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Some observations on Babuška and Brezzi theories*

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Summary. Some observations are made on abstract error estimates for Galerkin approximations based on Babuška-Brezzi conditions. A basic error estimate due to Babuška is sharpened by means of an identity that $\|P\| = \|I - P\|$ for any nontrivial idempotent operator P. Some remarks are also made on the Brezzi's theory for mixed variational problems and their Galerkin approximations.

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1. Introduction

In this short note we report some simple observations on two basic abstract theories for the (quasi-)optimal approximation property of Galerkin (or Petrov-Galerkin) methods for general variational problems. In Sect. 2, we show that the fundamental abstract error estimate for general Galerkin projections, due to Babuška [2] and Babuška and Aziz [3], can be improved to an optimal form. In Sect. 3, we briefly discuss a theory due to Brezzi [4] on mixed variational formulations and its relationship with the theory of Babuška. Our discussions are mainly motivated by an identity on nontrivial idempotent operators in Hilbert spaces which will be presented in Sect. 4.

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2. Babuška theory

Let U and V be two Hilbert spaces, with inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$ respectively. Let $\mathcal{B}(\cdot,\cdot) : U \times V \mapsto \mathbb{R}$ be a continuous bilinear form

$$\mathcal{B}(u,v) \le \|\mathcal{B}\| \|u\|_U \|v\|_V. \tag{1}$$

Consider the following variational problem: Find $u \in U$ such that

$$\mathcal{B}(u,v) = \langle f, v\rangle, \quad \forall v \in V, \tag{2}$$

where $f \in V^*$ (the space of continuous linear functionals on V) and $\langle\cdot,\cdot\rangle$ is the usual pairing between V^* and V.

A basic result, due to Babuška is that the problem (2) is well posed if and only if the following conditions hold (see [3], [4]):

$$\inf_{u\in U}\sup_{v\in V}\frac{\mathcal{B}(u,v)}{\|u\|_U\|v\|_V} > 0, \qquad \inf_{v\in V}\sup_{u\in U}\frac{\mathcal{B}(u,v)}{\|u\|_U\|v\|_V} > 0, \tag{3}$$

furthermore if (3) hold, then

$$\inf_{u\in U}\sup_{v\in V}\frac{\mathcal{B}(u,v)}{\|u\|_U\|v\|_V} = \inf_{v\in V}\sup_{u\in U}\frac{\mathcal{B}(u,v)}{\|u\|_U\|v\|_V} \equiv \alpha > 0, \tag{4}$$

and the unique solution of (2) satisfies

$$\|u\|_U \le \frac{\|f\|_{V^*}}{\alpha}.$$

The condition like (3) and (4) is often known as the Babuška-Brezzi condition or BB-condition in short.

Let $U_h \subset U$ and $V_h \subset V$ be two nontrivial subspaces of U and V respectively. We consider the following variational problem: Find $u_h \in U_h$ such that

$$\mathcal{B}(u_h,v_h) = \langle f, v_h\rangle, \quad \forall v_h \in V_h. \tag{5}$$

The solution u_h of this problem is often known as the Galerkin (or Petrov–Galerkin) approximation of u. Usually in applications U_h and V_h are finite dimensional and the subscript h is related to certain discretization parameters (such as grid size and polynomial degree). According to (3) we have that the problem (5) is uniquely solvable if and only if the following conditions hold:

$$\inf_{u_h\in U_h}\sup_{v_h\in V_h}\frac{\mathcal{B}(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} = \inf_{v_h\in V_h}\sup_{u_h\in U_h}\frac{\mathcal{B}(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} = \alpha_h > 0. \tag{6}$$

If U_h and V_h are finite dimensional the above two conditions are reduced to one. A fundamental result for Galerkin approximation is as follows.

