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BASIS OF FINITE ELEMENT METHODS FOR SOLID CONTINUA*

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SUMMARY

Finite element methods can be formulated from the variational principles in solid mechanics by relaxing the continuity requirements along the interelement boundaries. The combination of different variational principles and different boundary continuity conditions yields numerous types of approximate methods. This paper reviews and reinterprets the existing finite element methods and indicates other alternative schemes. Plate bending problems are used to compare the relative merits of the various methods.

INTRODUCTION

The objective of this article is to present a logic classification of the various approaches in the finite element analysis of solid continua. These approaches are associated with several variational principles in solid mechanics. Washizu's text[1] contains a detailed development of the variational principles. It will be shown that for finite element analyses many modified versions of the variational principles can be added.

Historically speaking, Courant's[2] approximate solution of St. Venant torsion problem should be considered as the first treatment of finite element analysis of solid continua. That problem was formulated by the principle of minimum potential energy, assuming a linear distribution of the warping function in each of the assemblage of triangular elements. Further insight into approximate solutions of boundary value problems was provided by Prager and Synge[3] by a geometric representation in function space. This procedure, which is often called the hypercircle method, was treated in detail in the text by Synge[4] and was further extended by other authors. The hypercircle method has been applied to finite element idealizations of solid continua. It can be used to make a quantitative estimate of the errors involved in the approximate solutions and may be looked upon as a geometric form of the well-known principles of minimum potential energy and minimum complementary energy.

Turner and his associates[5] first applied the technique of matrix displacement methods to plane stress problems, using triangular and rectangular elements. The stiffness matrices were derived by the so-called direct method, but the formulation was not based on the field equations

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of the entire elastic continuum. It was a number of years later that different types of variational principles in elasticity were discussed in connection with the development of more general finite element methods.[6–14] After a solid continuum is subdivided into an assemblage of discrete elements, displacement and/or stress fields are assumed within each element. The equations which result from the application of the variational principles are simultaneous algebraic (or matrix) equations which may have generalized displacements or generalized stresses or both, at the nodal points, as unknowns to be evaluated. Thus, the finite element methods are also classified generally as 'the displacement method', 'the force method' and the 'mixed method'.

However, such classifications are not entirely appropriate for the description of the analysis of solid continua. For example, there exist many types of finite element methods which employ nodal displacements as unknowns, but only one of which starts by assuming only displacement functions in the individual discrete elements. These methods should perhaps be labelled as the stiffness methods on account of their use of element stiffness matrices which relate the generalized element nodal forces and nodal displacements. Of these types of finite element methods, one is derived from the principle of minimum potential energy[6] and is based on the assumption of a displacement field continuous over the entire solid. It can thus be classified as a compatible model. The second method, which is derived from the principle of minimum complementary energy[10] and is based on an assumed equilibrium stress field, can be labelled as an equilibrium model. The third method is based on a modified complementary energy principle[9,14] for which compatible displacement functions are assumed along the interelement boundaries in addition to the assumed equilibrating stress field in each element. This method can thus be called a hybrid method. The fourth method can be derived from the Reissner's variational principle[15] based on an assumed displacement field which is continuous over the entire solid and assumed stress fields for individual elements. This method is called a mixed method.

It is obvious that a dual hybrid method can be formulated, for which equilibrating tractions are assumed along the interelement boundaries in addition to the assumed continuous displacement fields in each element. Yamamoto[13] proposed such a hybrid method in 1966, but Jones' earlier formulation[7] in 1964 may also be interpreted as this hybrid method. Finally, many types of finite element analyses can be formulated based on Reissner's principle, Herrmann's formulation of incompressible and nearly incompressible solids[11] and of bending of plates[12] are examples of mixed methods in finite element analysis. A special form of Reissner's principle for shells has also been incorporated in finite element analysis by Prato.[16]

A simultaneous application of the compatible and equilibrium models in finite element analysis allows the establishment of upper and lower bounds on the total strain energy[10,35] and hence serves as an assessment of the accuracy of the solutions. The variational formulation also provides a means to prove the convergence of the finite element methods.[14,17]

In the ordinary formulation in variational methods it is usually required that the assumed functions should possess derivatives which are continuous up to the highest order occurring in the corresponding Euler differential equations. In the finite element analysis we maintain this requirement within each discrete element but broaden the admissibility on the interelement boundary conditions to the degree that the functions shall possess continuous derivatives in such a manner that in addition to the fulfillment of the appropriate subsidiary conditions, the functional of the variational problem is defined. Since the number of interelement boundary is finite the functions corresponding to the extremum of the variational functional of the broadened admissibility will still be the exact solution of the differential equations for the continua. Thus, for the finite element analysis, the so called compatibility condition at the interelement boundary is defined by the requirement that the variational functional is definable. As a result, for different variational functional we may have different compatibility conditions. This will be demonstrated by the various finite element models discussed in the following sections.

PRINCIPLE OF MINIMUM POTENTIAL ENERGY AND THE COMPATIBLE MODEL

The principle of minimum potential energy may be stated as the vanishing of the variation of the total potential energy functional Π_p, which may be expressed as

$$\Pi_p = \int_V (\tfrac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \bar{F}_i u_i)\,\mathrm{d}V - \int_{S_\sigma} \bar{T}_i u_i\,\mathrm{d}S \tag{1}$$

In this expression,

ε_{ij} = strain tensor component
E_{ijkl} = elastic constants
V = volume
$\bar{F}_i$ = prescribed body force component
u_i = displacement
S = surface
$\bar{T}_i$ = prescribed surface traction
S_σ = portion of S over which the surface tractions are prescribed

In applying this principle, ε_{ij} is written in terms of the displacement u_i, and the variation of the functional should satisfy the displacement compatibility (continuity) conditions.

When a solid is divided into a finite number of discrete elements V_n, the potential energy functional may be written as

$$\Pi_p = \sum_n \left(\int_{V_n} [\tfrac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \bar{F}_i u_i]\,\mathrm{d}V - \int_{S_{\sigma_n}} \bar{T}_i u_i\,\mathrm{d}S \right) \tag{2}$$

where $S_{\sigma n}$ is the portion of the boundary of the nth element over which the surface tractions $\bar{T}_i$ are prescribed.

In applying the finite element method, approximate displacement functions u_i are represented by interpolating functions and generalized displacements at a finite number of nodal points of each element. The interpolating functions must be such that when the nodal displacements for two neighbouring elements are compatible, the displacements along the corresponding interelement boundary are compatible. In matrix form the assumed displacements may be written as

$$\mathbf{u} = [\mathbf{A}_q\ \mathbf{A}_r]\begin{Bmatrix}\mathbf{q}\\ \mathbf{r}\end{Bmatrix} \tag{3}$$

where $\mathbf{q}$ is the column matrix of element generalized displacements at boundary nodes where displacement compatibility with neighbouring elements is required and $\mathbf{r}$ is the column matrix of displacements either at an interior node or at a boundary node at which displacement compatibility with neighbouring elements is not required. Inclusion of the displacements $\mathbf{r}$ will not affect the boundary displacements which must be compatible with those of the neighbouring elements.

