FINITE ELEMENT METHODS FOR MAXWELL EQUATIONS

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1. Sobolev Spaces and Weak Formulations

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We introduce the Sobolev spaces

$$H(\operatorname{curl};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},$$
$$H(\operatorname{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \operatorname{div}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\}$$

The vector fields (E, H) belong to $H(\operatorname{curl};\Omega)$ while the flux (D, B) in $H(\operatorname{div};\Omega)$. We shall use the unified notation $H(d;\Omega)$ with $d = \operatorname{grad}, \operatorname{curl}$, or div. Note that $H(\operatorname{grad};\Omega)$ is the familiar $H^1(\Omega)$ space. The norm for $H(d;\Omega)$ is the graph norm

$$\|u\|_{d,\Omega} := \left(\|u\|^2 + \|du\|^2\right)^{1/2}.$$

We recall the integration by parts for vector functions below. Formally the boundary term is obtained by replacing the Hamilton operator by the unit outwards normal vector n. For example

$$\int_\Omega \nabla u \cdot \boldsymbol{\phi}\, \mathrm{d}x = -\int_\Omega u \nabla \cdot \boldsymbol{\phi}\, \mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} u \boldsymbol{\phi}\, \mathrm{d}S,$$
$$\int_\Omega \nabla \times \boldsymbol{u} \cdot \boldsymbol{\phi}\, \mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \nabla \times \boldsymbol{\phi}\, \mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \times \boldsymbol{u} \cdot \boldsymbol{\phi}\, \mathrm{d}S,$$
$$\int_\Omega \nabla \cdot \boldsymbol{u}\, \phi\, \mathrm{d}x = -\int_\Omega \boldsymbol{u} \cdot \nabla \phi\, \mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{u}\, \phi\, \mathrm{d}S.$$

The weak formulation is obtained by multiplying the original equation by a smooth test equation and applying the integration. The boundary condition will be discussed later. For time-harmonic Maxwell equations, the weak formulation for $\boldsymbol{E}$ is

$$(\mu^{-1}\nabla \times \boldsymbol{E}, \nabla \times \boldsymbol{\phi}) - \omega^2(\tilde{\epsilon}\boldsymbol{E}, \phi) = (\tilde{\boldsymbol{J}}, \phi) \quad \forall \boldsymbol{\phi} \in \mathcal{D}(\Omega). \tag{1}$$

And the equation for $\boldsymbol{H}$ is

$$(\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}, \nabla \times \boldsymbol{\phi}) - \omega^2(\mu\boldsymbol{H}, \phi) = (\tilde{\boldsymbol{J}}, \nabla \times \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \mathcal{D}(\Omega). \tag{2}$$

In (2) the source $\nabla \times \tilde{\boldsymbol{J}}$ is understood in the distribution sense and thus the differential operator is moved to the test function. The coefficient $\tilde{\epsilon}$ and the current $\tilde{\boldsymbol{J}}$ are in general complex functions and so are $\boldsymbol{E}, \boldsymbol{H}$.

The divergence constraint is build into the weak formulation when $\omega \neq 0$. For example $\operatorname{div}(\mu\boldsymbol{H}) = 0$ in the distribution sense can be obtained by applying div operator to the equation $\nabla \times (\tilde{\epsilon}^{-1}\nabla \times \boldsymbol{H}) - \omega^2\mu\boldsymbol{H} = \nabla \times \tilde{\boldsymbol{J}}$. When $\omega = 0$, we need to impose the constraint explicitly.

To simplify the discussion, we consider the following model problems:

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- Symmetric and positive definite problem:

$$\nabla \times (\alpha \nabla \times \boldsymbol{u}) + \beta \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \tag{3}$$

- Saddle point system:

$$\nabla \times (\alpha \nabla \times \boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \qquad \nabla \cdot (\beta \boldsymbol{u}) = 0 \quad \text{in } \Omega, \qquad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{4}$$

where α and β are uniformly bounded and positive coefficients. The right hand side $\boldsymbol{f}$ is divergence free, i.e. $\operatorname{div} \boldsymbol{f} = 0$ in the distribution sense.

2. Interface and Boundary Condition

For a vector $\boldsymbol{u} \in \mathbb{R}^3$ and a unit norm vector $\boldsymbol{n}$, we can decompose $\boldsymbol{u}$ into normal and tangential part as

$$\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} + \boldsymbol{n} \times (\boldsymbol{u} \times \boldsymbol{n}) = \boldsymbol{u}_n + \boldsymbol{u}_t.$$

The vector $\boldsymbol{u} \times \boldsymbol{n}$ is also on the tangent plane and orthogonal to the tangential component $\boldsymbol{u}_t$ which is a clockwise $90°$ rotation of $\boldsymbol{u}_t$ on the tangent plane. Consequently $\{\boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{u}_t, \boldsymbol{n}\}$ forms an orthogonal basis of $\mathbb{R}^3$; see Fig. 1.

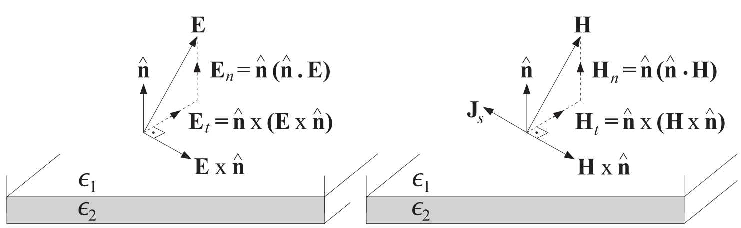

FIGURE 1. Field directions at boundary. Extract from *Electromagnetic Waves and Antennas* by Orfanidis [6].

The interface condition can be derived from the continuity requirement for piecewise smooth functions to be in $H(d;\Omega)$. Let $\Omega = K_1 \cup K_2 \cup S$ with interface $S = \bar{K}_1 \cap \bar{K}_2$. Let $u_i \in H(d; K_i)$. Define $u \in L^2(\Omega)$ as

$$u = \begin{cases} u_1 & x \in K_1, \\ u_2 & x \in K_2. \end{cases}$$

We can always define the derivative du in the distribution sense. To be a weak derivative, we need to verify

$$du = \begin{cases} du_1 & x \in K_1, \\ du_2 & x \in K_2. \end{cases}$$

To do so, let $\phi \in \mathscr{D}(\Omega)$, by the definition of the derivative of a distribution

$$\begin{aligned} \langle du, \phi \rangle &= \langle u, d^*\phi \rangle = (u_1, d^*\phi) + (u_2, d^*\phi) \\ &= (du_1, \phi) + (du_2, \phi) + \langle \gamma_S(u_1 - u_2), \phi \rangle_S, \end{aligned}$$

where d^* is the adjoint of d and γ_S is the correct restriction of functions on the interface depending on the differential operators. The negative sign in front of u_2 is from the fact the outwards normal direction of K_2 is opposite to that of K_1.

