2\mathrm{d}}(K)$, we have

$$\mathrm{tr}_E^{\mathrm{e}}\, \nabla\, \delta H_g^{2\mathrm{d}}(K) = \nabla\, \mathrm{tr}_H^{\mathrm{e}} \delta H_g^{2\mathrm{d}}(K) \subset \nabla\, H^{1\mathrm{d}}(\mathrm{e}) = W^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K).$$

Hence,

$$\mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(K) \subset W^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K).$$

On the other hand, since

$$\mathrm{tr}_E E^{2\mathrm{d}}(K)\big|_{\mathrm{e}} \subset \mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(K) \quad \forall \mathrm{e} \in \mathcal{E}(K),$$

we have

$$\dim \mathrm{tr}_E E^{2\mathrm{d}}(K) \leq \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim \mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(K) \leq \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim W^{1\mathrm{d}}(\mathrm{e}).$$

By (4.4), we have the above inequalities are indeed equalities. Hence,

$$\mathrm{tr}_E^{\mathrm{e}} E^{2\mathrm{d}}(K) = W^{1\mathrm{d}}(\mathrm{e}) \quad \forall \mathrm{e} \in \mathcal{E}(K). \tag{4.5}$$

This completes the proof of the second dimension count identity of property (ii) for a compatible exact sequence in Definition 2.5. The equalities (4.3) and (4.5) ensure that the sequence of traces $\mathrm{tr}^{\mathrm{e}}(\mathrm{S}^{2\mathrm{d}}(K)) = \mathrm{S}^{1\mathrm{d}}(\mathrm{e})$ is a compatible exact sequence, for all edges $\mathrm{e} \in \mathcal{E}(K)$. Thus, $\mathrm{S}^{2\mathrm{d}}(K)$ is a compatible exact sequence.

Finally, invoking [11, Cor. 3.2], we get the minimality of the sequence $\mathrm{S}^{2\mathrm{d}}(K)$ by the simple observation that

$$\overset{\circ}{H}{}^{2\mathrm{d}} = \overset{\circ}{H}{}_g^{2\mathrm{d}}, \quad \overset{\circ}{E}{}^{2\mathrm{d}} = \overset{\circ}{E}{}_g^{2\mathrm{d}}, \quad \overset{\circ}{W}{}^{2\mathrm{d}} = \overset{\circ}{W}{}_g^{2\mathrm{d}}.$$

The proof of Theorem 2.11 is now complete.

4.3. Proof of Theorem 2.13.

Proof. To simplify the notation, we set $H^{3\mathrm{d}}(K) := H_g^{3\mathrm{d}}(K) \oplus \delta H_g^{3\mathrm{d}}(K)$, $E^{3\mathrm{d}}(K) := E_g^{3\mathrm{d}}(K) \oplus \nabla\, \delta H_g^{3\mathrm{d}}(K) \oplus \delta E_g^{3\mathrm{d}}(K)$, $V^{3\mathrm{d}}(K) := V_g^{3\mathrm{d}}(K) \oplus \nabla\times \delta E_g^{3\mathrm{d}}(K)$, and $W^{3\mathrm{d}}(K) := W_g^{3\mathrm{d}}(K)$.

The exactness of the sequence $\mathrm{S}^{3\mathrm{d}}(K)$ comes directly from the exactness of the given sequence

$$H_g^{3\mathrm{d}}(K) \xrightarrow{\nabla} E_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\times} V_g^{3\mathrm{d}}(K) \xrightarrow{\nabla\cdot} W_g^{3\mathrm{d}}(K).$$

In view of Definition 2.6 of a compatible exact sequence, we need to prove that the three dimension count identities of property (ii) are satisfied, and that the sequence of traces on any face $\mathrm{f} \in \mathcal{F}(K)$ is a compatible exact sequence.

The proof of the first dimension count identity for the H^1-trace space $\mathrm{tr}_H H^{3\mathrm{d}}(K)$ and, consequently, that of the identity

$$\mathrm{tr}_H^{\mathrm{f}} H^{3\mathrm{d}}(K) = H^{2\mathrm{d}}(K) \quad \forall \mathrm{f} \in \mathcal{F}(K) \tag{4.6}$$

are omitted because they are very similar to those of the two-dimensional case; see details in the proof of Theorem 2.11.

Now, let us prove the third dimension count identity for the $H(\mathrm{div})$-trace space $\mathrm{tr}_V V^{3\mathrm{d}}(K)$ of property (ii) in Definition 2.6.

By properties (ii), (iii), and (iv) of $\delta E_g^{3\mathrm{d}}(K)$, we have

$$\begin{aligned} \dim \mathrm{tr}_V V^{3\mathrm{d}}(K) &= \dim V^{3\mathrm{d}}(K) - \dim \overset{\circ}{V}{}^{3\mathrm{d}}(K) \\ &= \dim V_g^{3\mathrm{d}}(K) + \dim \delta E_g^{3\mathrm{d}}(K) - \dim \nabla\times \overset{\circ}{V}{}^{3\mathrm{d}}(K) \\ &\quad - \dim\{v \in \overset{\circ}{V}{}^{3\mathrm{d}}(K) : \nabla\times v = 0\} \\ &= \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim W^{2\mathrm{d}}(\mathrm{f}) + \dim \overset{\circ}{W}{}_g^{3\mathrm{d}}(K) - \dim \nabla\times \overset{\circ}{V}{}^{3\mathrm{d}}(K) \\ &\geq \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim W^{2\mathrm{d}}(\mathrm{f}), \end{aligned} \tag{4.7}$$

where in the last inequality we used the fact that

$$\nabla\cdot \overset{\circ}{V}{}^{3\mathrm{d}}(K) \subset \overset{\circ}{W}{}^{3\mathrm{d}}(K) = \overset{\circ}{W}{}_g^{3\mathrm{d}}(K).$$

Now, by property (i) of Definition 2.12 of an $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible exact sequence we have $\mathrm{tr}_V^{\mathrm{f}} V_g^{3\mathrm{d}}(K) \subset W^{2\mathrm{d}}(\mathrm{f})$, and by property (i) of $\delta E_g^{3\mathrm{d}}(K)$ and the exactness of the sequence $\mathrm{S}^{2\mathrm{d}}(\mathrm{f})$, we have

$$\mathrm{tr}_V^{\mathrm{f}}\, \nabla\times \delta E_g^{3\mathrm{d}}(K) = \nabla\times \mathrm{tr}_E^{\mathrm{f}} \delta E_g^{3\mathrm{d}}(K) \subset \nabla\times E^{2\mathrm{d}}(\mathrm{f}) = W^{2\mathrm{d}}(\mathrm{f}).$$

Hence,

$$\mathrm{tr}_V^{\mathrm{f}} V^{3\mathrm{d}}(K) \subset W^{2\mathrm{d}}(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K).$$

This implies that

$$\dim \mathrm{tr}_V V^{3\mathrm{d}}(K) \leq \sum_{\mathrm{f}\in\mathcal{F}(K)} \mathrm{tr}_V^{\mathrm{f}} V^{3\mathrm{d}}(K) \leq \sum_{\mathrm{f}\in\mathcal{F}(K)} W^{2\mathrm{d}}(\mathrm{f}).$$

By (4.7), the above inequalities are indeed equalities, and we have

$$\mathrm{tr}_V^{\mathrm{f}} V^{3\mathrm{d}}(K) = W^{2\mathrm{d}}(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K), \tag{4.8}$$

$$\nabla \cdot \overset{\circ}{V}{}^{3\mathrm{d}}(K) = \overset{\circ}{W}{}^{3\mathrm{d}}(K). \tag{4.9}$$

This completes the proof of the third dimension count identity of property (ii) in Definition 2.6.