Taking the plane stress problem as an example, the column matrix is

$$\mathbf{u} = \{u(x,y),\ v(x,y)\} \tag{4}$$

and for a rectangular element with four corner nodes located at $(x,y) = (0,0)$, $(a,0)$, (a,b) and $(b,0)$

$$\mathbf{q} = \{u_1,v_1,u_2,v_2,u_3,v_3,u_4,v_4\} \tag{5}$$

(see Figure 1). To maintain the continuity in displacements along the interelement boundary, a bilinear interpolation may be used for the assumed displacement functions, i.e.

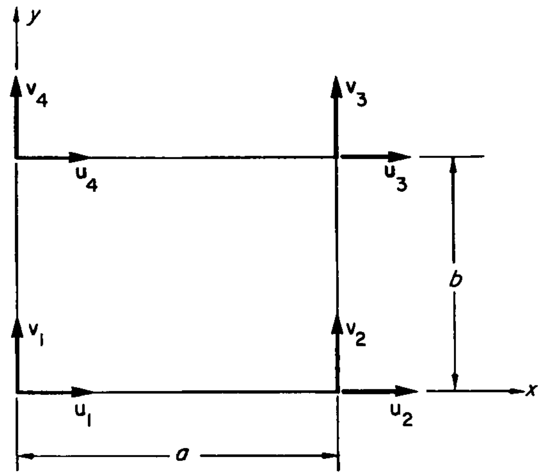

Figure 1. Plane stress rectangular element

$$\mathbf{A}_q = \left[\begin{array}{c|c|c|c|c|c|c|c} (a-x)(b-y) & 0 & x(b-y) & 0 & xy & 0 & (a-x)y & 0 \\ \hline 0 & (a-x)(b-y) & 0 & x(b-y) & 0 & xy & 0 & (a-x)y \end{array}\right] \tag{6}$$

If the centre of the element is introduced as an internal node (with nodal displacements u_c and v_c), then the displacement functions

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \sin\dfrac{\pi x}{a}\sin\dfrac{\pi y}{b} & 0 \\ 0 & \sin\dfrac{\pi x}{a}\sin\dfrac{\pi y}{b} \end{bmatrix} \begin{Bmatrix} u_c \\ v_c \end{Bmatrix} = \mathbf{A}_r\mathbf{r} \tag{7}$$

will not affect the boundary compatibility conditions.

The corresponding strain distribution is

$$\boldsymbol{\varepsilon} = [\mathbf{B}_q\ \mathbf{B}_r]\begin{Bmatrix} \mathbf{q} \\ \mathbf{r} \end{Bmatrix} \tag{8}$$

For the plane stress problem $\boldsymbol{\varepsilon} = \{\varepsilon_x,\varepsilon_y,\varepsilon_{xy}\}$ and $\mathbf{B}_q$ and $\mathbf{B}_r$ are obtained from derivatives of $\mathbf{A}_q$ and $\mathbf{A}_r$. Substituting equation (8) into equation (2), we obtain

$$\Pi_p = \sum_n(\tfrac{1}{2}\mathbf{q}^T\mathbf{k}_{qq}\mathbf{q} + \mathbf{r}^T\mathbf{k}_{rq}\mathbf{q} + \tfrac{1}{2}\mathbf{r}^T\mathbf{k}_{rr}\mathbf{r} - \overline{\mathbf{Q}}_q^T\mathbf{q} - \overline{\mathbf{Q}}_r^T\mathbf{r}) \tag{9}$$

where

$$\mathbf{k}_{qq} = \int_{Vn} \mathbf{B}_q^T \mathbf{E} \mathbf{B}_q \, \mathrm{d}V$$

$$\mathbf{k}_{rq} = \int_{Vn} \mathbf{B}_r^T \mathbf{E} \mathbf{B}_q \, \mathrm{d}V$$

$$\mathbf{k}_{rr} = \int_{Vn} \mathbf{B}_r^T \mathbf{E} \mathbf{B}_r \, \mathrm{d}V \tag{10}$$

$$\overline{\mathbf{Q}}_q = \int_{Vn} \mathbf{A}_q^T \overline{\mathbf{F}} \, \mathrm{d}V + \int_{S_{\sigma_n}} \mathbf{A}_{q_B}^T \overline{\mathbf{T}} \, \mathrm{d}S$$

$$\overline{\mathbf{Q}}_r = \int_{Vn} \mathbf{A}_r^T \overline{\mathbf{F}} \, \mathrm{d}V + \int_{S_{\sigma_n}} \mathbf{A}_{r_B}^T \overline{\mathbf{T}} \, \mathrm{d}S$$

In the above expressions, $\mathbf{E}$ is the elastic constant matrix, $\overline{\mathbf{F}}$ and $\overline{\mathbf{T}}$ are respectively the prescribed body and surface forces, and the subscript B is used to signify $\mathbf{A}_q$ and $\mathbf{A}_r$ evaluated along the boundary. Since the displacements $\mathbf{r}$ for different elements are independent, the stationary conditions with respect to their variations will yield

$$\mathbf{k}_{rq}\mathbf{q} + \mathbf{k}_{rr}\mathbf{r} - \overline{\mathbf{Q}}_r = \mathbf{0} \tag{11}$$

By solving equation (11) for the generalized coordinates $\mathbf{r}$ and substituting into equation (9), we obtain the following expression for the Π_p-functional

$$\Pi_p = \sum_n (\tfrac{1}{2}\mathbf{q}^T\mathbf{k}\mathbf{q} - \overline{\mathbf{Q}}^T\mathbf{q} + C_n) \tag{12}$$

where $\mathbf{k}$ and $\overline{\mathbf{Q}}$ are respectively the element stiffness matrix and equivalent nodal forces defined by

$$\mathbf{k} = \mathbf{k}_{qq} - \mathbf{k}_{rq}^T\mathbf{k}_{rr}^{-1}\mathbf{k}_{rq}$$

and

$$\overline{\mathbf{Q}} = \overline{\mathbf{Q}}_q - \mathbf{k}_{rq}\mathbf{k}_{rr}^{-1}\overline{\mathbf{Q}}_r \tag{13}$$

$$C_n = -\tfrac{1}{2}\overline{\mathbf{Q}}_r^T\mathbf{k}_{rr}^{-1}\overline{\mathbf{Q}}_r = \text{constant}$$

