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A BLOCK-CENTERED FINITE DIFFERENCE METHOD FOR THE DARCY–FORCHHEIMER MODEL*

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Abstract. A block-centered finite difference scheme is introduced to solve the nonlinear Darcy–Forchheimer equation, in which the velocity and pressure can be approximated simultaneously. The second-order error estimates for both pressure and velocity are established on a nonuniform rectangular grid. Numerical experiments using the scheme show the consistency of the convergence rates of our method with the theoretical analysis.

Key words. Darcy–Forchheimer model, block-centered finite difference, numerical analysis

AMS subject classifications. 65N06, 65N12, 65N15

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1. Introduction.

Darcy's flow in porous media is of great interest in many fields, such as oil recovery and groundwater pollution contamination. Darcy's law,

$$\mu K^{-1}\boldsymbol{u} + \nabla p = \rho g \nabla h,$$

describes the linear relationship between the Darcy velocity and the gradient of pressure. The law can be derived by a linear simplification of the momentum theorem. Here μ, ρ, g, and h represent the viscosity coefficient, the density of the fluid, the gravitational constant, and the depth of the domain, respectively. K is the permeability tensor. The above relationship is valid by experiments implemented with low velocity, small porosity, and permeability by Darcy in 1856; see [3] for example. A theoretical derivation of Darcy's law can be found in [7, 14].

In some cases, for example, when the velocity is high, a nonlinear relationship between velocity and the pressure gradient is developed, as suggested by Forchheimer in 1901 [3], by introducing a second-order term to reach a modified equation. The Forchheimer equation (or the Darcy–Forchheimer equation) takes the form

$$\mu K^{-1}\boldsymbol{u} + \beta\rho|\boldsymbol{u}|\boldsymbol{u} + \nabla p = \rho g \nabla h.$$

A theoretical derivation of Forchheimer's law can be found in [11].

Forchheimer's law mainly describes the inertial effects for high speed flow. The most important feature of Forchheimer's law is that it combines the monotonicity of the nonlinear term and the nondegeneracy of the Darcy part.

A mixed element method for the Forchheimer equation has been introduced by Girault and Wheeler [4]. They approximate the velocity by a piecewise constant function and pressure by the Crouzeix–Raviart element, and their mixed approximation is called the primal mixed element [10]. The convergence of their mixed element scheme

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was presented, and a first-order error estimate for velocity in the L^2 norm and a first-order error estimate for pressure in the H^1 norm were demonstrated. The numerical experiments using their method were carried out in [6], in which the convergence of the approximation was verified. A mixed element method for the time dependent problem was considered by Park [9], in which a semidiscrete scheme was proposed and the corresponding error was analyzed.

Recently we introduced a mixed element approximation for the stationary Darcy–Forchheimer equation [8], which is different from the scheme in [4]. The mixed formulation and the mixed elements we consider are the usually mixed formulation and mixed elements such as the Raviart–Thomas mixed element and the Brezzi–Douglas–Marini mixed element. We obtained the existence and uniqueness of the weak solution and established the optimal order error estimate. Numerical tests were also carried out accordingly.

The block-centered finite difference method can be thought of as the lowest order Raviart–Thomas mixed element method with proper quadrature formulation. The application of the finite difference enables us to approximate both the velocity and pressure with second-order accuracy for the linear elliptic problem (see [13], where a block-centered finite difference for the linear elliptic problem with a diagonal diffusion coefficient was considered). In [1] and [2] cell-centered finite differences for the linear elliptic problem with tensor diffusion coefficients were considered.

In this paper we present a blocked-centered finite difference method for the Darcy–Forchheimer equation, a kind of nonlinear elliptic problem for non-Darcy flow in porous media. One key problem in presenting the scheme is to give a proper approximation to the nonlinear diffusion coefficient. We demonstrate that the proposed scheme is second-order accurate for both velocity and pressure in the discrete L^2 norm on the nonuniform rectangular grid. These error estimates are superconvergent, in comparison with the results in [8]. Then we carry out some numerical experiments using the presented block-centered finite difference scheme.

The paper is organized as follows. In section 2 we give some notation. In section 3 we present the block-centered finite difference method for the one dimensional problem and give the corresponding numerical analysis. In section 4 we present the block-centered finite difference method for the two dimensional problem and give the corresponding numerical analysis. In section 5 some numerical experiments using the blocked-centered finite difference method are presented. The numerical results show that the convergence rates of our method are in agreement with the theoretical analysis.

Throughout the paper we will use C, with or without subscript, to denote a positive constant which can have different values in different situations.

2. The problem and some notation.

In this section we consider the following Darcy–Forchheimer problem describing non-Darcy flow in porous media:

$$\left\{\begin{array}{lrl} \text{(i)} & \mu K^{-1}\boldsymbol{u} + \beta\rho|\boldsymbol{u}|\boldsymbol{u} + \nabla p = \rho g \nabla h, & \boldsymbol{x} \in \Omega, \\ \text{(ii)} & \nabla\cdot\boldsymbol{u} = f, & \boldsymbol{x} \in \Omega, \\ \text{(iii)} & \boldsymbol{u}\cdot\boldsymbol{n} = f_N, & \boldsymbol{x} \in \partial\Omega, \end{array}\right. \tag{2.1}$$

with the compatibility condition

$$\int_\Omega f \, \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} f_N \, \mathrm{d}s. \tag{2.2}$$

Here p represents the pressure, while $\boldsymbol{u}$ represents the Darcy velocity of the fluid. For simplicity we suppose that $\Omega = (0,1)$ for the one dimensional problem, or $\Omega = (0,1)\times(0,1)$ for the two dimensional problem. $\boldsymbol{n}$ represents the unit exterior normal vector to the boundary of Ω, $|\cdot|$ denotes the Euclidean norm, and $|\boldsymbol{u}|^2 = \boldsymbol{u}\cdot\boldsymbol{u}$. ρ, μ, and β are scalar functions which represent the density of the fluid, its viscosity, and its dynamic viscosity, respectively. β is also referred to as the Forchheimer number. K is the permeability tensor function. For simplicity we suppose that $K = k\mathrm{I}$, where k is positive and I represents the unit matrix. $f(\boldsymbol{x}) \in L^2(\Omega)$, a scalar function, represents the source and sink of the systems. $\nabla h(\boldsymbol{x}) \in (L^2(\Omega))^d$, a vector function, is the gradient of the depth function $h(\boldsymbol{x}) \in H^1(\Omega)$. $f_N(\boldsymbol{x}) \in L^2(\partial\Omega)$, a scalar function, represents the Neumann boundary condition, or the flux through the boundary. Without loss of generality we suppose that $\nabla h(\boldsymbol{x}) = 0$ and $f_N = 0$.

With the above assumptions the problem (2.1) with compatibility condition (2.2) can be rewritten as

$$\begin{cases} \text{(i)} \quad \left(\dfrac{\mu}{k} + \beta\rho|\boldsymbol{u}|\right)\boldsymbol{u} + \nabla p = 0, \quad \boldsymbol{x}\in\Omega, \\ \text{(ii)} \quad \nabla\cdot\boldsymbol{u} = f, \quad \boldsymbol{x}\in\Omega, \\ \text{(iii)} \quad \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{x}\in\partial\Omega, \end{cases} \tag{2.3}$$

$$\int_\Omega f \, \mathrm{d}\boldsymbol{x} = 0. \tag{2.4}$$

We will derive the block-centered finite difference method for the above model problem. For this purpose we suppose there exist positive constants $\bar{a}$ and $\bar{C}$ such that

$$\bar{a} \le \frac{\mu}{k} \le \bar{C}, \qquad \bar{a} \le \beta\rho \le \bar{C}. \tag{2.5}$$

We use the partitions and notation as in [13]. For the one dimensional problem, the partition δ_x of $\Omega = (0,1)$ is defined as

$$\delta_x : 0 = x_{1/2} < x_{3/2} < \ldots < x_{N_x-1/2} < x_{N_x+1/2} = 1.$$

For $i = 1,\ldots,N_x$, define

$$\begin{aligned} &x_i = \frac{x_{i-1/2}+x_{i+1/2}}{2}, \\ &h_i = x_{i+1/2} - x_{i-1/2}, \qquad h = \max_i h_i, \\ &h_{i+1/2} = \frac{h_{i+1}+h_i}{2} = x_{i+1} - x_i, \\ &\Omega_i = (x_{i-1/2}, x_{i+1/2}). \end{aligned}$$

For a function $\theta(x)$, let θ_i, $\theta_{i+1/2}$ denote $\theta(x_i)$ and $\theta(x_{i+1/2})$, respectively. For discrete functions $\{\theta_i\}$ and $\{\theta_{i+1/2}\}$, define

$$[d_x\theta]_{i+1/2} = \frac{\theta_{i+1}-\theta_i}{h_{i+1/2}}, \qquad [D_x\theta]_i = \frac{\theta_{i+1/2}-\theta_{i-1/2}}{h_i}.$$

