A Mixed-Hybrid-Discontinuous Galerkin Finite Element Method for Convection-Diffusion Problems

Herbert Egger[*]   Joachim Schöberl[*]

We propose and analyse a new finite element method for convection diffusion problems based on the combination of a mixed method for the elliptic and a discontinuous Galerkin method for the hyperbolic part of the problem. The two methods are made compatible via hybridization and the combination of both is appropriate for the solution of intermediate convection-diffusion problems. By construction, the discrete solutions obtained for the limiting subproblems coincide with the ones obtained by the mixed method for the elliptic and the discontinuous Galerkin method for the limiting hyperbolic problem, respectively. We present a new type of analysis that explicitly takes into account the Lagrange-multipliers introduced by hybridization. The use of adequate energy norms allows to treat the purely diffusive, the convection dominated, and the hyperbolic regime in a unified manner. In numerical tests, we illustrated the efficiency of our approach and compare to results obtained with other methods for convection diffusion problems.

1 Introduction

In this paper we consider stationary convection-diffusion problems of the form

$$\mathrm{div}(-\epsilon \nabla u + \beta u) = f \quad \text{in } \Omega,$$
$$u = g_D \quad \text{on } \partial\Omega_D, \quad -\epsilon \frac{\partial u}{\partial n} + \beta_\nu u = g_N \quad \text{on } \partial\Omega_N, \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ consisting of a Dirichlet and a Neumann part, ϵ is a non-negative function and $\beta : \Omega \to \mathbb{R}^d$ is a d-dimensional vector field.

Similar problems arise in many applications, e.g., in the modeling of contaminant transport, in electro-hydrodynamics or macroscopic models for semiconductor devices. A feature that makes the numerical solution difficult is that often convection plays the dominant role in (1). In the case of vanishing diffusion, solutions of (1) will in general not be smooth, i.e., discontinuities are propagated along the characteristic direction β; nonlinear problems may even lead to discontinuities or blow-up in finite time when starting from smooth initial data. So appropriate numerical shemes for the convection dominated regime have to be able to deal with almost discontinuous solutions in an accurate but stable manner. Another property that is desireable to be reflected also on the discrete level is the conservation strucure inherent in the divergence form of (1).

Due to the variety of applications, there has been significant interest in the design and analysis of numerical schemes for convection dominated problems. Much work has been devoted to devise

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[*]Center for Computational Engineering Science, RWTH Aachen University, Germany

accurate and stable finite difference and finite volume methods for the solution of hyperbolic systems by means of appropriate upwind techniques including flux or slope limiters in the nonlinear case.

A different approach to the stable solution of (almost) hyperbolic problems is offered by discontinuous Galerkin methods, introduced originally for a linear hyperbolic equation in neutron transport [31, 26, 24]. Starting from the 70's, discontinuous Galerkin methods have been investigated intensively and applied to the solution of various linear and nonlinear hyperbolic and convection dominated elliptic problems with great success, cf. [7, 6, 1], and [16] for an overview and further references. Since in practical applications, convection respectively diffusion phenomena may dominate, in different parts of the computational domain, several attempts have been made to generalize discontinuous Galerkin methods also to elliptic problems [32, 29, 21], yielding numerical schemes very similar to interior penalty methods studied much earlier [28, 5, 2]. For further references on this topic and a unified analysis of several discontinuous Galerkin methods for elliptic problems, we refer to [4]. For discontinuous Galerkin methods applied to convection diffusion problems we refer to [14, 8, 12], and to [11] for a multiscale version. Two disadvantages of discontinuous Galerkin methods applied for problems with diffusion is that, compared to a standard conforming discretization, the overall number of unknowns is increased substantially and that the resulting linear systems are much less sparse.

Another very successful approach for the solution of convection dominated problems is the streamline diffusion method [22, 25], where standard conforming finite element discretizations are stabilized by adding in a conforming way an appropriate amount of artificial diffusion in streamline direction. This method is easy to implement and yield stable discretizations in many situations, but may lead to unphysically large layers near discontinuities and boundaries. For a comparison of high order discontinuous Galerkin and streamline diffusion methods, we refer to [20]. For an appropriate treatment of boundary layers via Nitsche's method, see [19]. In contrast to discontinuous Galerkin methods, the streamline diffusion method does not yield conservative discretizations.

Herre, we follow a different approach, namely the combination of upwind techniques used in discontinuous Galerkin methods for hyperbolic problems with conservative discretizations of mixed methods for elliptic problems. In order to make these two different methods compatible, we will utilize hybrid formulations for the mixed and the discontinuous Galerkin methods. It is well-known [3, 10, 15] that hybridization can be used for the efficient implementation of mixed finite elements for elliptic problems. Introducing the Lagrange-multipliers also in the discontinuous Galerkin methods allows us to couple both methods naturally and yields a stable mixed hybrid discontinuous Galerkin method with the following properties:

- For $\beta \equiv 0$ the numerical solution coincides with that of a mixed method, cf. [3, 10], and postprocessing techniques can be used to increase the accuracy of the solution.

- For $\epsilon \equiv 0$ the solution coincides with that obtained by a discontinuous Galerkin method for hyperbolic problems [26, 24].

- The intermediate convection-diffusion regime is treated automatically with no need to choose stabilization parameters.

For diffusion dominated regions, the stabilization can be omitted yielding a scheme that was studied numerically in 1D in [18]. Our analysis in Section 4.2 also includes this case. A particular advantage of our method is that it is formulated and can be implemented elementwise, i.e., it allows for static condensation on the element level yielding global systems for the Lagrange-multiplier only. In this way we arrive at global systems with less unknowns and much sparser stencils than other discontinuous Galerkin methods, while still utilizing the same upwind mechanisms.

Other extensions of mixed finite element methods to convection diffusion problems were considered in [13, 17].

The outline of this article is as follows: In Section 2, we review the hybrid formulation of the mixed method for the Poisson equation, and then introduce a hybrid version of the discontinuous

Galerkin method for the hyperbolic subproblem. The scheme for the intermediate convection-diffusion regime then results by a combination of the two methods for the limiting subproblems, and we show consistency and conservation of all three methods under consideration. Section 3 presents the main stability and boundedness estimates for the corresponding bilinear forms, and contains an a-priori error analysis in the energy norm with emphasis on the convection dominated regime. Details on super-convergence results and postprocessing for the diffusion dominated case are presented in Section 4. Results of numerical tests including a comparsion with the streamline diffusion method are presented in Section 5.

2 Hybrid mixed discontinuous Galerkin methods for convection diffusion problems

The aim of this section is to formulate the problem under consideration in detail and to fix the relevant notation and some basic assumptions. By introducing the *diffusive flux* $\sigma = -\epsilon \nabla u$ as a new variable, we rewrite (1) in mixed form

$$
\begin{aligned}
\sigma + \epsilon \nabla u = 0, \quad \mathrm{div}(\sigma + \beta u) = f \qquad \text{in } \Omega \\
u = g_D \qquad \text{on } \partial\Omega_D, \qquad -\epsilon \frac{\partial u}{\partial \nu} + \beta \nu u = g_N \qquad \text{on } \partial\Omega_N,
\end{aligned}
\tag{2}
$$

which will be the starting point for our considerations. Here and below, ν denotes the outward unit normal vector on the boundary of some domain. We refer to βu as the *convective flux* and call $\sigma + \beta u$ the *total flux*. Existence and uniqueness of a solution to (2) follows under standard assumptions on the coefficients. For ease of presentation, let us make some simplifying assumptions.

2.1 Basic assumptions and notations

We assume that Ω is a polyhedral domain and that $\partial\Omega_D = \partial\Omega$, i.e., $\partial\Omega_N = \emptyset$. Let $\mathcal{T}_h$ be a shape regular partition of Ω into simplices T and let $\mathcal{E}_h$ denote the set of facets E (element interfaces and element faces aligned to the boundary). We assume each T and E is generated by an affine map Φ_T or Φ_E from a corresponding reference element $\hat{T}$ respectively $\hat{E}$. With $\partial\mathcal{T}_h$ we denote the set of all element boundaries ∂T (with outward normal ν). Finally, by χ_S we denote the characteristic function of a set $S \subset \Omega$.

Regarding the coefficients, we assume for simplicity that $g_D = 0$ and that $\epsilon \geq 0$ is constant on elements $T \in \mathcal{T}_h$. Furthermore, the vector field β is assumed to be piecewise constant with continuous normal components across element interfaces, which implies that $\mathrm{div}\,\beta = 0$. Moreover, such a vectorfield β induces a natural splitting of element boundaries into inflow and outflow parts, i.e., we define the outflow boundary $\partial T^{out} := \{x \in \partial T : \beta\nu > 0\}$ and $\partial T^{in} = T \setminus \partial T^{out}$. The union of the element in-/outflow boundaries will be denoted by $\partial\mathcal{T}_h^{in}$ respectively $\partial\mathcal{T}_h^{out}$ and similarly, the symbols $\partial\Omega^{in}$ and $\partial\Omega^{out}$ are used for the in- and outflow regions of the boundary $\partial\Omega$.

For our analysis we will utilize the broken Sobolev spaces

$$
H^s(\mathcal{T}_h) := \{u : u \in H^s(T), \quad \forall T \in \mathcal{T}_h\}, \qquad s \geq 0,
$$

and for functions $u \in H^{s+1}$ we define $\nabla u \in [H^s(\mathcal{T}_h)]^d$ to be the piecewise gradient. In a natural manner we define the inner products

$$
(u, v)_T := \int_T uv\, dx \qquad \text{and} \qquad (u, v)_{\mathcal{T}_h} := \sum_{T \in \mathcal{T}_h} (u, v)_T,
$$

with the obvious modifications for vector valued functions. The norm induced by the volume integrals $(\cdot, \cdot)_{\mathcal{T}_h}$ is denoted by $\|u\|_{\mathcal{T}_h} := \sqrt{(u, u)_{\mathcal{T}_h}}$, and for piecewise constant α we define $\alpha(u, v)_{\mathcal{T}_h} := \sum_T (\alpha u, v)_T$ and $\alpha\|u\|_{\mathcal{T}_h} := \sqrt{\alpha^2(u, u)_{\mathcal{T}_h}}$. Norms and seminorms on the broken Sobolev spaces $H^s(\mathcal{T}_h)$ will be denoted by $\|\cdot\|_{s,\mathcal{T}_h}$ and $|\cdot|_{s,\mathcal{T}_h}$.

For the element interfaces we consider the function spaces

$$L^2(\mathcal{E}_h) := \{\mu : \mu \in L^2(E), \quad \forall E \in \mathcal{E}_h\}, \qquad \text{and}$$
$$L^2(\partial \mathcal{T}_h) := \{v : v \in L^2(\partial T), \quad \forall T \in \mathcal{T}_h\}.$$

Note that functions in $L^2(\partial \mathcal{T}_h)$ are double valued on element interfaces and may be considered as traces of elementwise defined functions. Moreover, we can identify $\mu \in L^2(\mathcal{E}_h)$ with a function $v \in L^2(\partial \mathcal{T}_h)$ by duplicating the values at element interfaces, so in this sense $L^2(\mathcal{E}_h) \subset L^2(\partial \mathcal{T}_h)$. For $u, v \in L_2(\partial \mathcal{T}_h)$ we denote integrals over element interfaces by

$$\langle \lambda, \mu \rangle_{\partial T} := \int_{\partial T} \lambda \mu \, ds \qquad \text{and} \qquad \langle \lambda, \mu \rangle_{\partial \mathcal{T}_h} := \sum_T \langle \lambda, \mu \rangle_{\partial T},$$

and the corresponding norms are denoted by $|u|_{\partial \mathcal{T}_h} := \sqrt{\langle u, u \rangle_{\partial \mathcal{T}_h}}$. Again, we write $\alpha \langle u, v \rangle_{\partial \mathcal{T}_h}$ with the meaning $\sum_{\partial T} \langle \alpha u, v \rangle_{\partial T}$.

