A MIXED FINITE ELEMENT METHOD FOR 2-nd ORDER ELLIPTIC PROBLEMS

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1. INTRODUCTION

Let Ω be a bounded open subset of R^n with a Lipshitz continuous boundary Γ. We consider the 2nd order elliptic model problem

(1.1) $$\begin{cases} -\Delta u = f & \text{in } \Omega , \\ u = 0 & \text{on } \Gamma , \end{cases}$$

where f is a given function of the space $L^2(\Omega)$. A variational form of problem (1.1), known as the *complementary energy principle*, consists in finding $\underset{\sim}{p} = \underset{\sim}{\text{grad}}\, u$ which minimizes the *complementary energy functional*

(1.2) $$I(\underset{\sim}{q}) = \frac{1}{2} \int_\Omega |\underset{\sim}{q}|^2 \, dx$$

over the affine manifold $\underset{\sim}{W}$ of vector-valued functions $\underset{\sim}{q} \in (L^2(\Omega))^n$ which satisfy the *equilibrium equation*

(1.3) $$\text{div}\, \underset{\sim}{q} + f = 0 \quad \text{in} \quad \Omega.$$

The use of complementary energy principle for constructing finite element discretizations of elliptic problems has been first advocated by Fraeijs de Veubeke [5],[6],[7]. The so-called *equilibrium method* consists first in constructing a finite-dimensional submanifold $\underset{\sim}{W}_h$ of $\underset{\sim}{W}$ and then in finding $\underset{\sim}{p}_h \in \underset{\sim}{W}_h$ which minimizes the complementary energy functional $I(\underset{\sim}{q})$ over the affine manifold $\underset{\sim}{W}_h$. For 2nd order elliptic problems, the numerical analysis of the equilibrium method has been

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made by Thomas [19],[20].

Now, we note that the practical construction of the submanifold $\underset{\sim}{W}_h$ is not in general a simple ptoblem since it requires a search for explicit solutions of the equilibrium equation (1.3) in the whole domain Ω.

In order to avoid the above difficulty, we can use a more general variational principle, known in elasticity theory as the *Hellinger-Reissner principle*, in which the constraint (1.3) has been removed at the expense however of introducing a Lagrange multiplier. This paper will be devoted to the study of a finite element method based on this variational principle. In fact, this so-called mixed method has been found very useful in some practical problems and refer to [17] for an application to the numerical solution of a nonlinear problem of radia tive transfer.

For some general results concerning mixed methods, we refer to Oden [12],[13], Oden & Reddy [14], Reddy [16]. Mixed methods for solving 4th order elliptic equations have been particularly analyzed: see Brezzi & Raviart [2], Ciarlet & Raviart [4], Johnson [9],[10],and Miyoshi [11]. For related results we refer also to Haslinger & Hlåvaĉek [8].

An outline of the paper is as follows. In § 2, we derive the mixed variational formulation of problem (1.1) and we define the related discrete elements, and in § 4, the error analysis of the associated finite element method is made. Finally, in § 5, we generalize the results of §§ 3,4 to mixed methods using rectangular elements.

Let us describe some of the notations used throughout this paper. Given an integer $m \geq 0$,

$$H^m(\Omega) = \{v \in L^2(\Omega);\ \partial^\alpha v \in L^2(\Omega),\ |\alpha| \leq m\}$$

denotes the usual Sobolev space provided the norm and semi-norm

$$\|v\|_{m,\Omega} = \Big(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha v|^2 dx\Big)^{\frac{1}{2}},\quad |v|_{m,\Omega} = \Big(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^2 dx\Big)^{\frac{1}{2}}.$$

Given a vector-valued function $\underset{\sim}{q} = (q_1,\dots\dots,q_n) \in (H^m(\Omega))^n$, we set:

$$\|\underset{\sim}{q}\|_{m,\Omega} = \Big(\sum_{i=1}^{n} \|q_i\|^2_{m,\Omega}\Big)^{\frac{1}{2}},\quad |\underset{\sim}{q}|_{m,\Omega} = \Big(\sum_{i=1}^{n} |q_i|^2_{m,\Omega}\Big)^{\frac{1}{2}}.$$

We denote by $H^{\frac{1}{2}}(\Gamma)$ the space of the traces $v|_\Gamma$ over Γ of the functions $v \in H^1(\Omega)$.

2. THE MIXED MODEL

In order to derive the appropriate variational form of problem (1.1), we introduce the space

$$(2.1) \qquad \underset{\sim}{H}(\mathrm{div}\ ;\ \Omega) = \{\underset{\sim}{q} \in (L^2(\Omega))^n\ ;\ \mathrm{div}\ \underset{\sim}{q} \in L^2(\Omega)\}$$

provided with the norm

$$(2.2) \qquad \|\underset{\sim}{q}\|_{\underset{\sim}{H}(\mathrm{div};\Omega)} = \left(\|\underset{\sim}{q}\|^2_{o,\Omega} + \|\mathrm{div}\ \underset{\sim}{q}\|^2_{o,\Omega}\right)^{\frac{1}{2}} .$$

Given a vector-valued function $\underset{\sim}{q} \in \underset{\sim}{H}(\mathrm{div}\ ;\ \Omega)$, we may define its normal component $\underset{\sim}{q} \cdot \underset{\sim}{\nu} \in H^{-\frac{1}{2}}(\Gamma)$ where $H^{-\frac{1}{2}}(\Gamma)$ is the dual space of $H^{\frac{1}{2}}(\Gamma)$ and $\underset{\sim}{\nu}$ is the unit outward normal along Γ. Moreover, we have Green's formula

$$(2.3) \qquad \forall\, v \in H^1(\Omega),\ \int_\Omega \{\underset{\sim\sim}{\mathrm{grad}}\, v \cdot \underset{\sim}{q} + v\, \mathrm{div}\, \underset{\sim}{q}\}dx = \int_\Gamma v\ \underset{\sim}{q}\cdot\underset{\sim}{\nu}\, d\gamma$$

where the integral $\int_\Gamma$ represents the duality between the spaces $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$.

We next define *problem* (P). Find a pair of functions $(\underset{\sim}{p},u) \in \underset{\sim}{H}(\mathrm{div}\ ;\ \Omega) \times L^2(\Omega)$ such that

$$(2.4) \qquad \forall \underset{\sim}{q} \in \underset{\sim}{H}(\mathrm{div}\ ;\ \Omega)\ ,\ \int_\Omega \underset{\sim}{p}\cdot\underset{\sim}{q}\, dx + \int_\Omega u\, \mathrm{div}\, \underset{\sim}{q}\, dx = 0\ ,$$

$$(2.5) \qquad \forall v \in L^2(\Omega)\ ,\ \int_\Omega v(\mathrm{div}\ \underset{\sim}{p} + f)dx = 0.$$

Theorem 1. The problem (P) has a unique solution $(\underset{\sim}{p},u) \in \underset{\sim}{H}(\mathrm{div}\ ;\ \Omega)\times L^2(\Omega)$. In addition, u is the solution of the problem (1.1) and we have (2.6) $\underset{\sim}{p} = \underset{\sim\sim}{\mathrm{grad}}\ u$.

Proof. Let us first check the uniqueness of the solution of problem (P). Hence, assume that $f = 0$; from (2.5), we get $\mathrm{div}\ \underset{\sim}{p} = 0$. Taking $\underset{\sim}{q} = \underset{\sim}{p}$ in (2.4), we obtain $p = 0$. Therefore, we have

$$(2.7) \qquad \forall q \in \underset{\sim}{H}(\mathrm{div}\ ;\ \Omega)\ ,\ \int_\Omega u\ \mathrm{div}\ \underset{\sim}{q}\ dx = 0$$

Now, let $w \in H^1(\Omega)$ be a function such that

$$\Delta w = u \quad \text{in} \quad \Omega.$$

Then, by choosing $\underset{\sim}{q} = \underset{\sim}{\text{grad}}\, w$ in (2.7), we get $u = 0$.