For discrete functions θ and τ, define the midpoint quadrature formula and the trapezoidal quadrature formula in Ω_i, $i = 1,\ldots,N_x$, as

$$\begin{aligned} &(\theta,\tau)_{M_x,\Omega_i} = h_i\theta_i\tau_i, \\ &(\theta,\tau)_{T_x,\Omega_i} = \frac{h_i}{2}[\theta_{i-1/2}\tau_{i-1/2} + \theta_{i+1/2}\tau_{i+1/2}]. \end{aligned}$$

For the two dimensional problem, the domain $\Omega = (0,1)\times(0,1)$ is partitioned by $\delta_x \times \delta_y$, where

$$\delta_x : 0 = x_{1/2} < x_{3/2} < \cdots < x_{N_x-1/2} < x_{N_x+1/2} = 1,$$
$$\delta_y : 0 = y_{1/2} < y_{3/2} < \cdots < y_{N_y-1/2} < y_{N_y+1/2} = 1.$$

For $i = 1,\ldots,N_x$ and $j = 1,\ldots,N_y$, define

$$\begin{aligned}
&x_i = \frac{x_{i-1/2}+x_{i+1/2}}{2},\\
&h_i = x_{i+1/2} - x_{i-1/2}, \qquad h = \max_i h_i,\\
&h_{i+1/2} = \frac{h_{i+1}+h_i}{2} = x_{i+1} - x_i,\\
&y_j = \frac{y_{j-1/2}+y_{j+1/2}}{2},\\
&k_j = y_{j+1/2} - y_{j-1/2}, \qquad k = \max_j k_j,\\
&k_{j+1/2} = \frac{k_{j+1}+k_j}{2} = y_{j+1} - y_j,\\
&\Omega_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2}),\\
&\Omega_{i+1/2,j} = (x_i, x_{i+1}) \times (y_{j-1/2}, y_{j+1/2}),\\
&\Omega_{i,j+1/2} = (x_{i-1/2}, x_{i+1/2}) \times (y_j, y_{j+1}).
\end{aligned}$$

For a function $\theta(x,y)$, let $\theta_{l,m}$ denote $\theta(x_l, y_m)$, where l may take values $i,\ i+1/2$ for nonnegative integers i, and m may take values $j,\ j+1/2$ for nonnegative integers j. For discrete functions with values at proper discrete points, define

$$\begin{aligned}
&[d_x\theta]_{i+1/2,j} = \frac{\theta_{i+1,j}-\theta_{i,j}}{h_{i+1/2}}, \qquad [D_x\theta]_{i,j} = \frac{\theta_{i+1/2,j}-\theta_{i-1/2,j}}{h_i},\\
&[d_y\theta]_{i,j+1/2} = \frac{\theta_{i,j+1}-\theta_{i,j}}{k_{j+1/2}}, \qquad [D_y\theta]_{i,j} = \frac{\theta_{i,j+1/2}-\theta_{i,j-1/2}}{k_j}.
\end{aligned}$$

Also define the discrete inner products and norms and seminorms as follows:

$$\begin{aligned}
&(\theta,\tau)_M = (\theta,\tau)_{M_x,M_y} = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} h_i k_j \theta_{i,j}\tau_{i,j},\\
&\|\theta\|_M^2 = (\theta,\theta)_{M_x,M_y},\\
&(\theta,\tau)_x = (\theta,\tau)_{T_x,M_y} = \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} h_{i-1/2} k_j \theta_{i-1/2,j}\tau_{i-1/2,j},\\
&(\theta,\tau)_y = (\theta,\tau)_{M_x,T_y} = \sum_{i=1}^{N_x}\sum_{j=2}^{N_y} h_i k_{j-1/2} \theta_{i,j-1/2}\tau_{i,j-1/2},\\
&\|\theta\|_x^2 = (\theta,\theta)_x, \qquad \|\theta\|_y^2 = (\theta,\theta)_y.
\end{aligned}$$

3. A block-centered finite difference method for the one dimensional problem. In this section we consider the block-centered finite difference method for the one dimensional problem. In this case the problem (2.3) with compatibility condition (2.4) can be written as

$$\text{(3.1)} \qquad \begin{cases} \text{(i)} & \left(\dfrac{\mu}{k} + \beta\rho|u|\right) u + \dfrac{\partial p}{\partial x} = 0 \quad \text{in } \Omega, \\ \text{(ii)} & \dfrac{\partial u}{\partial x} = f \quad \text{in } \Omega, \\ \text{(iii)} & u(0) = u(1) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$$\text{(3.2)} \qquad \int_0^1 f \; \mathrm{d}x = 0.$$

Define

$$\text{(3.3)} \qquad F_i = \frac{1}{h_i}\int_{\Omega_i} f dx = \frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i+1/2}} f dx.$$

Then f_i is a second-order approximation to F_i,

$$\text{(3.4)} \qquad f_i = F_i + O(h_i^2).$$

The block-centered finite difference approximations $U_{i+1/2}$ and P_i to $u(x_{i+1/2})$ and $p(x_i)$, respectively, are chosen so that

$$\text{(3.5)} \qquad [D_x U]_i = f_i, \qquad U_{1/2} = 0,$$

$$\text{(3.6)} \qquad \left(\frac{\mu}{k} + \beta\rho|U|\right)_{i+1/2} U_{i+1/2} = -[dxP]_{i+1/2}.$$

The pressure p and its approximation P are uniquely determined with the difference of a constant. To determine them uniquely we use the following condition:

$$\text{(3.7)} \qquad p(x_1) = P_1 = 0.$$

It is clear that with condition (3.7) the solution of the system (3.5) and (3.6) is unique.

Using (3.5) we get that

$$U_{i+1/2} = U_{i-1/2} + h_i f_i, \qquad U_{1/2} = 0.$$

Noting condition (2.4) and evaluation (3.4), it is easy to get that

$$U_{N_x+1/2} = U_{1/2} + \sum_{l=1}^{N_x} h_l f_l = \sum_{l=1}^{N_x} h_l (F_l + O(h_l^2)) = O(h^2).$$

We now verify that if u is sufficiently smooth, then U is a second-order approximation to u. Set the velocity error by

$$E^u_{i+1/2} = u_{i+1/2} - U_{i+1/2}.$$

THEOREM 3.1. *If u is sufficiently smooth, there exists a positive constant C independent of h such that*

$$\text{(3.8)} \qquad |E^u_{i+1/2}| = |u_{i+1/2} - U_{i+1/2}| \le Ch^2.$$

Proof. By (3.1), (3.4), (3.5), and the boundary condition we have that

$$
\begin{aligned}
E^u_{i+1/2} &= u_{i-1/2} - U_{i-1/2} + \int_{\Omega_i} f dx - \int_{\Omega_i} f_i dx \\
&= u_{1/2} - U_{1/2} + \sum_{l=1}^{i} \int_{\Omega_l} (f - f_l) dx \\
&= \sum_{l=1}^{i} h_l (F_l - f_l) = \sum_{l=1}^{i} O(h_l^{2+1}),
\end{aligned}
$$

which completes the proof. ∎

Set the pressure error as

$$E^p_i = p_i - P_i.$$

THEOREM 3.2. *If p and u are sufficiently smooth, then there exists a positive constant C independent of h such that*

$$|E^p_{i+1}| = |p_{i+1} - P_{i+1}| \le Ch^2. \tag{3.9}$$

Proof. From the first equation in (3.1) we have that

$$p_{i+1} - p_i = \int_{x_i}^{x_{i+1}} - \left(\frac{\mu}{k} + \beta\rho|u|\right) u dx.$$

Then combining this with (3.6) we have that

$$E^p_{i+1} = \int_{x_i}^{x_{i+1}} \left(- \left(\frac{\mu}{k} + \beta\rho|u|\right) u + \left(\frac{\mu}{k} + \beta\rho|U|\right)_{i+1/2} U_{i+1/2} \right) dx + E^p_i.$$

Using this inequality recursively we have that

$$
\begin{aligned}
E^p_{i+1} &= \sum_{l=1}^{i} \int_{x_l}^{x_{l+1}} \left(- \left(\frac{\mu}{k} + \beta\rho|u|\right) u + \left(\frac{\mu}{k} + \beta\rho|U|\right)_{l+1/2} U_{l+1/2} \right) dx \\
&= \sum_{l=1}^{i} - \int_{x_l}^{x_{l+1}} \left(\frac{\mu}{k} u - \left(\frac{\mu}{k} u\right)_{l+1/2} \right) dx - \sum_{l=1}^{i} \int_{x_l}^{x_{l+1}} \left(\frac{\mu}{k}\right)_{l+1/2} (u - U)_{l+1/2} dx \\
&\quad - \sum_{l=1}^{i} \int_{x_l}^{x_{l+1}} (\beta\rho|u|u - (\beta\rho|u|u)_{l+1/2}) dx \\
&\quad - \sum_{l=1}^{i} \int_{x_l}^{x_{l+1}} (\beta\rho)_{l+1/2} (|u_{l+1/2}|u_{l+1/2} - |U_{l+1/2}|U_{l+1/2}) dx \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

