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MULTISCALE HYBRID-MIXED METHOD*

RODOLFO ARAYA†, CHRISTOPHER HARDER‡, DIEGO PAREDES§, AND FRÉDÉRIC VALENTIN§

Abstract. This work presents a priori and a posteriori error analyses of a new multiscale hybrid-mixed method (MHM) for an elliptic model. Specially designed to incorporate multiple scales into the construction of basis functions, this finite element method relaxes the continuity of the primal variable through the action of Lagrange multipliers, while assuring the strong continuity of the normal component of the flux (dual variable). As a result, the dual variable, which stems from a simple postprocessing of the primal variable, preserves local conservation. We prove existence and uniqueness of a solution for the MHM method as well as optimal convergence estimates of any order in the natural norms. Also, we propose a face-residual a posteriori error estimator, and prove that it controls the error of both variables in the natural norms. Several numerical tests assess the theoretical results.

Key words. elliptic equation, mixed method, hybrid method, finite element, multiscale, porous media

AMS subject classifications. 65N12, 65N15, 65N30

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1. Introduction.

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded domain with polygonal boundary $\partial\Omega := \partial\Omega_D \cup \partial\Omega_N$, where $\partial\Omega_D$ and $\partial\Omega_N$ denote Dirichlet and Neumann boundaries, respectively. Consider the elliptic problem to find u such that

$$-\nabla\cdot(\mathcal{K}\nabla u) = f \quad \text{in } \Omega\,, \tag{1.1}$$

$$\mathcal{K}\,\nabla u\cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega_N\,, \quad u = g_D \quad \text{on } \partial\Omega_D\,, \tag{1.2}$$

where g_D and f are given regular functions, $\boldsymbol{n}$ is the outward normal vector of $\partial\Omega$. If $\partial\Omega_D = \emptyset$, we assume $\int_\Omega u = 0$ and $\int_\Omega f = 0$. The diffusion coefficient $\mathcal{K} = \{\mathcal{K}_{ij}\}$ is a symmetric tensor in $[L^\infty(\Omega)]^{d\times d}$ (with its usual meaning) which is assumed to be uniformly elliptic, i.e., there exist positive constants $c_{\min}$ and $c_{\max}$ such that

$$c_{\min}^2\,|\boldsymbol{\xi}|^2 \le \mathcal{K}_{ij}(\boldsymbol{x})\xi_i\xi_j \le c_{\max}^2\,|\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} = \{\xi_i\} \in \mathbb{R}^d,\, \boldsymbol{x} \in \bar{\Omega}\,, \tag{1.3}$$

where $|\cdot|$ is the Euclidean norm. The coefficient $\mathcal{K}$ is free to involve multiscale features as in [14] and [8], for instance.

It is often of interest to approximate both the primal variable $u \in H^1(\Omega)$ and the dual (flux) variable $\boldsymbol{\sigma} := -\mathcal{K}\,\nabla u \in H(\mathrm{div};\Omega)$ (these spaces having their usual

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†CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Concepción, Chile (raraya@ing-mat.udec.cl). This author was partially supported by CONICYT/Chile through FONDECYT project 1110551, Basal project CMM–CI2MA PFB–03, Anillo project ACT1118 (ANANUM), and Red Doctoral REDOC.CTA, MINEDUC project UCO1202 at Universidad de Concepción.

‡Mathematical and Computer Sciences, Metropolitan State University of Denver, Denver, CO 80217-3362 (harderc@msudenver.edu). This author was funded by CNPq/Brazil.

§Applied Mathematics Department, National Laboratory for Scientific Computing-LNCC, 25651-070 Petrópolis-RJ, Brazil (dparedes@lncc.br, valentin@lncc.br). These authors were funded by CONICYT/Chile, CNPq/Brazil, and CAPES/Brazil.

definitions). The standard approach is to substitute $\boldsymbol{\sigma}$ in (1.1)–(1.2) to yield a problem in mixed form. In the case of a heterogeneous coefficient $\mathcal{K}$, it is of particular interest to look for u and $\boldsymbol{\sigma}$ from the perspective of local problems as a way to collect fine-scale contributions in parallel. Such a viewpoint is featured in the works by Chen and Hou [7] and Arbogast [3]. A different approach, named the multiscale hybrid-mixed (MHM) method, was taken in [13]: u was sought as the solution of the elliptic equation in a weaker, broken space which relaxes continuity, allows reconstruction of the dual variable, and localizes computations. It was then shown in [13] that the framework provides a way to recover the aforementioned multiscale methods and to generalize them to higher-order approximations preserving consistency. In the present work, we focus on an analysis of the MHM method presented in [13], providing both a priori and a posteriori estimates. Although featured in a completely different framework, the current work shares common goals and similarities with some recent works such as the multiscale mortar finite element method [4] or the hybrid discontinuous Galerkin method [9], just to cite a few. In fact, such works also adopt a divide-and-conquer approach which ties local computations together through a global problem. On the other hand, an iterative process is involved in [4] which is not presented in our work. Also, those works rely on a dual-hybrid procedure, i.e., they hybridize the mixed version of problem (1.1)–(1.2) instead of the elliptic one. As a result, the Lagrange multipliers allow for relaxing the continuity of the flux and driving local problems as they prescribe Dirichlet boundary conditions at a local level. This feature also differentiates the MHM method from the aforementioned works.

For the sake of completeness, we now summarize the main points in deriving the MHM method. The starting point consists of stating problem (1.1)–(1.2) such that continuity on faces (hereafter this will refer to one-dimensional edges as well) is weakly enforced through the action of Lagrange multipliers. To this end, we introduce a family of regular triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω into elements K, with diameter h_K, and we set $h := \max_{K\in\mathcal{T}_h} h_K$. The collection of all faces F in the triangulation, with diameter h_F, is denoted $\mathcal{E}_h$. This set is decomposed into the set of internal faces $\mathcal{E}_0$, the set of faces on the Dirichlet boundary $\mathcal{E}_D$, and faces on the Neumann boundary $\mathcal{E}_N$. To each $F \in \mathcal{E}_h$, we associate a normal $\boldsymbol{n}$ taking care to ensure this is directed outward on $\partial\Omega$. For each $K \in \mathcal{T}_h$, we further denote by $\boldsymbol{n}^K$ the outward normal on ∂K, and let $\boldsymbol{n}_F^K := \boldsymbol{n}^K|_F$ for each $F \subset \partial K$.

We replace the original strong problem by the following weak formulation: *Find* $(\lambda, u) \in \Lambda \times V$ *such that*

$$(\mathcal{K}\nabla u,\, \nabla v)_{\mathcal{T}_h} + (\lambda\, \boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h} = (f,\, v)_{\mathcal{T}_h} \quad \text{for all } v \in V\,, \tag{1.4}$$

$$(\mu\, \boldsymbol{n}, [\![u]\!])_{\mathcal{E}_h} = (\mu, g_D)_{\mathcal{E}_D} \quad \text{for all } \mu \in \Lambda\,, \tag{1.5}$$

where we primarily work with the spaces $V := H^1(\mathcal{T}_h)$ (or $V := H^1(\mathcal{T}_h) \cap L^2_0(\Omega)$ in the case $\partial\Omega_D = \emptyset$) and

$$\Lambda := \left\{ \mu \in H^{-\frac{1}{2}}(\mathcal{E}_h) \,:\, \mu|_F = 0, \text{ for all } F \in \mathcal{E}_N \right\}.$$

Here, we adopt the notation $(\mu\, \boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h} := \sum_{K\in\mathcal{T}_h} (\mu\, \boldsymbol{n}\cdot \boldsymbol{n}^K, v)_{\partial K}$. We refer the reader to the definitions of the relevant broken spaces and further details on the notation $(\mu\, \boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h}$ in the appendix. However, we mention that $(\cdot,\cdot)_{\mathcal{T}_h}$ denotes a broken L^2 inner product which implicitly indicates summation over the set. Note problem (1.4)–(1.5) is the standard hybrid formulation from which the primal hybrid methods arise [16]. In the pure homogeneous Dirichlet case with $\mathcal{K}$ being the identity, such an approach is shown in [18] to be well posed with $\lambda \in H^{-1/2}(\mathcal{E}_h)$ and $u \in H^1(\Omega)$ being the solution to (1.1)–(1.2); the authors then propose inf-sup stable pairs of finite element subspaces.

We now characterize the solution of (1.4)–(1.5) as a collection of solutions of local problems which are pieced together using solutions to a global problem. To this end, we introduce the decomposition

$$V := V_0 \oplus V_0^{\perp}\,,$$

where V_0 corresponds to

$$V_0 := \{v \in V \,:\, v\,|_K \in \mathbb{P}_0(K),\ \text{for all}\ \ K \in \mathcal{T}_h\}\,,$$

and $\mathbb{P}_0(K)$ stands for the space of piecewise constants. The orthogonal complement in V corresponds to $V_0^{\perp} \equiv L_0^2(\mathcal{T}_h) \cap V$, and thus a function $v \in V$ admits the expansion $v = v_0 + v_0^{\perp}$ in terms of unique $v_0 \in V_0$ and $v_0^{\perp} := v - v_0 \in V_0^{\perp}$.

Next, we observe that by taking $(\mu, v) = (0, v_0^{\perp}\,|_K)$ in (1.4)–(1.5), a portion of the solution to problem (1.4)–(1.5) may be found locally in each element K. Indeed, the component $u_0^{\perp}$ of the exact solution can be expanded as

$$u_0^{\perp} = T\,\lambda + \hat{T}\,f\,, \tag{1.6}$$

where T and $\hat{T}$ are bounded linear operators determined by local problems and with value in $V_0^{\perp}$. To be precise, given $\mu \in \Lambda$, $T\,\mu\,|_K \in H^1(K) \cap L_0^2(K)$ is the unique solution of

$$(\mathcal{K}\nabla\, T\,\mu,\, \nabla w)_K = -\big(\mu\,\boldsymbol{n}\cdot\boldsymbol{n}^K,\, w\big)_{\partial K} \quad \text{for all } w \in H^1(K) \cap L_0^2(K)\,, \tag{1.7}$$

and given $q \in L^2(\Omega)$, $\hat{T}\,q\,|_K \in H^1(K) \cap L_0^2(K)$ is the unique solution of

$$(\mathcal{K}\nabla\, \hat{T}\,q,\, \nabla w)_K = (q,\, w)_K \quad \text{for all}\ \ w \in H^1(K) \cap L_0^2(K)\,. \tag{1.8}$$

Further properties of T and $\hat{T}$ are presented in Lemmas 8.1 and 8.2 in the appendix. Note that decomposition (1.6) provides us a way to eliminate the portion of the solution $u_0^{\perp}$ in terms of λ and f. We complete the computation of the exact solution u by selecting $(\mu, v) = (\mu, v_0)$ in (1.4)–(1.5) and solving the resulting global problem: *Find* $(\lambda, u_0) \in \Lambda \times V_0$ *such that*

$$(\lambda\,\boldsymbol{n}, [\![v_0]\!])_{\mathcal{E}_h} = (f, v_0)_{\mathcal{T}_h} \quad \text{for all } v_0 \in V_0\,, \tag{1.9}$$

$$(\mu\,\boldsymbol{n}, [\![u_0 + T\,\lambda]\!])_{\mathcal{E}_h} = (\mu, g_D)_{\mathcal{E}_D} - (\mu\,\boldsymbol{n}, [\![\hat{T}\,f]\!])_{\mathcal{E}_h} \quad \text{for all}\ \ \mu \in \Lambda\,. \tag{1.10}$$

It is worth mentioning that the dual variable

$$\boldsymbol{\sigma} = -\mathcal{K}\,\nabla(T\,\lambda + \hat{T}\,f)$$

belongs to the space $H(\mathrm{div};\Omega)$ since $\boldsymbol{\sigma}\cdot\boldsymbol{n}\,|_F$ is continuous across $F \in \mathcal{E}_h$ and $f \in L^2(\Omega)$ by assumption [6, page 95].

Interestingly, global problem (1.9)–(1.10) may be interpreted as a modified version of the mixed form of the elliptic problem (1.1)–(1.2). Indeed, owing to the identities (see [13]),

$$\begin{gathered}(\mu\,\boldsymbol{n}, [\![T\,\lambda]\!])_{\mathcal{E}_h} = -(\mathcal{K}\,\nabla T\,\lambda, \nabla T\,\mu)_{\mathcal{T}_h}, \quad (\lambda\,\boldsymbol{n}, [\![v_0]\!])_{\mathcal{E}_h} = -(\nabla\cdot(\mathcal{K}\,\nabla T\,\lambda), v_0)_{\mathcal{T}_h}\,, \\ (\mu\,\boldsymbol{n}, [\![\hat{T}\,f]\!])_{\mathcal{E}_h} = -(f, T\,\mu)_{\mathcal{T}_h}\,,\end{gathered} \tag{1.11}$$

we can write (1.9)–(1.10) in the following form: *Find* $(\lambda, u_0) \in \Lambda \times V_0$ *such that*

$$(\nabla\cdot(\mathcal{K}\,\nabla T\,\lambda), v_0)_{\mathcal{T}_h} = -(f, v_0)_{\mathcal{T}_h}\,, \tag{1.12}$$

$$(\mathcal{K}\,\nabla T\,\lambda, \nabla T\,\mu)_{\mathcal{T}_h} + (\nabla\cdot(\mathcal{K}\,\nabla T\,\mu), u_0)_{\mathcal{T}_h} = -(\mu, g_D)_{\mathcal{E}_D} - (f, T\,\mu)_{\mathcal{T}_h} \tag{1.13}$$

for all $(\mu, v_0) \in \Lambda \times V_0$.