Now, let us prove the second dimension count identity for the $H(\mathrm{curl})$-trace space $\mathrm{tr}_E E^{3\mathrm{d}}(K)$ of property (ii) in Definition 2.6. We use the following results of an exact sequence:

$$\dim H^{3\mathrm{d}}(K) - \dim E^{3\mathrm{d}}(K) + \dim V^{3\mathrm{d}}(K) - \dim W^{3\mathrm{d}}(K) = 1, \tag{4.10a}$$

$$\mathrm{tr}_E^{\mathrm{f}} E^{3\mathrm{d}}(K) \subset E^{2\mathrm{d}}(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K), \tag{4.10b}$$

$$\nabla \overset{\circ}{H}{}^{3\mathrm{d}}(K) = \{v \in \overset{\circ}{E}{}^{3\mathrm{d}}(K),\ \nabla \times v = 0\}, \tag{4.10c}$$

$$\nabla \times \overset{\circ}{E}{}^{3\mathrm{d}}(K) \subset \{v \in \overset{\circ}{V}{}^{3\mathrm{d}} :\ \nabla \cdot v = 0\}. \tag{4.10d}$$

Their proofs are trivial and hence omitted. The inclusion (4.10b) implies that

$$\begin{aligned} \dim \mathrm{tr}_E E^{3\mathrm{d}}(K) &\leq \sum_{\mathrm{e} \in \mathcal{E}(K)} \mathrm{tr}_E^{\mathrm{e}} E^{3\mathrm{d}}(K) + \sum_{\mathrm{v} \in \mathcal{V}(K)} \overset{\circ}{\overline{\mathrm{tr}_E^{\mathrm{f}} E^{3\mathrm{d}}}}(K) \\ &\leq \sum_{\mathrm{e} \in \mathcal{E}(K)} W^{1\mathrm{d}}(\mathrm{e}) + \sum_{\mathrm{v} \in \mathcal{V}(K)} \overset{\circ}{E}{}^{2\mathrm{d}}(\mathrm{f}). \end{aligned} \tag{4.11}$$

On the other hand, we have, by (4.10a) of an exact sequence, the equalities (4.9), (4.10c), and the inclusion (4.10d),

$$\begin{aligned} \dim \mathrm{tr}_E E^{3\mathrm{d}}(K) &= \dim E^{3\mathrm{d}}(K) - \dim \overset{\circ}{E}{}^{3\mathrm{d}}(K) \\ &= \dim E^{3\mathrm{d}}(K) - \dim \nabla \times \overset{\circ}{E}{}^{3\mathrm{d}}(K) - \dim\{v \in \overset{\circ}{E}{}^{3\mathrm{d}}(K) :\ \nabla \times v = 0\} \\ &= \dim H^{3\mathrm{d}}(K) + \dim V^{3\mathrm{d}}(K) - \dim W^{3\mathrm{d}}(K) - 1 \\ &\quad - \dim \nabla \times \overset{\circ}{E}{}^{3\mathrm{d}}(K) - \dim \nabla \overset{\circ}{H}{}^{3\mathrm{d}}(K) \\ &= \dim \mathrm{tr}_H H^{3\mathrm{d}}(K) + \dim \mathrm{tr}_V V^{3\mathrm{d}}(K) - 2 \\ &\quad + \dim \overset{\circ}{V}{}^{3\mathrm{d}}(K) - \dim \overset{\circ}{W}{}^{3\mathrm{d}}(K) - \dim \nabla \times \overset{\circ}{E}{}^{3\mathrm{d}}(K) \\ &\geq \dim \mathrm{tr}_H H^{3\mathrm{d}}(K) + \dim \mathrm{tr}_V V^{3\mathrm{d}}(K) - 2. \end{aligned}$$

By the first and third dimension count identities of property (ii) in Definition 2.6, we have the right-hand side of the above inequality is equal to

$$\begin{aligned} &\sum_{\mathrm{v} \in \mathcal{V}(K)} 1 + \sum_{\mathrm{e} \in \mathcal{E}(K)} (\dim \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e})) + \sum_{\mathrm{f} \in \mathcal{F}(K)} (\dim \overset{\circ}{H}{}^{2\mathrm{d}}(\mathrm{f}) + \dim W^{2\mathrm{d}}(\mathrm{f})) - 2 \\ &\quad = \sum_{\mathrm{v} \in \mathcal{V}(K)} 1 + \sum_{\mathrm{e} \in \mathcal{E}(K)} (\dim W^{1\mathrm{d}}(\mathrm{e}) - 1) + \sum_{\mathrm{f} \in \mathcal{F}(K)} (\dim \overset{\circ}{H}{}^{2\mathrm{d}}(\mathrm{f}) + \dim \overset{\circ}{W}{}^{2\mathrm{d}}(\mathrm{f}) + 1) - 2 \\ &\sum_{\mathrm{e} \in \mathcal{E}(K)} \dim W^{1\mathrm{d}}(\mathrm{e}) + \sum_{\mathrm{f} \in \mathcal{F}(K)} \dim \overset{\circ}{E}{}^{2\mathrm{d}}(\mathrm{f}) - I, \end{aligned}$$

where

$$I := \sum_{\mathrm{v}\in\mathcal{V}(K)} 1 - \sum_{\mathrm{e}\in\mathcal{E}(K)} 1 + \sum_{\mathrm{f}\in\mathcal{F}(K)} 1 - 2 = 0$$

by Euler's polyhedral formula. Hence,

$$\dim \mathrm{tr}_E E^{3\mathrm{d}}(K) \geq \sum_{\mathrm{e}\in\mathcal{E}(K)} \dim W^{1\mathrm{d}}(\mathrm{e}) + \sum_{\mathrm{f}\in\mathcal{F}(K)} \dim \overset{\circ}{E}{}^{2\mathrm{d}}(\mathrm{f}).$$

And the above inequality is indeed an equality by (4.11). Moreover, the inclusions in (4.10) are also equalities:

$$\mathrm{tr}_E^{\mathrm{f}} E^{3\mathrm{d}}(K) = E^{2\mathrm{d}}(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K), \tag{4.12a}$$

$$\nabla\times \overset{\circ}{E}{}^{3\mathrm{d}}(K) = \{v \in \overset{\circ}{V}{}^{3\mathrm{d}} : \nabla\cdot v = 0\}, \tag{4.12b}$$

and this completes the proof of the second dimension count identity of property (ii) in Definition 2.6. The equalities (4.6), (4.8), and (4.12a) imply that the sequence of traces $\mathrm{tr}^{\mathrm{f}}(\mathrm{S}^{3\mathrm{d}}(K)) = \mathrm{S}^{2\mathrm{d}}(\mathrm{f})$ for any face $\mathrm{f} \in \mathcal{F}(K)$ is a compatible exact sequence. Hence, $\mathrm{S}^{3\mathrm{d}}(K)$ is a compatible exact sequence.

Finally, invoking [11, Cor. 3.2], we get the minimality of the sequence $\mathrm{S}^{3\mathrm{d}}(K)$ by the simple observation that

$$\overset{\circ}{H}{}^{3\mathrm{d}} = \overset{\circ}{H}{}^{3\mathrm{d}}_g, \quad \overset{\circ}{E}{}^{3\mathrm{d}} = \overset{\circ}{E}{}^{3\mathrm{d}}_g, \quad \overset{\circ}{V}{}^{3\mathrm{d}} = \overset{\circ}{V}{}^{3\mathrm{d}}_g, \quad \overset{\circ}{W}{}^{3\mathrm{d}} = \overset{\circ}{W}{}^{3\mathrm{d}}_g.$$

This completes the proof of Theorem 2.13. ∎

5. Proofs of the applications in section 3.

In this section, we prove all the results of section 3 by applying Theorem 2.9 (for one dimension), Theorem 2.11 (for two dimensions), and Theorem 2.13 (for three dimensions).

5.1. The one-dimensionanl case.

The proof of Theorem 3.1 is a direct application of Theorem 2.9.

5.2. The two-dimensional case.

We first present the following result on exact sequences on the whole space $\mathbb{R}^2$. We give its proof in Appendix B.

LEMMA 5.1. *The following four sequences on $\mathbb{R}^2$ with coordinates (x, y) are exact for $k \geq 0$:*

$$\begin{array}{lcccccc}
\mathrm{S}^{2\mathrm{d}}_{1,\mathrm{k}}: & \mathcal{P}_{k+2} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_{k+1} & \xrightarrow{\nabla\times} & \mathcal{P}_k, \\
\mathrm{S}^{2\mathrm{d}}_{2,\mathrm{k}}: & \mathcal{P}_{k+1} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x}\times\widetilde{\mathcal{P}}_k & \xrightarrow{\nabla\times} & \mathcal{P}_k, \\
\mathrm{S}^{2\mathrm{d}}_{3,\mathrm{k}}: & \mathcal{Q}_k \oplus \{x^{k+1}, y^{k+1}\} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{Q}}_k \oplus \boldsymbol{x}\times\{x^k y^k\} & \xrightarrow{\nabla\times} & \mathcal{Q}_k, \\
\mathrm{S}^{2\mathrm{d}}_{4,\mathrm{k}}: & \mathcal{Q}_{k+1} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{Q}}_k \oplus \begin{pmatrix} y^{k+1}\mathcal{P}_k(x) \\ x^{k+1}\mathcal{P}_k(y) \end{pmatrix} & \xrightarrow{\nabla\times} & \mathcal{Q}_k.
\end{array}$$

The third sequence $\mathrm{S}^{3\mathrm{d}}_{2,\mathrm{k}}$ is new to the best of the authors' knowledge; the other three are well-known. Note that all four sequences have good "symmetry" in the sense that they are invariant under the coordinate permutation $(x, y) \longrightarrow (y, x)$.

Now, we are ready to prove the two-dimensional results of Theorem 3.2, Theorem 3.3, and Theorem 3.4 by applying Theorem 2.11.

Proof of Theorem 3.2. Let us fit the sequences $\mathrm{S}_{1,\mathrm{k}}^{\triangle}$ and $\mathrm{S}_{2,\mathrm{k}}^{\triangle}$ into the framework of Theorem 2.11.

