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ON THE ANGLE CONDITION IN THE FINITE ELEMENT METHOD*

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Abstract. The finite element procedure consists in finding an approximate solution in the form of piecewise linear functions, piecewise quadratic, etc. For two-dimensional problems, one of the most frequently used approaches is to triangulate the domain and find the approximate solution which is linear, quadratic, etc., in every triangle. A condition which is considered essential is that the angle of every triangle, independent of its size, should not be small. In this paper it is shown that the minimum angle condition is not essential. What is essential is the fact that no angle is too close to 180°.

1. Introduction. The finite element method has been widely studied, and it has become one of the most frequently used procedures for the numerical solution of partial differential equations. Numerous results have been obtained in recent years. We mention here [1] and the references quoted there, [2], [3], [4] and the references cited in these papers.

The finite element method consists in seeking an approximate solution in the form of piecewise linear functions, piecewise quadratics, etc. In two dimensions, one of the most commonly used approaches is to triangulate the domain (i.e., to cover the domain by triangles) and find the approximate solution which is linear, quadratic, etc., in every triangle. A condition which is considered essential is that the angle of any triangle, independent of its size, should not be small. This is called the minimum angle condition, which is required in all studies, e.g., [5], [6], [7], [2, p. 138] and many other papers. This condition has an analogue in higher dimensions (see [8]). In some sense this condition is very restrictive, especially when we seek to approximate functions which change more rapidly in one direction than in another direction.

The present paper shows that the minimum angle condition *is not essential.* What is essential is the fact that no angle is too close to 180°. In other words, the maximum angle condition is the proper condition in all 2-dimensional cases which have been considered.

The maximal angle condition in the case of piecewise linear approximation is referred to by Synge [10, p. 211]; however, it is not elaborated upon in any subsequent papers directly connected to the finite element method. The method of Synge appears to be hard to generalize. The approach in this paper is quite different and obviously may be generalized in many directions.

We shall show that this maximum angle condition is sufficient and also, in some sense necessary.[1]

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[1] After this paper was submitted, the authors learned of a report by P. Jamet, *Estimations d'erreur pour des elements finis droits presque degeneres*, CRM-447, Centre d'Etudes de Limeil, where similar results are discussed.

2. The sufficiency of the maximal angle condition. We shall not present here an abstract general version. Instead we prove a special theorem in two dimensions, where the underlying basic idea, which is simple, may be clearly presented without being encumbered by too many mathematical techniques. The generalization to higher dimensions does not present any difficulty.

Let, for $0<\alpha\leqq 1$,

$$T_h(\alpha)=\{(x_1,x_2)|0<x_1<h,\ 0<x_2<\alpha(h-x_1)\}$$

(i.e., $T_h(\alpha)$ is a right triangle (see Fig. 2.1)).

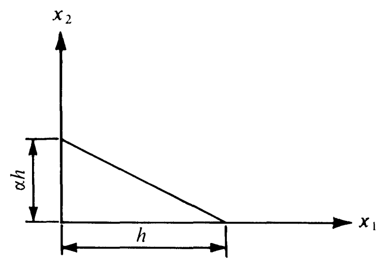

FIG. 2.1

We further let

$$T(\alpha)=T_h(\alpha)\quad \text{for } h=1$$

and

$$T=T_h(\alpha)\quad \text{for } h=\alpha=1.$$

Given $\Omega\subset R_2$, we define the usual Sobolev (Hilbert) spaces $H^l(\Omega)$ (l integer) with

$$\|u\|^2_{H^l(\Omega)}=\sum_{0\leqq\beta\leqq l}\|D^\beta u\|^2_{L_2(\Omega)}, \tag{2.1}$$

$$D^\beta=\frac{\partial^{\beta_1+\beta_2}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}},\quad \beta=(\beta_1,\beta_2),\quad |\beta|=\sum_{i=1}^{2}\beta_i,$$