In this work, we establish weak formulation (1.9)–(1.10) and its discrete version, the MHM method, are well posed (Theorem 3.2). We then show a best approximation result highlighting that the error only depends on the quality of the approximation on faces (Lemma 3.3), which we then use to prove that the MHM method provides optimal numerical approximations to the primal and dual variables in natural norms (Theorem 4.1). Furthermore, an a posteriori error estimator (see (5.1)–(5.3)) is precisely established in terms of the jump of the primal variable on the faces. Interestingly, such a face-based residual estimator is shown to control the natural norms of the primal and dual variables inside the whole computational domain (Theorem 5.2), revealing the effectivity and reliability of the estimator.

The paper is outlined as follows: The MHM finite element method is reviewed in section 2. Section 3 is dedicated to well-posedness of the method, and section 4 proposes a priori error estimates. The a posteriori error estimator is developed in section 5. Numerical results are then presented in section 6, followed by conclusions in section 7. Some auxiliary results are provided in the appendix.

2. The multiscale hybrid-mixed method. To present a finite element approximation to global problem (1.9)–(1.10), we shall only require a finite element space approaching Λ since the space V_0 is already discrete. At this point, we use a general approach of selecting a conforming finite subspace Λ_h of Λ, i.e.,

$$\Lambda_h \subset \Lambda \cap L^2(\mathcal{E}_h)\,, \tag{2.1}$$

making the mild assumption $\Lambda_0 \subseteq \Lambda_h$, where the space Λ_0 stands for

$$\Lambda_0 := \{\mu \in \Lambda \,:\, \mu\,|_F \in \mathbb{P}_0(F) \;\text{ for all } F \in \mathcal{E}_h\}\;.$$

Here $\mathbb{P}_0(F)$ denotes the space of constant polynomials over faces $F \in \mathcal{E}_h$. This assumption is key to establishing well-posedness. Observe that functions in Λ_h may be discontinuous at the vertices (or at the edges in the three-dimensional case), but are single valued along faces.

We now define the MHM method, which is built by using the subspace Λ_h in place of Λ. Given $\mu_h \in \Lambda_h$, find $T\mu_h\,|_K \in H^1(K) \cap L^2_0(K)$ such that it holds

$$(\mathcal{K}\nabla\,T\mu_h,\,\nabla w)_K = -\left(\mu_h\,\boldsymbol{n}\cdot\boldsymbol{n}^K,\,w\right)_{\partial K} \quad \text{for all } w \in H^1(K) \cap L^2_0(K)\,. \tag{2.2}$$

Then, using Λ_h in place of Λ in global problem (1.10) yields the following MHM method: *Find* $\left(\lambda_h, u_0^h\right) \in \Lambda_h \times V_0$ *such that*

$$(\lambda_h\,\boldsymbol{n}, [\![v_0]\!])_{\mathcal{E}_h} = (f, v_0)_{\mathcal{T}_h} \quad \text{for all } v_0 \in V_0\,, \tag{2.3}$$

$$(\mu_h\,\boldsymbol{n}, [\![u_0^h + T\lambda_h]\!])_{\mathcal{E}_h} = (\mu_h, g_D)_{\mathcal{E}_D} - (\mu_h\,\boldsymbol{n}, [\![\hat{T}f]\!])_{\mathcal{E}_h} \quad \text{for all } \mu_h \in \Lambda_h\,. \tag{2.4}$$

It is important to note that by assumption on the space Λ_h in (2.1), the jump terms in method (2.3)–(2.4) have a precise mathematical meaning.

Equivalently, we may express the MHM method in a mixed form through the use of identities (1.11): *Find* $\left(\lambda_h, u_0^h\right) \in \Lambda_h \times V_0$ *such that, for all* $(\mu_h, v_0) \in \Lambda_h \times V_0$*, it holds*

$$(\nabla\cdot(\mathcal{K}\,\nabla T\lambda_h), v_0)_{\mathcal{T}_h} = -(f, v_0)_{\mathcal{T}_h}\,, \tag{2.5}$$

$$(\mathcal{K}\,\nabla T\lambda_h, \nabla T\mu_h)_{\mathcal{T}_h} + \left(\nabla\cdot(\mathcal{K}\,\nabla T\mu_h), u_0^h\right)_{\mathcal{T}_h} = -\left(\mu_h, g_D\right)_{\mathcal{E}_D} - (f, T\mu_h)_{\mathcal{T}_h}\,. \tag{2.6}$$

Owing to the fact μ_h is an element of the finite element space Λ_h, $T\mu_h$ is seen as the linear combination of solutions of the problem (2.2) applied to each one of the basis functions spanning Λ_h with coefficients equal to the degrees of freedom of μ_h (see [13] for further details).

We close this section with several comments. Although we find that the mixed formulation of the Laplace problem is a consequence of the approach, we recall that the approach is built on an approximation of u. Therefore, we may interpret the approach as defining finite elements (i.e., basis functions and degrees of freedom) for which $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ is well-approximated. Also, an easy computation shows that method (2.3)–(2.4) (or (2.5)–(2.6)) is locally mass conservative, i.e.,

$$\int_K \nabla\cdot(\mathcal{K}\,\nabla(T\,\lambda_h + \hat{T}\,f)) = \int_K f \quad\Longleftrightarrow\quad \int_{\partial K} \lambda_h\,\boldsymbol{n}\cdot\boldsymbol{n}^K = \int_K f\,,$$

so that such a feature may be interpreted as the compatibility condition that is fulfilled by the local problems (1.8) and (2.2).

Also, it is worth noting that since u_0^h lies in the same space as u_0, the accuracy of u_0^h depends only on the best approximation of λ in Λ_h. In consequence, optimal convergence for $u_0^h + T\,\lambda_h + \hat{T}\,f$ and $\mathcal{K}\,\nabla(T\,\lambda_h + \hat{T}\,f)$ in the natural norms relies only on the capacity of λ_h to approximate λ. These statements are proved in the forthcoming sections 3–5, and numerically assessed in section 6.

The analysis in this work assumes that $T\,\lambda_h$ and $\hat{T}\,f$ are exactly known (see [13] for examples). In general, their numerical approximation is needed. This leads to a two-level methodology, where the functions $T\,\lambda_h$ and $\hat{T}\,f$ in (2.3)–(2.4) (equivalently, (2.5)–(2.6)) are replaced by their locally approximated discrete counterparts $T_h\,\lambda_h$ and $\hat{T}_h\,f$, where T_h and $\hat{T}_h$ approach T and $\hat{T}$, respectively, when the characteristic length of the submesh tends to zero (see [2] for an example of a two-level strategy with such a feature). Such computations may be performed either solving the elliptic problems (1.8) and (2.2) or, if local conformity in $H(\mathrm{div};K)$ is demanded, solving their mixed counterpart obtained from a recursive hybridization procedure. It is important to note that in either case, method (2.3)–(2.4) (or (2.5)–(2.6)) consists of the same number of degrees of freedom, with the local approximation appearing as a preprocessing step which is easily parallelized.

Finally, if we suppose f is regular (belonging to $H^1(\Omega)$, for instance), then the MHM method (2.3)–(2.4) maybe simplified by dropping the source term

$$(\mu_h\,\boldsymbol{n}, [\![\hat{T}f]\!])_{\mathcal{E}_h},$$

and using $u_0^h + T\,\lambda_h$ to approximate u. In fact, we prove that the induced consistency error stays controlled in section 4. As a result, we can completely disregard the local problem (1.8) in such cases.

3. Well-posedness and best approximation. In this section we show method (2.3)–(2.4) is well posed and provides a best approximation. First, we revisit an abstract result for mixed problems. Throughout the following sections, we will use C to denote an arbitrary positive constant that is independent of h but can change for each occurrence.

3.1. Abstract results. We consider the well-posedness of the following problem: *Find* $(u,p) \in W \times Q$ *such that*

$$B(u,p;v,q) = F(v,q) \quad \text{for all } (v,q) \in W \times Q, \tag{3.1}$$

where W and Q are reflexive Banach spaces equipped with the norms $\|.\|_W$ and $\|.\|_Q$, respectively. We assume here that the bounded bilinear and linear forms $B : (W \times Q) \times (W \times Q) \to \mathbb{R}$ and $F : W \times Q \to \mathbb{R}$ have the specific forms

$$\begin{aligned} B(u,p;v,q) &:= a(u,v) + b(v,p) + b(u,q)\,, \\ F(v,q) &:= f(v) + g(q)\,, \end{aligned}$$

where $a : W \times W \to \mathbb{R}$, $b : W \times Q \to \mathbb{R}$, $f : W \to \mathbb{R}$, and $g : Q \to \mathbb{R}$ are bounded bilinear and linear forms by assumption.

Defining a norm on $W \times Q$ by

$$\|(w,q)\|_{W\times Q} := \|w\|_W + \|q\|_Q\,,$$

problem (3.1) is well posed if and only if (i) the following surjectivity condition holds (with respect to the operator associated with $B(\cdot\,;\cdot)$): there exists a positive constant β such that

$$\inf_{(u,p)\in W\times Q} \sup_{(w,q)\in W\times Q} \frac{B(u,p;w,q)}{\|(u,p)\|_{W\times Q}\|(w,q)\|_{W\times Q}} \geq \beta\,, \tag{3.2}$$

and (ii) the following injectivity condition holds:

$$B(w,q;u,p) = 0 \text{ for all } (w,q) \in W \times Q \implies W \times Q \ni (u,p) = \mathbf{0}\,. \tag{3.3}$$

Above and hereafter we lighten notation and understand the supremum to be taken over sets excluding the zero function, even though this is not specifically indicated. It is well known (see [11, p. 101], for instance) that conditions (3.2) and (3.3) are satisfied given necessary and sufficient conditions on forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$. The sufficiency is revisited in the following lemma, in which we use an alternative proof to derive a more convenient constant β (in terms of its dependence on $\mathcal{K}$) than presented in [11] when W and Q are Hilbert spaces. We recall the norm of the operator $a(\cdot,\cdot)$ stands for

$$\|a\| := \sup_{v,w\in W} \frac{a(v,w)}{\|v\|_W\|w\|_W} < \infty\,. \tag{3.4}$$

LEMMA 3.1. *Let* $\mathcal{N} := \{w \in W \,:\, b(w,q) = 0, \text{ for all } q \in Q\}$, *and assume*

$$a(w,v) = 0 \text{ for all } w \in \mathcal{N} \implies \mathcal{N} \ni v = 0. \tag{3.5}$$

Moreover, suppose there exist positive constants c_a *and* c_b *such that*

$$c_a\|w\|_W \leq \sup_{v\in\mathcal{N}} \frac{a(w,v)}{\|v\|_W} \quad \text{for all } w \in \mathcal{N}, \tag{3.6}$$

$$c_b\|q\|_Q \leq \sup_{w\in W} \frac{b(w,q)}{\|w\|_W} \quad \text{for all } q \in Q. \tag{3.7}$$

Then, given $\beta = (2\max\{\frac{1}{c_b}+\frac{1}{c_a}(1+\frac{\|a\|}{c_b}), \frac{1}{c_b}(1+\|a\|(\frac{1}{c_b}+\frac{1}{c_a}(1+\frac{\|a\|}{c_b})))\})^{-1}$*, the bounded bilinear form* $B(\cdot;\cdot)$ *satisfies conditions* (3.2)–(3.3), *and problem* (3.1) *is well posed. If* W *and* Q *are Hilbert spaces then it holds*

$$\beta = \left(2\max\left\{\left(1+\frac{\|a\|}{c_a}\right), \frac{1}{c_b}\left(1+\frac{1}{c_a}\left(1+\frac{\|a\|}{c_a}\right)\right)\right\}\right)^{-1}.$$

Proof. In the case W and Q are reflexive Banach spaces, the proof may be found in [11, p. 101]. Next, without loss of generality, we work with W and Q as Hilbert spaces with inner products $(\cdot,\cdot)_W$ and $(\cdot,\cdot)_Q$, respectively, and we follow closely the proof in [20]. Let $A : W \to W$ and $B : W \to Q$ be the operators defined by

$$\begin{aligned} (Aw, v)_W &= a(w, v) \qquad \text{for all } w, v \in W,\\ (Bw, q)_Q &= b(w, q) \qquad \text{for all } w \in W,\ q \in Q. \end{aligned}$$

We see immediately that $\ker B = \mathcal{N}$. Letting $\Pi : W \to \mathcal{N}$ be the orthogonal projection and noting that the solution $u \in W$ to (3.1) may be (uniquely) decomposed as $u = u_{\mathcal{N}} + u^\star$, where $u_{\mathcal{N}} \in \mathcal{N}$ and $u^\star \in \mathcal{N}^\perp$, we write (3.1) as three separate statements [20]:

$$\begin{aligned} Bu^\star &= g,\\ (\Pi A)u_{\mathcal{N}} &= \Pi(f - Au^\star),\\ B^T p &= (I-\Pi)(I-P)(f - Au^\star). \end{aligned}$$

Here, $P = A\Pi(\Pi A)^{-1}\Pi$ and we note that each of these three statements above is a well-posed problem since the assumptions ensure that $\Pi A : \mathcal{N} \to \mathcal{N}$ and $B : \mathcal{N}^\perp \to Q$ are isomorphisms and $(I-\Pi)(I-P)(f - Au^\star)$ is clearly in the range of B^T. Furthermore, the assumptions guarantee that

$$\begin{aligned} \|B^{-1}\|_{\mathcal{L}(Q,\mathcal{N}^\perp)} = \|(B^T)^{-1}\|_{\mathcal{L}(\mathcal{N}^\perp,Q)} &\le \frac{1}{c_b},\\ \|\Pi A\|_{\mathcal{L}(\mathcal{N},\mathcal{N})} &\le \frac{1}{c_a}, \end{aligned}$$

where $\|.\|_{\mathcal{L}(X,Y)}$ represents the usual norm in the space of linear mappings acting on elements of X with values in Y. Also, since $P = P^2$, it follows that $\|I - P\| \le \frac{\|a\|}{c_a}$ (cf. [20]). Therefore, we have

$$\begin{aligned} \|u\|_W + \|p\|_Q &\le \|u^\star\|_W + \|u_{\mathcal{N}}\|_W + \|p\|_Q\\ &\le \|u^\star\|_W + \|f\|_W + \|Au^\star\|_W + \frac{\|a\|}{c_a}\Big[\|f\|_W + \|Au^\star\|_W\Big]\\ &\le \frac{1}{c_b}\left(1+\frac{1}{c_a}\left(1+\frac{\|a\|}{c_a}\right)\right)\|g\|_Q + \left(1+\frac{\|a\|}{c_a}\right)\|f\|_W. \end{aligned}$$