It is clear that

$$I_2 \le \sum_{l=1}^{i} \int_{x_l}^{x_{l+1}} \left(\frac{\mu}{k}\right)_{l+1/2} |E^u_{l+1/2}| dx \le Ch^2,$$

$$\begin{aligned} I_4 &= \sum_{l=1}^{i} h_{l+1}(\beta\rho)_{l+1/2}(|u_{l+1/2}|(u-U)_{l+1/2} - (|u|-|U|)_{l+1/2}U_{l+1/2}) \\ &\le \sum_{l=1}^{i} h_{l+1}(\beta\rho)_{l+1/2}(|u_{l+1/2}| + |U_{l+1/2}|)|(u-U)_{l+1/2}| \\ &\le Ch^2. \end{aligned}$$

Here we have used the fact that $|U_{l+1/2}| \le |u_{l+1/2}| + |E^u_{l+1/2}|$ is bounded and

$$|(|u|-|U|)_{l+1/2}| \le |(u-U)_{l+1/2}|.$$

Now we estimate $I_1 + I_3$:

$$\begin{aligned} I_1 + I_3 &= \sum_{l=1}^{i} -\int_{x_l}^{x_{l+1}} \left(\left(\frac{\mu}{k} + \beta\rho|u| \right) u - \left(\frac{\mu}{k} + \beta\rho|u| \right)_{l+1/2} u_{l+1/2} \right) dx \\ &= \sum_{l=1}^{i} \int_{x_l}^{x_{l+1}} (p'(x) - p'(x_{l+1/2}))dx. \end{aligned}$$

Similar to [13], using Taylor's expansion we can prove that

$$I_1 + I_3 \le Ch^2,$$

which completes the proof. ∎

4. A block-centered finite difference method for the two dimensional problem. In this section we consider the block-centered finite difference method for the two dimensional problem. In this case the problem (2.3) and the consistency condition (2.4) can be written as

$$\text{(4.1)} \qquad \begin{cases} \text{(i)} & \left(\frac{\mu}{k} + \beta\rho|\boldsymbol{u}| \right) \boldsymbol{u} + \nabla p = 0 \quad \text{in } \Omega = (0,1)\times(0,1), \\ \text{(ii)} & \nabla \cdot \boldsymbol{u} = f \quad \text{in } \Omega, \\ \text{(iii)} & \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$$\text{(4.2)} \qquad \int_\Omega f \ \mathrm{d}x\mathrm{d}y = 0.$$

Define

$$\text{(4.3)} \qquad F_{i,j} = \frac{1}{|\Omega_{i,j}|}\int_{\Omega_{i,j}} f dx = \frac{1}{|\Omega_{i,j}|}\int_{y_{j-1/2}}^{y_{j+1/2}} \int_{x_{i-1/2}}^{x_{i+1/2}} f dx dy.$$

Then $f_{i,j}$ is an approximation to $F_{i,j}$ with second-order accuracy

$$\text{(4.4)} \qquad f_{i,j} = F_{i,j} + O(h^2 + k^2).$$

For the definition of the scheme we make some preparations. For a discrete function $\{q_{i,j}\}$ with value on nodal points $\{x_{i,j}\}$, define a piecewise-constant function on Ω such that

$$\text{(4.5)} \qquad \Pi_h q(x,y) = q_{i,j}, \quad (x,y) \in \Omega_{i,j}.$$

For a pair of discrete functions $\{V^x_{i+1/2,j}\}$ and $\{V^y_{i,j+1/2}\}$ define the interpolant operator Π_2 as follows:

$$\Pi_2 V = (\Pi_x V^x, \Pi_y V^y), \tag{4.6}$$

where

$$\Pi_x V^x(x,y) = V^x_{i+1/2,j}, \quad (x,y) \in \Omega_{i+1/2,j}, \tag{4.7}$$

$$\Pi_y V^y(x,y) = V^y_{i,j+1/2}, \quad (x,y) \in \Omega_{i,j+1/2}. \tag{4.8}$$

Define $R(U,V)$ as the norm function for a vector (U,V):

$$R(U,V) = (U^2+V^2)^{\frac{1}{2}}. \tag{4.9}$$

Then define the square root average operators Q_x and Q_y as

$$[Q_x V]_{i+1/2,j} = \frac{1}{|\Omega_{i+1/2,j}|}\int_{\Omega_{i+1/2,j}} R(\Pi_x V^x, \Pi_y V^y)dxdy, \tag{4.10}$$

$$[Q_y V]_{i,j+1/2} = \frac{1}{|\Omega_{i,j+1/2}|}\int_{\Omega_{i,j+1/2}} R(\Pi_x V^x, \Pi_y V^y)dxdy. \tag{4.11}$$

Direct calculation shows that

$$\begin{aligned}&[Q_x V]_{i+1/2,j}\\ &= \frac{1}{4h_{i+1/2}}\Big\{h_i[R(V^x_{i+1/2,j}, V^y_{i,j+1/2}) + R(V^x_{i+1/2,j}, V^y_{i,j-1/2})]\\ &\quad + h_{i+1}[R(V^x_{i+1/2,j}, V^y_{i+1,j+1/2}) + R(V^x_{i+1/2,j}, V^y_{i+1,j-1/2})]\Big\},\end{aligned} \tag{4.12}$$

$$\begin{aligned}&[Q_y V]_{i,j+1/2}\\ &= \frac{1}{4k_{j+1/2}}\Big\{k_j[R(V^x_{i-1/2,j}, V^y_{i,j+1/2}) + R(V^x_{i+1/2,j}, V^y_{i,j+1/2})]\\ &\quad + k_{j+1}[R(V^x_{i-1/2,j+1}, V^y_{i,j+1/2}) + R(V^x_{i+1/2,j+1}, V^y_{i,j+1/2})]\Big\}.\end{aligned} \tag{4.13}$$

For simplicity we use the following notation:

$$a_1 = \frac{\mu}{k}, \quad a_2 = \beta\rho, \quad a(w) = a_1 + a_2 w. \tag{4.14}$$

We suppose that $a_2 = \beta\rho$ is a positive constant.

Denote the analytical velocity by $u = (u^x, u^y)$. The block-centered finite difference approximations $\{U^x_{i+1/2,j}\}$, $\{U^y_{i,j+1/2}\}$, and $\{P_{i,j}\}$ to $\{u^x(x_{i+1/2,j})\}$, $\{u^y(x_{i,j+1/2})\}$, and $\{p(x_{i,j})\}$, respectively, are chosen so that

$$[D_x U^x]_{i,j} + [D_y U^y]_{i,j} = f_{i,j}, \tag{4.15}$$

$$\left(a_{1,i+1/2,j} + a_2[Q_x U]_{i+1/2,j}\right)U^x_{i+1/2,j} = -[d_x P]_{i+1/2,j}, \tag{4.16}$$

$$\left(a_{1,i,j+1/2} + a_2[Q_y U]_{i,j+1/2}\right)U^y_{i,j+1/2} = -[d_y P]_{i,j+1/2}, \tag{4.17}$$

with boundary condition

$$U^x_{1/2,j} = 0,\ U^x_{N_x+1/2,j} = 0, \quad j = 0,\dots,N_y, \tag{4.18}$$

$$U^y_{i,1/2} = 0,\ U^y_{i,N_y+1/2} = 0, \quad i = 0,\dots,N_x. \tag{4.19}$$

The pressure p and its approximation P are uniquely determined with the difference of a constant. To determine them uniquely we use the following condition:

$$p(x_{1,1}) = P_{1,1} = 0. \tag{4.20}$$

Remark. From (4.16) and (4.17) we know that there exist two positive nonlinear discrete functions

$$\omega^x_{i+1/2,j} = \omega^x_{i+1/2,j}(|d_xP|, |d_yP|) = \frac{1}{\big(a_{1,i+1/2,j} + a_2[Q_xU]_{i+1/2,j}\big)} > 0,$$
$$\omega^y_{i,j+1/2} = \omega^y_{i,j+1/2}(|d_xP|, |d_yP|) = \frac{1}{\big(a_{1,i,j+1/2} + a_2[Q_yU]_{i,j+1/2}\big)} > 0,$$

which depend on $\{|[d_xP]_{i+1/2,j}|\}$ and $\{|[d_yP]_{i,j+1/2}|\}$, such that

$$U^x_{i+1/2,j} = -\omega^x_{i+1/2,j}[d_xP]_{i+1/2,j}, \quad U^y_{i,j+1/2} = -\omega^y_{i,j+1/2}[d_yP]_{i,j+1/2}.$$

Then from (4.15)

$$-\frac{[\omega^x d_xP]_{i+1/2,j} - [\omega^x d_xP]_{i-1/2,j}}{h_i} - \frac{[\omega^y d_yP]_{i,j+1/2} - [\omega^y d_yP]_{i,j-1/2}}{k_j} = f_{i,j}. \tag{4.21}$$

Equation (4.21) is a nonlinear finite difference scheme for the pressure equation. Then the approximate solution $\{P_{i,j}\}$ is uniquely determined under the condition (4.20). Consequently $\{U^x_{i+1/2,j}\}$ and $\{U^y_{i,j+1/2}\}$ are determined uniquely.