For the first sequence, $\mathrm{S}_{1,\mathrm{k}}^{\triangle}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{1\mathrm{d}}(\partial K) := \{\mathrm{S}^{1\mathrm{d}}(e) : \quad \mathcal{P}_{k+2}(e) \longrightarrow \mathcal{P}_{k+1}(e)\ \forall e \in \mathcal{E}(K)\},$$

and the $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{1,\mathrm{k}}^{\triangle}$ itself. We also have

$$\dim \delta H_g^{2\mathrm{d}} = 3(k+2) + k(k+1)/2 - (k+3)(k+4)/2 = 0,$$

hence $\delta H_g^{2\mathrm{d}}(K) = \{0\}$. Thus, $\mathrm{S}_{1,\mathrm{k}}^{\triangle}$ is a commuting exact sequence. Similarly, we conclude that $\mathrm{S}_{2,\mathrm{k}}^{\triangle}$ is also a commuting exact sequence. This completes the proof of Theorem 3.2. $\square$

Proof of Theorem 3.3. Let us fit the sequences, $\mathrm{S}_{1,\mathrm{k}}^{\square}$, $\mathrm{S}_{2,\mathrm{k}}^{\square}$, $\mathrm{S}_{3,\mathrm{k}}^{\square}$, and $\mathrm{S}_{4,\mathrm{k}}^{\square}$ into the framework of Theorem 2.11.

For the first sequence, $\mathrm{S}_{1,\mathrm{k}}^{\square}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{1\mathrm{d}}(\partial K) := \{\mathrm{S}^{1\mathrm{d}}(e) : \quad \mathcal{P}_{k+2}(e) \longrightarrow \mathcal{P}_{k+1}(e)\ \forall e \in \mathcal{E}(K)\},$$

and the $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{1,\mathrm{k}}^{2\mathrm{d}}$ in Lemma 5.1. We also have the space $\delta H_g^{2\mathrm{d}} := \delta H_{k+2}^{2,I}$ that satisfies all four properties of $\delta H_g^{2\mathrm{d}}$ in Theorem 2.11, in particular, we have

$$\dim \delta H_g^{2\mathrm{d}} = 4(k+2) + (k-1)k/2 - (k+3)(k+4)/2 = 2.$$

Hence, $\mathrm{S}_{1,\mathrm{k}}^{\square}$ is a commuting exact sequence.

The proof for the second sequence, $\mathrm{S}_{2,\mathrm{k}}^{\square}$, is identical to that for the first one.

For the third sequence, $\mathrm{S}_{3,\mathrm{k}}^{\square}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{1\mathrm{d}}(\partial K) := \{\mathrm{S}^{1\mathrm{d}}(e) : \quad \mathcal{P}_{k+1}(e) \longrightarrow \mathcal{P}_{k}(e)\ \forall e \in \mathcal{E}(K)\},$$

and the $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{3,\mathrm{k}}^{2\mathrm{d}}$ in Lemma 5.1. We also have the space $\delta H_g^{2\mathrm{d}} := \delta H_{k+1}^{2,I}$ that satisfies all four properties of $\delta H_g^{2\mathrm{d}}$ in Theorem 2.11, in particular, we have

$$\dim \delta H_g^{2\mathrm{d}} = 4(k+1) + (k-1)^2 - \big((k+1)^2 + 2\big) = 2.$$

Hence, $\mathrm{S}_{3,\mathrm{k}}^{\square}$ is a commuting exact sequence.

For the last sequence, $\mathrm{S}_{4,\mathrm{k}}^{\square}$, we have the set of complete trace sequences is

$$\mathrm{S}^{1\mathrm{d}}(\partial K) := \{\mathrm{S}^{1\mathrm{d}}(e) : \quad \mathcal{P}_{k+1}(e) \longrightarrow \mathcal{P}_{k}(e)\ \forall e \in \mathcal{E}(K)\},$$

and the $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{4,\mathrm{k}}^{2\mathrm{d}}$ in Lemma 5.1. We also have the space $\delta H_g^{2\mathrm{d}} := \{0\}$ that satisfies all four properties of $\delta H_g^{2\mathrm{d}}$ in Theorem 2.11, since

$$\dim \delta H_g^{2\mathrm{d}} = 4(k+1) + k^2 - (k+2)^2 = 0.$$

Hence, $\mathrm{S}_{4,\mathrm{k}}^{\square}$ is also a commuting exact sequence. This completes the proof of Theorem 3.3. $\square$

Proof of Theorem 3.4. Let us fit the sequences $\mathrm{S}_{1,\mathrm{k}}^{\mathrm{poly}}$ and $\mathrm{S}_{2,\mathrm{k}}^{\mathrm{poly}}$ into the framework of Theorem 2.11.

For the first sequence, $\mathrm{S}_{1,\mathrm{k}}^{\mathrm{poly}}$, we have the set of complete trace sequences is

$$\mathrm{S}^{1\mathrm{d}}(\partial K) := \{\mathrm{S}^{1\mathrm{d}}(e): \quad \mathcal{P}_{k+2}(e) \longrightarrow \mathcal{P}_{k+1}(e)\ \forall e \in \mathcal{E}(K)\},$$

and the $\mathrm{S}^{1\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{1,\mathrm{k}}^{2\mathrm{d}}$ in Lemma 5.1. The proof of the space $\delta H_g^{2\mathrm{d}} := \delta H_{k+2}^{2,II}$ satisfying all four properties of $\delta H_g^{2\mathrm{d}}$ in Theorem 2.11 is not trival. It is given in the proof of [14, Thm. 2.6].

The proof for the second sequence is identical to that for the first one. This completes the proof of Theorem 3.4. ∎

5.3. The three-dimensional case. As we did for the two-dimensional case, we first present the following result on exact sequences on the whole space $\mathbb{R}^3$. We give its proof in Appendix B.

LEMMA 5.2. *The following six sequences on $\mathbb{R}^3$ are exact for $k \geq 0$. The first two sequences are the famous polynomial de Rham sequences that contain polynomials of certain degree:*

$$\begin{array}{llclclcl}
\mathrm{S}_{1,\mathrm{k}}^{3\mathrm{d}}: & \mathcal{P}_{k+3} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_{k+2} & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{P}}_{k+1} & \xrightarrow{\nabla\cdot} & \mathcal{P}_k, \\
\mathrm{S}_{2,\mathrm{k}}^{3\mathrm{d}}: & \mathcal{P}_{k+1} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x} \times \widetilde{\boldsymbol{\mathcal{P}}}_k & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{P}}_k \oplus \boldsymbol{x}\widetilde{\mathcal{P}}_k & \xrightarrow{\nabla\cdot} & \mathcal{P}_k.
\end{array}$$

The next two sequences have spaces containing tensor-product polynomials of certain degree.

$$\begin{array}{llclclcl}
\mathrm{S}_{3,\mathrm{k}}^{3\mathrm{d}}: & \mathcal{Q}_k & \xrightarrow{\nabla} & \boldsymbol{\mathcal{Q}}_k & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{Q}}_k & \xrightarrow{\nabla\cdot} & \mathcal{Q}_k, \\
 & \oplus \left\{ \begin{array}{c} x^{k+1}, \\ y^{k+1}, \\ z^{k+1} \end{array} \right\} & & \oplus \boldsymbol{x} \times \left\{ \begin{array}{c} y^k z^k \nabla x, \\ z^k x^k \nabla y, \\ x^k y^k \nabla z \end{array} \right\} & & \oplus \boldsymbol{x}\, \{x^k y^k z^k\} & & \\
\mathrm{S}_{4,\mathrm{k}}^{3\mathrm{d}}: & \mathcal{Q}_{k+1} & \xrightarrow{\nabla} & \begin{pmatrix} \mathcal{P}_{k,k+1,k+1} \\ \mathcal{P}_{k+1,k,k+1} \\ \mathcal{P}_{k+1,k+1,k} \end{pmatrix} & \xrightarrow{\nabla\times} & \begin{pmatrix} \mathcal{P}_{k+1,k,k} \\ \mathcal{P}_{k,k+1,k} \\ \mathcal{P}_{k,k,k+1} \end{pmatrix} & \xrightarrow{\nabla\cdot} & \mathcal{Q}_k.
\end{array}$$

The last two sequences contain polynomials of certain degree in the (x, y)-plane, and have some spaces with tensor-product structure in the (x, z)- and (y, z)-plane.

$$\begin{array}{llclclcl}
\mathrm{S}_{5,\mathrm{k}}^{3\mathrm{d}}: & \mathcal{P}_{k|k} & \xrightarrow{\nabla} & \boldsymbol{\mathcal{P}}_{k|k} & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{P}}_{k|k} & \xrightarrow{\nabla\cdot} & \mathcal{P}_{k|k}, \\
 & \oplus \widetilde{\mathcal{P}}_{k+1}(x,y) \oplus \{z^{k+1}\} & & \oplus \begin{pmatrix} \begin{pmatrix} y \\ -x \end{pmatrix} \widetilde{\mathcal{P}}_k(x,y) \\ \widetilde{\mathcal{P}}_{k+1}(x,y)\, z^k \end{pmatrix} & & \oplus \begin{pmatrix} 0 \\ 0 \\ \widetilde{\mathcal{P}}_k(x,y)\, z^{k+1} \end{pmatrix} & & \\
\mathrm{S}_{6,\mathrm{k}}^{3\mathrm{d}}: & \mathcal{P}_{k+1|k+1} & \xrightarrow{\nabla} & \begin{pmatrix} \boldsymbol{N}_k(x,y) \otimes \mathcal{P}_{k+1}(z) \\ \mathcal{P}_{k+1|k} \end{pmatrix} & \xrightarrow{\nabla\times} & \begin{pmatrix} \boldsymbol{RT}_k(x,y) \otimes \mathcal{P}_k(z) \\ \mathcal{P}_{k|k+1} \end{pmatrix} & \xrightarrow{\nabla\cdot} & \mathcal{P}_{k|k}.
\end{array}$$

The third sequence, $\mathrm{S}_{3,\mathrm{k}}^{3\mathrm{d}}$, and the fifth sequence, $\mathrm{S}_{5,\mathrm{k}}^{3\mathrm{d}}$, are new to the best of the authors' knowledge; the other four are well-known. Note that all these six sequences, except the fifth, have good "symmetry" in the sense that they are invariant under any coordinate permutation. The fifth sequence is only invariant under the coordinate permutation $(x, y, z) \longrightarrow (y, x, z)$.