Then $u \in H(d;\Omega)$ if and only if

$$\begin{cases} u_1|_S = u_2|_S & \text{for } d = \mathrm{grad}\,, \\ \boldsymbol{n} \times \boldsymbol{u}_1|_S = \boldsymbol{n} \times \boldsymbol{u}_2|_S & \text{for } d = \mathrm{curl}\,, \\ \boldsymbol{n} \cdot \boldsymbol{u}_1|_S = \boldsymbol{n} \cdot \boldsymbol{u}_2|_S & \text{for } d = \mathrm{div}\,. \end{cases}$$

So for a function in $H(\mathrm{curl}\,;\Omega)$, its tangential component should be continuous across the interface and for a function in $H(\mathrm{div};\Omega)$, its normal component should be continuous. This will be the key of constructing finite element spaces for these Sobolev spaces.

When the interface S contains surface charge ρ_S and surface current J_S, the interface condition is for $\boldsymbol{H}$ and $\boldsymbol{D}$ is changed to

$$(\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \boldsymbol{n} = \boldsymbol{J}_S, \quad (\boldsymbol{D}_1 - \boldsymbol{D}_2) \cdot \boldsymbol{n} = \rho_S.$$

The interface condition for $\boldsymbol{H}$ can be build into the right hand side of the weak formulation (2) using a surface integral on S.

The boundary condition can be thought of as an interface condition when one side of the interface is the free space. The following are popular boundary conditions for Maxwell-type equations.

- If one side is a perfect conductor, then $\sigma = \infty$. By Ohm's law, to have a finite current, the electric field $\boldsymbol{E}$ should be zero. So we obtain the boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$ for a perfect conductor.
- Impedance boundary condition •1

$$n \times \boldsymbol{H} - \lambda \boldsymbol{E}_t = g.$$

•1 *more on this*

3. Traces

The trace of functions in $H(d;\Omega)$ is not simply the restriction of the function values since the differential operator div or curl only controls partial components of the vector function. The best way to look at the trace is, again, using integration by parts.

3.1. $H(\mathrm{div};\Omega)$ **space.** For functions $\boldsymbol{v} \in \boldsymbol{C}^1(\Omega), \phi \in C^1(\Omega)$ and Ω is a domain with a smooth boundary, we have the following integration by parts identity

$$\int_\Omega \mathrm{div}\, \boldsymbol{v} \phi \,\mathrm{d}x = -\int_\Omega \boldsymbol{v} \cdot \mathrm{grad}\, \phi \,\mathrm{d}x + \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{v}\, \phi \,\mathrm{dS}. \tag{5}$$

Then we relax the smoothness of functions and domains such that (5) still holds. First since for Lipschitz domains, the normal vector $\boldsymbol{n}$ of $\partial\Omega$ is well defined almost everywhere, we can assume Ω to be a bounded Lipschitz domain only. Second we only need $\boldsymbol{v} \in H(\mathrm{div};\Omega)$ and $\phi \in H^1(\Omega)$. Then (5) can be used to define the trace of $\boldsymbol{v} \in H(\mathrm{div};\Omega)$:

$$\langle \boldsymbol{n} \cdot \boldsymbol{v}, \gamma\phi \rangle_{\partial\Omega} := \int_\Omega \mathrm{div}\, \boldsymbol{v} \phi \,\mathrm{d}x + \int_\Omega \boldsymbol{v} \cdot \mathrm{grad}\, \phi \,\mathrm{d}x, \text{ for all } \phi \in H^1(\Omega). \tag{6}$$

In the left hand side of (6) we change from a boundary integral to an abstract duality action and $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator for H^1 functions. Since γ is an onto, $\gamma\phi$ will run over all $H^{1/2}(\partial\Omega)$ when ϕ run over $H^1(\Omega)$. That is $\boldsymbol{n} \cdot \boldsymbol{v}$ is a dual of $H^{1/2}(\partial\Omega)$. Note that $\partial(\partial\Omega) = 0$. So the right space for $\boldsymbol{n} \cdot \boldsymbol{v}$ is $H^{-1/2}(\partial\Omega)$. We summarize as the following theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Then the mapping $\gamma_n : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$ with $\gamma_n \boldsymbol{v} = \boldsymbol{n} \cdot \boldsymbol{v}|_{\partial\Omega}$ can be extended to a continuous linear map γ_n from $H(\operatorname{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$, namely*

$$\|\gamma_n \boldsymbol{v}\|_{-1/2,\partial\Omega} \lesssim \|\boldsymbol{v}\|_{\operatorname{div},\Omega}. \tag{7}$$

and the following Green's identity holds for functions $\boldsymbol{v} \in H(\operatorname{div};\Omega)$ and $\phi \in H^1(\Omega)$

$$\langle \gamma_n \boldsymbol{v}, \gamma\phi\rangle_{\partial\Omega} = \int_\Omega \operatorname{div} \boldsymbol{v} \phi \,\mathrm{d}x + \int_\Omega \boldsymbol{v} \cdot \operatorname{grad} \phi \,\mathrm{d}x. \tag{8}$$

The space $H_0(\operatorname{div};\Omega)$ can be defined as

$$H_0(\operatorname{div};\Omega) = \{\boldsymbol{v} \in H(\operatorname{div};\Omega) : \gamma_n \boldsymbol{v} = 0\}.$$

Proposition 3.2. *The trace operator γ_n from $H(\operatorname{div};\Omega)$ onto $H^{-1/2}(\partial\Omega)$ is surjective and there exists a continuous right inverse. Namely for any $g \in H^{-1/2}(\partial\Omega)$, there exists a function $\boldsymbol{v} \in H(\operatorname{div};\Omega)$ such that $\gamma_n \boldsymbol{v} = g$ in $H^{-1/2}(\partial\Omega)$ and $\|\boldsymbol{v}\|_{\operatorname{div};\Omega} \lesssim \|g\|_{-1/2,\partial\Omega}$.*

Proof. For a given $g \in H^{-1/2}(\partial\Omega)$, let $f = -|\Omega|^{-1}\langle g, 1\rangle$. We solve the Poisson equation

$$(\nabla p, \nabla \phi) = (f, \phi) + \langle g, \gamma\phi\rangle \quad \text{for all } \phi \in H^1(\Omega).$$

The existence and uniqueness of the solution $p \in H^1(\Omega)$ is ensured by the choice of f. By choosing $v \in H_0^1(\Omega)$, we conclude $-\Delta p = f$ in $L^2(\Omega)$, i.e. $\boldsymbol{v} = \nabla p$ is in $H(\operatorname{div};\Omega)$.

