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Galerkin and Finite Element Methods Stiffness and Mass matrices

D. Clouteau

Department of Mechanical and Civil Engineering Ecole Centrale Paris, France

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History of the FEM:

- B. Galerkin 1871-1945: St Petersburg, Civil and Mechanical Engineer. He created the "Galerkin's method" to compute approximate solution,
- W. Ritz 1878-1909: Göttingen, similar method in mathematical physics, with D. Hilbert.
- R. Courant 1888-1972: Göttingen \rightarrow New-York (triangular elements 1943)
- R. Clough 1920-: Berkeley, (FEM 1959) Seismic response of a Dam, *Dynamics of Structures*, 1975,
- O. Zienkiewicz 1921-: Swansea (FEM 1965), The Finite Element Method 1970.
- J. Argyris 1913-2004: Stuttgart, Civil Eng. \rightarrow Aerospace.

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The Galerkin's Method The basic idea

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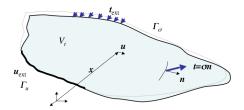
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- Take a set of displacement fields $\boldsymbol{w}_n(\boldsymbol{x})$,
- Look for an approximate solution u_h as :

$$oldsymbol{u}(oldsymbol{x},t) = \sum_{n=1}^{N} q_n(t) oldsymbol{w}_n(oldsymbol{x})$$

■ Use these fields $w_n(x)$ in the Virtual Power Principle to obtain a discrete dynamical system :

$$\mathbf{K}\mathbf{a} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{M}\ddot{\mathbf{a}} = \mathbf{f}$$

to be solved for the amplitudes $q_n(t)$

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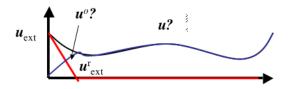
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Displacement Boundary conditions Linear case



- On Γ_u : $\boldsymbol{u} = \boldsymbol{u}_{\text{ext}} \neq 0$ but $\boldsymbol{w} = 0$ for the VPP.
- An auxiliary field $\boldsymbol{u}_{\text{ext}}^r(\boldsymbol{x},t)$ on V such that

$$\boldsymbol{u}_{\mathrm{ext}}^{r}(\boldsymbol{x},t) = \boldsymbol{u}_{\mathrm{ext}}(\boldsymbol{x},t) \qquad \boldsymbol{x} \in \Gamma_{u}$$

• A new unknown field $u^o = u - u^r_{\text{ext}} \in \mathbb{V}_o$ satisfying

$$\mathcal{P}_{ ext{kin}}^o(oldsymbol{w}) = \mathcal{P}_{ ext{int}}^o(oldsymbol{w}) + \mathcal{P}_{ ext{ext}}^o(oldsymbol{w}) + \mathcal{P}_{ ext{int}}^r(oldsymbol{w}) - \mathcal{P}_{ ext{kin}}^r(oldsymbol{w}) \qquad orall oldsymbol{w} \in \mathbb{V}_o$$

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The discrete dynamical system Using the Galerkin method

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- $\{w_n\}_{n=1,N}$ a basis of $\mathbb{V}_h \in \mathbb{V}_o$ (no need to be orthogonal for $(.,.)_{\mathbb{V}_o}$ but could be for others)
- \mathbf{u}_h the approximate solution on this basis

$$\boldsymbol{u}_h(\boldsymbol{x},t) = \sum_{n'=1}^N \boldsymbol{w}_{n'}(\boldsymbol{x}) q_{n'}(t) = \boldsymbol{W}(\boldsymbol{x}) \mathbf{q}(t),$$

■ The VPP with $\boldsymbol{w} = \boldsymbol{w}_n, \forall n \leq N$:

$$\mathcal{K}(\boldsymbol{u}, \boldsymbol{w}_n) + \mathcal{C}(\dot{\boldsymbol{u}}, \boldsymbol{w}_n) + \mathcal{M}(\ddot{\boldsymbol{u}}, \boldsymbol{w}_n) = \mathcal{P}(\boldsymbol{w}_n)$$

$$\sum_{n=1}^{N} \mathcal{K}(\boldsymbol{w}_{n'}, \boldsymbol{w}_n) q_{n'} + \mathcal{C}(\boldsymbol{w}_{n'}, \boldsymbol{w}_n) \dot{q}_{n'} + \mathcal{M}(\boldsymbol{w}_{n'}, \boldsymbol{w}_n) \ddot{q}_{n'} = \mathcal{P}(\boldsymbol{w}_n)$$

$$\mathbf{K}\mathbf{q} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}$$

The Stiffness and Mass matrices in the Galerkin method

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■ The symmetric positive $mass\ matrix\ \mathbf{M}$:

$$[\mathbf{M}]_{n'n} = \mathcal{M}(\boldsymbol{w}_{n'}, \boldsymbol{w}_n) = \int_{V} \rho \boldsymbol{w}_n \cdot \boldsymbol{w}_{n'} dV$$