The element nodal displacements $\mathbf{q}$ for different elements are not completely independent. Hence, a transformation is required to relate the element nodal displacements to a column of independent global displacements:

$$\{\mathbf{q}_1 \; \mathbf{q}_2 \; . \; . \; . \; \mathbf{q}_N\} = \mathbf{J}\mathbf{q}^* \tag{14}$$

where $\mathbf{J}$ includes the effect of transferring from local coordinates for individual elements to the global coordinates. When the coordinate transformation is not required, $\mathbf{J}$ is a Boolean matrix. The expression for Π_p can thus be written as

$$\Pi_p = \tfrac{1}{2}\mathbf{q}^{*T}\mathbf{K}\mathbf{q}^* - \mathbf{q}^{*T}\overline{\mathbf{Q}}^* + \sum_n C_n \tag{15}$$

where

and

$$\mathbf{K} = \mathbf{J}^T[\mathbf{k}_1\mathbf{k}_2 \ldots \mathbf{k}_N]\mathbf{J}$$
$$\overline{\mathbf{Q}}^* = \mathbf{J}^T\{\overline{\mathbf{Q}}_1\overline{\mathbf{Q}}_2 \ldots \overline{\mathbf{Q}}_N\}\mathbf{J} \tag{16}$$

are respectively the stiffness matrix of the assembled structure and the column matrix of applied generalized nodal forces consistent with the assumed displacement functions. Both the element stiffness matrix $\mathbf{k}$ and the assembled stiffness matrix $\mathbf{K}$ are positive semi-definite. However, if some of the generalized displacements are prescribed so that the remaining part of the partitioned matrix is positive definite, the expression for Π_p is

$$\Pi_p = \tfrac{1}{2}[\mathbf{q}_1^{*T}\bar{\mathbf{q}}_2^{*T}]\begin{bmatrix}\mathbf{K}_{11} & \mathbf{K}_{12}\\ \mathbf{K}_{12} & \mathbf{K}_{22}\end{bmatrix}\begin{Bmatrix}\mathbf{q}_1^*\\ \bar{\mathbf{q}}_2^*\end{Bmatrix} - [\mathbf{q}_1^*\bar{\mathbf{q}}_2^*]\begin{Bmatrix}\overline{\mathbf{Q}}_1^*\\ \mathbf{Q}_2^*\end{Bmatrix} \tag{17}$$

where $\mathbf{q}_1^*$ is unknown, $\bar{\mathbf{q}}_2^*$ is prescribed and $\overline{\mathbf{Q}}_1^*$ is presented, $\mathbf{Q}_2^*$ is unknown. Then the application of the minimum principle $\delta\Pi_p = 0$ will yield

$$\mathbf{K}_{11}\mathbf{q}_1^* = \mathbf{Q}_1^* = \mathbf{K}_{12}\bar{\mathbf{q}}_2^* \tag{18}$$

The minimum number of displacements which must be constrained is equal to the number of rigid body degrees of freedom of the structure.

The process of eliminating the additional generalized displacements which do not affect the compatibility with neighbouring elements is commonly referred to as the static condensation process. By the nature of the variational principle used, the assumed displacements do not satisfy the internal equilibrium equations. The effect of the additional displacement modes is to improve the satisfaction of these equations in the interior of the individual elements. However, the satisfaction of the equilibrium conditions along the interelement boundary will not be improved if the boundary displacement functions remain unimproved.

PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY-EQUILIBRIUM MODEL AND ASSUMED STRESS HYBRID MODEL

Both the equilibrium model and the assumed stress hybrid models for finite element analysis can be derived from the principle of minimum complementary energy, for which the functional to be varied is

$$\Pi_c = \int_V \tfrac{1}{2}C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}V - \int_{S_u} T_i\bar{u}_i\,\mathrm{d}S \tag{19}$$

In the above, C_{ijkl} is the elastic compliance tensor. The stress tensor satisfies the equilibrium conditions

$$\sigma_{ij,j} + \bar{F}_i = 0 \tag{20}$$

and is compatible with the prescribed boundary tractions over S_σ, and along the boundary S_u the displacements $\bar{u}_i$ are prescribed.

In applying the finite element method, the assumed stress field need not be continuous across the interelement boundaries, but equilibrium must be maintained for the surface tractions T_i, defined by

$$T_i = \sigma_{ij}\nu_j$$

where ν_j are the components of the unit vector normal to the boundary. Let two neighbouring elements 'a' and 'b' be isolated (Figure 2), and consider the boundary traction components $T_i^{(a)}(s)$ and $T_i^{(b)}(s)$ ($i = 1,2,3$) over the respective sides of the common boundary AB. (For simplicity, a plane stress example is used for illustration).

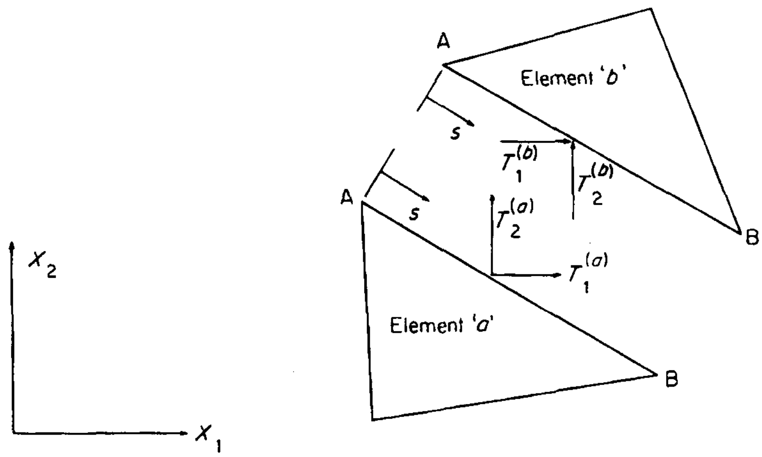

Figure 2. Equilibrium of boundary tractions along interelement boundary

The equilibrium conditions at the interelement boundary are given by

$$T_i^{(a)}(s) + T_i^{(b)}(s) = 0 \qquad i = 1,2,3 \tag{21}$$

Equations (21) may be considered as conditions of constraint and can be introduced by including Lagrange multiplier terms:

$$\int_{AB} \lambda_i(s)[T_i^{(a)}(s) + T_i^{(b)}(s)]\,\mathrm{d}s$$

or (22)

$$\int_{AB} \lambda_i T_i \,\mathrm{d}s\Big|_{\text{'a'}} + \int_{AB} \lambda_i T_i \,\mathrm{d}s\Big|_{\text{'b'}}$$

in the complementary energy functional to be varied. The Lagrange multipliers λ_i, which are functions of the surface coordinates, are to be treated as additional variables.