Note that $\langle \gamma_n \boldsymbol{v}, \gamma\phi\rangle = (\operatorname{div} \boldsymbol{v}, \phi) + (\boldsymbol{v}, \nabla\phi) = -(f,\phi) + (\nabla p, \nabla\phi) = \langle g, \gamma\phi\rangle$. Since $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective, we conclude $\gamma_n \boldsymbol{v} = g$ in $H^{-1/2}(\partial\Omega)$. That is we found a function $\boldsymbol{v} \in H(\operatorname{div};\Omega)$ such that $\gamma_n \boldsymbol{v} = g$.

From the stability of $-\Delta$ operator, we have

$$\|\boldsymbol{v}\| = \|\nabla p\| \lesssim \|f\| + \|g\|_{-1/2} \lesssim \|g\|_{-1/2}.$$

Together with the identity $\|\operatorname{div} \boldsymbol{v}\| = \|f\|$, we obtain the inequality $\|\boldsymbol{v}\|_{\operatorname{div};\Omega} \lesssim \|g\|_{-1/2,\partial\Omega}$. ∎

3.2. $H(\operatorname{curl};\Omega)$ **space.** Similarly we can use the integration by parts

$$\int_\Omega \operatorname{curl} \boldsymbol{v} \cdot \phi \,\mathrm{d}x = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\phi} \,\mathrm{d}x - \int_{\partial\Omega} (\boldsymbol{v} \times \boldsymbol{n}) \cdot \boldsymbol{\phi} \,\mathrm{d}S$$

to define the trace of $H(\operatorname{curl};\Omega)$. The trace only controls the tangential part of $\boldsymbol{v}|_{\partial\Omega}$.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain in $\mathbb{R}^3$ with unit outward normal $\boldsymbol{n}$. Then the mapping $\gamma_t : C^\infty(\bar{\Omega}) \to C^\infty(\partial\Omega)$ with $\gamma_t \boldsymbol{v} = \boldsymbol{v}|_{\partial\Omega} \times \boldsymbol{n}$ can be extended by continunity to a continuous linear map γ_t from $H(\operatorname{curl};\Omega)$ to $H^{-1/2}(\partial\Omega)$, namely*

$$\|\gamma_t \boldsymbol{v}\|_{-1/2,\partial\Omega} \lesssim \|\boldsymbol{v}\|_{\operatorname{curl},\Omega}. \tag{9}$$

and the following Green's identity holds for functions $\boldsymbol{v} \in H(\operatorname{curl};\Omega)$ and $\boldsymbol{\phi} \in \boldsymbol{H}^1(\Omega)$

$$\langle \gamma_t \boldsymbol{v}, \gamma\phi\rangle_{\partial\Omega} = \int_\Omega \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{\phi} \,\mathrm{d}x - \int_\Omega \operatorname{curl} \boldsymbol{v} \cdot \boldsymbol{\phi} \,\mathrm{d}x. \tag{10}$$

The trace γ_t from $H(\operatorname{curl};\Omega)$ to $H^{-1/2}(\partial\Omega)$, however, is not surjective since in (10) ϕ can be further extend from $H^1(\Omega)$ to $H(\operatorname{curl};\Omega)$. We write the boundary pair as

$$\langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi}\rangle = \langle \boldsymbol{v} \times \boldsymbol{n}, \boldsymbol{\phi}_t\rangle = \langle \gamma_t \boldsymbol{v}, \boldsymbol{\phi}_t\rangle. \tag{11}$$

The tangential component $\boldsymbol{\phi}_t$ will be still in $H^s(\operatorname{curl}_\Gamma, \Gamma)$ and the trace $\gamma_t \boldsymbol{v}$ is a rotation of the tangential component $\boldsymbol{v}_t$ and thus in $H^s(\operatorname{div}_\Gamma, \Gamma)$, where s is an appropriate index and $\operatorname{curl}_\Gamma, \operatorname{div}_\Gamma$ are the $\operatorname{curl}, \operatorname{div}$ operators on the boundary surface $\Gamma = \partial\Omega$, respectively. It turns out the index $s = -1/2$ and the duality pair for (11) is $H^{-1/2}(\operatorname{div}_\Gamma, \Gamma) - H^{-1/2}(\operatorname{curl}_\Gamma, \Gamma)$. The precise characterization is $\gamma_t : H(\operatorname{curl};\Omega) \to H^{-1/2}(\operatorname{div}_\Gamma, \Gamma)$ and this map is onto. Details can be found in the book [3] (page 58–60).

The space $H_0(\operatorname{curl};\Omega)$ can be defined as

$$H_0(\operatorname{curl};\Omega) = \{\boldsymbol{v} \in H(\operatorname{curl};\Omega) : \gamma_t \boldsymbol{v} = 0\}.$$

4. Well-posedness of Weak Formulations

Let $V = H_0(\operatorname{curl};\Omega)$. The weak formulation of (3) is: given an $\boldsymbol{f} \in \boldsymbol{L}^2$, find $\boldsymbol{u} \in V$ such that

$$(\alpha\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}) + (\beta\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in V. \tag{12}$$

In (12), the first term is obtained by integration by part (moving $\nabla\times$ in front of $\boldsymbol{\phi}$ to $\boldsymbol{u}$)

$$(\alpha\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{\phi}) = (\nabla \times (\alpha \times \nabla \times \boldsymbol{u}), \boldsymbol{\phi}) + (\alpha\nabla \times \boldsymbol{u}, \boldsymbol{n} \times \boldsymbol{\phi})_{\partial\Omega}$$

and chose the test function $\boldsymbol{\phi} \in V$ to remove the boundary term. The boundary condition for $\boldsymbol{u}$ is in Dirichlet type: $\boldsymbol{u} \times \boldsymbol{n} = 0$ on $\partial\Omega$.