It remains only to show that the pair ($\underset{\sim}{p} = \underset{\sim}{\text{grad}}\, u, u$) is a solution of problem (P), where u is the solution of problem (1.1). On the one hand, we have

$$\text{div}\, \underset{\sim}{p} + f = \Delta u + f = 0.$$

On the other hand, since $u = 0$ on Γ, we get by using the Green's formula

(2.8) $$\int_\Omega \{\underset{\sim}{p}\cdot\underset{\sim}{q} + u \,\text{div}\, \underset{\sim}{q}\}dx = \int_\Omega uq\cdot\nu \, d\gamma = 0$$

Remark 1. One can easily check that the solution (p,u) of problem (P) may be characterized as the unique seddle-point of the quadratic functional

$$L(\underset{\sim}{q},v) = I(\underset{\sim}{q}) + \int_\Omega v(\text{div}\, \underset{\sim}{q} + f)dx$$

over the space $\underset{\sim}{H}(\text{div} \,;\, \Omega) \times L^2(\Omega)$, i.e.,

$$\forall \underset{\sim}{q} \in \underset{\sim}{H}(\text{div} \,;\, \Omega),\ \forall v \in L^2(\Omega),\ L(\underset{\sim}{p},v) \le L(\underset{\sim}{p},u) \le L(\underset{\sim}{q},u)$$

Hence, the function u is the Lagrange multiplier associated with the constraint $\text{div}\, \underset{\sim}{p} + f = 0$.

Let us now introduce a general method of discretization of problem (1.1) based on the mixed variational formulation (2.4),(2.5). We are given two finite-dimensional spaces $\underset{\sim}{Q}_h$ and V_h such that

(2.8) $$\underset{\sim}{Q}_h \subset \underset{\sim}{H}(\text{div} \,;\, \Omega) \quad ; \quad V_h \subset L^2(\Omega).$$

Then we define *problem* (P_h) : Find a pair of functions ($\underset{\sim}{p}_h$, u_h) $\in \underset{\sim}{Q}_h \times V_h$ such that

(2.9) $$\forall\, \underset{\sim}{q}_h \in \underset{\sim}{Q}_h \,,\ \int_\Omega \underset{\sim}{p}_h\cdot\underset{\sim}{q}_h dx + \int_\Omega u_h \,\text{div}\, \underset{\sim}{q}_h \, dx = 0 \,,$$

(2.10) $$\forall\, v_h \in V_h \,,\ \int_\Omega v_h(\text{div}\, \underset{\sim}{p}_h + f)dx = 0 \,.$$

Using a general result of Brezzi [1 , Theorem 2.1] concerning the approximation of variational problems, we get the following

Theorem 2. Assume that

$$(2.11)\quad \left\{\begin{array}{l} \underset{\sim}{q}_h \in Q_h \\ \forall\, v_h \in V_h \,,\; \displaystyle\int v_h \operatorname{div} \underset{\sim}{q}_h \, dx = 0 \end{array}\right. \Rightarrow \operatorname{div} \underset{\sim}{q}_h = 0$$

and that there exists a constant $\alpha > 0$ such that

$$\forall\, v_h \in V_h \,,\; \sup_{\underset{\sim}{q}_h \in \underset{\sim}{Q}_h} \frac{\int_\Omega v_h \operatorname{div} \underset{\sim}{q}_h \, dx}{\|\underset{\sim}{q}_h\|_{\underset{\sim}{H}(\operatorname{div}\,;\,\Omega)}} \geq \alpha \|v_h\|_{o,\Omega} \,.$$

Then the problem (P_h) has a unique solution $(\underset{\sim}{p}_h \,, u_h) \in \underset{\sim}{Q}_h \times V_h$ and there exists a constant $\tau > 0$ which dependes only on α such that

$$(2.13)\quad \left\{\begin{array}{l} \|\underset{\sim}{p}-\underset{\sim}{p}_h\|_{\underset{\sim}{H}(\operatorname{div};\Omega)} + \|u-u_h\|_{o,\Omega} \leq \\ \leq \tau \left\{ \displaystyle\inf_{\underset{\sim}{q}_h \in Q_h} \|\underset{\sim}{p}-\underset{\sim}{q}_h\|_{\underset{\sim}{H}(\operatorname{div}\,;\,\Omega)} + \inf_{v_h \in V_h} \|u-v_h\|_{o,\Omega} \right\} . \end{array}\right.$$

Remark 2. Define the operator $\underset{\sim}{\nabla}_h \in L(V_h \,;\, \underset{\sim}{Q}_h)$ by

$$(2.14)\quad \forall\, v_h \in V_h \,,\; \forall\, \underset{\sim}{q}_h \in \underset{\sim}{Q}_h \,,\; \int_\Omega \underset{\sim}{\nabla}_h v_h \cdot \underset{\sim}{q}_h \, dx = - \int_\Omega v_h \operatorname{div} \underset{\sim}{q}_h \, dx \,.$$

Clearly, $\underset{\sim}{\nabla}_h$ can be viewed as an approximation of the operator $\underset{\sim}{\text{grad}}$.Now, the function u_h may be characterized as the unique solution of the following problem : Find $u_h \in V_h$ such that

$$(2.15)\quad \forall\, v_h \in V_h \,,\; \int_\Omega \underset{\sim}{\nabla}_h u_h \cdot \underset{\sim}{\nabla}_h v_h \, dx = \int_\Omega f v_h \, dx \,.$$

In fact, from the assumption (2.11) and (2.12), it follows that problem (2.15) has a unique solution $u_h \in V_h$. Moreover, it is readily seen that the pair $(\underset{\sim}{\nabla}_h u_h, u_h)$ is the solution of problem (P_h).

Since in general $V_h \not\subset H^1_o(\Omega)$, (2.15) is *non-conforming displacement* model for solving problem (1.1). For other non-conforming methods based on hybrid models, we refer to [15].

It remains to construct the finite-dimensional subspaces $\underset{\sim}{Q}_h$ and V_h of the spaces $\underset{\sim}{H}(\mathrm{div}\ ;\ \Omega)$ $L^2(\Omega)$ respectively so that they satisfy "good" approximation properties and the compatibility conditions (2.11) and (2.12) with a constant α independent of the parameter h.

For convenience, we shall assume in the sequel that $\bar{\Omega}$ is a bounded *polygon* of $\mathbb{R}^2$. We then establish a triangulation $\mathcal{K}_h$ of $\bar{\Omega}$ made up with triangles and parallelograms K whose diameters are $\leq h$. We begin by consrtuction finite-dimensional $\underset{\sim}{Q}_h$ of the space $\underset{\sim}{H}(\mathrm{div};\Omega)$. Given a finite element $K \in \mathcal{K}_h$, we denote by $\underset{\sim}{\nu}_K$ the unit outward normal along the boundary ∂K of K. Using the Green's formula (2.3) in each $K \in \mathcal{K}_h$, one can easily prove that a function $\underset{\sim}{q} \in (L^2(\Omega))^2$ belongs to the space $\underset{\sim}{H}(\mathrm{div}\ ;\ \Omega)$ if and only if the two following conditions hold :

(i) for all $K \in \mathcal{K}_h$, the restriction $\underset{\sim}{q}_{|K}$ of $\underset{\sim}{q}$ to the set K belongs to the space $\underset{\sim}{H}(\mathrm{div}\ ;\ K)$;

(ii) for any pair of adjacent elements K_1, $K_2 \in \mathcal{K}_h$, we have the reciprocity relation

$$(2.16) \qquad \underset{\sim}{q}_1 \cdot \underset{\sim}{\nu}_{K_1} + \underset{\sim}{q}_2 \cdot \underset{\sim}{\nu}_{K_2} = 0 \text{ on } K' = K_1 \subset K_2\ ,$$

where $\underset{\sim}{q}_i$ stands for $\underset{\sim}{q}_{|K_i}$, $i = 1,2$.

Hence the functions of $\underset{\sim}{Q}_h$ will be assumed to be smooth in each element $K \in \mathcal{K}_h$ and to satisfy the reciprocity conditions.

3. MIXED TRIANGULAR ELEMENTS

In this § , we shall assume that K is a triangle. With K and for any integer $k \geq 0$, we shall associate a space $\underset{\sim}{Q}_K$ of vector-valued functions $\underset{\sim}{q} \in \underset{\sim}{H}(\mathrm{div}\ ;\ K)$ such that :

(i) $\mathrm{div}\ \underset{\sim}{q}$ is a polynomial of degree $\leq k$;

(ii) the restriction of $\underset{\sim}{q} \cdot \underset{\sim}{\nu}_K$ to any side K' of K is a polynomial of degree $\leq k$.

