SIAM J. NUMER. ANAL.
Vol. 48, No. 2, pp. 498–523

A RESIDUAL-BASED A POSTERIORI ERROR ESTIMATOR FOR THE STOKES–DARCY COUPLED PROBLEM*

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Abstract. In this paper we develop an a posteriori error analysis of a new conforming mixed finite element method for the coupling of fluid flow with porous media flow. The flows are governed by the Stokes and Darcy equations, respectively, and the transmission conditions are given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law. The finite element subspaces consider Bernardi–Raugel and Raviart–Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for a Lagrange multiplier defined on the interface. We derive a reliable and efficient residual-based a posteriori error estimator for this coupled problem. The proof of reliability makes use of suitable auxiliary problems, diverse continuous inf-sup conditions satisfied by the bilinear forms involved, and local approximation properties of the Clément interpolant and Raviart–Thomas operator. On the other hand, Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions are the main tools for proving the efficiency of the estimator. Up to minor modifications, our analysis can be extended to other finite element subspaces yielding a stable Galerkin scheme.

Key words. a posteriori error analysis, Stokes equation, Darcy equation

AMS subject classifications. 65N15, 65N30, 74F10, 74S05

DOI. 10.1137/080727646

1. Introduction.

The development of appropriate numerical methods for the coupling of fluid flow (modeled by the Stokes equation) with porous media flow (modeled by the Darcy equation) has become a very active research area lately (see, e.g., [10], [20], [26], [31], [35], and the references therein). In particular, a new mixed variational formulation and associated mixed finite element methods have been derived in [31] and [26] for this coupled problem. The variational formulation employed in [31] and [26] consists of a primal approach in the fluid and a mixed approach in the porous medium. This means that only the original velocity and pressure unknowns are considered in the Stokes domain, whereas a further unknown (velocity) is added in the Darcy region. In addition, the corresponding interface conditions are given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law, which yields the introduction of the trace of the porous medium pressure as an additional Lagrange multiplier. The main results in [31] include well-posedness of the continuous formulation, and solvability, stability, and convergence of a nonconforming mixed finite element scheme. The nonconformity of the scheme in [31] arises from the fact that the above-mentioned Lagrange multiplier is approximated by piecewise constant functions, which are certainly not contained in the Sobolev space for the traces on the interface. A similar approach to [31] is studied in [20]. On the other hand, a slight

*Received by the editors June 18, 2008; accepted for publication (in revised form) February 16, 2010; published electronically April 28, 2010. This research was partially supported by the Institute for Computational Engineering and Sciences (ICES) of the University of Texas at Austin through the J. Tinsley Oden Faculty Fellowship Research Program, by FONDAP and BASAL projects CMM, Universidad de Chile, and by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción.

http://www.siam.org/journals/sinum/48-2/72764.html

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simplification of the model from [31], which consists of a porous medium entirely enclosed within a fluid region, is considered in [26], and a conforming mixed finite element method for the resulting coupled problem is introduced and analyzed there. In this case, the Lagrange multiplier on the interface is approximated by continuous piecewise linear elements, whereas Bernardi–Raugel, Raviart–Thomas, and piecewise constant elements are employed for the velocities and the pressures, respectively. The method proposed in [26] is the first which is conforming for the original mixed variational formulation proposed in [31]. The results in [26] have been recently extended in [27] and [28] to any pair of stable Stokes and Darcy elements and to the case of dual-mixed formulations in both domains.

Now, it is well known that in order to guarantee a good convergence behavior of most finite element solutions, especially under the eventual presence of singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities $\boldsymbol{\eta}$ that are expressed in terms of local indicators η_T defined on each element T of a given triangulation $\mathcal{T}$. The estimator $\boldsymbol{\eta}$ is said to be efficient (resp., reliable) if there exists $C_1 > 0$ (resp., $C_2 > 0$), independent of the mesh sizes, such that

$$C_1\,\boldsymbol{\eta} + \text{h.o.t.} \leq \|error\| \leq C_2\,\boldsymbol{\eta} + \text{h.o.t.},$$

where h.o.t. is a generic expression denoting one or several terms of higher order. One of the first papers concerning a posteriori error analysis of variational formulations with saddle-point structure is [37], where an estimator of explicit residual type was obtained for the Stokes problem. The technique employed in [37], which was also adopted in the related works [8] and [9], is based on the element residual method originally proposed in [7] and [19]. A unified analysis of this approach and its further application to the Stokes and Oseen equations were presented in [2] and [3], respectively. On the other hand, estimators based on residuals and on the solution of local problems, using Raviart–Thomas and Brezzi–Douglas–Marini spaces, are provided in [4] for elliptic partial differential equations of second order. The main novelty of [4] is the use of a Helmholtz decomposition to prove reliability and efficiency of the error estimators. In connection with Raviart–Thomas spaces, we may also refer the reader to [12], [14], and [25]. In particular, the Helmholtz decomposition technique is applied again in [14] and [25] to derive reliable and efficient residual-based a posteriori error estimators for the Poisson problem. Note that the analysis in [25] considers the case of mixed boundary conditions (cf. [6]). The extension of the results in [14] to the linear elasticity problem was developed in [16] and [32]. A comparison of four different kinds of error estimators for mixed finite element discretizations by Raviart–Thomas elements is given in [30]. More recently, energy norm a posteriori error estimates based on postprocessing were obtained in [33], and functional-type error estimates were presented in [34].

The purpose of this work is to derive a reliable and efficient residual-based a posteriori error estimator for the Stokes–Darcy coupled problem analyzed in [26]. Though one might think a priori that this should follow simply by combining the corresponding approaches already available for the Stokes and Darcy problems, the analysis below will show that this idea works only partially since further difficulties and several technical issues arise along the way. In this respect, it is important to remark that, on one hand, the transmission conditions stop us from splitting the analysis into the Stokes and Darcy parts, and, on the other hand, these conditions cannot be neglected since they also have to be incorporated into the resulting a posteriori error estimate. In spite of the above, we will observe that at some stages we will be able to deal, separately, with independent terms related to the Stokes and Darcy domains, respectively. The rest of the paper is organized as follows. In section 2 we recall from [26] the Stokes–Darcy coupled problem and its continuous and discrete mixed variational formulations. The kernel of the present work is given by sections 3 and 4, where we develop the a posteriori error analysis. In section 3 we employ auxiliary problems, suitable continuous inf-sup conditions, and local approximation properties of the Clément and Raviart–Thomas operators to derive a reliable residual-based a posteriori error estimator. A novel feature of our proof of reliability is the application of partial continuous inf-sup conditions of the bilinear form involving the Lagrange multipliers. The term partial refers here to intermediate inf-sup inequalities which are obtained in the proof of the full inf-sup condition. We remark that this new approach was motivated by the fact that the usual procedure of utilizing the full continuous inf-sup condition of that bilinear form would not work in this case. Also, it is worth mentioning that the derivation of the upper bounds for the components of the error needs to follow a specific ordering of the unknowns. In particular, a direct sum decomposition of the traces on the interface has to be considered in this analysis. Next, in section 4 we apply Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions to show the efficiency of the estimator. This proof benefits partially from the fact that, as a consequence of the independent terms mentioned above, some components of the a posteriori error estimator coincide with those obtained for the mixed formulation of the Darcy problem, whose corresponding efficiency estimates are already available in the literature (cf. [14], [23], [25]). Nevertheless, most of the terms, especially those related to the Stokes domain, seem not to have appeared in previous works, and hence the corresponding upper bounds are derived here.

Throughout the rest of the paper we utilize the standard terminology for Sobolev spaces. In particular, if S is an open set, its closure, or a Lipschitz continuous curve, and $r \in \mathbb{R}$, then $|\cdot|_{r,S}$ and $\|\cdot\|_{r,S}$ stand for the seminorm and norm in the Sobolev spaces $H^r(S)$, $[H^r(S)]^2$, and $[H^r(S)]^{2\times 2}$. Hereafter, given any normed space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. Also, we employ $\mathbf{0}$ as a generic null vector and use C and c, with or without subscripts, bars, tildes, or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2. The Stokes–Darcy coupled problem.

2.1. The model problem. Let Ω_2 be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ_2, and let Ω_1 be the annular region bounded by Γ_2 and another closed polygonal curve Γ_1 whose interior contains $\overline{\Omega}_2$. Then, the coupled problem consists of an incompressible viscous fluid occupying Ω_1, which flows back and forth across Γ_2 into a porous medium living in Ω_2 and saturated with the same fluid. Physically, we may think of Ω_2 as the cross-section of a three-dimensional porous medium, given, for instance, by a long cylinder parallel to the x_3-axis, which is inmersed in a viscous fluid. In what follows, $\mu > 0$ denotes the viscosity of the fluid and $\mathbf{K}$ is a symmetric and uniformly positive definite tensor in Ω_2 representing the permeability of the porous media divided by the viscosity. We assume that $\mathbf{K}$ is sufficiently smooth and that there exists $C > 0$ such that $\|\mathbf{K}(x)\, z\| \le C\,\|z\|$ for almost all $x \in \Omega_2$ and for all $z \in \mathbb{R}^2$. Then, the constitutive equations are given by the Stokes and Darcy laws, respectively, that is,

$$\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1) = -\, p_1\, \mathbf{I} + 2\,\mu\, \mathbf{e}(\mathbf{u}_1) \quad \text{in} \quad \Omega_1, \quad \text{and} \quad \mathbf{u}_2 = -\,\mathbf{K}\,\nabla\, p_2 \quad \text{in} \quad \Omega_2,$$

where $(\mathbf{u}_1, \mathbf{u}_2)$ and (p_1, p_2) denote the velocities and pressures in the corresponding domains, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, $\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)$ is the stress tensor, and $\mathbf{e}(\mathbf{u}_1) := \frac{1}{2}(\nabla \mathbf{u}_1 + (\nabla \mathbf{u}_1)^{\mathtt{t}})$ is the strain tensor. Hereafter, the superscript $^{\mathtt{t}}$ stands for the transpose matrix. Hence, given $\mathbf{f}_1 \in [L^2(\Omega_1)]^2$ and $f_2 \in L^2(\Omega_2)$ such that $\int_{\Omega_2} f_2 = 0$, the Stokes–Darcy coupled problem reads as follows: Find $(\mathbf{u}_1, \mathbf{u}_2)$ and (p_1, p_2) such that

$$\begin{aligned}
-\,\mathbf{div}\,\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1) &= \mathbf{f}_1 && \text{in} \quad \Omega_1,\\
\operatorname{div} \mathbf{u}_1 &= 0 && \text{in} \quad \Omega_1,\\
\mathbf{u}_1 &= \mathbf{0} && \text{on} \quad \Gamma_1,\\
\operatorname{div} \mathbf{u}_2 &= f_2 && \text{in} \quad \Omega_2,\\
\mathbf{u}_1 \cdot \boldsymbol{\nu} &= \mathbf{u}_2 \cdot \boldsymbol{\nu} && \text{on} \quad \Gamma_2,\\
\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu} + p_2\,\boldsymbol{\nu} &= -\,\frac{\mu}{\kappa}\,(\mathbf{u}_1 \cdot \mathbf{s})\,\mathbf{s} && \text{on} \quad \Gamma_2,
\end{aligned} \tag{2.1}$$

where $\mathbf{div}$ is the usual divergence operator div acting on each row of the tensor, $\boldsymbol{\nu}$ is the unit outward normal to Ω_1, $\mathbf{s}$ is the tangential vector on Γ_2, and $\kappa > 0$ is the friction constant. Note that the second transmission condition on Γ_2 can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu}) \cdot \boldsymbol{\nu} = -\, p_2 \quad \text{and} \quad (\boldsymbol{\sigma}_1(\mathbf{u}_1, p_1)\,\boldsymbol{\nu}) \cdot \mathbf{s} = -\,\frac{\mu}{\kappa}\,(\mathbf{u}_1 \cdot \mathbf{s}) \quad \text{on } \Gamma_2. \tag{2.2}$$

The first equation in (2.2) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman law, which establishes that the slip velocity along Γ_2 is proportional to the shear stress along Γ_2 (assuming, also, based on experimental evidence, that $\mathbf{u}_2 \cdot \mathbf{s}$ is negligible).

2.2. The mixed variational formulation. In order to introduce the variational formulation of (2.1), we now put $\Omega := \Omega_1 \cup \Gamma_2 \cup \Omega_2$ and define the spaces

$$\begin{aligned}
L^2_0(\Omega) &:= \left\{ q \in L^2(\Omega) : \quad \int_\Omega q = 0 \right\},\\
[H^1_{\Gamma_1}(\Omega_1)]^2 &:= \left\{ \mathbf{v}_1 \in [H^1(\Omega_1)]^2 : \ \mathbf{v}_1 = \mathbf{0} \ \text{on} \ \Gamma_1 \right\}, \quad \text{and}\\
H(\operatorname{div}; \Omega_2) &:= \left\{ \mathbf{v}_2 \in [L^2(\Omega_2)]^2 : \ \operatorname{div}\ \mathbf{v}_2 \in L^2(\Omega_2) \right\}.
\end{aligned}$$

In addition, we let

$$\mathbf{H} := [H^1_{\Gamma_1}(\Omega_1)]^2 \times H(\operatorname{div}; \Omega_2) \quad \text{and} \quad \mathbf{Q} := L^2_0(\Omega) \times H^{1/2}(\Gamma_2) \tag{2.3}$$

endowed with the product norms $\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}_1\|_{1,\Omega_1} + \|\mathbf{v}_2\|_{H(\operatorname{div};\Omega_2)}$ for all $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}$ and $\|(q, \xi)\|_{\mathbf{Q}} := \|q\|_{0,\Omega} + \|\xi\|_{1/2,\Gamma_2}$ for all $(q, \xi) \in \mathbf{Q}$. Also, we denote the global unknowns $\mathbf{u} := (\mathbf{u}_1, \mathbf{u}_2)$, $p := \left\{ \begin{smallmatrix} p_1 & \text{in} & \Omega_1, \\ p_2 & \text{in} & \Omega_2 \end{smallmatrix} \right.$ and introduce the Lagrange multiplier $\lambda := p_2 \quad \text{on} \quad \Gamma_2$. Hence, proceeding in the usual way (see, e.g., [31]), we find that the mixed variational formulation of (2.1) reads as follows: Find $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned}
\mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p, \lambda)) &= \mathbf{F}(\mathbf{v}) && \forall\, \mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H},\\
\mathbf{b}(\mathbf{u}, (q, \xi)) &= \mathbf{G}(q, \xi) && \forall\, (q, \xi) \in \mathbf{Q},
\end{aligned} \tag{2.4}$$

where $\mathbf{a} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$ are the bilinear forms defined by

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := 2\,\mu \int_{\Omega_1} \mathbf{e}(\mathbf{u}_1) : \mathbf{e}(\mathbf{v}_1) + \frac{\mu}{\kappa} \int_{\Gamma_2} (\mathbf{u}_1 \cdot \mathbf{s})\,(\mathbf{v}_1 \cdot \mathbf{s}) + \int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2 \cdot \mathbf{v}_2, \tag{2.5}$$

$$\mathbf{b}(\mathbf{v}, (q, \xi)) := - \int_{\Omega_1} q \operatorname{div} \mathbf{v}_1 - \int_{\Omega_2} q \operatorname{div} \mathbf{v}_2 + \langle \mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{v}_2 \cdot \boldsymbol{\nu}, \xi \rangle_{\Gamma_2}, \tag{2.6}$$

with $\langle \cdot, \cdot \rangle_{\Gamma_2}$ being the duality pairing of $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$ with respect to the $L^2(\Gamma_2)$-inner product, and $\mathbf{F} : \mathbf{H} \to \mathbb{R}$ and $\mathbf{G} : \mathbf{Q} \to \mathbb{R}$ are the linear functionals defined as

$$\mathbf{F}(\mathbf{v}) = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 \quad \forall\, \mathbf{v} \in \mathbf{H} \quad \text{and} \quad \mathbf{G}(q, \xi) = - \int_{\Omega_2} f_2\, q \quad \forall\, (q, \xi) \in \mathbf{Q}. \tag{2.7}$$

The well-posedness of the continuous formulation (2.4) follows from a straightforward application of [29, Chapter I, Theorem 4.1]. In fact, it is easy to see, thanks to the Cauchy–Schwarz inequality and the trace estimates in $[H^1(\Omega_1)]^2$ and $H(\operatorname{div}; \Omega_2)$ (see also [31, Lemmas 2.1 and 3.1] for details), that $\mathbf{a}$ and $\mathbf{b}$ are bounded. In what follows, $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$ denote the corresponding boundedness constants. In addition, the continuous inf-sup condition for $\mathbf{b}$ and the strong coerciveness of $\mathbf{a}$ on the null space $\mathbf{V}$ of $\mathbf{b}$, which reduces to

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{H} : \ \operatorname{div} \mathbf{v}_1 = \mathbf{0} \text{ in } \Omega_1, \ \operatorname{div} \mathbf{v}_2 = \mathbf{0} \text{ in } \Omega_2, \text{ and } \mathbf{v}_1 \cdot \boldsymbol{\nu} = \mathbf{v}_2 \cdot \boldsymbol{\nu} \text{ on } \Gamma_2\}, \tag{2.8}$$

are given in [26, Lemmas 2.1 and 2.2].