where β_1 are integers and $\|u\|_{L^2(\Omega)}$ denotes the usual norm in $L_2(\Omega)$, i.e.,

$$\|u\|^2_{L_2(\Omega)}=\int_\Omega u^2\,dx,\qquad dx=dx_1\,dx_2. \tag{2.2}$$

Now we define

$$\Xi_h(\alpha)=\left\{u\in H^1(T_h(\alpha)),\ \int_0^{\alpha h}u(0,x_2)\,dx_2=0\right\},$$

and

$$\mathcal{T}_h(\alpha)=\{u\in H^2(T_h(\alpha)),\ u(x_1^i,x_2^i)=0,\ i=1,2,3\},$$

where (x_1^i,x_2^i), $i=1,2,3$, denote the coordinates of the vertices of $T_h(\alpha)$. For $h=\alpha=1$, we omit writing the variables α and h. From well-known Sobolev imbedding theorems, we know that $\Xi_h(\alpha)$ (resp. $\mathcal{T}_h(\alpha)$) is a proper subspace of $H^1(T_h(\alpha))$ (resp. $H^2(T_h(\alpha))$.

Now we prove the following lemma.

LEMMA 2.1. *Let*

$$A^2 = \inf_{u \in \Xi} \frac{\int_T \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] dx}{\int_T u^2 \, dx}. \tag{2.3}$$

Then $A^2 > 0$.

Proof. If we let $H^1(T)/P_0$ denote the quotient space modulo the set of constant functions (see [9, § 1.7, p. 19]), then we have

$$\|u\|^2_{H^1(T)/P_0} \leqq \text{const.} \int_T \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] dx \tag{2.4}$$

(see [9, Thm. 1.6]).

Now let us assume that, on the contrary, $A = 0$. Then there exists a sequence $\{u_i\} \in \Xi$, $i = 1, 2, \cdots$, such that $\|u_i\|_{L_2(T)} = 1$ and

$$\int_T \left[\left(\frac{\partial u_i}{\partial x_1}\right)^2 + \left(\frac{\partial u_i}{\partial x_2}\right)^2\right] dx \to 0. \tag{2.5}$$

Therefore, using (2.4), we have $\|u_i\|_{H^1(T)/P_0} \to 0$.

Thus there exists a sequence $\{p_i\}$ of constant functions such that

$$\|u_i - p_i\|_{H^1(T)} \to 0. \tag{2.6}$$

On the other hand, $\{\|u_i\|_{H^1(T)}\}$ is a bounded sequence. Therefore $\|p_i\|_{H^1(T)} = \|p_i\|_{L_2(T)}$ is also bounded, and consequently there exist subsequences u_{ij} and p_{ij} such that $\|u_{i_j} - p_{i_j}\|_{H^1(T)} \to 0$, $p_{i_j} \to \bar{p}$, with $\bar{p}$ a constant function. Therefore

$$\|p_{i_j} - \bar{p}\|_{L_2(T)} \to 0,$$

and thus

$$\|u_{i_j} - \bar{p}\|_{H'(T)} \to 0.$$

From the imbedding theorem it follows that

$$\int_0^1 (u_{i_j}(0, x_2) - \bar{p})^2 \, dx_2 \to 0.$$

Since $u_{i_j} \in \Xi$, $\bar{p} = 0$. Therefore, $\|u_{i_j}\|_{H^1(T)} \to 0$, which is a contradiction because

$$\|u_{i_j}\|_{(H^1(T))} \geqq \|u_{i_j}\|_{L^2(T)} = 1.$$

The proof of Lemma 2.1 is complete.

LEMMA 2.2. *Let*

$$B^2(\alpha) = \inf_{u \in \mathcal{T}(\alpha)} \frac{\int_{T(\alpha)} \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right] dx}{\int_{T(\alpha)} \left[\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] dx}, \quad 0 < \alpha < 1. \tag{2.7}$$

Then $B(\alpha)\geqq A$. ($B(\alpha)$ is positive uniformly with respect to α.)

