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On the Finite Element Method

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Dedicated to Professor OTAKAR BORŮVKA on the occasion of his scientific jubilee.

1. Introduction

COURANT has suggested in [8] a finite difference method which is applicable to boundary value problems arising from variational problems. The finite difference equations are obtained by means of the Ritz method with trial functions that are piecewise linear over a triangulation of the plane. There are several papers dealing with this method. We name FRIEDRICHS and KELLER [12] and OGANESJAN [14][1]. Much more papers appeared in recent years in technical journals. The method has been called the finite element method (see the monograph [17] by ZIENKIEWICZ). It was developed originally as a concept of structural analysis and in contrast to the mathematicians the engineers have begun to use for approximation polynomials of higher degrees and have developed this method also for variational problems containing derivatives of the second order (see CLOUGH and TOCHER [7] and FRAEIJS DE VEUBEKE [9]).

In this paper we justify first a procedure by FRAEIJS DE VEUBEKE [10] which uses quadratic polynomials. Then we propose and justify a procedure using cubic polynomials. All these procedures concern variational problems containing first order derivatives only. In the last section we introduce and justify a procedure using fifth degree polynomials and applicable to variational problems containing second order derivatives. In all cases we derive asymptotic estimates of the discretization error. Some details of the computational technique and numerical results will be published in a subsequent paper.

2. Second Order Equations

Let Ω be a simply or multiply connected bounded domain in the x_1, x_2 plane with the boundary Γ. Γ consists of a finite number of simple closed curves Γ_i $(i=0, \ldots, r)$; $\Gamma_1, \ldots, \Gamma_r$ lie inside of Γ_0 and do not intersect. We consider the equation

$$Lu \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + cu = f(x_1, x_2) \tag{1}$$

[1] I am indebted to the referee for calling my attention to the following papers: AUBIN [1], CÉA [4], CIARLET [5], CIARLET, SCHULTZ and VARGA [6]. After having sent the manuscript to the editor there appeared a very interesting paper [3] by BIRKHOFF, SCHULTZ and VARGA.

in Ω. Here a_{ij}, c, f are continuous functions of (x_1, x_2) on $\bar{\Omega}$ and

$$a_{ij}(x_1, x_2) = a_{ji}(x_1, x_2), \quad \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geqq \mu \sum_{i=1}^{2} \xi_i^2, \tag{2}$$
$$\mu = \text{const} > 0, \qquad c(x_1, x_2) \geqq 0.$$

Two boundary value problems will be solved:

The Dirichlet problem

$$u|_{\Gamma} = 0 \tag{3}$$

and the third boundary value problem

$$\left[\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_j} \cos(\nu, x_i) + \sigma u\right]_{\Gamma} = 0. \tag{4}$$

In (4) ν is the outside normal to Γ and on Γ we assume

$$\sigma \geqq 0 \quad \text{on } \Gamma, \quad \sigma \not\equiv 0 \quad \text{if} \quad \min_{\bar{\Omega}} c = 0. \tag{5}$$

Under these assumptions both these problems are positive definite and the solution of the Dirichlet problem minimizes the functional

$$F_1(v) = \int_{\Omega} \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + c v^2 - 2 f v\right) dx \tag{6}$$

in the class of functions $\mathring{W}_2^{(1)}(\Omega)$ and the solution of the third boundary value problem minimizes

$$F_2(v) = \int_{\Omega} \left(\sum_{i,j=1}^{2} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + c v^2 - 2 f v\right) dx + \int_{\Gamma} \sigma v^2 \, d\Gamma \tag{7}$$

in the class $W_2^{(1)}(\Omega)$ (see, e.g., [13]). Here $W_2^{(k)}(\Omega)$ means the Sobolev space of functions having generalized derivatives up to the order k inclusive which belong to $L_2(\Omega)$. The norm is defined by

$$\|u\|^2_{W_2^{(k)}(\Omega)} = \sum_{|i| \leqq k} \|D^i u\|^2_{L_2(\Omega)}. \tag{8}$$

Here we use the notation $i = (i_1, i_2)$, $|i| = i_1 + i_2$, $D^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \partial x_2^{i_2}}$ and $\mathring{W}_2^{(k)}(\Omega)$ is the space of functions which we get by completing in the norm $\| \; \|_{W_2^{(k)}(\Omega)}$ the set of functions from $C^{(k)}(\Omega)$ with compact support in Ω.

As we want to use for approximation polynomials of higher order we restrict ourselves to the case that the boundary curves Γ_i $(i = 0, \ldots, r)$ are polygons. We consider all triangulations of $\bar{\Omega}$. To triangulate $\bar{\Omega}$ means to cover $\bar{\Omega}$ by a finite number of arbitrary triangles such that the open triangles are disjoint, the union of closed triangles is $\bar{\Omega}$ and any two adjacent triangles have a common side. To every triangulation we associate two parameters: h, ϑ. h is the largest side and ϑ the smallest angle of all triangles of the given triangulation.

To explain the procedure by Fraeijs de Veubeke let us consider a triangle T [2] with vertices P_1, P_2, P_3. Let Q_1, Q_2, Q_3 be the mid points of the sides of T and

[2] By T we denote both the triangle and its interior.

$p(x_1, x_2)$ be a quadratic polynomial

$$p(x_1, x_2) = \alpha_1 + \alpha_2 x_1 + \alpha_3 x_2 + \alpha_4 x_1^2 + \alpha_5 x_1 x_2 + \alpha_6 x_2^2.$$

We determine the six coefficients α_i by six conditions, namely that the polynomial $p(x_1, x_2)$ assumes given values at the nodes $P_1, P_2, P_3, Q_1, Q_2, Q_3$. We consider these values as parameters when no side of the triangle lies on the boundary Γ or when it lies and there is prescribed the natural boundary condition (4). In case of the Dirichlet condition (3) the values at the boundary nodes are equal to zero. It follows easily from Theorem 1 introduced later that the polynomial $p(x_1, x_2)$ is uniquely determined by these six conditions. We construct such polynomials for every triangle of the given triangulation and consider the spaces $\mathring{H}_2(\Omega)$ and $H_2(\Omega)$ of functions defined on every triangle by the corresponding polynomial ($\mathring{H}_2$ corresponds to the boundary condition (3) and H_2 to (4)). It is easy to show that every function from $\mathring{H}_2$ belongs to $\mathring{W}_2^{(1)}$ and every function from H_2 to $W_2^{(1)}$. These functions have piecewise continuous partial derivatives of the first order and we show that they assume the same values on the common sides of adjacent triangles which means that they are continuous. Let namely $P_j P_k$ be a common side of two triangles and p_1, p_2 be the corresponding polynomials. We can express the segment $P_j P_k$ by means of a parameter $s \in \langle 0, l \rangle$, l being the length of the side $P_j P_k$, and then both polynomials p_1, p_2 are quadratic polynomials of s assuming the same values for $s = 0, \frac{1}{2} l, l$. Hence it follows that the polynomials assume the same values on the side $P_j P_k$.