2.3. Remarks on the geometry. It is important to remark that, though the geometry described above was choosen to simplify the presentation, the case of a fluid flowing only across a part of the boundary of the porous medium does not really yield further complications for the analysis in the present paper. For instance, if we consider a fluid over the porous medium, then the boundary $\partial\Omega_1$ stays given by $\Gamma_1 \cup \Gamma_2$, but now with both curves meeting at their end-points, whereas a new piece of $\partial\Omega_2$, say, Γ, such that $\partial\Omega_2 = \Gamma_2 \cup \Gamma$, needs to be identified (see Figure 2.1). In this case, in addition to the equations given in the present section (which hold now with the notation introduced here), a boundary condition on Γ needs to be added. Following [22] and [31] (see also [21]), one usually considers the homogeneous Neumann condition,

$$\mathbf{u}_2 \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma, \tag{2.9}$$

which constitutes a no-flow assumption through Γ. In this way, having in mind the new geometry, the space $\mathbf{H}$ becomes now $[H^1_{\Gamma_1}(\Omega_1)]^2 \times H_0(\operatorname{div}; \Omega_2)$, where

$$H_0(\operatorname{div}; \Omega_2) := \Big\{ \mathbf{v}_2 \in H(\operatorname{div}; \Omega_2) : \quad \mathbf{v}_2 \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma \Big\}, \tag{2.10}$$

and $\mathbf{Q}$, $\mathbf{a}$, and $\mathbf{b}$ remain the same as before (cf. (2.3), (2.5), and (2.6)). In particular, it follows from (2.9) (see, e.g., [22, section 2]) that the restriction of $\mathbf{v}_2 \cdot \boldsymbol{\nu}$ to Γ_2 can be identified with an element of $H^{-1/2}(\Gamma_2)$, whereas $\mathbf{v}_1 \cdot \boldsymbol{\nu}|_{\Gamma_2} \in L^2(\Gamma_2) \subseteq H^{-1/2}(\Gamma_2)$ for each $\mathbf{v}_1 \in [H^1_{\Gamma_1}(\Omega_1)]^2$. As a consequence, the right space for the Lagrange multiplier λ is still $H^{1/2}(\Gamma_2)$, and hence the boundary term in the definition of $\mathbf{b}$ must again be understood as the duality pairing between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$. Therefore, the

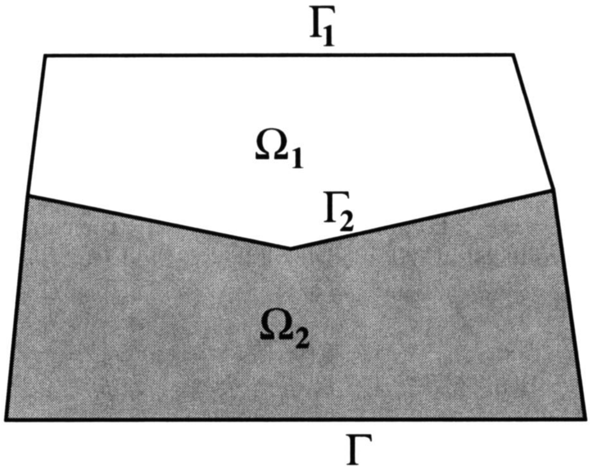

FIG. 2.1. *The fluid over the porous medium.*

proofs of the corresponding inf-sup conditions for **a** and **b** (see [22, section 4.2] for details) follow basically the same techniques applied in [26] and [31], thus confirming that no additional difficulties arise. We refer the reader to [22] for further details on this geometry and emphasize that only minor modifications would need to be incorporated into the forthcoming analysis.

We end this section by remarking that the only reason for restricting our attention here to two dimensions is the fact that the analysis in [26] assumed that case. We believe, however, that the results in [26] and the present paper should be extended without difficulties to the three-dimensional case.

2.4. The conforming mixed finite element method. We now let $\mathcal{T}_1$ and $\mathcal{T}_2$ be members of shape-regular families of triangulations, that is, satisfying the minimum angle condition, of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, by triangles T of diameter h_T, and we assume that the vertices of $\mathcal{T}_1$ and $\mathcal{T}_2$ coincide on the interface Γ_2. Also, we let $h := \max\{h_1, h_2\}$, where $h_i := \max\{\, h_T : T \in \mathcal{T}_i \,\}$ for each $i \in \{1, 2\}$. Then, for each $T \in \mathcal{T}_2$ we let $\mathrm{RT}_0(T)$ be the local Raviart–Thomas space of lowest order, and for each $T \in \mathcal{T}_1$ we let $\mathrm{BR}(T)$ be the local Bernardi–Raugel space (see [11], [29]). Hereafter, given S, an open set, its closure, or a Lipschitz continuous curve of $\mathbb{R}^2$, and a nonnegative integer k, $\mathbf{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$. Hence, we define the following finite element subspaces for the velocities and the pressure:

$$
\begin{aligned}
\mathbf{H}_{h_1} &:= \Big\{ \mathbf{v} \in [C(\bar{\Omega}_1)]^2 : \quad \mathbf{v}|_T \in \mathrm{BR}(T) \quad \forall\, T \in \mathcal{T}_1, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \Big\},\\
\mathbf{H}_{h_2} &:= \Big\{ \mathbf{v} \in H(\mathrm{div}\,;\Omega_2) : \quad \mathbf{v}|_T \in \mathrm{RT}_0(T) \quad \forall\, T \in \mathcal{T}_2 \Big\},\\
\mathbf{Q}_h &:= \Big\{ q \in L^2(\Omega) : \quad q|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_1 \cup \mathcal{T}_2 \Big\}, \quad \text{and}\\
\mathbf{Q}_{h,0} &:= \mathbf{Q}_h \cap L^2_0(\Omega).
\end{aligned}
$$

Next, we let $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be the partition of Γ_2 inherited from the triangulation $\mathcal{T}_2$ and introduce a second partition $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_m\}$ of Γ_2, also made of line segments, such that $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \dots, \tilde{\gamma}_m\}$ is a derefinement of $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. In other words, for each $j \in \{1, 2, \ldots, n\}$ there exists $i \in \{1, 2, \ldots, m\}$ such that $\gamma_j \subseteq \tilde{\gamma}_i$. Then, we set $\tilde{h} := \max\{|\tilde{\gamma}_j| : j \in \{1, \ldots, m\}\}$ and define the finite element subspace for the unknown $\lambda \in H^{1/2}(\Gamma_2)$ as

$$\mathbf{Q}_{\tilde{h}} := \Big\{ \xi \in C(\Gamma_2) : \quad \xi|_{\tilde{\gamma}_j} \in \mathbf{P}_1(\tilde{\gamma}_j) \quad \forall\, j \in \{1, \ldots, m\} \Big\}. \tag{2.11}$$

Then, denoting $\mathbf{H}_h := \mathbf{H}_{h_1} \times \mathbf{H}_{h_2}$ and $\mathbf{Q}_{h,\tilde{h}} := \mathbf{Q}_{h,0} \times \mathbf{Q}_{\tilde{h}}$, the conforming mixed finite element method introduced in [26] reads as follows: Find $(\mathbf{u}_h, (p_h, \lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$ such that

(2.12)
$$\begin{aligned} \mathbf{a}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{b}(\mathbf{v}_h, (p_h, \lambda_{\tilde{h}})) &= \mathbf{F}(\mathbf{v}_h) \qquad && \forall\, \mathbf{v}_h := (\mathbf{v}_{1,h}, \mathbf{v}_{2,h}) \in \mathbf{H}_h, \\ \mathbf{b}(\mathbf{u}_h, (q_h, \xi_{\tilde{h}})) &= \mathbf{G}(q_h, \xi_{\tilde{h}}) \qquad && \forall\, (q_h, \xi_{\tilde{h}}) \in \mathbf{Q}_{h,\tilde{h}}. \end{aligned}$$

The unique solvability, stability, and a priori error estimates for the Galerkin scheme (2.12) are established with full details in [26].

3. A residual-based a posteriori error estimator. We begin with some notation. Given $i \in \{1, 2\}$ and $T \in \mathcal{T}_i$, we let $\mathcal{E}(T)$ be the set of edges of T and denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_1 \cup \mathcal{T}_2$. Then we write

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_1) \cup \mathcal{E}_h(\Omega_1) \cup \mathcal{E}_h(\Gamma_2) \cup \mathcal{E}_h(\Omega_2),$$

where $\mathcal{E}_h(\Gamma_i) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_i\}$ and $\mathcal{E}_h(\Omega_i) := \{e \in \mathcal{E}_h : e \subseteq \Omega_i\}$, $i \in \{1, 2\}$. In what follows, h_e stands for the diameter of the edge $e \in \mathcal{E}_h$. Further, for each $e \in \mathcal{E}_h(\Gamma_2)$ we set $\tilde{h}_e := |\tilde{\gamma}_j|$, where $\tilde{\gamma}_j$ is the line segment containing edge e. Now, let $q \in L^2(\Omega_i)$ such that $q|_T \in C(T)$ for each $T \in \mathcal{T}_i$, and let $e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_i)$. We denote by $[q]$ the jump of q across e, that is, $[q] := (q|_T)|_e - (q|_{T'})|_e$, where T' is the other triangle of $\mathcal{T}_i$ having e as edge. Also, we fix a unit normal vector $\boldsymbol{\nu}_e := (\nu_1, \nu_2)^{\mathtt{t}}$ to the edge e, which points either inward T or inward T', and we let $\mathbf{s}_e := (-\nu_2, \nu_1)^{\mathtt{t}}$ be the corresponding fixed unit tangential vector along e. Then, given $\mathbf{v} \in [L^2(\Omega_i)]^2$ such that $\mathbf{v}|_T \in [C(T)]^2$ for each $T \in \mathcal{T}_i$, we let $[\mathbf{v} \cdot \mathbf{s}_e]$ be the tangential jump of $\mathbf{v}$ across e, that is, $[\mathbf{v} \cdot \mathbf{s}_e] := \{(\mathbf{v}|_T)|_e - (\mathbf{v}|_{T'})|_e\} \cdot \mathbf{s}_e$. In addition, for $\boldsymbol{\tau} \in [L^2(\Omega_i)]^{2\times 2}$ such that $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, we let $[\boldsymbol{\tau}\,\boldsymbol{\nu}_e]$ be the normal jump of $\boldsymbol{\tau}$ across e, that is, $[\boldsymbol{\tau}\,\boldsymbol{\nu}_e] := \{(\boldsymbol{\tau}|_T)|_e - (\boldsymbol{\tau}|_{T'})|_e\}\,\boldsymbol{\nu}_e$. From now on, when no confusion arises, we simply write $\mathbf{s}$ and $\boldsymbol{\nu}$ instead of $\mathbf{s}_e$ and $\boldsymbol{\nu}_e$, respectively. Finally, for sufficiently smooth scalar and vector fields q and $\mathbf{v} := (v_1, v_2)^{\mathtt{t}}$, respectively, we let $\mathbf{curl}\,(q) := (-\frac{\partial q}{\partial x_2}, \frac{\partial q}{\partial x_1})^{\mathtt{t}}$ and $\operatorname{curl}(\mathbf{v}) := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$. Next, given $(\mathbf{u}_h, (p_h, \lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$, the unique solution of (2.12), with $\mathbf{u}_h := (\mathbf{u}_{1,h}, \mathbf{u}_{2,h}) \in \mathbf{H}_{h_1} \times \mathbf{H}_{h_2}$, we define for each $T \in \mathcal{T}_1$ the a posteriori error indicator

(3.1)
$$\begin{aligned} \eta_{1,T}^2 := \|\operatorname{div} \mathbf{u}_{1,h}\|_{0,T}^2 + h_T^2\, \|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_1)} h_e\, \|[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\|_{0,e}^2 \\ + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_2)} \left\{ h_e \left\| \boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu} + \lambda_{\tilde{h}}\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,(\mathbf{u}_{1,h} \cdot \mathbf{s})\,\mathbf{s} \right\|_{0,e}^2 + \tilde{h}_e\, \|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{0,e}^2 \right\}, \end{aligned}$$

where

$$\boldsymbol{\sigma}_{1,h} := -\,p_h\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_{1,h}) \quad \text{on each} \quad T \in \mathcal{T}_1. \tag{3.2}$$

Similarly, for each $T \in \mathcal{T}_2$ we set

$$
\begin{aligned}
\eta_{2,T}^2 := & \|f_2 - \operatorname{div} \mathbf{u}_{2,h}\|_{0,T}^2 + h_T^2 \|\operatorname{curl}(\mathbf{K}^{-1}\mathbf{u}_{2,h})\|_{0,T}^2 \\
& + \sum_{e \in \mathcal{E}(T)\cap\mathcal{E}_h(\Omega_2)} h_e \|[\mathbf{K}^{-1}\mathbf{u}_{2,h}\cdot \mathbf{s}]\|_{0,e}^2 \\
& + \sum_{e \in \mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_2)} \left\{ h_e \left\| \mathbf{K}^{-1}\mathbf{u}_{2,h}\cdot\mathbf{s} + \frac{d\lambda_{\tilde{h}}}{d\mathbf{s}} \right\|_{0,e}^2 + \tilde{h}_e \|\mathbf{u}_{1,h}\cdot\boldsymbol{\nu} - \mathbf{u}_{2,h}\cdot\boldsymbol{\nu}\|_{0,e}^2 \right\} \\
& + h_T^2 \|\mathbf{K}^{-1}\mathbf{u}_{2,h}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_2)} h_e \|\lambda_{\tilde{h}} - p_h\|_{0,e}^2 .
\end{aligned}
\tag{3.3}
$$

Then, we introduce the global a posteriori error estimator

$$
\boldsymbol{\eta} := \left\{ \sum_{T\in\mathcal{T}_1} \eta_{1,T}^2 + \sum_{T\in\mathcal{T}_2} \eta_{2,T}^2 \right\}^{1/2} . \tag{3.4}
$$

The residual character of each term on the right-hand sides of (3.1) and (3.3) is quite clear. Also, we observe that some of the terms defining $\eta_{2,T}^2$ appear in the residual-based a posteriori error estimators for the mixed formulation of Darcy and related problems (see, e.g., [14], [25]). This fact will be utilized in section 4.