To further simplify the notation, we use the so-called M-index introduced in [16] and used in [15] to obtain various finite element spaces admitting M-decompositions in three dimensions. The definition of an M-index is given as follows.

DEFINITION 5.3 (The M-index). *The M-index of the space $V(K) \times W(K) \subset H(\mathrm{div}, K) \times H^1(K)$ is the number*

$$\begin{aligned} I_M(V \times W) := \ & \dim M(\partial K) - \dim \mathrm{tr}_V\{v \in V(K):\ \nabla\cdot v = 0\} \\ & - \dim \mathrm{tr}_H\{w \in W(K):\ \nabla w = 0\}, \end{aligned}$$

where

$$M(\partial K) = \{\mu \in L^2(\partial K) :\ \mu|_{\mathrm{f}} \in M(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K)\}$$

is a finite element space defined on the boundary ∂K *of a polyhedron* K*.*

Using the definition of an M-index, we have that the number in the right-hand side of property (iv) of $\delta E_g^{3\mathrm{d}}(K)$ in Theorem 2.13 is nothing but $I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}})$ with the trace space

$$M(\partial K) := \{\mu \in L^2(\partial K) :\ \mu|_{\mathrm{f}} \in W^{2\mathrm{d}}(\mathrm{f}) \quad \forall \mathrm{f} \in \mathcal{F}(K)\}. \tag{5.1}$$

That is,

$$\begin{aligned} I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}) = {} & \sum_{\mathrm{f}\in\mathcal{F}(K)} W^{2\mathrm{d}}(\mathrm{f}) + \dim \overset{\circ}{W}{}_g^{3\mathrm{d}}(K) \\ & + \dim \left\{ v \in \overset{\circ}{V}{}_g^{3\mathrm{d}}(K) : \nabla\cdot v = 0 \right\} - \dim V_g^{3\mathrm{d}}(K). \end{aligned} \tag{5.2}$$

To see this, we have

$$\begin{aligned} & \sum_{\mathrm{f}\in\mathcal{F}(K)} W^{2\mathrm{d}}(\mathrm{f}) + \dim \overset{\circ}{W}{}_g^{3\mathrm{d}}(K) + \dim \left\{ v \in \overset{\circ}{V}{}_g^{3\mathrm{d}}(K) : \nabla\cdot v = 0 \right\} - \dim V_g^{3\mathrm{d}}(K) \\ & = \dim M(\partial K) + \dim W_g^{3\mathrm{d}}(K) - 1 + \dim \left\{ v \in V_g^{3\mathrm{d}}(K) :\ \nabla\cdot v = 0 \right\} \\ & \qquad - \dim \mathrm{tr}_V \left\{ v \in V_g^{3\mathrm{d}}(K) :\ \nabla\cdot v = 0 \right\} - \dim V_g^{3\mathrm{d}}(K) \\ & = \dim M(\partial K) + \dim W_g^{3\mathrm{d}}(K) - 1 - \dim \mathrm{tr}_V \left\{ v \in V_g^{3\mathrm{d}}(K) :\ \nabla\cdot v = 0 \right\} \\ & \qquad - \dim \nabla\cdot V_g^{3\mathrm{d}}(K) \\ & = \dim M(\partial K) - \dim \mathrm{tr}_H \left\{ w \in W_g^{3\mathrm{d}}(K) :\ \nabla w = 0 \right\} \\ & \qquad\qquad - \dim \mathrm{tr}_V \left\{ v \in V_g^{3\mathrm{d}}(K) :\ \nabla\cdot v = 0 \right\} \\ & = I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}). \end{aligned}$$

From now on, the trace space $M(\partial K)$ will always be of the form (5.1) where the spaces $W^{2\mathrm{d}}(\mathrm{f})$ on each face vary in different locations.

Since the computation of the M-index for various polynomial spaces on the four reference polyhedra was given in [15], we can directly use those results to verify property (iv) of $\delta E_g^{3\mathrm{d}}(K)$.

Now, we are ready to prove the three-dimensional results of Theorem 3.5 to Theorem 3.8 by applying the general result of Theorem 2.13.

Proof of Theorem 3.5. Let us fit the sequences $\mathrm{S}_{1,\mathrm{k}}^{\triangle}$ and $\mathrm{S}_{2,\mathrm{k}}^{\triangle}$ into the framework of Theorem 2.13. Here the element

$$K = \{(x, y, z) : 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1,\ x + y + z < 1\}$$

is the reference tetrahedron with four square faces, six edges, and four vertices.

For the first sequence, $\mathrm{S}_{1,\mathrm{k}}^{\triangle}$, we have the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \{\mathrm{S}_{1,\mathrm{k}+1}^{\triangle}(\mathrm{f}) :\ \ \forall \mathrm{f} \in \mathcal{F}(K)\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}^{3\mathrm{d}}_{1,\mathrm{k}}$ in Lemma 5.2. We also have

$$\begin{aligned}\dim \delta H_g^{3\mathrm{d}} &= 4+6(k+2)+4\,(k+1)(k+2)/2+k(k+1)(k+2)/6\\&\quad-(k+4)(k+5)(k+6)/6\\&=0,\\ \dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}}\times W_g^{3\mathrm{d}})\\&=0,\end{aligned}$$

hence $\delta H_g^{3\mathrm{d}}=\{0\}$ and $\delta E_g^{3\mathrm{d}}=\{0\}$. So, $\mathrm{S}^{\triangle}_{1,\mathrm{k}}$ is a commuting exact sequence. Similarly, we conclude that $\mathrm{S}^{\triangle}_{2,\mathrm{k}}$ is also a commuting exact sequence. This completes the proof of Theorem 3.5. $\square$

Proof of Theorem 3.6. Let us fit the sequences $\mathrm{S}^{\boxtimes}_{1,\mathrm{k}}$, $\mathrm{S}^{\boxtimes}_{2,\mathrm{k}}$, $\mathrm{S}^{\boxtimes}_{3,\mathrm{k}}$, and $\mathrm{S}^{\boxtimes}_{4,\mathrm{k}}$ into the framework of Theorem 2.13. Here the element

$$K=\{(x,y,z): 0<x<1,\ 0<y<1,\ 0<z<1\}$$

is the reference cube with six square faces, twelve edges, and eight vertices.

For the first sequence $\mathrm{S}^{\boxtimes}_{1,\mathrm{k}}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K):=\{\mathrm{S}^{\square}_{1,\mathrm{k}+1}(\mathrm{f}):\ \ \forall \mathrm{f}\in\mathcal{F}(K)\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}^{3\mathrm{d}}_{1,\mathrm{k}}$ in Lemma 5.2. It is then easy to verify that $\delta H_g^{3\mathrm{d}}:=\delta H^{3,I}_{k+3}$ satisfies all four properties of $\delta H_g^{3\mathrm{d}}$, and $\delta E_g^{3\mathrm{d}}:=\delta E^{3,I}_{k+2}$ satisfies all four properties of $\delta E_g^{3\mathrm{d}}$ in Theorem 2.13. In particular, we have

$$\begin{aligned}\dim \delta H_g^{3\mathrm{d}} &= 8+12(k+2)+6\,k(k+1)/2+(k-2)(k-1)k/6\\&\quad-(k+4)(k+5)(k+6)/6\\&=3(k+4),\\ \dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}}\times W_g^{3\mathrm{d}}) = \ 3(k+2),\end{aligned}$$

where the last equality is due to [15, Thm. 2.12].