 \blacksquare The stiffness matrix **K**:

$$[\mathbf{K}]_{n'n} = (\mathcal{K}_e + \mathcal{K}_g - \mathcal{K}_i)(\boldsymbol{w}_{n'}, \boldsymbol{w}_n) = [\mathbf{K}_e + \mathbf{K}_g - \mathbf{K}_i]_{n'n}$$

■ The symetric positive elastic stiffness matrix \mathbf{K}_e :

$$[\mathbf{K}_{e}]_{n'n} = \int_{V} \boldsymbol{\sigma}(\boldsymbol{w}_{n'}) : \boldsymbol{\epsilon}(\boldsymbol{w}_{n}) dV = \int_{V} \operatorname{tr}(\boldsymbol{\epsilon}(\boldsymbol{w}_{n}) \boldsymbol{C}_{e} \boldsymbol{\epsilon}(\boldsymbol{w}_{n'})) dV$$
$$= \int_{V} (\lambda \operatorname{div}(\boldsymbol{w}_{n}) \operatorname{div}(\boldsymbol{w}_{n'}) + 2\mu \boldsymbol{\epsilon}(\boldsymbol{w}_{n}) : \boldsymbol{\epsilon}(\boldsymbol{w}_{n'})) dV$$

■ The positive symmetric damping matrix **D** and the skew-symmetric gyroscopic matrix \mathbf{C}_a :

$$[\mathbf{C}]_{n'n} = (\mathcal{C}_a + \mathcal{D}_g)(oldsymbol{w}_{n'}, oldsymbol{w}_{oldsymbol{n}}) = [\mathbf{C}_a + \mathbf{D}]_{ar{n}'n}$$

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The approximate eigenmodes Without gyroscopic terms

eigenmodes

■ The approximate eigenvectors $\hat{\mathbf{q}}_k$ and eigenfrequencies $\tilde{\omega}_k$:

$$\mathbf{K}\mathbf{q}_k = \tilde{\omega}_k^2 \mathbf{M} \mathbf{q}_k$$

Orthogonality:

$$\mathbf{q}_l^T \mathbf{M} \mathbf{q}_k = m_k \delta_{kl} \qquad \mathbf{q}_l^T \mathbf{K} \mathbf{q}_k = m_k \tilde{\omega}_k^2 \delta_{kl}$$

■ link with the true eigenfrequencies:

$$\tilde{\omega}_k \ge \omega_k$$

Which basis fields for the Galerkin Method? and foreseen difficulties

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- Two contradictory goals for a given accuracy :
 - Reducing the number of fields in the basis,
 - Using easy to compute displacement fields.
- Two remaining difficulties:
 - computing strains and stresses,
 - integrating over the domain.
- Two solutions:
 - Localized masses and coupling stiffnesses
 - The FE Method

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The Rayleigh Method

A discrete Dynamic model of a building

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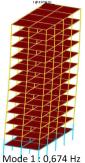
Hypotheses:

- Masses localized on the floors $m_i = m$,
- Horizontal displacement of the floors: N Degrees Of Freedom $q_i(t)$,
- \blacksquare One static mode due to inertial forces $\hat{\mathbf{q}}$
- Storey stiffness brought by columns: $k_i = k = n_c \frac{12EI}{k_a^3}$,
- The static solution :

$$\Delta \hat{q}_n = \hat{q}_n - \hat{q}_{n-1} = \frac{m}{k}(N+1-n),$$

■ The approximate natural frequency:

$$\omega_o = \frac{\mathcal{E}_e(\hat{\mathbf{q}})}{\mathcal{E}_{kin}(\hat{\mathbf{q}})} = \frac{k \|\Delta \hat{\mathbf{q}}\|^2}{m \|\hat{\mathbf{q}}\|^2}$$



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The Finite Element Method The objectives

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- Parametrize the geometry,
 - to perform integrations,
 - to define Basis function,
 - Solution: Divide the structure into simple cells (tetrahedrons, prisms, bricks, plates, beams...)
- Build basis functions
 - being continuous,
 - with simple derivatives,
 - Having given values on the boundary,
 - easy to integrate over the cells.
 - **Solution:** Polynomials on the cells
- Interpolating basis functions \Leftrightarrow Degrees of freedom q_n being the displacement field at some given nodes (not mandatory),

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Meshes and element shape functions Parametrizing the geometry