The result then follows by observing that the best possible constant β in (3.2) has the property

$$\beta \ge \left(2\max\left\{\left(1+\frac{\|a\|}{c_a}\right), \frac{1}{c_b}\left(1+\frac{1}{c_a}\left(1+\frac{\|a\|}{c_a}\right)\right)\right\}\right)^{-1}. \qquad \square$$

3.2. Well-posedness of the MHM method. First, we express (1.9)–(1.10) such that it fits in the abstract form (3.1). To this end, we define the bilinear forms $a : \Lambda \times \Lambda \to \mathbb{R}$ and $b : \Lambda \times V \to \mathbb{R}$ by

$$a(\lambda, \mu) := (\mu\, \boldsymbol{n}, [\![T\,\lambda]\!])_{\mathcal{E}_h}\,, \quad b(\lambda, v) := (\lambda\, \boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h}\,,$$

and, thereby, problem (1.9)–(1.10) reads as follows: *Find* $(\lambda, u_0) \in \Lambda \times V_0$ *such that*

$$B(\lambda, u_0; \mu, v_0) = F(\mu, v_0) \quad \text{for all } (\mu, v_0) \in \Lambda \times V_0\,, \tag{3.8}$$

where

$$\begin{aligned} B(\lambda, u_0; \mu, v_0) &:= a(\lambda, \mu) + b(\mu, u_0) + b(\lambda, v_0)\,, \\ F(\mu, v_0) &:= (f, v_0)_{\mathcal{T}_h} - (\mu\, \boldsymbol{n}, [\![\hat{T}\, f]\!])_{\mathcal{E}_h} + (\mu, g_D)_{\mathcal{E}_D}\,. \end{aligned}$$

The MHM method (2.3)–(2.4) is written similarly: *Find* $(\lambda_h, u_0^h) \in \Lambda_h \times V_0$ *such that*

$$B(\lambda_h, u_0^h; \mu_h, v_0) = F(\mu_h, v_0) \quad \text{for all } (\mu_h, v_0) \in \Lambda_h \times V_0\,. \tag{3.9}$$

In order to introduce a norm on $\Lambda \times V_0$, we first define a norm on $H(\mathrm{div};\Omega)$ and a norm on V, respectively, as follows:

$$\|\boldsymbol{\sigma}\|^2_{\mathrm{div}} := \sum_{K\in\mathcal{T}_h} \left(\|\boldsymbol{\sigma}\|^2_{0,K} + d^2_\Omega\, \|\nabla\cdot \boldsymbol{\sigma}\|^2_{0,K}\right), \tag{3.10}$$

$$\|v\|^2_V := \sum_{K\in\mathcal{T}_h} \left(d_\Omega^{-2}\, \|v\|^2_{0,K} + \|\nabla v\|^2_{0,K}\right), \tag{3.11}$$

where d_Ω is the diameter of Ω. Next, we define the quotient norm on Λ,

$$\|\mu\|_\Lambda := \inf_{\substack{\boldsymbol{\sigma}\in H(\mathrm{div};\Omega) \\ \boldsymbol{\sigma}\cdot\boldsymbol{n}=\mu \text{ on } \partial K,\, K\in\mathcal{T}_h}} \|\boldsymbol{\sigma}\|_{\mathrm{div}}. \tag{3.12}$$

Interestingly, from the definition of norms (3.11) and (3.12), the following equivalence holds (see Lemma 8.3 in the appendix): Given $\mu \in \Lambda$,

$$\frac{\sqrt{2}}{2}\, \|\mu\|_\Lambda \le \sup_{v\in V} \frac{b(\mu, v)}{\|v\|_V} \le \|\mu\|_\Lambda\,, \tag{3.13}$$

which has the immediate consequence that $b(\cdot,\cdot)$ is a bounded bilinear form, as is $a(\cdot,\cdot)$ since by definition $a(\lambda, \mu) = b(\mu, T\,\lambda)$. Finally, using (3.11) and (3.12), we equip the space $\Lambda \times V_0$ with the following norm of $\Lambda \times V$,

$$\|(\mu, v_0)\|_{\Lambda\times V} := \|\mu\|_\Lambda + \|v_0\|_V\,. \tag{3.14}$$

In the sequel, we will make use of the following tensor norm on $\mathcal{K}$,

$$\|\mathcal{K}\|_\infty := \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} \max_{|\boldsymbol{\xi}|=1} (\mathcal{K}(\boldsymbol{x})\, \boldsymbol{\xi}, \boldsymbol{\xi})^{1/2}, \quad \boldsymbol{\xi} \in \mathbb{R}^d\,.$$

By the properties (1.3), the tensor $\mathcal{K}$ is invertible at each point $\boldsymbol{x}$ (its inverse tensor denoted by $\mathcal{K}^{-1}$) and it holds

$$c_{\min} \le \|\mathcal{K}\|_\infty \le c_{\max}\,. \tag{3.15}$$

We shall also make extensive use of the following value,

$$\kappa := \frac{c_{\max}}{c_{\min}}\,, \tag{3.16}$$

and note that if the entries of $\mathcal{K}$ are constant functions, then κ is simply the condition number of $\mathcal{K}$. We are ready to present the well-posedness result.

THEOREM 3.2. *Suppose Λ_l is an arbitrary subspace of Λ. Then, given (λ, u_0), $(\mu, v_0) \in \Lambda_l \times V_0$, it holds*

$$B(\lambda, u_0; \mu, v_0) \leq \bar{C}\, \|(\lambda, u_0)\|_{\Lambda\times V}\, \|(\mu, v_0)\|_{\Lambda\times V}\,, \tag{3.17}$$

where $\bar{C} = \max\{2\frac{\kappa}{c_{\min}}, 1\}$. Moreover, under the assumption $\Lambda_0 \subseteq \Lambda_l$, it follows that

$$\sup_{(\mu,v_0)\in\Lambda_l\times V_0} \frac{B(\lambda, u_0; \mu, v_0)}{\|(\mu, v_0)\|_{\Lambda\times V}} \geq \beta\, \|(\lambda, u_0)\|_{\Lambda\times V} \quad \textit{for all } (\lambda, u_0) \in \Lambda_l \times V_0\,, \tag{3.18}$$

where $\beta = (2\max\{(1 + 2\,\kappa^2), C\,(1 + c_{\max}(1 + 2\,\kappa^2))\})^{-1}$, C is a positive constant independent of h and $\mathcal{K}$, and

$$B(\lambda, u_0;\, \mu, v_0) = 0 \quad \textit{for all } (\lambda, u_0) \in \Lambda_l \times V_0 \implies \Lambda_l \times V_0 \ni (\mu, v_0) = \mathbf{0}\,. \tag{3.19}$$

Hence, problems (3.8) *and* (3.9) *are well posed.*

Proof. First, we prove (3.17). Since by definition $a(\lambda, \mu) = b(\mu, T\lambda)$, it follows by the equivalence result (3.13), Lemmas 8.3 and 8.1 in the appendix, and the definition of norm (3.14) that

$$\begin{aligned}
B(\lambda, u_0; \mu, v_0) &= b(\mu, T\,\lambda + u_0) + b(\lambda, v_0)\\
&\leq \sup_{w\in V} \frac{b(\mu, w)}{\|w\|_V} \|T\,\lambda + u_0\|_V + \sup_{w\in V} \frac{b(\lambda, w)}{\|w\|_V} \|v_0\|_V\\
&\leq \|\mu\|_\Lambda (\|T\,\lambda\|_V + \|u_0\|_V) + \|\lambda\|_\Lambda \|v_0\|_V\\
&\leq 2\frac{\kappa}{c_{\min}} \|\mu\|_\Lambda \|\lambda\|_\Lambda + \|\mu\|_\Lambda \|u_0\|_V + \|\lambda\|_\Lambda \|v_0\|_V\,,
\end{aligned}$$

and result (3.17) follows immediately. Observe that in the process of proving (3.17), we have also established $a(\lambda, \mu) \leq 2\frac{\kappa}{c_{\min}} \|\lambda\|_\Lambda \|\mu\|_\Lambda$, so that we conclude from (3.4),

$$\|a\| \leq 2\frac{\kappa}{c_{\min}}. \tag{3.20}$$

To prove (3.18) and (3.19), we establish the conditions of Lemma 3.1. Define $\mathcal{N} := \{\mu \in \Lambda_l \,:\, b(\mu, v_0) = 0 \text{ for all } v_0 \in V_0\}$. It follows by the identity (1.11) that for arbitrary $\mu \in \mathcal{N}$, $\nabla\cdot(\mathcal{K}\,\nabla\, T\mu) = 0$. Using (1.3), we get

$$\begin{aligned}
-a(\mu, \mu) &= (\mathcal{K}^{-1}\,\mathcal{K}\,\nabla\, T\mu, \mathcal{K}\,\nabla\, T\mu)_{\mathcal{T}_h}\\
&\geq \sum_{K\in\mathcal{T}_h} \frac{1}{c_{\max}}\, \|\mathcal{K}\,\nabla\, T\mu\|_{0,K}^2\\
&\geq \frac{1}{c_{\max}}\, \|\mu\|_\Lambda^2\,, \qquad (3.21)
\end{aligned}$$

where we also used the definition (3.12) of norm $\|.\|_\Lambda$. Therefore, the operator $-a(\cdot,\cdot)$ is coercive on $\mathcal{N}$, which verifies (3.5) and (3.6) of Lemma 3.1.

Next, we choose arbitrary $v_0 \in V_0$, and let $\boldsymbol{\sigma}^*$ be the function in the lowest-order Raviart–Thomas finite element space [17] such that $(\nabla\cdot \boldsymbol{\sigma}^*, v_0)_{\mathcal{T}_h} \geq c_b \, \|\boldsymbol{\sigma}^*\|_{\text{div}} \|v_0\|_V$, where c_b is a positive constant independent of the functions $\boldsymbol{\sigma}^*$ and v_0. Defining $\mu^* := \boldsymbol{\sigma}^* \cdot \boldsymbol{n}$, it then follows by identities (1.11) and the definition of the norm (3.12) that $b(\mu^*, v_0) \geq c_b \, \|\mu^*\|_\Lambda \|v_0\|_V$. Having verified all conditions of Lemma 3.1 with $c_a = \frac{1}{c_{\max}}$ and c_b, noting (3.20), the inf-sup constant β is

$$\beta = \left(2 \max \left\{ (1 + 2\,\kappa^2), \frac{1}{c_b} (1 + c_{\max}(1 + 2\,\kappa^2)) \right\} \right)^{-1},$$

where we note that c_b is independent of h and $\mathcal{K}$. ∎

3.3. Best approximation estimates. Standard theory implies the MHM method (3.9) is strongly consistent and provides a best approximation result, as pointed out in the next lemma. Interestingly, the result shows that the quality of approximation depends only on the space Λ_h.

LEMMA 3.3. *Let* $(\lambda, u_0) \in \Lambda \times V_0$ *and* $(\lambda_h, u_0^h) \in \Lambda_h \times V_0$ *be the solutions of* (3.8) *and* (3.9)*, respectively. Under the assumptions of Theorem* 3.2*, the following results hold:*

$$B(\lambda - \lambda_h, u_0 - u_0^h; \mu_h, v_0) = 0 \quad \textit{for all } (\mu_h, v_0) \in \Lambda_h \times V_0\,, \tag{3.22}$$

and

$$\|(\lambda - \lambda_h, u_0 - u_0^h)\|_{\Lambda\times V} \leq \frac{\bar{C}}{\beta} \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda\,, \tag{3.23}$$

with $\bar{C}$ *and* β *being the continuity and inf-sup constants from Theorem* 3.2*, respectively.*

Proof. The first result follows directly from the definition of problems (3.8) and (3.9). As for (3.23), Cèa's lemma [20] implies

$$\|(\lambda - \lambda_h, u_0 - u_0^h)\|_{\Lambda\times V} \leq \frac{\bar{C}}{\beta} \inf_{(\mu_h, v_0) \in \Lambda_h \times V_0} \|(\lambda - \mu_h, u_0 - v_0)\|_{\Lambda\times V}\,,$$

so that the result follows by observing u_0 is best approximated in V_0 by taking $v_0 = u_0$. ∎

As a result of the consistency of the MHM method, its solution fulfills the local divergence constraint exactly, as shown in the next result. Hereafter, we shall make use of the following characterizations of the exact and numerical solutions u and u_h,

$$u = u_0 + T\,\lambda + \hat{T}\,f \quad \text{and} \quad u_h = u_0^h + T\,\lambda_h + \hat{T}\,f\,,$$

where $(\lambda, u_0) \in \Lambda \times V_0$ and $(\lambda_h, u_0^h) \in \Lambda_h \times V_0$ solve (3.8) and (3.9), respectively.