We begin by introducing the space $\hat{\underset{\sim}{Q}}$ associated with the unit right triangle $\hat{K}$ in the (ξ,η)-plane whose vertices are $\hat{a}_1 = (1,0)$, $\hat{a}_2 = (0,1)$, $\hat{a}_3 = (0,0)$. Let us first give some notations. We denote by P_k the space of all polynomials of degree $\leq k$ in the two variables ξ,η and by $\hat{S}_k$ the space of all functions defined over $\partial\hat{K}$ whose restrictions to any side $\hat{K}'$ of $\hat{K}$ are polynomials of degree $\leq k$. Given a point $\hat{x} = (\xi,\eta)$ of R^2, we denote by $\lambda_i = \lambda_i(\hat{x})$, $1 \leq i \leq 3$, the barycentric coordinates of $\hat{x}$ with respect to the vertices $\hat{a}_i$ of $\hat{K}$.

Now, the space $\hat{\underset{\sim}{Q}}$ is required to satisfy the following properties:

(3.1) $$(P_k)^2 \subset \hat{\underset{\sim}{Q}}\ ;$$

(3.2) $$\dim(\hat{\underset{\sim}{Q}}) = (k+1)(k+3)\ ;$$

(3.3) $$\forall \hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}\ ,\ \mathrm{div}\ \hat{\underset{\sim}{q}} = \frac{\partial \hat{q}_2}{\partial \xi} + \frac{\partial \hat{q}_2}{\partial \eta} \in P_k\ ;$$

(3.4) $$\forall \hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}\ ,\ \hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}} \in \hat{S}_k \quad (\text{where } \hat{\underset{\sim}{\nu}} \text{ stands for } \underset{\sim}{\nu}_{\hat{K}})\ ;$$

(3.5) $$\hat{\underset{\sim}{Q}}_o = \{\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}\ ;\ \mathrm{div}\ \hat{\underset{\sim}{q}} = 0\} \subset (P_k)^2\ .$$

Lemma 1. *Assume that the conditions* (3.2)-(3.5) *hold. Then a function* $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ *is uniquely determined by:*

(a) *the values of* $\hat{\underset{\sim}{q}} \cdot \underset{\sim}{\nu}$ *at* (k+1) *distinct points of each side* $\hat{K}'$ *of* $\hat{K}$;

(b) *the moments of order* $\leq k-1$ *of* $\hat{\underset{\sim}{q}}$, i.e.,

$$\int_{\hat{K}} \hat{q}_i \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3}\, d\hat{x}\ ,\ i = 1,2,\ a_1 + \alpha_2 + \alpha_3 = k-1.$$

Proof. Since by (3.2) the number of degrees of freedom (a),(b) is equal to the dimension of the space $\hat{\underset{\sim}{Q}}$, it is sufficient to prove that a function $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ which satisfies the two conditions:

(3.6) $\quad \hat{q} \cdot \nu = 0$ at $(k+1)$ distinct points of each side $\hat{K}'$ of $\hat{K}$,

(3.7) $\quad \int_{\hat{K}} \hat{q}_i \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3}\, dx = 0,\ i = 1,2,\ \alpha_1 + \alpha_2 + \alpha_3 = k-1$

must vanish identically. In fact, conditions (3.4) and (3.6) imply $\hat{q} \cdot \hat{\nu} = 0$ on $\partial\hat{K}$. Hence, using (3.7) and applying the Green's formula (2.3) in $\hat{K}$, we obtain for all $\hat{\varphi} \in P_k$

$$\int_{\hat{K}} \hat{\varphi}\ \mathrm{div}\ \hat{q}\ d\hat{x} = - \int_{\hat{K}} \mathrm{grad}\ \hat{\varphi} \cdot \hat{q}\ d\hat{x} + \int_{\partial\hat{K}} \hat{\varphi}\ \hat{q} \cdot \hat{\nu}\ d\hat{\gamma} = 0\ .$$

Since, by (3.3), div $\hat{q} \in P_k$, we get div $\hat{q} = 0$ so that $\hat{q} \in \hat{Q}_o$.

Now, it follows from (3.5) that there exists a polynomial $\hat{w} \in P_{k+1}$ uniquely determined up to an additive constant such that

$$\hat{q} = \mathrm{curl}\ \hat{w} = \left(\frac{\partial \hat{w}}{\partial \eta} , - \frac{\partial \hat{w}}{\partial \xi} \right)\ .$$

Note that $\hat{q} \cdot \hat{\nu} = \frac{\partial \hat{w}}{\partial \tau} = 0$ on $\partial\hat{K}$, where $\frac{\partial}{\partial \hat{\tau}}$ stands for the tangential derivative along $\partial\hat{K}$. Thus we may assume that $\hat{w} = 0$ on $\partial\hat{K}$ and we may write

$$\hat{w} = \lambda_1 \lambda_2 \lambda_3 \hat{z}, \quad \hat{z} \in P_{k-2} \quad (\hat{z} = 0 \quad \text{for } k = 0,1)\ .$$

Using again (3.7), we obtain for any $\hat{r} \in (P_{k-1})^2$

$$0 = \int_{\hat{K}} \hat{q} \cdot \hat{r}\ d\hat{x} = \int_{\hat{K}} \mathrm{curl}\ \hat{w} \cdot \hat{r}\ d\hat{x} = \int_{\hat{K}} \hat{w}\ \mathrm{curl}\ \hat{r}\ d\hat{x} = \int_{\hat{K}} \lambda_1 \lambda_2 \lambda_3 \hat{z}\ \mathrm{curl}\ \hat{r}\ d\hat{x},$$

where curl $\hat{r} = \frac{\partial \hat{r}_2}{\partial \xi} - \frac{\partial \hat{r}_1}{\partial \eta} \in P_{k-2}$. Clearly, we can choose $\hat{r}$ so that $\hat{z} =$ curl $\hat{r}$ and then

$$\int_{\hat{K}} \lambda_1 \lambda_2 \lambda_3 \hat{z}^2\, d\hat{x} = 0\ .$$

Therefore, we get $\hat{z} = 0$ so that $\hat{w} = 0$ and $\hat{q} = \mathrm{curl}\ \hat{w} = 0$.

Remark 3. As regards the degrees of freedom of a function $\hat{q} \in \hat{Q}$, one could have equivalently specified the moments of order $\leq k$

$$\int_{\hat{K}'} \hat{\varphi}\ \hat{q} \cdot \hat{\nu}\ d\hat{\gamma}\ , \quad \hat{\varphi} \in P_k$$

of $\hat{q}\cdot\hat{\nu}$ on the side $\hat{K}'$ instead of its values at (k+1) distinct points of $\hat{K}'$.

Let us give some examples of spaces $\hat{Q}$.

Example 1. Let $k \geq 0$ be an *even* integer; we define $\hat{Q}$ to be the space of all functions $\hat{q}$ of the form

$$(3.8)\quad \begin{cases} \hat{q}_1 = \mathrm{pol}_k(\xi,\eta) + \alpha_0\xi^{k+1} + \alpha_1\xi^k + \ldots + \alpha_{\frac{k}{2}}\,\xi^{\frac{k}{2}+1}\eta^{\frac{k}{2}} \\ \hat{q}_2 = \mathrm{pol}_k(\xi,\eta) + \beta_0\eta^{k+1} + (\beta_1\xi\eta^k + \ldots + \beta_{\frac{k}{2}}\,\xi^{\frac{k}{2}}\eta^{\frac{k}{2}+1} \end{cases}$$

with

$$(3.9)\qquad \sum_{i=0}^{\frac{k}{2}} (-1)^i(\alpha_i - \beta_i) = 0$$

In (3.8), $\mathrm{pol}_k(\xi,\eta)$ denotes any polynomial of degree k in the two variable ξ,η. Clearly, conditions (3.1),(3.2) hold. Next $\hat{q}\cdot\hat{\nu}$ is obviously a polynomial of degree $\leq k$ on each side $\xi = 0$ and $\eta = 0$ of $\hat{K}$. On the other hand, it follows from (3.9) that $\hat{q}\cdot\hat{\nu}$ is also a polynomial of degree $\leq k$ on the side $\xi + \eta = 1$. Finally, we have

$$\mathrm{div}\ \hat{q} = \mathrm{pol}_{k-1}(\xi,\eta) + \sum_{i=0}^{\frac{k}{2}} (k+1-i)(\alpha_i\xi^{k-i}\eta^i + \beta_i\xi^i\eta^{k-i}) \in P_k$$

so that $\mathrm{div}\ \hat{q} = 0$ implies

$$\begin{cases} \alpha_i = \beta_i = 0\ ,\ 0 \leq i \leq \frac{k}{2} - 1\ , \\ \alpha_{\frac{k}{2}} + \beta_{\frac{k}{2}} = 0 \end{cases}$$

and, by the condition (3.9)

$$\alpha_i = \beta_i = 0 \quad , \quad 0 \leq i \leq \frac{k}{2} \quad ,$$

Hence, hypotheses (3.1)-(3.5) hold.