Proof. Let Q be a mapping of $\mathcal{T}(\alpha)$ onto $\mathcal{T}$ such that

$$Q(u)=U(x_1,x_2),\qquad (x_1,x_2)\in T,$$

where

$$U(x_1,x_2)=u(x_1,\alpha x_2),\qquad (x_1,\alpha x_2)\in T(\alpha)$$

Then obviously we have

$$B^2(\alpha)=\inf_{U\in\mathcal{T}}\frac{\int_T\left[\left(\frac{\partial^2U}{\partial x_1^2}\right)^2+2\alpha^{-2}\left(\frac{\partial^2U}{\partial x_1\partial x_2}\right)^2+\alpha^{-4}\left(\frac{\partial^2U}{\partial x_2^2}\right)^2\right]dx}{\int_T\left[\left(\frac{\partial U}{\partial x_1}\right)^2+\alpha^{-2}\left(\frac{\partial U}{\partial x_2}\right)^2\right]dx},\tag{2.8}$$

and therefore, since $0<\alpha\leqq 1$, we have

$$B^2(\alpha)\geqq\inf_{U\in\mathcal{T}}\frac{\int_T\left\{\left(\frac{\partial^2U}{\partial x_1^2}\right)^2+\left(\frac{\partial^2U}{\partial x_1\partial x_2}\right)^2+\alpha^{-2}\left[\left(\frac{\partial^2U}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2U}{\partial x_2^2}\right)^2\right]\right\}dx}{\int_T\left[\left(\frac{\partial U}{\partial x_1}\right)^2+\alpha^{-2}\left(\frac{\partial U}{\partial x_2}\right)^2\right]dx}.\tag{2.9}$$

Now letting $\partial U/\partial x_2=w$, since $U\in H^2(T)$, we have $w\in H^1(T)$ *and* $w\in\Xi$ because $u\in\mathcal{T}$. Therefore from Lemma 2.1 we obtain

$$\begin{aligned}\alpha^{-2}\int_T\left[\left(\frac{\partial^2U}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2U}{\partial x_2^2}\right)^2\right]dx&=\alpha^{-2}\int_T\left[\left(\frac{\partial w}{\partial x_1}\right)^2+\left(\frac{\partial w}{\partial x_2}\right)^2\right]dx\\&\geqq\alpha^{-2}A^2\int_T w^2\,dx\\&=\alpha^{-2}A^2\int_T\left(\frac{\partial U}{\partial x_2}\right)^2dx.\end{aligned}\tag{2.10}$$

In a similar fashion, by interchanging the role of x_1 and x_2, we obtain

$$\int_T\left[\left(\frac{\partial^2U}{\partial x_1^2}\right)^2+\left(\frac{\partial^2U}{\partial x_1\partial x_2}\right)^2\right]dx\geqq A^2\int_T\left(\frac{\partial U}{\partial x_1}\right)^2dx.\tag{2.11}$$

Therefore we have

$$B^2(\alpha)\geqq\inf_{U\in\mathcal{T}}\frac{A^2\left[\int_T\left[\left(\frac{\partial U}{\partial x_1}\right)^2+\alpha^{-2}\left(\frac{\partial U}{\partial x_2}\right)^2\right]dx\right]}{\int_T\left[\left(\frac{\partial U}{\partial x_1}\right)^2+\alpha^{-2}\left(\frac{\partial U}{\partial x_2}\right)^2\right]dx}\geqq A^2,$$

and the lemma is proved.

In a similar way as that for Lemma 2.1, the following lemma may be proved.

LEMMA 2.3. *Let*

$$\bar{A}^2 = \inf_{u \in \mathcal{T}} \frac{\int_T \left[\left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right] dx}{\int_T u^2 \, dx}. \tag{2.12}$$

Then

$$\bar{A} > 0. \tag{2.13}$$

The following lemma is also easily proved.