Theorem 1 (Babuška and Aziz, 1972 [3]). *Let (1), (3) and (6) hold. Then*

$$\|u-u_h\|_U \leq \left[\mathbf{1}+\frac{\|\mathcal{B}\|}{\alpha_h}\right]\inf_{w_h\in U_h}\|u-w_h\|_U. \tag{7}$$

As a consequence if the BB-conditions (6) are satisfied uniformly with respect to the parameter h, namely $\alpha_h \geq \alpha_0 > 0$ for some α_0 independent of h, then u_h is uniform (w.r.t. h), quasi-optimal approximation of u, namely

$$\|u-u_h\|_U \leq \left[\mathbf{1}+\frac{\|\mathcal{B}\|}{\alpha_0}\right]\inf_{w_h\in U_h}\|u-w_h\|_U. \tag{8}$$

While the estimate (8) is good enough in most applications, it is not aesthetically pleasing because of the presence of additional constant "$\mathbf{1}$" in its right hand side. It is not difficult to see that the constant "$\mathbf{1}$" can indeed be removed in the special case when $U=V$, $U_h=V_h$ and $\mathcal{B}(\cdot,\cdot)$ is symmetric. The main purpose of this paper is to show that the constant "$\mathbf{1}$" can in fact be removed in general.

Theorem 2. *Let (1), (3) and (6) hold. Then*

$$\|u-u_h\|_U \leq \frac{\|\mathcal{B}\|}{\alpha_h}\inf_{w_h\in U_h}\|u-w_h\|_U. \tag{9}$$

Proof. Consider the mapping $\Pi_h : U \mapsto U_h$ defined as $P_h u = u_h$. Using the fact that under the conditions of the theorem the problem (5) has a unique solution it is easy to see that this mapping is linear and idempotent, namely $P_h^2 = P_h$. The new twist in our proof is the identity

$$\|P_h\|_{\mathcal{L}(U,U)} = \|I-P_h\|_{\mathcal{L}(U,U)}, \tag{10}$$

which can be traced back to T. Kato [7] (see also Lemma 5 below). Applying this identity we get

$$\begin{aligned}\|u-u_h\|_U &= \|(I-P_h)(u-w_h)\|_U \leq \|I-P_h\|_{\mathcal{L}(U,U)}\|(u-w_h)\|_U\\ &= \|P_h\|_{\mathcal{L}(U,U)}\|u-w_h\|_U,\end{aligned}$$

where $w_h \in U_h$ is arbitrary. By (6) and (1) we get

$$\|P_h u\|_U \leq \frac{1}{\alpha_h}\sup_{v_h\in V_h}\frac{\mathcal{B}(u_h,v_h)}{\|v_h\|_V} = \frac{1}{\alpha_h}\sup_{v_h\in V_h}\frac{\mathcal{B}(u,v_h)}{\|v_h\|_V} \leq \frac{\|\mathcal{B}\|}{\alpha_h}\|u\|_U,$$

and the desired estimate (9) follows. □

3. Brezzi theory

Consider the mixed variational problem:

$$\begin{cases} a(u,v) + b(v,p) = \langle f, v\rangle, \ \forall v \in V, \\ b(u,q) \qquad\qquad\ = \langle g, q\rangle, \ \forall q \in Q, \end{cases} \tag{11}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous bilinear forms

$$\begin{aligned} &a(\cdot,\cdot) : V \times V \mapsto \mathbb{R};\ a(u,v) \le \|a\|\|u\|_V\|v\|_V,\ \forall u \in V, \forall v \in V, \\ &b(\cdot,\cdot) : V \times Q \mapsto \mathbb{R};\ b(v,q) \le \|b\|\|v\|_V\|q\|_Q,\ \ \forall v \in V, \forall q \in Q, \end{aligned}$$

and $f \in V^*$, $g \in Q^*$. A special theory was developed by Brezzi [4] for this type of problems. We shall now discuss about this theory.