For the sake of completeness, here we present a proof of property (iii) for $\delta E_g^{3\mathrm{d}}(K)$, which is a bit more difficult to verify than the other properties. Given a function $p\in\widetilde{\mathcal{P}}_{k+1}(y,z)$, we have

$$\nabla\times(x\,p\,(y\,\nabla\,z-z\,\nabla\,y))=(k+3)p\,x\,\nabla\,x-p\,y\,\nabla\,y-p\,z\,\nabla\,z.$$

This means that the function $\nabla\times(x\,p\,(y\,\nabla\,z-z\,\nabla\,y))\in\nabla\times\delta E_g^{3\mathrm{d}}(K)$ has normal trace equal to *zero* on the three faces $x=0$, $y=0$, and $z=0$ of the unit cube K, and has normal trace equals to $(k+3)p\in\widetilde{\mathcal{P}}_{k+1}(y,z)$ on the face $x=1$. Similar results hold for

$$\begin{aligned}\nabla\times(y\,q\,(z\,\nabla\,x-x\,\nabla\,z))&=-q\,x\,\nabla\,x+(k+3)q\,y\,\nabla\,y-q\,z\,\nabla\,z,\\ \nabla\times(z\,r\,(x\,\nabla\,y-y\,\nabla\,x))&=-r\,x\,\nabla\,x-r\,y\,\nabla\,y+(k+3)r\,z\,\nabla\,z,\end{aligned}$$

with $q\in\widetilde{\mathcal{P}}_{k+1}(z,x)$ and $r\in\widetilde{\mathcal{P}}_{k+1}(x,y)$. Using this fact, we have that any function in $\nabla\times\delta E_g^{3\mathrm{d}}(K)$ has a normal trace equal to *zero* on one face and is a polynomial of degree $k+1$ on its opposite (parallel) face for at least one pair of parallel faces. On the other hand, if a function in $V_g^{3\mathrm{d}}(K) = \boldsymbol{\mathcal{P}}_{k+1}$ has normal trace equal to *zero* on one face, the normal trace on its opposite face must be a polynomial of degree no greater than k. Hence, $\mathrm{tr}_V\, \nabla\times \delta E_g^{3\mathrm{d}}(K) \cap \mathrm{tr}_V V_g^{3\mathrm{d}}(K) = \{0\}$. This implies property (iii) of $\delta E_g^{3\mathrm{d}}(K)$.

Thus, $\mathrm{S}^{\boxtimes}_{1,\mathrm{k}}$ is a commuting exact sequence.

The proof for the second sequence, $\mathrm{S}^{\boxtimes}_{2,\mathrm{k}}$, is identical to that for the first one.

For the third sequence, $\mathrm{S}^{\boxtimes}_{3,\mathrm{k}}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \{\mathrm{S}^{\square}_{3,\mathrm{k}}(\mathrm{f}) : \quad \forall \mathrm{f} \in \mathcal{F}(K)\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}^{3\mathrm{d}}_{3,\mathrm{k}}$ in Lemma 5.2. It is then trivial to verify that $\delta H_g^{3\mathrm{d}} := \delta H_{k+1}^{3,II}$ satisfies all four properties of $\delta H_g^{3\mathrm{d}}$, and $\delta E_g^{3\mathrm{d}} := \delta E_{k+1}^{3,II}$ satisfies all four properties of $\delta E_g^{3\mathrm{d}}$ in Theorem 2.13. In particular, we have, for $k \geq 1$,

$$\begin{aligned}
\dim \delta H_g^{3\mathrm{d}} &= 8 + 12k + 6(k-1)^2 + (k-1)^3 - \big((k+1)^3 + 3\big)\\
&= 9,\\
\dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}) = \ 6,
\end{aligned}$$

where the last equality is due to [15, Thm. 2.11]. So, $\mathrm{S}^{\boxtimes}_{3,\mathrm{k}}$ is a commuting exact sequence.

For the last sequence, $\mathrm{S}^{\boxtimes}_{4,\mathrm{k}}$, we have the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \{\mathrm{S}^{\square}_{4,\mathrm{k}}(\mathrm{f}) : \quad \forall \mathrm{f} \in \mathcal{F}(K)\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}^{3\mathrm{d}}_{4,\mathrm{k}}$ in Lemma 5.2. We also have that the space $\delta H_g^{3\mathrm{d}} := \{0\}$ satisfies all four properties of $\delta H_g^{3\mathrm{d}}$, and $\delta E_g^{3\mathrm{d}} := \{0\}$ satisfies all four properties of $\delta E_g^{3\mathrm{d}}$ in Theorem 2.13 since

$$\begin{aligned}
\dim \delta H_g^{3\mathrm{d}} &= 8 + 12k + 6k^2 + k^3 - (k+2)^3\\
&= 0,\\
\dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}) = \ 0.
\end{aligned}$$

Thus, $\mathrm{S}^{\boxtimes}_{4,\mathrm{k}}$ is a commuting exact sequence. This completes the proof of Theorem 3.6. ∎

Proof of Theorem 3.7. Let us fit the sequences $\mathrm{S}^{\boxslash}_{1,\mathrm{k}}$, $\mathrm{S}^{\boxslash}_{2,\mathrm{k}}$, $\mathrm{S}^{\boxslash}_{3,\mathrm{k}}$, and $\mathrm{S}^{\boxslash}_{4,\mathrm{k}}$ into the framework of Theorem 2.13. Here the element

$$K = \{(x,y,z) : 0 < x,\ 0 < y,\ 0 < z < 1,\ x + y < 1\}$$

is the reference prism with five faces (two triangular faces and three square faces), nine edges, and six vertices.

For the first sequence, $\mathrm{S}^{\boxslash}_{1,\mathrm{k}}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \left\{ \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : \quad \begin{matrix} \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) = \mathrm{S}^{\triangle}_{1,\mathrm{k}+1}(\mathrm{f}) & \text{if f is a triangle,} \\ \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) = \mathrm{S}^{\square}_{1,\mathrm{k}+1}(\mathrm{f}) & \text{if f is a square} \end{matrix} \right\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}^{3\mathrm{d}}_{1,\mathrm{k}}$ in Lemma 5.2. It is then trivial to verify that $\delta H_g^{3\mathrm{d}} := \delta H_{k+3}^{3,III}$ satisfies all four properties of $\delta H_g^{3\mathrm{d}}$, and $\delta E_g^{3\mathrm{d}} := \delta E_{k+2}^{3,III}$ satisfies all four properties of $\delta E_g^{3\mathrm{d}}$ in Theorem 2.13. In particular, we have

$$\begin{aligned}\dim \delta H_g^{3\mathrm{d}} &= 6 + 9(k+2) + (k+1)(k+2) + 3\,k(k+1)/2 \\ &\quad + (k-1)k(k+1)/6 - (k+4)(k+5)(k+6)/6 \\ &= k+6, \\ \dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}) = \ k+3,\end{aligned}$$

where the last equality is due to [15, Thm. 2.8]. Thus, $\mathrm{S}_{1,\mathrm{k}}^{\boxdot}$ is a commuting exact sequence.

The proof for the second sequence, $\mathrm{S}_{2,\mathrm{k}}^{\boxdot}$, is identical to that for the first one.

For the third sequence, $\mathrm{S}_{3,\mathrm{k}}^{\boxdot}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \left\{ \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : \quad \begin{array}{ll} \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) = \mathrm{S}_{2,\mathrm{k}}^{\triangle}(\mathrm{f}) & \text{if f is a triangle,} \\ \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) = \widetilde{\mathrm{S}}_{3,\mathrm{k}}^{\square}(\mathrm{f}) & \text{if f is a square} \end{array} \right\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{5,\mathrm{k}}^{3\mathrm{d}}$ in Lemma 5.2. It is then trivial to verify that $\delta H_g^{3\mathrm{d}} := \delta H_{k+1}^{3,III}$ satisfies all four properties of $\delta H_g^{3\mathrm{d}}$, and $\delta E_g^{3\mathrm{d}} := \delta E_{k+1}^{3,III}$ satisfies all four properties of $\delta E_g^{3\mathrm{d}}$ in Theorem 2.13. In particular, we have, for $k \geq 1$,

$$\begin{aligned}\dim \delta H_g^{3\mathrm{d}} &= 6 + 9k + (k-1)k + 3(k-1)^2 + (k-2)(k-1)^2/2 \\ &\quad - \big((k+1)^2(k+2)/2 + k + 3\big) \\ &= k+4, \\ \dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}) = \ k+2,\end{aligned}$$

where the last equality is due to [15, Thm. 2.7]. Thus, $\mathrm{S}_{3,\mathrm{k}}^{\boxdot}$ is a commuting exact sequence.