We now verify that if the analytical solutions u and p are sufficiently smooth, (U^x, U^y) is a second-order approximation to (u^x, u^y). For this purpose we present some lemmas first.

The next lemma can be proved similarly to [13]; see the appendix therein.

LEMMA 4.1. *If p and u are sufficiently smooth, then there exist $\{\tilde{P}_{i,j}\}$, $\{\tilde{U}^x_{i+1/2,j}\}$, and $\{\tilde{U}^y_{i,j+1/2}\}$ such that*

$$\begin{cases} (a(|u|))_{i+1/2,j}\tilde{U}^x_{i+1/2,j} = -[d_x\tilde{P}]_{i+1/2,j}, \\ (a(|u|))_{i,j+1/2}\tilde{U}^y_{i,j+1/2} = -[d_y\tilde{P}]_{i,j+1/2}, \end{cases} \tag{4.22}$$

and with the following approximate properties:

$$\begin{cases} |p_{i,j} - \tilde{P}_{i,j}| = O(h^2 + k^2), \\ |u^x_{i+1/2,j} - \tilde{U}^x_{i+1/2,j}| = O(h^2 + k^2), \\ |u^y_{i,j+1/2} - \tilde{U}^y_{i,j+1/2}| = O(h^2 + k^2). \end{cases} \tag{4.23}$$

LEMMA 4.2. *If p and u are sufficiently smooth, then there exist $\{\tilde{\tilde{P}}_{i,j}\}$, $\{\tilde{\tilde{U}}^x_{i+1/2,j}\}$, and $\{\tilde{\tilde{U}}^y_{i,j+1/2}\}$ such that*

$$\begin{cases} a_{i+1/2,j}([Q_xu]_{i+1/2,j})\tilde{\tilde{U}}^x_{i+1/2,j} = -[d_x\tilde{\tilde{P}}]_{i+1/2,j}, \\ a_{i,j+1/2}([Q_yu]_{i,j+1/2})\tilde{\tilde{U}}^y_{i,j+1/2} = -[d_y\tilde{\tilde{P}}]_{i,j+1/2}, \end{cases} \tag{4.24}$$

and with the following approximate properties:

$$
\left\{
\begin{aligned}
&|p_{i,j} - \tilde{\tilde{P}}_{i,j}| &&= O(h^2 + k^2),\\
&|u^x_{i+1/2,j} - \tilde{\tilde{U}}^x_{i+1/2,j}| &&= O(h^2 + k^2),\\
&|u^y_{i,j+1/2} - \tilde{\tilde{U}}^y_{i,j+1/2}| &&= O(h^2 + k^2).
\end{aligned}
\right.
\tag{4.25}
$$

Proof. By (4.23) we have that

$$
(u^x - \tilde{U}^x)_{i+1/2,j} = O(h^2 + k^2), \quad (u^y - \tilde{U}^y)_{i,j+1/2} = O(h^2 + k^2).
$$

Using (4.22) we have that

$$
\left\{
\begin{aligned}
&a_{i+1/2,j}([Q_x u]_{i+1/2,j})\tilde{U}^x_{i+1/2,j}\\
&\quad = -[d_x\tilde{P}]_{i+1/2,j} + (a_{i+1/2,j}([Q_x u]_{i+1/2,j}) - (a(|u|))_{i+1/2,j})u^x_{i+1/2,j}\\
&\qquad + O(h^2 + k^2),\\
&a_{i,j+1/2}([Q_y u]_{i,j+1/2})\tilde{U}^y_{i,j+1/2}\\
&\quad = -[d_y\tilde{P}]_{i,j+1/2} + (a_{i,j+1/2}([Q_y u]_{i,j+1/2}) - (a(|u|))_{i,j+1/2})u^y_{i,j+1/2}\\
&\qquad + O(h^2 + k^2).
\end{aligned}
\right.
\tag{4.26}
$$

Now we estimate $a([Q_x u]_{i+1/2,j}) - a(|u_{i+1/2,j}|)$:

$$
a_{i+1/2,j}([Q_x u]_{i+1/2,j}) - (a(|u|))_{i+1/2,j} = a_2([Q_x u]_{i+1/2,j} - |u_{i+1/2,j}|).
$$

We have used the superscripts x and y to distinguish the velocity on the x- and y-directions. To avoid confusion from now on we use (ξ,η) to denote the coordinate in a two dimensional domain.

There are four parts in the definition of $[Q_x u]_{i+1/2,j}$. We estimate $R(u^x_{i+1/2,j}, u^y_{i,j+1/2})$ first. For simplicity define

$$
r(\xi,\eta) = R(u^x_{i+1/2,j}, u^y(\xi,\eta)) = \left((u^x_{i+1/2,j})^2 + (u^y(\xi,\eta))^2\right)^{\frac{1}{2}}.
$$

Direct calculation shows that

$$
\begin{aligned}
\frac{\partial r(\xi,\eta)}{\partial \xi} &= \frac{u^y(\xi,\eta)}{r(\xi,\eta)}\frac{\partial u^y(\xi,\eta)}{\partial \xi},\\
\frac{\partial r(\xi,\eta)}{\partial \eta} &= \frac{u^y(\xi,\eta)}{r(\xi,\eta)}\frac{\partial u^y(\xi,\eta)}{\partial \eta},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 r(\xi,\eta)}{\partial \xi^2} &= \frac{(u^x_{i+1/2,j})^2}{r(\xi,\eta)^3}\left(\frac{\partial u^y(\xi,\eta)}{\partial \xi}\right)^2 + \frac{u^y(\xi,\eta)}{r(\xi,\eta)}\frac{\partial^2 u^y(\xi,\eta)}{\partial \xi^2},\\
\frac{\partial^2 r(\xi,\eta)}{\partial \eta^2} &= \frac{(u^x_{i+1/2,j})^2}{r(\xi,\eta)^3}\left(\frac{\partial u^y(\xi,\eta)}{\partial \eta}\right)^2 + \frac{u^y(\xi,\eta)}{r(\xi,\eta)}\frac{\partial^2 u^y(\xi,\eta)}{\partial \eta^2},\\
\frac{\partial^2 r(\xi,\eta)}{\partial \xi\partial \eta} &= \frac{(u^x_{i+1/2,j})^2}{r(\xi,\eta)^3}\frac{\partial u^y(\xi,\eta)}{\partial \xi}\frac{\partial u^y(\xi,\eta)}{\partial \eta} + \frac{u^y(\xi,\eta)}{r(\xi,\eta)}\frac{\partial^2 u^y(\xi,\eta)}{\partial \xi\partial \eta}.
\end{aligned}
$$

By Taylor's expansion, there exists a constant $\theta_1 \in (0,1)$ such that

$$
\begin{aligned}
r(x_i, y_{j+1/2}) &= r(x_{i+1/2}, y_j) - \frac{\partial r(x_{i+1/2}, y_j)}{\partial \xi}\frac{h_i}{2} + \frac{\partial r(x_{i+1/2}, y_j)}{\partial \eta}\frac{k_j}{2}\\
&\quad + \left(-\frac{h_i}{2}\frac{\partial}{\partial \xi} + \frac{k_j}{2}\frac{\partial}{\partial \eta}\right)^2 r\left(x_{i+1/2} - \frac{\theta_1 h_i}{2}, y_j + \frac{\theta_1 k_j}{2}\right).
\end{aligned}
\tag{4.27}
$$

It is clear that

$$\begin{aligned}r(x_i, y_{j+1/2}) &= R(u^x_{i+1/2,j}, u^y_{i,j+1/2}),\\ r(x_{i+1/2}, y_j) &= |u^x_{i+1/2,j}|,\\ \frac{\partial r(x_{i+1/2}, y_j)}{\partial \xi} &= \frac{u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \xi},\\ \frac{\partial r(x_{i+1/2}, y_j)}{\partial \eta} &= \frac{u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \eta}.\end{aligned}$$