COROLLARY 3.4. *Let* $(\lambda, u_0) \in \Lambda \times V_0$ *and* $(\lambda_h, u_0^h) \in \Lambda_h \times V_0$ *be the solutions of* (3.8) *and* (3.9)*, respectively. The following result holds:*

$$\nabla\cdot(\mathcal{K}\,\nabla\,u_h) = \nabla\cdot(\mathcal{K}\,\nabla\,u) \quad \textit{in } \Omega\,. \tag{3.24}$$

Proof. Given $v_0 \in V_0$ and $K \in \mathcal{T}_h$, we select $(\mu, v_0) = (0, v_0\,|_K)$ in (3.8) and (3.9). Then, from identities (1.11) the continuous and the discrete solutions u and

u_h, respectively, satisfy

$$\begin{aligned}\int_K \nabla\cdot(\mathcal{K}\,\nabla\, u_h)\, v_0 &= -\int_{\partial K} \lambda_h\, \boldsymbol{n}\cdot\boldsymbol{n}^K\, v_0\\ &= -\int_K f\; v_0\\ &= \int_K \nabla\cdot(\mathcal{K}\,\nabla\, u)\, v_0\end{aligned}$$

and the result follows by observing that $\nabla\cdot(\mathcal{K}\,\nabla(u-u_h))\,|_K \in \mathbb{R}$ for all $K \in \mathcal{T}_h$. ◻

Remark 1. If the contribution $\hat{T}\,f$, which is present in u_h, is not exactly available (and computed from a two-level method) then result (3.24) must be weakened to

$$\Pi_K \nabla\cdot(\mathcal{K}\,\nabla\, u_h) \;=\; \Pi_K \nabla\cdot(\mathcal{K}\,\nabla\, u) \quad \text{for all } K \in \mathcal{T}_h\,, \tag{3.25}$$

Π_K being the local L^2 projection onto the constant space, i.e, $\Pi_K\, v := \frac{1}{|K|}\int_K v$. ◻

From Lemma 3.3, we next provide estimates in natural norms. Some results make use of the assumption that problem (1.1)–(1.2) has smoothing properties (see [11, Definition 3.14] for details).

LEMMA 3.5. *Let* $(\lambda, u_0) \in \Lambda \times V_0$ *and* $(\lambda_h, u_0^h) \in \Lambda_h \times V_0$ *be the solutions of* (3.8) *and* (3.9)*, respectively. Then, it holds*

$$\|u_0 - u_0^h\|_{0,\Omega} \le \frac{\bar{C}\, d_\Omega}{\beta} \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda\,, \tag{3.26}$$

$$\|\mathcal{K}\nabla(u - u_h)\|_{\mathrm{div}} \le \sqrt{2}\,\kappa\, \frac{\bar{C}}{\beta} \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda\,, \tag{3.27}$$

$$\|u - u_h\|_{0,\Omega} \le \left(1 + \frac{2\,\kappa}{c_{\min}}\right) \frac{\bar{C}\, d_\Omega}{\beta} \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda\,, \tag{3.28}$$

where $\bar{C}$ *and* β *are the continuity and the inf-sup constants from Theorem* 3.2*, respectively. Furthermore, if problem* (1.1)–(1.2) *has smoothing properties, there exist positive constants* C*, independent of* h *and* $\mathcal{K}$*, such that*

$$\|u - u_h\|_{0,\Omega} \le C\, \frac{\bar{C}^2}{\beta\, c_{\min}}\, h \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda\,, \tag{3.29}$$

$$\|u_0 - u_0^h\|_{0,\Omega} \le C\, \frac{\bar{C}\,(\bar{C} + \kappa)}{\beta\, c_{\min}}\, h \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda\,. \tag{3.30}$$

Proof. Result (3.26) follows directly from the best approximation result of Lemma 3.3. Next, note that

$$\begin{aligned} u - u_h &= (u_0 + T\,\lambda + \hat{T}\,f) - (u_0^h + T\,\lambda_h + \hat{T}\,f)\\ &= (u_0 - u_0^h) + T\,(\lambda - \lambda_h)\,. \end{aligned} \tag{3.31}$$

Therefore, Lemma 8.1 implies $\|\mathcal{K}\,\nabla(u-u_h)\|_{\mathrm{div}} \le \sqrt{2}\,\kappa\,\|\lambda - \lambda_h\|_\Lambda$, so that result (3.27) follows from Lemma 3.3. From (3.31) and Lemma 8.1, we observe

$$\begin{aligned}\|u - u_h\|_\Omega &\le \|u_0 - u_0^h\|_{0,\Omega} + d_\Omega\,\|T(\lambda - \lambda_h)\|_V\\ &\le \|u_0 - u_0^h\|_{0,\Omega} + \frac{2\, d_\Omega\,\kappa}{c_{\min}}\,\|\lambda - \lambda_h\|_\Lambda\,,\end{aligned}$$

and estimate (3.28) results from (3.27) and Lemma 3.3.

To prove result (3.29), we employ a duality argument. Define $e := u - u_h$ and suppose that $(\gamma, w_0) \in \Lambda \times V_0$ satisfies

$$B(\mu, v_0; \gamma, w_0) = (T\,\mu + v_0, e)_{\mathcal{T}_h} \quad \text{for all } (\mu, v_0) \in \Lambda \times V_0. \tag{3.32}$$

The problem of finding such a (γ, w_0) is the adjoint to problem (3.8) with homogenous Dirichlet boundary condition prescribed on $\partial\Omega$, and the right-hand side rewritten using identities (1.11). Furthermore, define $(\gamma_0, w_0^h) \in \Lambda_0 \times V_0$ by the finite-dimensional adjoint problem

$$B(\mu_0, v_0; \gamma_0, w_0^h) = (T\,\mu_0 + v_0, e)_{\mathcal{T}_h} \quad \text{for all } (\mu_0, v_0) \in \Lambda_0 \times V_0. \tag{3.33}$$

Both (3.32) and (3.33) have unique solutions by Theorem 3.2 and the symmetry of the problem statements. Under the assumption that problem (1.1)–(1.2) has smoothing properties, we observe that the solution $w := w_0 + T\,\gamma + \hat{T}e$ has extra regularity since $f = e \in L^2(\Omega)$, and there is a positive constant C (depending only on Ω) such that $\|w\|_{2,\Omega} \leq \frac{C}{c_{\min}} \|e\|_{0,\Omega}$. From this, Lemma 3.3, and the interpolation estimate (a particular case of result (4.2))

$$\inf_{\mu_0 \in \Lambda_0} \|\gamma - \mu_0\|_\Lambda \leq C\,h\,\|w\|_{2,\Omega}\,,$$

where C is a positive constant independent of h and $\mathcal{K}$, we get

$$\begin{aligned} \|(\gamma - \gamma_0, w_0 - w_0^h)\|_{\Lambda\times V} &\leq C\,h\|w\|_{2,\Omega} \\ &\leq \frac{C}{c_{\min}}\,h\,\|e\|_{0,\Omega}\,. \end{aligned}$$

Thus, by definition (3.32) of (γ, w_0), the consistency result of Lemma 3.3, the continuity result of Theorem 3.2, and the best approximation result of Lemma 3.3, we find

$$\begin{aligned} \|e\|_{0,\Omega}^2 &= (e,e)_{\mathcal{T}_h} \\ &= (T\,(\lambda - \lambda_h) + (u_0 - u_0^h), e)_{\mathcal{T}_h} \\ &= B(\lambda - \lambda_h, u_0 - u_0^h; \gamma, w_0) \\ &= B(\lambda - \lambda_h, u_0 - u_0^h; \gamma - \gamma_0, w_0 - w_0^h) \\ &\leq \bar{C}\,\|(\lambda - \lambda_h, u_0 - u_0^h)\|_{\Lambda\times V}\|(\gamma - \gamma_0, w_0 - w_0^h)\|_{\Lambda\times V} \\ &\leq \frac{\bar{C}^2}{\beta}\,\frac{C}{c_{\min}}\,h \inf_{\mu_h \in \Lambda_h} \|\lambda - \mu_h\|_\Lambda \|e\|_{0,\Omega}\,, \end{aligned}$$

which establishes (3.29). As for (3.30), using the triangle inequality, the local Poincaré inequality (8.3), and Lemma 8.1, it holds

$$\begin{aligned} \|u_0 - u_0^h\|_{0,\Omega} &\leq \|u - u_h\|_{0,\Omega} + \|T\,(\lambda - \lambda_h)\|_{0,\Omega} \\ &\leq \|u - u_h\|_{0,\Omega} + C\,h\,\|T\,(\lambda - \lambda_h)\|_V \\ &\leq \|u - u_h\|_{0,\Omega} + C\,\frac{2\,\kappa}{c_{\min}}\,h\,\|\lambda - \lambda_h\|_\Lambda\,, \end{aligned}$$

and the result follows from (3.29) and Lemma 3.3. ∎

As a corollary to the previous lemma, we can establish bounds which indicate the impact on the best approximation results of ignoring $\hat{T}\,f$. This requires the projection $\Pi : V \to V_0$ defined such that for $v \in V$, $\Pi\,v\,|_K = \Pi_K\,v$ for all $K \in \mathcal{T}_h$.

COROLLARY 3.6. *Let* $(\lambda, u_0) \in \Lambda \times V_0$ *and* $(\lambda_h, u_0^h) \in \Lambda_h \times V_0$ *be the solutions of* (3.8) *and* (3.9), *respectively. There exists* C *such that*

$$\|\mathcal{K}\nabla(u - T\,\lambda_h)\|_{\rm div} \le C\left(\inf_{\mu_h\in\Lambda_h} \|\lambda - \mu_h\|_\Lambda + \|f - \Pi\, f\|_{0,\Omega}\right), \tag{3.34}$$

$$\|u - (u_0^h + T\,\lambda_h)\|_{0,\Omega} \le C\left(\inf_{\mu_h\in\Lambda_h} \|\lambda - \mu_h\|_\Lambda + h\|f - \Pi\, f\|_{0,\Omega}\right). \tag{3.35}$$

Moreover, if problem (1.1)–(1.2) *has smoothing properties, it holds*

$$\|u - (u_0^h + T\,\lambda_h)\|_{0,\Omega} \le C\,h\left(\inf_{\mu_h\in\Lambda_h} \|\lambda - \mu_h\|_\Lambda + \|f - \Pi\, f\|_{0,\Omega}\right). \tag{3.36}$$

Proof. First, by the triangle inequality we get

$$\|\mathcal{K}\,\nabla(u - T\,\lambda_h)\|_{\rm div} \le \|\mathcal{K}\,\nabla(u - u_h)\|_{\rm div} + \|\mathcal{K}\,\nabla\,\hat{T}\,f\|_{\rm div},$$

so that Lemmas 3.5 and 8.2 imply result (3.34). Similarly, from the local Poincaré inequality (8.3), we see

$$\begin{aligned}\|u - (u_0^h + T\,\lambda_h)\|_{0,\Omega} &\le \|u - u_h\|_{0,\Omega} + \|\hat{T}\,f\|_{0,\Omega}\\ &\le \|u - u_h\|_{0,\Omega} + C\,h\,\|\mathcal{K}\,\nabla\hat{T}\,f\|_V\,,\end{aligned}$$

from which the result (3.35) follows by result (3.28) of Lemmas 3.5 and 8.2. If problem (1.1)–(1.2) has smoothing properties, we use (3.29) of Lemma 3.5 instead, which yields (3.36). ∎

4. A priori error estimates. Note that the result in Lemma 3.3 holds for any finite element space Λ_h under the assumption $\Lambda_0 \subseteq \Lambda_h$. As such, the MHM method (3.9) achieves optimal convergence given by the best approximation properties of Λ_h. In this section, we consider the approximation properties of the subspace

$$\Lambda_h \equiv \Lambda_l := \{\mu \in \Lambda : \mu|_F \in \mathbb{P}_l(F), \text{ for all } F \in \mathcal{E}_h\}\,, \tag{4.1}$$

where $l \ge 0$. Supposing $1 \le k \le l+1$, we follow closely [18] (see [10] for a $h-p$ version) to show that, given $w \in H^{k+1}(\Omega)$, there exists C such that

$$\inf_{\mu_l\in\Lambda_l} \|\lambda - \mu_l\|_\Lambda \le C\,h^k\|w\|_{k+1,\Omega}\,, \tag{4.2}$$

where $\lambda = -\mathcal{K}\,\nabla w\cdot\boldsymbol{n}$. This approximation property implies the convergence rates of the following theorem.

THEOREM 4.1. *Let* $(\lambda, u_0) \in \Lambda \times V_0$ *and* $(\lambda_l, u_0^h) \in \Lambda_l \times V_0$ *be the solutions of* (3.8) *and* (3.9), *respectively. Assume* $u \in H^{k+1}(\Omega)$, *where* $1 \le k \le l+1$. *Then, there exist positive constants* C, *independent of* h *and* $\mathcal{K}$, *such that*

$$\|(\lambda - \lambda_l, u_0 - u_0^h)\|_{\Lambda\times V} \le C\,\frac{\bar{C}}{\beta}\,h^k\,\|u\|_{k+1,\Omega}\,, \tag{4.3}$$

$$\|\mathcal{K}\nabla(u - u_h)\|_{\rm div} \le C\,\kappa\,\frac{\bar{C}}{\beta}\;h^k\,\|u\|_{k+1,\Omega}\,, \tag{4.4}$$

$$\|u - u_h\|_{0,\Omega} \le C\left(1 + \frac{\kappa}{c_{\min}}\right)\frac{\bar{C}}{\beta}\,h^k\,\|u\|_{k+1,\Omega}\,, \tag{4.5}$$

$$\|u - u_0^h\|_{0,\Omega} \le C\left(1 + \frac{\bar{C}\,(\bar{C} + \kappa)}{\beta\,c_{\min}}\right)h\,\|u\|_{k+1,\Omega}\,, \tag{4.6}$$

where $\bar{C}$ and β are the continuity and the inf-sup constants from Theorem 3.2*, respectively. Moreover, if problem* (1.1)–(1.2) *has smoothing properties, the following estimates hold:*

$$\|u - u_h\|_{0,\Omega} \leq C \, \frac{\bar{C}^2}{\beta \, c_{\min}} \, h^{k+1} \, \|u\|_{k+1,\Omega} \,, \tag{4.7}$$

$$\|u_0 - u_0^h\|_{0,\Omega} \leq C \, \frac{\bar{C} \, (\bar{C} + \kappa)}{\beta \, c_{\min}} \, h^{k+1} \, \|u\|_{k+1,\Omega} \,. \tag{4.8}$$

Proof. Result (4.3) follows using estimate (4.2) in the best approximation result of Lemma 3.3, and results (4.4)–(4.5), (4.7)–(4.8) follow using estimate (4.2) in, respectively, (3.27)–(3.30) of Lemma 3.5. Finally, we arrive at estimate (4.6) using the triangle inequality, $u_0 = \Pi \, u$ with the approximation property of Π, and (4.8) with $h \leq d_\Omega$, as follows:

$$\begin{aligned} \|u - u_0^h\|_{0,\Omega} &\leq \|u - u_0\|_{0,\Omega} + \|u_0 - u_0^h\|_{0,\Omega} \\ &\leq C \left(1 + \frac{\bar{C} \, (\bar{C} + \kappa)}{\beta \, c_{\min}}\right) h \, \|u\|_{k+1,\Omega} \,. \qquad \square \end{aligned}$$

As a corollary to the previous theorem, we prove the influence of ignoring $\hat{T} \, f$ on the best approximation results.