The following theorem is the main result of this section.

THEOREM 3.1. *Let $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ and $(\mathbf{u}_h,(p_h,\lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$ be the unique solutions of* (2.4) *and* (2.12), *respectively. Then, there exists $C_{\mathtt{rel}} > 0$, independent of h and $\tilde{h}$, such that*

$$
\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}} + \|(p-p_h,\lambda-\lambda_{\tilde{h}})\|_{\mathbf{Q}} \le C_{\mathtt{rel}}\,\boldsymbol{\eta}. \tag{3.5}
$$

The proof of Theorem 3.1 is separated into the four parts given by the next subsections of the present section. For this purpose, and among other issues to be discussed later, we need to consider that $H^{1/2}(\Gamma_2) = H_0^{1/2}(\Gamma_2)\oplus\mathbb{R}$, where $H_0^{1/2}(\Gamma_2) := \left\{ \xi \in H^{1/2}(\Gamma_2) : \langle 1,\xi\rangle_{\Gamma_2} = 0 \right\}$. In particular, $\lambda - \lambda_{\tilde{h}} \in H^{1/2}(\Gamma_2)$ is rewritten as $\lambda - \lambda_{\tilde{h}} = (\lambda-\lambda_{\tilde{h}})_0 + c$, with $(\lambda-\lambda_{\tilde{h}})_0 \in H_0^{1/2}(\Gamma_2)$ and $c := \frac{1}{|\Gamma_2|}\langle 1, \lambda-\lambda_{\tilde{h}}\rangle_{\Gamma_2} \in \mathbb{R}$. Hence, in order to conclude the reliability estimate (3.5), we found that the upper bounds of the components of the total error must be derived according to the following order: $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}$, $\|(\lambda-\lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$, $\|p-p_h\|_{0,\Omega}$, and $\|\lambda-\lambda_{\tilde{h}}\|_{1/2,\Gamma_2}$.

3.1. Upper bound for $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}$. We begin with the following preliminary estimate.

LEMMA 3.2. *There exists $C_1 > 0$, independent of h and $\tilde{h}$, such that*

$$
\begin{aligned}
C_1\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}} \le & \sup_{\substack{\mathbf{v}\in\mathbf{V}\\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{a}(\mathbf{u}-\mathbf{u}_h,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}} \\
& + \|\operatorname{div}\mathbf{u}_{1,h}\|_{0,\Omega_1} + \|\mathbf{f}_2 - \operatorname{div}\mathbf{u}_{2,h}\|_{0,\Omega_2} + \|\mathbf{u}_{1,h}\cdot\boldsymbol{\nu} - \mathbf{u}_{2,h}\cdot\boldsymbol{\nu}\|_{-1/2,\Gamma_2}.
\end{aligned}
\tag{3.6}
$$

Proof. We let $\bar{\mathbf{F}} \in \mathbf{H}'$ and $\bar{\mathbf{G}} \in \mathbf{Q}'$ be the functionals given by $\bar{\mathbf{F}}(\mathbf{v}) = \mathbf{0}\ \forall\,\mathbf{v}\in\mathbf{H}$ and $\bar{\mathbf{G}}(q,\xi) := \mathbf{b}(\mathbf{u}-\mathbf{u}_h,(q,\xi))\ \forall (q,\xi)\in\mathbf{Q}$, and we let $(\bar{\mathbf{u}},(\bar{p},\bar{\lambda})) \in \mathbf{H}\times\mathbf{Q}$ be the unique solution of (2.4) with $\bar{\mathbf{F}}$ and $\bar{\mathbf{G}}$ instead of $\mathbf{F}$ and $\mathbf{G}$. It is clear that

$$
\|\bar{\mathbf{u}}\|_{\mathbf{H}} \le \bar{C}\,\|\bar{\mathbf{G}}\|_{\mathbf{Q}'}. \tag{3.7}
$$

Now, according to the second equation of (2.4) and the definitions of $\mathbf{G}$ (cf. (2.7)) and $\mathbf{b}$ (cf. (2.6)), we find that

$$\begin{aligned}\bar{\mathbf{G}}(q,\xi) &:= \mathbf{G}(q,\xi) - \mathbf{b}(u_h,(q,\xi))\\ &= \int_{\Omega_1} q \operatorname{div} \mathbf{u}_{1,h} - \int_{\Omega_2} q\,(f_2 - \operatorname{div} \mathbf{u}_{2,h}) - \langle \mathbf{u}_{1,h}\cdot\boldsymbol{\nu} - \mathbf{u}_{2,h}\cdot\boldsymbol{\nu}, \xi\rangle_{\Gamma_2},\end{aligned}$$

which, together with (3.7), implies that

$$\|\bar{\mathbf{u}}\|_{\mathbf{H}} \le C\left\{\|\operatorname{div} \mathbf{u}_{1,h}\|_{0,\Omega_1} + \|\mathbf{f}_2 - \operatorname{div} \mathbf{u}_{2,h}\|_{0,\Omega_2} + \|\mathbf{u}_{1,h}\cdot\boldsymbol{\nu} - \mathbf{u}_{2,h}\cdot\boldsymbol{\nu}\|_{-1/2,\Gamma_2}\right\}. \tag{3.8}$$

On the other hand, using the second equation of (2.4) and the definition of $\bar{\mathbf{G}}$, we see that $(\mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}})$ belongs to $\mathbf{V}$, the kernel of $\mathbf{b}$, and, hence, the coerciveness estimate provided in [26, Lemma 2.2] yields

$$\begin{aligned}\alpha\,\|\mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}}\|_{\mathbf{H}}^2 &\le \mathbf{a}(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}}) - \mathbf{a}(\bar{\mathbf{u}}, \mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}})\\ &\le \mathbf{a}(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}}) + \|\mathbf{a}\|\,\|\bar{\mathbf{u}}\|_{\mathbf{H}}\,\|\mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}}\|_{\mathbf{H}}.\end{aligned}$$

Then, dividing the above inequality by $\|\mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}}\|_{\mathbf{H}}$ and taking the supremum on $\mathbf{V}$, we deduce that

$$\alpha\,\|\mathbf{u}-\mathbf{u}_h-\bar{\mathbf{u}}\|_{\mathbf{H}} \le \sup_{\substack{\mathbf{v}\in\mathbf{V}\\ \mathbf{v}\neq\mathbf{0}}} \frac{\mathbf{a}(\mathbf{u}-\mathbf{u}_h,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}} + \|\mathbf{a}\|\,\|\bar{\mathbf{u}}\|_{\mathbf{H}},$$

which, combined with the triangle inequality and (3.8), leads to (3.6). □

We now aim to estimate the supremum in (3.6). To this end, we first observe from the first equation of (2.4) that $\mathbf{a}(\mathbf{u},\mathbf{v}) = \int_{\Omega_1} \mathbf{f}_1\cdot\mathbf{v}_1\ \forall\,\mathbf{v} := (\mathbf{v}_1,\mathbf{v}_2)\in\mathbf{V}$. Similarly, from the first equation of (2.12) we obtain $\mathbf{a}(\mathbf{u}_h,\mathbf{v}_h) = \int_{\Omega_1}\mathbf{f}_1\cdot\mathbf{v}_{1,h} - \mathbf{b}(\mathbf{v}_h,(p_h,\lambda_{\tilde{h}}))$ $\forall\,\mathbf{v}_h := (\mathbf{v}_{1,h},\mathbf{v}_{2,h})\in\mathbf{H}_h$. It follows that

$$\mathbf{a}(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) = \int_{\Omega_1}\mathbf{f}_1\cdot(\mathbf{v}_1-\mathbf{v}_{1,h}) - \mathbf{a}(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h) - \mathbf{b}(\mathbf{v}-\mathbf{v}_h,(p_h,\lambda_{\tilde{h}})).$$

Equivalently, denoting $\widehat{\mathbf{v}}_1 := \mathbf{v}_1 - \mathbf{v}_{1,h}$ and $\widehat{\mathbf{v}}_2 := \mathbf{v}_2 - \mathbf{v}_{2,h}$, we can write

$$\mathbf{a}(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) = E_1(\widehat{\mathbf{v}}_1) + E_2(\widehat{\mathbf{v}}_2) \qquad \forall\,\mathbf{v}\in\mathbf{V}, \forall\,\mathbf{v}_h := (\mathbf{v}_{1,h},\mathbf{v}_{2,h})\in\mathbf{H}_h, \tag{3.9}$$

where

$$E_1(\widehat{\mathbf{v}}_1) := \int_{\Omega_1}\mathbf{f}_1\cdot\widehat{\mathbf{v}}_1 - \mathbf{a}(\mathbf{u}_h,(\widehat{\mathbf{v}}_1,\mathbf{0})) - \mathbf{b}((\widehat{\mathbf{v}}_1,\mathbf{0}),(p_h,\lambda_{\tilde{h}})) \tag{3.10}$$

and

$$E_2(\widehat{\mathbf{v}}_2) := -\,\mathbf{a}(\mathbf{u}_h,(\mathbf{0},\widehat{\mathbf{v}}_2)) - \mathbf{b}((\mathbf{0},\widehat{\mathbf{v}}_2),(p_h,\lambda_{\tilde{h}})). \tag{3.11}$$

In what follows we bound $E_1(\widehat{\mathbf{v}}_1)$ and $E_2(\widehat{\mathbf{v}}_2)$ for suitable choices of $\mathbf{v}_{1,h}\in\mathbf{H}_{h_1}$ and $\mathbf{v}_{2,h}\in\mathbf{H}_{h_2}$, respectively. To this end, we will apply the Clément interpolation operators $I_{1,h}: H^1(\Omega_1)\to X_{1,h}$ and $I_{2,h}: H^1(\Omega_2)\to X_{2,h}$ (cf. [18]), where

$$X_{i,h} := \left\{v\in C(\bar{\Omega}_i): \quad v|_T\in\mathbf{P}_1(T)\quad \forall\,T\in\mathcal{T}_i\right\} \qquad \forall\,i\in\{1,2\}.$$

Also, we will make use of a vectorial version of $I_{1,h}$, say, $\mathbf{I}_{1,h} : [H^1(\Omega_1)]^2 \to [X_{1,h}]^2$, which is defined componentwise by $I_{1,h}$. In addition, it is important to remark from [18] that $I_{1,h}$ can be defined so that $I_{1,h}(v) \in X_{1,h} \cap H^1_{\Gamma_1}(\Omega_1)$ for all $v \in H^1_{\Gamma_1}(\Omega_1)$.

The following lemma establishes the local approximation properties of $I_{i,h}$.

LEMMA 3.3. *For each* $i \in \{1,2\}$ *there exist constants* $c_1, c_2 > 0$*, independent of* h_i*, such that for all* $v \in H^1(\Omega_i)$ *there hold*

$$\|v - I_{i,h}(v)\|_{0,T} \le c_1\, h_T\, \|v\|_{1,\Delta(T)} \quad \forall\, T \in \mathcal{T}_i, \quad \textit{and}$$

$$\|v - I_{i,h}(v)\|_{0,e} \le c_2\, h_e^{1/2}\, \|v\|_{1,\Delta(e)} \quad \forall\, e \in \mathcal{E}_h,$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_i : T' \cap T \neq \emptyset\}$ *and* $\Delta(e) := \cup\{T' \in \mathcal{T}_i : T' \cap e \neq \emptyset\}$.

Proof. See [18]. □

The upper bounds for $E_1(\widehat{\mathbf{v}}_1)$ and $E_2(\widehat{\mathbf{v}}_2)$ are provided in the next two lemmas. In particular, the proof of Lemma 3.5, which gives the bound for $E_2(\widehat{\mathbf{v}}_2)$, follows some of the arguments employed for the a posteriori error analysis, via mixed finite element methods, of the Poisson problem (see, e.g., [14], [25]).

LEMMA 3.4. *For each* $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}$*, define* $\widehat{\mathbf{v}}_1 := \mathbf{v}_1 - \mathbf{v}_{1,h}$*, with* $\mathbf{v}_{1,h} := \mathbf{I}_{1,h}(\mathbf{v}_1)$*. Then, there exists* $C > 0$*, independent of* h *and* $\tilde{h}$*, such that*

$$|E_1(\widehat{\mathbf{v}}_1)| \le C \left\{ \sum_{T \in \mathcal{T}_1} \widehat{\eta}_{1,T}^2 \right\}^{1/2} \|\mathbf{v}_1\|_{1,\Omega_1} \qquad \forall\, \mathbf{v} \in \mathbf{V}, \tag{3.12}$$

where

$$\begin{aligned} \widehat{\eta}_{1,T}^2 := {} & h_T^2\, \|\mathbf{f}_1 + \mathbf{div}\, \boldsymbol{\sigma}_{1,h}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_1)} h_e\, \|[\boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu}]\|_{0,e}^2 \\ & + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_2)} h_e \left\| \boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu} + \lambda_{\tilde{h}}\, \boldsymbol{\nu} + \frac{\mu}{\kappa}\, (\mathbf{u}_{1,h} \cdot \mathbf{s})\, \mathbf{s} \right\|_{0,e}^2, \end{aligned} \tag{3.13}$$

and $\boldsymbol{\sigma}_{1,h}$ *is given by* (3.2).