$\mathring{H}_2$ and H_2 are finite dimensional subspaces of $\mathring{W}_2^{(1)}$ and $W_2^{(1)}$, respectively. The approximate solutions $U(x_1, x_2)$ of our boundary value problems are determined as that functions from $\mathring{H}_2$ or H_2 which minimize (6) in the class $\mathring{H}_2$ and (7) in the class H_2, respectively. The existence and uniqueness of $U(x_1, x_2)$ follows immediately if we realize that the functionals (6) and (7) without the linear term $-2\int_\Omega f v\, dx$, considered in the classes $\mathring{H}_2$ and H_2, respectively, are positive definite quadratic forms of the nodal values.

We want to find an estimate for the discretization error $u(x_1, x_2) - U(x_1, x_2)$ in the norm $\| \|_{W_2^{(1)}(\Omega)}$ proving thus the convergence of the method. The argumentation goes in the same lines as in [12] and [16]. Consider first the Dirichlet problem. It is well known (see, e.g., [13]) that if u is the solution and $v \in \mathring{W}_2^{(1)}$ then

$$D(v-u) = F_1(v) - F_1(u)$$

where $D(z) = D(z, z)$ and

$$D(\varphi, \psi) = \int_\Omega \left[\sum_{i,j=1}^{2} a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j} + c\,\varphi\psi \right] dx.$$

Hence we have

$$D(U-u) = F_1(U) - F_1(u) = \min_{v \in \mathring{H}_2} F_1(v) - F_1(u) = \min_{v \in \mathring{H}_2} D(v-u) \leq D(u - \tilde{u})$$

where $\tilde{u}$ is the function from $\mathring{H}_2$ assuming the same values at nodal points as the solution $u(x_1, x_2)$.

Now to get an estimate for the discretization error it is sufficient to derive an estimate for the difference $u-\tilde{u}$. We use the following Theorem 1 which we prove later.

Theorem 1. Let the function $\varphi(x_1, x_2)$ be continuous on a closed triangle $\overline{T}$ and have bounded derivatives of the third order in the interior of T,

$$|D^i \varphi(x_1, x_2)| \leqq M_3, \quad |i|=3.$$

Further let φ vanish at the vertices P_1, P_2, P_3 and at the mid points Q_1, Q_2, Q_3. Then it holds in $\overline{T}$

$$\left|\frac{\partial \varphi(x_1, x_2)}{\partial x_j}\right| \leqq \frac{2}{\sin\alpha} M_3 c^2, \quad j=1, 2, \qquad |\varphi(x_1, x_2)| \leqq M_3 c^3, \tag{9}$$

where c is the greatest side and α the smallest angle of the triangle T.

Assume that the solution $u(x_1, x_2)$ has bounded derivatives of the third order, M_3 being the bound. Applying Theorem 1 for every triangle of the triangulation and for $\varphi=u-\tilde{u}$, taking into account that $D^i\varphi = D^i u$ for $|i|=3$ and remembering the meaning of the parameters h, ϑ we get

$$[D(U-u)]^{\frac{1}{2}} \leqq C \frac{1}{\sin\vartheta} M_3 h^2$$

where the constant C does not depend on the triangulation. Using first the assumptions (2) and then Friedrichs inequality we come easily to the following statement:

If the solution $u(x_1, x_2)$ has bounded derivatives of the third order in Ω,

$$|D^i u(x_1, x_2)| \leqq M_3, \quad |i|=3, \tag{10}$$

then it holds

$$\|u-U\|_{W_2^{(1)}(\Omega)} \leqq C \frac{1}{\sin\vartheta} M_3 h^2 \tag{11}$$

where the constant C does not depend on the triangulation.

(11) means that if we refine the triangulation in such a way that $\vartheta \geqq \vartheta_0 > 0$ the approximate solution $U(x_1, x_2)$ converges to the exact solution in the norm $\| \ \|_{W_2^{(1)}(\Omega)}$, the convergence being of the order h^2. This is the same rate of convergence as in the case of the usual second order finite difference scheme. However we need only to assume the boundedness of the derivatives of the third order whereas in the case of the usual finite difference scheme the boundedness of the fourth order derivatives is required.

Concerning the boundary condition (4) the estimate (11) remains true and only minor changes in the proof are necessary. Instead of the form $D(z)$ we have

$$D_1(z) = D(z) + \int_\Gamma \sigma z^2 d\Gamma.$$

If $\min_{\overline{\Omega}} c(x_1, x_2) = c_0 > 0$ then

$$D_1(z) \geqq \mu \int_\Omega \left[\left(\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2\right] dx + c_0 \int_\Omega z^2 dx$$

and from $D_1(U-u) \leqq D_1(\tilde{u}-u)$ we get (11) immediately. If $\min\limits_{\bar{\Omega}} c(x_1, x_2) = 0$ then from our assumption (5) it follows that $\sigma \geqq \sigma_0 > 0$ on a part $\overline{\Gamma}$ of Γ for σ_0 being sufficiently small. Then

$$D_1(z) \geqq \mu \int\limits_{\Omega} \left[\left(\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{\partial z}{\partial x_2}\right)^2\right] dx + \sigma_0 \int\limits_{\dot{\Gamma}} z^2 \, d\Gamma.$$

Using the generalised Friedrichs inequality (see [11], footnote 13a) we again get (11).

We turn our attention to the approximation by cubic polynomials. A general cubic polynomial in two variables has ten coefficients. Thus we must impose ten conditions. We suggest two possibilities: a) to prescribe the values of the polynomial and its first derivatives at the vertices P_1, P_2, P_3 and as the tenth condition the value of the polynomial at the center of gravity P_0 of the triangle. b) to prescribe the values of the polynomial at the vertices, at the center of gravity and at six points lying on the sides of the triangle and dividing them in three equal parts. We shall discuss the first case only.