Consider for instance the case k = 0. Then a function $\hat{q} \in \hat{Q}$ is of the form

$$(3.10)\qquad \begin{cases} \hat{q}_1 = a_0 + a_1\xi \\ \hat{q}_2 = b_0 + b_1\eta \end{cases} \quad , \quad a_1 = b_1 \quad ,$$

and by Lemma 1, the degrees of freedom of $\hat{\underset{\sim}{q}}$ may be chosen as the values of $\hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\nu}}$ at the midpoints of the sides of the triangle $\hat{K}$.

Example 2. Now, let $k \geq 1$ be an *odd* integer; we then define $\hat{\underset{\sim}{Q}}$ to be the space of all functions $\hat{\underset{\sim}{q}}$ of the form

$$(3.11)\qquad \begin{cases} \hat{q}_1 = \mathrm{pol}_k(\xi,\eta) + \alpha_o\xi^{k+1}+\alpha_1\xi^k\eta+\ldots+\alpha_{\frac{k+1}{2}}\xi^{\frac{k+1}{2}}\eta^{\frac{k+1}{2}} , \\ \\ \hat{q}_2 = \mathrm{pol}_k(\xi,\eta) + \beta_o\eta^{k+1}+\beta_1\xi\eta^k+\ldots+\beta_{\frac{k+1}{2}}\xi^{\frac{k+1}{2}}\eta^{\frac{k+1}{2}} , \end{cases}$$

with

$$(3.12)\qquad \sum_{i=0}^{\frac{k+1}{2}} (-1)^i\alpha_i = \sum_{i=0}^{\frac{k+1}{2}} (-1)^i\beta_i = 0 .$$

Here again, one can easily check that conditions (3.1)-(3.5) hold. For $k = 1$, a function $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ is of the form

$$(3.13)\qquad \begin{cases} \hat{q}_1 = a_o + a_1\xi + a_2\eta + a_3\xi(\xi+\eta) , \\ \\ \hat{q}_2 = b_o + b_1\xi + b_2\eta + b_3\eta(\xi+\eta) , \end{cases}$$

and, by Lemma 1, the degrees of freedom of $\hat{\underset{\sim}{q}}$ may be chosen as the values of $\hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\nu}}$ at two distinct points of each side of $\hat{K}$ (for the Gauss-Legendre points) and as the mean value

$$\frac{1}{\mathrm{mes}(\hat{K})}\int_{\hat{K}} \hat{\underset{\sim}{q}}\, d\hat{x} = \frac{1}{2}\int_{\hat{K}} \hat{\underset{\sim}{q}}\, d\hat{x}$$

of $\hat{\underset{\sim}{q}}$ over $\hat{K}$.

Next, consider any triangle K in the (x_1, x_2)-plane whose vertices are denoted by a_i, $1\leq i\leq 3$. We set :

$$(3.14)\qquad h_K = \text{diameter of } K,$$

$$(3.15)\qquad \rho_K = \text{diameter of the inscribed circle in } K.$$

Let $F_K : \hat{x} \to F_K(\hat{x}) = B_K\hat{x} + b_K$, $B_K \in \mathcal{L}(\mathbb{R}^2)$, $b_K \in \mathbb{R}^2$, be the unique affine invertible mapping such that

$$F_K(\hat{a}_i) = a_i \ , \ 1 \leq i \leq 3 .$$

With any *scalar* function $\hat{\varphi}$ defined on $\hat{K}$ (resp. on $\partial\hat{K}$), we associate the function φ defined on K (resp. on ∂K) by

$$(3.16) \qquad \varphi = \hat{\varphi} \circ F_K^{-1} \quad (\hat{\varphi} = \varphi \circ F_K) .$$

On the other hand, with any *vector-valued* function $\hat{\underset{\sim}{q}} = (\hat{q}_1 , \hat{q}_2)$ defined on $\hat{K}$, we associate the function $\underset{\sim}{q}$ defined on K by

$$(3.17) \qquad \underset{\sim}{q} = \frac{1}{J_K} B_K \hat{\underset{\sim}{q}} \circ F_K^{-1} \quad (\hat{\underset{\sim}{q}} = J_K B_K^{-1} \underset{\sim}{q} \circ F_K) ,$$

where $J_K = \det(B_K)$. We shall *constantly* use in the sequel the one-to-one *correspondences* $\hat{\varphi} \longleftrightarrow \varphi$ $\hat{\underset{\sim}{q}} \leftrightarrow \underset{\sim}{q}$

The choice of the transformation (3.17) is based on the followong standard result.

Lemma 2. *For any function* $\hat{\underset{\sim}{q}} \in (H^1(\hat{K}))^2$, *we have*:

$$(3.18) \qquad \forall\ \hat{\varphi} \in L^2(\hat{K}) , \int_{\hat{K}} \hat{\varphi} \operatorname{div} \hat{\underset{\sim}{q}}\, d\hat{x} = \int_K \varphi \operatorname{div} \underset{\sim}{q}\, dx ,$$

$$(3.19) \qquad \forall\ \hat{\varphi} \in L^2(\partial\hat{K}) , \int_{\partial\hat{K}} \hat{\varphi}\ \hat{\underset{\sim}{q}}\cdot\hat{\underset{\sim}{\nu}}\, d\hat{\gamma} = \int_{\partial K} \varphi\ \underset{\sim}{q}\cdot\underset{\sim}{\nu}_K\, d\gamma .$$

For the proof, see [18] for instance. We shall also need,

Lemma 3. *We have for any integer* $\ell \geq 0$:

$$(3.20) \qquad \forall\ \hat{\varphi} \in H^{\ell}(\hat{K}) , \ |\hat{\varphi}|_{\ell,\hat{K}} \leq \|B_K\|^{\ell} |J_K|^{-\frac{1}{2}} |\varphi|_{\ell,K} ,$$

$$(3.21) \qquad \forall\ \hat{\underset{\sim}{q}} \in (H^{\ell}(\hat{K}))^2 , \ |\hat{\underset{\sim}{q}}|_{\ell,\hat{K}} \leq \|B_K\|^{2} \|B_K^{-1}\| \ |J_K|^{\frac{1}{2}} |\underset{\sim}{q}|_{\ell,K}$$

where $\|B_K\|$ (resp. $\|B_K^{-1}\|$) *denotes the spectral norm of* B_K (resp. B_K^{-1}).

Proof. The inequality (3.20) has been derived in [3 , inequality (4.15)]. By using (3.17), the inequality (3.21) can be obtained in a very similar way.

Now, with the triangle K, we associate the space

$$(3.22) \qquad \underset{\sim}{Q}_K = \{\underset{\sim}{q} \in \underset{\sim}{H}(\operatorname{div} ; K) \ ; \ \hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}\}$$

Assume that conditions (3.3) and (3.4) hold. Then, by Lemma 2, the functions $\underset{\sim}{q}$ of the space $\underset{\sim}{Q}_K$ satisfy the desired properties (i) and (ii).