Theorem 3. *[Brezzi [4]] The variational problem (11) is well posed if and only if the following BB-conditions hold*

$$\inf_{u\in V_0}\sup_{v\in V_0}\frac{a(u,v)}{\|u\|_V\|v\|_V} = \inf_{v\in V_0}\sup_{u\in V_0}\frac{a(u,v)}{\|u\|_V\|v\|_V} \equiv \alpha > 0, \tag{12}$$

where $V_0 = \{v \in V : b(v,q) = 0,$ *for all* $q \in Q\}$, *and*

$$\inf_{q\in Q}\sup_{v\in V}\frac{b(v,q)}{\|v\|_V\|q\|_Q} \equiv \beta > 0. \tag{13}$$

Furthermore, under the conditions of (13) and (12), the unique solution $(u,p) \in V \times Q$ *of (11) satisfies*

$$\|(u,p)\|_{V\times Q} \le \mathcal{K}(\alpha^{-1}, \beta^{-1}, \|a\|)\|(f,g)\|_{V^*\times Q^*}, \tag{14}$$

where $\mathcal{K}(\cdot,\cdot,\cdot)$ *is a function which is increasing in each variable.*

Let us now discuss the relationship between the Brezzi theory and Babuška theory. Setting $\mathcal{B}((u,p),(v,q)) = a(u,v) + b(v,p) + b(u,q)$, then (11) is obviously equivalent to the following problem

$$\mathcal{B}((u,p),(v,q)) = \langle f, v\rangle + \langle g, q\rangle, \quad \forall (v,q) \in V \times Q. \tag{15}$$

Then, following Babuška theory, this mixed variational problem (11) is well-posed if and only if the following BB-conditions hold:

$$\begin{aligned} &\inf_{(u,p)\in V\times Q}\ \sup_{(v,q)\in V\times Q}\frac{\mathcal{B}((u,p),(v,q))}{\|(u,p)\|_{V\times Q}\|(v,q)\|_{V\times Q}} = \\ &\inf_{(v,q)\in V\times Q}\ \sup_{(u,p)\in V\times Q}\frac{\mathcal{B}((u,p),(v,q))}{\|(u,p)\|_{V\times Q}\|(v,q)\|_{V\times Q}} \equiv \gamma > 0, \end{aligned} \tag{16}$$

where

$$\|(v,q)\|^2_{V\times Q} = \|v\|^2_V + \|q\|^2_Q, \quad \forall (v,q) \in V \times Q.$$

Combining Babuška theory and Brezzi theory, we see that the BB-conditions (16) are equivalent to BB-conditions (12) and (13) and the main constants resulted from these two theories are related by

$$\gamma \geq \frac{1}{\mathcal{K}(\alpha^{-1}, \beta^{-1}, \|a\|)}. \tag{17}$$

In view of the above relation, we are interested in obtaining sharp estimate for $\mathcal{K}$ in terms of α and β. Let us demonstrate now, by using the identity for idempotent operator, we are able to derive some interesting estimate (see (20) below).

The arguments we shall use here have much in common with those given in the original pioneering work of Brezzi [4] (see also Arnold [1] for another interesting and elegant argument), but we pay more attention to the quantitative estimates for underlying constants.

To begin with our derivation, we first define $A : V \mapsto V$ and $B : V \mapsto Q$, $B^* : Q \mapsto V$:

$$\begin{aligned} &(Au, v)_V = a(u, v), \\ &(Bv, q)_Q = (v, B^*q)_V = b(v, q), \quad \forall u \in V,\ \forall v \in V,\ \forall q \in Q. \end{aligned}$$

With an abuse of notation, let $f \in V$ and $g \in Q$ be the Riesz representations of the original $f \in V^*$ and $g \in Q^*$ respectively, we then have

$$\begin{aligned} Au + B^*p &= f, \\ Bu &= g. \end{aligned}$$

We note that $V_0 = \ker(B)$ and the condition (13) means that B^* is injective. Hence B is surjective and $B : V_0^\perp \mapsto Q$ (the restriction of B) and $B^* : Q \mapsto V_0^\perp$ are isomorphic and

$$\|B^{-1}\|_{\mathcal{L}(Q, V_0^\perp)} = \|(B^*)^{-1}\|_{\mathcal{L}(V_0^\perp, Q)} = \beta^{-1}. \tag{18}$$