Proof. Let $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}$ and define $\widehat{\mathbf{v}}_1 := \mathbf{v}_1 - \mathbf{v}_{1,h}$, with $\mathbf{v}_{1,h} := \mathbf{I}_{1,h}(\mathbf{v}_1)$. The definition of the local Bernardi–Raugel space $\mathrm{BR}(T)$ (cf. [11]) implies that $[X_{1,h} \cap H^1_{\Gamma_1}(\Omega_1)]^2 \subseteq \mathbf{H}_{h_1}$, and hence $\mathbf{v}_{1,h} := \mathbf{I}_{1,h}(\mathbf{v}_1) \in \mathbf{H}_{h_1}$. Thus, applying the approximation properties from Lemma 3.3, we deduce that

$$\|\widehat{\mathbf{v}}_1\|_{0,T} \le c_1\, h_T\, \|\mathbf{v}_1\|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_1, \tag{3.14}$$

and

$$\|\widehat{\mathbf{v}}_1\|_{0,e} \le c_2\, h_e^{1/2}\, \|\mathbf{v}_1\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h(\Omega_1) \cup \mathcal{E}_h(\Gamma_2). \tag{3.15}$$

Now, according to the definitions of $\mathbf{a}$ (cf. (2.5)) and $\mathbf{b}$ (cf. (2.6)), it follows from (3.10) that

$$\begin{aligned} E_1(\widehat{\mathbf{v}}_1) = {} & \int_{\Omega_1} \mathbf{f}_1 \cdot \widehat{\mathbf{v}}_1 - 2\,\mu \int_{\Omega_1} \mathbf{e}(\mathbf{u}_{1,h}) : \mathbf{e}(\widehat{\mathbf{v}}_1) - \frac{\mu}{\kappa} \int_{\Gamma_2} (\mathbf{u}_{1,h} \cdot \mathbf{s})\, (\widehat{\mathbf{v}}_1 \cdot \mathbf{s}) \\ & + \int_{\Omega_1} p_h\, \mathrm{div}\, \widehat{\mathbf{v}}_1 - \langle \widehat{\mathbf{v}}_1 \cdot \boldsymbol{\nu}, \lambda_{\tilde{h}} \rangle_{\Gamma_2}. \end{aligned}$$

Then, noting that $p_h \operatorname{div} \widehat{\mathbf{v}}_1 = p_h \mathbf{I} : \nabla \widehat{\mathbf{v}}_1$, using the symmetry of $\mathbf{e}(\cdot)$, and recalling the definition of $\boldsymbol{\sigma}_{1,h}$ (cf. (3.2)), we find that

$$\int_{\Omega_1} p_h \operatorname{div} \widehat{\mathbf{v}}_1 - 2\mu \int_{\Omega_1} \mathbf{e}(\mathbf{u}_{1,h}) : \mathbf{e}(\widehat{\mathbf{v}}_1) = - \int_{\Omega_1} \boldsymbol{\sigma}_{1,h} : \nabla \widehat{\mathbf{v}}_1,$$

which yields

$$E_1(\widehat{\mathbf{v}}_1) = \int_{\Omega_1} \mathbf{f}_1 \cdot \widehat{\mathbf{v}}_1 - \int_{\Omega_1} \boldsymbol{\sigma}_{1,h} : \nabla \widehat{\mathbf{v}}_1 - \frac{\mu}{\kappa} \int_{\Gamma_2} (\mathbf{u}_{1,h} \cdot \mathbf{s})\, (\widehat{\mathbf{v}}_1 \cdot \mathbf{s}) - \langle \widehat{\mathbf{v}}_1 \cdot \boldsymbol{\nu}, \lambda_{\tilde{h}} \rangle_{\Gamma_2}. \tag{3.16}$$

In this way, integrating by parts on each $T \in \mathcal{T}_1$ and observing that $\widehat{\mathbf{v}}_1$ vanishes on Γ_1, we deduce that

$$\begin{aligned} &\int_{\Omega_1} \mathbf{f}_1 \cdot \widehat{\mathbf{v}}_1 - \int_{\Omega_1} \boldsymbol{\sigma}_{1,h} : \nabla \widehat{\mathbf{v}}_1 \\ &= \sum_{T \in \mathcal{T}_1} \int_T \left\{ \mathbf{f}_1 + \mathbf{div}\, \boldsymbol{\sigma}_{1,h} \right\} \cdot \widehat{\mathbf{v}}_1 - \sum_{e \in \mathcal{E}_h(\Omega_1)} \int_e \widehat{\mathbf{v}}_1 \cdot [\boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu}] - \sum_{e \in \mathcal{E}_h(\Gamma_2)} \int_e \widehat{\mathbf{v}}_1 \cdot \boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu}, \end{aligned}$$

which, replaced back into (3.16), gives

$$\begin{aligned} E_1(\widehat{\mathbf{v}}_1) = &\sum_{T \in \mathcal{T}_1} \int_T \left\{ \mathbf{f}_1 + \mathbf{div}\, \boldsymbol{\sigma}_{1,h} \right\} \cdot \widehat{\mathbf{v}}_1 - \sum_{e \in \mathcal{E}_h(\Omega_1)} \int_e \widehat{\mathbf{v}}_1 \cdot [\boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu}] \\ &- \sum_{e \in \mathcal{E}_h(\Gamma_2)} \int_e \left\{ \boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu} + \lambda_{\tilde{h}}\, \boldsymbol{\nu} + \frac{\mu}{\kappa} (\mathbf{u}_{1,h} \cdot \mathbf{s})\, \mathbf{s} \right\} \cdot \widehat{\mathbf{v}}_1. \end{aligned}$$

Finally, applying the Cauchy–Schwarz inequality, the estimates (3.14) and (3.15), the fact that the numbers of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, and some algebraic manipulations, we conclude from the above equation the estimate (3.12). □

LEMMA 3.5. *For each* $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}$, *define* $\widehat{\mathbf{v}}_2 := \mathbf{v}_2 - \mathbf{v}_{2,h}$, *with* $\mathbf{v}_{2,h} := \mathbf{curl}\,(I_{2,h}(\varphi))$, *where* $\varphi \in H^1(\Omega_2)$ *is such that* $\mathbf{v}_2 = \mathbf{curl}\,(\varphi)$. *Then, there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$|E_2(\widehat{\mathbf{v}}_2)| \le C \left\{ \sum_{T \in \mathcal{T}_2} \widehat{\eta}_{2,T}^2 \right\}^{1/2} \|\mathbf{v}_2\|_{H(\operatorname{div};\Omega_2)} \qquad \forall\, \mathbf{v} \in \mathbf{V}, \tag{3.17}$$

where

$$\begin{aligned} \widehat{\eta}_{2,T}^2 := &\; h_T^2\, \|\operatorname{curl}(\mathbf{K}^{-1}\, \mathbf{u}_{2,h})\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_2)} h_e\, \|[\mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \mathbf{s}]\|_{0,e}^2 \\ &+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_2)} h_e \left\| \mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \mathbf{s} + \frac{d\lambda_{\tilde{h}}}{d\mathbf{s}} \right\|_{0,e}^2. \end{aligned} \tag{3.18}$$

Proof. Let $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}$. The definition of $\mathbf{V}$ (cf. (2.8)) ensures that $\operatorname{div} \mathbf{v}_2 = 0$ in Ω_2, and hence, since Ω_2 is connected, there exists $\varphi \in H^1(\Omega_2)$ with $\int_{\Omega_2} \varphi = 0$, such that $\mathbf{v}_2 = \mathbf{curl}\,(\varphi)$. Then we let $\varphi_h := I_{2,h}(\varphi)$ and define $\mathbf{v}_{2,h} := \mathbf{curl}\,(\varphi_h)$. It follows that $\mathbf{v}_{2,h} \in \mathbf{H}_{h_2}$, $\operatorname{div} \mathbf{v}_{2,h} = 0$ in Ω_2, and $\widehat{\mathbf{v}}_2 := \mathbf{v}_2 - \mathbf{v}_{2,h} = \mathbf{curl}\, \widehat{\varphi}$, with $\widehat{\varphi} := \varphi - \varphi_h$. Then, applying again the approximation properties from Lemma 3.3, we obtain that

$$\|\widehat{\varphi}\|_{0,T} \le c_1\, h_T\, \|\varphi\|_{1,\Delta(T)} \qquad \forall\, T \in \mathcal{T}_2, \tag{3.19}$$

and

$$\|\widehat{\varphi}\|_{0,e} \le c_2\, h_e^{1/2}\, \|\varphi\|_{1,\Delta(e)} \qquad \forall\, e \in \mathcal{E}_h(\Omega_2) \cup \mathcal{E}_h(\Gamma_2). \tag{3.20}$$

Now, according to the definitions of $\mathbf{a}$ (cf. (2.5)) and $\mathbf{b}$ (cf. (2.6)), and using that $\operatorname{div} \widehat{\mathbf{v}}_2 = 0$ in Ω_2, we find from (3.11) that

$$E_2(\widehat{\mathbf{v}}_2) = -\int_{\Omega_2} \mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \widehat{\mathbf{v}}_2 + \langle \widehat{\mathbf{v}}_2 \cdot \boldsymbol{\nu}, \lambda_{\tilde{h}} \rangle_{\Gamma_2}. \tag{3.21}$$

Then, replacing $\widehat{\mathbf{v}}_2$ by $\mathbf{curl}\, \widehat{\varphi}$, noting that $\mathbf{curl}\, \widehat{\varphi} \cdot \boldsymbol{\nu} = -\frac{d\widehat{\varphi}}{d\mathbf{s}}$, and integrating by parts on each $T \in \mathcal{T}_2$ and on Γ_2, we deduce that

$$\begin{aligned} E_2(\widehat{\mathbf{v}}_2) &= \sum_{T \in \mathcal{T}_2} \int_T \operatorname{curl}(\mathbf{K}^{-1}\, \mathbf{u}_{2,h})\, \widehat{\varphi} + \sum_{e \in \mathcal{E}_h(\Omega_2)} \int_e [\mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \mathbf{s}]\, \widehat{\varphi} \\ &\quad + \sum_{e \in \mathcal{E}_h(\Gamma_2)} \int_e \left\{ \mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \mathbf{s} + \frac{d\lambda_{\tilde{h}}}{d\mathbf{s}} \right\} \widehat{\varphi}. \end{aligned} \tag{3.22}$$

In this way, applying the Cauchy–Schwarz inequality, the estimates (3.19) and (3.20), the fact that the numbers of triangles in $\Delta(T)$ and $\Delta(e)$ are bounded, and the inequality $\|\varphi\|_{1,\Omega_2} \le c\, |\varphi|_{1,\Omega_2} = c\, \|\mathbf{curl}\,(\varphi)\|_{0,\Omega_2} = c\, \|\mathbf{v}_2\|_{0,\Omega_2} = c\, \|\mathbf{v}_2\|_{H(\operatorname{div};\Omega_2)}$, we conclude from (3.22) the required estimate (3.17). $\square$

The bound for the supremum in (3.6) (cf. Lemma 3.2) can be established now.

LEMMA 3.6. *There exists $C > 0$, independent of h and $\tilde{h}$, such that*

$$\sup_{\substack{\mathbf{v} \in \mathbf{V} \\ \mathbf{v} \neq \mathbf{0}}} \frac{\mathbf{a}(\mathbf{u} - \mathbf{u}_h, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}} \le C \left\{ \sum_{T \in \mathcal{T}_1} \widehat{\eta}_{1,T}^2 + \sum_{T \in \mathcal{T}_2} \widehat{\eta}_{2,T}^2 \right\}^{1/2},$$

where $\widehat{\eta}_{1,T}$ and $\widehat{\eta}_{2,T}$ are defined as in Lemmas 3.4 and 3.5, respectively.

Proof. The proof follows straightforwardly from identity (3.9) and the estimates (3.12) (cf. Lemma 3.4) and (3.17) (cf. Lemma 3.5). $\square$

In order to complete the upper bound for $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}}$ in terms of local quantities, it remains to estimate the boundary term $\|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Gamma_2}$. For this purpose, we apply a technical result given in [15]. In fact, taking $q_h = 0$ in the second equation of the discrete scheme (2.12), we find that $(\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu})$ is $L^2(\Gamma_2)$-orthogonal to the continuous piecewise linear functions on the partition $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_m\}$ of Γ_2. Hence, applying [15, Theorem 2] and recalling that $\{\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_m\}$ is a derefinement of $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, we deduce that

$$\|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Gamma_2}^2 \le C(\tilde{h}) \sum_{e \in \mathcal{E}_h(\Gamma_2)} \tilde{h}_e\, \|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{0,e}^2,$$

where

$$C(\tilde{h}) := C_0 \left\{ \log(1 + K) \right\}^{1/2}, \quad K := \max\{K^-, K^+\},$$

$$K^- := \max\left\{ \frac{|\tilde{\gamma}_j|}{|\tilde{\gamma}_{j-1}|} : \quad j = \overline{2, m} \right\}, \quad K^+ := \max\left\{ \frac{|\tilde{\gamma}_j|}{|\tilde{\gamma}_{j+1}|} : \quad j = \overline{1, m-1} \right\},$$

and $C_0 > 0$ is independent of h and $\tilde{h}$. Then, thanks to the local quasi-uniformity of $\mathcal{T}_2$, which follows from its shape-regularity, we deduce that $C(\tilde{h})$ is bounded above by a constant $\tilde{C} > 0$, independent of h and $\tilde{h}$, whence

$$\|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{-1/2,\Gamma_2}^2 \leq \tilde{C} \sum_{e \in \mathcal{E}_h(\Gamma_2)} \tilde{h}_e \, \|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{0,e}^2. \tag{3.23}$$

We are now in a position to provide the full a posteriori estimate for $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}}$.

THEOREM 3.7. *There exists $C > 0$, independent of h and $\tilde{h}$, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} \leq C \left\{ \sum_{T \in \mathcal{T}_1} \overline{\eta}_{1,T}^2 + \sum_{T \in \mathcal{T}_2} \overline{\eta}_{2,T}^2 \right\}^{1/2},$$

where $\overline{\eta}_{1,T}^2 = \eta_{1,T}^2$ (cf. (3.1)), that is,

$$\overline{\eta}_{1,T}^2 := \|\operatorname{div} \mathbf{u}_{1,h}\|_{0,T}^2 + h_T^2 \, \|\mathbf{f}_1 + \mathbf{div}\, \boldsymbol{\sigma}_{1,h}\|_{0,T}^2 + \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_1)} h_e \, \|[\boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu}]\|_{0,e}^2$$

$$+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_2)} \left\{ h_e \, \left\| \boldsymbol{\sigma}_{1,h}\, \boldsymbol{\nu} + \lambda_{\tilde{h}}\, \boldsymbol{\nu} + \frac{\mu}{\kappa}\, (\mathbf{u}_{1,h} \cdot \mathbf{s})\, \mathbf{s} \right\|_{0,e}^2 + \tilde{h}_e \, \|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{0,e}^2 \right\},$$

and

$$\overline{\eta}_{2,T}^2 := \|f_2 - \operatorname{div} \mathbf{u}_{2,h}\|_{0,T}^2 + h_T^2 \, \|\operatorname{curl}(\mathbf{K}^{-1}\, \mathbf{u}_{2,h})\|_{0,T}^2$$

$$+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_2)} h_e \, \|[\mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \mathbf{s}]\|_{0,e}^2$$

$$+ \sum_{e \in \mathcal{E}(T) \cap \mathcal{E}_h(\Gamma_2)} \left\{ h_e \, \left\| \mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \mathbf{s} + \frac{d\lambda_{\tilde{h}}}{d\mathbf{s}} \right\|_{0,e}^2 + \tilde{h}_e \, \|\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}\|_{0,e}^2 \right\}.$$

Proof. The proof is a straightforward consequence of Lemmas 3.2 and 3.6 and the estimate (3.23). □

3.2. Upper bound for $\|(\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\tilde{h}})_0\|_{1/2,\Gamma_2}$. The derivation of the upper bound for $\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$ requires the partial continuous inf-sup condition for $\mathbf{b}$ given by the following lemma.