Let us now turn to the formulation of appropriate finite element spaces. We start from piecewise polynomials on the reference elements, and define the finite element spaces via appropriate mappings, cf. [9]. By $\mathcal{P}_k(\hat{T})$ respectively $\mathcal{P}_k(\hat{F})$, we denote the set of all polynomials of order $\leq k$, and by $RT_k(\hat{T}) := \mathcal{P}_k(\hat{T}) \oplus \vec{x} \cdot \mathcal{P}_k(\hat{T})$ we denote the Raviart-Thomas (-Nedelec) element, cf. [30, 27] and [10]. Here the symbol $\oplus$ is used to denote the union of two vector spaces. For our finite element methods we will utilize the following functions spaces:

$$\Sigma_h := \{\tau_h \in [L_2(\Omega)]^d : \tau_h|_T = \frac{1}{\det \Phi_T'} \Phi_T' \hat{\tau} \circ \Phi_T^{-1}, \quad \hat{\tau} \in RT_k(\hat{T})\},$$
$$\mathcal{V}_h := \{vh \in L_2(\Omega) : v_h|_T = \hat{v} \circ \Phi_T^{-1}, \quad \hat{v} \in \mathcal{P}_k(T)\},$$
$$\mathcal{M}_h := \{\mu_h \in L_2(\mathcal{E}_h) : \mu|_E = \hat{\mu} \circ \Phi_E^{-1}, \mu = 0 \text{ on } \partial \Omega, \quad \hat{\mu} \in \mathcal{P}_k(\hat{E})\}.$$

For convenience we will sometimes use the notation $\mathcal{W}_h := \Sigma_h \times \mathcal{V}_h \times \mathcal{M}_h$. Since we assumed that our elements T are generated by affine maps Φ_T, the finite element spaces could be defined equivalently as the appropriate polynomial spaces on the mapped triangles, cf. [10]. This would however complicate a generalization to non affine elements.

Let us now turn to the formulation of the finite element methods. We will start by recalling the hybrid mixed formulation for the elliptic subproblem ($\beta \equiv 0$) and then introduce a hybrid version for the discontinuous Galerkin method for the hyperbolic subproblem ($\epsilon \equiv 0$). The scheme for the intermediate convection diffusion problem then results by simply adding up the bilinear and linear forms of the limiting subproblems.

2.2 Diffusion

For $\beta \equiv 0$ equation (2) reduces to the mixed form of the Dirichlet problem

$$\sigma = -\epsilon \nabla u, \qquad \text{div} \sigma = f, \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial \Omega, \tag{3}$$

and the corresponding (dual) mixed variational problem reads

$$\frac{1}{\epsilon}(\sigma, \tau)_{\mathcal{T}_h} - (u, \text{div} \tau)_{\mathcal{T}_h} = 0 \qquad\qquad \forall \tau \in H(\text{div}, \Omega)$$
$$(\text{div} \sigma, v)_{\mathcal{T}_h} = (f, v)_{\mathcal{T}_h} \qquad\qquad \forall v \in L^2(\Omega).$$

While a conforming discretization of (3) allows to easily obtain conservation also on the discrete level, it also has some disadvantages: The resulting linear system is a saddlepoint problem and involves considerably more degrees of freedom than a standard (primal) H^1 conforming discretization of (3). Both difficulties can be overcome by hybridization, cf. [3, 10, 15]. Let us shortly sketch the main ideas: Instead of requiring the discrete fluxes to be in $H(\text{div}, \Omega)$, one can use completely discontinuous piecewise polynomial ansatz functions, and ensure continuity of the normal fluxes

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over element interfaces by adding appropriate constraints. The corresponding discretized variational problem reads

$$\frac{1}{\epsilon}(\sigma_h, \tau_h)_{\mathcal{T}_h} - (u_h, \mathrm{div}\tau_h)_{\mathcal{T}_h} + \langle \lambda_h, \tau_h \nu \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \tau_h \in \Sigma_h$$
$$(\mathrm{div}\sigma_h, v_h)_{\mathcal{T}_h} = (f, v)_{\mathcal{T}_h}, \qquad \forall v_h \in \mathcal{V}_h$$
$$\langle \sigma_h \nu, \mu_h \rangle_{\partial \mathcal{T}_h} = 0, \qquad \forall \mu_h \in \mathcal{M}_h.$$

Note that the choice of finite element spaces allows to eliminate the dual and primal variables on the element level, yielding a global (positive definite) system for the Lagrange multipliers only. The global system has an optimal sparsity pattern and information on the Lagrange multipliers can further be used to obtain better reconstructions by local postprocessing. We refer to [3, 10, 33] for further discussion of these issues, and come back to postprocessing later in Section 4.

After integration by parts, we arrive at the following hybrid mixed finite element method.

Method 1 (Diffusion) *Find $(\sigma_h, u_h, \lambda_h) \in \Sigma_h \times \mathcal{V}_h \times \mathcal{M}_h$ such that*

$$\mathcal{B}_D(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h) = \mathcal{F}_D(\tau_h, v_h, \mu_h) \tag{4}$$

for all $\tau_h \in \Sigma_h$, $v_h \in \mathcal{V}_h$ and $\mu_h \in \mathcal{M}_h$, where $\mathcal{B}_D$ and $\mathcal{F}_D$ are defined by

$$\mathcal{B}_D(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h) :=$$
$$\frac{1}{\epsilon}(\sigma_h, \tau_h)_{\mathcal{T}_h} + (\nabla u_h, \tau_h)_{\mathcal{T}_h} + \langle \lambda_h - u_h, \tau_h \nu \rangle_{\partial \mathcal{T}_h} + (\sigma_h, \nabla v_h)_{\mathcal{T}_h} + \langle \sigma_h \nu, \mu_h - v_h \rangle_{\partial \mathcal{T}_h}, \tag{5}$$

and

$$\mathcal{F}_D(\tau_h, v_h, \mu_h) := -(f, v_h)_{\mathcal{T}_h}. \tag{6}$$

We only mention that the case $\epsilon = 0$ on some elements T can be allowed in principle; for these elements the term $\frac{1}{\epsilon}(\sigma_h, \tau_h)_T$ just has to be interpreted as $\sigma_h|_T \equiv 0$.

Remark 1 Let $\Sigma := [H^1(\mathcal{T}_h)]^d$, $\mathcal{V} := H^1(\mathcal{T}_h)$ and $\mathcal{M} := \{\mu \in L^2(\mathcal{E}_h) : \mu = 0 \text{ on } \partial\Omega\}$, and let $\mathcal{W} := \Sigma \times \mathcal{V} \times \mathcal{M}$ denote the continuous analogue to $\mathcal{W}_h$. The above bilinear form is then defined for all $(\sigma, u, \lambda; \tau_h, v_h, \mu_h) \in \mathcal{W} \oplus \mathcal{W}_h \times \mathcal{W}_h$. This property will be used below to show consistency of the method and obtain Galerkin orthogonality. In fact, $\mathcal{B}_D$ could be defined uniquely even for all $(\sigma, u, \lambda) \in H(\mathrm{div}; \Omega) \times H^1(\mathcal{T}_h) \times L^2(\mathcal{E}_h)$ by just omitting the terms with μ_h.

Method 1 is algebraically equivalent to the conforming $RT_k \times \mathcal{P}_k$ discretization of the dual mixed formulation of (3), and can be seen as a pure implementation trick. Below, we will analyze Method 1 in a somewhat non standard way, including the gradient of the primal variable and the Lagrange multipliers explicitly in the energy norm. This kind of analysis is quite close to that of of discontinuous Galerkin methods for elliptic problems, and allows us to investigate the mixed method together with the discontinuous Galerkin method for the hyperbolic subproblem in a uniform framework.

2.3 Convection

By setting $\epsilon \equiv 0$ in (2), we arrive at the limiting hyperbolic problem

$$\mathrm{div}(\beta u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega^{in}. \tag{7}$$

Multiplying (7) by a test function $v \in H^1(\mathcal{T}_h)$, and adding stabilization (see (36)), we obtain the discontinuous Galerkin method for hyperbolic problems [31, 26, 24]

$$(\mathrm{div}(\beta u), v)_{\mathcal{T}_h} + \langle \beta_\nu(u^+ - u), v \rangle_{\partial \mathcal{T}_h^{in}} = (f, v)_{\mathcal{T}_h},$$

where $u^+ := u|_{\partial T^+}$ denotes the upwind value and T^+ is the upwind element, i.e., the element attached to E where $\beta \cdot \nu_T \geq 0$. To incorporate the boundary condition, we define $u^+ = 0$ on $\partial\Omega^{in}$. After integration by parts, and noting that $u = u^+$ on ∂T^{out} we obtain

$$(u, \beta \nabla v)_{\mathcal{T}_h} - \langle \beta_\nu u^+, v \rangle_{\partial \mathcal{T}_h^{in}} - \langle \beta_\nu u, v \rangle_{\partial \mathcal{T}_h^{out}} = -(f, v)_{\mathcal{T}_h}.$$

In order to make the discontinuous Galerkin method compatible with the hybrid mixed method formulated in the previous section, let us introduce the upwind value as a new variable $\lambda := u^+$, and let us define the the symbol

$$\{\lambda/u\} := \begin{cases} \lambda, & E \subset \partial T^{in} \\ u, & E \subset \partial T^{out}, \end{cases}$$

for all $T \in \mathcal{T}_h$. Note that $\lambda = \{\lambda/u\} = u^+$ on both sides of E, so $\{\lambda/u\}$ is just a new characterization of the upwind value. After discretization we now arrive at the following hybrid version of the discontinuous Galerkin method.

Method 2 (Convection) *Find $(u_h, \lambda_h) \in \mathcal{V}_h \times \mathcal{M}_h$ such that*

$$\mathcal{B}_C(u_h, \lambda_h; v_h, \mu_h) = \mathcal{F}_C(v_h, \mu_h) \tag{8}$$

for all $(v_h, \mu_h) \in \mathcal{V}_h \times \mathcal{M}_h$ with

$$\mathcal{B}_C(u_h, \lambda_h; v_h, \mu_h) := (u_h, \beta \nabla v_h)_{\mathcal{T}_h} + \langle \beta_\nu \{\lambda_h/u_h\}, \mu_h - v_h \rangle_{\partial \mathcal{T}_h}, \tag{9}$$

and

$$\mathcal{F}_C(v_h, \mu_h) := -(f, v)_{\mathcal{T}_h} \tag{10}$$

By construction, Method 2 is algebraically equivalent to the classical discontinuous Galerkin method. This can easily be seen by testing with $\mu_h = \chi_E$ which yields that $\lambda_h = u_h^+$ on the element interfaces. All terms of the bilinear form are again defined elementwise, which allows us to use static condensation on the element level. Moreover, as in the case of pure diffusion, the bilinear form $\mathcal{B}_C$ can be extended onto $\mathcal{W} \oplus \mathcal{W}_h \times \mathcal{W}_h$, which then allows to derive consistency and use Galerkin orthogonality arguments. On facets E where $\beta_\nu = 0$, the Lagrange multiplier is not uniquely defined, and we set $\lambda = 0$ there.

2.4 Convection-diffusion regime

Let us now return to the original convection diffusion problem and consider the system

$$\sigma + \epsilon \nabla u = 0, \quad \mathrm{div}(\sigma + \beta u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{11}$$

Since we used the same spaces for the discretization of the elliptic and hyperbolic subproblems, the two hybrid methods can be coupled in a very natural way by simply adding up their bilinear and linear forms. This yields the following hybrid mixed discontinuous Galerkin method for the intermediate convection diffusion regime.