Again we consider the prescribed values as parameters when no side of the triangle lies on the boundary Γ or when it lies and we have the boundary condition (4). In case of the Dirichlet condition (3) the values of the polynomial at boundary nodes must be equal to zero whereas the first derivatives must satisfy the condition

$$\frac{\partial p}{\partial x_1} \cos\alpha + \frac{\partial p}{\partial x_2} \sin\alpha = 0$$

where α is the angle of the corresponding side with the x_1-axe. This simply means that $\partial p/\partial s = 0$ at the boundary node, $\partial/\partial s$ being the derivative in the direction of the side. It follows from Theorem 2 introduced below that the polynomial $p(x_1, x_2)$ is uniquely determined by these ten conditions. We construct such polynomials for every triangle of the given triangulation and consider the spaces $\mathring{H}_3(\Omega)$ and $H_3(\Omega)$ of functions defined on every triangle by the corresponding polynomial. Again it is true that $\mathring{H}_3 \subset \mathring{W}_2^{(1)}$ and $H_3 \subset W_2^{(1)}$. Let namely $P_j P_k$ be the common side of two triangles. The corresponding polynomials are cubic polynomials of a parameter $s \in \langle 0, l \rangle$ and they have the same values and the same derivative with respect to s at the endpoints of the interval $\langle 0, l \rangle$. Hence they assume the same values on $P_j P_k$. $\mathring{H}_3$ and H_3 are finite dimensional subspaces of $\mathring{W}_2^{(1)}$ and $W_2^{(1)}$, respectively. The approximate solutions $U(x_1, x_2)$ of our boundary value problems are determined as that functions from $\mathring{H}_3$ and H_3 which minimize (6) in the class $\mathring{H}_3$ and (7) in the class H_3, respectively. Again the existence and uniqueness of $U(x_1, x_2)$ follow in the same way as before.

To get an estimate for the discretization error $u(x_1, x_2) - U(x_1, x_2)$ we use the following

Theorem 2. Let the function $\varphi(x_1, x_2)$ be continuously differentiable on a closed triangle $\overline{T}$ and have bounded derivatives of the fourth order in the interior:

$$|D^i \varphi(x_1, x_2)| \leqq M_4, \quad |i| = 4.$$

Further let φ and its first derivatives vanish at the vertices of T and let φ vanish at the center of gravity of T. Then it holds in $\bar{T}$

$$(12)\qquad \left|\frac{\partial\varphi(x_1, x_2)}{\partial x_j}\right| \leqq \frac{5}{\sin\alpha} M_4 c^3, \quad j=1,2, \qquad |\varphi(x_1, x_2)| \leqq \frac{3}{\sin\alpha} M_4 c^4$$

where c is the greatest side and α the smallest angle of the triangle T.

A consequence of this theorem is the following statement:

If the solution $u(x_1, x_2)$ has bounded derivatives of the fourth order in Ω,

$$|D^i u(x_1, x_2)| \leqq M_4, \qquad |i| = 4,$$

then it holds

$$(13)\qquad \|u - U\|_{W_2^{(1)}(\Omega)} \leqq C \frac{1}{\sin\vartheta} M_4 h^3$$

where the constant C does not depend on the triangulation.

The convergence is of the order h^3 under the same assumptions which in case of the usual second order finite difference scheme ensure the convergence of the order h^2.

Let us remark at the end that the estimates (11) and (13) can be proved in a similar way also for the Neumann problem and for the mixed problem where the condition (3) is prescribed on a part of the boundary and on the remainder of it there is prescribed the condition (4).

3. Auxiliary Lemmas and Proofs of Theorem 1 and 2

We first introduce the notation for the elements of the triangle T. We denote the angles at the vertices P_1, P_2 and P_3 by α, β and γ, respectively. We choose the notation of the vertices in such a way that $\alpha \leqq \beta \leqq \gamma$. The sides are denoted by $c = P_1P_2$, $a = P_2P_3$, $b = P_3P_1$. From $\alpha \leqq \beta \leqq \gamma$ it follows by elementary considerations

$$(14)\qquad \begin{gathered} a \leqq b \leqq c, \\ \frac{1}{\sin\gamma} \leqq \frac{1}{\sin\beta} \leqq \frac{1}{\sin\alpha}, \\ \frac{c}{b} < 2. \end{gathered}$$

Now we state some very simple lemmas. Only one of them will be proved.

Lemma 1. Let s_1, s_2 be two directions containing an angle ω. Let $\frac{\partial\varphi(P)}{\partial s_1} = k_1$, $\frac{\partial\varphi(P)}{\partial s_2} = k_2$, P being a point in the (x_1, x_2)-plane. Then

$$\left|\frac{\partial\varphi(P)}{\partial x_j}\right| \leqq \frac{|k_1| + |k_2|}{|\sin\omega|}, \qquad j = 1, 2$$

and if $0 < \omega < \frac{1}{3}\pi$ then

$$\left|\frac{\partial\varphi(P)}{\partial s}\right| \leqq \frac{2\sqrt{3}}{3} \max|k_j|$$

where s is any direction lying inside the acute angle formed by s_1 and s_2.

In the following lemmas $g(s)$ means always a function of a real parameter $s \in \langle 0, l \rangle$, continuous on $\langle 0, l \rangle$ and having a derivative of the second, third or fourth order.

Lemma 2. Let $g(0)=\eta_1$, $g(l)=\eta_2$ and $|g''(s)| \leqq K_2$ in $(0, l)$. Then

$$|g(s)| \leqq \max|\eta_j| + \tfrac{1}{8} K_2 l^2.$$

Lemma 3. Let $g(0)=\eta_1$, $g(\frac{1}{2}l)=\eta_2$, $g(l)=\eta_3$ and $|g'''(s)| \leqq K_3$ in $(0, l)$. Then

$$|g(s)| \leqq \frac{5}{4} \max|\eta_j| + \frac{\sqrt{3}}{6^3} K_3 l^3,$$

$$|g'(s)| \leqq \frac{8}{l} \max|\eta_j| + \frac{1}{4} K_3 l^2.$$

Lemma 4. Let $g(0)=\eta_1$, $g'(0)=k_1$, $g(l)=\eta_2$ and $|g'''(s)| \leqq K_3$ in $(0, l)$. Then

$$|g(s)| \leqq \max|\eta_j| + \tfrac{1}{4}|k_1|\, l + \tfrac{1}{4} K_3 l^3.$$

Lemma 5. Let $g(0)=\eta_1$, $g(l)=\eta_2$, $g'(0)=k_1$, $g'(l)=k_2$ and $|g^{(4)}(s)| \leqq K_4$ in $(0, l)$. Then

$$|g(s)| \leqq \max|\eta_j| + \frac{1}{4} l \max|k_j| + \frac{1}{16.24} K_4 l^4,$$