LEMMA 3.8. *There exists $\beta_1 > 0$ such that*

$$\sup_{\substack{\mathbf{v}_2 \in H(\operatorname{div};\Omega_2) \\ \mathbf{v}_2 \neq \mathbf{0} \\ \operatorname{div} \mathbf{v}_2 = 0}} \frac{\mathbf{b}((\mathbf{0}, \mathbf{v}_2), (q, \xi))}{\|\mathbf{v}_2\|_{H(\operatorname{div};\Omega_2)}} \geq \beta_1 \, \|\xi\|_{1/2,\Gamma_2} \quad \forall\, (q, \xi) \in L_0^2(\Omega) \times H_0^{1/2}(\Gamma_2). \tag{3.24}$$

Proof. The proof follows by using some of the arguments from [26, Lemma 2.1]. We omit further details here. □

We remark here that the novelty of (3.24) lies in the fact that it does not hold on the whole space $\mathbf{Q} := L_0^2(\Omega) \times H^{1/2}(\Gamma_2)$ but only on the subspace $L_0^2(\Omega) \times H_0^{1/2}(\Gamma_2)$. To this respect, we recall from the proof of [26, Lemma 2.1] that, in the case of $\mathbf{Q}$, it was shown that there exist β_1, $\beta_2 > 0$ such that

$$\sup_{\substack{\mathbf{v}_2 \in H(\operatorname{div};\Omega_2) \\ \mathbf{v}_2 \neq \mathbf{0}}} \frac{\mathbf{b}((\mathbf{0}, \mathbf{v}_2), (q, \xi))}{\|\mathbf{v}_2\|_{H(\operatorname{div};\Omega_2)}} \geq \beta_1 \, \|\xi\|_{1/2,\Gamma_2} - \beta_2 \, \|q\|_{0,\Omega} \quad \forall (q, \xi) \in \mathbf{Q}. \tag{3.25}$$

However, because of the second term on the right-hand side of (3.25), this inequality cannot be employed to estimate either $\|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2}$ or $\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$ unless one first bounds $\|p - p_h\|_{0,\Omega}$. This fact motivated the derivation of (3.24), which is utilized below to bound $\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$. Then, another partial continuous inf-sup condition for $\mathbf{b}$ will be employed in section 3.3 to estimate $\|p - p_h\|_{0,\Omega}$, and finally, a slight modification of (3.25) will yield the upper bound for $\|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2}$ in section 3.4.

The identity given by the following lemma, which will also be needed to bound the term $\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$, follows easily from the first equations of the continuous and discrete formulations (2.4) and (2.12).

LEMMA 3.9. *Let* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\mathbf{u}_h, (p_h, \lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$ *be the unique solutions of* (2.4) *and* (2.12), *respectively. Then, for all* $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{H}$ *and for all* $\mathbf{v}_h := (\mathbf{v}_{1,h}, \mathbf{v}_{2,h}) \in \mathbf{H}_h$ *there holds*

(3.26)
$$\mathbf{b}(\mathbf{v}, (p - p_h, \lambda - \lambda_{\tilde{h}})) = -\mathbf{a}(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + \int_{\Omega_1} \mathbf{f}_1 \cdot \widehat{\mathbf{v}}_1 - \mathbf{a}(\mathbf{u}_h, \widehat{\mathbf{v}}) - \mathbf{b}(\widehat{\mathbf{v}}, (p_h, \lambda_{\tilde{h}})),$$

where $\widehat{\mathbf{v}} = (\widehat{\mathbf{v}}_1, \widehat{\mathbf{v}}_2) := (\mathbf{v}_1 - \mathbf{v}_{1,h}, \mathbf{v}_2 - \mathbf{v}_{2,h})$.

We now proceed to bound $\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$. A direct application of (3.24) gives

$$\beta_1 \, \|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2} \le \sup_{\substack{\mathbf{v}_2 \in H(\mathrm{div}\,;\Omega_2) \\ \mathbf{v}_2 \neq \mathbf{0} \\ \mathrm{div}\, \mathbf{v}_2 = 0}} \frac{\mathbf{b}((\mathbf{0}, \mathbf{v}_2), (p - p_h, (\lambda - \lambda_{\tilde{h}})_0))}{\|\mathbf{v}_2\|_{H(\mathrm{div}\,;\Omega_2)}}. \tag{3.27}$$

Then, for each $\mathbf{v}_2 \in H(\mathrm{div}\,;\Omega_2)$, $\mathbf{v}_2 \neq \mathbf{0}$, such that $\mathrm{div}\, \mathbf{v}_2 = 0$, we obviously have $\langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} = 0$, which yields $\langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, (\lambda - \lambda_{\tilde{h}})_0 \rangle_{\Gamma_2} = \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, \lambda - \lambda_{\tilde{h}} \rangle_{\Gamma_2}$. It follows that $\mathbf{b}((\mathbf{0}, \mathbf{v}_2), (p - p_h, (\lambda - \lambda_{\tilde{h}})_0)) = \mathbf{b}((\mathbf{0}, \mathbf{v}_2), (p - p_h, \lambda - \lambda_{\tilde{h}}))$, and hence, (3.27) becomes

$$\beta_1 \, \|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2} \le \sup_{\substack{\mathbf{v}_2 \in H(\mathrm{div}\,;\Omega_2) \\ \mathbf{v}_2 \neq \mathbf{0} \\ \mathrm{div}\, \mathbf{v}_2 = 0}} \frac{\mathbf{b}((\mathbf{0}, \mathbf{v}_2), (p - p_h, \lambda - \lambda_{\tilde{h}}))}{\|\mathbf{v}_2\|_{H(\mathrm{div}\,;\Omega_2)}}. \tag{3.28}$$

We now employ the identity provided by Lemma 3.9. In fact, taking $\mathbf{v} := (\mathbf{0}, \mathbf{v}_2) \in \mathbf{H}$ with $\mathbf{v}_2 \neq \mathbf{0}$ and $\mathrm{div}\, \mathbf{v}_2 = 0$ in Ω_2, and $\mathbf{v}_h := (\mathbf{0}, \mathbf{v}_{2,h}) \in \mathbf{H}_h$, we obtain from (3.26) that

(3.29)
$$\mathbf{b}((\mathbf{0}, \mathbf{v}_2), (p - p_h, \lambda - \lambda_{\tilde{h}})) = -\mathbf{a}(\mathbf{u} - \mathbf{u}_h, (\mathbf{0}, \mathbf{v}_2)) + E_2(\widehat{\mathbf{v}}_2) \qquad \forall\, \mathbf{v}_{2,h} \in \mathbf{H}_{h_2},$$

where, according to (3.11) and the definitions of $\mathbf{a}$ (cf. (2.5)) and $\mathbf{b}$ (cf. (2.6)),

$$E_2(\widehat{\mathbf{v}}_2) := -\mathbf{a}(\mathbf{u}_h, (\mathbf{0}, \widehat{\mathbf{v}}_2)) - \mathbf{b}((\mathbf{0}, \widehat{\mathbf{v}}_2), (p_h, \lambda_{\tilde{h}})) = -\int_{\Omega_2} \mathbf{K}^{-1}\, \mathbf{u}_{2,h} \cdot \widehat{\mathbf{v}}_2 + \langle \widehat{\mathbf{v}}_2 \cdot \boldsymbol{\nu}, \lambda_{\tilde{h}} \rangle_{\Gamma_2}.$$

Next, we exactly follow the proof of Lemma 3.5 to show that $\mathbf{v}_{2,h} \in \mathbf{H}_{h_2}$ can be chosen so that

$$|E_2(\widehat{\mathbf{v}}_2)| \le C \left\{ \sum_{T \in \mathcal{T}_2} \widehat{\eta}_{2,T}^2 \right\}^{1/2} \|\mathbf{v}_2\|_{H(\mathrm{div}\,;\Omega_2)} \qquad \forall\, \mathbf{v} \in \mathbf{V}, \tag{3.30}$$

where $\widehat{\eta}_{2,T}$ is defined by (3.18).

In this way, replacing (3.29) into (3.28) and applying the boundedness of $\mathbf{a}$ and estimate (3.30), we conclude that

$$\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2} \leq C \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}}^2 + \sum_{T \in \mathcal{T}_2} \widehat{\eta}_{2,T}^2 \right\}^{1/2}. \tag{3.31}$$

More precisely, we can establish the following theorem.

THEOREM 3.10. *There exists $C > 0$, independent of h and $\tilde{h}$, such that*

$$\|(\lambda - \lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2} \leq C \left\{ \sum_{T \in \mathcal{T}_1} \overline{\eta}_{1,T}^2 + \sum_{T \in \mathcal{T}_2} \overline{\eta}_{2,T}^2 \right\}^{1/2},$$

where $\overline{\eta}_{1,T}^2$ and $\overline{\eta}_{2,T}^2$ are as defined in Theorem 3.7.

Proof. The proof is a direct consequence of estimate (3.31), Theorem 3.7, and the fact that the terms defining $\widehat{\eta}_{2,T}^2$ are part of $\overline{\eta}_{2,T}^2$. □

3.3. Upper bound for $\|p - p_h\|_{0,\Omega}$. The derivation of the upper bound for $\|p - p_h\|_{0,\Omega}$ requires the partial continuous inf-sup condition for $\mathbf{b}$ provided by the following lemma.

LEMMA 3.11. *There exists $\beta_2 > 0$ such that*

$$\sup_{\substack{\mathbf{v} \in \mathbf{H},\, \mathbf{v} \neq \mathbf{0} \\ \mathbf{v}_1 \cdot \boldsymbol{\nu} = \mathbf{v}_2 \cdot \boldsymbol{\nu} \text{ on } \Gamma_2}} \frac{\mathbf{b}(\mathbf{v}, (q, \xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta_2 \|q\|_{0,\Omega} \qquad \forall\, (q, \xi) \in L_0^2(\Omega) \times H^{1/2}(\Gamma_2). \tag{3.32}$$

Proof. The proof is given in the second part of the proof of [26, Lemma 2.1] (see [26, (2.4)]). We omit details here. □

Then, we apply (3.32) with $(q, \xi) = (p - p_h, \lambda - \lambda_{\tilde{h}})$ and obtain

$$\beta_2 \|p - p_h\|_{0,\Omega} \leq \sup_{\substack{\mathbf{v} \in \mathbf{H},\, \mathbf{v} \neq \mathbf{0} \\ \mathbf{v}_1 \cdot \boldsymbol{\nu} = \mathbf{v}_2 \cdot \boldsymbol{\nu} \text{ on } \Gamma_2}} \frac{\mathbf{b}(\mathbf{v}, (p - p_h, \lambda - \lambda_{\tilde{h}}))}{\|\mathbf{v}\|_{\mathbf{H}}}. \tag{3.33}$$

Now, given $\mathbf{v} \in \mathbf{H}$, $\mathbf{v} \neq \mathbf{0}$, with $\mathbf{v}_1 \cdot \boldsymbol{\nu} = \mathbf{v}_2 \cdot \boldsymbol{\nu}$ on Γ_2, we let $\mathbf{w}_2 \in [H^1(\Omega_2)]^2$ such that $\operatorname{div} \mathbf{w}_2 = \operatorname{div} \mathbf{v}_2$ in Ω_2 and

$$\|\mathbf{w}_2\|_{1,\Omega_2} \leq C \|\operatorname{div} \mathbf{v}_2\|_{0,\Omega_2}. \tag{3.34}$$

It follows that

$$\begin{aligned} \mathbf{b}(\mathbf{v}, (p - p_h, \lambda - \lambda_{\tilde{h}})) &= - \int_{\Omega_1} (p - p_h) \operatorname{div} \mathbf{v}_1 - \int_{\Omega_2} (p - p_h) \operatorname{div} \mathbf{w}_2 \\ &= \mathbf{b}((\mathbf{v}_1, \mathbf{w}_2), (p - p_h, \lambda - \lambda_{\tilde{h}})) - \langle \mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{w}_2 \cdot \boldsymbol{\nu}, \lambda - \lambda_{\tilde{h}} \rangle_{\Gamma_2}. \end{aligned} \tag{3.35}$$

In addition, we also deduce that

$$\langle \mathbf{w}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} = \int_{\Omega_2} \operatorname{div} \mathbf{w}_2 = \int_{\Omega_2} \operatorname{div} \mathbf{v}_2 = \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} = \langle \mathbf{v}_1 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2},$$

which shows that $\mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{w}_2 \cdot \boldsymbol{\nu}$ belongs to $H_0^{-1/2}(\Gamma_2)$, and hence

$$\langle \mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{w}_2 \cdot \boldsymbol{\nu}, \lambda - \lambda_{\tilde{h}} \rangle_{\Gamma_2} = \langle \mathbf{v}_1 \cdot \boldsymbol{\nu} - \mathbf{w}_2 \cdot \boldsymbol{\nu}, (\lambda - \lambda_{\tilde{h}})_0 \rangle_{\Gamma_2}. \tag{3.36}$$

On the other hand, thanks to the identity (3.26) (cf. Lemma 3.9), we have that

$$\begin{aligned}\mathbf{b}((\mathbf{v}_1,\mathbf{w}_2),(p-p_h,\lambda-\lambda_{\tilde{h}})) &= -\,\mathbf{a}(\mathbf{u}-\mathbf{u}_h,(\mathbf{v}_1,\mathbf{w}_2)) + \int_{\Omega_1}\mathbf{f}_1\cdot\widehat{\mathbf{v}}_1 \\ &\quad -\mathbf{a}(\mathbf{u}_h,(\widehat{\mathbf{v}}_1,\widehat{\mathbf{w}}_2)) - \mathbf{b}((\widehat{\mathbf{v}}_1,\widehat{\mathbf{w}}_2),(p_h,\lambda_{\tilde{h}})) \qquad \forall\,(\mathbf{v}_{1,h},\mathbf{w}_{2,h})\in\mathbf{H}_h,\end{aligned} \tag{3.37}$$

where $\widehat{\mathbf{v}}_1 := \mathbf{v}_1-\mathbf{v}_{1,h}$ and $\widehat{\mathbf{w}}_2 := \mathbf{w}_2-\mathbf{w}_{2,h}$.