COROLLARY 4.2. *Let $(\lambda, u_0) \in \Lambda \times V_0$ and $(\lambda_l, u_0^h) \in \Lambda_l \times V_0$ be the solutions of* (3.8) *and* (3.9)*, respectively. Under the assumption of Theorem* 4.1*, there exists C such that*

$$\|\mathcal{K}\nabla(u - T \, \lambda_l)\|_{\mathrm{div}} \leq C \left(h^k \, \|u\|_{k+1,\Omega} + \|f - \Pi \, f\|_{0,\Omega} \right), \tag{4.9}$$

$$\|u - (u_0^h + T \, \lambda_l)\|_{0,\Omega} \leq C \left(h^k \, \|u\|_{k+1,\Omega} + h \, \|f - \Pi \, f\|_{0,\Omega} \right). \tag{4.10}$$

Moreover, if problem (1.1)–(1.2) *has smoothing properties, it follows that*

$$\|u - (u_0^h + T \, \lambda_l)\|_{0,\Omega} \leq C \left(h^{k+1} \, \|u\|_{k+1,\Omega} + h \, \|f - \Pi \, f\|_{0,\Omega} \right). \tag{4.11}$$

Proof. The result is a direct application of Corollary 3.6 along with (4.2). $\square$

The previous corollary indicates that, in the case of lowest-order interpolation (i.e., $\Lambda_h \equiv \Lambda_0$), excluding $\hat{T} \, f$ from the numerical solution does not weaken convergence rates when $f \in H^1(\Omega)$. Consequently, we may disregard the contribution associated with $\hat{T} \, f$ in the MHM method in such cases, which brings the desirable feature of avoiding any computation related to local problem (1.8). To see this clearly, consider the inconsistent MHM method defined by ignoring the term $-(\mu_l \, \boldsymbol{n}, [\![\hat{T} \, f]\!])_{\mathcal{T}_h}$. Such a method reads as follows: *Find $(\bar{\lambda}_l, \bar{u}_0^h) \in \Lambda_l \times V_0$ such that*

$$B(\bar{\lambda}_l, \bar{u}_0^h; \mu_l, v_0) \, = \, (f, v_0)_{\mathcal{T}_h} + (\mu_l, g_D)_{\mathcal{E}_D} \quad \text{for all } (\mu_l, v_0) \in \Lambda_l \times V_0 \,. \tag{4.12}$$

The next estimates show that the induced consistency error remains smaller than the leading error for the lowest-order interpolation.

THEOREM 4.3. *Let $(\lambda, u_0) \in \Lambda \times V_0$ and $(\bar{\lambda}_l, \bar{u}_0^h) \in \Lambda_l \times V_0$ be the solutions of* (3.8) *and* (4.12)*, respectively. Under the assumption of Theorem* 4.1*, there exists C*

such that

$$\|(\lambda - \bar{\lambda}_l, u_0 - \bar{u}_0^h)\|_{\Lambda\times V} \le C\left(h^k\,\|u\|_{k+1,\Omega} + h\,\|f - \Pi\, f\|_{0,\Omega}\right), \tag{4.13}$$

$$\|\mathcal{K}\nabla(u - T\,\bar{\lambda}_l)\|_{\rm div} \le C\left(h^k\,\|u\|_{k+1,\Omega} + \|f - \Pi\, f\|_{0,\Omega}\right), \tag{4.14}$$

$$\|u - (\bar{u}_0^h + T\,\bar{\lambda}_l)\|_{0,\Omega} \le C\left(h^k\,\|u\|_{k+1,\Omega} + h\,\|f - \Pi\, f\|_{0,\Omega}\right). \tag{4.15}$$

Proof. Clearly, the inconsistent MHM method (4.12) is well posed. Furthermore, since the method is defined from the consistent method (3.9) by removing the term $-(\mu_l\,\boldsymbol{n}, [\![\hat{T}\,f]\!])_{\mathcal{T}_h} = (f, T\,\mu_l)_{\mathcal{T}_h}$ (see (1.11)), the first Strang lemma (e.g., [11, p. 95]) implies there is a constant C such that

$$\begin{aligned}
\|(\lambda - \bar{\lambda}_l, u_0 - \bar{u}_0^h)\|_{\Lambda\times V} &\le C\left[\inf_{(\mu_l, v_0)\in\Lambda_l\times V_0}\|(\lambda - \mu_l, u_0 - v_0)\|_{\Lambda\times V}\right.\\
&\qquad\left.+\sup_{(\mu_l,v_0)\in\Lambda_l\times V_0}\frac{|(f, T\,\mu_l)_{\mathcal{T}_h}|}{\|(\mu_l, v_0)\|_{\Lambda\times V}}\right]\\
&\le C\left[\inf_{\mu_l\in\Lambda_l}\|\lambda - \mu_l\|_{\Lambda} + \sup_{(\mu_l,v_0)\in\Lambda_l\times V_0}\frac{|(f, T\,\mu_l)_{\mathcal{T}_h}|}{\|(\mu_l, v_0)\|_{\Lambda\times V}}\right],
\end{aligned}$$

where we used $v_0 = u_0$. Now, using $T\,\mu_l\,|_K \in L_0^2(K)$, the Cauchy–Schwarz inequality, the local Poincaré inequality (8.3), and Lemma 8.1, it follows that

$$\begin{aligned}
|(f, T\,\mu_l)_{\mathcal{T}_h}| &= \left|\sum_{K\in\mathcal{T}_h}(f - \Pi\, f, T\,\mu_l)_{\mathcal{T}_h}\right|\\
&\le \|f - \Pi\, f\|_{0,\Omega}\|T\,\mu_l\|_{0,\Omega}\\
&\le C\,h\,\|f - \Pi\, f\|_{0,\Omega}\|T\,\mu_l\|_V\\
&\le C\,h\,\|f - \Pi\, f\|_{0,\Omega}\|\mu_l\|_{\Lambda},
\end{aligned}$$

and we find result (4.13) from (4.2). From Lemmas 8.1 and 8.2, we get

$$\begin{aligned}
\|\mathcal{K}\,\nabla(u - T\,\bar{\lambda}_l)\|_{\rm div} &\le \|\mathcal{K}\,\nabla T\,(\lambda - \bar{\lambda}_l)\|_{\rm div} + \|\mathcal{K}\,\nabla\hat{T}\,f\|_{\rm div}\\
&\le C(\|\lambda - \bar{\lambda}_l\|_{\Lambda} + \|f - \Pi\, f\|_{0,\Omega}),
\end{aligned}$$

and result (4.14) follows using (4.13). As for result (4.15), we make use of the triangle inequality and Lemmas 8.1 and 8.2, to obtain

$$\begin{aligned}
\|u - (\bar{u}_0^h - T\,\bar{\lambda}_l)\|_{0,\Omega} &\le \|u_0 - \bar{u}_0^h\|_{0,\Omega} + \|T\,(\lambda - \bar{\lambda}_l)\|_{0,\Omega}\\
&\le \|u_0 - \bar{u}_0^h\|_{0,\Omega} + C\,\|T\,(\lambda - \bar{\lambda}_l)\|_V\\
&\le \|u_0 - \bar{u}_0^h\|_{0,\Omega} + C\,\|\lambda - \bar{\lambda}_l\|_{\Lambda},
\end{aligned}$$

and the result follows from (4.13). ∎

5. A posteriori error estimates. Recalling that $u_h = u_0^h + T\,\lambda_h + \hat{T}\,f$, let us define the residual on faces as follows:

$$R_F := \begin{cases} -\frac{1}{2}[\![u_h]\!], & F \in \mathcal{E}_0\,, \\ (g_D - u_h)\,\boldsymbol{n}, & F \in \mathcal{E}_D\,, \\ \mathbf{0}, & F \in \mathcal{E}_N\,, \end{cases} \tag{5.1}$$

where we assume (for simplicity) that $g_D\,|_F \in \mathbb{P}_l(F)$ for all $F \in \mathcal{E}_D$. Also, set

$$\eta_F := \frac{c_l\,c_{\min}}{h_F^{1/2}}\,\|R_F\|_{0,F}\,, \tag{5.2}$$

where $c_{\min}$ is defined in (1.3) and c_l is a positive constant depending on l, but independent of $\mathcal{K}$ and h, left to be fixed in the next section. The error estimator is

$$\eta := \left[\sum_{K\in\mathcal{T}_h} \eta_K^2\right]^{1/2}, \quad \eta_K^2 := \sum_{F\subset\partial K} \eta_F^2\,. \tag{5.3}$$

Before heading to the main result of this section, we need an auxiliary result.

LEMMA 5.1. *There exists $\chi_h \in V$ satisfying*

$$(\mu\,\boldsymbol{n}, [\![\chi_h]\!])_{\mathcal{E}_h} = -(\mu\,\boldsymbol{n}, [\![u_h]\!])_{\mathcal{E}_h} + (\mu, g_D)_{\mathcal{E}_D} \quad \textit{for all } \mu \in \Lambda\,, \tag{5.4}$$

and a positive constant C, independent of h and $\mathcal{K}$, such that

$$\|\chi_h\|_V \le \frac{C\,\kappa}{c_l\,\min\{1, c_{\min}^2\}}\,\eta\,.$$

Proof. Let $(\bar{\chi}_h, \bar{\xi}_h) \in V \times \Lambda$ be the solution of the following hybrid problem

$$\begin{aligned} \frac{1}{d_\Omega^2}(\bar{\chi}_h, v)_{\mathcal{T}_h} + (\mathcal{K}\nabla\bar{\chi}_h, \nabla v)_{\mathcal{T}_h} + (\bar{\xi}_h\,\boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h} &= \frac{1}{d_\Omega^2}(u_h, v)_{\mathcal{T}_h} - (\nabla\cdot(\mathcal{K}\nabla u_h), v)_{\mathcal{T}_h} \\ &\qquad \text{for all } v \in V, \\ (\mu\,\boldsymbol{n}, [\![\bar{\chi}_h]\!])_{\mathcal{E}_h} &= (\mu, g_D)_{\mathcal{E}_D} \quad \text{for all } \mu \in \Lambda\,. \end{aligned}$$

Observe that $\bar{\chi}_h \in H^1(\Omega)$ is unique (using the arguments from [17]) and satisfies $\frac{1}{d_\Omega^2}\bar{\chi}_h - \nabla\cdot(\mathcal{K}\nabla\bar{\chi}_h) = \frac{1}{d_\Omega^2}u_h - \nabla\cdot(\mathcal{K}\nabla u_h) \in L^2(\Omega)$, $\bar{\chi}_h = g_D$ on $\partial\Omega_D$, $\mathcal{K}\nabla\bar{\chi}_h\cdot\boldsymbol{n} = 0$ on $\partial\Omega_N$, and $\mathcal{K}\nabla\bar{\chi}_h\cdot\boldsymbol{n}\,|_F = -\bar{\xi}_h$ on $F \in \mathcal{E}_0$. Thereby, $\bar{\chi}_h \in H^{3/2}(\Omega)$ from standard regularity results (cf. [12]), and then $\bar{\xi}_h \in L^2(\mathcal{E}_h)$. Setting $\chi_h := \bar{\chi}_h - u_h$ and $\xi_h := \bar{\xi}_h + \mathcal{K}\nabla u_h\cdot\boldsymbol{n}$ and using the fact that $u_h\,|_K \in H^{3/2}(K)$ (cf. [12]), it holds that $(\chi_h, \xi_h) \in H^{3/2}(\mathcal{T}_h) \times L^2(\mathcal{E}_h)$ satisfies

$$\begin{aligned} \frac{1}{d_\Omega^2}(\chi_h, v)_{\mathcal{T}_h} + (\mathcal{K}\nabla\chi_h, \nabla v)_{\mathcal{T}_h} + (\xi_h\,\boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h} &= 0 \quad \text{for all } v \in V\,, \\ (\mu\,\boldsymbol{n}, [\![\chi_h]\!])_{\mathcal{E}_h} &= -(\mu\,\boldsymbol{n}, [\![u_h]\!])_{\mathcal{E}_h} + (\mu, g_D)_{\mathcal{E}_D} \quad \text{for all } \mu \in \Lambda\,. \end{aligned}$$

As a result, the product $(\xi_h\,\boldsymbol{n}, [\![u_h]\!])_F$, for all $F \in \mathcal{E}_0$, is meaningful, and from the