For the last sequence, $\mathrm{S}_{4,\mathrm{k}}^{\boxdot}$, we have the set of complete trace sequences is

$$\mathrm{S}^{2\mathrm{d}}(\partial K) := \left\{ \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) : \quad \begin{array}{ll} \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) = \mathrm{S}_{2,\mathrm{k}}^{\triangle}(\mathrm{f}) & \text{if f is a triangle,} \\ \mathrm{S}^{2\mathrm{d}}(\mathrm{f}) = \mathrm{S}_{4,\mathrm{k}}^{\square}(\mathrm{f}) & \text{if f is a square} \end{array} \right\},$$

and the given $\mathrm{S}^{2\mathrm{d}}(\partial K)$-admissible sequence is $\mathrm{S}_{6,\mathrm{k}}^{3\mathrm{d}}$ in Lemma 5.2. We also have that the space $\delta H_g^{3\mathrm{d}} := \{0\}$ satisfies all four properties of $\delta H_g^{3\mathrm{d}}$, and $\delta E_g^{3\mathrm{d}} := \{0\}$ satisfies all four properties of $\delta E_g^{3\mathrm{d}}$ in Theorem 2.13 since

$$\begin{aligned}\dim \delta H_g^{3\mathrm{d}} &= 6 + 9k + (k-1)k + 3\,k^2 + (k-1)k^2/2 \\ &\quad - (k+2)^2(k+3)/2 \\ &= 0, \\ \dim \delta E_g^{3\mathrm{d}} &= I_M(V_g^{3\mathrm{d}} \times W_g^{3\mathrm{d}}) = \ 0.\end{aligned}$$

Thus, $\mathrm{S}_{4,\mathrm{k}}^{\boxdot}$ is a commuting exact sequence. This completes the proof of Theorem 3.7. ∎

Proof of Theorem 3.8. Let us fit the sequences $\mathrm{S}_{1,\mathrm{k}}^{\blacktriangle}$, $\mathrm{S}_{2,\mathrm{k}}^{\blacktriangle}$, $\mathrm{S}_{3,\mathrm{k}}^{\blacktriangle}$, and $\mathrm{S}_{4,\mathrm{k}}^{\blacktriangle}$ into the framework of Theorem 2.13. Here the element

$$K = \{(x,y,z) : 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1,\ x+z < 1,\ y+z < 1\}$$

is the reference pyramid with five faces (four triangular faces and one square face), eight edges, and five vertices.

For the first sequence, $\mathrm{S}^{\Delta}_{1,\mathrm{k}}$, we have that the set of complete trace sequences is

$$\mathrm{S}^{\mathrm{2d}}(\partial K) := \left\{ \mathrm{S}^{\mathrm{2d}}(\mathrm{f}) : \quad \begin{array}{ll} \mathrm{S}^{\mathrm{2d}}(\mathrm{f}) = \mathrm{S}^{\triangle}_{1,\mathrm{k}+1}(\mathrm{f}) & \text{if f is a triangle,} \\ \mathrm{S}^{\mathrm{2d}}(\mathrm{f}) = \mathrm{S}^{\square}_{1,\mathrm{k}+1}(\mathrm{f}) & \text{if f is a square} \end{array} \right\},$$

and the given $\mathrm{S}^{\mathrm{2d}}(\partial K)$-admissible sequence is $\mathrm{S}^{\mathrm{3d}}_{1,\mathrm{k}}$ in Lemma 5.2. It is then trivial to verify that $\delta H^{\mathrm{3d}}_g := \delta H^{3,IV}_{k+3}$ satisfies all four properties of δH^{3d}_g, and $\delta E^{\mathrm{3d}}_g := \delta E^{3,IV}_{k+2}$ satisfies all four properties of δE^{3d}_g in Theorem 2.13. In particular, we have

$$\begin{aligned} \dim \delta H^{\mathrm{3d}}_g &= 5 + 8(k+2) + 2(k+1)(k+2) + k(k+1)/2 \\ &\quad + (k-1)k(k+1)/6 - (k+4)(k+5)(k+6)/6 \\ &= 2\,k + 5, \\ \dim \delta E^{\mathrm{3d}}_g &= I_M(V^{\mathrm{3d}}_g \times W^{\mathrm{3d}}_g) = \ 3, \end{aligned}$$

where the last equality is due to [15, Thm. 2.6]. Thus, $\mathrm{S}^{\Delta}_{1,\mathrm{k}}$ is a commuting exact sequence.

The proofs for the other three sequences are similar to that for the first one, and hence are omitted. This completes the proof of Theorem 3.8. ∎

6. Conclusion.

We presented a systematic construction of commuting exact sequences on polygonal/polyhedral elements. The systematic construction is applied to the reference triangle, reference square, and general polygon in two dimensions; and to the reference tetrahedron, reference cube, reference prism, and reference pyramid in three dimensions. In this way, we obtain concrete commuting exact sequences. We also obtain *mapped* commuting exact sequences on a two-dimensional hybrid mesh consisting of triangles and parallelograms, and *nonmapped* commuting exact sequences on a general two-dimensional polygonal mesh. Finally, we obtain *mapped* commuting exact sequences on a three-dimensional hybrid mesh with elements obtained via affine mapping from one of the four reference polyhedra. The construction of basis functions for our four families of commuting exact sequences on hybrid polyhedral meshes constitutes the subject of ongoing work.

As already pointed out in [15], finding stable pairs of finite element spaces defining mixed methods for the diffusion problem on general polyhedral elements is a very complicated task, mainly due to the need of characterizing the *solenoidal bubble space*

$$\{v \in \overset{\circ}{V}{}^{\mathrm{3d}}_g(K) : \quad \nabla\cdot v = 0\},$$

that is, the subspace of V^{3d}_g containing divergence-free functions with zero normal trace on the boundary. Finding exact sequences with a commuting diagram property shares exactly the same difficulty. However, Theorem 2.13 does shed light on a promising approach to carry out the construction. In particular, we can take the given exact sequence to be $\mathrm{S}^{\mathrm{3d}}_{1,\mathrm{k}}$, which is a $\mathrm{S}^{\mathrm{2d}}(\partial K)$-admissible exact sequence for the set of commuting trace sequences

$$\mathrm{S}^{\mathrm{2d}}(\partial K) := \{\mathrm{S}^{\mathrm{poly}}_{1,\mathrm{k}}(\mathrm{f}) : \quad \forall \mathrm{f} \in \mathcal{F}(K)\},$$

or to be $\mathrm{S}^{\mathrm{3d}}_{2,\mathrm{k}}$, which is a $\mathrm{S}^{\mathrm{2d}}(\partial K)$-admissible exact sequence for the set of commuting trace sequences

$$\mathrm{S}^{\mathrm{2d}}(\partial K) := \{\mathrm{S}^{\mathrm{poly}}_{2,\mathrm{k}}(\mathrm{f}) : \quad \forall \mathrm{f} \in \mathcal{F}(K)\}.$$

Then, the only task left is to find the spaces $\delta H_g^{3\mathrm{d}}(K)$ and $\delta E_g^{3\mathrm{d}}(K)$ that satisfy the properties described in Theorem 2.13. The actual construction of commuting exact sequences on a general polyhedron is currently under way.

Appendix A: the harmonic interpolators. Here we rewrite in our notation the definition of the harmonic interpolators (in one-, two-, and three-space dimensions) introduced in [12]. To simplify the notation, we let the domain of these interpolations be the space of smooth fields. However, we only need these spaces to be regular enough such that the related differential and trace operators used in the harmonic interpolators make sense; see [13, sect. 2.3] for the related spaces with minimal regularity.

One-dimensional harmonic interpolators. Let $K \in \mathbb{R}$ be a segment. We denote by Π_H^1 and Π_W^1 the harmonic interpolators associated to H^1- and L^2-fields, respectively. We denote by

$$H^{1\mathrm{d}}(K) \longrightarrow W^{1\mathrm{d}}(K)$$

the corresponding compatible exact sequence.

The harmonic interpolators $\Pi_H^1 \times \Pi_W^1 :\ C^\infty(K) \times C^\infty(K) \to\ H^{1\mathrm{d}}(K) \times W^{1\mathrm{d}}(K)$ are defined by the following equations:

$$\begin{aligned} \Pi_H^1 u(\mathrm{v}) &= u(\mathrm{v}) && \forall \mathrm{v} \in \mathcal{V}(K), \\ (\nabla\, \Pi_H^1 u, v)_K &= (\nabla\, u, v)_K && \forall v \in \nabla\, \overset{\circ}{H}{}^{1\mathrm{d}}(K), \end{aligned}$$

and

$$(\Pi_W^1 u, v)_K = (u, v)_K \quad \forall v \in \overset{\circ}{W}{}^{1\mathrm{d}}(K) \oplus \mathcal{P}_0(K).$$

Two-dimensional harmonic interpolators. Let $K \in \mathbb{R}^2$ be a polygon. We denote by Π_H^2, Π_E^2, and Π_W^2 the harmonic interpolators associated to H^1-, $H(\mathrm{curl})$-, and L^2-fields, respectively. We denote by

$$H^{2\mathrm{d}}(K) \longrightarrow E^{2\mathrm{d}}(K) \longrightarrow W^{2\mathrm{d}}(K)$$

the corresponding compatible exact sequence.