Hence, replacing (3.36) and (3.37) into (3.35) and employing the definitions (3.10) and (3.11), we can write

$$\begin{aligned}\mathbf{b}(\mathbf{v},(p-p_h,\lambda-\lambda_{\tilde{h}})) &= -\;\mathbf{a}(\mathbf{u}-\mathbf{u}_h,(\mathbf{v}_1,\mathbf{w}_2)) \\ &\quad +\langle\mathbf{v}_1\cdot\boldsymbol{\nu}-\mathbf{w}_2\cdot\boldsymbol{\nu},(\lambda-\lambda_{\tilde{h}})_0\rangle_{\Gamma_2} + E_1(\widehat{\mathbf{v}}_1)+E_2(\widehat{\mathbf{w}}_2),\end{aligned} \tag{3.38}$$

where

$$E_1(\widehat{\mathbf{v}}_1) = \int_{\Omega_1}\mathbf{f}_1\cdot\widehat{\mathbf{v}}_1 - \mathbf{a}(\mathbf{u}_h,(\widehat{\mathbf{v}}_1,\mathbf{0})) - \mathbf{b}((\widehat{\mathbf{v}}_1,\mathbf{0}),(p_h,\lambda_{\tilde{h}})) \quad \forall\,\mathbf{v}_{1,h}\in\mathbf{H}_{h_1}, \tag{3.39}$$

and

$$E_2(\widehat{\mathbf{w}}_2) = -\mathbf{a}(\mathbf{u}_h,(\mathbf{0},\widehat{\mathbf{w}}_2)) - \mathbf{b}((\mathbf{0},\widehat{\mathbf{w}}_2),(p_h,\lambda_{\tilde{h}})) \qquad \forall\,\mathbf{w}_{2,h}\in\mathbf{H}_{h_2}. \tag{3.40}$$

In what follows we estimate the terms on the right-hand side of (3.38). Using (3.34) we easily find that

$$\begin{aligned}&\left|\,\mathbf{a}(\mathbf{u}-\mathbf{u}_h,(\mathbf{v}_1,\mathbf{w}_2)) + \langle\mathbf{v}_1\cdot\boldsymbol{\nu}-\mathbf{w}_2\cdot\boldsymbol{\nu},(\lambda-\lambda_{\tilde{h}})_0\rangle_{\Gamma_2}\,\right| \\ &\qquad \le C\left\{\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}} + \|(\lambda-\lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}\right\}\|\mathbf{v}\|_{\mathbf{H}}.\end{aligned} \tag{3.41}$$

Note that the upper bounds for $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{H}}$ and $\|(\lambda-\lambda_{\tilde{h}})_0\|_{1/2,\Gamma_2}$ are already available in Theorems 3.7 and 3.10, respectively.

Now, proceeding exactly as in the proof of Lemma 3.4, that is, choosing $\mathbf{v}_{1,h} := \mathbf{I}_{1,h}(\mathbf{v}_1)$, we deduce the existence of $C>0$, independent of h and $\tilde{h}$, such that

$$|E_1(\widehat{\mathbf{v}}_1)| \le C\left\{\sum_{T\in\mathcal{T}_1}\widehat{\theta}_{1,T}^2\right\}^{1/2}\|\mathbf{v}_1\|_{1,\Omega_1} \qquad \forall\,\mathbf{v}\in\mathbf{H}, \tag{3.42}$$

where $\widehat{\theta}_{1,T}^2 = \widehat{\eta}_{1,T}^2$ (cf. (3.13)).

In order to bound $|\,E_2(\widehat{\mathbf{w}}_2)\,|$ we need to introduce the usual Raviart–Thomas interpolation operator $\Pi_h : [H^1(\Omega_2)]^2 \to \mathbf{H}_{h_2}$ (see [13], [36]), which, given $\boldsymbol{\tau}\in[H^1(\Omega_2)]^2$, is characterized by the identities

$$\operatorname{div}(\Pi_h(\boldsymbol{\tau})) = \mathcal{P}_h(\operatorname{div}(\boldsymbol{\tau})) \quad\text{and}\quad \int_e \Pi_h(\boldsymbol{\tau})\cdot\boldsymbol{\nu} = \int_e \boldsymbol{\tau}\cdot\boldsymbol{\nu} \quad \forall\,\text{edge } e \text{ of } \mathcal{T}_2, \tag{3.43}$$

where $\mathcal{P}_h$ is the $L^2(\Omega_2)$-orthogonal projector onto the piecewise constant functions on Ω_2. It is well known (see, e.g.,[13], [36], and [24, Theorem 4.5]) that Π_h satisfies the approximation properties

$$\|\boldsymbol{\tau}-\Pi_h(\boldsymbol{\tau})\|_{0,T} \le c\,h_T\,\|\boldsymbol{\tau}\|_{1,T} \qquad \forall\,T\in\mathcal{T}_2, \quad \forall\,\boldsymbol{\tau}\in[H^1(\Omega_2)]^2, \quad\text{and} \tag{3.44}$$

$$\|\boldsymbol{\tau}\cdot\boldsymbol{\nu}-\Pi_h(\boldsymbol{\tau})\cdot\boldsymbol{\nu}\|_{0,e} \le c\,h_e^{1/2}\,\|\boldsymbol{\tau}\|_{1,T_e} \quad \forall\,\text{edge } e \text{ of } \mathcal{T}_2, \quad \forall\,\boldsymbol{\tau}\in[H^1(\Omega_2)]^2, \tag{3.45}$$

where T_e is a triangle of $\mathcal{T}_2$ containing e on its boundary.

LEMMA 3.12. *Let* $\widehat{\mathbf{w}}_2 := \mathbf{w}_2 - \mathbf{w}_{2,h}$ *with* $\mathbf{w}_{2,h} = \Pi_h(\mathbf{w}_2) \in \mathbf{H}_{h_2}$. *Then, there exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$|E_2(\widehat{\mathbf{w}}_2)| \leq C \left\{ \sum_{T \in \mathcal{T}_2} \widehat{\theta}^2_{2,T} \right\}^{1/2} \|\mathbf{v}_2\|_{H(\mathrm{div}\,;\Omega_2)}, \tag{3.46}$$

where

$$\widehat{\theta}^2_{2,T} := h_T^2 \|\mathbf{K}^{-1}\mathbf{u}_{2,h}\|^2_{0,T} + \sum_{e \in \mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_2)} h_e \left\|\lambda_{\tilde{h}} - p_h\right\|^2_{0,e}. \tag{3.47}$$

Proof. Applying the approximation properties (3.44) and (3.45), we find that for each $T \in \mathcal{T}_2$ and for each $e \in \mathcal{E}_h(\Omega_2) \cup \mathcal{E}_h(\Gamma_2)$ there hold

$$\|\widehat{\mathbf{w}}_2\|_{0,T} \leq c_1\, h_T\, \|\mathbf{w}_2\|_{1,T} \quad \text{and} \quad \|\widehat{\mathbf{w}}_2 \cdot \boldsymbol{\nu}\|_{0,e} \leq c_2\, h_e^{1/2}\, \|\mathbf{w}_2\|_{1,T_e}. \tag{3.48}$$

Now, according to the definitions of $\mathbf{a}$ (cf. (2.5)) and $\mathbf{b}$ (cf. (2.6)), and using that $\widehat{\mathbf{w}}_2 \cdot \boldsymbol{\nu} \in L^2(\Gamma_2)$, which follows from the fact that $\mathbf{w}_2 \in [H^1(\Omega_2)]^2$ and $\mathbf{w}_{2,h} \cdot \boldsymbol{\nu}$ is piecewise constant on Γ_2, we obtain from (3.40) and (3.43), noting that p_h is also piecewise constant on Γ_2, that

$$E_2(\widehat{\mathbf{w}}_2) = -\int_{\Omega_2} \mathbf{K}^{-1}\,\mathbf{u}_{2,h} \cdot \widehat{\mathbf{w}}_2 + \sum_{e \in \mathcal{E}_h(\Gamma_2)} \int_e \left(\lambda_{\tilde{h}} - p_h\right) \widehat{\mathbf{w}}_2 \cdot \boldsymbol{\nu}. \tag{3.49}$$

In this way, the Cauchy–Schwarz inequality and the estimates (3.48) imply that

$$|\, E_2(\widehat{\mathbf{w}}_2)\,| \leq \sum_{T \in \mathcal{T}_2} c_1 h_T \|\mathbf{K}^{-1}\,\mathbf{u}_{2,h}\|_{0,T}\, \|\mathbf{w}_2\|_{1,T} + \sum_{e \in \mathcal{E}_h(\Gamma_2)} c_2 h_e^{1/2}\, \|\lambda_{\tilde{h}} - p_h\|_{0,e}\, \|\mathbf{w}_2\|_{1,T_e},$$

which, together with (3.34), yields (3.46)–(3.47). ∎

Consequently, replacing (3.38) into (3.33) and employing the estimates (3.41), (3.42), and (3.46), we conclude that

$$\|p - p_h\|^2_{0,\Omega} \leq C \left\{ \|\mathbf{u} - \mathbf{u}_h\|^2_{\mathbf{H}} + \|(\lambda - \lambda_{\tilde{h}})_0\|^2_{1/2,\Gamma_2} + \sum_{T \in \mathcal{T}_1} \widehat{\theta}^2_{1,T} + \sum_{T \in \mathcal{T}_2} \widehat{\theta}^2_{2,T} \right\}. \tag{3.50}$$

In other words, we have the following result.

THEOREM 3.13. *There exists* $C > 0$, *independent of* h *and* $\tilde{h}$, *such that*

$$\|p - p_h\|_{0,\Omega} \leq C \left\{ \sum_{T \in \mathcal{T}_1} \eta^2_{1,T} + \sum_{T \in \mathcal{T}_2} \eta^2_{2,T} \right\}^{1/2},$$

where $\eta^2_{1,T}$ *and* $\eta^2_{2,T}$ *are defined by* (3.1) *and* (3.3), *respectively.*

Proof. The proof follows directly from (3.50), the definitions of $\widehat{\theta}^2_{1,T}$ (cf. (3.13)) and $\widehat{\theta}^2_{2,T}$ (cf. (3.47)), and the estimates provided by Theorems 3.7 and 3.10. ∎

3.4. Upper bound for $\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_{\tilde{h}}\|_{1/2,\Gamma_2}$. In order to derive the upper bound of $\|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2}$, we require the partial continuous inf-sup condition for $\mathbf{b}$ established by the following lemma.

LEMMA 3.14. *There exist β_3, $\beta_4 > 0$ such that for each $(q,\xi) \in L^2_0(\Omega) \times H^{1/2}(\Gamma_2)$*

(3.51)
$$\sup_{\substack{\mathbf{v}_2 \in H(\mathrm{div};\Omega_2),\, \mathbf{v}_2 \neq \mathbf{0} \\ \mathrm{div}\,\mathbf{v}_2 \in \mathbb{R}}} \frac{\mathbf{b}((\mathbf{0},\mathbf{v}_2),(q,\xi))}{\|\mathbf{v}_2\|_{H(\mathrm{div};\Omega_2)}} \geq \beta_3 \|\xi\|_{1/2,\Gamma_2} - \beta_4 \|q\|_{0,\Omega}.$$

Proof. Essentially, this inequality is derived in the first part of the proof of [26, Lemma 2.1] (see [26, (2.3)]). We omit the details here. □

We now apply (3.51) to $(q,\xi) = (p - p_h, \lambda - \lambda_{\tilde{h}})$ and obtain

(3.52)
$$\beta_3 \|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2} \leq \beta_4 \|p - p_h\|_{0,\Omega} + \sup_{\substack{\mathbf{v}_2 \in H(\mathrm{div};\Omega_2)\setminus\{\mathbf{0}\} \\ \mathrm{div}\,\mathbf{v}_2 \in \mathbb{R}}} \frac{\mathbf{b}((\mathbf{0},\mathbf{v}_2),(p-p_h,\lambda-\lambda_{\tilde{h}}))}{\|\mathbf{v}_2\|_{H(\mathrm{div};\Omega_2)}}.$$

Note that the upper bound for $\|p - p_h\|_{0,\Omega}$ is already available in Theorem 3.13. Hence, in order to bound the supremum on the right-hand side of (3.52), we consider $\mathbf{v}_2 \in H(\mathrm{div};\Omega_2)$, $\mathbf{v}_2 \neq \mathbf{0}$, such that $\mathrm{div}\,\mathbf{v}_2 \in \mathbb{R}$, and use the identity (3.26) (cf. Lemma 3.9) to obtain

$$\mathbf{b}((\mathbf{0},\mathbf{v}_2),(p-p_h,\lambda-\lambda_{\tilde{h}})) = -\mathbf{a}(\mathbf{u}-\mathbf{u}_h,(\mathbf{0},\mathbf{v}_2)) + E_2(\widehat{\mathbf{v}}_2) \quad \forall\, \mathbf{v}_{2,h} \in \mathbf{H}_{h_2}, \tag{3.53}$$

where $\widehat{\mathbf{v}}_2 = \mathbf{v}_2 - \mathbf{v}_{2,h}$ and

$$E_2(\widehat{\mathbf{v}}_2) = -\int_{\Omega_2} \mathbf{K}^{-1}\,\mathbf{u}_{2,h} \cdot \widehat{\mathbf{v}}_2 + \int_{\Omega_2} p_h \,\mathrm{div}\,\widehat{\mathbf{v}}_{2,h} + \langle \widehat{\mathbf{v}}_2 \cdot \boldsymbol{\nu}, \lambda_{\tilde{h}} \rangle_{\Gamma_2}.$$

Since $\mathrm{div}\,\mathbf{v}_2$ is constant in Ω_2 and $\boldsymbol{\nu}$ points inside Ω_2, we find that

$$\mathrm{div}\,\mathbf{v}_2 = -\frac{1}{|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2} \qquad \text{in} \quad \Omega_2. \tag{3.54}$$

Then we define $\mathbf{w}_2 := \mathbf{v}_2 + \mathbf{z}_2$, with $\mathbf{z}_2(\mathbf{x}) := \frac{1}{2|\Omega_2|} \langle \mathbf{v}_2 \cdot \boldsymbol{\nu}, 1 \rangle_{\Gamma_2}\, \mathbf{x}\ \forall\, \mathbf{x} \in \Omega_2$, and observe that $\mathrm{div}\,\mathbf{w}_2 = 0$ in Ω_2. Thus, there exists $\varphi \in H^1(\Omega_2)$ such that $\int_{\Omega_2} \varphi = 0$ and $\mathbf{w}_2 = \mathbf{curl}\,(\varphi)$. In addition, it is easy to see that there holds

$$\|\varphi\|_{1,\Omega_2} \leq C\, |\varphi|_{1,\Omega_2} = \|\mathbf{w}_2\|_{0,\Omega_2} = C\, \|\mathbf{w}_2\|_{H(\mathrm{div};\Omega_2)} \leq C\, \|\mathbf{v}_2\|_{H(\mathrm{div};\Omega_2)}. \tag{3.55}$$

Next, we let $\varphi_h := I_{2,h}(\varphi)$, introduce $\mathbf{w}_{2,h} := \mathbf{curl}\,(\varphi_h)$, observe that $\mathbf{z}_2$ belongs to the finite element subspace $\mathbf{H}_{h_2}$, and define $\mathbf{v}_{2,h} := \mathbf{w}_{2,h} - \mathbf{z}_2 \in \mathbf{H}_{h_2}$. It follows that $\widehat{\mathbf{v}}_2 := \mathbf{v}_2 - \mathbf{v}_{2,h} = \mathbf{w}_2 - \mathbf{w}_{2,h} =: \widehat{\mathbf{w}}_2 = \mathbf{curl}\,(\varphi - \varphi_h)$, and hence

$$E_2(\widehat{\mathbf{v}}_2) = E_2(\widehat{\mathbf{w}}_2) = -\int_{\Omega_2} \mathbf{K}^{-1}\,\mathbf{u}_{2,h} \cdot \widehat{\mathbf{w}}_2 + \langle \widehat{\mathbf{w}}_2 \cdot \boldsymbol{\nu}, \lambda_{\tilde{h}} \rangle_{\Gamma_2}.$$

Therefore, proceeding exactly as in the proof of Lemma 3.5, and using the estimate (3.55), we can show, with $\widehat{\eta}_{2,T}$ defined by (3.18), that

$$|E_2(\widehat{\mathbf{v}}_2)| = |E_2(\widehat{\mathbf{w}}_2)| \leq C \left\{ \sum_{T \in \mathcal{T}_2} \widehat{\eta}^{\,2}_{2,T} \right\}^{1/2} \|\mathbf{v}_2\|_{H(\mathrm{div}\,;\Omega_2)}. \tag{3.56}$$

In this way, replacing (3.53) into (3.52), and employing the boundedness of $\mathbf{a}$, the estimate (3.56), the definition of $\widehat{\eta}^{\,2}_{2,T}$ (cf. (3.18)), and the bounds provided by Theorems 3.13 and 3.7, we can establish the following result.