Method 3 (Convection diffusion) *Find $(\sigma_h, u_h, \lambda_h) \in (\Sigma_h, \mathcal{V}_h, \mathcal{M}_h)$ such that*

$$\mathcal{B}(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h) = \mathcal{F}(\sigma_h, u_h, \lambda_h) \tag{12}$$

for all $\tau_h \in \Sigma_h$, $v_h \in \mathcal{V}_h$ and $\mu_h \in \mathcal{M}_h$, where $\mathcal{B}$ and $\mathcal{F}$ are defined by

$$\begin{aligned} \mathcal{B}(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h) := \\ \tfrac{1}{\epsilon}(\sigma_h, \tau_h)_{\mathcal{T}_h} + (\nabla u_h, \tau_h)_{\mathcal{T}_h} + \langle \lambda_h - u_h, \tau_h \nu \rangle_{\partial \mathcal{T}_h} \\ + (\sigma_h + \beta u_h, \nabla v_h)_{\mathcal{T}_h} + \langle \sigma_h \nu + \beta_\nu \{\lambda_h/u_h\}, \mu_h - v_h \rangle_{\partial \mathcal{T}_h}, \end{aligned} \tag{13}$$

and

$$\mathcal{F}(\tau_h, v_h) := -(f, v_h)_{\mathcal{T}_h}. \tag{14}$$

By testing with $\mu_h = \chi_E$ for $E \in \mathcal{E}_h$ we obtain that $\sigma_h \nu_E + \beta \nu_E \{\lambda_h/u_h\}$ is continuous across element interfaces. Here, ν_E denotes the unit normal vector on E with fixed orientation. Thus λ_h and $\sigma_h \nu_E + \beta \nu_E \{\lambda_h/u_h\}$ have unique values on the element interfaces and can be considered as discrete traces for u and the total flux $\sigma + \beta u$.

2.5 Consistency and conservation

Before we turn to a detailed analysis of the finite element methods 1-3, let us summarize two important properties, which follow almost directly from the corresponding properties of the mixed respectively the discontinuous Galerkin method for limiting subproblems. For sake of completeness we sketch the proofs in the present framework.

Proposition 1 (Consistency) *The methods 1-3 are consistent, i.e., let u denote the solution of the problem* (3), (7), *respectively* (11) *and define $\sigma = -\epsilon \nabla u$ and $\lambda = u$. Then the corresponding variational equations* (4), (8) *and* (12) *hold, if σ_h, u_h, λ_h are replaced by σ, u and λ.*

PROOF. Method 1: Let u denote the solution of (3), and make the substitutions as mentioned in the proposition. Then, we obtain by testing the bilinear form $\mathcal{B}_D$ with $(\tau_h, 0, 0)$

$$\begin{aligned} \mathcal{B}_D(&-\epsilon \nabla u, u, u; \tau_h, 0, 0) \\ &= -(\nabla u, \tau_h)_{\mathcal{T}_h} + (\nabla u, \tau_h)_{\mathcal{T}_h} - \langle u - u, \tau_h \nu \rangle_{\partial \mathcal{T}_h \setminus \partial \Omega} - \langle u, \tau_h \nu \rangle_{\partial \Omega} \\ &= -\langle u, \tau_h \nu \rangle_{\partial \Omega} = 0. \end{aligned}$$

Next we test with $(0, v_h, 0)$ and integrate by parts to recover

$$\mathcal{B}_D(-\epsilon \nabla u, u, u; 0, v_h, 0) = -(\mathrm{div}(-\epsilon \nabla u), v_h)_{\mathcal{T}_h} = -(f, v_h)_{\mathcal{T}_h},$$

which follows since u is the solution of (3). Finally, testing with $(0, 0, \mu_h)$ we obtain

$$\mathcal{B}_D(-\epsilon \nabla u, u, u; 0, 0, \mu_h) = \langle -\epsilon \frac{\partial u}{\partial n}, \mu_h \rangle_{\partial \mathcal{T}_h} = 0,$$

which holds since $\mathrm{div}(\epsilon \nabla u) = f \in L^2$ implies $\epsilon \nabla u \in H(\mathrm{div}; \Omega)$ and thus the normal flux $-\epsilon \frac{\partial u}{\partial n}$ is continuous across element interfaces. Note that at this point we formally require some extra regularity, e.g., $u \in H^1(\Omega) \cap H^{3/2+\varepsilon}(\mathcal{T}_h)$ or $\sigma = -\epsilon \nabla u \in L^s(\Omega)$ for some $s > 2$, in order to ensure that the moments $\langle \epsilon \frac{\partial u}{\partial n}, \mu_h \rangle$ are well-defined for $\mu_h \in \mathcal{M}_h$, cf. [10]. As already mentioned in Remark 1 this extra regularity assumption can be dropped by appropriately defining B_D. Summarizing we have shown that Method 1 is consistent.

Next, consider Method 2 and let u denote the solution of (7). Substituting u for u_h and λ_h in (8) - (10) and testing with $(v_h, 0)$ we obtain after integration by parts

$$\mathcal{B}_C(u, u; v_h, 0) = (\mathrm{div}(\beta u), v_h)_{\mathcal{T}_h} - \langle \beta_\nu u, v_h \rangle_{\partial \Omega^{in}} = -(f, v_h)_{\mathcal{T}_h}.$$

Now test with $(0, \mu_h)$, then we have

$$\mathcal{B}_C(u, u; 0, \mu_h) = \langle \beta_\nu u, \mu_h \rangle_{\partial \mathcal{T}_h} = 0,$$

since u and μ_h are single valued and β_ν appear two times with different signs for each element interface. Thus we have proven consistency of Method 2.

Finally, Method 3 is consistent as it is the sum of two consistent methods. $\qquad\square$

While consistency is a key ingredient for the derivation of a-priori error estimates, conservation is a property of the discrete methods which is desired for physical reasons, since it inhibits unphysical increase of mass or total charge. This is particularly important for time dependent problems. If a finite element scheme allows to test with piecewise constant functions, conservation can be shown to hold locally (for each element) as well as globally as long as the discrete fluxes are single valued on element interfaces.

Proposition 2 (Conservation) *The methods 1-3 are locally and globally conservative.*

PROOF. Let us first show the local conservation of Method 1 by testing (4) with $(0, \chi_T, 0)$. This yields

$$-(f, 1)_T = \mathcal{B}_D(u_h, \lambda_h, \sigma_h; 0, \chi_T, 0) = -\langle \sigma_h \nu, 1 \rangle_{\partial T},$$

that is, the total flux over an element boundary equals the sum of internal sources, and hence the method is locally conservative. By testing with $(0, 0, \chi_E)$ for some $E \in \mathcal{E}_h$, we obtain continuity of the normal fluxes $\sigma_h \nu$ across element interfaces, and so the scheme is also globally conservative.

Now to Method 2: Testing with $(\chi_T, 0)$ we get

$$(f, 1)_T = \mathcal{B}_C(u_h, \lambda_h; \chi_T, 0) = \langle \beta_\nu \lambda_h, 1 \rangle_{\partial T^{in}} + \langle \beta_\nu u_h, 1 \rangle_{\partial T^{out}},$$

so the total flux over the element boundaries equals the sum of internal sources and fluxes over the boundary of the domain. Note that $\beta_\nu \{\lambda_h / u_h\}$ defines our unique flux on element interfaces. Now let $E \in \mathcal{E}_h$ such that $E = \partial T_1^{out} \cap \partial T_2^{in}$. By testing with $(0, \chi_E)$, we obtain

$$0 = \mathcal{B}_C(u_h, \lambda_h; 0, \chi_E) = \langle \beta_\nu \{\lambda_h / u_h\}, 1 \rangle_{\partial T_1^{out}} + \langle \beta_\nu \{\lambda_h / u_h\}, 1 \rangle_{\partial T_2^{in}}$$
$$= \langle \beta_\nu u_h, 1 \rangle_{\partial T_1^{out}} + \langle \beta_\nu \lambda_h, 1 \rangle_{\partial T_2^{in}},$$

so the total outflow over a facet on one element levels the inflow over the same facet on the neighboring element.

Finally, Method 3 is conservative as it is the sum of two conservative methods. □

3 A priori error analysis

As already mentioned previously, our analysis of the hybrid methods under consideration is inspired by that of discontinuous Galerkin methods [24, 4], in particular we will utilize similar mesh dependent energy norms for proving stability and boundedness of the bilinear- and linear forms. We will show stability of Method 1 in the norm

$$\||(\tau, v, \mu)\||_D := \left(\frac{1}{\epsilon} \|\tau\|_{\mathcal{T}_h}^2 + \epsilon \|\nabla v\|_{\mathcal{T}_h}^2 + \frac{\epsilon}{h} |\lambda - u|_{\partial \mathcal{T}_h}^2 \right)^{1/2}, \tag{15}$$

and stability of Method 2 will be analysed with respect to the norm

$$\||(u, \lambda)\||_C := \left(\frac{h}{|\beta|} \|\beta \nabla u\|_{\mathcal{T}_h}^2 + |\beta_\nu| \, |\lambda - u|_{\partial \mathcal{T}_h}^2 \right)^{1/2}. \tag{16}$$

Here by $|\beta|$ and $|\beta_\nu|$ we understand appropriate bounds for β respectively β_ν on single elements or facets. Note, that for $\epsilon \sim h\beta$ (the crossover from diffusion dominated to convection dominated regime) all terms in (15) and (16) scale uniformly with respect to ϵ, β and h. For proving the boundedness of the bilinear forms we require slightly different norms

$$\||(\tau, v, \mu)\||_{D,*} := \left(\||(\tau, v, \mu)\||_D^2 + \frac{h}{\epsilon} |\tau \nu|_{\partial \mathcal{T}_h}^2 \right)^{1/2}, \tag{17}$$

and

$$\||(u, \lambda)\||_{C,*} := \left(\frac{|\beta|}{h} \|u\|_{\mathcal{T}_h}^2 + |\beta_\nu| \, |\{\lambda / u\}|_{\partial \mathcal{T}_h}^2 \right)^{1/2}. \tag{18}$$

These norms scale again in the same manner with respect to h, ϵ and β as their counterparts (15) and (16), and therefor it can be shown easily that the additional terms do not disturb the approximation.

3.1 Pure diffusion - Method 1

Below we will require the following preprapatory result.

Lemma 1 *Let $v_h \in \mathcal{V}_h$ and $\mu_h \in \mathcal{M}_h$ be given. Then there exists a unique solution $\tilde{\tau} \in \Sigma_h$ defined elementwise by the variational problems*

$$(\tilde{\tau}, p)_T = (\nabla v_h, p)_T, \qquad\qquad \forall p \in [\mathcal{P}_{k-1}(T)]^d$$
$$\langle \tilde{\tau}\nu, q \rangle_{\partial T} = \langle \mu_h, q \rangle_{\partial T}, \qquad\qquad \forall q \in \mathcal{P}_k(\partial T).$$

Moreover, there exists a constant c_I only depending on the shape of the elements such that

$$\|\tilde{\tau}\|_{\mathcal{T}_h} \leq c_I \left(\|\nabla v_h\|_{\mathcal{T}_h}^2 + h\,|\mu_h|_{\partial \mathcal{T}_h}^2 \right)^{1/2} \tag{19}$$

holds.

PROOF. The existence of a unique solution $\tilde{\tau}$ follows with standard arguments, and the norm estimate then follows by the usual scaling argument and the equivalence of norms on finite dimensional spaces, cf. [10] for details. $\qquad\square$

Since the estimate (19) uses an inverse inequality, the constant c_I depends on the shapes of the elements. Lemma 1 now allows us to construct a suitable test function for establishing the following stability estimate.

Proposition 3 (Stability) *There exists a positive constant c_D independent of the meshsize h such that the estimate*

$$\sup_{(\tau_h, v_h, \mu_h) \neq 0} \frac{\mathcal{B}_D(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h)}{\||(\tau_h, v_h, \mu_h)\||_D} \geq c_D \; \||(\sigma_h, u_h, \lambda_h)\||_D, \tag{20}$$

holds for all $(\sigma_h, u_h, \lambda_h) \in \Sigma_h \times \mathcal{V}_h \times \mathcal{M}_h$.