When all the interelement boundaries have been considered, the complementary energy functional may be written as:

$$\Pi = \sum_n \left(\int_{V_n} \tfrac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}V - \int_{S_n} \lambda_i T_i \,\mathrm{d}S - \int_{S_{u_n}} T_i \bar{u}_i \,\mathrm{d}S \right) \tag{23}$$

where S_{u_n} is the boundary of V_n where the displacements are prescribed and S_n is the interelement boundary of V_n. By taking the variation of Π with respect to σ_{ij} and λ_i, it is easily shown that the Lagrange multipliers λ_i are equal to u_i, the displacements along the interelement boundary, i.e.,

$$\Pi = \sum_n \left(\int_{V_n} \tfrac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl}\,\mathrm{d}V - \int_{S_n} T_i u_i \,\mathrm{d}S - \int_{S_{u_n}} T_i \bar{u}_i \,\mathrm{d}S \right) \tag{24}$$

In the finite element solution, the assumed approximate functions for the stresses σ_{ij} are divided into two parts. The first part, which consists of a finite number of parameters β, should satisfy the homogeneous equations of equilibrium, while the second part is a particular solution of the equations of equilibrium with the prescribed body forces. In matrix form, the stresses σ_{ij} are expressed as

$$\boldsymbol{\sigma} = \mathbf{P}\boldsymbol{\beta} + \mathbf{P}_F\boldsymbol{\beta}_F \tag{25}$$

where $\boldsymbol{\beta}$ is unknown and $\mathbf{P}_F\ \boldsymbol{\beta}_F$ is determined from the particular solution. For elements which contain boundaries with prescribed surface tractions, some of the β's in the first term will also be prescribed. In this case, all the prescribed β's are put in the second term.

Both the equilibrium and the hybrid methods can be derived from equation (24), but the distinction lies in the treatment of the surface integrals along the interelement boundaries. For the equilibrium model, the surface tractions T_i along each boundary are represented uniquely by the generalized loads $\mathbf{Q}$ pertaining to the boundary and can be written as

$$\mathbf{T} = \boldsymbol{\phi}\mathbf{Q} \tag{26}$$

For example, when $\mathbf{Q}$ are the values of T_i at a finite number of boundary points, $\boldsymbol{\phi}$ represents the corresponding interpolation functions. Since the surface tractions are also related to the assumed stress distribution, they can also be expressed as

$$\mathbf{T} = \mathbf{R}\boldsymbol{\beta} + \mathbf{R}_F\boldsymbol{\beta}_F \tag{27}$$

Thus, there is a unique relation between $\mathbf{Q}$ and $\boldsymbol{\beta}$ of the form

$$\mathbf{Q} = \mathbf{G}^T\boldsymbol{\beta} + \mathbf{G}_F^T\boldsymbol{\beta}_F \tag{28}$$

The corresponding element generalized displacements $\mathbf{q}$ are defined by

$$\mathbf{Q}^T\mathbf{q} = \int_{\partial V_n} T_i u_i \,\mathrm{d}S = \mathbf{Q}^T\int_{\partial V_n} \boldsymbol{\phi}^T\mathbf{u}_B \,\mathrm{d}S \tag{29}$$

or

$$\mathbf{q} = \int_{\partial V_n} \boldsymbol{\phi}^T\mathbf{u}_B \,\mathrm{d}S \tag{30}$$

where

$$\partial V_n = S_n + S_{\sigma_n} + S_{u_n}$$

is the entire boundery of V_n. Thus, the generalized displacement is a weighted integral of the boundary displacements, and to maintain the same $\mathbf{q}$ for two neighbouring elements does not guarantee compatibility along the entire boundary. It is seen that, in an equilibrium model, each generalized displacement $\mathbf{q}$ is common to only two neighbouring elements.

For the hybrid method, the approximate displacements along the interelement boundaries are represented by interpolation functions and the generalized displacements $\mathbf{q}$ at a finite number of boundary nodes

$$\mathbf{u}_B = \mathbf{L}\mathbf{q} \tag{31}$$

Since the interpolating functions $\mathbf{L}$ are applied to the individual boundary segments, they are relatively easy to construct so that interelement compatibility is maintained. Unlike the equilibrium model, the generalized displacements $\mathbf{q}$ may be referred to corner nodes where more than two elements are connected.

The corresponding generalized nodal forces are defined again by

$$\mathbf{q}^T\mathbf{Q} = \int_{\partial V_n} T_i u_i \, dS \tag{32}$$

or

$$\mathbf{Q} = \int_{\partial V_n} \mathbf{L}^T\mathbf{T} \, dS \tag{33}$$

Thus, the generalized nodal force is a weighted integral of the boundary tractions, and to maintain equilibrium at a node does not guarantee equilibrium over the entire boundary. Another significant difference from the equilibrium model is that the stresses and the boundary displacements are independently assumed. Hence, the number of parameters in $\boldsymbol{\beta}$ and the number of generalized displacements $\mathbf{q}$ can be chosen independently in the hybrid model.

Finally, since the hybrid method relies on assumed boundary displacements, the prescribed boundary stresses no longer constitute a restrained boundary condition. In this case the functional Π can be written more conveniently as

$$\Pi = \sum_n \left(\int_{V_n} \tfrac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl} \, dV - \int_{\partial V_n} T_i u_i \, dS + \int_{S_{\sigma_n}} \bar{T}_i u_i \, dS \right) \tag{34}$$

where

$$\partial V_n = S_n + S\sigma_n + S_{u_n}$$

is the entire boundary of V_n, and $u_i = \bar{u}_i$ on S_{u_n}.

Substitution of equations (25), (28) and (29) into (24) and substitution of equations (25), (27) and (31) into (34) both yield equations of the form

$$\Pi = \sum_n (\tfrac{1}{2}\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{H}_F\boldsymbol{\beta}_F - \boldsymbol{\beta}^T\mathbf{G}\mathbf{q} + \mathbf{S}^T\mathbf{q} + B_n) \tag{35}$$

where

$$\begin{aligned} \mathbf{H} &= \int_{V_n} \mathbf{P}^T\mathbf{C}\mathbf{P} \, dV \\ \mathbf{H}_F &= \int_{V_n} \mathbf{P}^T\mathbf{C}\mathbf{P}_F \, dV \\ \mathbf{B}_n &= \tfrac{1}{2}\boldsymbol{\beta}_F^T \int_{V_n} \mathbf{P}_F^T\mathbf{C}\mathbf{P}_F \, dV \boldsymbol{\beta}_F \end{aligned} \tag{36}$$

and $\mathbf{C}$ is the elastic compliance matrix. For the equilibrium model, $\mathbf{G}$ and $\mathbf{G}_F$ are defined by equation (28). For the hybrid model they are given by

$$\mathbf{G} = \int_{\partial V_n} \mathbf{R}\mathbf{L} \, dS; \qquad \mathbf{G}_F = \int_{\partial V_n} \mathbf{R}_F^T\mathbf{L} \, dS \tag{37}$$

The vector $\mathbf{S}^T$ is also different for the two models

$$\begin{aligned} \mathbf{S}^T &= -\boldsymbol{\beta}_F^T\mathbf{G}_F \qquad \text{(equilibrium model)} \\ \mathbf{S}^T &= -\boldsymbol{\beta}_F^T\mathbf{G}_F + \int_{S_{\sigma_n}} \bar{\mathbf{T}}^T\mathbf{L} \, dS \qquad \text{(hybrid)} \end{aligned} \tag{38}$$

where $\bar{\mathbf{T}}$ are the prescribed boundary tractions.