THEOREM 3.15. *There exists $C > 0$, independent of h and $\tilde{h}$, such that*

$$\|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2} \le C \left\{ \sum_{T\in\mathcal{T}_1} \eta_{1,T}^2 + \sum_{T\in\mathcal{T}_2} \eta_{2,T}^2 \right\}^{1/2},$$

where $\eta_{1,T}^2$ and $\eta_{2,T}^2$ are defined by (3.1) *and* (3.3)*, respectively.*

Finally, the proof of Theorem 3.1 follows directly from Theorems 3.7, 3.13, and 3.15.

4. Efficiency of the a posteriori error estimator. In what follows we let $\mathcal{P}_{0,T}$ be the $[L^2(T)]^2$-orthogonal projection onto $[\mathbf{P}_0(T)]^2$ with respect to the inner product $\langle \mathbf{f}, \mathbf{g}\rangle := \int_T \psi_T\, \mathbf{f}\cdot\mathbf{g}\ \forall\, \mathbf{f},\, \mathbf{g} \in [L^2(T)]^2$, where ψ_T is the usual triangle-bubble function (see the paragraph right before Lemma 4.3 for further details on ψ_T).

The following theorem is the main result of this section.

THEOREM 4.1. *Let $(\mathbf{u}, (p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ and $(\mathbf{u}_h, (p_h, \lambda_{\tilde{h}})) \in \mathbf{H}_h \times \mathbf{Q}_{h,\tilde{h}}$ be the unique solutions of* (2.4) *and* (2.12)*, respectively, and assume that $\mathbf{f}_1$ is sufficiently smooth. Then, there exists $C_{\mathtt{eff}} > 0$, independent of h and $\tilde{h}$, such that*

$$C_{\mathtt{eff}}\ \boldsymbol{\eta} \le \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}} + \|(p - p_h, \lambda - \lambda_{\tilde{h}})\|_{\mathbf{Q}} + \text{h.o.t.}, \tag{4.1}$$

where h.o.t. *denotes the higher order terms given by $\sum_{T\in\mathcal{T}_1} h_T^2\, \|\mathbf{f}_1 - \mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2$.*

In order to prove (4.1) we need the following preliminary result, which basically follows by applying integration by parts backwardly in the formulation (2.4).

THEOREM 4.2. *Let $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$ be the unique solution of* (2.4)*. Then* $\operatorname{div}\mathbf{u}_1 = 0$ *in* Ω_1, $\operatorname{div}\mathbf{u}_2 = f_2$ *in* Ω_2*, and* $\mathbf{u}_2\cdot\boldsymbol{\nu} = \mathbf{u}_1\cdot\boldsymbol{\nu}$ *on* Γ_2*. In addition, defining* $p_1 := p|_{\Omega_1}$ $p_2 := p|_{\Omega_2}$*, and* $\boldsymbol{\sigma}_1 := 2\,\mu\,\mathbf{e}(\mathbf{u}_1) - p_1\,\mathbf{I}$*, there hold* $p_2 \in H^{1+\delta}(\Omega_2)$ *for some* $\delta > 0$, $\lambda = p_2$ *on* Γ_2, $\mathbf{div}\,\boldsymbol{\sigma}_1 = -\,\mathbf{f}_1$ *in* Ω_1 *(which yields* $\boldsymbol{\sigma}_1 \in H(\mathbf{div};\Omega_1)$*), and* $\boldsymbol{\sigma}_1\boldsymbol{\nu} + \lambda\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,(\mathbf{u}_1\cdot\mathbf{s})\,\mathbf{s} = 0$ *on* Γ_2.

Since $\operatorname{div}\mathbf{u}_1 = 0$ in Ω_1 and $\operatorname{div}\mathbf{u}_2 = f_2$ in Ω_2, we easily find that

$$\|\operatorname{div}\mathbf{u}_{1,h}\|_{0,T} = \|\operatorname{div}\mathbf{u}_1 - \operatorname{div}\mathbf{u}_{1,h}\|_{0,T} \le C\,|\mathbf{u}_1 - \mathbf{u}_{1,h}|_{1,T} \qquad \forall\, T\in\mathcal{T}_1 \tag{4.2}$$

and

$$\|f_2 - \operatorname{div}\mathbf{u}_{2,h}\|_{0,T} = \|\operatorname{div}\mathbf{u}_2 - \operatorname{div}\mathbf{u}_{2,h}\|_{0,T} \le C\,\|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{H(\operatorname{div};T)} \quad \forall\, T\in\mathcal{T}_2. \tag{4.3}$$

In order to derive the upper bounds for the remaining terms defining the global a posteriori error estimator $\boldsymbol{\eta}$ (cf. (3.4)), we proceed similarly as in [14] and [16] (see also [23]), and apply Helmholtz decomposition, inverse inequalities, and the localization technique based on triangle-bubble and edge-bubble functions. To this end, we now recall some notation and introduce further preliminary results. Given $\mathcal{T} \in \{\mathcal{T}_1, \mathcal{T}_2\}$, $T\in\mathcal{T}$, and $e\in\mathcal{E}(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions, respectively (see (1.5) and (1.6) in [39]). In particular, ψ_T satisfies $\psi_T \in \mathbf{P}_3(T)$, $\operatorname{supp}(\psi_T) \subseteq T$, $\psi_T = 0$ on ∂T, and $0 \le \psi_T \le 1$ in T. Similarly, $\psi_e|_T \in \mathbf{P}_2(T)$, $\operatorname{supp}(\psi_e) \subseteq w_e := \cup\{T'\in\mathcal{T} : e\in\mathcal{E}(T')\}$, $\psi_e = 0$ on $\partial T\backslash e$, and $0\le\psi_e\le 1$ in w_e. We also recall from [38] that, given $k\in\mathbb{N}\cup\{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p)\in\mathbf{P}_k(T)$ and $L(p)|_e = p$ $\forall p\in\mathbf{P}_k(e)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of ψ_T, ψ_e, and L are collected in the following lemma.

LEMMA 4.3. *Given $k\in\mathbb{N}\cup\{0\}$, there exist positive constants c_1, c_2, and c_3, depending only on k and the shape-regularity of the triangulations (minimum angle* *condition), such that for each triangle T and $e \in \mathcal{E}(T)$ there hold*

$$\|q\|_{0,T}^2 \le c_1\,\|\psi_T^{1/2}\,q\|_{0,T}^2 \qquad \forall\, q \in \mathbf{P}_k(T), \tag{4.4}$$

$$\|p\|_{0,e}^2 \le c_2\,\|\psi_e^{1/2}\,p\|_{0,e}^2 \qquad \forall\, p \in \mathbf{P}_k(e), \quad \textit{and} \tag{4.5}$$

$$\|\psi_e^{1/2}\,L(p)\|_{0,T}^2 \le c_3\,h_e\,\|p\|_{0,e}^2 \qquad \forall\, p \in \mathbf{P}_k(e). \tag{4.6}$$

Proof. See [38, Lemma 1.3]. □

The inverse estimate for polynomials (cf. [17, Theorem 3.2.6]) will also be used.

The following four lemmas provide the corresponding upper bounds for the remaining terms defining $\eta_{1,T}^2$ (cf. (3.1)).

LEMMA 4.4. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $T \in \mathcal{T}_1$ there holds*

$$h_T^2\|\mathbf{f}_1+\mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}^2 \le c\left\{\|p-p_h\|_{0,T}^2+|\mathbf{u}_1-\mathbf{u}_{1,h}|_{1,T}^2+h_T^2\|\mathbf{f}_1-\mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2\right\}. \tag{4.7}$$

Proof. Given $T \in \mathcal{T}_1$, we first observe, according to the definitions of the subspaces $\mathbf{H}_{h_1}$ and $\mathbf{Q}_{h,0}$, that $\mathbf{div}\,\boldsymbol{\sigma}_{1,h}$ is constant on T. Hence, using the triangle inequality, we obtain that

$$h_T^2\|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}^2 \le 2h_T^2\left\{\|\mathbf{f}_1 - \mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2 + \|\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T}^2\right\}. \tag{4.8}$$

Now, applying the estimate (4.4), the orthogonality condition satisfied by $\mathcal{P}_{0,T}$, and the fact that $\mathbf{div}\,\boldsymbol{\sigma}_1 = -\,\mathbf{f}_1$ in Ω_1, we find that

$$\begin{aligned}\|\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T}^2 &\le c_1\,\|\psi_T^{1/2}\,\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T}^2\\ &= -\,c_1\int_T \nabla\big(\psi_T\,\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\big) : (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1),\end{aligned}$$

where integration by parts was employed in the last equality. Then, the Cauchy–Schwarz inequality, the inverse estimate (cf. [17, Theorem 3.2.6]), the fact that $0 \le \psi_T \le 1$, the definitions of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_{1,h}$, and the triangle inequality imply that

$$\begin{aligned}\|\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T}^2 &\le c_1\,|\psi_T\,\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})|_{1,T}\;\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_{1,h}\|_{0,T}\\ &\le C\,h_T^{-1}\,\|\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T}\,\big\{\,\|p - p_h\|_{0,T} + |\mathbf{u}_1 - \mathbf{u}_{1,h}|_{1,T}\,\big\},\end{aligned}$$

and, hence, dividing by $\|\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T}$, we arrive at

$$\|\mathcal{P}_{0,T}(\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h})\|_{0,T} \le C\,h_T^{-1}\,\big\{\,\|p - p_h\|_{0,T} + |\mathbf{u}_1 - \mathbf{u}_{1,h}|_{1,T}\,\big\}. \tag{4.9}$$

In this way, (4.8) and (4.9) yield (4.7). □

LEMMA 4.5. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $e \in \mathcal{E}_h(\Omega_1)$ there holds*

$$h_e\|[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\|_{0,e}^2 \le c\sum_{T\subseteq\omega_e}\left\{\|p - p_h\|_{0,T}^2 + |\mathbf{u}_1 - \mathbf{u}_{1,h}|_{1,T}^2 + h_T^2\|\mathbf{f}_1 - \mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2\right\}, \tag{4.10}$$

where ω_e is the union of the two triangles in $\mathcal{T}_1$ having e as an edge.

Proof. Since $\boldsymbol{\sigma}_1 \in H(\mathbf{div}\,;\Omega_1)$ (cf. Theorem 4.2), it is clear that $[\boldsymbol{\sigma}_1\boldsymbol{\nu}] = \mathbf{0}$ on each $e \in \mathcal{E}_h(\Omega_1)$. Then, given a particular $e \in \mathcal{E}_h(\Omega_1)$, we apply (4.5) and the integration by parts formula on each $T \subseteq \omega_e$, to obtain

$$\|\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e}^2 \le c_2\,\|\,\psi_e^{1/2}\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e}^2 = c_2 \int_e \psi_e\,\mathbf{L}([\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}])\cdot[(\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_1)\boldsymbol{\nu}]$$

$$= c_2 \sum_{T\subseteq\omega_e}\left\{\int_T \mathbf{div}\,(\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_1)\cdot\psi_e\,\mathbf{L}([\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]) + \int_T (\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_1) : \nabla(\psi_e\,\mathbf{L}([\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]))\right\}.$$

Next, employing the Cauchy–Schwarz inequality, the inverse estimate (cf. [17, Theorem 3.2.6]), and the fact that $\mathbf{div}\,\boldsymbol{\sigma}_1 = -\,\mathbf{f}_1$ in Ω_1 (cf. Theorem 4.2), we get

$$\|\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e}^2$$
$$\le C \sum_{T\subseteq\omega_e}\left\{\|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T} + h_T^{-1}\,\|\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_1\|_{0,T}\right\}\|\psi_e\,\mathbf{L}([\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}])\|_{0,T}.$$

Using that $0 \le \psi_e \le 1$ and applying the estimate (4.6), we see that

$$\|\psi_e\,\mathbf{L}([\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}])\|_{0,T} \le \|\psi_e^{1/2}\,\mathbf{L}([\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}])\|_{0,T} \le c_3^{1/2}\,h_e^{1/2}\,\|\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e},$$

which, replacing it back into the above inequality and dividing by $\|\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e}$, gives

$$\|\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e} \le C\,h_e^{1/2} \sum_{T\subseteq\omega_e}\left\{\|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T} + h_T^{-1}\,\|\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_1\|_{0,T}\right\}.$$

Hence, noting that $h_e \le h_T$, we deduce that

$$h_e\,\|\,[\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu}]\,\|_{0,e}^2 \le C \sum_{T\subseteq\omega_e}\left\{h_T^2\,\|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}^2 + \|\boldsymbol{\sigma}_{1,h}-\boldsymbol{\sigma}_1\|_{0,T}^2\right\}. \tag{4.11}$$

Finally, (4.11), the efficiency estimate (4.7) (cf. Lemma 4.4), and the definitions of $\boldsymbol{\sigma}_{1,h}$ and $\boldsymbol{\sigma}_1$ imply (4.10) and complete the proof. □

Before establishing the following lemma, we need to recall a discrete trace inequality. Indeed, there exists $c > 0$, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_1 \cup \mathcal{T}_2$ and $e \in \mathcal{E}(T)$, there holds

$$\|v\|_{0,e}^2 \le c\left\{h_e^{-1}\,\|v\|_{0,T}^2 + h_e\,|v|_{1,T}^2\right\} \qquad \forall\, v \in H^1(T). \tag{4.12}$$

For a proof of inequality (4.12) we refer the reader to [1, Theorem 3.10] (see also [5, (2.4)] or [40, Lemma 3.2]).