PROOF. Let us start with testing the bilinear form (5) with $(\sigma_h, -u_h, -\lambda_h)$, which yields

$$\mathcal{B}_D(\sigma_h, u_h, \lambda_h; \sigma_h, -v_h, -\mu_h) = \tfrac{1}{\epsilon}\|\sigma_h\|_{\mathcal{T}_h}^2.$$

Now let $\tilde{\tau}$ be defined as in Lemma 1 with μ_h replaced by $\frac{\epsilon}{h}(\lambda_h - u_h)$ and ∇v_h replaced by $\epsilon \nabla u_h$, so that

$$\|\tilde{\tau}\|_{\mathcal{T}_h} \leq c_I \left(\tfrac{\epsilon^2}{h}|\lambda_h - u_h|_{\partial\mathcal{T}_h}^2 + \epsilon^2 \|\nabla u_h\|_{\mathcal{T}_h}^2 \right)^{1/2} \tag{21}$$

holds with constant c_I independent of the meshsize h. For $\gamma > 0$ we then obtain

$$\begin{aligned}
\mathcal{B}_D&(\sigma_h, u_h, \lambda_h; \gamma\tilde{\tau}, 0, 0) \\
&= \gamma\tfrac{1}{\epsilon}(\sigma_h, \tilde{\tau})_{\mathcal{T}_h} + \gamma(\nabla u_h, \tilde{\tau})_{\mathcal{T}_h} + \gamma\langle \lambda_h - u_h, \tilde{\tau}\rangle_{\partial\mathcal{T}_h} \\
&\geq -\tfrac{1}{2\epsilon}\|\sigma_h\|_{\mathcal{T}_h}^2 - \tfrac{\gamma^2}{2\epsilon}\|\tilde{\tau}\|_{\mathcal{T}_h}^2 + \gamma(\epsilon\|\nabla u_h\|_{\mathcal{T}_h}^2 + \tfrac{\epsilon}{h}|\lambda_h - u_h|_{\partial\mathcal{T}_h}^2) \\
&\geq -\tfrac{1}{2\epsilon}\|\sigma_h\|_{\mathcal{T}_h}^2 + (\gamma - \frac{c_I\gamma^2}{2})(\epsilon\|\nabla u_h\|_{\mathcal{T}_h}^2 + \tfrac{\epsilon}{h}|\lambda_h - u_h|_{\partial\mathcal{T}_h}^2),
\end{aligned}$$

where we used (21) for the last estimate. The assertion of the proposition now follows by choosing $\gamma = 1/c_I$ and combining the estimates for the two choices of test functions. $\qquad\square$

We want to emphasize at this point that ellipticity holds regardless of the value of c_I in (21); only the ellipticity constant c_D depends on c_I and thus on the quality of the mesh. This is certainly an advantage of the mixed discretization in comparison to, e.g., the symmetric interior penalty method and also some other discontinuous Galerkin methods for elliptic problems, where sufficiently much penalization has to be added in order to ensure stability, see [4] for details.

After using a Galerkin orthogonality in the analysis below, we will need boundedness of $\mathcal{B}_D$ on the larger space $\mathcal{W} \oplus \mathcal{W}_h \times \mathcal{W}_h$.

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Proposition 4 (Boundedness) *There exists a constant C_D independent of h such that the estimate*

$$|\mathcal{B}_D(\sigma, u, \lambda; \tau_h, v_h, \mu_h)| \leq C_D \,\|\!|\!| (\sigma, u, \lambda) \|\!|\!|_{D,*} \,\|\!|\!| (\tau_h, v_h, \mu_h) \|\!|\!|_D \tag{22}$$

holds for all $(\sigma, u, \lambda) \in \mathcal{W} \oplus \mathcal{W}_h$ and $(\tau_h, v_h, \mu_h) \in \mathcal{W}_h$.

PROOF. We consider only the term $\langle \lambda - u, \tau_h \nu \rangle_{\partial \mathcal{T}_h}$ in detail. Using the Cauchy-Schwarz and a discrete trace inequality $|\tau_h \nu|_{\partial T} \leq \frac{c}{\sqrt{h}} \|\tau_h\|_T$, we obtain $|\langle \lambda - u, \tau_h \nu \rangle_{\partial T}| \leq \frac{c}{\sqrt{h}} |\lambda - u|_{\partial \mathcal{T}_h} \|\tau_h\|_T$. The result then follows by standard estimates for the remaining terms and summing up over all elements. $\qquad\square$

The above dicrete trace inequality cannot be used for the term involving $\sigma \nu$, since $\sigma \in \mathcal{W} \otimes \mathcal{W}_h$. Therefore an additional term appears in the norm $\|\!|\!| \cdot \|\!|\!|_{D,*}$.

3.2 Pure convection - Method 2

Since Method 2 is equivalent to the discontinuous Galerkin method for hyperbolic problems, our analysis is carried out in a similar manner to that presented in [24].

Proposition 5 (Stability) *There exists a constant c_C independent of the meshsize h such that the estimate*

$$\sup_{(v_h, \mu_h)} \frac{\mathcal{B}_C(u_h, \lambda_h; v_h, \mu_h)}{\|\!|\!| (v_h, \mu_h) \|\!|\!|_C} \geq c_C \,\|\!|\!| (u_h, \lambda_h) \|\!|\!|_C \tag{23}$$

holds for all $(u_h, \lambda_h) \in \mathcal{V}_h \times \mathcal{M}_h$.

PROOF. We start by choosing test functions $v_h = -u_h$ and $\mu_h = -\lambda_h$. Since $\mathrm{div}\beta = 0$ we have $(u_h, \beta \nabla u_h)_T = \frac{1}{2} \langle \beta_\nu u_h, u_h \rangle_{\partial T}$ on each element, and thus

$$\mathcal{B}_C(u_h, \lambda_h; -u_h, -\lambda_h)$$
$$= -\frac{1}{2} \langle \beta_\nu u_h, u_h \rangle_{\partial \mathcal{T}_h} + \langle \beta_\nu \{\lambda_h / u_h\}, u_h \rangle_{\partial \mathcal{T}_h} - \langle \beta_\nu \{\lambda_h / u_h\}, \lambda_h \rangle_{\partial \mathcal{T}_h}$$
$$= (1) + (2) + (3) = (*).$$

Recall that λ_h equals 0 on $\partial \Omega$, and let us rearrange the terms (1)-(3) in the following way:

$$(1) = -\frac{1}{2} \langle \beta_\nu u_h, u_h \rangle_{\partial \mathcal{T}_h} = \frac{1}{2} |\beta_\nu| |u_h|^2_{\partial \mathcal{T}_h^{in}} - \frac{1}{2} |\beta_\nu| |u_h|^2_{\partial \mathcal{T}_h^{out}},$$
$$(2) = \langle \beta_\nu \{\lambda_h / u_h\}, u_h \rangle_{\partial \mathcal{T}_h} = |\beta_\nu| |u_h|^2_{\partial \mathcal{T}_h^{out}} - |\beta_\nu| \langle \lambda_h, u_h \rangle_{\partial \mathcal{T}_h^{in}},$$
$$(3) = -\langle \beta_\nu \{\lambda_h / u_h\}, \lambda_h \rangle_{\partial \mathcal{T}_h} = |\beta_\nu| |\lambda_h|^2_{\partial \mathcal{T}_h^{in}} - |\beta_\nu| \langle \lambda_h, u_h \rangle_{\partial \mathcal{T}_h^{out}}.$$

Now let T_1, T_2 denote two elements sharing the facet $E = \partial T_1^{out} \cap \partial T_2^{in}$. Since λ_h is single valued on E by definition, we have $\lambda_h|_{\partial T_1^{out}} = \lambda_h|_{\partial T_2^{in}}$, that means we can shift the terms only involving the Lagrange multiplier between neighboring elements. Summing up, we obtain

$$(*) = \frac{1}{2} |\beta_\nu| |\lambda_h - u_h|^2_{\partial \mathcal{T}_h}.$$

Let us now include a second term in the stability estimate by testing the bilinear form with $v_h = -\gamma \frac{h}{|\beta|} \beta \nabla u_h$ for some $\gamma \geq 0$, which yields

$$\mathcal{B}_C(u_h, \lambda_h; v_h, 0) = -\frac{\gamma h}{|\beta|} (u_h, \beta \nabla (\beta \nabla u_h))_{\mathcal{T}_h} + \frac{\gamma h}{|\beta|} \langle \beta_\nu \{\lambda_h / u_h\}, \beta \nabla u_h \rangle_{\partial \mathcal{T}_h}$$
$$= \frac{\gamma h}{|\beta|} \|\beta \nabla u_h\|^2_{\mathcal{T}_h} + \frac{\gamma h}{|\beta|} \langle \beta_\nu (\lambda_h - u_h), \beta \nabla u_h \rangle_{\partial \mathcal{T}_h^{in}}$$
$$\geq c \gamma \left(\frac{h}{|\beta|} \|\beta \nabla u_h\|^2_{\mathcal{T}_h} - |\beta_\nu| |\lambda_h - u_h|^2_{\partial \mathcal{T}_h} \right).$$

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For the last estimate we used Youngs' inequality and a discrete trace inequality. The result now follows by choosing $\gamma = \frac{1}{4c}$ and combining the estimates for the two different test functions. Note that by inverse inequalities and due to our scaling of v_h with $h/|\beta|$, it follows that $\||(v_h, 0)\||_C \leq C \, \||(u_h, 0)\||_C$ with a constant C independent of the meshsize. $\qquad\square$

Proposition 6 (Boundedness) *There exists a constant C_C independent of h such that the estimate*

$$|\mathcal{B}_C(u, \lambda; v_h, \mu_h)| \leq C_C \, \||(u, \lambda)\||_{C,*} \, \||(v_h, \mu_h)\||_C \tag{24}$$

holds for all $u \in \mathcal{V} \oplus \mathcal{V}_h$, $\lambda \in \mathcal{M} \oplus \mathcal{M}_h$, and $(v_h, \mu_h) \in \mathcal{V}_h \times \mathcal{M}_h$.

PROOF. The assertion follows directly from the definition of the norms and the Cauchy-Schwarz inequality. $\qquad\square$

3.3 Convection-diffusion - Method 3

Due to the structure of Method 3 as the combination of Methods 1 and 2, the stability and boundedness of the bilinear form (13) follows almost directly from the corresponding properties of the bilinear forms for the limiting subproblems. The appropriate norms for the analysis of Method 3 are given by

$$\||(\sigma_h, u_h, \lambda_h)\|| = \left(\||(\sigma_h, u_h, \lambda_h)\||_D^2 + \||(u_h, \lambda_h)\||_C^2 \right)^{1/2} \tag{25}$$

and

$$\||(\sigma, u, \lambda)\||_* = \left(\||(\sigma, u, \lambda)\||_{D,*}^2 + \||(u, \lambda)\||_{C,*}^2 \right)^{1/2} \tag{26}$$

i.e., they are just assembled from the norms used for the analysis of the elliptic and hyperbolic subproblems. Note that all terms in the norm scale appropriately, e.g., in the diffusion dominated case ($|\beta| h \leq \epsilon$) the terms comming from the convective part can be absorbed by the terms stemming from the stability of the diffusion part. Let us now state the properties of $\mathcal{B}$ in detail.

Proposition 7 (Stability) *There exists a positive constant c_B not depending on the meshsize h such that*

$$\sup_{(\tau_h, v_h, \mu_h)} \frac{\mathcal{B}(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h)}{\||(\tau_h, v_h, \mu_h)\||} \geq c_B \, \||(\sigma_h, u_h, \lambda_h)\|| \tag{27}$$

holds for all $(\sigma_h, u_h, \lambda_h) \in \Sigma_h \times \mathcal{V}_h \times \mathcal{M}_h$.