The stationary conditions of the functional given by equation (35) with respect to variations of $\boldsymbol{\beta}$ and $\mathbf{q}$ then yield

$$\mathbf{H}\boldsymbol{\beta} + \mathbf{H}_F\boldsymbol{\beta}_F - \mathbf{G}\mathbf{q} = 0 \tag{39}$$

and

$$\sum_n (\boldsymbol{\beta}^T\mathbf{G} - \mathbf{S}^T)\delta\mathbf{q} = 0 \tag{40}$$

By solving for $\boldsymbol{\beta}$ from equation (39) and substituting back into equation (35), we can express the functional Π in terms of the generalized displacements $\mathbf{q}$ only, i.e.,

$$\Pi = -\sum_n (\tfrac{1}{2}\mathbf{q}^T\mathbf{k}\mathbf{q} - \bar{\mathbf{Q}}^T\mathbf{q} + C_n) \tag{41}$$

where

$$\begin{aligned} \mathbf{k} &= \mathbf{G}^T\mathbf{H}^{-1}\mathbf{G} = \text{element stiffness matrix} \\ \bar{\mathbf{Q}} &= \mathbf{G}^T\mathbf{H}^{-1}\mathbf{H}_F\boldsymbol{\beta}_F + \mathbf{S} \end{aligned} \tag{42}$$

and

$$C_n = \tfrac{1}{2}\boldsymbol{\beta}_F^T\mathbf{H}_F^T\mathbf{H}^{-1}\mathbf{H}_F\boldsymbol{\beta}_F - B_n$$

Equation (41) is of the same form as equation (12). Hence, it also leads to the same matrix equation as given by equation (18).

Inspection of equation (40) reveals that if the total number of assumed stress modes $\boldsymbol{\beta}$ of all elements is smaller than the total number of unknown displacements $\mathbf{q}$, there will be, in general, no solution for the β's. Even if there is a solution for the β's in equation (40), the unknown q's cannot be determined uniquely from equation (39) since in such cases the number of unknown q's in at least some of the elements is necessarily greater than the number of β's. Let M be the total number of stress modes and N, the total number of generalized displacements, of which at least l must be constrained, l being the number of rigid body degrees of freedom of the structure. Then the requirement for the assumed stress modes is $M \geqslant N - l$. It is obvious that such requirements on the minimum number of assumed stress modes can always be met by the hybrid model because the stress modes and boundary displacements are assumed independently. However, this requirement creates the chief difficulty of some of the equilibrium models, for which the number of β's and the number of q's for a given element are uniquely related.

It can be shown[18] that the requirement stated above is to avoid any kinematical deformation modes. Indeed, with the equilibrium model an individual element may contain kinematic deformation modes itself and hence must be sufficiently constrained in order to prevent kinematical motion.

A remark should be made about the stress distribution corresponding to the prescribed body forces. Tong and Pian[14] have shown that when the stress modes are so chosen that the polynomials in $\mathbf{P}\boldsymbol{\beta}$ are complete polynomials of at least the same degree as those in $\mathbf{P}_F\boldsymbol{\beta}_F$, then the finite element solutions will be independent of the forming of the particular solutions.

HYBRID METHOD BASED ON ASSUMED DISPLACEMENT FIELD

In applying the principle of minimum potential energy to finite element analysis, we may assume the displacement functions independently for each individual element, while the interelement compatibility conditions may be introduced by including Lagrange multiplier terms in the functional of equation (2), i.e.,

$$\Pi = \sum_n \left[\int_{V_n} (\tfrac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \bar{F}_i u_i)\,dV - \int_{S_n} \lambda_i u_i\,dS - \int_{S_{\sigma_n}} \bar{T}_i u_i\,dS \right] \tag{43}$$

Obviously, the Lagrange multipliers λ_i here are the interelement boundary tractions T_i and a variational functional which is the dual of equation (34) can be written as

$$\Pi = \sum_n \left[\int_{V_n} (\tfrac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \bar{F}_i u_i)\,dV - \int_{\partial V_n} T_i u_i\,dS + \int_{S_{u_n}} T_i \bar{u}_i\,dS \right] \tag{44}$$

where $Ti = \bar{T}i$ on S_{σ_n}.

In the finite element analysis, if the assumed displacement fields in the individual elements do not guarantee compatibility at the interelement boundaries, it is sufficient to represent the Lagrange multipliers λ_i (i.e., the boundary tractions T_i) approximately in terms of a finite number of parameters which may be interpreted as the generalized nodal forces $\mathbf{Q}$:

$$\mathbf{T} = \mathbf{\Phi Q} \tag{45}$$

The corresponding generalized nodal displacements are defined by

$$\mathbf{Q}^T\mathbf{q} = \int_{\partial V_n} T_i u_i\,dS \tag{46}$$

or

$$\mathbf{q} = \int_{\partial V_n} \mathbf{\Phi u}_B\,dS \tag{47}$$

Thus, the generalized nodal displacement is a weighted integral of the boundary displacements, and to maintain compatibility of the generalized displacements does not guarantee compatibility of the entire boundary. Similar to the other hybrid method, the displacement field and the boundary tractions are independently assumed. Hence, the number of parameters in $\mathbf{u}$ and the number of generalized forces $\mathbf{Q}$ can be chosen independently.

Several remarks should be made on the present hybrid model.

1. When the assumed displacement functions in the interior of each element include all the rigid body modes (which do not enter into the strain energy expression) the corresponding generalized forces around an element will be related by the static equations of equilibrium.

2. Similar to the formulation of the assumed stress hybrid model we can conclude that the total number of displacement modes of all elements should be larger than the total number of interelement generalized forces (redundant forces). The physical interpretation is that the inadequacy in the degree of freedom in displacements means the overconstraining of the displacement.

3. The final matrix equations can be set up with redundant boundary forces as unknowns. Like the matrix force method the choice of redundant forces should be such that the matrix equations are not ill conditioned.