Let $\varPi : V \mapsto V_0$ be the orthogonal projection. Then, by (13), $A_0 \equiv \varPi A : V_0 \mapsto V_0$ is an isomorphism satisfying

$$\|A_0^{-1}\| = \alpha^{-1}. \tag{19}$$

By means of these isomorphic properties, there are unique $u_1 \in V_0^\perp$, $u_0 \in V_0$ and $p \in Q$ satisfying

$$\begin{aligned} Bu_1 &= g, \\ A_0 u_0 &= \varPi(f - Au_1), \\ B^*p &= f - Au = (I - \varPi)(I - P)(f - Au_1), \end{aligned}$$

where $u = u_0 + u_1$ and $P = A\varPi A_0^{-1}\varPi$. Obviously (u, p) is the desired solution of (11). Note that P is idempotent and hence, by Lemma 5 (see

below), $\|I-P\| = \|P\| \le \alpha^{-1}\|a\|$. Thus, by (18) and (19), we deduce that (14) is satisfied with
(20)

$$\mathcal{K}^2 = \frac{1}{2}\left(\kappa_{11}+\kappa_{22}+\sqrt{(\kappa_{11}-\kappa_{22})^2+4\kappa_{12}^2}\right) \le \kappa_{12}+\max(\kappa_{11},\kappa_{22}).$$

where, with $\kappa = \beta^{-1}\|a\|$,

$$\kappa_{11} = \alpha^{-2}(1+\kappa^2), \quad \kappa_{22} = \kappa^2\kappa_{11}+\beta^{-2}, \quad \kappa_{12} = \kappa\kappa_{11}.$$

We have thus established the well-posedness of the problem (11) under the BB-conditions (12) and (13) together with an explicit estimate for $\mathcal{K}$ given by (20).

We shall now briefly discuss the Galerkin approximation for (11). We consider two nontrivial finite dimensional subspaces $V_h \subset V$ and $Q_h \subset Q$ and the following variational problem:

$$\begin{cases} a(u_h,v_h)+b(v_h,p_h) = \langle f,v_h\rangle \ \forall v_h \in V_h, \\ b(u_h,q_h) \qquad\qquad\qquad = \langle g,q_h\rangle \ \forall q_h \in Q_h. \end{cases} \tag{21}$$

Theorem 4. *Let* $V_{h,0} = \{v_h \in V_h : b(v_h,q_h) = 0, \quad \forall q \in Q_h\}$ *and assume that the following BB-conditions hold*

$$\inf_{u_h\in V_{h,0}} \sup_{v_h\in V_{h,0}} \frac{a(u_h,v_h)}{\|u_h\|_V\|v_h\|_V} \equiv \alpha_h > 0, \tag{22}$$

and

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{b(v_h,q_h)}{\|v_h\|_V\|q_h\|_Q} \equiv \beta_h > 0. \tag{23}$$

Then the discrete problem (21) is well-posed and

$$\|(u-u_h,p-p_h)\|_{V\times Q} \le \\ (\|a\|+\|b\|)\mathcal{K}(\alpha_h^{-1},\beta_h^{-1},\|a\|) \inf_{(v_h,q_h)\in V_h\times Q_h} \|(u-v_h,p-q_h)\|_{V\times Q}.$$

Furthermore if $\alpha_h \ge \alpha_0$ *and* $\beta_h \ge \beta_0$ *for some positive constants* α_0 *and* β_0*, then*

$$\|(u-u_h,p-p_h)\|_{V\times Q} \le \\ (\|a\|+\|b\|)\mathcal{K}(\alpha_0^{-1},\beta_0^{-1},\|a\|) \inf_{(v_h,q_h)\in V_h\times Q_h} \|(u-v_h,p-q_h)\|_{V\times Q}.$$

We would like to remark that the above approximation result is a direct consequence of Theorem 2, Theorem 3, (17) and the obvious estimate that $\|\mathcal{B}\| \leq \|a\|+\|b\|$. In some of the existing works, another approach in proving Theorem 4 is considered (see [5], [6])) and some additional arguments are needed, first to establish estimate for $u-u_h$ and then for $p-p_h$. This more refined analysis can be interesting in some applications (for example, when the BB-conditions are not uniformly satisfied), but it may not be necessary in general.