LEMMA 2.4. *Let*

$$\bar{B}^2(\alpha) = \inf_{u \in \mathcal{T}_\alpha} \frac{\int_{T(\alpha)} \left[\left(\frac{\partial^2 u}{\partial x_1} \right)^2 + 2 \left(\frac{\partial u}{\partial x_1 \partial x_2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right] dx}{\int_{T(\alpha)} u^2 \, dx}, \qquad 0 < \alpha \leqq 1. \tag{2.14}$$

Then

$$\bar{B}(\alpha) \geqq \bar{A}. \tag{2.15}$$

($\bar{A}$ *is positive independently of* α).

Let for $u \in H^2(T_h(\alpha))$, Ru be a linear function on $T_h(\alpha)$ which coincides with u at the vertices of $T_h(\alpha)$. Further let $Zu = u - Ru$. Z is obviously a mapping of $H^2(T_h(\alpha))$ onto $\mathcal{T}_h(\alpha)$.

THEOREM 2.1. *Let*

$$C_h^2(\alpha) = \inf_{u \in H^2(T_h(\alpha))} \frac{\|u\|^2_{H^2(T_h(\alpha))}}{\|Zu\|^2_{H^1(T_h(\alpha))}}. \tag{2.16}$$

Then

$$C_h(\alpha) \geqq h^{-1}(\bar{A}^{-2}h^2 + A^{-2})^{-1/2}, \qquad 0 < h \leqq 1 \tag{2.17}$$

(*uniformly with respect to* α, $0 < \alpha \leqq 1$).

Proof. (i) We have

$$\|u\|^2_{H^2(T_h(\alpha))} = \int_{T_h(\alpha)} \left[\left(\frac{\partial^2 Zu}{\partial x_1^2} \right)^2 + 2 \left(\frac{\partial^2 Zu}{\partial x_1 \partial x_2} \right)^2 + \left(\frac{\partial^2 Zu}{\partial x_2^2} \right)^2 \right] dx + \|u\|^2_{H^1(T_h(\alpha))},$$

and therefore

$$C_h^2(\alpha) \geqq \inf_{u \in \mathcal{T}_h(\alpha)} \frac{\int_{T_h(\alpha)} \left[\left(\frac{\partial^2 u}{\partial x_1^2} \right)^2 + 2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_2} \right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 \right] dx}{\int_{T_h(\alpha)} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right] dx + \int_{T_h(\alpha)} u^2 \, dx}. \tag{2.18}$$

(ii) On the other hand, Lemmas 2.2 and 2.4 yield, for any $u \in \mathcal{T}_h(\alpha)$,

$$\int_{T_h(\alpha)} u^2\,dx \leqq \bar{A}^{-2}h^4 \int_{T_h(\alpha)} \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right] dx, \tag{2.19}$$

$$\begin{aligned}\int_{T_h(\alpha)} \left[\left(\frac{\partial u^2}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2\right] dx \\ \leqq A^{-2}h^2 \int_{T_h(\alpha)} \left[\left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2\right] dx,\end{aligned} \tag{2.20}$$

and therefore

$$C_h^2(\alpha) \geqq \frac{1}{\bar{A}^{-2}h^2 + A^{-2}}\, h^{-2}. \tag{2.21}$$

At the beginning of this section we introduced the right triangle $T_h(\alpha)$. Now we define a general triangle $T_h(\alpha, \gamma)$ such that it is the image of $T_h(\alpha)$ under the linear transformation Φ_γ given by

$$\Phi_\gamma(x_1, x_2) = (\xi_1, \xi_2),$$

where

$$\xi_1 = x_1 + \gamma x_2, \qquad \xi_2 = x_2$$

(see Fig. 2.2.)