REISSNER'S VARIATIONAL PRINCIPLE—MIXED METHOD

In the principle of minimum potential energy, the functional Π_p is in terms of the displacement field u_i; in the principle of minimum complementary energy, the functional Π_c is in terms of the stress field σ_{ij}. In Reissner's variational principle, the functional Π_R is in terms of both u_i and σ_{ij}. Reissner's Principle states the stationary property of Π_R where

$$\Pi_R = \int_V [-B(\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i}) - \bar{F}_i u_i]\,\mathrm{d}V - \int_{S_\sigma} \bar{T}_i u_i \mathrm{d}S - \int_{S_u} T_i(u_i - \bar{u}_i)\mathrm{d}S \quad (48)$$

where $B(\sigma_{ij})$ is the complementary energy density, but it is not a minimum principle. It can be derived by applying appropriate conditions of constraint (by means of Lagrange multiplier) to either the potential energy principle or the complementary energy principle.[1]*

In the finite element analysis, stress and displacement are assumed separately for each individual element. The question is what are the continuity requirements for the stresses and the displacements at the interelement boundaries so that the functional Π_R is defined. For simplicity, let us consider a two-dimensional problem, and choose a local coordinate system N-s denoting the normal and tangential directions of an interelement boundary. We only have to study the second terms of the first integral of equation (48), since it is the only term involving differentiation. This term can be written as

$$\int [\sigma_N u_{N,N} + \sigma_{Ns}(u_{N,s} + u_{s,N}) + \sigma_s u_{s,s}]\,\mathrm{d}A$$

If the displacements have discontinuity across the boundary, the differentiation along the normal direction will yield a delta function. But the above integral is still defined if the stresses are continuous across the boundary. However, if the displacement is continuous, the above integral will be defined even if the stresses are discontinuous. Therefore, for the two-dimensional problem there are four possible requirements for the interelement compatibility:

1. Both normal and shear stress;
2. The normal stress and the tangential displacement;
3. The normal displacement and the tangential stress; and
4. Both normal and tangential displacements are continuous along each of the interelement boundaries.

Similarly, we can find that there are eight kinds of possible requirements of continuity for a three dimensional problem (because there are two shear stresses and two tangential displacements). Based on different requirements, equation (48) can be written in different forms which are all appropriate for the application of the finite element analysis. We can write

$$\Pi_R = \sum_n \left\{ \int_{V_n} [-B(\sigma_{ij}) + \tfrac{1}{2}\sigma_{ij}(u_{i,j} + u_{j,i}) - \bar{F}_i u_i]\,\mathrm{d}V - A_n - \int_{S_{\sigma_n}} \bar{T}_i u_i \,\mathrm{d}S - \int_{S_{u_n}} T_i(u_i - u_i)\,\mathrm{d}S \right\} \quad (49)$$

* There are other examples in the development of one variational principle from the other using the Lagrange multiplier technique. For incompressible or nearly incompressible material, the finite element method by either the minimum potential or the minimum complementary energy principle cannot be used. Herrmann[11] has suggested a variational principle which can be used for these special situations and which contains the displacements and the mean stress as the primary variables. Tong[19] has developed another variational principle based on the assumed stress approach to handle the same situations.

in which, the term A_n arises from possible jump function of the derivatives of u_i across the interelement boundaries. For example,

$$A_n = \int_{\partial V_n} T_i u_i \,\mathrm{d}S \tag{50}$$

where $T_i = \sigma_{ij} \nu_j$, if only the normal and shear stresses are continuous;

$$A_n = \int_{\partial V_n} T_N u_N \,\mathrm{d}S \tag{51}$$

if only the normal stress T_N is continuous;

$$A_n = 0 \tag{52}$$

if both displacements are continuous; and

$$A_n = \int_{\partial V_n} (T_{s_1} u_{s_1} + T_{s_2} u_{s_2}) \,\mathrm{d}S$$

if both shear stresses and the normal displacement are continuous.

When the energy functional of equations (49) and (52) is used in the finite element formulation, the stresses σ_{ij}, which need not satisfy the equilibrium condition is expressed as

$$\boldsymbol{\sigma} = \mathbf{P}\boldsymbol{\beta} \tag{53}$$

in an individual element, and the displacements, by interpolation functions and nodal generalized coordinates as

$$\mathbf{u} = \mathbf{A}\mathbf{q} \tag{54}$$

If the boundary displacements can be made to satisfy the prescribed values, equation (49) may be expressed in the form

$$\Pi_R = \sum_n (-\tfrac{1}{2}\boldsymbol{\beta}^T\mathbf{H}\boldsymbol{\beta} + \boldsymbol{\beta}^T\mathbf{G}\mathbf{q} - S^T\mathbf{q}) \tag{55}$$

where

$$\mathbf{H} = \int_{V_n} \mathbf{P}^T\mathbf{C}\mathbf{P}\,\mathrm{d}V$$

$$\mathbf{G} = \int_{V_n} \mathbf{P}^T\mathbf{A}\,\mathrm{d}V \tag{56}$$

$$\mathbf{S}^T = \int_{V_n} \bar{\mathbf{F}}\mathbf{A}\,\mathrm{d}V + \int_{S_{\sigma n}} \bar{\mathbf{T}}\mathbf{A}_B\,\mathrm{d}S$$

It is seen that equation (55) is of the same form as equation (35) which is associated with the assumed stress hybrid model. Thus, the condition will lead to the same matrix equation as given by equation (18). The application of the Reissner's principle in this form, however, does not provide particular advantage over the compatible model since the interelement boundary compatibility is still required.

It is obvious that the application of the Reissner's principle in the form of equations (49) and (50) will lead to equations similar to that of the assumed displacement hybrid method. As a matter of fact, in the use of equations (49) and (50) for finite element analysis, conditions on the continuity of stresses can be further relaxed by assuming the stresses σ_{ij} in the interior of each element and T_i on the interelement boundary independently. In carrying out the variation with respect to both σ_{ij} and T_i it automatically gives the relation between $\sigma_{ij}\nu_j$ and T_i at the interelement boundary (of course, only approximately in finite element analysis). Then this formulation does bear some resemblence to that given by Jones.[7]

The use of the Reissner's principle, however, has its merit in the finite element analysis of plates and shells.[12,16,20] In the case of plate bending, for example, the energy integral Π_R takes the form

$$\begin{aligned}\Pi_R = &\int_A [-B(M^{\alpha\beta}) - M^{\alpha\beta} w_{,\alpha\beta} - pw]\,\mathrm{d}A \\ &+ \int_{S_\sigma} (\overline{M}_n w_{,n} + \overline{M}_{ns} w_{,s} - \overline{Q}_n w)\,\mathrm{d}s \\ &+ \int_{S_u} [M_n(w_{,n} - \overline{w}_{,n}) + M_{ns}(w_{,s} - \overline{w}_{,s}) - Q_n(w - \overline{w})]\,\mathrm{d}s\end{aligned} \tag{57}$$

where the Greek index takes the value of 1 or 2. The sign convention for the displacements and stresses is shown in Figure 3.