PROOF. We will show the inf-sup stability by testing with the functions used in the previous stability estimates, i.e., $\tau_h = \sigma_h + \alpha\tilde{\tau}$, $v_h = -u_h + \gamma\frac{h}{|\beta|}\beta\nabla u_h$, $\mu_h = -\lambda_h$. In view of Proposition 3 and 5, it only remains to estimate the additional term comming from the test function $\gamma\frac{h}{|\beta|}\beta\nabla u_h$ inserted in the diffusion bilinear form, viz.

$$\mathcal{B}_D(\sigma_h, u_h, \lambda_h; 0, \gamma\frac{h}{|\beta|}\beta\nabla u_h, 0) = -\gamma\frac{h}{|\beta|}(\sigma_h, \nabla(\beta\nabla u_h)) + \gamma\frac{h}{|\beta|}\langle\sigma_h\nu, \beta\nabla u_h\rangle$$

$$= \gamma\frac{h}{|\beta|}(\mathrm{div}\sigma_h, \beta\nabla u_h) \geq -\gamma\frac{h}{|\beta|}\|\mathrm{div}\sigma_h\|\|\beta\nabla u_h\|$$

$$\geq -c\gamma\left(\frac{1}{\epsilon}\|\sigma_h\|^2 + \epsilon\|\nabla u_h\|^2\right) \geq -c\gamma \, \||(\sigma_h, u_h, \lambda_h)\||_D^2.$$

This term can be absorbed by the stability estimate for the diffusion problem as long as γ is chosen to be sufficiently small. Note that γ does not depend on h, ϵ or β, i.e., the stability constant c_B does not depend on these parameters. $\qquad\square$

The boundedness of the bilinear form follows directly by combining the two results for the limiting subproblems.

Corollary 1 (Boundedness) *There exists a constant C_B independent of the meshsize h such that*

$$|\mathcal{B}(\sigma, u, \lambda; \tau_h, v_h, \mu_h)| \leq C_B \ |||\,(\sigma, u, \lambda)\,|||_* \ |||(\tau_h, v_h, \mu_h)|||\qquad(28)$$

holds for all $(\sigma, u, \lambda) \in \mathcal{W} \oplus \mathcal{W}_h$ and $(\tau_h, v_h, \mu_h) \in \mathcal{W}_h$.

As a last ingredient for deriving the a-priori error estimates, we have to establish some approximation properties of our finite dimensional spaces with respect to the norms under consideration.

3.4 Interpolation operators and approximation properties

Let us start by introducing appropriate interpolation operators and then recall some basic interpolation error estimates. For $T \in \mathcal{T}_h$, $E \in \mathcal{E}_h$, and functions $u \in L^2(T)$, $\lambda \in L^2(E)$ we define the local L^2 projections $\Pi_k^T u$ and $\Pi_k^E \lambda$ by

$$(u - \Pi_k^T u, v_h)_T = 0, \qquad \forall v_h \in \mathcal{P}_k(T),$$

respectively

$$(\lambda - \Pi_k^E \lambda, \mu_h)_E = 0, \qquad \forall \mu_h \in \mathcal{P}_k(E).$$

These interpolation operators satisfy the following error estimates, cf [9].

Lemma 2 *Let Π_k^T and Π_k^E be defined as above. Then the estimates*

$$\|u - \Pi_k^T u\|_T \leq Ch^s |u|_{s,T}, \qquad\qquad 0 \leq s \leq k+1,$$
$$\|\nabla(u - \Pi_k^T u)\|_T \leq Ch^s |u|_{s+1,T}, \qquad\qquad 0 \leq s \leq k,$$
$$\|u - \Pi_k^T u\|_{\partial T} + \|u - \Pi_k^E u\|_{\partial T} \leq Ch^{s+1/2} |u|_{s+1,T}, \qquad\qquad 0 \leq s \leq k,$$

hold with a constants C independent of h.

The corresponding interpolation operators for functions on $\mathcal{T}_h$ respectively $\mathcal{E}_h$ are defined elementwise and are denoted by the same symbols.

For the flux function σ we utilize the Raviart-Thomas interpolant defined by

$$(\sigma - \Pi_k^{RT}\sigma, p_h)_T = 0, \qquad\qquad \forall p_h \in [\mathcal{P}_{k-1}(T)]^d,$$
$$((\sigma - \Pi_k^{RT}\sigma)\nu, \mu_h)_E = 0, \qquad\qquad \forall \mu_h \in \mathcal{P}_k(E),\, E \subset \partial T.$$

In order to make moments of $\sigma\nu$ be well-defined on single facets E one has to require some extra regularity, e.g., $\sigma \in H(\mathrm{div}, T) \cap L^s(T)$ for some $s > 2$ or $\sigma \in H^{1/2+\varepsilon}(T)$, cf. [10]. Under such an assumption, the following interpolation error estimates hold [10, 34].

Lemma 3 *Let Π_k^{RT} be defined as above. Then the estimates*

$$\|\sigma - \Pi_k^{RT}\sigma\|_T + h^{1/2}\|(\sigma - \Pi_k^{RT}\sigma)\nu\|_{\partial T} \leq Ch^s |\sigma|_{s,T}, \qquad\qquad 1/2 < s \leq k+1,$$
$$\|\mathrm{div}(\sigma - \Pi_k^{RT}\sigma)\|_T \leq Ch^s |\mathrm{div}\sigma|_{s,T}, \qquad\qquad 1 \leq s \leq k+1.$$

hold with constant C independent of h.

Applying these results elementwise, we immediately obtain the following interpolation error estimates for the mesh dependent norms used above.

Proposition 8 *Let $u \in H^1(\Omega) \cap H^{3/2+\varepsilon}(\mathcal{T}_h)$, and set $\sigma := -\epsilon\nabla u$. Then*

$$|||(\sigma - \Pi_k^{RT}\sigma, u - \Pi_k^T u, \lambda - \Pi_k^{RT} u)|||_{D,*} \leq Ch^s \sqrt{\epsilon}|u|_{s+1,\mathcal{T}_h}, \qquad 1/2 < s \leq k, \qquad(29)$$

and for $u \in H^1(\Omega)$ there holds

$$|||(u - \Pi_k^T u, \lambda - \Pi_k^{RT} u)|||_{C,*} \leq Ch^{s+1/2}\sqrt{|\beta|}|u|_{s+1,\mathcal{T}_h}, \qquad 0 \leq s \leq k, \qquad(30)$$

with constants C not depending on u or h. The same estimates hold if the $*$-norms are replaced by their counterparts without $*$.

Remark 2 The estimates of Proposition 8 hold with obvious modifications, if the smoothness s or the polynomial degree k vary locally. We assume uniform polynomial degree and smoothness only for ease of notation here.

The interpolation error estimate (29) is suboptimal regarding the approximation capabilities of the flux interpolant. In fact by Lemma 3 one can obtain

$$\frac{1}{\sqrt{\epsilon}}\|\sigma - \Pi_k^{RT}\sigma\| \leq Ch^s\sqrt{\epsilon}|u|_{s+1,\mathcal{T}_h} \qquad \text{for } 1/2 < s \leq k+1,$$

so the best possible rate is h^{k+1} instead of h^k as for $\|\!|\!|\cdot\|\!|\!|_D$ in (29). We will use this fact in Section 4 to derive super convergence results for the primal variable u_h.

3.5 A priori error estimates

The error of the finite element approximation can be decomposed into an approximation error and a discrete error. Let $(\sigma_h, u_h, \lambda_h)$ denote the discrete solution of (12), and let u be the solution of (11) and define $\sigma := -\epsilon\nabla u$. Then we have

$$\begin{aligned}
\|\!|\!|\,(\sigma - \sigma_h, u - u_h, u - \lambda_h)\,\|\!|\!| & \qquad (31)\\
& \leq \|\!|\!|(\sigma - \Pi_k^{RT}\sigma, u - \Pi_k^T u, u - \Pi_k^E u)\,\|\!|\!| + \|\!|\!|\,(\Pi_k^{RT}\sigma - \sigma_h, \Pi_k^T u - u_h, \Pi_k^E u - \lambda_h)\,\|\!|\!|.
\end{aligned}$$

Using stability and boundedness of the bilinear form, and applying Galerkin orthogonality, the second term can now also be estimated by the interpolation error.

Proposition 9 *Let $(\sigma_h, u_h, \lambda_h) \in \mathcal{W}_h$ denote the solution of (12), and let $u \in H^1(\Omega) \cap H^{3/2+\varepsilon}(\mathcal{T}_h)$ be the solution of the convection diffusion problem (11). Then there exists a constant C independent of the meshsize h such that the estimate*

$$\|\!|\!|(\Pi_k^{RT}(-\epsilon\nabla u) - \sigma_h, \Pi_k^T u - u_h, \Pi_k^E u - \lambda_h)\|\!|\!| \leq Ch^s(\sqrt{\epsilon} + h^{1/2}\sqrt{|\beta|})|u|_{s+1,\mathcal{T}_h}$$

holds for $1/2 < s \leq k$.

PROOF. Let us define $\sigma = -\epsilon\nabla u$, $\lambda = u$. By application of the stability estimate (20), Galerkin orthogonality and the boundedness (22) of the bilinear form, we obtain

$$\begin{aligned}
c_B \,\|\!|\!|\,(\Pi_k^{RT}\sigma - \sigma_h, \Pi_k^T u - u_h, \Pi_k^E u - \lambda_h)\|\!|\!| & \\
& \leq \sup_{(\tau_h, v_h, \mu_h) \neq 0} \mathcal{B}(\Pi_k^{RT}\sigma - \sigma_h, \Pi_k^T u - u_h, \Pi_k^E u - \lambda_h; \tau_h, v_h, \mu_h)/\|\!|\!|\,(\tau_h, v_h, \mu_h)\|\!|\!| \\
& = \sup_{(\tau_h, v_h, \mu_h) \neq 0} \mathcal{B}(\Pi_k^{RT}\sigma - \sigma, \Pi_k^T u - u, \Pi_k^E u - u; \tau_h, v_h, \mu_h)/\|\!|\!|\,(\tau_h, v_h, \mu_h)\|\!|\!| \\
& \leq C_B \,\|\!|\!|\,(\Pi_k^{RT}\sigma - \sigma, \Pi_k^T u - u, \Pi_k^E u - u)\,\|\!|\!|_*.
\end{aligned}$$

The assertion follows directly from (29). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The complete error estimate can now be derived by combining (31) and Proposition 8.

Theorem 1 (Energy norm estimate) *Let $(\sigma_h, u_h, \lambda_h)$ be the finite element solution of Method 3, and let $u \in H^1(\Omega) \cap H^{3/2+\varepsilon}(\mathcal{T}_h)$ denote the solution of (11) and $\sigma := -\epsilon\nabla u$. Then*

$$\|\!|\!|(\sigma - \sigma_h, u - u_h, u - \lambda_h)\|\!|\!| \leq Ch^s(\sqrt{\epsilon} + h^{1/2}\sqrt{|\beta|})|u|_{s+1,\mathcal{T}_h}$$

holds for $1/2 < s \leq k$ with constant C independent of the meshsize h.

In the convection dominated case, the error estimate coincides with the well known error estimates for the discontinuous Galerkin and the streamline diffusion method for hyperbolic problems, cf. [24, 25].

Corollary 2 *Let $\epsilon \leq |\beta| h$ on each element, and let the conditions of Theorem 1 hold. Then the estimate*

$$\||(\sigma - \sigma_h, u - u_h, u - \lambda_h)\|| \leq Ch^{s+1/2}\sqrt{|\beta|}|u|_{s+1, \mathcal{T}_h}$$

holds for $1/2 < s \leq k$ with constant C independent of the parameters ϵ, β and h.

This estimate holds in particular for the limiting hyperbolic problem ($\epsilon \equiv 0$) in which case $\sigma = \sigma_h \equiv 0$ and $\||(\tau, v, \mu)\|| = \||(v, \mu)\||_C$, and so Method 3 collapses with Method 2, i.e., the discontinuous Galerkin method for hyperbolic problems.

In analogy to standard error estimates for mixed methods for the Poisson problem, we obtain the following convergence result in the diffusion dominated regime.

Corollary 3 *Let $\epsilon \geq |\beta| h$ and let the conditions of Theorem 1 hold. Then the estimate*

$$\||(\sigma - \sigma_h, u - u_h, \lambda - \lambda_h)\|| \leq Ch^s\sqrt{\epsilon}|u|_{s+1, \mathcal{T}_h}$$

holds for $1/2 < s \leq k$ with constant C independent of ϵ, β and h. Moreover, we have $\||\cdot\||_D \sim \||\cdot\||$.