Then from (4.27) we have that

$$\begin{aligned}R(u^x_{i+1/2,j}, u^y_{i,j+1/2}) = |u^x_{i+1/2,j}|&\\ -\frac{u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \xi}\frac{h_i}{2} &+ \frac{u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \eta}\frac{k_j}{2}\\ &+ \left(-\frac{h^i}{2}\frac{\partial}{\partial \xi} + \frac{k_j}{2}\frac{\partial}{\partial \eta}\right)^2 r\left(x_{i+1/2} - \frac{\theta_1 h_i}{2}, y_j + \frac{\theta_1 k_j}{2}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left(R(u^x_{i+1/2,j}, u^y_{i,j+1/2}) - |u^x_{i+1/2,j}|\right) u^x_{i+1/2,j}&\\ = -\frac{u^x_{i+1/2,j}u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \xi}\frac{h_i}{2} &+ \frac{u^x_{i+1/2,j}u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \eta}\frac{k_j}{2}\\ + u^x_{i+1/2,j}\left(-\frac{h_i}{2}\frac{\partial}{\partial \xi} + \frac{k_j}{2}\frac{\partial}{\partial \eta}\right)^2 &r\left(x_{i+1/2} - \frac{\theta_1 h_i}{2}, y_j + \frac{\theta_1 k_j}{2}\right).\end{aligned} \tag{4.28}$$

Similarly, there exists a constant $\theta_2 \in (0, 1)$ such that

$$\begin{aligned}\left(R(u^x_{i+1/2,j}, u^y_{i,j-1/2}) - |u^x_{i+1/2,j}|\right) u^x_{i+1/2,j}&\\ = -\frac{u^x_{i+1/2,j}u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \xi}\frac{h_i}{2} &- \frac{u^x_{i+1/2,j}u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \eta}\frac{k_j}{2}\\ + u^x_{i+1/2,j}\left(-\frac{h_i}{2}\frac{\partial}{\partial \xi} - \frac{k_j}{2}\frac{\partial}{\partial \eta}\right)^2 &r\left(x_{i+1/2} - \frac{\theta_2 h_i}{2}, y_j - \frac{\theta_2 k_j}{2}\right).\end{aligned} \tag{4.29}$$

Summing (4.28) and (4.29) we get that

$$\begin{aligned}\left(\frac{R(u^x_{i+1/2,j}, u^y_{i,j+1/2}) + R(u^x_{i+1/2,j}, u^y_{i,j-1/2})}{2} - |u^x_{i+1/2,j}|\right) u^x_{i+1/2,j}&\\ = \frac{u^x_{i+1/2,j}u^y_{i+1/2,j}}{|u_{i+1/2,j}|}\frac{\partial u^y_{i+1/2,j}}{\partial \xi}\left(-\frac{h_i}{2}\right) + O(h^2 + k^2).&\end{aligned} \tag{4.30}$$

The first term on the right-hand side of (4.30) takes the value of the function

$$G(\xi, \eta) = \frac{u^x(\xi, \eta)u^y(\xi, \eta)}{(u^x(\xi, \eta)^2 + u^y(\xi, \eta)^2)^{\frac{1}{2}}}\frac{\partial u^y(\xi, \eta)}{\partial \xi}$$

at point $(x_{i+1/2}, y_j)$. Direct calculation shows that the first-order derivatives of $G(\xi, \eta)$ are bounded, and so

$$G(x_{i+1/2}, y_j) = G(x_i, y_j) + O(h).$$

Then

$$\left(\frac{R(u^x_{i+1/2,j}, u^y_{i,j+1/2}) + R(u^x_{i+1/2,j}, u^y_{i,j-1/2})}{2} - |u^x_{i+1/2,j}|\right) u^x_{i+1/2,j} = \frac{u^x_{i,j} u^y_{i,j}}{|u_{i,j}|} \frac{\partial u^y_{i,j}}{\partial \xi} \left(-\frac{h_i}{2}\right) + O(h^2 + k^2). \tag{4.31}$$

Similarly we get that

$$\left(\frac{R(u^x_{i+1/2,j}, u^y_{i+1,j+1/2}) + R(u^x_{i+1/2,j}, u^y_{i+1,j-1/2})}{2} - |u^x_{i+1/2,j}|\right) u^x_{i+1/2,j} = \frac{u^x_{i+1,j} u^y_{i+1,j}}{|u_{i+1,j}|} \frac{\partial u^y_{i+1,j}}{\partial \xi} \frac{h_{i+1}}{2} + O(h^2 + k^2). \tag{4.32}$$

According to the definition of $[Q_x u]_{i+1/2}$, noticing (4.31) and (4.32), we have that

$$\begin{aligned}
&([Q_x u]_{i+1/2,j} - |u_{i+1/2,j}|) u^x_{i+1/2,j} \\
&\quad = \frac{u^x_{i+1/2,j}}{4h_{i+1/2}} \Big\{ h_i \left(R(u^x_{i+1/2,j}, u^y_{i,j+1/2}) + R(u^x_{i+1/2,j}, u^y_{i,j-1/2}) - 2|u_{i+1/2,j}| \right) \\
&\quad\quad + h_{i+1} \left(R(u^x_{i+1/2,j}, u^y_{i+1,j+1/2}) + R(u^x_{i+1/2,j}, u^y_{i+1,j-1/2}) - 2|u_{i+1/2,j}| \right) \Big\} \\
&\quad = \frac{1}{4h_{i+1/2}} \left\{ -\frac{u^x_{i,j} u^y_{i,j}}{|u_{i,j}|} \frac{\partial u^y_{i,j}}{\partial \xi} h_i^2 + \frac{u^x_{i+1,j} u^y_{i+1,j}}{|u_{i+1,j}|} \frac{\partial u^y_{i+1,j}}{\partial \xi} h_{i+1}^2 \right\} + O(h^2 + k^2).
\end{aligned}$$

Then

$$\begin{aligned}
&(a_{i+1/2,j}([Q_x u]_{i+1/2,j}) - (a(|u|))_{i+1/2,j}) u^x_{i+1/2,j} \\
&\quad = a_2([Q_x u]_{i+1/2,j} - |u_{i+1/2,j}|) u^x_{i+1/2,j} \\
&\quad = \frac{a_2}{4h_{i+1/2}} \left\{ -\frac{u^x_{i,j} u^y_{i,j}}{|u_{i,j}|} \frac{\partial u^y_{i,j}}{\partial \xi} h_i^2 + \frac{u^x_{i+1,j} u^y_{i+1,j}}{|u_{i+1,j}|} \frac{\partial u^y_{i+1,j}}{\partial \xi} h_{i+1}^2 \right\} + O(h^2 + k^2).
\end{aligned} \tag{4.33}$$

Similarly we can get

$$\begin{aligned}
&(a_{i,j+1/2}([Q_y u]_{i,j+1/2}) - (a(|u|))_{i,j+1/2}) u^y_{i,j+1/2} \\
&\quad = \frac{a_2}{4k_{j+1/2}} \left\{ -\frac{u^x_{i,j} u^y_{i,j}}{|u_{i,j}|} \frac{\partial u^x_{i,j}}{\partial \eta} k_j^2 + \frac{u^x_{i,j+1} u^y_{i,j+1}}{|u_{i,j+1}|} \frac{\partial u^x_{i,j+1}}{\partial \eta} k_{j+1}^2 \right\} + O(h^2 + k^2).
\end{aligned} \tag{4.34}$$

Define

$$\tilde{\tilde{P}}_{i,j} = \tilde{P}_{i,j} - \frac{a_2}{4} \frac{u^x_{i,j} u^y_{i,j}}{|u_{i,j}|} \frac{\partial u^y_{i,j}}{\partial \xi} h_i^2 - \frac{a_2}{4} \frac{u^x_{i,j} u^y_{i,j}}{|u_{i,j}|} \frac{\partial u^x_{i,j}}{\partial \eta} k_j^2;$$

then

$$\tilde{\tilde{P}}_{i,j} - p_{i,j} = O(h^2 + k^2).$$

From (4.26), (4.33), and (4.34) we have the following form:

$$\left\{\begin{array}{l} a_{i+1/2,j}([Q_x u]_{i+1/2,j})[\tilde{U}^x_{i+1/2,j} + O(h^2+k^2)] = -[d_x \tilde{\tilde{P}}]_{i+1/2,j}, \\ a_{i,j+1/2}([Q_y u]_{i,j+1/2})[\tilde{U}^y_{i,j+1/2} + O(h^2+k^2)] = -[d_y \tilde{\tilde{P}}]_{i,j+1/2}. \end{array}\right. \tag{4.35}$$

Then define the parts $[\tilde{U}^x_{i+1/2,j} + O(h^2+k^2)]$, $[\tilde{U}^y_{i,j+1/2} + O(h^2+k^2)]$ as $\tilde{\tilde{U}}^x_{i+1/2,j}$, $\tilde{\tilde{U}}^y_{i,j+1/2}$, respectively; we get (4.24) and the error estimate (4.25). Here for simplicity we did not give the exact form of $O(h^2+k^2)$.