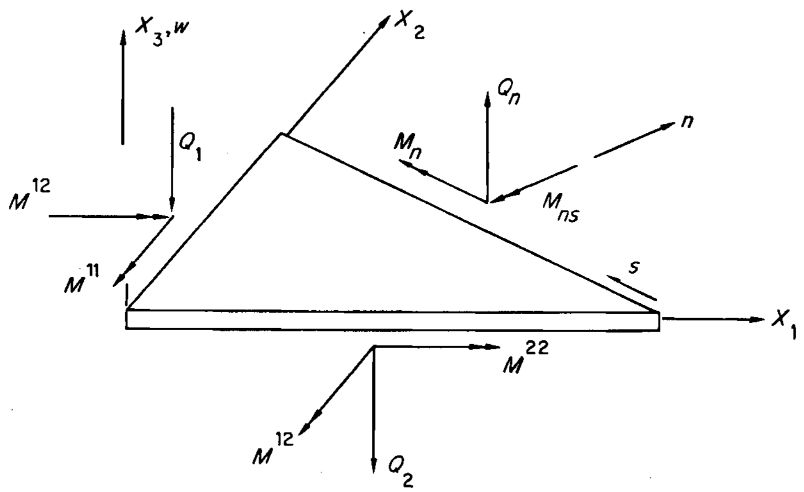

Figure 3. Sign convention for plate bending problem

We want to know what are the continuity requirements for the moments and the displacements at the interelement boundaries. Similar to our previous treatment of three dimensional solid we also only have to consider the second term in the first integral here. Choosing the local *n-s* coordinate system at an interelement boundary, this term can be written as

$$-\int_A [M_n w_{,nn} + M_{ns} w_{,ns} + M_s w_{,ss}]\,\mathrm{d}A$$

We see that only a differentiation along the normal direction may create a delta function (because in the finite element analysis smooth function is used for w within each element), hence, the above integral is definable if any of the following sets of conditions is satisfied:

1. Continuity of w and $w_{,n}$
2. Continuity of w and M_n
3. Continuity of M_{ns}, M_n and the derivatives of M_n.

In the first case, the functional form of equation (57) can be applied directly to the finite element analysis, i.e., an integral over the entire area in equation (57) is equal to the sum of the integrals over each individual element. In the second case, because of the allowance of discontinuity of w, n, Π_R can be written as

$$\Pi_R = \sum_m \left\{ \int_{A_m} [-B(M^{\alpha\beta}) - M^{\alpha\beta}w_{,\alpha\beta} - pw]\,dA + \int_{S_m} M_n w_{,n}\,ds \right.$$

$$+ \int_{S_{\sigma_m}} (\overline{M}_n w_{,n} + \overline{M}_{ns} w_{,s} - \overline{Q}_n w)\,ds \tag{58}$$

$$\left. + \int_{S_{u_m}} [M_n(w_{,n} - \overline{w}_{,n}) + M_{ns}(w_{,s} - \overline{w}_{,s}) - Q_n(w - \overline{w})]\,ds \right\}$$

In case (3), because of the allowance of the discontinuity of w itself across the interelement boundaries, by using the properties of a delta function and its derivative, equation (57) can be written as

$$\Pi_R = \sum_m \left\{ \int_{A_m} [-B(M^{\alpha\beta}) - M^{\alpha\beta}w_{,\alpha\beta} - pw]\,dA \right.$$

$$+ \int_{S_m} (M_n w_{,n} + M_{ns} w_{,s})\,ds - \int_{S_m} (M_{n,n} + M_{ns,s})w\,ds \tag{59}$$

$$+ \int_{S_{\sigma_m}} (\overline{M}_n w_{,n} + \overline{M}_{ns} w_{,s} - \overline{Q}_n w)\,ds$$

$$\left. + \int_{S_{u_m}} [M_n(w_{,n} - \overline{w}_{,n}) + M_{ns}(w_{,s} - \overline{w}_{,s}) - Q_n(w - \overline{w})]\,ds \right\}$$

In the application to finite element method case 2 has special advantage over the strictly displacement method. Here only the continuity of w and M_n are required over interelement boundaries, thus the interpolation function can be easily constructed. In particular, w can be assumed as linear function along an interelement boundary so it can have the same form as that of the in-plane displacements. This will be most convenient when the problem under investigation consists of plates joining at angles.

Herrmann's papers[12,20] on finite element analyses for plate bending problems involves a different version of the energy functional, when equation (58) is integrated by parts, we obtain

$$\Pi_R = \sum_m \left\{ \int_{A_m} [-B(M^{\alpha\beta}) + w_{,\alpha} M^{\alpha\beta}{}_{,\beta} - pw] \, \mathrm{d}A - \int_{S_m} M_{ns} w_{,s} \, \mathrm{d}s \right. \tag{60}$$

$$+ \int_{S_{\sigma_m}} [(\overline{M}_n - M_n) w_{,n} + (\overline{M}_{ns} - M_{ns}) w_{,s} - \overline{Q}_n w] \, \mathrm{d}s$$

$$\left. - \int_{S_{u_m}} [M_n \overline{w}_{,n} + M_{ns} \overline{w}_{,s} - Q_n (w - \overline{w})] \, \mathrm{d}s \right\}$$

Herrmann's second plate bending paper[20] makes use of the above equations with further requirements that the prescribed normal moment and transverse deflection boundary conditions are satisfied while in his first paper[12] all the assumed stress couples $M^{\alpha\beta}$ are continuous along the interelement boundary, hence the second integral of equation (60) does not appear. In this case the required conditions are, of course, more than is necessary.

It should be pointed out, finally, that both cases (2) and (3) can be modified by relaxing the requirements on M_n and on M_{ns}, M_n and $M_{n,n}$ at the interelement boundary respectively such that they are independent of the assumed $M^{\alpha\beta}$ in the interior of the individual elements. The relations between $M^{\alpha\beta}$ and the boundary moments are, of course, derived by the variational process.

In the finite element analysis of shells there exist even more versions by the Reissner's variational principle. For example, in a special formulation the stress resultants $N^{\alpha\beta}$ can be avoided and the three components of displacements u, v and w, and the three stress couples $M^{\alpha\beta}$ are used as the primary variables. One admissible interelement boundary condition is the continuity in u, v, w and in the normal moment M_n.[16]

PLATE BENDING ANALYSIS

The plate bending problem is the only problem that has been analysed by most of the types of finite element methods outlined above. Thus, it is chosen for the illustrative examples.