$$|g'(s)| \leqq \frac{3}{l} \max|\eta_j| + \max|k_j| + \frac{1}{24} K_4 l^3.$$

Proof. Let

$$p(\xi) = \eta_1[1-3\xi^2+2\xi^3] + \eta_2[3\xi^2-2\xi^3] + k_1 l[\xi-2\xi^2+\xi^3] + k_2 l[-\xi^2+\xi^3]$$

and set $g(s)=p(l^{-1}s)+\psi(s)$. The polynomial $p(l^{-1}s)$ is an Hermite interpolation polynomial. We have

$$\psi(0)=\psi'(0)=\psi(l)=\psi'(l)=0, \qquad |\psi^{(4)}(s)| \leqq K_4$$

and from the remainder theorem for Hermite interpolation (see, e.g., [2]) we get $|\psi(s)| \leqq \frac{1}{16.24} K_4 l^4$. Concerning $\psi'(s)$ there exists $\xi_1 \in (0, l)$ such that $\psi'(\xi_1)=0$. From the remainder theorem for Lagrange interpolation we have

$$|\psi'(s)| \leqq \tfrac{1}{6} K_4 |x(x-\xi_1)(x-l)| \leqq \tfrac{1}{6} K_4 \tfrac{1}{4} l^2 |x-\xi_1| \leqq \tfrac{1}{24} K_4 l^3.$$

As

$$|p(\xi)| \leqq \max|\eta_j| + \tfrac{1}{4} l \max|k_j|, \qquad |p'(\xi)| \leqq 3 \max|\eta_j| + \max|k_j|$$

the lemma follows.

Lemma 6. Let $g(0)=\eta_1$, $g'(0)=k_1$, $g(\frac{2}{3}l)=\eta_2$, $g(l)=\eta_3$ and $|g^{(4)}(s)| \leqq K_4$ in $(0, l)$. Then

$$|g'(s)| \leqq \frac{27}{2l} \max|\eta_j| + |k_1| + \frac{1}{18} K_4 l^3.$$

Proof of Theorem 1. In the triangle T it holds

$$\left|\frac{\partial^3 \varphi}{\partial s_1\, \partial s_2\, \partial s_3}\right| \leqq 2\sqrt{2} M_3 \tag{15}$$

where $\partial/\partial s_j$ means a derivation in the direction s_j and s_j $(j=1, 2, 3)$ are arbitrary directions. Let us choose $\varepsilon>0$ and let us construct a triangle $P_1'P_2'P_3'$ lying inside T with sides a', b', c' which are parallel to the sides of T and lie in a distance δ. We have $a', b', c'<c$ and from continuity of φ it follows that $|\varphi(P_i')|\leqq\varepsilon$ and $|\varphi(Q_i')|\leqq\varepsilon$ if δ is sufficiently small (Q_1', Q_2' and Q_3' are the mid points of the sides $P_1'P_2'$, $P_2'P_3'$ and $P_3'P_1'$). We consider the function φ on the side $P_2'P_3'$ as a function of a parameter $s\in\langle 0, a'\rangle$. Let $\partial/\partial s_1$ be the derivative in the direction of the side $P_2'P_3'$. By (15) we have $|\partial^3\varphi/\partial s_1^3|\leqq 2\sqrt{2}M_3$. We apply Lemma 3 for $g=\varphi|_{P_2'P_3'}$ and $l=a'<c$ so that $K_3=2\sqrt{2}M_3$, $\max|\eta_j|\leqq\varepsilon$. Thus

$$\left|\frac{\partial\varphi(P_2')}{\partial s_1}\right|\leqq\frac{8}{a'}\varepsilon+\frac{\sqrt{2}}{2}M_3c^2.$$

In the same way we get

$$\left|\frac{\partial\varphi(P_2')}{\partial s_2}\right|\leqq\frac{8}{c'}\varepsilon+\frac{\sqrt{2}}{2}M_3c^2\leqq\frac{8}{a'}\varepsilon+\frac{\sqrt{2}}{2}M_3c^2$$

where s_2 is the direction of the side $P_2'P_1'$. Hence by Lemma 1

$$\left|\frac{\partial\varphi(P_2')}{\partial x_j}\right|\leqq\frac{16\varepsilon}{a'\sin\beta}+\frac{\sqrt{2}M_3}{\sin\beta}c^2\leqq\frac{16\varepsilon}{a'\sin\alpha}+\frac{\sqrt{2}M_3}{\sin\alpha}c^2.$$

The same inequality holds for the vertex P_3':

$$\left|\frac{\partial\varphi(P_3')}{\partial x_j}\right|\leqq\frac{16\varepsilon}{a'\sin\alpha}+\frac{\sqrt{2}M_3}{\sin\alpha}c^2.$$

From the last two inequalities and by Lemma 2 $\left(g=\left.\frac{\partial\varphi}{\partial x_j}\right|_{P_2'P_3'},\ K_2=2M_3,\ l=a'<c\right)$ we obtain

$$\left|\frac{\partial\varphi(P')}{\partial x_j}\right|\leqq\frac{16\varepsilon}{a'\sin\alpha}+\frac{\sqrt{2}M_3}{\sin\alpha}c^2+\frac{1}{4}M_3c^2$$

where P' is an arbitrary point of the side $P_2'P_3'$. For the vertex P_1' it also holds

$$\left|\frac{\partial\varphi(P_1')}{\partial x_j}\right|\leqq\frac{16\varepsilon}{a'\sin\alpha}+\frac{\sqrt{2}M_3}{\sin\alpha}c^2.$$

Now let P' be the point of the side $P_2'P_3'$ which lies on the line going through P_1' and P, P being an arbitrary point from the interior of the triangle $P_1'P_2'P_3'$. Applying again Lemma 2 $\left(g=\left.\frac{\partial\varphi}{\partial x_j}\right|_{P_1'P'},\ K_2=2M_3,\ l=P_1'P'\leqq c'<c\right)$ we get from the last two inequalities

$$\left|\frac{\partial\varphi(P)}{\partial x_j}\right|\leqq\frac{16}{a'\sin\alpha}\varepsilon+\frac{\sqrt{2}M_3}{\sin\alpha}c^2+\frac{1}{2}M_3c^2$$

and letting $\varepsilon\to 0+$

$$\left|\frac{\partial\varphi(P)}{\partial x_j}\right|\leqq\frac{2\sqrt{2}+\sin\alpha}{2\sin\alpha}M_3c^2\leqq\frac{2\sqrt{2}+\frac{1}{2}\sqrt{3}}{2\sin\alpha}M_3c^2\leqq\frac{2}{\sin\alpha}M_3c^2.$$

This is the first inequality in Theorem 1. An estimate for the function φ itself could be won from the estimates of the first derivatives. A better result, the second inequality in Theorem 1, follows from this lemma.