The harmonic interpolators $\Pi_H^2 \times \Pi_E^2 \times \Pi_W^2 :\ C^\infty(\bar{K}) \times C^\infty(K; \mathbb{R}^2) \times C^\infty(K) \to H^{2\mathrm{d}}(K) \times E^{2\mathrm{d}}(K) \times W^{2\mathrm{d}}(K)$ are defined by the following equations:

$$\begin{aligned} \Pi_H^2 u(\mathrm{v}) &= u(\mathrm{v}) && \forall \mathrm{v} \in \mathcal{V}(K), \\ \big(\nabla\, \mathrm{tr}_H^{\mathrm{e}}(\Pi_H^2 u), v\big)_{\mathrm{e}} &= (\nabla\, \mathrm{tr}_H^{\mathrm{e}}(u), v)_{\mathrm{e}} && \forall v \in \nabla\, \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}), \forall \mathrm{e} \in \mathcal{E}(K), \\ (\nabla\, \Pi_H^2 u, v)_K &= (\nabla\, u, v)_K && \forall v \in \nabla\, \overset{\circ}{H}{}^{2\mathrm{d}}(K), \end{aligned}$$

and

$$\begin{aligned} \big(\mathrm{tr}_E^{\mathrm{e}}(\Pi_E^2 u), v\big)_{\mathrm{e}} &= (\mathrm{tr}_E^{\mathrm{e}}(u), v)_{\mathrm{e}} && \forall v \in \overset{\circ}{W}{}^{1\mathrm{d}}(\mathrm{e}) \oplus \mathcal{P}_0(\mathrm{e}), \forall \mathrm{e} \in \mathcal{E}(K), \\ (\nabla\times \Pi_E^2 u, v)_K &= (\nabla\times u, v)_K && \forall v \in \nabla\times \overset{\circ}{E}{}^{2\mathrm{d}}(K), \\ (\Pi_E^2 u, v)_K &= (u, v)_K && \forall v \in \{v \in \overset{\circ}{E}{}^{2\mathrm{d}}(K) :\ \nabla\times v = 0\}, \end{aligned}$$

and

$$(\Pi_W^2 u, v)_K = (u, v)_K \quad \forall v \in \overset{\circ}{W}{}^{2\mathrm{d}}(K) \oplus \mathcal{P}_0(K).$$

Three-dimensional harmonic interpolators. Let $K \in \mathbb{R}^3$ be a polyhedron. We denote by Π_H^3, Π_E^3, Π_V^3, and Π_W^3 the harmonic interpolators associated to H^1-, $H(\text{curl})$-, $H(\text{div})$- and L^2-fields, respectively. We denote by

$$H^{3\mathrm{d}}(K) \longrightarrow E^{3\mathrm{d}}(K) \longrightarrow V^{3\mathrm{d}}(K) \longrightarrow W^{3\mathrm{d}}(K)$$

the corresponding compatible exact sequence.

The harmonic interpolators $\Pi_H^3 \times \Pi_E^3 \times \Pi_V^3 \times \Pi_W^3 : \; C^\infty(\bar{K}) \times C^\infty(\bar{K};\mathbb{R}^3) \times C^\infty(\bar{K};\mathbb{R}^3) \times C^\infty(\bar{K}) \to \; H^{3\mathrm{d}}(K) \times E^{3\mathrm{d}}(K) \times V^{3\mathrm{d}}(K) \times W^{3\mathrm{d}}(K)$ are defined by the following equations:

$$\begin{aligned}
\Pi_H^3 u(\mathrm{v}) &= u(\mathrm{v}) && \forall \mathrm{v} \in \mathcal{V}(K),\\
\big(\nabla\, \mathrm{tr}_H^{\mathrm{e}}(\Pi_H^3 u), v\big)_{\mathrm{e}} &= (\nabla\, \mathrm{tr}_H^{\mathrm{e}}(u), v)_{\mathrm{e}} && \forall v \in \nabla \overset{\circ}{H}{}^{1\mathrm{d}}(\mathrm{e}), \forall \mathrm{e} \in \mathcal{E}(K),\\
\big(\nabla\, \mathrm{tr}_H^{\mathrm{f}}(\Pi_H^3 u), v\big)_{\mathrm{f}} &= \big(\nabla\, \mathrm{tr}_H^{\mathrm{f}}(u), v\big)_{\mathrm{f}} && \forall v \in \nabla \overset{\circ}{H}{}^{2\mathrm{d}}(\mathrm{f}), \forall \mathrm{f} \in \mathcal{F}(K),\\
\big(\nabla\, \Pi_H^3 u, v\big)_K &= (\nabla\, u, v)_K && \forall v \in \nabla \overset{\circ}{H}{}^{3\mathrm{d}}(K),
\end{aligned}$$

and

$$\begin{aligned}
\big(\mathrm{tr}_E^{\mathrm{e}}(\Pi_E^3 u), v\big)_{\mathrm{e}} &= (\mathrm{tr}_E^{\mathrm{e}}(u), v)_{\mathrm{e}} && \forall v \in \overset{\circ}{W}{}^{1\mathrm{d}}(\mathrm{e}) \oplus \mathcal{P}_0(K), \forall \mathrm{e} \in \mathcal{E}(K),\\
\big(\nabla\times \mathrm{tr}_E^{\mathrm{f}}(\Pi_E^3 u), v\big)_{\mathrm{f}} &= \big(\nabla\times \mathrm{tr}_E^{\mathrm{f}}(u), v\big)_{\mathrm{f}} && \forall v \in \nabla\times \overset{\circ}{E}{}^{2\mathrm{d}}(\mathrm{f}), \forall \mathrm{f} \in \mathcal{F}(K),\\
\big(\mathrm{tr}_E^{\mathrm{f}}(\Pi_E^3 u), v\big)_{\mathrm{f}} &= \big(\mathrm{tr}_E^{\mathrm{f}}(u), v\big)_{\mathrm{f}} && \forall v \in \{v \in \overset{\circ}{E}{}^{2\mathrm{d}}(\mathrm{f}) :\; \nabla\times v = 0\}, \forall \mathrm{f} \in \mathcal{F}(K),\\
\big(\nabla\times \Pi_E^3 u, v\big)_K &= (\nabla\times u, v)_K && \forall v \in \nabla\times \overset{\circ}{E}{}^{3\mathrm{d}}(K),\\
\big(\Pi_E^3 u, v\big)_K &= (u, v)_K && \forall v \in \{v \in \overset{\circ}{E}{}^{3\mathrm{d}}(K) :\; \nabla\times v = 0\},
\end{aligned}$$

and

$$\begin{aligned}
\big(\mathrm{tr}_V^{\mathrm{f}}(\Pi_V^3 u), v\big)_{\mathrm{f}} &= \big(\mathrm{tr}_V^{\mathrm{f}}(u), v\big)_{\mathrm{f}} && \forall v \in \overset{\circ}{W}{}^{2\mathrm{d}}(\mathrm{f}) \oplus \mathcal{P}_0(\mathrm{f}), \forall \mathrm{f} \in \mathcal{F}(K),\\
\big(\nabla\cdot \Pi_V^3 u, v\big)_K &= (\nabla\cdot u, v)_K && \forall v \in \nabla\cdot \overset{\circ}{V}{}^{3\mathrm{d}}(K),\\
\big(\Pi_V^3 u, v\big)_K &= (u, v)_K && \forall v \in \{v \in \overset{\circ}{V}{}^{3\mathrm{d}}(K) :\; \nabla\cdot v = 0\},
\end{aligned}$$

and

$$(\Pi_W^3 u, v)_K = (u, v)_K \quad \forall v \in \overset{\circ}{W}{}^{3\mathrm{d}}(K) \oplus \mathcal{P}_0(K).$$

Appendix B: proofs of sequence exactness.

Proof of Lemma 5.1. We only provide a sketch of the proof since it is very simple. The exactness of these four sequences can be proven using exactly the same argument by showing that the differential operators map the previous function space into the next one, the curl operator is surjective, and the following dimension count holds

$$\dim H^{2\mathrm{d}}(K) - \dim E^{2\mathrm{d}}(K) + \dim W^{2\mathrm{d}}(K) = 1,$$

where $H^{2\mathrm{d}}(K)$, $E^{2\mathrm{d}}(K)$, $W^{2\mathrm{d}}(K)$ are the related spaces for the sequence. This completes the sketch of the proof.

Proof of Lemma 5.2. The exactness of the first two sequences, $\mathrm{S}^{2\mathrm{d}}_{1,\mathrm{k}}$ and $\mathrm{S}^{2\mathrm{d}}_{2,\mathrm{k}}$, is well-known; see [4] for an elegant proof in arbitrary space dimensions which includes the three-dimensional result as a special case; it uses the *Koszul complex* and the *homotopy formula*.

Next, we prove that the last two sequences, $\mathrm{S}^{2\mathrm{d}}_{5,\mathrm{k}}$ and $\mathrm{S}^{2\mathrm{d}}_{6,\mathrm{k}}$, are exact. We omit the proofs for the third and fourth sequences since they are similar to those we are going to present now.