The well-posedness of (12) is trivial since the bilinear form is equivalent to the inner product of $H(\operatorname{curl};\Omega)$ and thus the existence and uniqueness of the solution can be obtained by the Riesz representation theorem. The stability constant, however, will be proportional to $1/|\beta|$ and thus below up as $|\beta| \to 0$. We will revisit this issue after we have discussed the saddle point formulation.

For the saddle point formulation of Maxwell equation (4), the natural Sobolev space for $\boldsymbol{u}$ is again $V = H_0(\operatorname{curl};\Omega)$ and the bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\nabla \times \boldsymbol{u}, \nabla \times \boldsymbol{v}), \qquad \text{for } \boldsymbol{u}, \boldsymbol{v} \in H_0(\operatorname{curl};\Omega),$$

which induces an operator $A : V \to V'$, $\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v})$.

As a function in $H(\operatorname{curl};\Omega)$ space, however, the divergence operator cannot be applied directly to $\boldsymbol{u}$. It should be understood in the weak sense, i.e.,

$$-(\operatorname{div}^w \boldsymbol{u}, q) = (\boldsymbol{u}, \operatorname{grad} q) \quad \forall q \in Q := H_0^1(\Omega).$$

We define the bilinear form

$$b(\boldsymbol{v}, q) = (\boldsymbol{v}, \operatorname{grad} q) = -(\operatorname{div}^w \boldsymbol{v}, q), \qquad \text{for } \boldsymbol{v} \in H_0(\operatorname{curl};\Omega), q \in H_0^1(\Omega)$$

which induces operator $B : V \to Q'$ as $\langle B\boldsymbol{u}, q\rangle = b(\boldsymbol{u}, q)$ for all $q \in H_0^1(\Omega)$ and $B' : Q \to V'$ as the dual of B. To impose the constraint, a Lagrangian multiplier $p \in H_0^1(\Omega)$ should be introduced.

We arrive at the saddle point formulation of (4): given $\boldsymbol{f} \in V'$, find $\boldsymbol{u} \in V, p \in Q$ s.t.

$$\begin{pmatrix} A & B' \\ B & O \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ p \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}. \tag{13}$$

The well-posedness of the saddle point system (13) is a consequence of the inf-sup condition of B and the coercivity of A in the null space $X = \ker(B) = H_0(\operatorname{curl};\Omega) \cap \ker(\operatorname{div}^w)$; see *Chapter: Inf-sup conditions for operator equations*.

Lemma 4.1. *We have the inf-sup condition*

$$\inf_{p\in Q} \sup_{\boldsymbol{v}\in V} \frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\operatorname{curl}} |p|_1} = 1. \tag{14}$$

Proof. Here we follow the convention in the Stokes equation to write out the formulation on B. It is more natural, however, to show the adjoint $B' = \operatorname{grad}$ is injective. We can interpret

$$\|\nabla p\|_{V'} = \sup_{\boldsymbol{v}\in V} \frac{\langle B\boldsymbol{v}, p\rangle}{\|\boldsymbol{v}\|_{\mathrm{curl}}} = \sup_{\boldsymbol{v}\in V} \frac{\langle \boldsymbol{v}, \nabla p\rangle}{\|\boldsymbol{v}\|_{\mathrm{curl}}},$$

and it suffices to prove

$$\|\nabla p\|_{V'} = \|\nabla p\|. \tag{15}$$

First by the Cauchy-Schwarz inequality and the definition of the curl norm, we have $\|\nabla p\|_{V'} \leq \|\nabla p\|$. To prove the inequality in the opposite direction, we simply chose $\boldsymbol{v} = \operatorname{grad} p$. Then $\langle B\boldsymbol{v}, p\rangle = |p|_1^2$ and $\|\boldsymbol{v}\|_{\mathrm{curl}} = \|\boldsymbol{v}\| = |p|_1$. Therefore $\|\nabla p\|_{V'} \geq \|\nabla p\|$ by the definition of sup. □

The coercivity in the null space can be derived from the following Poincaré-type inequality.

Lemma 4.2 (Poincaré inequality. Lemma 3.4 and Theorem 3.6 in [2])**.** *When Ω is simply connected and $\partial\Omega$ consists of only one component, we have*

$$\|\boldsymbol{v}\| \lesssim \|\operatorname{curl} \boldsymbol{v}\| \quad \textit{for any } \boldsymbol{v} \in X. \tag{16}$$

A heuristic argument for the above Poincaré inequality is: using $-\Delta \boldsymbol{u} = \operatorname{grad}\operatorname{div}\boldsymbol{u} + \operatorname{curl}\operatorname{curl}\boldsymbol{u}$, we get $\|\boldsymbol{u}\|_1 \eqsim \|\operatorname{curl}\boldsymbol{u}\|$ for $\boldsymbol{u} \in X$. Together with the Poincaré inequality $\|\boldsymbol{u}\| \lesssim \|\boldsymbol{u}\|_1$, we get the desired result. The difficulty to make this argument rigorous is the boundary condition. For $\boldsymbol{u} \in H_0(\operatorname{curl};\Omega)$, only the tangential component is zero.

A sketch of a proof is: show that $\operatorname{curl} : X \to H := H_0(\operatorname{div};\Omega) \cap \ker(\operatorname{div})$ is one-to-one and continuous. Then by the open mapping theorem, the inverse is also continuous which leads to (16). For each $\boldsymbol{\psi} \in H$, i.e., $\operatorname{div}\boldsymbol{\psi} = 0$, with the assumption of the domain Ω, there exists a vector potential $\boldsymbol{v}$ such that $\boldsymbol{\psi} = \operatorname{curl}\boldsymbol{v}$, which is not unique. But if we further require $\operatorname{div}\boldsymbol{v} = 0$ and impose boundary condition $\boldsymbol{v}\times\boldsymbol{n} = 0$, then the potential is unique.

Another approach is through the compact embedding. We can show

Lemma 4.3. *For a Lipschitz polyhedron domain Ω, there exists a constant $s \in (1/2, 1]$ depending only on Ω such that $X \hookrightarrow \boldsymbol{H}^s(\Omega)$ and consequently X is compactly imbedded in $L^2(\Omega)^3$.*

With the compact embedding, we can mimic the proof for H^1-type Poincaré inequality to get (16).

Now we revisit the stability of (12). We further require $\operatorname{div}\boldsymbol{f} = 0$. We consider the stability in the space X for which we can apply Poincaé inequality to obtain a coercivity in dependent of β. We can then obtain stability

$$\|\nabla\times\boldsymbol{u}\| \lesssim \|\boldsymbol{f}\|$$

with a constant depending only on α and constant in the Poincaré inequality.