Lemma 7. Let the function $\psi(x_1, x_2)$ be continuous on a closed triangle $\overline{T}$ and have bounded derivatives of the third order in the interior. Further let $\psi(P_i) = \eta_i$, $\psi(Q_i) = \zeta_i$. Then it holds on $\overline{T}$

$$|\psi(x_1, x_2)| \leqq 6\eta + M_3 c^3, \quad \eta = \max(|\eta_i|, |\zeta_i|). \tag{16}$$

Proof. We again consider the triangle $P_1' P_2' P_3'$. Let $\psi(P_i') = \eta_i'$, $\psi(Q_i') = \zeta_i'$. If δ is sufficiently small then $|\eta_i - \eta_i'| \leqq \varepsilon$, $|\zeta_i - \zeta_i'| \leqq \varepsilon$. By means of Lemma 3 we easily show that on the boundary of the triangle $P_1' P_2' P_3'$ it holds

$$|\psi| \leqq \frac{5}{4}\eta' + \frac{2\sqrt{6}}{6^3} M_3 c^3, \quad \eta' = \max(|\eta_i'|, |\zeta_i'|). \tag{17}$$

By Lemma 3 we again get

$$\left|\frac{\partial\psi(P_1')}{\partial s_1}\right| \leqq \frac{8}{b'}\eta' + \frac{\sqrt{2}}{2} M_3 c^2, \quad \left|\frac{\partial\psi(P_1')}{\partial s_2}\right| \leqq \frac{8}{c'}\eta' + \frac{\sqrt{2}}{2} M_3 c^2,$$

s_1 and s_2 now being the directions of the sides $P_1' P_3'$ and $P_1' P_2'$, respectively. By the second part of Lemma 1 it follows from these inequalities that

$$\left|\frac{\partial\psi(P_1')}{\partial s}\right| \leqq \frac{16\sqrt{3}}{3} \frac{\eta'}{b'} + \frac{\sqrt{6}}{3} M_3 c^2, \tag{18}$$

s being any direction lying in the angle α. Let P' be the point on the side $P_2' P_3'$ which lies on the line going through P_1' and P. By (17), (18) and Lemma 4 we obtain

$$|\psi(P)| \leqq \frac{5}{4}\eta' + \frac{2\sqrt{6}}{6^3} M_3 c^3 + \frac{4\sqrt{3}}{3} \frac{c}{b'}\eta' + \frac{\sqrt{6}}{12} M_3 c^3 + \frac{\sqrt{2}}{2} M_3 c^3$$

and letting $\varepsilon \to 0+$

$$|\psi(P)| \leqq \left(\frac{5}{4} + \frac{4\sqrt{3}}{3} \cdot \frac{c}{b}\right)\eta + \left(\frac{2\sqrt{6}}{6^3} + \frac{\sqrt{6}}{12} + \frac{\sqrt{2}}{2}\right) M_3 c^3 \leqq 6\eta + M_3 c^3.$$

Proof of Theorem 2. We consider the triangle $P_1' P_2' P_3'$ and choose δ so small that $|D^i \varphi(P_j')| \leqq \varepsilon$, $|i| \leqq 1$, $j = 1, 2, 3$ and $|\varphi(P_0')| \leqq \varepsilon$ where P_0' is the center of gravity of the triangle $P_1' P_2' P_3'$. By Lemma 5 applied to $g = \varphi|_{P_2' P_3'}$ ($l = a' < c$, $K_4 = 4M_4$) it holds

$$|\varphi(Q_2')| \leqq \varepsilon\left(1 + \frac{1}{4}c\right) + \frac{1}{96} M_4 c^4, \quad \left|\frac{\partial\varphi(Q_2')}{\partial s_1}\right| \leqq \varepsilon\left(\frac{3}{a'} + 1\right) + \frac{1}{6} M_4 c^3,$$

s_1 being the direction of the side $P_2' P_3'$. Let s_2 be the direction of the line joining P_1' with Q_2'. As $\left|\frac{\partial\varphi(P_1')}{\partial x_j}\right| \leqq \varepsilon$ we have $\left|\frac{\partial\varphi(P_1')}{\partial s_2}\right| \leqq \sqrt{2}\,\varepsilon$. Applying Lemma 6 to $g = \varphi|_{P_1' Q_2'}$ ($l = P_1' Q_2' < c$, $K_4 = 4M_4$) we get

$$\left|\frac{\partial\varphi(Q_2')}{\partial s_2}\right| \leqq \varepsilon\left[\frac{27}{2 P_1' Q_2'}\left(1 + \frac{1}{4}c\right) + \sqrt{2}\right] + \left(\frac{9c}{64 P_1' Q_2'} + \frac{2}{9}\right) M_4 c^3.$$

By Lemma 1 it follows

$$\left|\frac{\partial\varphi(Q_2')}{\partial x_j}\right| \leqq \frac{1}{\sin\sigma}\left\{\varepsilon\left[\frac{27}{2 P_1' Q_2'}\left(1 + \frac{1}{4}c\right) + \sqrt{2} + \frac{3}{a'} + 1\right] + \left(\frac{9c}{64 P_1' Q_2'} + \frac{2}{9} + \frac{1}{6}\right) M_4 c^3\right\}$$

where σ is the angle between the directions s_1 and s_2. As $\beta<\sigma<\pi-\gamma$ it holds $\frac{1}{\sin\sigma}<\frac{1}{\sin\alpha}$ and letting $\varepsilon\to 0+$ we obtain

$$\left|\frac{\partial\varphi(Q_2)}{\partial x_j}\right|\leqq\frac{1}{\sin\alpha}\left(\frac{9c}{64\,\overline{P_1Q_2}}+\frac{7}{18}\right)M_4c^3.$$

By elementary considerations it is easy to prove that $\frac{c}{\overline{P_1Q_2}}<2$. Hence

$$\left|\frac{\partial\varphi(Q_2)}{\partial x_j}\right|\leqq\frac{1}{\sin\alpha}\left(\frac{9}{32}+\frac{7}{18}\right)M_4c^3.$$

The same inequality is true for the mid points Q_1, Q_3. Setting $\psi=\partial\varphi/\partial x_j$ and applying Lemma 7 we get

$$\left|\frac{\partial\varphi}{\partial x_j}\right|\leqq\frac{1}{\sin\alpha}\cdot\frac{193}{48}M_4c^3+M_4c^3\leqq\frac{5}{\sin\alpha}M_4c^3.$$