We complete the proof of the lemma. ∎

LEMMA 4.3. *Let $\boldsymbol{x}, \boldsymbol{h} \in R^d$. The vector-valued function $\boldsymbol{f} : R^d \to R^d$ is defined as $\boldsymbol{f}(\boldsymbol{x}) = |\boldsymbol{x}|\boldsymbol{x}$. Then there exists a positive constant C_1 such that*

$$C_1(|\boldsymbol{x}| + |\boldsymbol{x}+\boldsymbol{h}|)|\boldsymbol{h}|^2 \le (\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x})) \cdot \boldsymbol{h}. \tag{4.36}$$

Proof. The inequality can be concluded by using Taylor's expansion. Its proof is similar to those in [5, 8, 12]. ∎

LEMMA 4.4. *For any functions $\boldsymbol{V} = (V^x, V^y)$ and $\boldsymbol{W} = (W^x, W^y)$ we have that*

$$(|\boldsymbol{V}|\boldsymbol{V} - |\boldsymbol{W}|\boldsymbol{W}, \boldsymbol{V} - \boldsymbol{W}) \ge 0, \tag{4.37}$$

and further we have that

$$(a(|\boldsymbol{V}|)\boldsymbol{V} - a(|\boldsymbol{W}|)\boldsymbol{W}, \boldsymbol{V} - \boldsymbol{W}) \ge \bar{a}||\boldsymbol{V} - \boldsymbol{W}||^2. \tag{4.38}$$

Here the constant $\bar{C}$ is defined as in (2.5).

Proof. Using Lemma 4.3, we easily get

$$\int_\Omega (|\boldsymbol{V}|\boldsymbol{V} - |\boldsymbol{W}|\boldsymbol{W}) \cdot (\boldsymbol{V} - \boldsymbol{W})\, \mathrm{d}\boldsymbol{x} \ge C_1 \int_\Omega (|\boldsymbol{V}| + |\boldsymbol{W}|)|\boldsymbol{V} - \boldsymbol{W}|^2\, \mathrm{d}\boldsymbol{x},$$

which completes the proof of (4.37).

By the definition of $a(|\boldsymbol{V}|)$ we can get (4.38) directly. ∎

LEMMA 4.5. *If p and u are sufficiently smooth, then there exist $\hat{P}_{i,j}$, $\hat{U}^x_{i+1/2,j}$, and $\hat{U}^y_{i,j+1/2}$ such that*

$$\left\{\begin{array}{lcl} a_{i+1/2,j}([Q_x \hat{U}]_{i+1/2,j})\hat{U}^x_{i+1/2,j} & = & -[d_x \hat{P}]_{i+1/2,j}, \\ a_{i,j+1/2}([Q_y \hat{U}]_{i,j+1/2})\hat{U}^y_{i,j+1/2} & = & -[d_y \hat{P}]_{i,j+1/2}, \end{array}\right. \tag{4.39}$$

and with the following approximate properties:

$$\left\{\begin{array}{lcl} |p_{i,j} - \hat{P}_{i,j}| & = & O(h^2+k^2), \\ ||\Pi_2(u - \hat{U})|| & = & O(h^2+k^2). \end{array}\right. \tag{4.40}$$

Proof. Let $\hat{P}_{i,j} = \tilde{\tilde{P}}_{i,j}$, and define $\hat{U}^x_{i+1/2,j}$ and $\hat{U}^y_{i,j+1/2}$ as follows:

$$\left\{\begin{array}{l} a_{i+1/2,j}([Q_x \hat{U}]_{i+1/2,j})\hat{U}^x_{i+1/2,j} = a_{i+1/2,j}([Q_x u]_{i+1/2,j})\tilde{\tilde{U}}^x_{i+1/2,j}, \\ a_{i,j+1/2}([Q_y \hat{U}]_{i,j+1/2})\hat{U}^y_{i,j+1/2} = a_{i,j+1/2}([Q_x u]_{i,j+1/2})\tilde{\tilde{U}}^y_{i,j+1/2}. \end{array}\right. \tag{4.41}$$

Then from Lemma 4.2 we know that (4.39) and the first evaluation of (4.40) hold.

From the definition of $\hat{U}^x_{i+1/2,j}$, we have that, for any discrete function $\{V^x_{i+1/2,j}\}$,

$$\int_{\Omega_{i+1/2,j}} a_{i+1/2,j}([Q_x\hat{U}])\Pi_x\hat{U}^x\Pi_x V^x dxdy = \int_{\Omega_{i+1/2,j}} a_{i+1/2,j}([Q_x u])\Pi_x\tilde{\tilde{U}}^x\Pi_x V^x dxdy.$$

From the definitions of Q_x and Π_2 we can rewrite the above equation as follows:

$$\begin{aligned}\int_{\Omega_{i+1/2,j}} & a_{i+1/2,j}(|\Pi_2\hat{U}|)\Pi_x\hat{U}^x\Pi_x V^x dxdy \\ &= \int_{\Omega_{i+1/2,j}} a_{i+1/2,j}(|\Pi_2 u|)\Pi_x\tilde{\tilde{U}}^x\Pi_x V^x dxdy.\end{aligned} \tag{4.42}$$

Similarly, for any discrete function $\{V^y_{i,j+1/2}\}$ we have that

$$\begin{aligned}\int_{\Omega_{i,j+1/2}} & a_{i,j+1/2}(|\Pi_2\hat{U}|)\Pi_y\hat{U}^y\Pi_y V^y dxdy \\ &= \int_{\Omega_{i,j+1/2}} a_{i,j+1/2}(|\Pi_2 u|)\Pi_y\tilde{\tilde{U}}^y\Pi_y V^y dxdy.\end{aligned} \tag{4.43}$$

For discrete functions denote the inner product by

$$\begin{aligned}\hat{a}(\Pi_2 U, \Pi_2 V) \equiv & \sum_{ij}\int_{\Omega_{i+1/2,j}} a_{1,i+1/2,j}\Pi_x U^x\Pi_x V^x dxdy \\ & + \sum_{ij}\int_{\Omega_{i,j+1/2}} a_{1,i,j+1/2}\Pi_y U^y\Pi_y V^y dxdy \\ & = (a_1\Pi_x U^x, \Pi_x V^x)_x + (a_1\Pi_y U^y, \Pi_y V^y)_y.\end{aligned} \tag{4.44}$$

Summing (4.42) and (4.43) for all possible $\Omega_{i+1/2,j}$ and $\Omega_{i,j+1/2}$, we have that

$$\begin{aligned}&\hat{a}(\Pi_2\hat{U}, \Pi_2 V) + (a_2|\Pi_2\hat{U}|\Pi_2\hat{U}, \Pi_2 V) \\ &= \hat{a}(\Pi_2\tilde{\tilde{U}}, \Pi_2 V) + (a_2|\Pi_2 u|\Pi_2\tilde{\tilde{U}}, \Pi_2 V).\end{aligned} \tag{4.45}$$

Moving the first term on the right-hand side to the left-hand side and subtracting $(a_2|\Pi_2\tilde{\tilde{U}}|)\Pi_2\tilde{\tilde{U}}, \Pi_2 V)$ on both sides of (4.45), we have that

$$\begin{aligned}&\hat{a}(\Pi_2(\hat{U}-\tilde{\tilde{U}}), \Pi_2 V) + (a_2(|\Pi_2\hat{U}|\Pi_2\hat{U} - |\Pi_2\tilde{\tilde{U}}|\Pi_2\tilde{\tilde{U}}), \Pi_2 V) \\ &= (a_2(|\Pi_2 u| - |\Pi_2\tilde{\tilde{U}}|)\Pi_2\tilde{\tilde{U}}, \Pi_2 V).\end{aligned} \tag{4.46}$$

Let $V^x = \hat{U}^x - \tilde{\tilde{U}}^x$ and $V^y = \hat{U}^y - \tilde{\tilde{U}}^y$; using Lemma 4.4 we have that

$$\begin{aligned}&\bar{a}||\Pi_2(\hat{U}-\tilde{\tilde{U}})||^2 \\ &\le \hat{a}(\Pi_2(\hat{U}-\tilde{\tilde{U}}), \Pi_2(\hat{U}-\tilde{\tilde{U}})) + a_2(|\Pi_2\hat{U}|\Pi_2\hat{U} - |\Pi_2\tilde{\tilde{U}}|\Pi_2\tilde{\tilde{U}}, \Pi_2(\hat{U}-\tilde{\tilde{U}})) \\ &= a_2((|\Pi_2 u| - |\Pi_2\tilde{\tilde{U}}|)\Pi_2\tilde{\tilde{U}}, \Pi_2(\hat{U}-\tilde{\tilde{U}})) \\ &\le C(h^2+k^2)\|\Pi_2(\hat{U}-\tilde{\tilde{U}})||.\end{aligned} \tag{4.47}$$

Then

$$||\Pi_2(\hat{U} - \Pi_2\tilde{\tilde{U}})|| \leq C(h^2 + k^2). \tag{4.48}$$

Combining this with the error estimate of $u - \tilde{\tilde{U}}$ completes the proof. ∎