5. Finite Element Methods for Maxwell Equations

In this section we first present two finite element spaces for Maxwell equations, discuss the interpolation error, and give convergence analysis of finite element methods for Maxwell equations using these spaces.

5.1. **Edge Elements.** We describe two types of edge elements developed by Nédélec [4, 5] in 1980s. We also briefly mention the implementation of these elements in MATLAB and recommend the readers to do the project *Project: Edge Finite Element Method for Maxwell-type Equations.*

5.1.1. *First family: lowest order.* For the k-th edge $\boldsymbol{e}_k$ with vertices (i,j) and the direction from i to j, the basis $\boldsymbol{\phi}_k$ and corresponding degree of freedom $l_k(\cdot)$ are

$$\boldsymbol{\phi}_k = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i,$$

$$l_k(\boldsymbol{v}) = \int_{e_k} \boldsymbol{v}\cdot\boldsymbol{t}\, ds \approx \frac{1}{2}[\boldsymbol{v}(i)+\boldsymbol{v}(j)]\cdot \boldsymbol{e}_k,$$

where the approximation is exact when $\boldsymbol{v}\cdot\boldsymbol{t}$ is linear.

We verify the duality $l_k(\boldsymbol{\phi}_k)=1$ as follows

$$\boldsymbol{\phi}_k(i)\cdot\boldsymbol{e}_k = \nabla\lambda_j\cdot\boldsymbol{e}_k = \int_{e_k}\nabla\lambda_j\cdot\boldsymbol{t}\,ds = \lambda_j(j)-\lambda_j(i) = 1$$

$$\boldsymbol{\phi}_k(j)\cdot\boldsymbol{e}_k = \nabla\lambda_i\cdot\boldsymbol{e}_k = \int_{e_k}\nabla\lambda_i\cdot\boldsymbol{t}\,ds = \lambda_i(j)-\lambda_i(i) = -1,$$

and consequently $l_k(\boldsymbol{\phi}_k)=1$.

If we change the integral to another edge (m,n). If $(m,n)\cap(i,j)=\varnothing$, then $\lambda_i|_{e_{mn}} = \lambda_j|_{e_{mn}} = 0$. Without loss of generality, consider $m=i$ and $n\notin\{i,j\}$. Then in the basis $\boldsymbol{\phi}_k$ either $\nabla\lambda_j\cdot\boldsymbol{t}_{mn}=0$ or $\lambda_j|_{e_{mn}}=0$ and therefore $\boldsymbol{\phi}_k\cdot\boldsymbol{t}_{mn}=0$. This verifies $l_i(\boldsymbol{\phi}_k)=0$ for $i\neq k$.

The lowest order edge element is

$$\mathrm{NE}^0 = \mathrm{span}\{\boldsymbol{\phi}_k, k=1,2,\cdots,6\}$$

which is a linear polynomial. For a 2D triangle, the formulae for the basis is the same and three basis functions on a triangle is shown below We also show three basis function

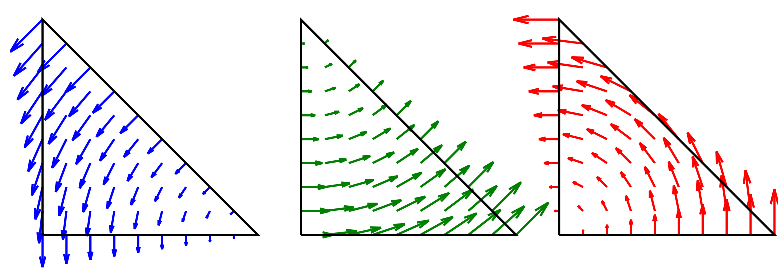

FIGURE 2. Basis of NE^0 in a triangle.

associated to three edges on one face in a tetrahedron in Fig. 3. Notice that the vector field $\boldsymbol{\phi}_k$ of edge k is orthogonal to other edges.

The lowest order element NE^0 is not $\mathcal{P}_1^3$ whose dimension is $4\times 3=12$. In other words, the lowest order edge element is an incomplete linear polynomial space and can only reproduce constant vector. From the approximation point of view, the L^2 error can be only first order. Nevertheless the $H(\mathrm{curl}\,)$ norm is still first order.

Be careful on the orientation of edges. For each edge in the triangulation, we need to assign a unique orientation and will be called the global orientation. The orientation in one tetrahedron or one triangle (called local orientation) may not be consistent with

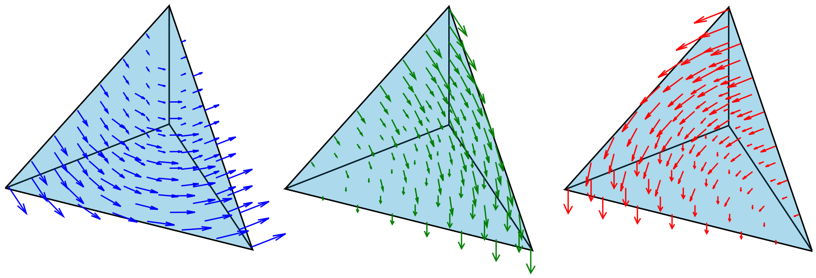

FIGURE 3. Three basis of NE^0 associated to three edges on one face in a tetrahedron.

the global one. In 3-D, locally we still use $i < j$ as the orientation. In 2-D, the local edges are orientated counterclockwise. For corresponding data structure, we refer to the documentation of $i$fem: Lowest Order Edge Element in 3D.

The necessary data structure for the edge element can be obtained via

```
[elem2edge,edge,elem2edgeSign] = dof3edge(elem);
```

In the output `elem2edge` is the element-wise pointer from `elem` to `edge` which records all edges with ordering `edge(k,1)<edge(k,2)`. The inconsistency of local edges to global edges is in `elem2edgeSign`.