To prove the remaining inequality let us consider the parallel to the x_1-axe going through an arbitrary point $P\in T$. One of the intersections of this line with the boundary of T, say $\overline{P}$, lies in a distance not greater than $\frac{1}{2}c$ from P. Thus from

$$\varphi(P)=\varphi(\overline{P})+\int_{\bar{x}_1}^{x_1}\frac{\partial\varphi}{\partial x_1}\,dx_1$$

it follows

$$|\varphi(P)|\leqq|\varphi(\overline{P})|+\frac{1}{2}c\,\frac{5}{\sin\alpha}M_4c^3.$$

The values of φ on the boundary of T can be estimated by means of Lemma 5, the bound being $\frac{1}{96}M_4c^4$. The last inequality gives

$$|\varphi(P)|\leqq\frac{1}{\sin\alpha}\left(\frac{5}{2}+\frac{1}{96}\sin\alpha\right)M_4c^4\leqq\frac{3}{\sin\alpha}M_4c^4.$$

4. Fourth Order Equations

For the sake of brevity we restrict ourselves to the Dirichlet problem for the biharmonic equation:

$$\Delta^2u=f(x_1,x_2),\qquad (x_1,x_2)\in\Omega,\tag{19}$$

$$u|_\Gamma=\left.\frac{\partial u}{\partial\nu}\right|_\Gamma=0.\tag{20}$$

However, we remind that the procedure and results, which will be proved, remain the same for more general equations and for other boundary conditions which do not make any troubles. This is certainly an advantage of the finite element method against the usual finite difference method.

The solution $u(x_1,x_2)$ of the Dirichlet problem (19), (20) minimizes the functional

$$F_3(v)=\int_\Omega\left[\left(\frac{\partial^2v}{\partial x_1^2}\right)^2+2\left(\frac{\partial^2v}{\partial x_1\,\partial x_2}\right)^2+\left(\frac{\partial^2v}{\partial x_2^2}\right)^2-2f(x_1,x_2)\,v\right]dx\tag{21}$$

in the class $\mathring{W}_2^{(2)}(\Omega)$. If we want to approximate the solution in the triangles by polynomials we must choose such polynomials that the resulting trial functions are not only continuous in $\bar{\Omega}$ but they have continuous derivatives in $\bar{\Omega}$. They will then belong to $\mathring{W}_2^{(2)}$ as their second order derivatives will be piecewise continuous. To this end let us consider a polynomial of the fifth degree

$$p(x_1, x_2) = \alpha_1 + \alpha_2 x_1 + \cdots + \alpha_{20} x_1 x_2^4 + \alpha_{21} x_2^5.$$

To determine such a polynomial we need 21 conditions. We choose them in the following way: we prescribe the values $D^i p(P_j)$, $|i| \leqq 2$, $j = 1, 2, 3$, and the values $\frac{\partial p(Q_i)}{\partial \nu}$, $j = 1, 2, 3$. P_j and Q_j have the same meaning as before, i.e. P_j are vertices of the triangle T and Q_j are midpoints of the sides; ν is the normal to the boundary of T. It follows easily from Theorem 3 introduced later that the polynomial $p(x_1, x_2)$ is uniquely determined by these values. We show that the functions defined in $\bar{\Omega}$ and equal in the triangles to the corresponding polynomials are continuously differentiable and have piecewise continuous second order derivatives in $\bar{\Omega}$. It is sufficient to show that on a common side $P_j P_k$ of two triangles the corresponding polynomials assume the same values and the same value of the normal derivative. Now if we consider the polynomials on the side $P_j P_k$ as functions of a parameter $s \in \langle 0, l \rangle$ we know that they assume the same values at the ends of the interval $\langle 0, l \rangle$ and the first and second derivatives with respect to s at the ends of $\langle 0, l \rangle$ are the same. As they are polynomials of the fifth degree in s they assume the same values on the whole side $P_j P_k$. The normal derivatives assume the same values at the points $s = 0, \frac{1}{2} l, l$ of the interval $\langle 0, l \rangle$ and they assume the same value of the derivative with respect to s at the ends of $\langle 0, l \rangle$ because at the ends of $\langle 0, l \rangle$ the second order derivatives of the polynomials have the same values. Thus being polynomials of the fourth degree in s they assume the same values in $\langle 0, l \rangle$.

We consider the values $D^i p(P_j)$ and $\partial p(Q_j)/\partial \nu$ ($|i| \leqq 2$, $j = 1, 2, 3$) as parameters. Only in the case of boundary nodes the situation is different. We must take into account the boundary conditions. After a simple analysis which we do not carry out here we would find out that some of the boundary parameters are equal to zero and some are multiples of another boundary parameter. Denote by $\mathring{H}_5(\Omega)$ the class of functions defined on $\bar{\Omega}$ which on the particular triangles of the given triangulation are equal to just introduced polynomials. $\mathring{H}_5$ is a finite dimensional subspace of $\mathring{W}_2^{(2)}$. The approximate solution $U(x_1, x_2)$ is defined as that function from $\mathring{H}_5$ which minimizes the functional (21) in the class $\mathring{H}_5$. We can prove as before the existence and uniqueness of $U(x_1, x_2)$.

To find an estimate for the discretization error let us introduce the notation

$$J_0(z) = \int_\Omega \left[\left(\frac{\partial^2 z}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 z}{\partial x_2^2}\right)^2\right] dx.$$

Then it holds (see [13])

$$\begin{aligned} J_0(U - u) &= F_3(U) - F_3(u) = \min_{v \in \mathring{H}_5} F_3(v) - F_3(u) \\ &= \min_{v \in \mathring{H}_5} J_0(v - u) \leqq J_0(u - \tilde{u}) \end{aligned}$$

where $\tilde{u}$ is the function from $\overset{\circ}{H}_5$ such that the values of $D^i\tilde{u}$ for $|i|\leqq 2$ at the vertices of the triangles and the values $\partial\tilde{u}/\partial\nu$ at the midpoints of the sides are the same as those of the solution $u(x_1, x_2)$. To estimate $J_0(u-\tilde{u})$ we use the following

Theorem 3. Let $\varphi(x_1, x_2)$ be twice continuously differentiable on the closed triangle $\overline{T}$ and have bounded derivatives of the sixth order in the interior of T,

$$|D^i\varphi(x_1, x_2)|\leqq M_6, \quad |i|=6.$$

Further let

$$D^i\varphi(P_j)=\frac{\partial\varphi(Q_j)}{\partial\nu}=0 \quad \text{for} \quad |i|\leqq 2, \; j=1,2,3.$$

Then it holds in $\overline{T}$

$$|D^i\varphi(x_1, x_2)|\leqq\frac{C}{(\sin\alpha)^{|i|}}M_6 c^{6-|i|}, \quad |i|\leqq 4,$$

where α is the smallest angle and c the greatest side of the triangle T and C is a constant independent of the triangle T and the function φ [3].