Cauchy–Schwarz inequality and a scaling argument (cf. [6, p. 111]), we get

$$
\begin{aligned}
&\min\left\{1, c_{\min}^2\right\} \|\chi_h\|_V^2 \\
&\quad \le \left[\frac{1}{d_\Omega^2}(\chi_h, \chi_h)_{\mathcal{T}_h} + (\mathcal{K}\nabla\chi_h, \nabla\chi_h)_{\mathcal{T}_h}\right] \\
&\quad = -(\xi_h\, \boldsymbol{n}, [\![\chi_h]\!])_{\mathcal{E}_h} \\
&\quad = (\xi_h\, \boldsymbol{n}, [\![u_h]\!])_{\mathcal{E}_h} - (\xi_h, g_D)_{\mathcal{E}_D} \\
&\quad \le \sum_{F\in\mathcal{E}_0} \|\xi_h\|_{0,F} \|[\![u_h]\!]\|_{0,F} + \sum_{F\in\mathcal{E}_D} \|\xi_h\|_{0,F} \|g_D - u_h\|_{0,F} \\
&\quad \le \left[\sum_{F\in\mathcal{E}_h} h_F\, \|\xi_h\|_{0,F}^2\right]^{1/2} \left[\sum_{F\in\mathcal{E}_0} \frac{1}{h_F} \|[\![u_h]\!]\|_{0,F}^2 + \sum_{F\in\mathcal{E}_D} \frac{1}{h_F} \|g_D - u_h\|_{0,F}^2\right]^{1/2} \\
&\quad \le \frac{1}{c_{\min} c_l} \left[\sum_{F\in\mathcal{E}_h} h_F\, \|\mathcal{K}\nabla\chi_h \cdot \boldsymbol{n}\|_{0,F}^2\right]^{1/2} \eta \\
&\quad \le \frac{C\,\kappa}{c_l} \|\chi_h\|_V\, \eta\,,
\end{aligned}
$$

and the result follows. ∎

Hereafter, we shall make use of the following norm on $H(\mathrm{div};\Omega)$:

$$
\|\boldsymbol{\sigma}\|_{\mathrm{div},h}^2 := \sum_{K\in\mathcal{T}_h} \left(\|\boldsymbol{\sigma}\|_{0,K}^2 + h_K^2\, \|\nabla\cdot\boldsymbol{\sigma}\|_{0,K}^2\right), \tag{5.5}
$$

and, also, of the following locally defined norm: Given $F \in \mathcal{E}_h$, we set

$$
\|v\|_{V,\omega_F}^2 := \sum_{K\in\omega_F} \left(h_K^{-2} \|v\|_{0,K}^2 + \|\nabla v\|_{0,K}^2\right),
$$

where ω_F is either the set of (two) elements K, $K' \in \mathcal{T}_h$ such that $K \cap K' = \{F\}$ if $F \in \mathcal{E}_0$, or corresponds to K if $F \subset \partial K \cap \mathcal{E}_h/\mathcal{E}_0$. We are ready to establish the a posteriori error estimate, showing the reliability and efficiency of the error estimator.

THEOREM 5.2. *Let η be defined in* (5.3), *and assume $u \in V$ and $\mathcal{K}\,\nabla u \in H(\mathrm{div};\Omega)$. There exist positive constants C, independent of h and $\mathcal{K}$, such that*

$$
\|\mathcal{K}\,\nabla(u - u_h)\|_{\mathrm{div},h} + c_{\min}\, \|u - u_h\|_V \le C\, \frac{\max\{c_{\min}, \kappa\}\, \kappa}{\beta\, \min\{1, c_{\min}^2\}\; c_l}\, \eta\,, \tag{5.6}
$$

where β is the inf-sup constant in Theorem 3.2. *Moreover, given $F \in \mathcal{E}_h$, it holds*

$$
\eta_F \le C\, c_{\min}\, \|u - u_h\|_{V,\omega_F}\,. \tag{5.7}
$$

Proof. We establish the result (5.6) first. Take $(\mu, v_0) \in \Lambda \times V_0$. From Lemma 8.1 (with $\mathcal{K}$ taken as the identity matrix) we conclude the existence of a function $\boldsymbol{\sigma}^* \in H(\mathrm{div};\Omega)$ with the property $\boldsymbol{\sigma}^* \cdot \boldsymbol{n} = \mu$, and $\|\boldsymbol{\sigma}^*\|_{\mathrm{div}} \le \sqrt{2}\, \|\mu\|_\Lambda$. Next, from (3.8) and (3.9), Lemma 5.1, the Cauchy–Schwarz inequality, the definition of norms in (3.10) and (3.11), integrating by parts, and from Lemma 5.1 again, we get

$$\begin{aligned}
B(\lambda-\lambda_h,u_0-u_0^h;\mu,v_0) &= B(\lambda-\lambda_h,u_0-u_0^h;\mu,0) = (\mu\,\boldsymbol{n},[\![u-u_h]\!])_{\mathcal{E}_h} \qquad (5.8)\\
&= -(\mu\,\boldsymbol{n},[\![u_h]\!])_{\mathcal{E}_h} + (\mu,g_D)_{\mathcal{E}_D}\\
&= (\mu\,\boldsymbol{n},[\![\chi_h]\!])_{\mathcal{E}_h}\\
&= \sum_{K\in\mathcal{T}_h}(\boldsymbol{\sigma}^*\cdot\boldsymbol{n}^K,\chi_h)_{\partial K}\\
&= \sum_{K\in\mathcal{T}_h}(\nabla\cdot\boldsymbol{\sigma}^*,\chi_h)_K + (\boldsymbol{\sigma}^*,\nabla\chi_h)_K \qquad (5.9)\\
&\le \|\boldsymbol{\sigma}^*\|_{\mathrm{div}}\|\chi_h\|_V \qquad (5.10)\\
&\le \frac{C\sqrt{2}\,\kappa}{c_l\,\min\{1,c_{\min}^2\}}\,\|\mu\|_\Lambda\,\eta\,, \qquad (5.11)
\end{aligned}$$

where C is a positive constant independent of h and $\mathcal{K}$. It then follows by Theorem 3.2 and definition (3.14) of $\|(\cdot,\cdot)\|_{\Lambda\times V}$ that

$$\begin{aligned}
\|(\lambda-\lambda_h,u_0-u_0^h)\|_{\Lambda\times V} &\le \frac{1}{\beta}\sup_{(\mu,v_0)\in\Lambda\times V_0}\frac{B(\lambda-\lambda_h,u_0-u_0^h;\mu,v_0)}{\|(\mu,v_0)\|_{\Lambda\times V}}\\
&\le \frac{C\sqrt{2}\,\kappa}{\beta\,c_l\,\min\{1,c_{\min}^2\}}\,\eta\,. \qquad (5.12)
\end{aligned}$$

Since $u-u_h = T\,(\lambda-\lambda_h)+u_0-u_0^h$, Lemma 8.1 and the definition of $\|(\cdot,\cdot)\|_{\Lambda\times V}$ imply

$$\begin{aligned}
\|\mathcal{K}\,\nabla(u-u_h)\|_{\mathrm{div},h} &\le \|\mathcal{K}\,\nabla(u-u_h)\|_{\mathrm{div}} = \|\mathcal{K}\,\nabla T\,(\lambda-\lambda_h)\|_{\mathrm{div}}\\
&\le \sqrt{2}\,\kappa\,\|\lambda-\lambda_h\|_\Lambda \le \sqrt{2}\,\kappa\,\|(\lambda-\lambda_h,u_0-u_0^h)\|_{\Lambda\times V}
\end{aligned}$$

and

$$\begin{aligned}
\|u-u_h\|_V &\le \|u_0-u_0^h\|_V + \|T\,(\lambda-\lambda_h)\|_V \le \|u_0-u_0^h\|_V + \frac{2\,\kappa}{c_{\min}}\|\lambda-\lambda_h\|_\Lambda\\
&\le \frac{2}{c_{\min}}\max\{c_{\min},\kappa\}\,\|(\lambda-\lambda_h,u_0-u_0^h)\|_{\Lambda\times V}\,.
\end{aligned}$$

Therefore, summing up both previous estimates we get

$$\begin{aligned}
\|\mathcal{K}\,\nabla(u-u_h)\|_{\mathrm{div},h} + c_{\min}\|u-u_h\|_V &\le 2\,\max\{c_{\min},\kappa\}\;\|(\lambda-\lambda_h,u_0-u_0^h)\|_{\Lambda\times V}\\
&\le C\,\frac{\max\{c_{\min},\kappa\}\,\kappa}{\beta\,\min\{1,c_{\min}^2\}\;c_l}\,\eta\,,
\end{aligned}$$

and result (5.6) follows. Now, we turn to proving (5.7). Given a face $F\in\mathcal{E}_h$, let $\mu^*\in\Lambda$ be defined such that $\mu^*\boldsymbol{n}|_F = R_F$ and $\mu^*\boldsymbol{n}|_{F'} = \mathbf{0}$ for $\mathcal{E}_h \ni F' \neq F$. It follows by (5.8) and $R_F\in[L^2(F)]^d$ (with its usual meaning) that

$$\|R_F\|_{0,F}^2 \le 2\,(R_F,[\![u-u_h]\!])_F \le 2\,\|R_F\|_{0,F}\|[\![u-u_h]\!]\|_{0,F}\,,$$

and thus, the local trace inequality (8.4) and mesh regularity imply

$$\begin{aligned}
\|R_F\|_{0,F} &\leq 2\,\|[\![u-u_h]\!]\|_{0,F} \\
&\leq C \sum_{K\in\omega_F} \left[h_K^{-1}\|u-u_h\|_{0,K}^2 + h_K\|\nabla\,(u-u_h)\|_{0,K}^2\right]^{1/2} \\
&\leq C\,h_F^{1/2} \sum_{K\in\omega_F} \left[h_K^{-2}\|u-u_h\|_{0,K}^2 + \|\nabla(u-u_h)\|_{0,K}^2\right]^{1/2} \\
&= C\,h_F^{1/2}\|u-u_h\|_{V,\omega_F}\,.
\end{aligned}$$

Multiplying both sides by $c_l\,c_{\min}$, result (5.7) follows from definition (5.2) of η_F. ∎

We now show stronger control of the L^2-norm holds assuming extra regularity.

COROLLARY 5.3. *Let η be defined in* (5.3) *and suppose problem* (1.1)–(1.2) *has smoothing properties. Also, assume that the conditions of Theorem* 5.2 *hold. Then, there exists a positive constant C, independent of h and $\mathcal{K}$, such that*

$$\|K\,\nabla(u-u_h)\|_{\mathrm{div},h} + c_{\min}\left(h^{-1}\|u-u_h\|_{0,\Omega} + \left(\sum_{K\in\mathcal{T}_h}\|\nabla(u-u_h)\|_{0,K}^2\right)^{1/2}\right) \leq\; C\,\kappa\,\frac{\max\{c_{\min},\kappa\} + \frac{\bar{C}^2}{\beta\,c_{\min}}}{\beta\,\min\{1,c_{\min}^2\}\;c_l}\,\eta\,,$$

where β and $\bar{C}$ are the inf-sup and the continuity constant in Theorem 3.2, *respectively.*

Proof. Using (3.29) of Lemma 3.5 and (5.12), we establish that

$$h^{-1}\|u-u_h\|_{0,\Omega} \leq C\,\frac{\bar{C}^2}{\beta\,c_{\min}}\|\lambda-\lambda_h\|_\Lambda \leq C\,\frac{\bar{C}^2\kappa}{\beta^2\,c_{\min}\min\{1,c_{\min}^2\}\;c_l}\,\eta\,,$$

where C is a positive constant independent of h and $\mathcal{K}$, and the result follows from Theorem 5.2. ∎

We close this section with some important comments. First, if f is assumed piecewise constant in each $K\in\mathcal{T}_h$, then the estimator η_F is driven by the simplified face-residual terms

$$R_F := \begin{cases} -\frac{1}{2}\,[\![u_0^h + T\,\lambda_h]\!], & F\in\mathcal{E}_0\,,\\ (g_D - u_0^h - T\,\lambda_h)\,\boldsymbol{n}, & F\in\mathcal{E}_D\,,\\ \mathbf{0}, & F\in\mathcal{E}_N\,, \end{cases} \tag{5.13}$$

as $\hat{T}\,f$ vanishes according to (1.8). More generally, from the trace inequality (8.4)

$$\|\hat{T}\,f\|_{0,F} \leq C\,\left[\frac{1}{h_K}\|\hat{T}\,f\|_{0,K}^2 + h_K\|\nabla\hat{T}\,f\|_{0,K}^2\right]^{1/2}$$

and, since $\hat{T}\,f\in L_0^2(K)$, the Poincaré and trace inequalities (8.3) and (8.4), respectively, imply

$$\|[\![\hat{T}\,f]\!]\|_{0,F} \leq C\,h_K^{1/2}\|\nabla\hat{T}\,f\|_{0,K} \leq C\,h_K^{1/2}\|f-\Pi_K\,f\|_{0,K}\,,$$

where we used Lemma 8.2. Consequently, the error is also bounded by the estimator given in (5.13) added to $[\sum_{K\in\mathcal{T}_h} h_K \|f - \Pi_K f\|_{0,K}^2]^{1/2}$, which corresponds to a higher-order term if f is regular and l is low. Finally, if $u \in H^{k+1}(\Omega)$ with $1 \le k \le l+1$, and $l \ge 0$ is the degree of the polynomial interpolation, then the estimator η satisfies the following estimate:

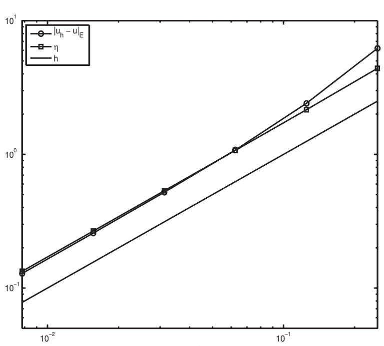
$$\eta \le C\, h^k\, |u|_{k+1,\Omega}\,.$$

6. Numerical results. As the a priori estimates have already been verified in [13], this section is dedicated to the validation of the a posteriori error estimates. We present three illuminating numerical experiments computed using the `triangle` software [19] to perform mesh adaptations. In all cases, the domain is a unit square which is decomposed into triangles. The first numerical test aims at validating theoretical results, while the second and third ones deal with the capacity of the MHM method and the a posteriori estimator to handle problems with singularities. One of these has a jumping coefficient, while the other is the quarter five-spot problem. The latter, in spite of lying outside the scope of current theoretical framework, is investigated to demonstrate the robustness of the MHM method and its associated error estimator.