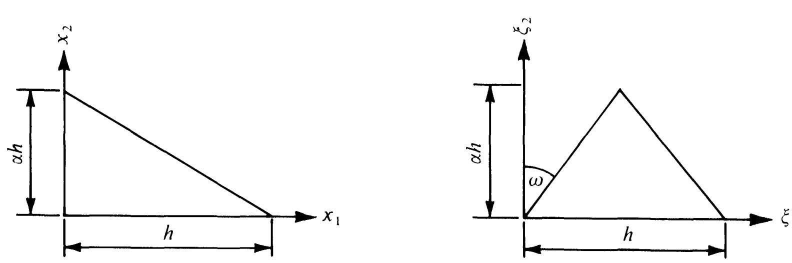

FIG. 2.2. $|\omega| < \pi/2$, $\gamma = \tan \omega$.

The Jacobian of this mapping is given by

$$J = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}, \qquad J^{-1} = \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix}.$$

By the use of usual techniques (see, e.g., [2, p. 138] and [6]), we obtain the following theorem.

THEOREM 2.2. *Let*

$$C_h^2(\alpha, \gamma) = \inf_{u \in H^2(T_h(\alpha,\gamma))} \frac{\|u\|^2_{H^2(T_h(\alpha,\gamma))}}{\|Zu\|_{H^1(T_h(\alpha,\gamma))}}. \tag{2.22}$$

Then for any $h \leqq 1$,

$$C_h(\alpha, \gamma) \geqq h^{-1}\Psi(\gamma), \tag{2.23}$$

where $\Psi(\gamma)$ is a continuous positive and finite function defined for all $-\infty<\gamma<\infty$. (For $|\gamma|<\gamma_0$, we have $\Psi(\gamma)\geqq\Psi(\gamma_0)>0$).

Thus far we have considered only the case when one side of the triangle was along the x_i-axis. We observe that since the norms in $H^l(\Omega)$ are invariant with respect to the rotation of the coordinates, Theorem 2.2 holds also for a triangle in general position.

Consider a general triangle T as shown in Fig. 2.3,

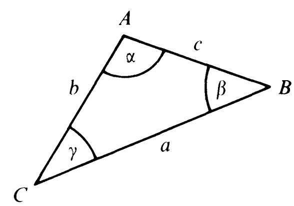

FIG. 2.3

and assume that α is the maximum angle, obviously $\pi>\alpha\geqq\pi/3$. Further, let h be the length of the maximimal side of T. Let us associate with any triangle two real numbers (α, h). Then Theorem 2.2 implies that

$$\|u-Ru\|_{H^1(T)}\leqq\Gamma(\alpha)h\|u\|_{H^2(T)}, \tag{2.24}$$

where $\Gamma(\alpha)((\pi/3)\leqq\alpha<\pi)$ is an increasing finite function.

Let τ be a triangulation of the entire two-dimensional space R_2; i.e., we cover R_2 by triangles which are either disjoint or have a common vertex or a common side. We associate with every triangulation two parameters α and h, h is the largest side and α is the largest angle of all the triangles in the given triangulation. Let $V_\tau\subset H^1(R_2)$ be the space of all functions which are linear in every triangle of our triangulation τ. The following result holds.

THEOREM 2.3.

$$\inf_{v\in V_\tau}\|u-v\|_{H^1(R_2)}\leqq\Gamma(\alpha)h\|u\|_{H^2(R_2)}. \tag{2.25}$$

The above theorem is an immediate consequence of inequality (2.24).

We proved Theorem 2.3 only for the case $H^l(R_2)$, $l=1.2$. However, using well-known extension theorems (see, e.g., [3, p. 30]), Theorem 2.3 can easily be extended to more general domains.

Remark 2.1. There is an essential difference between Theorem 2.3, and analogous theorems in the literature (see, e.g., [2, p. 138], [6], [7], [8] and others). The inequality (2.25) is the same, but in the papers cited above, α is the minimum angle of the triangles, whereas in our case it is the maximum angle. To obtain a uniform estimate, Theorem 2.3 requires that $\alpha\leqq\alpha_0<\pi$, where α_0 is independent of h. The analogous theorems in the literature require that the minimum angle be bounded below independently of h. Our theorems allow one angle to be arbitrarily small, but no two angles may be arbitrarily small. In § 3 we show that if two angles are small, then (2.25) does not generally hold.