Concerning the approximation of smooth vector-valued functions $\underset{\sim}{q}$ by functions of the space $\underset{\sim}{Q}_K$, we have

Theorem 3. Assume that the conditions (3.1)-(3.5) hold and let the space $\underset{\sim}{Q}_K$ be defined as in (3.22). Then there exist an operator $\underset{\sim}{\pi}_K \in \mathcal{L}((H^1(K))^2;$ and a constant $C > 0$ independent of K such that:

(i) for each side K' of K and for all $\varphi \in P_k$,

$$(3.23)\qquad \int_{K'} (\underset{\sim}{\pi}_K \underset{\sim}{q} - \underset{\sim}{q})\cdot \underset{\sim}{\nu}_K \varphi \, d\gamma = 0 ,$$

(ii) for all function $\underset{\sim}{q} \in (H^{k+1}(K))^2$ with div $\underset{\sim}{q} \in H^{k+1}(K)$,

$$(3.24)\qquad \|\underset{\sim}{\pi}_K \underset{\sim}{q} - \underset{\sim}{q}\|_{\underset{\sim}{H}(\mathrm{div};K)} \leq C \frac{h_K^{k+2}}{\rho_K} \left(|\underset{\sim}{q}|_{k+1,K} + |\mathrm{div}\, \underset{\sim}{q}|_{k+1,K}\right)$$

Proof. Given a function $\hat{\underset{\sim}{q}} \in (H^1(\hat{K}))^2$, there exists by Lemma 1 and Remark ~~1~~ 3 a unique function $\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ such that

$$(3.25)\qquad \forall\, \hat{\varphi} \in P_k ,\ \int_{\hat{K}'} (\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} - \hat{\underset{\sim}{q}})\cdot \hat{\underset{\sim}{\nu}}\, \hat{\varphi}\, d\hat{\gamma} = 0 \text{ for each side } \hat{K}' \text{ of } \hat{K} ,$$

$$(3.26)\qquad \forall\, \hat{\underset{\sim}{r}} \in (P_{K-1})^2,\ \int_{\hat{K}} (\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} - \hat{\underset{\sim}{q}})\cdot \hat{\underset{\sim}{r}}\, d\hat{x}$$

It follows from (3.1) that $\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} = \hat{\underset{\sim}{q}}$ for all $\hat{\underset{\sim}{q}} \in (P_K)^2$. Then, by applying Lemma 7 of [3] in vector form, we get for all $\hat{\underset{\sim}{q}} \in (H^{k+1}(\hat{K}))^2$

$$(3.27)\qquad \|\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} - \hat{\underset{\sim}{q}}\|_{o,\hat{K}} \leq c_1 |\hat{\underset{\sim}{q}}|_{k+1,\hat{K}}$$

for some constant $c_1 = c_1(\hat{K})$. On the other hand, using (3.25),(3.26) and the Green's formula, we obtain for all $\hat{\varphi} \in P_k$

$$\int_{\hat{K}} \mathrm{div}(\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} - \hat{\underset{\sim}{q}})\hat{\varphi}\, d\hat{x} = - \int_{\hat{K}} (\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} - \hat{\underset{\sim}{q}})\cdot \underset{\sim\sim\sim}{\mathrm{grad}}\, \hat{\varphi}\, dx + \int_{\partial\hat{K}} (\hat{\underset{\sim}{q}} - \hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}})\cdot \hat{\underset{\sim}{\nu}}\, \hat{\varphi}\, d\hat{\gamma} = 0.$$

Hence $\mathrm{div}(\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}})$ is the orthogonal projection in $L^2(\hat{K})$ of div $\hat{\underset{\sim}{q}}$ upon P_K. Then, assuming that div $\hat{\underset{\sim}{q}} \in H^{k+1}(\hat{K})$ and applying again [3, Lemma 7], we obtain for some constant $c_2 = c_2(\hat{K})$

$$(3.28)\qquad \|\mathrm{div}(\hat{\underset{\sim}{\pi}} \hat{\underset{\sim}{q}} - \hat{\underset{\sim}{q}})\|_{o,\hat{K}} \leq c_2\ |\mathrm{div}\, \hat{\underset{\sim}{q}}|_{k+1,\hat{K}} \cdot$$

Define now the operator $\underset{\sim}{\pi}_K$ by

$$\forall\ \underset{\sim}{q} \in (H^1(K))^2,\quad \widehat{\underset{\sim}{\pi}_K \underset{\sim}{q}} = \hat{\underset{\sim}{\pi}}\, \hat{\underset{\sim}{q}}\ .$$

Clearly, (3.23) follows from (3.25) and Lemma 2. Since

$$\underset{\sim}{\pi}_K \underset{\sim}{q}-\underset{\sim}{q} = \frac{1}{J_K} B_K(\hat{\underset{\sim}{\pi}}\, \hat{\underset{\sim}{q}}-\hat{\underset{\sim}{q}}) \circ F_K^{-1}\ ,$$

we have

$$\|\underset{\sim}{\pi}_K \underset{\sim}{q}-\underset{\sim}{q}\|_{o,K} \leq \|B_K\|\,|J_K|^{-\frac{1}{2}}\, \|\hat{\underset{\sim}{\pi}}\, \hat{\underset{\sim}{q}}-\hat{\underset{\sim}{q}}\|_{o,\hat{K}}\ .$$

Thus, by using inequalities (3.27) and (3.21) for $\ell = k+1$, we get for all $\underset{\sim}{q} \in (H^{k+1}(K))^2$

$$\|\underset{\sim}{\pi}_K \underset{\sim}{q}-\underset{\sim}{q}\|_{o,K} \leq c_1 \|B_K\|^{k+2} \|B_K^{-1}\|\, |\underset{\sim}{q}|_{k+1,K}\ . \tag{3.29}$$

Finally, from (3.18) we have

$$\operatorname{div}(\underset{\sim}{\pi}_K \underset{\sim}{q}-\underset{\sim}{q}) = \frac{1}{J_K} (\operatorname{div}(\hat{\underset{\sim}{\pi}}\, \hat{\underset{\sim}{q}}-\hat{\underset{\sim}{q}})) \circ F_K^{-1}$$

so that

$$\|\operatorname{div}(\underset{\sim}{\pi}_K \underset{\sim}{q}-\underset{\sim}{q})\|_{o,K} = |J_K|^{-\frac{1}{2}}\, \|\operatorname{div}(\hat{\underset{\sim}{\pi}}\, \hat{\underset{\sim}{q}}-\hat{\underset{\sim}{q}})\|_{o,\hat{K}}\ .$$

Therefore, noticing that

$$\operatorname{div} \hat{\underset{\sim}{q}} = J_K (\widehat{\operatorname{div} \underset{\sim}{q}})$$

and applying the inequalities (3.28) and (3.20) (with $\ell = k+1$ and $\varphi = \operatorname{div} q$), we obtain when $\operatorname{div} \underset{\sim}{q} \in H^{k+1}(K)$

$$\|\operatorname{div}(\underset{\sim}{\pi}_K \underset{\sim}{q}-\underset{\sim}{q})\|_{o,K} \leq c_2 \|B_K\|^{k+1} |\operatorname{div} \underset{\sim}{q}|_{k+1,K}\ . \tag{3.30}$$

Since, by [15 , Lemma 2] , we have

$$\|B_K\| \leq \frac{h_K}{\rho_{\hat{K}}}\ ,\quad \|B_K^{-1}\| \leq \frac{h_{\hat{K}}}{\rho_K}\ , \tag{3.31}$$

the desired inequality (3.24) follows from (3.29) and (3.30).

4. ERROR BOUNDS

Assume that $\mathcal{K}_h$ is a triangulation of $\bar{\Omega}$ made up with triangles K whose diameters are $\leq$ h. We now introduce the space

$$(4.1) \qquad \underset{\sim}{Q}_h = \{\underset{\sim}{q}_h \in \underset{\sim}{H}(\mathrm{div}\,;\Omega)\ ;\ \forall\, K \in \mathcal{K}_h,\ \underset{\sim}{q}_h|_K \in \underset{\sim}{Q}_K\}$$

where, for all $K \in \mathcal{K}_h$, the space $\underset{\sim}{Q}_K$ is defined as in (3.22).