Applying Theorem 3 to the function $\varphi=u-\tilde{u}$ and taking into account that $D^i(u-\tilde{u})=D^iu$ for $|i|=6$ we easily get an estimate

$$[J_0(U-u)]^{\frac{1}{2}}\leqq\frac{C}{\sin^2\vartheta}M_6 h^4,$$

M_6 being now the bound for $|D^iu|$, $|i|=6$. Using twice Friedrichs inequality we obtain the same estimate for $\|u-U\|_{W_2^{(2)}}$. We come to the following statement:

If the solution $u(x_1, x_2)$ has bounded derivatives of the sixth order in Ω, $|D^iu|\leqq M_6$ for $|i|=6$, then it holds

$$\|u-U\|_{W_2^{(2)}(\Omega)}\leqq\frac{C}{\sin^2\vartheta}M_6 h^4 \tag{22}$$

where the constant C does not depend on the triangulation. By Sobolev's lemma [4] *it also holds*

$$\max_{\overline{\Omega}}|u-U|\leqq\frac{C}{\sin^2\vartheta}M_6 h^4. \tag{23}$$

5. Some Lemmas and Proof of Theorem 3

First we introduce Sobolev's lemma in a form which we need. By $\widetilde{W}_2^{(k)}(\Omega)$ we denote the space consisting of functions which together with all generalized derivatives of the order k belong to $L_2(\Omega)$. The norm is given by

$$\|u\|^2_{\widetilde{W}_2^{(k)}(\Omega)}=\|u\|^2_{L_2(\Omega)}+\sum_{|i|=k}\|D^iu\|^2_{L_2(\Omega)}.$$

Sobolev's lemma. Let $0\leqq m<k-1$ and $u(x_1, x_2)\in\widetilde{W}_2^{(k)}(\Omega)$. Then $u(x_1, x_2)\in C^{(m)}(\overline{\Omega})$ and

$$\max_{(x_1,x_2)\in\overline{\Omega},\,|i|\leqq m}|D^iu(x_1, x_2)|\leqq C\|u\|_{\widetilde{W}_2^{(k)}(\Omega)} \tag{24}$$

where the constant C does not depend on $u(x_1, x_2)$.

[3] It is possible to get a numerical value of C; we do not try it as we do not need it.

[4] See next section.

Sobolev's lemma is usually stated for the spaces $W_2^{(k)}$ (actually for more general spaces $W_p^{(k)}$). But for domains which are starlike with respect to a disc, and we shall use Sobolev's lemma for our domain Ω and the triangle T only, the spaces $\widetilde{W}_2^{(k)}$ and $W_2^{(k)}$ consist of the same set of functions and the norms are equivalent (see [15]).

Next we state without proofs three lemmas of the same character as lemmas 2—6. For further applications we may suppose $l \leqq 1$.

Lemma 8. Let $g^{(j)}(0)=\eta_1^{(j)}$, $g^{(j)}(l)=\eta_2^{(j)}$, $j=0,1,2$, and $|g^{(6)}(s)|\leqq K_6$ in $(0,l)$. Then

$$|g(s)| \leqq C_1 \max(|\eta_1^{(j)}|, |\eta_2^{(j)}|) + \frac{1}{2^6 \cdot 6!} K_6 l^6,$$

$$|g'(s)| \leqq C_2 l^{-1} \max(|\eta_1^{(j)}|, |\eta_2^{(j)}|) + \frac{1}{2^4 \cdot 5!} K_6 l^5$$

where the constants C_1, C_2 are absolute constants, i.e. in this case constants independent on $g(s)$ and l [5].

Lemma 9. Let $g^{(j)}(0)=\eta_1^{(j)}$, $g^{(j)}(l)=\eta_2^{(j)}$, $j=0,1$, $g(\frac{1}{2}l)=\eta_3$ and $|g^{(5)}(s)|\leqq K_5$ in $(0,l)$. Then

$$|g(s)| \leqq C_3 \max(|\eta_1^{(j)}|, |\eta_2^{(j)}|, |\eta_3|) + \frac{4\sqrt{5}}{10^3 \cdot 5!} K_5 l^5,$$

$$|g''(s)| \leqq C_4 l^{-2} \max(|\eta_1^{(j)}|, |\eta_2^{(j)}|, |\eta_3|) + \frac{1}{6} K_5 l^3.$$

Lemma 10. Let $g^{(j)}(0)=\eta_1^{(j)}$, $j=0,\ldots,3$, $g^{(j)}(l)=\eta_2^{(j)}$, $j=0,1$ and $|g^{(6)}(s)|\leqq K_6$ in $(0,l)$. Then

$$|g(s)| \leqq C_5 \max\{\max|\eta_1^{(j)}|, \max|\eta_2^{(j)}|\} + \frac{1}{5\cdot 3^8} K_6 l^6.$$

Proof of Theorem 3. We construct again a triangle $P_1' P_2' P_3'$ lying inside T with sides a', b', c' which are parallel to the sides of T and lie in a distance δ. We choose δ so small that $|D^i \varphi(P_j')| \leqq \varepsilon/2$, $|i| \leqq 2$, $j=1,2,3$. We consider the function $g=\varphi|_{P_1' P_2'}$ as a function of a parameter $s \in \langle 0, c'\rangle$. As $\left|\frac{\partial^6 \varphi}{\partial l_1 \ldots \partial l_6}\right| \leqq 2^3 M_6$ for arbitrary directions l_j $(j=1,\ldots,6)$ it holds $|g^{(6)}(s)| \leqq 2^3 M_6$ and obviously

$$\left|\frac{d^j g(0)}{ds^j}\right| \leqq \varepsilon, \quad \left|\frac{d^j g(c')}{ds^j}\right| \leqq \varepsilon, \quad j=0,1,2.$$

By Lemma 8 we have on $P_1' P_2'$

$$|\varphi| \leqq C_1 \varepsilon + \frac{1}{2^3 \cdot 6!} M_6 c'^6, \quad \left|\frac{\partial \varphi}{\partial s}\right| \leqq C_2 \varepsilon \frac{1}{c'} + \frac{1}{2 \cdot 5!} c'^5.$$

Letting $\varepsilon \to 0+$ we see that on the side $P_1 P_2$ it holds

$$|\varphi| \leqq \frac{1}{2^3 \cdot 6!} M_6 c^6,$$

$$\left|\frac{\partial \varphi}{\partial s}\right| \leqq \frac{1}{2 \cdot 5!} M_6 c^5. \tag{25}$$

[5] In the sequel we shall denote by C_j absolute constants, i.e. constants independent on functions considered, on the interval $\langle 0, l\rangle$ and on the triangle T.