6.1. An analytical solution. This numerical test assesses the theoretical aspects of the method presented in the previous sections. We consider an analytical solution $u(x,y) = \cos(2\,\pi\,x)\cos(2\,\pi\,y)$ and prescribe the corresponding boundary conditions and right-hand side. To study the reliability and efficiency of the estimator (5.3), consider the following effectivity index $E_f := \frac{\eta}{|u-u_h|_E}$, where the corresponding values of c_l are 3, 7, 18, 50, for $l = 0, 1, 2, 3$, respectively, and

$$|u-u_h|_E := \|\mathcal{K}\nabla(u-u_h)\|_{\mathrm{div},h} + c_{\min}\left(h^{-1}\|u-u_h\|_{0,\Omega} + \left(\sum_{K\in\mathcal{T}_h}\|\nabla(u-u_h)\|_{0,K}^2\right)^{1/2}\right).$$

First, we set $\mathcal{K} = I$, where I is the identity matrix, and illustrate the results in Figure 6.1, with $l = 0$ and $l = 3$, on a sequence of structured triangular meshes. In both cases the effectivity index is close to 1. We also vary $\mathcal{K} = \alpha\, I$ with $\alpha \in \mathbb{R}$ ranging from 10^{-6} to 10^{6}, and we investigate the effectivity index with respect to the value of $\mathcal{K}$ (see Figure 6.2). We observe that the results match perfectly with the theoretical order of convergence (linear when $l = 0$ and fourth order for $l = 3$) and we verify that the effectivity index stays close to one and independent of h and $\mathcal{K}$ in all cases. Analogous results also arise using $l = 1$ and $l = 2$ by modifying the values of c_l accordingly.



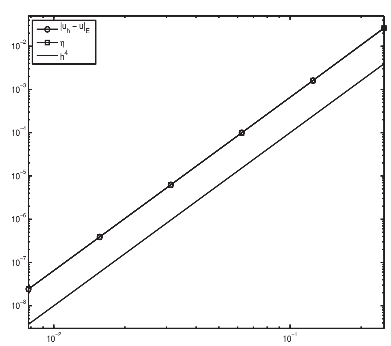


FIG. 6.1. *Convergence curves for $\mathcal{K} = I$ with $l = 0$ (left) and $l = 3$ (right).*

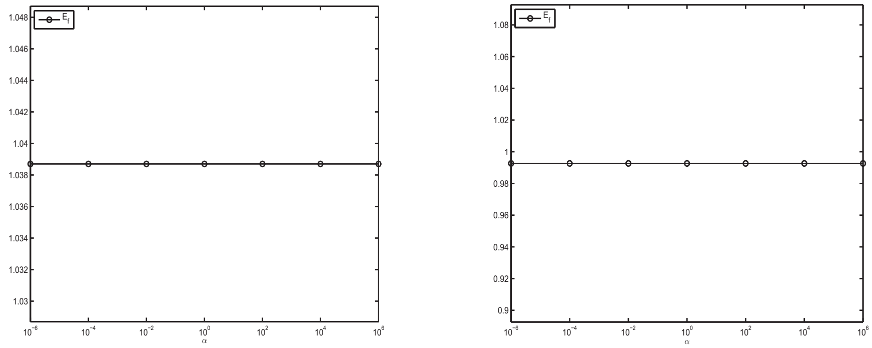


FIG. 6.2. *The effectivity index shows independence with respect to $\mathcal{K} = \alpha I$ (on the finest mesh). Here $l = 0$ (left) and $l = 3$ (right).*

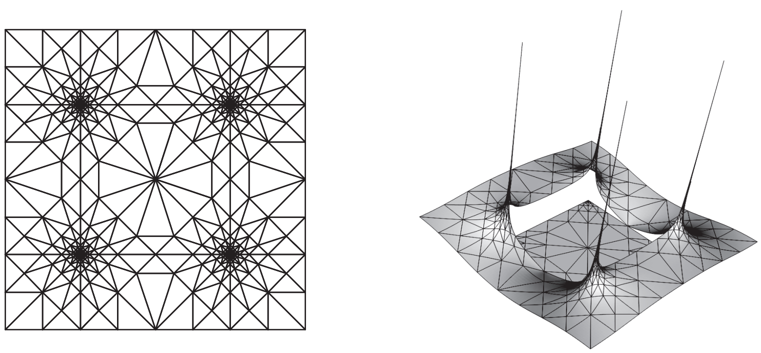


FIG. 6.3. *The final adapted mesh (848 elements) with $l = 2$ (left) and the surface of $|\boldsymbol{\sigma}_h|$ (right).*

6.2. A discontinuous coefficient case. We now consider performance of the MHM method and the a posteriori estimator (5.3) in the presence of discontinuous coefficients. We let $\mathcal{K} = 10^{-6} I$ in a square of area 0.25 centered at the barycenter of the unit-square domain, and take $\mathcal{K} = I$ elsewhere. Dirichlet conditions of $u = 1$ and $u = 0$ are used on the left- and right-hand sides of the square, respectively, with homogeneous Neumann conditions on the top and bottom. It is worth mentioning that the performance on this test motivates the use of the MHM method in oil recovery applications where different permeabilities are present. Figure 6.3 presents the final adapted mesh obtained using $l = 2$, the initial mesh had only two elements. We see that the mesh has been adapted to capture the singularities at the corners of the square area having $\mathcal{K} = 10^{-6} I$. Observe also that the estimator has led faces to be aligned with this square area, thereby allowing accurate approximation of the flux ($\boldsymbol{\sigma}_h := -\mathcal{K}\,\nabla u_h$) between the regions with different $\mathcal{K}$. In fact, consider, again, Figure 6.3 (right), which shows the absolute value of the flux variable. We see a very good approximation, with great performance across the interface between the regions with different coefficients.

6.3. The five-spot problem. The quarter five-spot problem is of practical importance in oil recovery and serves as one of the main benchmarks to validate the stability and accuracy of numerical methods for the Darcy model. This problem is now addressed considering $\mathcal{K} = I$ in a unit-square domain, with injection and production wells modeled by Dirac deltas. Figure 6.4 presents the final adapted meshes adopting

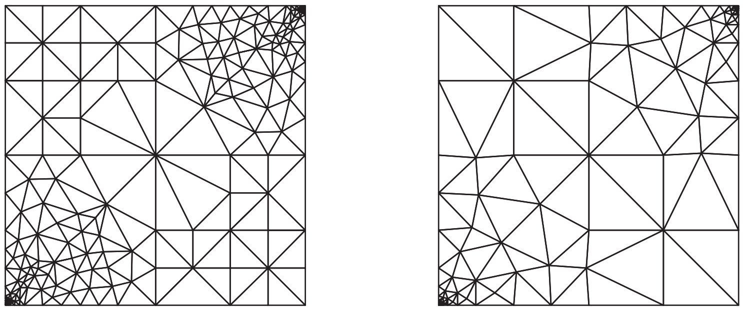


FIG. 6.4. *The final adapted mesh (356 elements) with $l = 0$ (left) and the final adapted mesh (184 elements) with $l = 2$ (right).*

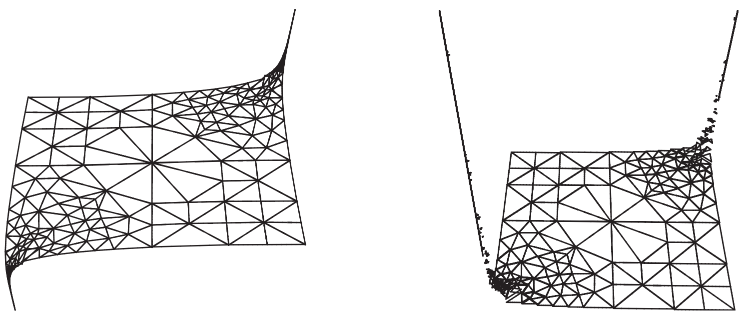


FIG. 6.5. *Surfaces of u_h (left) and $|\boldsymbol{\sigma}_h|$ (right) on the final adapted mesh. Here $l = 0$.*

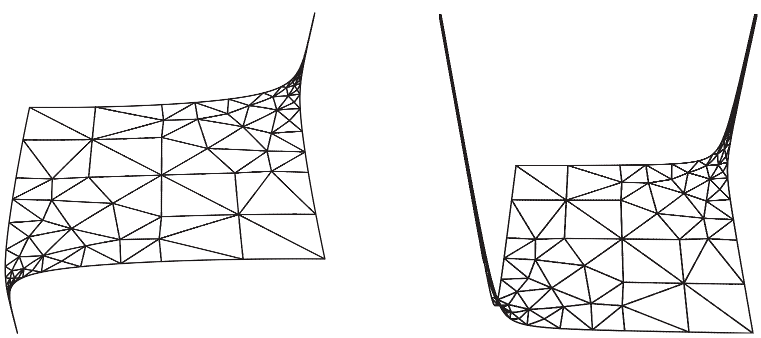


FIG. 6.6. *Surfaces of u_h (left) and $|\boldsymbol{\sigma}_h|$ (right) on the final adapted mesh. Here $l = 2$.*

$l = 0$ and $l = 2$ on faces. As expected, mesh refinement is concentrated around wells, and we see that the use of higher-order approximation on faces ($l = 2$) lowers the number of elements required to achieve the same precision when compared to the case $l = 0$. This is illustrated in Figures 6.5 and 6.6, in which we show surfaces of the primal and dual variables, i.e. u_h and $\boldsymbol{\sigma}_h$, respectively, on these adapted meshes. Overall, the results show that the MHM method and its associated error estimator deal perfectly with problems which lie outside the theory in which they were developed.

7. Conclusion. The MHM method, first presented in [13] as a consequence of a hybridization procedure, emerges as a method that naturally incorporates multiple scales while providing solutions with high-order precision in the $H^1(\mathcal{T}_h)$ and $H(\mathrm{div};\Omega)$

spaces for the primal and dual (or flux) variables, respectively. The analysis results in a priori estimates showing optimal convergence in natural norms and provides a face-based a posteriori estimator. Regarding the latter, we prove that reliability and efficiency hold with respect to natural norms. Although the computation of local problems is embedded in the upscaling procedure, they are completely independent and thus may be obtained using parallel computation facilities. Also interesting is that the flux variable preserves the local conservation property using a simple postprocessing of the primal variable. Overall, the aforementioned features stem from a new family of inf-sup stable pairs of approximation spaces based on the simplest space (i.e., piecewise constant functions) and face-based interpolations. Numerical tests have assessed the theoretical results, showing in particular the great performance of the proposed a posteriori estimator. Thereby, we conclude that the MHM method, which is naturally shaped to be used in parallel computing environments, appears to be a highly competitive option to handle realistic multiscale boundary value problems with precision on coarse meshes.

8. Appendix. Throughout this work, we use the following broken Sobolev spaces

$$
\begin{aligned}
H^m(\mathcal{T}_h) &:= \left\{ v \in L^2(\Omega) \,:\, v|_K \in H^m(K),\, K \in \mathcal{T}_h \right\},\\
H^{\frac{1}{2}}(\mathcal{E}_h) &:= \left\{ \mu \in \Pi_{K\in\mathcal{T}_h} H^{\frac{1}{2}}(\partial K) \,:\, \exists\, v \in H^1(\mathcal{T}_h) \text{ s.t. } \mu|_{\partial K} = v|_{\partial K},\, K \in \mathcal{T}_h \right\},\\
H^{-\frac{1}{2}}(\mathcal{E}_h) &:= \left\{ \mu \in \Pi_{K\in\mathcal{T}_h} H^{-\frac{1}{2}}(\partial K) \,:\, \exists\, \boldsymbol{\sigma} \in H(\mathrm{div};\Omega) \text{ s.t. } \mu\,|_{\partial K} = \boldsymbol{\sigma}\cdot\boldsymbol{n}|_{\partial K},\, K \in \mathcal{T}_h \right\},
\end{aligned}
$$

where we identify $H^{\frac{1}{2}}(\partial K) := \{\mu \in L^2(\partial K) \,:\, \exists\, v \in H^1(K) \text{ s.t. } \mu = v|_{\partial K},\, K \in \mathcal{T}_h\}$, and $H^{-\frac{1}{2}}(\partial K)$ is its dual space.