LEMMA 4.6. *Let* $\{V^x_{i+1/2,j}\}, \{V^y_{i,j+1/2}\}$, $\{W^x_{i+1/2,j}\}, \{W^y_{i,j+1/2}\}$, $\{q^x_{i,j}\}$, *and* $\{q^y_{i,j}\}$ *be discrete functions with* $W^x_{1/2,j} = W^x_{N_x+1/2,j} = W^y_{i,1/2} = W^y_{i,N_y+1/2} = 0$ *and*

$$\left\{\begin{array}{lcl} [\omega^x V^x]_{i+1/2,j} & = & -[d_x q^x]_{i+1/2,j}, \\ [\omega^y V^y]_{i,j+1/2} & = & -[d_y q^y]_{i,j+1/2}, \end{array}\right. \tag{4.49}$$

where ω^x *and* ω^y *are generic discrete functions. Then the following hold:*

$$(\omega^x V^x, W^x)_x = (q^x, D_x W^x)_M, \quad (\omega^y V^y, W^y)_y = (q^y, D_y W^y)_M. \tag{4.50}$$

Proof. Summing by parts, using condition (4.49) we can achieve the desired results. ∎

Now we give the error estimates for $u^x - U^x$ and $u^y - U^y$.

THEOREM 4.7. *If the solutions* u *and* p *are sufficiently smooth, then there exists a positive constant* C *independent of* h *and* k *such that*

$$||u^x - U^x||_x + ||u^y - U^y||_y \leq C(h^2 + k^2). \tag{4.51}$$

Proof. Define the auxiliary one dimensional finite difference approximations $U^{1Dx}_{i+1/2,j}$, $P^{1Dx}_{i,j}$, $U^{1Dy}_{i,j+1/2}$, and $P^{1Dy}_{i,j}$ to $u^x_{i+1/2,j}$, $p_{i,j}$, $u^y_{i,j+1/2}$, and $p_{i,j}$, respectively, by

$$\left\{\begin{array}{l} a_{i+1/2,j}([Q_x U^{1Dx}]_{i+1/2,j}) U^{1Dx}_{i+1/2,j} = -[d_x P^{1Dx}]_{i+1/2,j}, \\ [D_x U^{1Dx}]_{i,j} = \dfrac{1}{|\Omega_{i,j}|}\displaystyle\int_{\Omega_{i,j}} \left(f - \frac{\partial u^y}{\partial \eta}\right) d\xi d\eta, \quad U^{1Dx}_{1/2,j} = 0, \end{array}\right. \tag{4.52}$$

$$\left\{\begin{array}{l} a_{i,j+1/2}([Q_x U^{1Dy}]_{i,j+1/2}) U^{1Dy}_{i,j+1/2} = -[d_y P^{1Dy}]_{i,j+1/2}, \\ [D_y U^{1Dy}]_{i,j} = \dfrac{1}{|\Omega_{i,j}|}\displaystyle\int_{\Omega_{i,j}} \left(f - \frac{\partial u^x}{\partial \xi}\right) d\xi d\eta, \quad U^{1Dy}_{i,1/2} = 0. \end{array}\right. \tag{4.53}$$

Since

$$\int_0^1 \left(f - \frac{\partial u^y}{\partial \eta}\right) d\xi = \int_0^1 \left(\frac{\partial u^x}{\partial \xi}\right) d\xi = u^x(1,\eta) - u^x(0,\eta) = 0,$$

we have

$$\sum_i h_i \frac{1}{|\Omega_{i,j}|}\int_{\Omega_{i,j}} \left(f - \frac{\partial u^y}{\partial \eta}\right) d\xi d\eta = \frac{1}{k_j}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_0^1 \left(f - \frac{\partial u^y}{\partial \eta}\right) d\xi d\eta = 0. \tag{4.54}$$

Similarly

$$\sum_j k_j \frac{1}{|\Omega_{i,j}|}\int_{\Omega_{i,j}} \left(f - \frac{\partial u^x}{\partial \xi}\right) d\xi d\eta = 0. \tag{4.55}$$

From the equations in (4.52) and (4.53) we know that U^{1Dx} and U^{1Dy} are well defined. Then the equations for P^{1Dx} and P^{1Dy} become linear; therefore, P^{1Dx} and P^{1Dy} are also well defined.

By the error estimate theorem in the last section we have that

$$\|u^x - U^{1Dx}\|_x \le Ch^2, \quad \|u^y - U^{1Dy}\|_y \le Ck^2. \tag{4.56}$$

To get the desired results it suffices to estimate $(U^x - U^{1Dx})$ and $(U^y - U^{1Dy})$.

From (4.15), (4.52), and (4.53) we have that

$$[D_x(U^x - U^{1Dx}) + D_y(U^y - U^{1Dy})]_{i,j} = 0 \tag{4.57}$$

for all (i,j) such that $(x_i, y_j) \in \Omega$.

Also from conditions (4.54) and (4.55) we have that

$$U^{1Dx}_{N_x+1/2,j} = 0, \quad U^{1Dy}_{i,N_y+1/2} = 0. \tag{4.58}$$

From Lemma 4.4, assumption (2.5), and the Schwarz inequality

$$\begin{aligned}
\bar{a}||\Pi_2(U^{1D} - U)||^2 &= \bar{a}(||U^{1Dx} - U^x||_x^2 + ||U^{1Dy} - U^y||_y^2) \\
&\le \hat{a}(\Pi_2(U^{1D} - U), \Pi_2(U^{1D} - U)) + (a_2(|U^{1D}|U^{1D} - |U|U), U^{1D} - U) \\
&= \left(a(Q_x U^{1Dx})\Pi_x U^{1Dx} - a(Q_x U)\Pi_x U^x, \Pi_x(U^{1Dx} - U^x)\right)_x \\
&\quad + \left(a(Q_y U^{1Dy})\Pi_y U^{1Dy} - a(Q_y U^y)\Pi_y U^y, \Pi_y(U^{1Dy} - U^y)\right)_y \\
&= \left(-d_x P^{1Dx} + d_x P, U^{1Dx} - U^x\right)_x + \left(-d_y P^{1Dy} + d_y P, U^{1Dy} - U^y\right)_y .
\end{aligned} \tag{4.59}$$

From (4.52), (4.53), and (4.58) we have that

$$\begin{aligned}
(U^{1Dx} - U)_{1/2,j} &= (U^{1Dx} - U)_{N_x+1/2,j} = 0, \\
(U^{1Dy} - U)_{i,1/2} &= (U^{1Dy} - U)_{i,N_y+1/2} = 0.
\end{aligned}$$

Therefore, by (4.52), (4.53), (4.57), and the definition (4.44) we have that

$$\begin{aligned}
&\left(-d_x P^{1Dx} + d_x P, U^{1Dx} - U^x\right)_x + \left(-d_y P^{1Dy} + d_y P, U^{1Dy} - U^y\right)_y \\
&= \left(P^{1Dx} - P, D_x[U^{1Dx} - U^x]\right)_M + \left(P^{1Dy} - P, D_y[U^{1Dy} - U^y]\right)_M \\
&= (P^{1Dx} - \hat{P}, D_x[U^{1Dx} - U^x])_M - (P^{1Dy} - \hat{P}, D_y[U^{1Dy} - U^y])_M \\
&= -(d_x P^{1Dx} - d_x \hat{P}, U^{1Dx} - U^x)_M + (d_y P^{1Dy} - d_y \hat{P}, U^{1Dy} - U^y)_M \\
&= (a(Q_x U^{1Dx})U^{1Dx} - a(Q_x \hat{U})\hat{U}^x, U^{1Dx} - U^x)_x \\
&\quad + (a(Q_y U^{1Dy})U^{1Dy} - a(Q_y \hat{U}^y)\hat{U}^y, U^{1Dy} - U^y)_y \\
&= \hat{a}(|\Pi_2(U^{1D} - \hat{U}), \Pi_2(U^{1D} - U)) \\
&\quad + (a_2(|\Pi_2 U^{1D}|\Pi_2 U^{1D} - |\Pi_2 \hat{U}|\Pi_2 \hat{U}), \Pi_2(U^{1D} - U)).
\end{aligned} \tag{4.60}$$

From (4.59) and (4.60) we have that

$$\begin{aligned}
||\Pi_2(U^{1D} - U)||^2 &\le C||\Pi_2(U^{1D} - \hat{U})||^2 \\
&= C(||U^{1Dx} - u^x + u^x - \hat{U}^x||_x^2 + ||U^{1Dy} - u^y + u^y - \hat{U}^y||_y^2) \\
&\le C(h^2 + k^2)^2.
\end{aligned}$$

Combining this with (4.56) completes the proof. □

Next we give the error estimate for pressure. Using the duality technique introduced in [13] we can get the following lemma; see proof of Theorem 4.2 in [13].