The same estimates are true for the other sides. As $\frac{\partial \varphi(Q_j)}{\partial \nu} = 0$ $(j = 1, 2, 3)$ we obtain from the last estimate and Lemma 1 that

$$\left|\frac{\partial \varphi(Q_j)}{\partial x_k}\right| \leqq \frac{1}{2 \cdot 5!} M_6 c^5, \quad j = 1, 2, 3, \quad k = 1, 2. \tag{26}$$

Now by translation of the origin of coordinates we achieve that the vertex P_1 lies in the origin. Let α_1 and α_2 be the angles between the side $P_1 P_2$ and $P_1 P_3$, respectively, and the x_1-axe. Then $|\alpha_2 - \alpha_1| = \alpha$ and we can assume that $\alpha = \alpha_2 - \alpha_1$. Let us introduce new coordinates ξ_1, ξ_2 by the linear transformation

$$\xi_1 = \frac{1}{c \sin \alpha}(x_1 \sin \alpha_2 - x_2 \cos \alpha_2), \qquad \xi_2 = \frac{1}{b \sin \alpha}(- x_1 \sin \alpha_1 + x_2 \cos \alpha_1)$$

so that

$$x_1 = c \cos \alpha_1 \cdot \xi_1 + b \cos \alpha_2 \cdot \xi_2, \qquad x_2 = c \sin \alpha_1 \cdot \xi_1 + b \sin \alpha_2 \cdot \xi_2.$$

The triangle $\overline{T}$ is mapped on the triangle $\overline{T}_1$ with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and the points Q_j are transformed in the mid points of the sides, i.e. in the points $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(0, \frac{1}{2})$. We denote the new vertices by R_j and the new mid points of the sides by S_j. Let us consider the function $\psi(\xi_1, \xi_2) = \frac{\varphi(x_1, x_2)}{c^6 M_6}$. This function is four times continuously differentiable on $\overline{T}_1$ as it belongs to $\widetilde{W}_2^{(6)}(T_1)$ and it holds

$$\begin{aligned} &|D^i \psi(\xi_1, \xi_2)| \leqq 8, \quad |i| = 6, \quad (\xi_1, \xi_2) \in T_1, \\ &D^i \psi(R_j) = 0, \quad |i| \leqq 2, \quad j = 1, 2, 3. \end{aligned} \tag{27}$$

Further, from (25) and (26) we have

$$|\psi| \leqq \frac{1}{2^3 \cdot 6!} \tag{28}$$

on the boundary of T_1 and

$$\left|\frac{\partial \psi(S_j)}{\partial \xi_k}\right| \leqq \frac{\sqrt{2}}{2 \cdot 5!}, \quad j = 1, 2, 3; \quad k = 1, 2. \tag{29}$$

We shall prove that it follows from these estimates

$$|\psi(\xi_1, \xi_2)| \leqq C_6 \quad \text{for} \quad (\xi_1, \xi_2) \in \overline{T}_1. \tag{30}$$

Then from (27), (30) and from Sobolev's lemma ($k = 6$ so that $0 \leqq m < 5$) we get $|D^i \psi(\xi_1, \xi_2)| \leqq C_7$, $|i| \leqq 4$. If we return to the function $\varphi(x_1, x_2)$ we see that it holds

$$|D^i \varphi(x_1, x_2)| \leqq \frac{C_8}{(\sin \alpha)^{|i|}} M_6 c^{6-|i|}, \quad |i| \leqq 4, \quad (x_1, x_2) \in \overline{T}.$$

To prove (30) let us consider a triangle $R_1' R_2' R_3'$ lying inside T_1 with sides parallel to the sides of T in a distance δ. We choose δ so small that $|D^i \psi(R_j')| \leqq \varepsilon$, $|i| \leqq 2$, $\left|\frac{\partial \psi(S_j')}{\partial \xi_k}\right| \leqq \varepsilon + \frac{\sqrt{2}}{2 \cdot 5!}$, $k = 1, 2$ and we consider the function $g = \left.\frac{\partial \psi}{\partial \xi_1}\right|_{R_1' R_2'}$. Obviously $s = \xi_1$, $|g^{(j)}(0)| \leqq \varepsilon$, $|g^{(j)}(l)| \leqq \varepsilon$ for $j = 0, 1$, $\left|g\left(\frac{1}{2} l\right)\right| \leqq \varepsilon + \frac{\sqrt{2}}{2 \cdot 5!}$ by (29) and $|g^{(5)}(\xi_1)| \leqq 8$ by (27). Using Lemma 9 and letting $\varepsilon \to 0+$ we get

$$\left|\frac{\partial^3 \psi(\xi_1, 0)}{\partial \xi_1^3}\right| \leqq C_4 \frac{\sqrt{2}}{2 \cdot 5!} + \frac{8}{6} = C_9.$$

Considering the function $g = \frac{\partial \psi}{\partial \xi_1}\Big|_{R_1' R_3'}$, we prove $\left|\frac{\partial^3 \psi(0, \xi_2)}{\partial \xi_1 \partial \xi_2^2}\right| \leqq C_{10}$. These estimates are simultaneously true in the point $R_1 = (0, 0)$ and can be proved also for the remaining derivatives of the third order so that it holds

$$|D^i \psi(0, 0)| \leqq C_{11}, \quad |i| = 3. \tag{31}$$

Further by similar arguments it can be proved that

$$\left|\frac{\partial \psi(\xi_1, \xi_2)}{\partial \xi_k}\right| \leqq C_{12} \quad \text{on} \quad R_2 R_3, \quad k = 1, 2. \tag{32}$$

Now let R be an arbitrary point of T_1 and R_4 be the point of the side $R_2 R_3$ which lies on the line going through the origin R_1 and the point R. The length of the segment $R_1 R_4$ is not greater than 1. Let us consider the function $g = \psi|_{R_1 R_4}$ as a function of a parameter s. It holds $g(0) = g'(0) = g''(0) = 0$, $|g^{(3)}(0)| \leqq 2\sqrt{2}\, C_{11}$ by (31), $|g(l)| \leqq \frac{1}{2^3 \cdot 6!}$ by (28), $|g'(l)| \leqq \sqrt{2}\, C_{12}$ by (32), $|g^{(6)}(s)| \leqq 2^3 \cdot 8$ by (27) and $l \leqq 1$. Hence (30) follows from Lemma 10.

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