Clearly, this estimate holds also for Method 1 in the case of pure diffusion. Let us remark once again that all terms in the a-priori error estimates are defined locally, so the smoothness index s and the polynomial degree k can vary locally, allowing for hp-adaptivity.

4 Super convergence and postprocessing for diffusion dominated problems

The best possible rate for $\frac{1}{\sqrt{\epsilon}}\|\sigma - \sigma_h\|$ guaranteed by Theorem 1 and Corollary 3 is h^k, which is one order suboptimal regarding the interpolation error estimate of Lemma 3. It is well-known however that in the purely elliptic case, the optimal rate h^{k+1} can be obtained by a refined analysis, and we will derive corresponding results below. Since we consider the case of dominating diffusion in this section, we assume for ease of notation that $\epsilon \equiv 1$ in the sequel.

4.1 Refined analysis for pure diffusion

Although the estimate (29) is optimal concerning the approximation error with respect to the norm $\||\cdot\||_D$, we can obtain better error estimates for $\sigma = -\nabla u$, i.e., we will show that $\|\sigma - \sigma_h\|$ depends only on the interpolation error $\|\sigma - \Pi_k^{RT}\sigma\|$, and thus optimal convergence for σ_h can be expected. We refer to [3, 10, 33] for corresponding results in the mixed framework.

Proposition 10 *Let $(\sigma_h, u_h, \lambda_h)$ denote the solution of (4), and let u, $\sigma := -\nabla u$ be the solution of problem (3). Then*

$$\||(\sigma_h - \sigma, u_h - \Pi_k^T u, \lambda_h - \Pi_k^E u)\||_D \leq Ch^s|u|_{s+1, \Omega} \tag{32}$$

holds for $1/2 < s \leq k + 1$ with constant C independent of h.

PROOF. Let us first consider the following term.

$$\begin{aligned}
B_D(\Pi_k^{RT}\sigma &- \sigma, \Pi_k^T u - u, \Pi_k^E u - u; \tau_h, v_h, \lambda_h) \\
&= (\Pi_k^{RT}\sigma - \sigma, \tau_h)_{\mathcal{T}_h} - (\Pi_k^T u - u, \text{div}\tau_h)_{\partial\mathcal{T}_h} + \langle \Pi_k^E u - u, \tau_h\nu \rangle_{\partial\mathcal{T}_h} \\
&\quad + (\text{div}(\Pi_k^{RT}\sigma - \sigma), v_h)_{\mathcal{T}_h} + \langle (\Pi_k^{RT}\sigma - \sigma)\nu, \mu_h \rangle_{\partial\mathcal{T}_h} \\
&= (\Pi_k^{RT}\sigma - \sigma, \tau_h)_{\mathcal{T}_h},
\end{aligned}$$

where the last equality follows from the definition of the interpolants. Then in the same manner as in the proof of Proposition 9 we obtain

$$c_D \||(\Pi_k^{RT}\sigma - \sigma_h, \Pi_k^T u - u_h, \Pi_k^E u - \lambda_h)\||_D \leq \|\Pi_k^{RT}\sigma - \sigma\|_{\mathcal{T}_h},$$

and the statement follows by application of the triangle inequality and the interpolation error estimate (29). □

Note that for the modified error (32), the best possible rate now is h^{k+1}, which is optimal in view of the interpolation error estimates. As we show next, the estimates for $(\Pi_k^T u - u_h)$ and $(\Pi_k^E u - \lambda_h)$ can even be improved if we assume that the domain Ω is convex, cf. [33] for similar results in the mixed framework.

Proposition 11 *Let Ω be convex and $u \in H^1(\Omega) \cap H^{3/2+\varepsilon}(\mathcal{T}_h)$ be the solution of (3). Moreover, let u_h denote the discrete solution obtained by Method 1. Then the estimate*

$$\|\Pi_k^T u - u_h\|_0 \le C h^{s+1} \begin{cases} |u|_{s+2,\mathcal{T}_h}, & k = 0 \\ |u|_{s+1,\mathcal{T}_h}, & k > 0 \end{cases} \tag{33}$$

holds for $1/2 < s \le k+1$ (resp. $0 \le s \le 1$ for $k = 0$). If in addition f is piecewise constant, then

$$\|\Pi_0^T u - u_h\|_0 \le C h^{s+1} |u|_{s+1,\mathcal{T}_h} \tag{34}$$

holds also for $k = 0$.

PROOF. Let $\phi \in H_0^1(\Omega)$ denote the solution of the Poisson equation $\Delta\phi = \Pi_k^T u - u_h$ with homogeneous Dirichlet conditions, and let $z := \nabla\phi$. Due to convexity of Ω we have

$$\|\phi\|_{2,\Omega} \le c \, \|\Pi_k^T u - u_h\|_0 \qquad \text{and} \qquad \|\phi - \Pi_k^T \phi\| \le c \, h^{\min(k+1,2)} \|\Pi_k^T u - u_h\|_0.$$

Using the definition of ϕ and z we obtain

$$\begin{aligned}
\|\Pi_k^T u - u_h\|_0^2 &= (\Pi_k^T u - u_h, \operatorname{div} z) = (\Pi_k^T u - u_h, \operatorname{div}(\Pi_k^{RT} z)) = (\sigma - \sigma_h, \Pi_k^{RT} z) \\
&= (\sigma - \sigma_h, \Pi_k^{RT} z - \nabla\phi) - (\operatorname{div}(\sigma - \sigma_h), \phi - \Pi_k^T \phi) \\
&\le \|\sigma - \sigma_h\|_0 \|\Pi_k^{RT} z - \nabla\phi\|_0 + \|\operatorname{div}(\sigma - \sigma_h)\|_0 \|\phi - \Pi_k^T \phi\|_0.
\end{aligned}$$

The first estimate now follows by Lemma 3. If f is piecewise polynomial of order k then $\operatorname{div}(\sigma - \sigma_h) \equiv 0$, so the last term in the above estimate vanishes and we conclude the second assertion. □

4.2 The diffusion dominated case

Let us show now that similar results still hold in the presence of convection as long as diffusion is sufficiently dominating. In this case, we can discretize the convective term without upwind stabilization, and we therefore consider the following bilinear form instead of (9)

$$\mathcal{B}_C^{NU}(u_h, \lambda_h; v_h, \mu_h) := (u_h, \beta \nabla v_h)_{\mathcal{T}_h} + \langle \beta_\nu \lambda_h, \mu_h - v_h \rangle_{\partial\mathcal{T}_h}. \tag{35}$$

Such a discretization for the convective part was investigated numerically but not analysed previously in [18] for a 1D problem. There, the authors conjectured that this discretization already introduces some stabilization, which is not the case as is clear from our analysis.

Consistency and conservation: Substituting the continuous solution u for u_h and λ_h in (35) we obtain after integration by parts that

$$-(\operatorname{div}(\beta u), v_h)_{\mathcal{T}_h} + \langle \beta_\nu u, \mu_h \rangle_{\partial\mathcal{T}_h} = -(\operatorname{div}(\beta u), v_h)_{\mathcal{T}_h} = (-f, v_h)_{\mathcal{T}_h},$$

so the bilinear form $\mathcal{B}_C^{NU}$ is consistent. The scheme is also conservative, since the flux $\beta_\nu \lambda_h$ in (35) is single valued on element interfaces. Moreover, we have

$$\mathcal{B}_C(u_h, \lambda_h; v_h, \mu_h) = \mathcal{B}_C^{NU}(u_h, \lambda_h; v_h, \mu_h) + |\beta_\nu| \langle \lambda_h - u_h, \mu_h - v_h \rangle_{\partial\mathcal{T}_h^{out}}, \tag{36}$$

which clarifies what kind of upwind was used for the discontinuous Galerkin stabilization in (9).

Stability: Testing the bilinear form $\mathcal{B}_C^{NU}$ with $v_h = -u_h$ and $\mu_h = -\lambda_h$, we obtain

$$
\begin{aligned}
\mathcal{B}_C^{NU}(u_h, \lambda_h; -u_h, -\lambda_h) &= -(u_h, \beta \nabla u_h)_{\mathcal{T}_h} - \langle \beta_\nu \lambda_h, \lambda_h - u_h \rangle_{\partial \mathcal{T}_h} \\
&= -\frac{1}{2} \langle \beta_\nu u_h, u_h \rangle_{\partial \mathcal{T}_h} - \langle \beta_\nu \lambda_h, \lambda_h - u_h \rangle_{\partial \mathcal{T}_h} \\
&= \frac{1}{2} |\beta_\nu| |u_h - \lambda_h|^2_{\partial \mathcal{T}_h^{in}} - \frac{1}{2} |\beta_\nu| |u_h - \lambda_h|^2_{\partial \mathcal{T}_h^{out}}
\end{aligned}
$$

Note that by adding stabilization the stabilization term $|\beta_\nu| |u_h - \lambda_h|^2_{\partial \mathcal{T}_h^{out}}$, the last term becomes strictly positive, i.e.,

$$
\begin{aligned}
\mathcal{B}_C(u_h, \lambda_h; -u_h, -\lambda_h) &= \mathcal{B}_C^{NU}(u_h, \lambda_h; -u_h, -\lambda_h) + |\beta_\nu| |\lambda_h - u_h|^2_{\partial \mathcal{T}_h^{out}} \\
&= \frac{1}{2} |\beta_\nu| |\lambda_h - u_h|^2_{\partial \mathcal{T}_h},
\end{aligned}
$$

and we recover the first part of the stability estimate of Proposition 5.

Following the approach for the convection dominated case, we now consider the following method for for the diffusion dominated regime, cf. also [18]:

Method 4 (no upwind) *Find* $(\sigma_h, u_h, \lambda_h) \in \mathcal{W}_h$ *such that*

$$
\mathcal{B}^{NU}(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h) = \mathcal{F}(v_h, \mu_h), \tag{37}
$$

holds for all $(\tau_h, v_h, \mu_h) \in \mathcal{W}_h$ *where* $\mathcal{B}^{NU} := \mathcal{B}_D + \mathcal{B}_C^{NU}$.

For the proof of stability of the bilinear form B^{NU}, we require that the convection is sufficiently small. A suffecent condition is given by

$$
|\beta_\nu| |\lambda_h - u_h|^2_{\partial \mathcal{T}_h} \leq c_D \, |\!|\!| (\sigma_h, u_h, \lambda_h) |\!|\!|^2_D, \qquad \forall (\sigma_h, u_h, \lambda_h) \in \mathcal{W}_h. \tag{38}
$$

Remark 3 Recall that the stability constant c_D and thus the validity of condition (38) depends only on the constant of an inverse inequality and thus on the shape of the elements. Moreover, since both norms are defined elementwise, it is possible to decide for each element separately if stabilization should be added or not. Clearly, (38) can be shown to hold if $|\beta| h \leq c_T \epsilon$ is valid on each element with constant c_T only depending on the shape of the individual elements.

Using (38) as the characterization of dominating diffusion, we can now prove the following stability result.

Proposition 12 *Let* (38) *be valid. Then the estimate*

$$
\sup_{(\tau_h, v_h, \mu_h) \neq 0} \frac{\mathcal{B}^{NU}(\sigma_h, u_h, \lambda_h; \tau_h, v_h, \mu_h)}{|\!|\!| (\tau_h, v_h, \mu_h) |\!|\!|_D} \geq \frac{c_D}{2} \, |\!|\!| (\sigma_h, u_h, \lambda_h) |\!|\!|^2_D \tag{39}
$$

holds for all $(\sigma_h, u_h, \lambda_h) \in \mathcal{W}_h$ *with* c_D *denoting the stability constant of Proposition 3.*

Since the convective terms can be absorbed by the diffusion terms, the boundedness result of Corollary 1 applies with $|\!|\!| \cdot |\!|\!|_{(*)}$ replaced by $|\!|\!| \cdot |\!|\!|_{D,(*)}$. Using the stability estimate (39), the following a-priori error estimate is obtained in a similar manner as Proposition 10 for the purely elliptic case.