For the easy of computing curl-curl matrix, we also list the curl of the basis

$$\nabla \times \boldsymbol{\phi}_k = 2\nabla\lambda_i \times \nabla\lambda_j,$$

and present a geometry interpretation. The barycentric coordinate λ_i is a linear polynomial and the face f_i is on the zero level set of λ_i. Therefore $\nabla\lambda_i$ is an inward normal direction of face f_i opposite to vertex i. Write $\nabla\lambda_i = \|\nabla\lambda_i\|\boldsymbol{n}_i$ with $\boldsymbol{n}_i$ the unit inwards normal direction of f_i. So the direction of the vector $\nabla\lambda_i \times \nabla\lambda_j$ is $\boldsymbol{n}_i \times \boldsymbol{n}_j$ which is the edge vector of the conjugate edge of k. Here we defined the conjugate edge, indexed by k', as the edge formed by the other two vertices, i.e., e_k and $e_{k'}$ have no intersection and the direction of $e_{k'}$ is "opposite" to e_k; see Fig. **??** •2 . With these notation, we get

•2 *a figure here*

$$\nabla \times \boldsymbol{\phi}_k = 2\|\nabla\lambda_i\|\|\nabla\lambda_j\|\boldsymbol{t}_{k'}. \tag{17}$$

The volume and gradient of barycentric coordinates can be obtained by the subroutine

```
[Dlambda,volume] = gradbasis3(node,elem).
```

To enumerate 6 edges in one tetrahedron, we use

```
locEdge = [1 2; 1 3; 1 4; 2 3; 2 4; 3 4]
```

and two `for` loops `for k=1:6` and `for l=1:6`. Two vertices of the k-edge can be obtained by `locEdge(k,:)`. The unit edge vector $\boldsymbol{t}$ can be computed from `edge` and use `elem2edge` to get the index. By using the global edge vector in (17), no sign correction needed. For variable coefficients, simply replace the volume $|\tau|$ by the weighted one, i.e. $\int_\tau \alpha \,\mathrm{d}x$.

The mass matrix $(\boldsymbol{\phi}_k, \boldsymbol{\phi}_l)$ can be computed using the integral formulae of barycentric coordinates:

$$\int_\tau \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\,\mathrm{d}x = \frac{\alpha_1!\alpha_2!\alpha_3!\alpha_4!6}{(\sum_i \alpha_i + 3)!}|\tau|. \tag{18}$$

The computation of (negative) weak divergence of an edge element, i.e., the local matrix B_τ, is essentially $(\phi_k, \nabla\lambda_i)$ which is a linear combination of the entry $(\nabla\lambda_i, \nabla\lambda_j)$. Instead of computing $6\times 4 = 24$ times inner product, one can compute a 4×4 SPD matrix with 9 times inner product and the rest is much cheaper addition and substraction.

5.1.2. *Second family: linear polynomial.* In addition to $\boldsymbol{\phi}_k$, for each edge, we add one more basis

$$\boldsymbol{\psi}_k = \lambda_i\nabla\lambda_j + \lambda_j\nabla\lambda_i,$$
$$l_k^1(\boldsymbol{v}) = 3\int_{e_k} \boldsymbol{v}\cdot\boldsymbol{t}(\lambda_i-\lambda_j)\,ds \approx \frac{1}{2}[\boldsymbol{v}(i)-\boldsymbol{v}(j)]\cdot\boldsymbol{e}_k.$$

The approximation is obtained by the Simpson's rule with the fact $\lambda_i - \lambda_j = 0$ at the middle point, which is exact when $\boldsymbol{v}\cdot\boldsymbol{t}$ is linear. Obviously $\{l_k(\cdot), l_k^1(\cdot), k=1,2,\cdots,6\}$ are linear independent. We then show it is dual to $\{\boldsymbol{\phi}_k, \boldsymbol{\psi}_k\}$

The Simpson's rule is exact for $l_k^1(\boldsymbol{\psi}_k)$ and thus

$$l_k^1(\boldsymbol{\psi}_k) = \frac{1}{2}\left[\boldsymbol{\psi}_k\cdot\boldsymbol{e}_{ij}(i) - \boldsymbol{\psi}_k\cdot\boldsymbol{e}_{ij}(j)\right] = \frac{1}{2}\left[(\lambda_i-\lambda_j)(i) - (\lambda_i-\lambda_j)(j)\right] = 1.$$

The verification of $\boldsymbol{\psi}_k\cdot\boldsymbol{e}_l = 0$, for $l\neq k$, is similar as before. Therefore $\{l_k^1\}$ is a dual basis of $\{\boldsymbol{\psi}_k\}$.

We need to verify one more duality

$$l_k(\boldsymbol{\psi}_l) = 0, \quad l_k^1(\boldsymbol{\phi}_l) = 0, \quad \forall l = 1,2,\cdots,6.$$

We only need to worry about $l=k$ since $\boldsymbol{\psi}_k\cdot\boldsymbol{t}_l = \boldsymbol{\phi}_k\cdot\boldsymbol{t}_l = 0$ if $k\neq l$. Notice that $\boldsymbol{\psi}_k\cdot\boldsymbol{t}_k$ is odd (respect to the middle point) and thus the integral is zero. Similarly $\boldsymbol{\phi}_k\cdot\boldsymbol{t}_k = 1$ and thus $l_k^1(\boldsymbol{\phi}_k) = 0$.

The lowest order second family of edge element is

$$\mathrm{NE}^1 = \mathrm{span}\{\boldsymbol{\phi}_k, \boldsymbol{\psi}_k, k=1,2,\cdots,6\},$$

which is a full linear polynomial and will reproduce linear polynomials. Therefore the L^2-norm of error will be second order. The $H(\mathrm{curl})$ norm, however, is still first order since $\boldsymbol{\psi}_k = \nabla(\lambda_i\lambda_j)$ and $\nabla\times\boldsymbol{\psi}_k = 0$ has no contribution to the approximation of curl. Plot of $\boldsymbol{\psi}_k$ in a triangle is

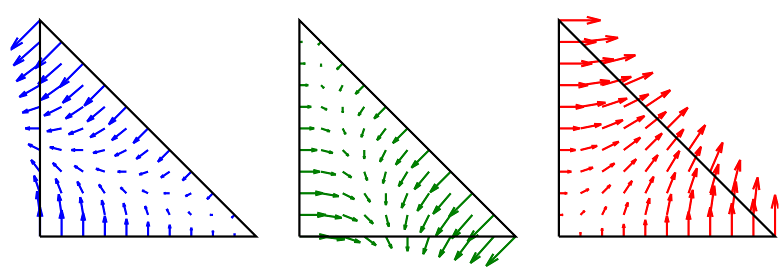

FIGURE 4. Basis vectors $\boldsymbol{\psi}_k$ of NE^1 in a triangle.