The degrees of freedom of a function $\underset{\sim}{q}_h \in \underset{\sim}{Q}_h$ are easily determined; they can be chosen as

(i) the values of $\underset{\sim}{q}_h \cdot \nu_K$, at (k+1) distinct points of each side K' of the triangulation $\mathcal{K}_h$;

(ii) the moments of order $\leq$ k-1 of $\underset{\sim}{q}_h$ over each triangle $K \in \mathcal{K}_h$.

On the other hand, for any $\underset{\sim}{q}_h \in \underset{\sim}{Q}_h$ and any $K \in \mathcal{K}_h$, we have $(\mathrm{div}\,\underset{\sim}{q}_h)_{|K} \in P_k$. Hence, a natural choice for the space V_h is given by

$$(4.2) \qquad V_h = \{v_h \in L^2(\Omega)\ ;\ \forall\, K \in \mathcal{K}_h\ ,\ v_{h|K} \in P_k\}\ ,$$

so that condition (2.11) is automatically satisfied.

Note that the function $v_h \in V_h$ do not satisfy any continuity constraint at the interelement boundaries.

Now, in order to apply Theorem 2, the essential step consists in proving that the compatibility condition (2.12) holds with a constant α independent of h. In fact, we want to show that, for any function $v_h \in V_h$, there exists a function $\underset{\sim}{q}_h \in \underset{\sim}{Q}_h$ such that

$$(4.3) \qquad \mathrm{div}\,\underset{\sim}{q}_h = v_h \text{ in } \Omega$$

and

$$(4.4) \qquad \|\underset{\sim}{q}_h\|_{\underset{\sim}{H}(\mathrm{div}\,;\Omega)} \leq C\|v_h\|_{0,\Omega}\ ,$$

where the constant C is independent of h. For the proof, we need some technical preliminary results.

Let K a triangle of $\mathcal{K}_h$; we denote by $S_{k,\partial K}$ the space of all functions defined over ∂K whose restrictions to any side K' of K are polynomials of degree $\leq$ k.

Lemma 4. *Let there be given functions* $v \in P_k$ *and* $\mu \in S_{k,\partial K}$ *such that*

$$(4.5) \qquad \int_K v\,dx = \int_{\partial K} \mu\,d\gamma\ .$$

Assume that conditions (3.2)-(3.5) *hold. Then there exists a function* $\underset{\sim}{q} \in \underset{\sim}{Q}_K$ *such that*

$$(4.6)\qquad \begin{cases} \operatorname{div} \underset{\sim}{q} = v & \textit{in } K , \\ \underset{\sim}{q} \cdot \underset{\sim}{\nu}_K = \mu & \textit{on } \partial K , \end{cases}$$

and

$$(4.7)\qquad \|\underset{\sim}{q}\|_{\underset{\sim}{H}(\operatorname{div}\, ;\, K)} \leq C\left(\|v\|^2_{o,K} + \frac{h_K^2}{\rho_K}\|u\|^2_{o,\partial K}\right)^{\frac{1}{2}}$$

where the constant C *is independent of* K.

Proof. Let $\hat{v}_1 \in P_k$ and $\hat{\mu}_1 \in \hat{S}_k$ be functions such that

$$(4.8)\qquad \int_{\hat{K}} \hat{v}_1 \, d\hat{x} = \int_{\partial\hat{K}} \hat{\mu}_1 \, d\hat{\gamma}$$

Then the Neumann problem

$$(4.9)\qquad \begin{cases} \Delta\hat{w} = \hat{v}_1 & \text{in } \hat{K} , \\ \dfrac{\partial\hat{w}}{\partial\hat{\nu}} = \hat{\mu}_1 & \text{on } \partial\hat{K}, \end{cases}$$

has a solution $\hat{w} \in H^1(\hat{K})$ which is unique up to an additive constant. Moreover, there exists a constant $c_1 = c_1(\hat{K}) > 0$ such that

$$(4.10)\qquad |\hat{w}|_{1,\hat{K}} \leq c_1\left(\|\hat{v}_1\|^2_{o,\hat{K}} + \|\hat{\mu}_1\|^2_{o,\partial\hat{K}}\right)^{\frac{1}{2}}$$

Now, by Lemma 1, there exists a unique function $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ such that

$$\begin{cases} \forall\, \hat{r} \in (P_{k-1})^2 , \displaystyle\int_{\hat{K}} (\hat{\underset{\sim}{q}} - \underset{\sim}{\operatorname{grad}}\, \hat{w}) \cdot \hat{r}\, d\hat{x} = 0 , \\ \hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}} = \hat{\mu}_1 \quad \text{on } \partial\hat{K} . \end{cases}$$

From (4.9) and the Green's formula, it follows that

$$\forall\, \hat{\varphi} \in P_k , \int_{\hat{K}} \hat{\varphi} \operatorname{div} \hat{\underset{\sim}{q}}\, d\hat{x} = - \int_{\hat{K}} \hat{\underset{\sim}{q}} \cdot \underset{\sim}{\operatorname{grad}}\, \hat{\varphi}\, d\hat{x} + \int_{\partial\hat{K}} \hat{\varphi}\, \hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}}\, d\hat{\gamma} =$$

$$= - \int_{\hat{K}} \underset{\sim}{\text{grad}}\ \hat{w}\ \underset{\sim}{\text{grad}}\ \hat{\varphi}\ d\hat{x} + \int_{\partial\hat{K}} \hat{\varphi}\ \frac{\partial \hat{w}}{\partial \hat{\nu}}\ d\hat{\gamma} = \int_{\hat{K}} \hat{\varphi}\ \Delta\hat{w}\ d\hat{x}\ ,$$

so that div $\hat{\underset{\sim}{q}}$ is the orthogonal projection in $L^2(\hat{K})$ of $\Delta\hat{w}$ upon P_k. Hence, we get

$$(4.11) \qquad \begin{cases} \operatorname{div} \hat{\underset{\sim}{q}} = \hat{v}_1 \text{ in } \hat{K}_1\ , \\ \\ \hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}} = \hat{\mu}_1 \text{ on } \partial\hat{K}. \end{cases}$$

On the other hand, it follows from (4.10) that

$$(4.12) \qquad \|\hat{\underset{\sim}{q}}\|_{o,\hat{K}} \leq c_2 \left(\|\hat{v}_1\|^2_{o,\hat{K}} + \|\hat{\mu}_1\|^2_{o,\partial\hat{K}}\right)^{\frac{1}{2}}$$

for some constant $c_2 = c_2(\hat{K}) > 0$.

Now, let $K \in \mathcal{K}_h$; with the functions $v \in P_k$ and $\mu \in S_{k,\partial K}$ such that (4.5) holds, we associate the functions $\hat{v}_1 \in P_k$ and $\hat{\mu}_1 \in \hat{S}_k$ defined by

$$(4.13) \qquad \begin{cases} \forall\ \hat{\varphi} \in P_k\ ,\ \displaystyle\int_{\hat{K}} \hat{v}_1 \hat{\varphi}\ d\hat{x} = \int_K v\ \varphi\ dx\ , \\ \\ \forall\ \hat{\varphi} \in \hat{S}_k\ ,\ \displaystyle\int_{\partial\hat{K}} \hat{\mu}_1 \hat{\varphi}\ d\hat{\gamma} = \int_{\partial K} \mu\ \varphi\ d\gamma\ . \end{cases}$$

Clearly we have (4.8) and there exists a function $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ such that (4.11) and (4.12) hold. We next define $\underset{\sim}{q} \in \underset{\sim}{Q}_K$ by

$$(4.14) \qquad \underset{\sim}{q} = \frac{1}{J_K}\ B_K\ \hat{\underset{\sim}{q}} \circ F_K^{-1}\ ,$$

so that, by (4.11) and Lemma 2, we get (4.6).