LEMMA 4.6. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $e \in \mathcal{E}_h(\Gamma_2)$ there holds*

(4.13)
$$h_e\,\left\|\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu} + \lambda_{\tilde{h}}\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,(\mathbf{u}_{1,h}\cdot\mathbf{s})\,\mathbf{s}\right\|_{0,e}^2$$
$$\le c\left\{\|p - p_h\|_{0,T}^2 + \|\mathbf{u}_1 - \mathbf{u}_{1,h}\|_{1,T}^2 + h_e\,\|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2 + h_T^2\,\|\mathbf{f}_1 - \mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2\right\},$$

where T is the triangle of $\mathcal{T}_1$ having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Gamma_2)$ we let $\chi_e := \boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu} + \lambda_{\tilde{h}}\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,(\mathbf{u}_{1,h}\cdot\mathbf{s})\,\mathbf{s}$ on e. Then, applying (4.5), using that $\boldsymbol{\sigma}_1\,\boldsymbol{\nu} + \lambda\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,(\mathbf{u}_1\cdot\mathbf{s})\,\mathbf{s} = \mathbf{0}$ on Γ_2 (cf. Theorem 4.2), recalling that $\psi_e = 0$ on $\partial T\backslash e$ (T being the triangle of $\mathcal{T}_1$ having e as an edge), and integrating by parts on T, we obtain that

$$\begin{aligned}\|\chi_e\|_{0,e}^2 &\le c_2\,\|\psi_e^{1/2}\,\chi_e\|_{0,e}^2 = c_2\int_e \psi_e\,\chi_e\cdot\Big\{\boldsymbol{\sigma}_{1,h}\,\boldsymbol{\nu} + \lambda_{\tilde{h}}\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,(\mathbf{u}_{1,h}\cdot\mathbf{s})\,\mathbf{s}\Big\}\\ &= c_2\int_T \nabla(\psi_e\,\mathbf{L}(\chi_e)) : (\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1) + c_2\int_T \psi_e\,\mathbf{L}(\chi_e)\cdot\mathbf{div}\,(\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1)\\ &\quad + c_2\int_e \psi_e\,\chi_e\cdot\Big\{(\lambda_{\tilde{h}} - \lambda)\,\boldsymbol{\nu} + \frac{\mu}{\kappa}\,((\mathbf{u}_{1,h} - \mathbf{u}_1)\cdot\mathbf{s})\,\mathbf{s}\Big\}.\end{aligned}$$

Hence, the Cauchy–Schwarz inequality, the inverse estimate (cf. [17, Theorem 3.2.6]), the inequality $0 \le \psi_e \le 1$, and the fact that $\mathbf{div}\,\boldsymbol{\sigma}_1 = -\,\mathbf{f}_1$ in Ω_1 (cf. Theorem 4.2) imply from the above equation that

$$\begin{aligned}\|\chi_e\|_{0,e}^2 &\le C\Big\{h_T^{-1}\,\|\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1\|_{0,T} + \|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}\Big\}\,\|\psi_e\,\mathbf{L}(\chi_e)\|_{0,T}\\ &\quad + C\Big\{\|\lambda_{\tilde{h}} - \lambda\|_{0,e} + \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,e}\Big\}\,\|\chi_e\|_{0,e}.\end{aligned}$$

Now, applying the estimate (4.6), we see that $\|\psi_e\,\mathbf{L}(\chi_e)\|_{0,T} \le c_3^{1/2}\,h_e^{1/2}\,\|\,\chi_e\,\|_{0,e}$, which, employing it in the above inequality and dividing by $\|\,\chi_e\,\|_{0,e}$, yields

$$\begin{aligned}\|\chi_e\|_{0,e} &\le C\,h_e^{1/2}\Big\{h_T^{-1}\,\|\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1\|_{0,T} + \|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}\Big\}\\ &\quad + C\Big\{\|\lambda_{\tilde{h}} - \lambda\|_{0,e} + \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,e}\Big\}.\end{aligned}$$

It follows, using that $h_e \le h_T$, and applying the discrete trace inequality (4.12) to $\|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,e}^2$, that

$$\begin{aligned}h_e\,\|\chi_e\|_{0,e}^2 &\le C\Big\{\|\boldsymbol{\sigma}_{1,h} - \boldsymbol{\sigma}_1\|_{0,T}^2 + h_T^2\,\|\mathbf{f}_1 + \mathbf{div}\,\boldsymbol{\sigma}_{1,h}\|_{0,T}^2\Big\}\\ &\quad + C\Big\{h_e\,\|\lambda_{\tilde{h}} - \lambda\|_{0,e}^2 + \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,T}^2 + h_e^2\,|\mathbf{u}_{1,h} - \mathbf{u}_1|_{1,T}^2\Big\}.\end{aligned} \tag{4.14}$$

Finally, it is not difficult to see that (4.14), the efficiency estimate (4.7) (cf. Lemma 4.4), and the definitions of $\boldsymbol{\sigma}_{1,h}$ and $\boldsymbol{\sigma}_1$ imply (4.13). We omit further details. □

LEMMA 4.7. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $e \in \mathcal{E}_h(\Gamma_2)$ there holds*

$$\begin{aligned}\tilde{h}_e\,\|\,\mathbf{u}_{1,h}\cdot\boldsymbol{\nu} - \mathbf{u}_{2,h}\cdot\boldsymbol{\nu}\,\|_{0,e}^2 &\le c\Big\{\|\mathbf{u}_1 - \mathbf{u}_{1,h}\|_{0,T_1}^2 + h_{T_1}^2\,|\mathbf{u}_1 - \mathbf{u}_{1,h}|_{1,T_1}^2\\ &\quad + \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T_2}^2 + h_{T_2}^2\,\|\mathrm{div}\,(\mathbf{u}_2 - \mathbf{u}_{2,h})\|_{0,T_2}^2\Big\},\end{aligned} \tag{4.15}$$

where T_1 and T_2 are the triangles of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, having e as an edge.

Proof. Given $e \in \mathcal{E}_h(\Gamma_2)$, we let T_1 and T_2 be the triangles of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, having e as an edge, and define $\chi_e := \mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}$ on e. Then, applying (4.5), using that $\mathbf{u}_2 \cdot \boldsymbol{\nu} = \mathbf{u}_1 \cdot \boldsymbol{\nu}$ on Γ_2 (cf. Theorem 4.2), recalling that $\psi_e = 0$ on $\partial T_2 \backslash e$, and integrating by parts on T_2, we get

$$\begin{aligned}\|\chi_e\|_{0,e}^2 &\le c_2 \, \|\psi_e^{1/2} \, \chi_e\|_{0,e}^2 = c_2 \int_e \psi_e \, \chi_e \, (\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_{2,h} \cdot \boldsymbol{\nu}) \\ &= c_2 \int_e \psi_e \, \chi_e \, (\mathbf{u}_{1,h} \cdot \boldsymbol{\nu} - \mathbf{u}_1 \cdot \boldsymbol{\nu}) \\ &\quad + c_2 \left\{ \int_{T_2} \nabla(\psi_e \, L(\chi_e)) \cdot (\mathbf{u}_2 - \mathbf{u}_{2,h}) + \int_{T_2} \psi_e \, L(\chi_e) \, \mathrm{div}\,(\mathbf{u}_2 - \mathbf{u}_{2,h}) \right\}.\end{aligned}$$

Now, the Cauchy–Schwarz inequality, the inverse estimate (cf. [17, Theorem 3.2.6]), and the fact that $0 \le \psi_e \le 1$ yield

$$\begin{aligned}\|\chi_e\|_{0,e}^2 &\le c_2 \, \|\chi_e\|_{0,e} \, \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,e} \\ &\quad + C \left\{ h_{T_2}^{-1} \, \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T_2} + \|\mathrm{div}\,(\mathbf{u}_2 - \mathbf{u}_{2,h})\|_{0,T_2} \right\} \|\psi_e \, L(\chi_e)\|_{0,T_2}.\end{aligned}$$

But, using that $0 \le \psi_e \le 1$ and applying the estimate (4.6), we see that

$$\|\psi_e \, L(\chi_e)\|_{0,T_2} \le \|\psi_e^{1/2} \, L(\chi_e)\|_{0,T_2} \le c_3^{1/2} \, h_e^{1/2} \, \|\chi_e\|_{0,e}.$$

Hence, similarly as in the previous proofs, and using that $\tilde{h}_e \le c \, h_e$, which follows from the fact that shape-regular meshes are locally quasi-uniform, and that $h_e \le h_{T_2}$, we deduce that

$$\tilde{h}_e \|\chi_e\|_{0,e}^2 \le C \left\{ h_e \|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,e}^2 + \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T_2}^2 + h_{T_2}^2 \|\mathrm{div}\,(\mathbf{u}_2 - \mathbf{u}_{2,h})\|_{0,T_2}^2 \right\},$$

which, applying again the discrete trace inequality (4.12) to $\|\mathbf{u}_{1,h} - \mathbf{u}_1\|_{0,e}^2$, and noting that there also holds $h_e \le h_{T_1}$, yields (4.15). □

We now provide the upper bounds for the remaining six terms defining $\eta_{2,T}^2$ (cf. (3.3)). The following four lemmas deal with the terms involving $\mathbf{u}_{2,h}$. Their proofs, which make use of a Helmholtz decomposition of the error $\mathbf{u}_2 - \mathbf{u}_{2,h}$, are already available in the literature (see, e.g., [14], [23]). From now on we assume, without loss of generality, that $\mathbf{K}^{-1} \, \mathbf{u}_{2,h}$ is polynomial on each $T \in \mathcal{T}_2$. Otherwise, additional higher order terms, given by the errors arising from suitable polynomial approximations, should appear in the bounds below.

LEMMA 4.8. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $T \in \mathcal{T}_2$ there holds*

$$h_T^2 \, \|\mathrm{curl}\,(\mathbf{K}^{-1} \, \mathbf{u}_{2,h})\|_{0,T}^2 \le c \, \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T}. \tag{4.16}$$

Proof. See [14, Lemma 6.1]. □

LEMMA 4.9. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $e \in \mathcal{E}_h(\Omega_2)$ there holds*

$$h_e \, \| \, [\mathbf{K}^{-1} \, \mathbf{u}_{2,h} \cdot \mathbf{s}] \, \|_{0,e}^2 \le c \sum_{T \subseteq \omega_e} \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T}^2, \tag{4.17}$$

where ω_e is the union of the two triangles in $\mathcal{T}_2$ having e as an edge.

Proof. See [14, Lemma 6.2]. □

LEMMA 4.10. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $T \in \mathcal{T}_2$ there holds*

$$h_T^2 \, \|\mathbf{K}^{-1} \mathbf{u}_{2,h}\|_{0,T}^2 \leq c \left\{ \|p - p_h\|_{0,T}^2 + h_T^2 \, \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T}^2 \right\}. \tag{4.18}$$

Proof. See [23, Lemma 5.5] or [14, Lemma 6.3]. □

LEMMA 4.11. *There exists $c > 0$, independent of h and $\tilde{h}$, such that*

$$\sum_{e \in \mathcal{E}_h(\Gamma_2)} h_e \left\| \mathbf{K}^{-1} \mathbf{u}_{2,h} + \frac{d\lambda_{\tilde{h}}}{ds} \right\|_{0,e}^2 \leq c \left\{ \|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2}^2 + \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,\Omega_2}^2 \right\}. \tag{4.19}$$

Proof. This proof makes use of the fact that $\nabla p_2 = -\mathbf{K}^{-1} \mathbf{u}_2$ in Ω_2 and $\lambda = p_2$ on Γ_2 (cf. Theorem 4.2). We omit further details and refer the reader to [23, Lemma 5.7]. □

The upper bound for the term of $\eta_{2,T}^2$ involving p_h is given in what follows.

LEMMA 4.12. *There exists $c > 0$, independent of h and $\tilde{h}$, such that for each $e \in \mathcal{E}_h(\Gamma_2)$ there holds*

$$h_e \|\lambda_{\tilde{h}} - p_h\|_{0,e}^2 \leq c \left\{ \|p - p_h\|_{0,T}^2 + h_T^2 \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T}^2 + h_e \|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2 \right\}, \tag{4.20}$$

where T is the triangle of $\mathcal{T}_2$ having e as an edge.

Proof. Since p_h is piecewise constant and $\nabla p = -\mathbf{K}^{-1} \mathbf{u}_2$ in Ω_2 (cf. Theorem 4.2), we first deduce that for each $T \subseteq \omega_e$ there holds

$$\begin{aligned} h_T^2 \, |p - p_h|_{1,T}^2 &= h_T^2 \, \|\nabla p\|_{0,T}^2 = h_T^2 \, \|\mathbf{K}^{-1} \mathbf{u}_2\|_{0,T}^2 \\ &\leq 2 \left\{ h_T^2 \, \|\mathbf{K}^{-1} \mathbf{u}_2 - \mathbf{K}^{-1} \mathbf{u}_{2,h}\|_{0,T}^2 + h_T^2 \, \|\mathbf{K}^{-1} \mathbf{u}_{2,h}\|_{0,T}^2 \right\} \\ &\leq C \left\{ h_T^2 \, \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T}^2 + h_T^2 \, \|\mathbf{K}^{-1} \mathbf{u}_{2,h}\|_{0,T}^2 \right\}. \end{aligned} \tag{4.21}$$

Next, since $\lambda = p_2$ on Γ_2 (cf. Theorem 4.2), for each $e \in \mathcal{E}_h(\Gamma_2)$ there holds

$$h_e \, \| \, \lambda_{\tilde{h}} - p_h \, \|_{0,e}^2 = h_e \, \| \, (p - p_h) + (\lambda_{\tilde{h}} - \lambda) \, \|_{0,e}^2 \leq 2 \, h_e \left\{ \|p - p_h\|_{0,e}^2 + \|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2 \right\},$$

which, employing the discrete trace inequality (4.12) and the estimate (4.21), yields

$$\begin{aligned} h_e \, \| \, \lambda_{\tilde{h}} - p_h \, \|_{0,e}^2 &\leq C \left\{ \|p - p_h\|_{0,T}^2 + h_T^2 \, |p - p_h|_{1,T}^2 + h_e \, \|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2 \right\} \\ \leq C \bigg\{ \|p - p_h\|_{0,T}^2 &+ h_T^2 \, \|\mathbf{u}_2 - \mathbf{u}_{2,h}\|_{0,T}^2 + h_T^2 \, \|\mathbf{K}^{-1} \mathbf{u}_{2,h}\|_{0,T}^2 + h_e \, \|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2 \bigg\}. \end{aligned}$$

Finally, (4.20) follows from the above inequality and the efficiency estimate (4.18). □

Concerning the term $h_e \, \|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2$ appearing in (4.20) (see also (4.13)), we observe that

$$\sum_{e \in \mathcal{E}_h(\Gamma_2)} h_e \, \|\lambda - \lambda_{\tilde{h}}\|_{0,e}^2 \leq h \, \|\lambda - \lambda_{\tilde{h}}\|_{0,\Gamma_2}^2 \leq C \, h \, \|\lambda - \lambda_{\tilde{h}}\|_{1/2,\Gamma_2}^2. \tag{4.22}$$

We have thus completed the derivation of the efficiency estimates for all the terms defining the a posteriori error indicators $\eta_{1,T}^2$ and $\eta_{2,T}^2$. Therefore, as a straightforward consequence of (4.2), (4.3), Lemmas 4.4–4.12, and (4.22), we conclude the existence of a constant $C > 0$, independent of h and $\tilde{h}$, such that

$$\boldsymbol{\eta}^2 \leq C \left\{ \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{H}}^2 + \|(p - p_h, \lambda - \lambda_{\tilde{h}})\|_{\mathbf{Q}}^2 + \sum_{T \in \mathcal{T}_1} h_T^2 \|\mathbf{f}_1 - \mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2 \right\}. \tag{4.23}$$

In particular, if $\mathbf{f}_1|_T \in [H^1(T)]^2$ for each $T \in \mathcal{T}_1$, then there holds

$$\sum_{T \in \mathcal{T}_1} h_T^2 \|\mathbf{f}_1 - \mathcal{P}_{0,T}(\mathbf{f}_1)\|_{0,T}^2 \leq C\, h^4 \sum_{T \in \mathcal{T}_1} \|\mathbf{f}_1\|_{1,T}^2 .$$

This inequality and (4.23) yield (4.1), which completes the proof of Theorem 4.1.

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