Theorem 2.3 shows that the usual minimum angle condition is not essential when piecewise linear functions are used. The underlying idea is applicable in general. We first briefly discuss the situation for the piecewise quadratic approximation. In this case, a quadratic polynomial is completely characterized by its 6 values at the vertices and at the midpoints of the sides (see Fig. 2.4).

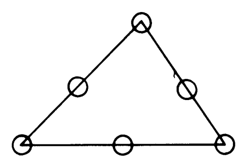

FIG. 2.4

The essential step in the proof of the desired theorem is the estimate

$$E(\alpha)=\inf_{u\in V(\alpha)}\frac{\displaystyle\int_{T(\alpha)}\left[\left(\frac{\partial^3 u}{\partial x_1^3}\right)^2+3\left(\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right)^2+3\left(\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right)^2+\left(\frac{\partial^3 u}{\partial x_2^3}\right)^2\right]dx}{\displaystyle\int_{T(\alpha)}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]dx},\tag{2.26}$$

where $V(\alpha)\subset H^3(T(\alpha))$ consists of functions which vanish at the vertices and at the midpoints of the sides of $T(\alpha)$. To obtain this estimate, we need a lemma analogous to Lemma 2.1, namely the following.

LEMMA 2.5. *Let*

$$F^2=\inf_{u\in W}\frac{\displaystyle\int_T\left[\left(\frac{\partial^2 u}{\partial x_1}\right)^2+2\left(\frac{\partial^2 u}{\partial x_1\partial x_2}\right)^2+\left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 dx\right.}{\displaystyle\int_T u^2\,dx},\tag{2.27}$$

where W is the space of all functions such that

$$\int_0^{1/2} u(0,x_2)\,dx_2=0,\tag{2.28}$$

$$\int_{1/2}^{1} u(0,x_2)\,dx_2=0,\tag{2.29}$$

$$\int_0^{1/2} u(\tfrac{1}{2},x_2)\,dx_2=0.\tag{2.30}$$

Then $F>0$.

The proof is very similar to the proof of Lemma 2.1. The essential part is the fact that any linear function satisfying (2.28)–(2.30) is the zero function. Consider

$$E(\alpha)=\inf_{u\in V}\frac{\displaystyle\int_T\left[\left(\frac{\partial^3 u}{\partial x_1^3}\right)^2+2\alpha^{-2}\left(\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right)^2+\alpha^{-4}\left(\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right)^2+\alpha^{-2}\left(\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right)^2+2\alpha^{-4}\left(\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right)^2+\alpha^{-6}\left(\frac{\partial^3 u}{\partial x_2^3}\right)^2\right]dx}{\displaystyle\int_T\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\alpha^{-2}\left(\frac{\partial u}{\partial x_2}\right)^2\right]dx},$$

Letting $w=\partial u/\partial x_1$ and $v=\partial u/\partial x_2$ and using Lemma 2.5, we may show that

$$E(\alpha)>E_0>0\quad\text{independently of }\alpha.$$

In a very similar manner, it is possible to furnish the analogous of the other results.

Other cases such as cubic and quartic approximations, etc., may be dealt with similarly. In fact, the approach for obtaining the approximation result is always the same; i.e., one first proves the analogues of Lemmas 2.1 and 2.4. An essential step is to determine the proper number of conditions which are the analogues of (2.28)–(2.30). In the case of quadratic approximation, we need three conditions since $\partial u/\partial x_1$ and $\partial u/\partial x_2$ are linear. In the case of cubic approximation, we need 6 conditions. In this case, we show what these conditions are. To this end, we study the cubic element on a right triangle when values and first derivatives of the functions are prescribed in the vertices together with the integral (over the triangle) of the second mixed derivative. It is easy to check that these ten conditions determine uniquely the cubic polynomial.