It remains only to show the estimate (4.7). We get from (4.12) and (4.14)

$$(4.15) \qquad \|\underset{\sim}{q}\|^2_{o,K} \leq c_2^2\ \|B_K\|^2\ |J_K|^{-1} \left(\|\hat{v}_1\|^2_{o,\hat{K}} + \|\mu_1\|^2_{o,\partial\hat{K}}\right)\ .$$

Since $\hat{v}_1 = |J_K|\, v \circ F_K$, we obtain

$$(4.16) \qquad \|\hat{v}_1\|^2_{o,\hat{K}} = |J_K|\ \|v\|^2_{o,K}\ .$$

On the other hand, let $\hat{K}'$ be a side of $\hat{K}$ and let $K' = F_K(\hat{K}')$. Since the superficial measures of $\hat{K}'$ and K' are remated by

$$\text{meas}(K') \leq \|B_K^{-1}\| \, |J_K| \, \text{meas}(\hat{K}')$$

we obtain

$$\|\hat{\mu}_1\|^2_{o,\hat{K}'} \leq \|B_K^{-1}\| \, |J_K| \, \|\mu\|^2_{o,K'} \ . \tag{4.17}$$

By combining the inequalities (4.15) - (4.17), we get

$$\|\underset{\sim}{q}\|^2_{o,K} \leq c_2^2 \|B_K\|^2 \left(\|v\|^2_{o,K} + \|B_K^{-1}\| \, \|\mu\|^2_{o,\partial K} \right) \ . \tag{4.18}$$

Therefore, the desired inequality follows from (4.18) and (3.31).

Let us next introduce the space

$$M_h = \{ \mu_h \in \prod_{K \in \mathcal{K}_h} S_{k,\partial K} \ ; \ \mu_h|_{\partial K_1} + \mu_h|_{\partial K_2} = 0 \quad \text{on } K_1 \cap K_2 \tag{4.19}$$

for every pair of adjacent triangles K_1, $K_2 \in \mathcal{K}_h\}$.

We consider a *regular family* $(\mathcal{K}_h)$ of triangulations of $\bar{\Omega}$ in the sense of [3], in that there exists a constant $\sigma > 0$ independent of h such that

$$\max_{K \in \mathcal{K}_h} \frac{h_K}{\rho_K} \leq \sigma \ . \tag{4.20}$$

Lemma 5. *Let there be given spaces* V_h *and* M_h *defined as in* (4.2) *and* (4.19) *which are associated with a regular family of triangulations. Then, with any function* $v_h \in V_h$ *, we can associate a function* $\mu_h \in M_h$ *such that for all* $K \in \mathcal{K}_h$

$$\int_K v_h \, dx = \int_{\partial K} \mu_h \, d\gamma \tag{4.21}$$

and

$$\Big(\sum_{K \in \mathcal{K}_h} h_K \|\mu_h\|^2_{o,\partial K} \Big)^{\frac{1}{2}} \leq C \|v_h\|_{o,\Omega} \ , \tag{4.22}$$

where the constant $C > 0$ *is independent of* h .

Proof. We shall construct the function μ_h by using a hybrid finite element method as it has been described and studied in [15]. Hence, the proof of the Lemma will depend heavily upon the results of [15]. We first define the space

$$X_h = \{\varphi_h \in L^2(\Omega) \; ; \; \forall K \in \mathcal{K}_h \, , \; \varphi_{h|K} \in P_{k+2}\}$$

provided with the norm

$$\|\varphi_h\|_{X_h} = \left\{ \sum_{K \in \mathcal{K}_h} \left(|\varphi_h|^2_{1,K} + h_K^{-2} \|\varphi_h\|^2_{0,K} \right) \right\}^{\frac{1}{2}} .$$

Next, we set :

$$a(\varphi_h, \psi_h) = \sum_{K \in \mathcal{K}_h} \int_K \underset{\sim}{\operatorname{grad}} \, \varphi_h \cdot \underset{\sim}{\operatorname{grad}} \, \psi_h dx, \qquad \varphi_h, \psi_h \in X_h \, ,$$

$$b(\varphi_h, \mu_h) = - \sum_{K \in \mathcal{K}_h} \int_{\partial K} \varphi_h \mu_h d\gamma \quad , \qquad \varphi_h \in X_h, \mu_h \in M_h \, .$$

Then, by using [15, Theorem 2 and Lemmas 2,3,4], there exists a unique pair of functions $(\varphi_h \, , \, \mu_h) \in X_h \times M_h$ such that

$$(4.23) \quad \forall \, \psi_h \in X_h \, , \; a(\varphi_h \, , \, \psi_h) + b(\psi_h \, , \, \mu_h) = \int_\Omega v_h \psi_h dx \, ,$$

$$(4.24) \quad \forall \, \rho_h \in M_h \, , \; b(\varphi_h \, , \, \rho_h) = 0.$$

By choosing in (4.23)

$$\psi_h = \text{characteristic function of the set } K \in \mathcal{K}_h$$

we get (4.21) for all $K \in \mathcal{K}_h$.

Now, in order to prove the inequality (4.22), we introduce the following subspace of the space X_h

$$Y_h = \left\{ \psi_h \in X_h \; ; \; \forall \rho_h \in M_h \, , \; b(\psi_h \, , \, \rho_h) = 0 \right\}$$

Clearly, the function $\varphi_h \in Y_h$ may be characterized as the solution of

$$\forall \, \psi_h \in Y_h \, , \; a(\varphi_h \, , \, \psi_h) = \int_\Omega v_h \psi_h dx \, .$$

Therefore, we get

$$a(\varphi_h , \varphi_h) \leq \|v_h\|_{o,\Omega}\|\varphi_h\|_{o,\Omega} .$$

By, [15, inequality (6.18)], we have the discrete analogue of the Poincaré-Friedrichs inequality :

$$\forall\ \psi_h \in Y_h\ ,\ \|\psi_h\|_{o,\Omega} \leq c_1\, a(\psi_h , \psi_h)^{\frac{1}{2}} ,$$

where the constant c_1 is independent of h. Hence we obtain

$$(4.25)\qquad a(\varphi_h , \varphi_h)^{\frac{1}{2}} \leq c_1 \|v_h\|_{o,\Omega} .$$

Next, it follows from (4.23) and (4.25) that

$$(4.26)\qquad b(\psi_h , \mu_h) \leq (\|\psi_h\|_{o,\Omega} + c_1\, a(\psi_h , \psi_h)^{\frac{1}{2}})\|v_h\|_{o,\Omega} \leq$$

$$\leq c_2 \|\psi_h\|_{X_h} \|v_h\|_{o,\Omega} ,$$

where c_2 is a constant independent of h. Thus, the inequality (4.22) follows from (4.26) and the following inequality

$$\Big(\sum_{K \in \mathcal{K}_h} h_K \|\mu_h\|^2_{o,\partial K}\Big)^{\frac{1}{2}} \leq c_3 \sup_{\psi_h \in X_h} \frac{b(\psi_h,\mu_h)}{\|\psi_h\|_{X_h}} ,\ c_3 = c_3(\Omega) .$$

(cfr. [15 , inequality (6.29)]).

We are now able to state

Theorem 4. Let there be given spaces $\underset{\sim}{Q}_h$ and V_h defined as in (4.1) and (4.2), which are associated with a regular family of triangulations. Assume in addition that the conditions (3.2)-(3.5) hold.Then, with any function $v_h \in V_h$, we can associate a function $\underset{\sim}{q}_h \in \underset{\sim}{Q}_h$ which satisfies the conditions (4.3), (4.4) with a constant $C > 0$ independent of h.

Proof. Let v_h be a function in V_h. By Lemma 5, we construct a function $\mu_h \in M_h$ such that the conditions (4.21) and (4.22) hold.
Next, using Lemma 4, there exists a function $\underset{\sim}{q}_h \in (L^2(\Omega))^2$ such that for all $K \in \mathcal{K}_h$, we have :

$$\begin{cases} \underset{\sim}{q}_{h|K} \in \underset{\sim}{Q}_K , \\ \operatorname{div}(\underset{\sim}{q}_{h|K}) = v_{h|K} , \\ (\underset{\sim}{q}_{h|K})\cdot \underset{\sim}{\nu}_K = \mu_{h|\partial K} . \end{cases}$$

Since $\mu_h \in M_h$, the reciprocity conditions (2.15) hold so that $\underset{\sim}{q}_h \in \underset{\sim}{Q}_h$ and div $\underset{\sim}{q}_h = v_h$ in Ω. Moreover, it follows from (4.7) and (4.20) that

$$(4.27) \qquad \|\underset{\sim}{q}_h\|^2_{\underset{\sim}{H}(\mathrm{div};\Omega)} \leq C^2 \left(\|v_h\|^2_{o,\Omega} + \sigma \sum_{K \in \mathcal{K}_h} h_K \|\mu_h\|^2_{o,\partial K}\right).$$

Combining the inequalities (4.22) and (4.27), we obtain the inequality (4.4).