4. An identity for nontrivial idempotent operator

For completeness, we shall now describe a general result related to the identity (10) and include a (new) proof. This result can be traced back to Kato [7] and a more general result can be found in Zikatanov [8].

Lemma 5. *Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \mapsto H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then the following identity holds*

$$\|P\|_{\mathcal{L}(H,H)} = \|I-P\|_{\mathcal{L}(H,H)}. \tag{24}$$

Proof. We first prove the theorem when $\dim H = 2$. Then both P and $I-P$ have to be rank 1, namely $Pv = (b,v)_H a$ and $(I-P)v = (d,v)_H c$ for some fixed nonzero $a,b,c,d \in H$ satisfying $(a,b)_H = (c,d)_H = 1$ and for all $v \in H$ we also have

$$v = Pv + (I-P)v = (b,v)_H\, a + (d,v)_H\, c.$$

A simple manipulation of the above identities yields that

$$\|a\|_H^2\|b\|_H^2 = \|c\|_H^2\|d\|_H^2 = 1-(a,c)_H(b,d)_H.$$

The desired identity then follows because of the following obvious relations:

$$\|P^*P\|_{\mathcal{L}(H,H)} = \|a\|_H^2\|b\|_H^2 \quad \text{and}$$
$$\|(I-P)^*(I-P)\|_{\mathcal{L}(H,H)} = \|c\|_H^2\|d\|_H^2.$$

In general, for any given $x \in H$ such that $\|x\| = 1$, we consider a subspace $X = \operatorname{span}\{x,\ Px\}$. We note that X is invariant with respect to P and $I-P$. If $\dim X = 1$, then we must have $(I-P)x = 0$. If $\dim X = 2$, we have from two dimensional result just proved, $\|(I-P)x\|_X \leq \|P\|_X$. In any case, we have

$$\|(I-P)x\|_H = \|(I-P)x\|_X \leq \|P\|_{\mathcal{L}(X,X)} \leq \|P\|_{\mathcal{L}(H,H)},$$

which implies $\|I-P\|_{\mathcal{L}(H,H)} \leq \|P\|_{\mathcal{L}(H,H)}$. Similarly $\|P\|_{\mathcal{L}(H,H)} \leq \|I-P\|_{\mathcal{L}(H,H)}$. This completes the proof. $\square$

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References

1. D. N. Arnold: Discretization by finite elements and a model parameter problem. Numer. Math. **37**, 405–421 (1981)
2. I. Babuška: The finite element method with Lagrangian multipliers. Numer. Math. **20**, 179–192 (1973)
3. I. Babuška, A. K. Aziz: Lectures on the mathematical foundations of the finite element method. University of Maryland, College Park, Washington DC, 1972. Technical Note BN-748
4. F. Brezzi: On the existence, uniqueness and approximation of saddle–point problems arising from Lagrange multipliers. R.A.I.R.O Anal. Numer. **R2**, 129–151 (1974)
5. F. Brezzi, M. Fortin: Mixed and hybrid finite element methods. New York: Springer 1991
6. V. Girault, P.-A. Raviart: Finite element methods for Navier–Stokes equations. Berlin: Springer 1986. Theory and algorithms
7. T. Kato: Estimation of iterated matrices with application to von Neumann condition. Numer. Math. **2**, 22–29 (1960)
8. L. Zikatanov: Singular values of idempotent operators. Technical report, (preprint), (2000)