To better understand the behavior of functions in $H^1(\mathcal{T}_h)$ on $\mathcal{E}_h$, we introduce the notion of jump $[\![\,\cdot\,]\!]$ and average value $\{\cdot\}$ (see [5]); given a function $v \in H^1(\mathcal{T}_h)$, these are defined on face $F = \partial K_1 \cap \partial K_2 \in \mathcal{E}_0$ by

$$
[\![v]\!]|_F := v^{K_1}|_F\, \boldsymbol{n}_F^{K_1} + v^{K_2}|_F\, \boldsymbol{n}_F^{K_2}, \qquad \{v\}\,|_F := \frac{1}{2}\left(v^{K_1}|_F + v^{K_2}|_F\right),
$$

where $v^{K_i} \in H^1(K_i)$, $i \in \{1,2\}$. Furthermore, we define the jump and average values of vector-valued functions $\boldsymbol{\sigma} \in [H^1(\mathcal{T}_h)]^d$, respectively, by

$$
[\![\boldsymbol{\sigma}]\!]|_F := \boldsymbol{\sigma}^{K_1}|_F\cdot \boldsymbol{n}_F^{K_1} + \boldsymbol{\sigma}^{K_2}|_F\cdot \boldsymbol{n}_F^{K_2}, \qquad \{\boldsymbol{\sigma}\}\,|_F := \frac{1}{2}\left(\boldsymbol{\sigma}^{K_1}|_F + \boldsymbol{\sigma}^{K_2}|_F\right).
$$

For faces $F \in \mathcal{E}_D \cup \mathcal{E}_N$ with incident triangle K, we define the jump of a scalar function and average value of a vector-valued function by $[\![v]\!]|_F := v|_F^K\, \boldsymbol{n}_F^K$ and $\{\boldsymbol{\sigma}\}\,|_F := \boldsymbol{\sigma}^K|_F$. An important identity holds regarding these values:

$$
\sum_{K\in\mathcal{T}_h} \left(\boldsymbol{\sigma}^K\cdot \boldsymbol{n}^K, v^K\right)_{\partial K} = \left(\{\boldsymbol{\sigma}\}, [\![v]\!]\right)_{\mathcal{E}_h} + \left([\![\boldsymbol{\sigma}]\!], \{v\}\right)_{\mathcal{E}_0}, \tag{8.1}
$$

where $(\cdot,\cdot)_{\mathcal{E}_h}$ and $(\cdot,\cdot)_{\mathcal{E}_0}$ implicitly indicate summation over the respective sets $\mathcal{E}_h$ and $\mathcal{E}_0$. Here and throughout this work, we understand $(\cdot,\cdot)_{\partial K}$ in the sense of a product of duality so that given $\mu \in H^{-\frac{1}{2}}(\partial K)$, $(\mu, v)_{\partial K}$ makes sense for arbitrary $v \in H^{\frac{1}{2}}(\partial K)$. In the case $\boldsymbol{\sigma} \in [H^1(\mathcal{T}_h)]^d \cap H(\mathrm{div};\Omega)$, it holds from (8.1) with $\Lambda \ni \mu := \boldsymbol{\sigma}\cdot \boldsymbol{n}$,

$$\sum_{K\in\mathcal{T}_h} \left(\mu\, \boldsymbol{n}\cdot \boldsymbol{n}^K, v^K\right)_{\partial K} = (\mu\, \boldsymbol{n}, [\![v]\!])_{\mathcal{E}_h}\,. \tag{8.2}$$

In the general case $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$, the right-hand side of the above equivalence may lose its mathematical meaning. Nonetheless, since the right-hand side is suggestive of the action of the left-hand side (which continues to be valid mathematically), we adopt it as a formal notation throughout this work when $\boldsymbol{\sigma}$ belongs to $H(\mathrm{div};\Omega)$. Also, we shall need some auxiliary results such as the optimal local Poincaré inequality (on convex domains): For $v \in H^1(K) \cap L^2_0(K)$ it holds [15]

$$\|v\|_{0,K} \le \frac{h_K}{\pi} \|\nabla v\|_{0,K}\,, \tag{8.3}$$

and the local trace inequality (cf. [1, Thm. 3.10]): Given $v \in H^1(K)$ there exists a C, such that

$$\|v\|_{0,\partial K} \,\le\, C \left(\frac{1}{h_K}\|v\|^2_{0,K} + h_K\|\nabla v\|^2_{0,K}\right)^{1/2}\,. \tag{8.4}$$

Next, we prove some of the auxiliary results which were used in previous sections.

Lemma 8.1. *Let $\mu \in \Lambda$ and suppose $\mathcal{K} \in [L^\infty(\Omega)]^{d\times d}$ is symmetric positive definite. Define $T : \Lambda \to V$ such that for each $K \in \mathcal{T}_h$, $T\,\mu\,|_K \in H^1(K)\cap L^2_0(K)$ is the unique solution of*

$$(\mathcal{K}\,\nabla\, T\,\mu,\, \nabla w)_K = -\left(\mu\, \boldsymbol{n}\cdot \boldsymbol{n}^K,\, w\right)_{\partial K} \quad \text{for all } w \in H^1(K)\cap L^2_0(K)\,.$$

Then, T is a bounded linear operator satisfying the following bounds:

$$\|\mathcal{K}\,\nabla T\,\mu\|_{\mathrm{div}} \le \sqrt{2}\,\kappa\,\|\mu\|_\Lambda\,, \tag{8.5}$$

$$\|T\,\mu\|_V \le 2\frac{\kappa}{c_{\min}}\,\|\mu\|_\Lambda\,. \tag{8.6}$$

Proof. By definition (3.10) of $\|.\|_{\mathrm{div}}$, the fact $\nabla\cdot(\mathcal{K}\nabla T\,\mu)\,|_K \in \mathbb{R}$ with the identities of (1.11) implies

$$\begin{aligned}
\|\mathcal{K}\,\nabla T\,\mu\|^2_{\mathrm{div}} &= \sum_{K\in\mathcal{T}_h}\left[\|\mathcal{K}\,\nabla T\,\mu\|^2_{0,K} + d^2_\Omega\,\|\nabla\cdot(\mathcal{K}\,\nabla T\,\mu)\|^2_{0,K}\right]\\
&\le \sum_{K\in\mathcal{T}_h}\left[(\mathcal{K}\,\nabla T\,\mu, \|\mathcal{K}\|_\infty \nabla T\,\mu)_K + d^2_\Omega\,(\nabla\cdot(\mathcal{K}\,\nabla T\,\mu), \nabla\cdot(\mathcal{K}\,\nabla T\,\mu))_K\right]\\
&\le \sum_{K\in\mathcal{T}_h} -(\mu\,\boldsymbol{n}\cdot\boldsymbol{n}^K, c_{\max}\,T\,\mu + d^2_\Omega\,\nabla\cdot(\mathcal{K}\,\nabla T\,\mu))_{\partial K}\,,
\end{aligned}$$

where we used (3.15). Therefore, since $c_{\max}\,T\,\mu + d^2_\Omega\,\nabla\cdot(\mathcal{K}\,\nabla T\,\mu) \in V$, it follows by the local Poincaré inequality (8.3) and the fact $\nabla\cdot(\mathcal{K}\,\nabla T\,\mu)|_K \in \mathbb{R}$,

$$\begin{aligned}
\|\mathcal{K}\nabla T\,\mu\|_{\mathrm{div}}^2 &\le \sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V}\left[\sum_{K\in\mathcal{T}_h}\Big(d_\Omega^{-2}\|c_{\max}T\,\mu + d_\Omega^2\nabla\cdot(\mathcal{K}\,\nabla T\,\mu)\|_{0,K}^2\right.\\
&\qquad\qquad\left.+\|c_{\max}\nabla T\,\mu\|_{0,K}^2\Big)\right]^{1/2}\\
&\le \sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V}\left[\sum_{K\in\mathcal{T}_h}\Big(2\,d_\Omega^{-2}\,c_{\max}^2\|T\,\mu\|_{0,K}^2 + 2\,d_\Omega^2\|\nabla\cdot(\mathcal{K}\,\nabla T\,\mu)\|_{0,K}^2\right.\\
&\qquad\qquad\left.+c_{\max}^2\|\nabla T\,\mu\|_{0,K}^2\Big)\right]^{1/2}\\
&\le \sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V}\left[\sum_{K\in\mathcal{T}_h}\Big(\frac{(2+\pi^2)\,c_{\max}^2}{\pi^2\,c_{\min}^2}\|\mathcal{K}\,\nabla T\,\mu\|_{0,K}^2\right.\\
&\qquad\qquad\left.+2\,d_\Omega^2\|\nabla\cdot(\mathcal{K}\,\nabla T\,\mu)\|_{0,K}^2\Big)\right]^{1/2}.
\end{aligned}$$

Then, using the definition of κ in (3.16), we get

$$\|\mathcal{K}\,\nabla T\,\mu\|_{\mathrm{div}} \le \sqrt{2}\,\kappa\,\sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V}\,. \tag{8.7}$$

Now, choose arbitrary $v\in V$, and suppose that $\boldsymbol{\sigma}\in H(\mathrm{div};\Omega)$ satisfies the property $\boldsymbol{\sigma}\cdot\boldsymbol{n}^K|_{\partial K} = \mu\,\boldsymbol{n}\cdot\boldsymbol{n}^K$ for $\mu\in\Lambda$. It follows by (8.2), Green's Theorem, and the Cauchy–Schwarz inequality that

$$\begin{aligned}
\sum_{F\in\mathcal{E}_h}(\mu\,\boldsymbol{n},[\![v]\!])_F &= \sum_{K\in\mathcal{T}_h}(\mu\,\boldsymbol{n}\cdot\boldsymbol{n}^K,v)_{\partial K} = \sum_{K\in\mathcal{T}_h}(\boldsymbol{\sigma}\cdot\boldsymbol{n}^K,v)_{\partial K}\\
&= \sum_{K\in\mathcal{T}_h}\left[(\nabla\cdot\boldsymbol{\sigma},v)_K + (\boldsymbol{\sigma},\nabla v)_K\right]\\
&\le \sum_{K\in\mathcal{T}_h}\left[d_\Omega\,\|\nabla\cdot\boldsymbol{\sigma}\|_{0,K}\,d_\Omega^{-1}\,\|v\|_{0,K} + \|\boldsymbol{\sigma}\|_{0,K}\|\nabla v\|_{0,K}\right] \le \|\boldsymbol{\sigma}\|_{\mathrm{div}}\|v\|_V.
\end{aligned}$$

Then, by the definition of supremum, it follows that

$$\sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V} = \sup_{v\in V}\frac{(\mu\,\boldsymbol{n},[\![v]\!])_{\mathcal{E}_h}}{\|v\|_V} \le \|\boldsymbol{\sigma}\|_{\mathrm{div}}\,.$$

Since $\boldsymbol{\sigma}$ was arbitrarily taken, the inequality above and the definition of infimum imply

$$\sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V} \le \|\mu\|_\Lambda\,, \tag{8.8}$$

and result (8.5) follows immediately replacing the result above in (8.7). The bound (8.6) follows using the Poincaré inequality (8.3) and result (8.5). ∎

LEMMA 8.2. *Let $q\in L^2(\Omega)$ and suppose $\mathcal{K}\in[L^\infty(\Omega)]^{d\times d}$ is symmetric positive definite. Define $\hat{T}: L^2(\Omega)\to V$ such that for each $K\in\mathcal{T}_h$, $\hat{T}\,q\,|_K\in H^1(K)\cap L_0^2(K)$ is the unique solution of*

$$(\mathcal{K}\,\nabla\,\hat{T}\,q,\,\nabla w)_K = (q,w)_K \quad \text{for all } w\in H^1(K)\cap L_0^2(K)\,. \tag{8.9}$$

Then, $\hat{T}$ is a bounded linear operator satisfying the following bounds:

$$\|\mathcal{K}\,\nabla\,\hat{T}\,q\|_{\mathrm{div}} \leq \sqrt{2}\,d_\Omega\,\kappa\,\|q - \Pi\,q\|_{0,\Omega}\,, \tag{8.10}$$

$$\|\hat{T}\,q\|_V \leq 2\,d_\Omega\,\frac{\kappa}{c_{\min}}\,\|q - \Pi\,q\|_{0,\Omega}\,. \tag{8.11}$$

Proof. First, we establish (8.10). Note that (3.15), the fact $\hat{T}\,q\,|_K \in L^2_0(K) \cap H^1(K)$, and the Cauchy–Schwarz and the local Poincaré inequality (8.3), and $h_K \leq d_\Omega$ imply

$$\begin{aligned}\|\mathcal{K}\,\nabla\,\hat{T}\,q\|^2_{0,K} &\leq \|\mathcal{K}\|_\infty(\mathcal{K}\,\nabla\,\hat{T}\,q, \nabla\,\hat{T}\,q)_K \\ &\leq c_{\max}\,(q, \hat{T}\,q)_K = c_{\max}\,(q - \Pi_K\,q, \hat{T}\,q)_K \\ &\leq c_{\max}\,\|q - \Pi_K\,q\|_{0,K}\|\hat{T}\,q\|_{0,K} \\ &\leq \frac{\kappa}{\pi}\,d_\Omega\,\|q - \Pi_K\,q\|_{0,K}\|\mathcal{K}\,\nabla\,\hat{T}\,q\|_{0,K}\,.\end{aligned}$$

Furthermore, it holds from (8.9) that $-\nabla\cdot(\mathcal{K}\,\nabla\,\hat{T}q)\,|_K = q - \Pi_K\,q$. Therefore, by definition (3.10) of $\|.\|_{\mathrm{div}}$, and observing that $1 \leq \kappa$, we get

$$\begin{aligned}\|\mathcal{K}\,\nabla\,\hat{T}\,q\|^2_{\mathrm{div}} &= \sum_{K\in\mathcal{T}_h}\left[\|\mathcal{K}\,\nabla\,\hat{T}\,q\|^2_{0,K} + d^2_\Omega\,\|q - \Pi_K\,q\|^2_{0,K}\right] \\ &\leq 2\,d^2_\Omega\,\max\left\{\left(\frac{\kappa}{\pi}\right)^2, 1\right\}\;\|q - \Pi\,q\|^2_{0,\Omega}\,,\end{aligned}$$

from which the bound (8.10) follows immediately. The bound (8.11) follows using the local Poincaré inequality (8.3) and the result (8.10). ◻

LEMMA 8.3. *Suppose $\mu \in \Lambda$. It follows that*

$$\frac{\sqrt{2}}{2}\,\|\mu\|_\Lambda \leq \sup_{v\in V}\frac{b(\mu, v)}{\|v\|_V} \leq \|\mu\|_\Lambda.$$

Proof. Choose arbitrary $\mu \in \Lambda$. The left-hand bound follows from equation (8.7) (with $\mathcal{K}$ as the identity matrix) to establish there exists $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$ with the properties that $\boldsymbol{\sigma}\cdot\boldsymbol{n}|_{\partial K} = \mu|_{\partial K}$ and $\frac{\sqrt{2}}{2}\|\boldsymbol{\sigma}\|_{\mathrm{div}} \leq \sup_{v\in V}\frac{b(\mu,v)}{\|v\|_V}$. The right-hand bound is (8.8) in the proof of Lemma 8.1. ◻

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