We now have our main result.

Theorem 5. We assume that $u \in H^{k+2}(\Omega)$ and $\Delta u \in H^{k+1}(\Omega)$ for some integer $k \geq 0$. Let there be given spaces $\underset{\sim}{Q}_h$ and V_h defined as in (4.1)-(4.2), which are associated with a regular family of triangulations. We assume in addition that the conditions (3.1)-(3.5) hold. Then problem (P_h) has a unique solution and there exists a constant $\mathcal{K}$ independent of h such that

$$(4.28) \qquad \|\underset{\sim}{p}-\underset{\sim}{p}_h\|_{\underset{\sim}{H}(\mathrm{div};\Omega)} + \|u-u_h\|_{o,\Omega} \leq \mathcal{K}\, h^{k+1} \left(|u|_{k+1,\Omega}+|u|_{k+2,\Omega}+|\Delta u|_{k+1,\Omega}\right).$$

Proof. Let $v_h \in V_h$; by the previous theorem, we have

$$\sup_{\underset{\sim}{q}_h \in \underset{\sim}{Q}_h} \frac{\int_\Omega v_h \,\mathrm{div}\, \underset{\sim}{q}_h \, dx}{\|\underset{\sim}{q}_h\|_{\underset{\sim}{H}(\mathrm{div}\,;\,\Omega)}} \geq \frac{1}{C} \|v_h\|_{o,\Omega}$$

so that the hypothesis (2.12) holds with $\alpha = \frac{1}{C}$. Thus, by Theorem 2, it remains only to evaluate the quantities

$$\inf_{\underset{\sim}{q}_h \in \underset{\sim}{Q}_h} \|\underset{\sim}{p}-\underset{\sim}{q}_h\|_{\underset{\sim}{H}(\mathrm{div};\Omega)} \quad \text{and} \quad \inf_{v_h \in V_h} \|u-v_h\|_{o,\Omega} .$$

On the one hand, by using Theorem 3, we define $\Pi_h \, \underset{\sim}{p} \in (L^2(\Omega))^2$ by

$$\forall\, K \in \mathcal{K}_h , \quad \underset{\sim}{\pi}_h\, p|_K = \underset{\sim}{\pi}_K(\underset{\sim}{p}|_K) .$$

It follows from (3.23) that the reciprocity relations (2.15) hold so that $\underset{\sim}{\pi}_h \in \underset{\sim}{Q}_h$. Next, we deduce from (3.24) and (4.20) that for some constant c_1 independent of h

$$(4.29) \qquad \|\underset{\sim}{p}-\underset{\sim}{\pi}_h p\|_{\underset{\sim}{H}(\mathrm{div};\Omega)} \leq c_1 h^{k+1} \left(|u|_{k+2,\Omega} + |\Delta u|_{k+1,\Omega}\right).$$

On the other hand, a straightforward application of [3, Theorem 5] gives for some constant c_2 independent of h

(4.30) $$\inf_{v_h \in V_h} \|u-v_h\|_{o,\Omega} \leq c_2 h^{k+1} |u|_{k+1,\Omega} .$$

Then, inequality (4.28) follows from inequalities (2.11),(4.29) and (4.30).

5. MIXED QUADRILATERAL ELEMENTS

We shall briefly discuss the case of quadrilateral elements. As for triangular elements, we begin by introducing the space $\hat{\underset{\sim}{Q}}$ associated with the unit square $\hat{K} = [0,1]^2$ in the (ξ,η)-plane. Given two integers $k, \ell \geq 0$, let us denote by $P_{k,\ell}$ the space of all polynomials in the two variables ξ,η of the form

(5.1) $$P(\xi,\eta) = \sum_{i=0}^{k} \sum_{j=0}^{\ell} c_{ij}\, \xi^i \eta^j \quad , \quad c_{ij} \in \mathbb{R} .$$

Now we define the space $\hat{\underset{\sim}{Q}}$ by

(5.2) $$\hat{\underset{\sim}{Q}} = \left\{ \hat{\underset{\sim}{q}} = (\hat{q}_1 , \hat{q}_2) ; \ \hat{q}_1 \in P_{k+1,k} , \ \hat{q}_2 \in P_{k,k+1} \right\}$$

Note that, for $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$, we have :

(i) $$\operatorname{div} \hat{\underset{\sim}{q}} = \frac{\partial \hat{q}_1}{\partial \xi} + \frac{\partial \hat{q}_2}{\partial \eta} \in P_{k,k} \quad ;$$

(ii) the restriction of $\hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}}$ to any side $\hat{K}'$ of $\hat{K}$ is a polynomial of degree $\leq k$.

One can prove

Lemma 6. *A function* $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ *is uniquely determined by:*

(a) *the values of* $\hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}}$ *at* (k+1) *distinct points of each side* $\hat{K}'$ *of* $\hat{K}$ *;*

(b) *the quantities*

$$\int_{\hat{K}} \hat{q}_1 \xi^i \eta^j d\hat{x} \ , \quad 0 \leq i \leq k-1 \ , \quad 0 \leq j \leq k \ ,$$

$$\int_{\hat{K}} \hat{q}_2 \xi^i \eta^j d\hat{x} \ , \quad 0 \leq i \leq k \ , \quad 0 \leq j \leq k-1$$

The proof goes along the same lines of that of Lemma 1.

Consider for instance the case k = 0. A Function $\hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}}$ is of the form

$$(5.3)\qquad \begin{cases} \hat{q}_1 = a_o + a_1 \xi \quad , \\ \\ \hat{q}_2 = b_o + b_1 \eta \quad , \end{cases}$$

and by Lemma 6, the degrees of freedom of $\hat{\underset{\sim}{q}}$ may be choosen as the values of $\hat{\underset{\sim}{q}} \cdot \hat{\underset{\sim}{\nu}}$ at the midpoints of the sides of the square $\hat{K}$.

Next, let K be a parallelogram in the (x_1, x_2)-plane. There exists an affine invertible mapping $F_K : \hat{x} \to F_K(\hat{x}) = B_K\hat{x} + b_K$, auch that $K = F_K(\hat{K})$. With K, we associate the space

$$(5.4)\qquad \underset{\sim}{Q}_K = \left\{ \underset{\sim}{q} : K \to \mathbb{R}^2 \ ; \ \underset{\sim}{q} = \frac{1}{J_K} B_K \hat{q} \circ F_K^{-1} , \ \hat{\underset{\sim}{q}} \in \hat{\underset{\sim}{Q}} \right\} .$$

Let $\underset{\sim}{q} \in \underset{\sim}{Q}_K$; the restriction of $\underset{\sim}{q} \cdot \underset{\sim}{\nu}_K$ to any side K' of the quadrilateral K is a polynomial of degree $\leq k$.

Assume now that $\mathcal{H}_h$ is a triangulation of $\bar{\Omega}$ made up with parallelograms K whose diameters are $\leq$ h. We set :

$$(5.5)\qquad \underset{\sim}{Q}_h = \left\{ \underset{\sim}{q}_h \in \underset{\sim}{H}(\mathrm{div} \ ; \ \Omega) \ ; \ \forall\, K \in \mathcal{H}_h \ , \ \underset{\sim}{q}_{h|K} \in \underset{\sim}{Q} \right\} .$$

Note that, for any $\underset{\sim}{q}_h \in \underset{\sim}{Q}_h$ and any $K \in \mathcal{H}_h$, we have

$$(\mathrm{div}\ \underset{\sim}{q}_h)_{|K} \circ F_K \in P_{k,k} \ .$$

So we set

$$(5.6)\qquad V_h = \left\{ v_h \in L^2(\Omega) \ ; \ \forall\, K \in \mathcal{H}_h \ , \ v_{h|K} \circ F_K \in P_{k,k} \right\} .$$

By using the techniques of §§ 3,4, one can similarly prove that problem (P_h) has a unique solution $(\underset{\sim}{p}_h , u_h) \in \underset{\sim}{Q}_h \times V_h$ and that the error bound (4.28) still holds.

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