The analysis of this case leads to the study of the subspace $V\subset H^4(T)$, ($T=T_h(\alpha)$ with $h=\alpha=1$) of functions u which vanish together with their first derivatives at the vertices and, in addition,

$$\int_T\frac{\partial^2 u}{\partial x_1\partial x_2}\,dx=0.$$

To proceed analogously as before, we need 6 conditions which are satisfied by $w=\partial u/\partial x_1$ (resp. $\partial u/\partial x_2$), $u\in V$ and such that the only quadratic polynomial which satisfies all of them is zero.

The following conditions will give what we need:

$$w=0\quad\text{at the vertices of the triangle (3 conditions),}$$

$$\int_0^1 w(x_1,0)\,dx_1=0,$$

$$\int_0^1\frac{\partial w}{\partial x_2}(x_1,0)\,dx_1=0,$$

$$\int_T\frac{\partial w}{\partial x_2}(x_1,x_2)\,dx=0.$$

The analysis is now a simple repetition of the arguments used previously.

3. The maximum angle condition is essential. We show the essentiality of the maximum angle condition by constructing an example. To this end, let

$$\Omega=\{(x_1, x_2)\,\big|\; |x_1|<1, |x_2|<1\},$$

and let us consider a triangulation of Ω as shown in Fig. 3.1, where $1/(2H)$ and $N=1/h$ are integers.

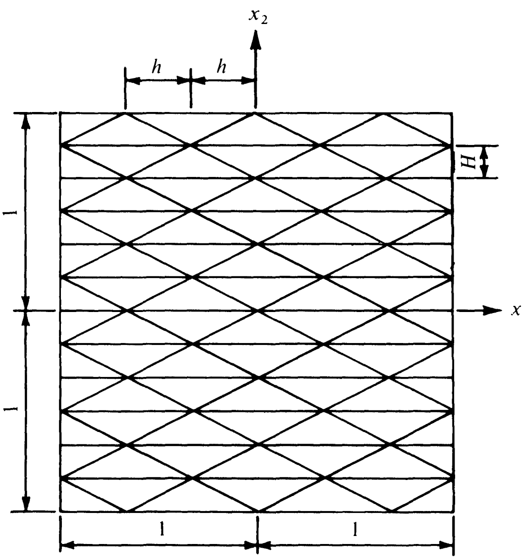

FIG. 3.1

Let us assume that $C^{(1)}h^{\beta}<H\leqq C^{(2)}h^{\beta}$ and $0<C^{(1)}<C^{(2)}<\infty$. Then the maximum and minimum angle conditions are satisfied for $\beta=1$. Let $V_h\subset H^1(\Omega)$ be the space of functions which are linear in every triangle. Then for $\beta=1$, we have

$$\inf_{v\in V_h}\|u-v\|_{H^1(\Omega)}\leqq Ch\|u\|_{H^2(\Omega)}, \tag{3.1}$$

where C depends neither on h nor on u.

Now we show that (3.1) is not valid when $\beta\geqq 5$. In this case, obviously the maximum and minimum angle condition is violated. In order to show that (3.1) does not hold, we choose a special u, namely $u_0=x_1^2$.

Suppose, on the contrary, that (3.1) holds. Thus we can find, for every h (resp. N), a function $v_h\in V_h$ such that

$$\|u_0-v_h\|_{H^1(\Omega)}\leqq Ch. \tag{3.2}$$

Let us single out a triangle τ of our triangulation (see Fig. 3.2)

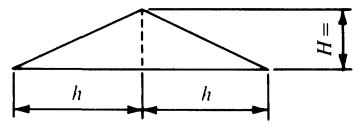

FIG. 3.2

Since $\partial u_o/\partial x_2 = 0$, we have from (3.2)

$$\int_\tau \left(\frac{\partial v_h}{\partial x_2}\right)^2 dx \leqq C^2 h^2. \tag{3.3}$$