LEMMA 4.8. *For the discrete solution defined above, there exists a positive constant C independent of h and k such that*

$$\|P - \hat{P}\|_M^2 \le C(\|U^x - \hat{U}^x\|_x^2 + \|U^y - \hat{U}^y\|_y^2). \tag{4.61}$$

THEOREM 4.9. *Under the condition of Theorem* 4.7, there holds

$$||p - P||_M = O(h^2 + k^2). \tag{4.62}$$

Proof. From Lemma 4.8 we have that

$$||P - \hat{P}||_M^2 \le C(\|U^x - \hat{U}^x\|_x^2 + \|U^y - \hat{U}^y\|_y^2) \le C(h^2 + k^2)^2. \tag{4.63}$$

Combining this with Lemma 4.5 completes the proof. ∎

Remark. When $\beta\rho$ is not a constant, using the present proof we cannot get Lemma 4.5 and then cannot get the second-order error estimates. We will consider the problem next.

5. Numerical experiment. In this section we carry out some numerical experiments using the block-centered finite difference scheme in the two dimensional region. For simplicity, the region is selected as unit square; i.e., $\Omega = [0, 1] \times [0, 1]$. The permeability is a constant, and the viscosity, density, and Forchheimer number β are also constants. For simplicity, take $\mu = 2, K = 4, \rho = 1$, and $\beta = 5$.

We test two examples to verify the convergence. In a nonuniform rectangular grid the locations of u^x, u^y, and p are illustrated in Figure 5.1.

The initial partition is 10×10 grid. And then the grid is refined four times. Two different types of fine grid generation are used.

Case A. *Uniform refinement.* Each time the nonequidistant grid is bisected in each direction to generate the next grid.

Case B. *Uniform refinement with random perturbation.* Based on Case A, the refined grid is further randomly perturbed to generate the next grid.

The a priori error estimates in discrete L^2-norms and convergence rates for velocity and pressure are listed in Tables 5.1–5.4 and plotted in Figures 5.2 and 5.3. In the tables the ratios between the maximum and minimum meshsizes in each dimension, i.e., $\frac{h_{max}}{h_{min}}$ and $\frac{k_{max}}{k_{min}}$, are listed to illustrate the nonuniformity of the grid.

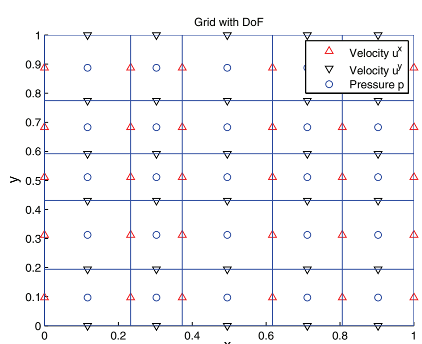

FIG. 5.1. *Location of u^x, u^y, and p.*

TABLE 5.1
Error and convergence rates for Example 1 in Case A.

Partition	Error($\|\boldsymbol{v}\|_{0,2}$)	Rate	Error($\|p\|_{0,2}$)	Rate	$\frac{h_{max}}{h_{min}}$	$\frac{k_{max}}{k_{min}}$
10×10	8.3931E-3	-	1.4529E-2	-	4.6597	2.8856
20×20	2.1773E-3	-1.9441	3.5711E-3	-2.0251	4.6597	2.8856
40×40	5.7292E-4	-1.9255	8.8994E-4	-2.0048	4.6597	2.8856
80×80	1.4520E-4	-1.9802	2.2230E-4	-2.0013	4.6597	2.8856
160×160	3.6682E-5	-1.9848	5.5563E-5	-2.0003	4.6597	2.8856

TABLE 5.2
Error and convergence rates for Example 1 in Case B.

Partition	Error($\|\boldsymbol{v}\|_{0,2}$)	Rate	Error($\|p\|_{0,2}$)	Rate	$\frac{h_{max}}{h_{min}}$	$\frac{k_{max}}{k_{min}}$
10×10	5.2570E-3	-	9.5456E-3	-	1.5461	3.1463
20×20	1.7093E-3	-1.6233	2.3770E-3	-2.0036	1.6626	3.9961
40×40	4.9948E-4	-1.7757	6.1672E-4	-1.9459	2.2831	4.9802
80×80	1.3141E-4	-1.9265	1.5622E-4	-1.9809	2.7609	5.0732
160×160	3.4446E-5	-1.9318	4.0046E-5	-1.9638	3.1673	5.6819

TABLE 5.3
Error and convergence rates for Example 2 in Case A.

Partition	Error($\|\boldsymbol{v}\|_{0,2}$)	Rate	Error($\|p\|_{0,2}$)	Rate	$\frac{h_{max}}{h_{min}}$	$\frac{k_{max}}{k_{min}}$
10×10	3.5154E-2	-	1.3692E-2	-	2.2061	3.2754
20×20	7.8141E-3	-2.1714	3.9470E-3	-1.7965	2.2061	3.2754
40×40	2.1000E-3	-1.8962	9.9522E-4	-1.9881	2.2061	3.2754
80×80	5.3720E-4	-1.9669	2.5039E-4	-1.9909	2.2061	3.2754
160×160	1.3534E-4	-1.9889	6.2720E-5	-1.9972	2.2061	3.2754

Example 1. The analytical solution is as follows:

$$\begin{cases} p(x,y) = (x-x^2)(y-y^2), \\ \boldsymbol{u}(x,y) = (\sin\pi x\cos\pi y, \cos\pi x \sin\pi y)^T. \end{cases}$$

The Neumann boundary condition and the right-hand-side term are determined according to the analytical solution.

The numerical results are listed in Figure 5.2 and Tables 5.1 and 5.2.

Example 2. The analytical solution is as follows:

$$p(x,y) = \arctan 10(x+y-1), \qquad \boldsymbol{u}(x,y) \ = \ (-y,x)^T.$$

The Neumann boundary condition and the right-hand-side term are determined according to the analytical solution.

The numerical results are listed in Figure 5.3 and Tables 5.3 and 5.4.

From Figures 5.2 and 5.3 and Tables 5.1, 5.2, 5.3, and 5.4 we can see that the block-centered finite difference approximations for pressure and velocity have the second-order accuracy in discrete L^2 norms. Concerning fine grid generation, the results of Case A exactly coincide with the second order accuracy, while Case B roughly converges in second order, and this can be viewed as the effect of the random perturbation in the generation of the fine generation. These results are consistent with the error estimates in Theorems 4.7 and 4.9.

TABLE 5.4
Error and convergence rates for Example 2 in Case B.

Partition	Error($\|\boldsymbol{v}\|_{0,2}$)	Rate	Error($\|p\|_{0,2}$)	Rate	$\frac{h_{max}}{h_{min}}$	$\frac{k_{max}}{k_{min}}$
10×10	3.5656E-2	-	1.8637E-2	-	5.0619	2.4768
20×20	9.8392E-3	-1.8594	5.0206E-3	-1.8942	6.0081	2.7031
40×40	2.8969E-3	-1.7645	1.3319E-3	-1.9148	6.7309	3.6329
80×80	8.8228E-4	-1.7153	3.5777E-4	-1.8965	7.7049	4.6325
160×160	2.3420E-4	-1.9135	9.4923E-5	-1.9142	8.9007	5.1970

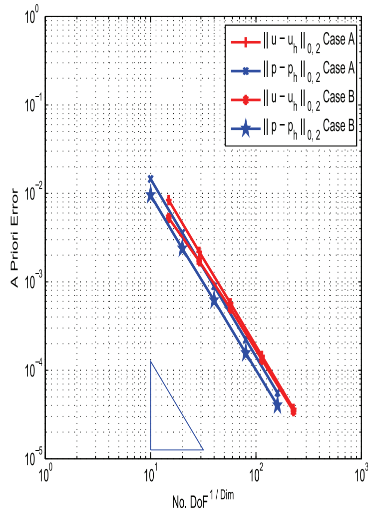

FIG. 5.2. *Convergence rates of Example* 1 *(the tangent of the triangle is* 2*).*

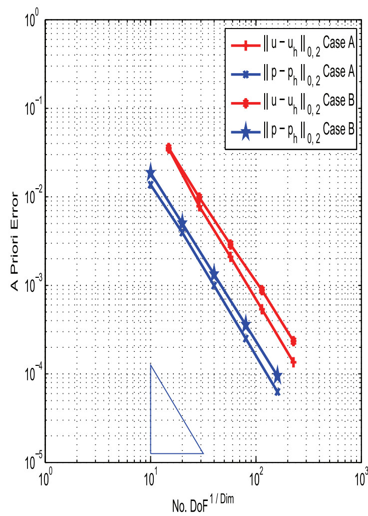

FIG. 5.3. *Convergence rates of Example* 2 *(the tangent of the triangle is* 2*).*

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