For NE^1, the curl-curl matrix is simply the zero extension of that of NE^0 (6×6 to 12×12). The 12×12 local mass matrix can be computed by loops of $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$. For the B matrix, now the space for the Lagrange multiplier is the quadratic Lagrange element. The 12×10 local matrix can be computed with the help of mass matrix since the gradient of edge bubble functions is a scaling of basis function $\boldsymbol{\psi}_k$.

The global finite element space is obtained by gluing piecewise one. Using the barycentric coordinate in each tetrahedron, for an edge, the basis $\boldsymbol{\phi}_k, \boldsymbol{\psi}_k$ can be extend to all tetrahedron surrounding this edge. Given a triangulation $\mathcal{T}$, let $\mathcal{E}$ be the edge set of $\mathcal{T}$. Define

$$\mathrm{NE}^0(\mathcal{T}) = \mathrm{span}\{\boldsymbol{\phi}_e, e \in \mathcal{E}\},$$
$$\mathrm{NE}^1(\mathcal{T}) = \mathrm{span}\{\boldsymbol{\phi}_e, \boldsymbol{\psi}_e, e \in \mathcal{E}\}.$$

To show the obtained spaces are indeed in $H(\mathrm{curl}\,;\Omega)$, it suffices to verify the tangential continuity of the piecewise polynomials. Given a triangular face f, in one tetrahedron, we label the vertex opposite to f as x_f and the corresponding barycentric coordinate will be denoted by λ_f. For an edge e using x_f as an vertex, the corresponding basis $\boldsymbol{\phi}_e$ or $\boldsymbol{\psi}_e$ is a linear combination of $\lambda_i \nabla\lambda_f$ and $\lambda_f \nabla \lambda_i$. Restrict to f, $\lambda_f|_f = 0$ and $\nabla\lambda_f \times n_f = 0$ since $\nabla\lambda_f$ is a norm vector of f. Therefore we showed that $\boldsymbol{\phi}_e|_f \times n_f = \boldsymbol{\psi}_e|_f \times n_f = 0$ for edges e containing n_f. Therefore for $\boldsymbol{v} \in \mathrm{NE}^0(\mathcal{T})$ or $\mathrm{NE}^1(\mathcal{T})$, the trace $\boldsymbol{v}|_f \times n_f$ depends only on the basis function of edges of f which is the ideal continuity of a $H(\mathrm{curl}\,;\Omega)$ function.

5.2. **Interpolation Error Estimate.** We consider the canonical interpolation to the edge element space. Given a triangulation $\mathcal{T}_h$ with mesh size h. Define $I_h^{\mathrm{curl}} : V \cap \mathrm{dom}(I_h^{\mathrm{curl}}) \to \mathrm{NE}^0(\mathcal{T}_h)$ as follows: given a function $\boldsymbol{u} \in V$, define $\boldsymbol{u}_I = I_h^{\mathrm{curl}}\boldsymbol{u} \in \mathrm{NE}^0(\mathcal{T}_h)$ by matching the d.o.f.

$$l_e(I_h^{\mathrm{curl}}\boldsymbol{u}) = l_e(\boldsymbol{u}) \quad \forall e \in \mathcal{E}_h(\mathcal{T}_h).$$

Namely

$$\boldsymbol{u}_I = \sum_{e\in\mathcal{E}_h} \left(\int_e \boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s\right)\boldsymbol{\phi}_e$$

For the second family edge element space, add $l_e^1(\cdot)$ and $\boldsymbol{\psi}_e$.

The L^2-norm error estimate of $\boldsymbol{u} - \boldsymbol{u}_I$ is relatively easy. Restrict to one element τ, as the operator $I - I_h^{\mathrm{curl}}$ preserves constant vectors, by the Bramble-Hilbert lemma lemma, we obtain

$$\|\boldsymbol{u} - \boldsymbol{u}_I\|_{0,\tau} \lesssim h_\tau |\boldsymbol{u}|_1. \tag{19}$$

For the second family edge element space, the operator $I - I_h^{\mathrm{curl}}$ preserves linear polynomial and thus second order error estimate can be obtained.

Exercise 5.1. In one tetrahedron τ, verify I_h^{curl} to $\mathrm{NE}^0(\tau)$ will preserve constant vector and to $\mathrm{NE}^1(\tau)$ linear vectors.

For the error $\nabla\times(\boldsymbol{u} - \boldsymbol{u}_I)$, if we want to use Bramble-Hilbert lemma, we need to introduce the Piola transformation to connecting the curl operators $\nabla\times$ and $\hat{\nabla}\times$ in the current element and reference element. Instead we introduce the lowest order face element for $H(\mathrm{div};\Omega)$ and use the commuting diagram to change to the estimate of L^2-error.

Give a face f_l formed by vertices $[i,j,k]$, we introduce a basis vector

$$\boldsymbol{\phi}_l = 2(\lambda_i\nabla\lambda_j\times\nabla\lambda_k + \lambda_j\nabla\lambda_k\times\nabla\lambda_i + \lambda_k\nabla\lambda_i\times\nabla\lambda_j),$$

and the corresponding degree of freedom

$$l_{f_l}(\boldsymbol{v}) = \int_{f_l}\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S \approx \boldsymbol{v}(\boldsymbol{c})\cdot\boldsymbol{n}_{f_l}|f_l|,$$

where the approximation is exact for linear polynomial $\boldsymbol{v}$.

Exercise 5.2. [Face element]

(1) Verify $\{l_{f_i}, i = 1,2,3,4\}$ is a dual basis of $\{\boldsymbol{\phi}_{f_j}, j = 1,2,3,4\}$.
(2) For a triangle in 2D, the degree of freedom remains the same. Please write out the basis functions. A plot of this basis can be found in Fig. 5.

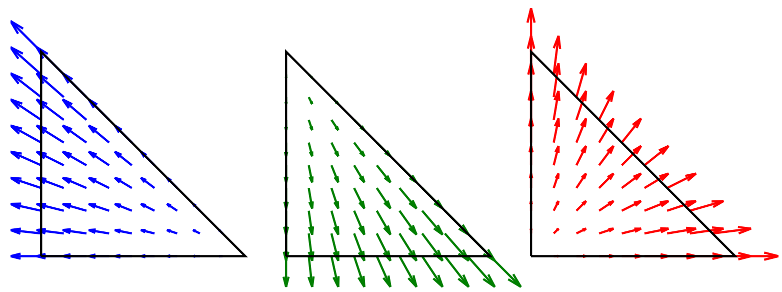

FIGURE 5. Basis vectors $\boldsymbol{\phi}_k$ of RT^0 in a triangle.