Proposition 13 *Let condition* (38) *be valid and* $(\sigma_h, u_h, \lambda_h)$ *denote the solution of Method 4. Moreover, let* $u \in H^1(\Omega) \cap H^{3/2+\varepsilon}(\mathcal{T}_h)$ *denote the solution of problem* (11), *and set* $\sigma := -\nabla u$. *Then*

$$
|\!|\!| (\sigma_h - \sigma, u_h - \Pi_k^T u, \lambda_h - \Pi_k^E u) |\!|\!|_D \leq C h^s |u|_{s+1,\Omega}
$$

holds for all $1/2 < s \leq k+1$ *with constant* C *independent of* h.

PROOF. In view of Proposition 10 we only have to ensure that the convective term does not disturb the estimate. Following the proof of Proposition 10, i.e., testing with the same test functions as there, we obtain the additional term

$$\mathcal{B}_C^{NU}(\Pi_k^T u - u, \Pi_k^E u - u; v_h, \mu_h) = (\Pi_k^T u - u, \beta \nabla v_h)_{\mathcal{T}_h} + \langle \beta_\nu(\Pi_k^E u - u), \mu_h - v_h \rangle_{\partial \mathcal{T}_h} = 0,$$

since $\beta \nabla v_h \in \mathcal{P}_k(T)$ on each element, and $\beta_\nu(\mu_h - v_h) \in \mathcal{P}_k(E)$ for each facet. The result now follows along the lines of the proof of Proposition 10. □

Proposition 13 allows us to derive a superconvergence estimate for $\|\Pi_k^T u - u_h\|_{\mathcal{T}_h}$ like in the purely elliptic case.

Proposition 14 *Let Ω be convex and u be the solution of (11) with β satisfying (38). Moreover, let u_h denote the discrete solution of Method 4. Then*

$$\|\Pi_k^T u - u_h\|_{\mathcal{T}_h} \leq Ch^{s+1} \begin{cases} |u|_{s+2,\mathcal{T}_h} & k = 0, \\ |u|_{s+1,\mathcal{T}_h} & k > 0, \end{cases}$$

holds for $1/2 < s \leq k+1$ respecively $0 \leq s \leq 1$ in case $k = 0$.

PROOF. By means of Proposition 13, the result follows in the same way as Proposition 11. □

Due to the lack of a condition $\mathrm{div}(\sigma - \sigma_h) \equiv 0$, which is valid in the purely elliptic case, we can not obtain (34) here. So in the lowest order case, superconvergence holds only under some additional smoothness of the solution u.

4.3 Postprocessing

The super convergence results of the previous section can now be utilized to construct better approximations $\tilde{u}_h \in \mathcal{P}_{k+1}(\mathcal{T}_h)$ by local postprocessing. Here, we follow an approach proposed by Stenberg [33] for the mixed discretization of the Poisson equation (3), and construct our postprocessed solution from the approximations of the primal and the dual variable. Alternative approaches based on the Lagrange multipliers can be found in [3, 10].

Let us define $\tilde{u}_h \in \mathcal{P}_{k+1}(\mathcal{T}_h)$ elementwise by the variational problems

$$(\nabla u_h^*, \nabla v)_T = -(\sigma_h, \nabla v)_T, \qquad \forall v \in \mathcal{P}_{k+1}(T) : (v, 1)_T = 0$$
$$(u_h^*, 1)_E = (u_h, 1)_T.$$

Then the following order optimal error estimate holds.

Proposition 15 *Let Ω be convex and u denote the solution of (11) with (38) being valid. Moreover, let $(\sigma_h, u_h, \lambda_h)$ be the solution of Method (4) and u_h^* be defined as above. Then*

$$\|\nabla(u_h^* - u)\|_{\mathcal{T}_h} \leq Ch^s |u|_{s+1,\mathcal{T}_h}$$

and

$$\|u_h^* - u\|_{\mathcal{T}_h} \leq Ch^{s+1} \begin{cases} |u|_{s+2,\mathcal{T}_h}, & k = 0 \\ |u|_{s+1,\mathcal{T}_h}, & k > 0, \end{cases}$$

for all $1/2 < s \leq k+1$ with constant C independent of the meshsize h. For $k = 0$, the second estimate holds for $0 \leq s \leq 1$.

PROOF. Let $\tilde{u}_h \in H^1(\Omega) \cap \mathcal{P}_{k+1}(\mathcal{T}_h)$ denote the finite element solution of the standard H^1 conforming finite element method applied to the solution of (3). Then $\|\nabla(u - \tilde{u}_h)\|_{\mathcal{T}_h} \leq Ch^s |u|_{s+1,\mathcal{T}_h}$ for $0 \leq s \leq k+1$. Moreover, $\|u - \tilde{u}_h\| \leq Ch^{s+1}|u|_{s+1,\mathcal{T}_h}$ for $0 \leq s \leq k+1$, since we assumed convexity of Ω and $f \in L_2$. Now define $\tilde{v}_h := (I - \Pi_0^T)(\tilde{u}_h - u_h^*)$. Then

$$\|\nabla \tilde{v}_h\|_T^2 = (\nabla(I - \Pi_0^T)(\tilde{u}_h - u_h^*), \nabla \tilde{v}_h)_T = (\nabla(\tilde{u}_h - u_h^*), \nabla \tilde{v}_h)_T$$
$$= (\nabla(\tilde{u}_h - u), \nabla \tilde{v}_h)_T + (\nabla u + \sigma_h, \nabla \tilde{v}_h)_T$$
$$\leq \|\nabla \tilde{v}_h\|_T (\|\nabla(u - \tilde{u}_h)\|_T + \|\sigma_h + \nabla u\|)_T.$$

Summing up over all elements and using the estimates for $(u - \tilde{u}_h)$ and Proposition 13 yields

$$\begin{aligned}
\|\nabla(u - u_h^*)\|_{\mathcal{T}_h} &\leq \|\nabla(u - \tilde{u}_h)\|_{\mathcal{T}_h} + \|\nabla(\tilde{u}_h - u_h^*)\|_{\mathcal{T}_h} \\
&= \|\nabla(u - \tilde{u}_h)\|_{\mathcal{T}_h} + \|\nabla \tilde{v}_h\|_{\mathcal{T}_h} \\
&\leq Ch^s |u|_{s+1, \mathcal{T}_h},
\end{aligned}$$

which already is the first part of the result. In order to establish the L^2 estimate, note that by $\Pi_0^T \tilde{v}_h = 0$ we obtain $\|\tilde{v}_h\|_T \leq Ch\|\nabla \tilde{v}_h\|_T$ via an inverse inequality. Hence

$$\begin{aligned}
\|u - u_h^*\|_T &\leq \|u - \tilde{u}_h\|_T + \|\tilde{u}_h - u_h^*\|_T \\
&\leq \|u - \tilde{u}_h\|_T + \|\tilde{v}_h\|_T + \|\Pi_0^T(\tilde{u}_h - u_h^*)\|_T \\
&= \|u - \tilde{u}_h\|_T + \|\tilde{v}_h\|_T + \|\Pi_0^T(\tilde{u}_h - u)\|_T + \|\Pi_0^T(u - u_h)\|_T.
\end{aligned}$$

Summing up over all elements, and using that

$$\|\Pi_0^T(\tilde{u}_h - u)\|_{\mathcal{T}_h} \leq \|\tilde{u}_h - u\|_{\mathcal{T}_h} \leq Ch^{s+1}|u|_{s+1, \mathcal{T}_h}$$

and Proposition 14, we conclude the L^2 estimate. $\qquad\square$

Remark 4 In the purely elliptic case ($\beta \equiv 0$) with f piecewise constant, we can obtain the optimal estimate $\|u - u_h^*\| \leq h^{s+1}|u|_{s+1, \mathcal{T}_h}$ also for the case $k = 0$ by using the estimate (34) instead of Proposition 14.

5 Implementation and numerical tests

Let us now illustrate the theoretical results derived in the previous section by some numerical tests. As a model problem, let us consider

$$\begin{aligned}
-\epsilon \Delta u + \beta \nabla u &= f \quad \text{in} \quad \Omega := (0,1)^2 \\
u &= g \quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{40}$$

where ϵ and β are constant on the whole domain. Since for the limiting hyperbolic problem our method is equivalent to the discontinuous Galerkin method, we will compare our results mainly to those obtained by the streamline diffusion method [22, 25, 23]. For a detailed comparison of hp-versions of the streamline diffusion method with discontinuous Galerkin methods for first order hyperbolic problems we refer to [20].

The variational form of the streamline diffusion method is formally derived by using $v + \alpha\beta\nabla v$ as a test function in the variational formulation of (40). Assuming $g = 0$ for simplicity, this yields

Method 5 (streamline diffusion) *Find* $u \in H_0^1(\Omega) \cap H^2(\mathcal{T}_h)$ *such that*

$$\begin{aligned}
\epsilon(\nabla u, \nabla v)_{\mathcal{T}_h} &+ (\beta\nabla u, v)_{\mathcal{T}_h} + \alpha[-\epsilon(\Delta u, \beta\nabla v)_{\mathcal{T}_h} + (\beta\nabla u, \beta\nabla v)_{\mathcal{T}_h}] \\
&= (f, v)_{\mathcal{T}_h} + \alpha(f, \beta\nabla v).
\end{aligned}$$

In order to obtain stability of the method, the stabilization parameter has to be choses appropriately, depending on the shape of the elements in the mesh. Typically, the stabilization parameter is in the order of $h/|\beta|$, where h is the local mesh size. For higher order methods, also the polynomial degree influences the choice of α, cf. [20]. For our numerical tests we us

$$\alpha = \max\{h|\beta| - 2\epsilon, 0\}/|\beta|^2.$$

In this way, stabilization is "turned off" in the diffusion dominant regime. In a similar manner, we add edge stabilization (36) to our mixed hybrid discontinuous Galerkin method with a factor $\frac{\alpha}{h} = \max\{h|\beta_\nu| - 2\epsilon, 0\}/(h|\beta_\nu|)$.

5.1 Numerical tests

With a first example, we want to illustrate that our method is capable of dealing with boundary layers near outflow boundaries very well, and in contrast to the streamline diffusion method, we do not obtain large layers. In a second example, we then show that even without any kind of shock stabilization, discontinuities can be treated rather well even for almost hyperbolic problems. As a general remark we would like to emphasize that we always compare our method using polynomials of order k with the streamline diffusion method using polynomials of degree $k+1$. Thus, formally the approximation properties of our finite element spaces is one order less. However, as our numerical results indicate, this affects the results only in the diffusion dominated case, where according to our theory we can increase the approximations by local postprocessing. This is illustrated in Example 3.

Example 1: In the first test we set $g = 0$ and

$$f = \beta_1[y + (e^{\beta_2 y/\epsilon} - 1)/(1 - e^{1/\epsilon})] + \beta_2[1 + (e^{\beta_1 x/\epsilon} - 1)/(1 - e^{\beta_1/\epsilon})].$$

For $\epsilon > 0$, the exact solution to (40) is then given by

$$u(x, y) = [x + (e^{\beta_1 x/\epsilon} - 1)/(1 - e^{\beta_1/\epsilon})] \cdot [y + (e^{\beta_2 y/\epsilon} - 1)/(1 - e^{\beta_2/\epsilon})],$$

i.e., the solution has boundary layers at the top and right outflow boundaries. For a numerical study, we set $\epsilon = 0.01$ and $\beta = (2, 1)$, and then solve the problem numerically for various meshsizes h and polynomial degrees k. Table 1 displays the errors of the numerical solutions obtained with Method 3 and the streamline diffusion method.

		streamline diffusion			mixed hybrid DG		
h	ne	k=1	k=2	k=3	k=0	k=1	k=2
0.25	32	0.21	0.19	0.19	0.065	0.040	0.033
0.125	128	0.15	0.13	0.13	0.048	0.036	0.025
0.0625	512	0.097	0.088	0.088	0.040	0.026	0.014
0.03125	2048	0.056	0.050	0.050	0.032	0.014	0.0052

Table 1: L^2 errors obtained for Example 1 on uniform meshes with meshsize h and ne elements using order k polynomials.