We prove the exactness by showing that the following three identities hold

$$\mathbb{R} = \mathrm{Ker}_{\nabla} H^{3\mathrm{d}}, \tag{B.1a}$$

$$\nabla\, H^{3\mathrm{d}} = \mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}}, \tag{B.1b}$$

$$\nabla\cdot V^{3\mathrm{d}} = W^{3\mathrm{d}}, \tag{B.1c}$$

and that the following dimension count identity holds

$$\dim H^{3\mathrm{d}} - \dim E^{3\mathrm{d}} + \dim V^{3\mathrm{d}} - \dim W^{3\mathrm{d}} = 1. \tag{B.1d}$$

The other equality, namely $\nabla\times E^{3\mathrm{d}} = \mathrm{Ker}_{\nabla\cdot} V^{3\mathrm{d}}$, is a direct consequence of the above results. To see this, we have

$$\begin{aligned}
\dim \nabla\times E^{3\mathrm{d}} &= \dim E^{3\mathrm{d}} - \dim \mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}} \\
&= \dim E^{3\mathrm{d}} - \dim \nabla\, H^{3\mathrm{d}} && \text{by (B.1b)} \\
&= \dim E^{3\mathrm{d}} - \dim H^{3\mathrm{d}} + 1 && \text{by (B.1a)} \\
&= \dim V^{3\mathrm{d}} - \dim W^{3\mathrm{d}} && \text{by (B.1d)} \\
&= \dim V^{3\mathrm{d}} - \dim \nabla\cdot V^{3\mathrm{d}} && \text{by (B.1c)} \\
&= \dim \mathrm{Ker}_{\nabla\cdot} V^{3\mathrm{d}}.
\end{aligned}$$

We start with the verification for the sequence $\mathrm{S}^{2\mathrm{d}}_{6,\mathrm{k}}$. The first equality (B.1a) is trivially satisfied. We also have

$$\nabla\cdot \begin{pmatrix} 0 \\ 0 \\ \mathcal{P}_k(x,y) \oplus \mathcal{P}_{k+1}(z) \end{pmatrix} = \mathcal{P}_k(x,y) \oplus \mathcal{P}_k(z),$$

which implies the third equality (B.1c). The dimension equality (B.1d) is also easy to verify by the fact that

$$\begin{aligned}
\dim H^{3\mathrm{d}} &= \dim \mathcal{P}_{k+1|k+1} \\
&= (k+2)^2(k+3)/2, \\
\dim E^{3\mathrm{d}} &= \dim \boldsymbol{N}_k(x,y)\cdot \dim \mathcal{P}_{k+1}(z) + \dim \mathcal{P}_{k+1|k} \\
&= 3(k+1)(k+2)(k+3)/2, \\
\dim V^{3\mathrm{d}} &= \dim \boldsymbol{RT}_k(x,y)\cdot \dim \mathcal{P}_k(z) + \dim \mathcal{P}_{k|k+1} \\
&= (k+1)^2(k+3) + (k+1)(k+2)^2/2, \\
\dim W^{3\mathrm{d}} &= \dim \mathcal{P}_{k|k} \\
&= (k+1)^2(k+2)/2.
\end{aligned}$$

Now, we are left to prove the identity (B.1b). Since $\nabla\, H^{3\mathrm{d}} \subset \mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}}$, we just need to show that $\mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}} \subset \nabla\, H^{3\mathrm{d}}$. To this end, let p be a function in $\mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}}$.

Since $E^{3\mathrm{d}} \subset \boldsymbol{\mathcal{P}}_{2k+2}$, we have $p = \nabla q$ for a scalar polynomial function $q \in \mathcal{P}_{2k+3}$. We have

$$\begin{pmatrix} \partial_x p \\ \partial_y p \end{pmatrix} \in \boldsymbol{N}_k(x,y) \otimes \mathcal{P}_{k+1}(z), \quad \text{and } \partial_z p \in \mathcal{P}_{k+1}(x,y) \otimes \mathcal{P}_k(z)$$

Now, we write q in terms of a polynomial in z with its coefficients being polynomials in x and y:

$$q = \sum_{\alpha=0}^{2k+3} f_\alpha(x,y) z^\alpha.$$

We have

$$\partial_z q = \sum_{\alpha=1}^{2k+3} \alpha f_\alpha(x,y) z^{\alpha-1} \in \mathcal{P}_{k+1}(x,y) \otimes \mathcal{P}_k(z).$$

This implies that $f_\alpha(x,y) \in \mathcal{P}_{k+1}(x,y)$ for $1 \le \alpha \le k+1$, and $f_\alpha(x,y) = 0$ for $\alpha \ge k+2$. Hence,

$$q = f_0(x,y) + \sum_{\alpha=1}^{k+1} f_\alpha(x,y) z^\alpha.$$

Since $f_\alpha(x,y) \in \mathcal{P}_{k+1}(x,y)$ for $1 \le \alpha \le k+1$ and $H^{3\mathrm{d}} := \mathcal{P}_{k+1}(x,y) \otimes \mathcal{P}_{k+1}(z)$, we have

$$\sum_{\alpha=1}^{k+1} f_\alpha(x,y) z^\alpha \in H^{3\mathrm{d}}.$$

Then, we have

$$\begin{pmatrix} \partial_x f_0(x,y) \\ \partial_y f_0(x,y) \end{pmatrix} \in \boldsymbol{N}_k(x,y),$$

which implies $f_0(x,y) \in \mathcal{P}_{k+1}(x,y)$, hence $q \in H^{3\mathrm{d}}$. This completes the proof of the equality (B.1b). Hence the sequence $\mathrm{S}^{2\mathrm{d}}_{6,\mathrm{k}}$ is exact.

Now, we prove equalities (B.1) for the sequence $\mathrm{S}^{2\mathrm{d}}_{5,\mathrm{k}}$. The first equality (B.1a) is trivially satisfied. The third equality is due to the fact that

$$\nabla \cdot \begin{pmatrix} \mathcal{P}_k(x,y) \oplus \mathcal{P}_k(z) \\ 0 \\ \widetilde{\mathcal{P}}_k(x,y) \oplus \mathcal{P}_{k+1}(z) \end{pmatrix} = \mathcal{P}_k(x,y) \oplus \mathcal{P}_k(z).$$

The dimension equality (B.1d) is also easy to verify by the fact that

$$\begin{aligned} \dim H^{3\mathrm{d}} &= k+3+\dim \mathcal{P}_{k|k}, \\ \dim E^{3\mathrm{d}} &= 2k+3+\dim \boldsymbol{\mathcal{P}}_{k|k}, \\ \dim V^{3\mathrm{d}} &= k+1+\dim \boldsymbol{\mathcal{P}}_{k|k}, \\ \dim W^{3\mathrm{d}} &= \dim \mathcal{P}_{k|k}. \end{aligned}$$

Now, we are left to prove the identity (B.1b). Again, we prove that $\mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}} \subset \nabla H^{3\mathrm{d}}$. Since the spaces in $\mathrm{S}^{2\mathrm{d}}_{5,\mathrm{k}}$ are included in the related spaces in $\mathrm{S}^{2\mathrm{d}}_{6,\mathrm{k}}$, which is an exact sequence, we have any function $p \in \mathrm{Ker}_{\nabla\times} E^{3\mathrm{d}}$ is a gradient of a function $q \in \mathcal{P}_{k+1|k+1}$. We have

$$\begin{pmatrix} \partial_x p \\ \partial_y p \end{pmatrix} \in \begin{pmatrix} \mathcal{P}_{k|k} \\ \mathcal{P}_{k|k} \end{pmatrix} \oplus \begin{pmatrix} y \\ -x \end{pmatrix} \widetilde{\mathcal{P}}_k(x,y), \quad \text{and } \partial_z p \in \mathcal{P}_{k|k} \oplus \widetilde{\mathcal{P}}_{k+1} z^k.$$

Now, we show that the function q is actually a function in the space

$$H^{3\mathrm{d}} = \mathcal{P}_{k|k} \oplus \widetilde{\mathcal{P}}_{k+1}(x,y) \oplus \{z^{k+1}\}.$$

Again, we express q as a polynomial of the variable z with coefficients polynomials of x and y:

$$q = \sum_{\alpha=0}^{k+1} f_\alpha(x,y) z^\alpha,$$

where $f_\alpha(x,y) \in \mathcal{P}_{k+1}(x,y)$. Using the fact that $\partial_z q \in \mathcal{P}_{k|k} \oplus \widetilde{\mathcal{P}}_{k+1} z^k$, we immediately get $f_\alpha(x,y) \in \mathcal{P}_k(x,y)$ for $1 \le \alpha \le k$. Moreover, since $\partial_x q \in \mathcal{P}_{k|k} \oplus y\widetilde{\mathcal{P}}_k(x,y)$, we have $\partial_x f_{k+1}(x,y) = 0$. Similarly, $\partial_y f_{k+1}(x,y) = 0$. This implies that $f_{k+1}(x,y)$ is a constant. Hence, $q \in H^{3\mathrm{d}}$ as desired. This completes the proof that $\mathrm{S}^{2\mathrm{d}}_{5,\mathrm{k}}$ is an exact sequence and completes the proof of Theorem 5.2. ∎

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