We define the lowest order face element space

$$\mathrm{RT}^0(\tau) = \mathrm{span}\{\boldsymbol{\phi}_{f_j}, j = 1,2,3,4\}$$

and the global version

$$\mathrm{RT}^0(\mathcal{T}) = \mathrm{span}\{\boldsymbol{\phi}_f, f \in \mathcal{F}(\mathcal{T})\},$$

where $\mathcal{F}(\mathcal{T})$ is the set of all faces of a triangulation $\mathcal{T}$.

Given a triangulation $\mathcal{T}_h$ with mesh size h. Define $I_h^{\mathrm{div}} : V \to \mathrm{RT}^0(\mathcal{T}_h)$ as follows: given a function $\boldsymbol{u} \in V$, define $\boldsymbol{u}_I = I_h^{\mathrm{div}}\boldsymbol{u} \in \mathrm{RT}^0(\mathcal{T}_h)$ by matching the d.o.f.

$$l_f(I_h^{\mathrm{div}}\boldsymbol{u}) = l_f(\boldsymbol{u}) \quad \forall f \in \mathcal{F}_h(\mathcal{T}_h).$$

Namely

$$\boldsymbol{u}_I = \sum_{f\in\mathcal{F}_h} \left(\int_e \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S\right)\boldsymbol{\phi}_f$$

We verify the crucial commuting property

$$\nabla\times I_h^{\mathrm{curl}}\boldsymbol{u} = I_h^{\mathrm{div}}\nabla\times\boldsymbol{u}$$

by the Stokes' theorem and the definition of interpolation operators:

$$\begin{aligned}\int_f I_h^{\mathrm{div}}(\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}_f\,\mathrm{d}S &= \int_f (\nabla\times\boldsymbol{u})\cdot\boldsymbol{n}_f\,\mathrm{d}S = \int_{\partial f}\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s\\ &= \int_{\partial f} I_h^{\mathrm{curl}}\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}s = \int_f(\nabla\times I_h^{\mathrm{curl}}\boldsymbol{u})\cdot\boldsymbol{n}_f\,\mathrm{d}S\end{aligned}$$

Then as I_h^{div} preserves the constant vector, we obtain

$$\|\nabla\times(\boldsymbol{u}-\boldsymbol{u}_I)\|_{0,\tau} = \|(I - I_h^{\mathrm{div}})\nabla\times\boldsymbol{u}\|_{0,\tau} \lesssim h_\tau|\nabla\times\boldsymbol{u}|_{1,\tau}$$

The commuting diagram can be extended to the whole sequence and summarized in the figure below

Exercise 5.3. Prove the commuting diagram shown in Fig. 6.

$$
\begin{tikzcd}
H^1 \arrow[r, "\mathrm{grad}"] \arrow[d, "I_h^{\mathrm{grad}}"] & H(\mathrm{curl}) \arrow[r, "\mathrm{curl}"] \arrow[d, "I_h^{\mathrm{curl}}"] & H(\mathrm{div}) \arrow[r, "\mathrm{div}"] \arrow[d, "I_h^{\mathrm{div}}"] & L^2 \arrow[d, "I_h^0"] \\
P^1 \arrow[r, "\mathrm{grad}"] & NE^0 \arrow[r, "\mathrm{curl}"] & RT^0 \arrow[r, "\mathrm{div}"] & P^0
\end{tikzcd}
$$

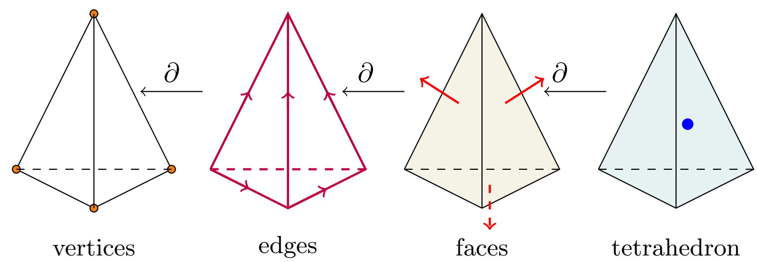

FIGURE 6. Commuting diagram of finite element spaces.

Remark 5.4. The domain of the canonical interpolation $I_h^{\mathrm{curl}}, I_h^{\mathrm{div}}$ are smooth subspace of $H(\mathrm{curl}\,;\Omega)$ or $H(\mathrm{div};\Omega)$, respectively. For example, vven for a H^1 function $\boldsymbol{u}$, the trace $\boldsymbol{u}$ restricted on an edge is not well defined. The arguments above require the function smooth enough. Quasi-interpolation, which relaxes the smoothness of the function and preserves the nice commuting diagram, have been constructed recently.

5.3. **Well-posedness of Discrete Saddle Point Problem.** For finite element approximation, we chose edge element space $V_h \subset H_0(\mathrm{curl}\,;\Omega)$ and define the subspace $X_h = V_h \cap \ker(\mathrm{div}_h)$. By the exact sequence, $V_h = X_h \oplus \mathrm{grad}\,(S_h)$. The bilinear form $a(\cdot,\cdot)$ is not well defined on V_h but it is on the subspace X_h due to the following discrete Poincaré inequality. Note that $X_h \not\subset X$, it is not a simple consequence of the Poincaré inequality in Lemma 4.2

Lemma 5.5 (Discrete Poincaré inequality)**.** *For $v_h \in X_h$,*

$$\|v_h\| \lesssim \|\mathrm{curl}\, v_h\|.$$

A systematic way of proving the Poincare inequality is using the exact sequence in both continuous and discrete level and use the bounded operator; See Arnold, Falk and Winther [1].

For the discrete problem, $f_h \in X_h'$ i.e. discrete divergence free $(f_h, \mathrm{grad}\, v_h) = 0$ for all $v_h \in S_h$. Then $A^{-1} : X_h' \to X_h$ is well defined and the following stability result holds:

$$\|u_h\|_{\mathrm{curl}} \lesssim \|f_h\|_{X_h'}.$$

Discrete inf-sup condition is easy. Just take ∇p_h.

Then we obtain the first order error estimate:

$$
\begin{aligned}
\|\nabla \times (u - u_h)\| + \|\nabla (p - p_h)\| &\lesssim \|\nabla \times (u - u_I)\| + \|\nabla (p - p_I)\| \\
&\lesssim h\left(|\nabla \times u|_1 + |p|_2\right).
\end{aligned}
$$

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