The exact solution is almost bilinear away from the boundary layers. Therefore one cannot expect to gain much from further increasing the polynomial degree. Note however, that in the mixed-hybrid-DG(0) method, the solution u is approximated only by piecewise constant functions. Nevertheless, the error is smaller than the one obtained by any of the streamline-diffusion(k) methods. This indicates that most of the error in the streamline diffusion method actually stems from adding the stabilization term. This also explains why the streamline-diffusion(k) methods do not give better results for increased polynomial degree, while our mixed-hybrid-DG method does.

Since in our example the location of boundary layers is determined a-priori, one should of course use local mesh refinement towards the outflow boundaries. The exact solution using adaptive grids are plotted in Figure 1. In Table 2 we summarise the numerical results obtained on adaptive meshes. In order to obtain an error of less than 0.051 we can use the streamline-diffusion(2) method with 898 elements, the mixed-hybrid-DG(0) method with 320 elements or the mixed-hybrid-DG(1) method with only 8 elements. Similar results are also obtained in the second Example below. If we continue to locally refine the mesh towards the boundary layer, the problem becomes diffusion dominant (at least in the boundary layers), and the streamline-diffusion(k+1) method gives better results then the mixed-hybrid-DG(k) method. In the current example, this happens at about 5000 elements. In the diffusion dominant region, we can however improve the solution obtained by our hybrid method by local postprocessing, cf. Example 3 below.
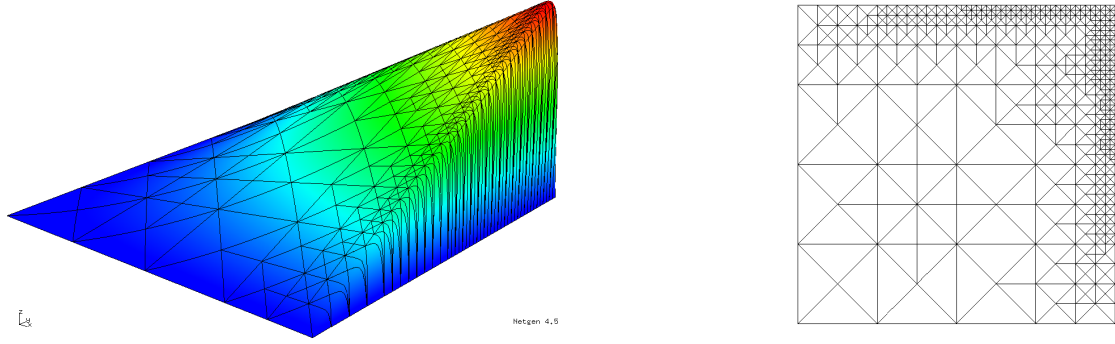
Figure 1: Example 1: exact solution and locally adapted mesh with 878 elements.

		streamline diffusion			mixed hybrid DG		
h	ne	k=1	k=2	k=3	k=0	k=1	k=2
0.125	46	0.16	0.16	0.16	0.061	0.038	0.026
0.0625	125	0.12	0.11	0.11	0.058	0.027	0.015
0.03125	320	0.080	0.078	0.078	0.043	0.016	0.0066
0.015625	878	0.054	0.051	0.046	0.032	0.0080	0.0022

Table 2: L^2 error of numerical solutions obtained with the streamline-diffusion(k) and the mixed-hybrid-DG(k) method for adaptively refined mesh with ne elements and minimal meshsize h.

Example 2: For a second test, we set $f = 0$, $\beta = (2, 1)$ as before, and $\epsilon = 10^{-6}$, so we are dealing with an (almost) hyperbolic problem. Additionally, we introduce a discontinuity in the boundary conditions, i.e., we set $u(0, y) = H(y - 0.5)$ on the left inflow boundary ($H(\cdot)$ denotes the Heavyside function), and we set $u = 0$ on the remaining part of the boundary. The exact solution for $\epsilon = 0$ (the boundary conditions at the outflow boundaries have to be omitted in this case) is given by

$$u(x, y) = \begin{cases} 1 & y > 0.5(1 + x) \\ 0 & \text{else.} \end{cases}$$

Below we use the solution of the purely hyperbolic problem also for calculation of the numerical errors of the finite element solutions. Again, we solve on uniform meshes (not aligned to the discontinuity) and compare the solutions obtained with Method 3 and the streamline upwind method for different polynomial degrees.

		streamline diffusion			mixed hybrid DG		
h	ne	k=1	k=2	k=3	k=0	k=1	k=2
0.25	32	0.33	0.25	0.23	0.22	0.15	0.11
0.125	128	0.25	0.19	0.18	0.18	0.11	0.087
0.0625	512	0.18	0.14	0.13	0.15	0.090	0.069
0.03125	2048	0.13	0.10	0.096	0.12	0.070	0.053

Table 3: L^2 errors of streamline-diffusion(k) and mixed-hybrid-DG(k) method for uniformly refined meshes with mehssize h and ne elements for polynomial degree k.

Since the exact solution is piecewise constant, increasing the polynomial degree can only yield improvements of the approximations in the boundary and internal layers. As in the previous

example, we can highly improve the performance by using locally adapted meshes. The corresponding numerical results are listed in Table 4. In Figure 2 we plot solutions obtained with the

		streamline diffusion			mixed hybrid DG		
h	ne	k=1	k=2	k=3	k=0	k=1	k=2
0.25	15	0.37	0.26	0.24	0.23	0.15	0.11
0.125	54	0.27	0.21	0.18	0.20	0.11	0.10
0.0625	140	0.20	0.15	0.14	0.16	0.093	0.082
0.03125	368	0.14	0.11	0.10	0.13	0.075	0.064

Table 4: L^2 errors of streamline-diffusion(k) and mixed-hybrid-DG(k) method for adaptively refined meshes with ne elements and minimal meshsize h for polynomial degree k.

streamline-diffusion(3) and mixed-hybrid-DG(2) method on an appropriate adaptive mesh. Both
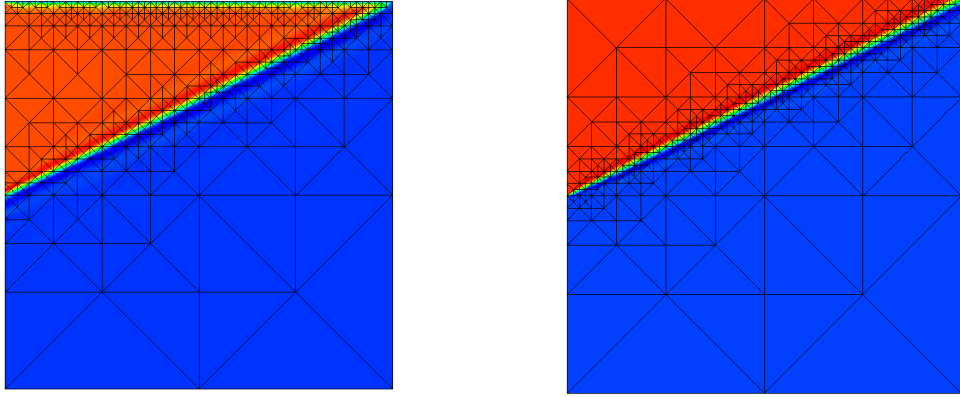

Figure 2: Streamline-diffusion(3) and mixed-hybrid-DG(2) solution obtained on appropriate adapted meshes with 878 respectively 798 elements.

solutions resolve the discontinuity at the interior layer rather stably and oscillations are located only within the elements touching the discontinuity. Moreover, although the mesh is not aligned with the flow direction β, the numerical diffusion is very small and the jump of the solution stays within one element throughout the domain. Note also that the mixed hybrid DG method deals with the outflow boundary in a more natural way than the streamline diffusion method, so no mesh refinement is needed at the boundary layers.

Let us now turn to a diffusion dominated problem an illustrate the increase in accuracy obtained by local post-processing discussed in Section 4.3.

Example 3: Set $\beta = (2, 1)$, $\epsilon = 0.1$ and $f = 1$ in (40), and let $u = 0$ at the boundary. As an approximation for the true solution we take the conforming finite element solution with polynomial degree 8. We solve problem (40) with Method 3 and 4 respectively, and compare the numerical results with those obtained by the streamline diffusion method. As outlined above, we decrease the stabilization parameter with the meshsize such that no stabilization is added for $|\beta|h < 2\epsilon$. The results obtained for various mesh sizes and polynomial degrees are summarized in Table 5.

Since the problems become diffusion dominant for a meshsize of $h \sim 0.1$, the streamline-diffusion(k+1) methods give better results than the hybrid(k) methods in the last two lines of Table 5. For improving the approximation for the mixed hybrid DG method in that case, we now apply local postprocessing as discussed in Section 4. In Table 6 we list the results obtained with the

		streamline diffusion		mixed hybrid DG	
h	ne	k=2	k=3	k=1	k=2
0.25	32	0.071	0.071	0.020	0.0064
0.125	128	0.022	0.021	0.0078	0.0015
0.0625	512	0.00056	0.000046	0.0024	0.00025
0.03125	2048	0.000076	0.0000032	0.00065	0.000034

Table 5: L^2 errors of streamline-diffusion(k) and mixed-hybrid-DG(k) method for uniform meshes with ne elements and meshsize h for polynomial degrees k.

mixed-hybrid DG method and postprocessing, and we compare with the L^2 best approximation errors for the according finite element spaces.

		postprocessed		l2-projection	
h	ne	k=2	k=3	k=2	k=3
0.25	32	0.011	0.0031	0.0063	0.0019
0.125	128	0.0025	0.00043	0.0015	0.00027
0.0625	512	0.00042	0.000037	0.00025	0.000023
0.03125	2048	0.000057	0.0000026	0.000034	0.0000016

Table 6: L^2 errors of postprocessed solution of the mixed-hybrid-DG(k-1) method and the best piecewise polynomial approximation of order k on uniform meshes with ne elements and meshsize h.

Throughout our numerical experiments, the error of the postprocessed solution is always in the order of 1-3 times the best approximation error. Moreover, the mixed hybrid DG method with postprocesing always yielded slightly more accurate results than the standard confoming finite element method (which agrees with the streamline diffusion method in the diffusion dominating regime) with the corresponding polynomial degree.

5.2 Concluding remarks

In this paper we proposed a new finite element method for convection diffusion problems based on a mixed discretization for the elliptic part and a discontinuous Galerkin formulation for the convective part. The two methods are made compatible via hybridization, and the Lagrange multipliers play an essential role for the stabilization of the method and throuhgout the analysis.

Like other discontinuous Galerkin methods, but in contrast to the streamline diffusion method, the presented scheme is locally and globaly conservative, which makes it a natural candidate for problems where conservation is important, e.g., for time dependent problems. Moreover, the treatment of boundary layers is very natural and allows a seamless change from convection dominated to purely hyperbolic regimes, where the outflow boundary conditions just disappear also in the numerical scheme. In the hyperbolic limit, our method corresponds to (a hybrid version of) the classical discontinuous Galerkin method and thus inherits the stabilizing features of discontinuous Galerkin methods for hyperbolic problems.

The (hybrid) mixed methods allows a more natural treatment of elliptic operators than the discontinuous Galerkin methods. In particular, the the discretization of diffusion terms does not increase the stencil of the scheme. In contrast to the streamline diffusion method and to several variants of discontinuous Galerkin methods, the stabilization parameter is always smaller or equal to 1, and inverse inequalities respectively the shape of the triangles only affect the stability constant. In the diffusion dominated regime, we can apply local postprocessing in order to improve the approximation of the primal variable by one order.

A particular advantage of our method from a computational point of view is that it is formulated and can be implemented purely element wise. This allows static condensation of the primal and flux variables on the element level, and only the Lagrange multipliers appear in the global system. Thus the presentend mixed hybrid discontinuous Galerkin method has smaller stencils as well as fewer degrees of freedom than standard discontinuous Galerkin methods, but still provides the same stability.

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