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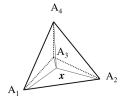
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- The nodes with coordinates $x_{A=1,N_d}$
- The elements (tetrahedrons) $V_{E=1,N_e}$ with their local nodes $(A_i)_{i=1,n_E}$,
- Local parametrization of $x \in V_E$:



$$x = \sum_{i=1}^{4} w_i(x) x_{A_i}$$
 $0 \le w_i \le 1$ $\sum_{i=1}^{4} w_i = 1$

• $w_i(\mathbf{x})$ are linear interpolating shape functions $(w_i(\mathbf{x}_i) = \delta_{ij})$

$$w_i(oldsymbol{x}) = rac{(oldsymbol{x}_{A_k} - oldsymbol{x}_{A_j}) \wedge (oldsymbol{x}_{A_l} - oldsymbol{x}_{A_j}) \cdot (oldsymbol{x} - oldsymbol{x}_{A_j})}{6V_E} = rac{h_i(oldsymbol{x})}{H_i}$$

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Nodal shape functions Parametrizing the geometry

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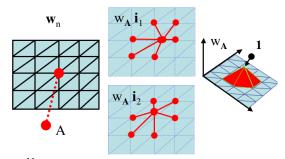
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- $V_A = \bigcup_{e=1}^{N_A} V_{E_A^e}$ the support the shape function,
- The scalar shape function

$$w_A(\boldsymbol{x}) = w_{E_A^e}(\boldsymbol{x})\delta_{AA_i}, \qquad \boldsymbol{x} \in V_{E_A^e}$$

- The displacement basis function $w_{Aa}(x) = w_A(x)i_a$
- Boundary condition $x_A \notin \Gamma_u$
- w_{Aa} are continuous and piecewise differentiable.



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Local matrix assembling The tetrahedron element

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■ The strain tensor:

$$oldsymbol{\epsilon}(oldsymbol{w}_i(oldsymbol{x})oldsymbol{i}_a) = rac{oldsymbol{i}_a \otimes_s oldsymbol{n}_i}{H_i}$$

■ The stress tensor:

$$\sigma(\boldsymbol{w}_i(\boldsymbol{x})\boldsymbol{i}_a) = \frac{\lambda \boldsymbol{i}_a \cdot \boldsymbol{n}_i \boldsymbol{I}_d + 2\mu \boldsymbol{i}_a \otimes_s \boldsymbol{n}_i}{H_i}$$

■ The local stiffness matrix:

$$[K]_{iajb}^{E} = \int_{V_{E}} \boldsymbol{\sigma}(\boldsymbol{w}_{i}(\boldsymbol{x})\boldsymbol{i}_{a}) : \boldsymbol{\epsilon}(\boldsymbol{w}_{j}(\boldsymbol{x})\boldsymbol{i}_{b})dV$$

$$= V_{E} \frac{\lambda \boldsymbol{i}_{a} \cdot \boldsymbol{n}_{i}\boldsymbol{i}_{b} \cdot \boldsymbol{n}_{j} + \mu(\delta_{ab}\boldsymbol{n}_{i} \cdot \boldsymbol{n}_{j} + \boldsymbol{n}_{i} \cdot \boldsymbol{i}_{a}\boldsymbol{n}_{i} \cdot \boldsymbol{i}_{b})}{H_{i}H_{i}}$$

■ The mass matrix:

$$[M]_{iajb}^{E} = \delta_{ab} \int_{V_{E}} \rho \boldsymbol{w}_{i}(\boldsymbol{x}) \boldsymbol{w}_{j}(\boldsymbol{x}) dV = \delta_{ab} \int_{V_{E}} \rho \frac{h_{i}h_{j}}{H_{i}H_{j}} dV$$

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Global FE matrices Assembling and properties

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• Global assembling, axample of the *mass matrix*:

$$[M]_{AaBb} = \sum_{E=1}^{N_E} \sum_{i \le n_E; j \le n_E} \delta_{AA_i} \delta_{BA_j} [M]_{iajb}^E$$

■ Properties:

- \blacksquare All matrices but \mathbf{C}_a are symmetric
- \mathbf{C}_a is skew-symmetric,
- Huge matrices: 300K nodes leads to 10¹² terms and 8To of memory in full storage for each,
- All matrices are sparse: to be accounted in the storage (Profil, sparse, Element by element 50Mo).

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- What's new?
 - A general procedure leading to a discrete dynamical model:

$$(\mathbf{K}_e + \mathbf{K}_g - \mathbf{K}_i)\mathbf{q} + (\mathbf{C}_a + \mathbf{D})\dot{\mathbf{q}} + \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f}$$

- an efficient and versatile procedure to assemble this system: Finite Element Method
- What's left?
 - How to solve the discrete model?
 - Which level of refinement?
 - What about beams and plates?