On the other hand, $\partial v_h/\partial x_2$ is constant, and hence we have

$$\left|\frac{\partial v_h}{\partial x_2}\right|^2 \leqq \bar{C} h^{1-\beta}. \tag{3.4}$$

Now let

$$\eta^h(x_1, x_2) = u_0(x_1, x_2) - v_h(x_1, x_2),$$
$$\varphi^h(x_1, x_2) = \tfrac{1}{2}[\eta^h(x_1+h, x_2) + \eta^h(x_1-h, x_2)] - \eta^h(x_1, x_2).$$

Then we obviously have

$$\varphi^h(x_1, x_2) = h^2 - \Psi^h(x_1, x_2), \tag{3.5}$$

where

$$\Psi^h(x_1, x_2) = \tfrac{1}{2}[v_h(x_1+h, x_2) + v_h(x_1-h, x_2)] - v_h(x_1, x_2).$$

The function $\Psi^h(ih, x_2)$ (i an integer) as a function of x_2 is obviously piecewise linear, and we have

$$\Psi^h(ih, jH) = 0, \tag{3.6}$$

for all j even (resp. odd) integer when i is odd (resp. even).

On the other hand, (3.4) together with (3.6) yields

$$|\Psi^h(ih, x_2)| \leqq C_2 h^{(1+\beta)/2}, \tag{3.7}$$

where C_2 is independent of h. Thus we have

$$\varphi^h(ih, x_2) \geqq C_3 h^2, \tag{3.8}$$

for sufficiently small h, and $C_3 > 0$ is independent of h.

Letting

$$Z_i^h = \int_{-1}^{+1} \eta(ih, x_2)\, dx_2, \qquad i = -\frac{1}{h}+1, \cdots, +\frac{1}{h}-1,$$
$$\theta_i^h = \int_{-1}^{+1} \varphi^h(ih, x_2)\, dx_2, \qquad i = -\frac{1}{h}+1, \cdots, +\frac{1}{h}-1,$$

we obtain

$$\theta_i \geqq C_3 h^2 \qquad (C_3 > 0, \text{ independently of } h)$$

and

(3.9)
$$Z^h_{i-1} - 2Z^h_i + Z^h_{i+1} = 2\theta^h_i$$

In addition, (3.2) and a well-known imbedding theorem give, for $-1/h \leqq i \leqq 1/h$,

(3.10)
$$|Z^h_i| \leqq C_4 h.$$

Define

(3.11)
$$w^h_i = C_4 h + C_3((ih)^2 - 1)$$

and

(3.12)
$$\xi^h_i = w^h_i - Z^h_i.$$

Then

$$\xi^h_{-1/h} \geqq 0, \qquad \xi^h_{1/h} \geqq 0$$

and

$$\xi^h_{i-1} - 2\xi^h_i + \xi^h_{i+1} \leqq 0.$$

Therefore

$$\xi^h_i \geqq 0 \quad \text{for all } i.$$

Thus

(3.13)
$$Z^h_0 \leqq -C_3 + C_4 h$$

for all h, which is obviously in contradiction with (3.10). Therefore we have shown that (3.2) cannot hold.

Remark 3.1. Let us change the triangulation shown in Fig. 3.1 to the one shown in Fig. 3.3 below. This triangulation obviously violates the minimum angle condition but satisfies the maximum angle condition. Therefore, as shown in previous section, (3.2) holds.

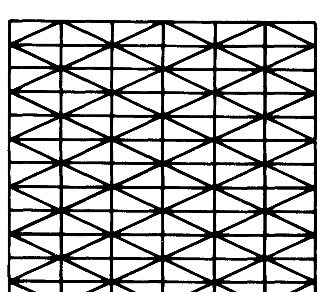

FIG. 3.3

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