Compatible models

The earliest attempts in finite element analysis for plate bending problems involved only rectangular elements[6] and the solutions obtained by the assumed displacement method were found to converge to the exact solutions even though the displacement functions for each element did not completely satisfy the interelement boundary compatibility conditions. However, when triangular plate elements were tried[21] it became evident that the non-compatible finite element solution could lead to quite erroneous results. Thus a number of investigators have developed displacement compatible models for plate analysis. For rectangular elements Bogner, Fox and Schmit[22] presented solutions using double 4th degree Hermite interpolation polynomials for elements with 16 degree of freedom, i.e. with w, $w_{,x}$, $w_{,y}$ and $w_{,xy}$ at each corner as generalized coordinates. In such a case, the displacement w and the normal slope $w_{,n}$ all vary as cubic functions along each edge, hence the interelement compatibility is satisfied. The paper also includes the formulation of 36 degree of freedom rectangular element with 9 generalized coordinates at each corner. Deák and Pian[23] used a interpolation function which is defined differently in four subregions and formulated a compatible rectangular element with 12 degree of freedom, i.e. with w, $w_{,x}$ $w_{,y}$ and $w_{,xy}$ at each corner as generalized coordinates. In this case, the displacement w varies as a cubic function while the normal slope $w_{,n}$ varies linearly along each edge in order to satisfy the interelement compatibility condition. For triangular elements, Clough and Tocher[21] obtained a compatible interpolation function which has three generalized coordinates at each vertex. For the interelement compatibility the displacement w must vary again as a cubic function and the normal slope $w_{,n}$ a linear function, along each edge. It turns out that such interpolation function must also be defined differently in three subregions. Bazeley and others[24] also achieved the formulation of a compatibile triangular element by applying certain constraining condition to a formulation which violates the normal slope compatibility. Finally, Fraeijs de Veubeke[25,26] suggested a compatible quadrilateral element which has 16 degree of freedom, with the transverse displacement and two-rotations and each corner and the normal rotation at the mid-point of each edge as generalized coordinates.

Assumed Stress Hybrid Model

The assumed stress hybrid model has been used by Pian[27] for rectangular plate element and by Severn and Taylor[28] for both rectangular and right triangular plate elements. For the hybrid model, the stress distribution $\boldsymbol{\sigma}$ is assumed inside each element while the displacement function $\mathbf{u}_B$ is only needed along the boundary. In plate bending analysis, the displacements to be specified along each edge are the lateral displacement w and the normal slope $w_{,n}$. Since these displacements are functions of the boundary coordinates s only, they can be expressed easily in terms of one-dimensional Hermite interpolation functions. In formulations in both references 27 and 28, three generalized coordinates are used at each corner of the elements, hence, w varies as a cubic-function while $w_{,n}$ as a linear function along each edge. In a recent study[29] a 16 degree of freedom rectangular element has been used and 4th order Hermite interpolation functions are used for both w and $w_{,n}$ along each edge.

The assumed equilibrating stress modes can be expressed most conveniently in terms of polynomials with undetermined parameters $\boldsymbol{\beta}$. Figure 4 is a sketch of the deformed rectangular element with its edges lining up with the x and y axes. The stresses to be assumed are the bending moments M_x, M_y and M_{xy}. The transverse shears Q_x and Q_y must be derived from the following equilibrium equations:

$$\begin{aligned} M_{x,x} + M_{xy,y} &= Q_x \\ M_{xy,x} + M_{y,y} &= Q_y \\ Q_{x,x} + Q_{y,y} &= -p \end{aligned}$$

where $p(x,y)$ is the intensity of the distributed lateral loading. The first term $\mathbf{P}\boldsymbol{\beta}$ in the assumed stress function, equation (25), concerns the homogeneous equilibrium equations. When all linear terms for the moments are included, there are 9 β's and if all quadratic and cubic terms are included there are 17 β's and 27 β's respectively. In reference 28, quadratic functions for moments were used while in references 27 and 29, the effect of different degrees of stress approximation were investigated.

Equilibrium Model

The use of the equilibrium model for plate bending problems will be illustrated by the triangular element shown in Figure 5. In the analysis by Fraeijs de Veubeke and Sander[30,31] the distributions of bending and twisting moments are assumed to be linear. The boundary forces that must be in equilibrium at the interelement boundary are the normal moment M_n and the equivalent transverse shear force $V_n = Q_n - M_{ns,s}$ where s is along the edge and M_{ns} is the twisting moment. In addition to these forces a corner load Z_i equal to $(M_{ns})^+_{,s_i} - (M_{ns})^-_{,s_i}$ must be introduced whenever the twisting moment M_{ns} is finite at the corner. In the present case we use 1, 2 and 3 to label the 3 vertices and the corresponding opposite sides of the triangle. Since the distributions in moments are linear while the distributions of transverse shearing forces are

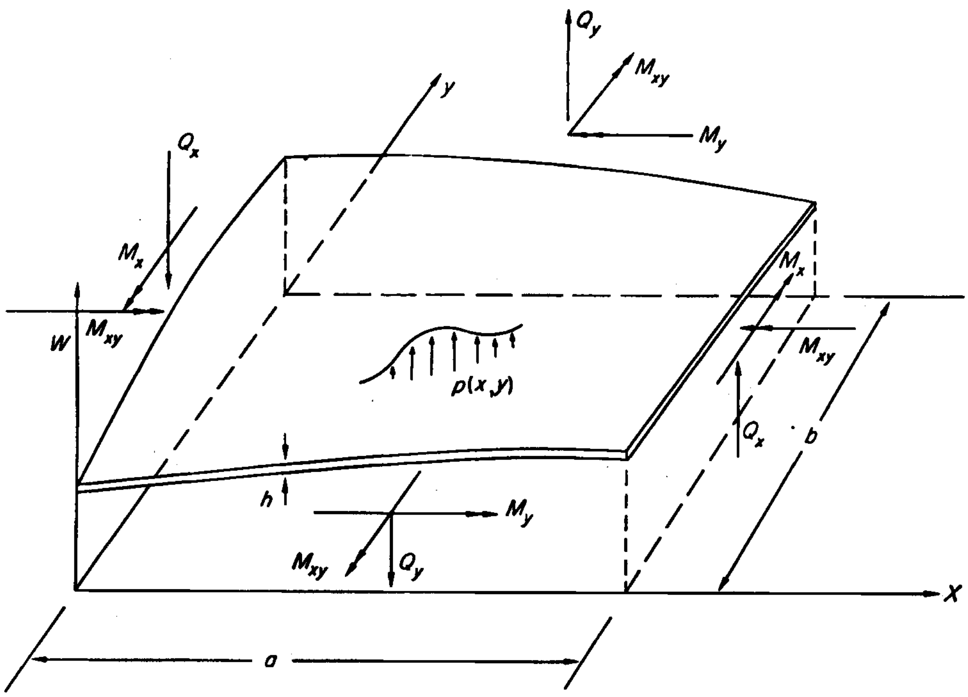

Figure 4. Boundary tractions of a rectangular plate element

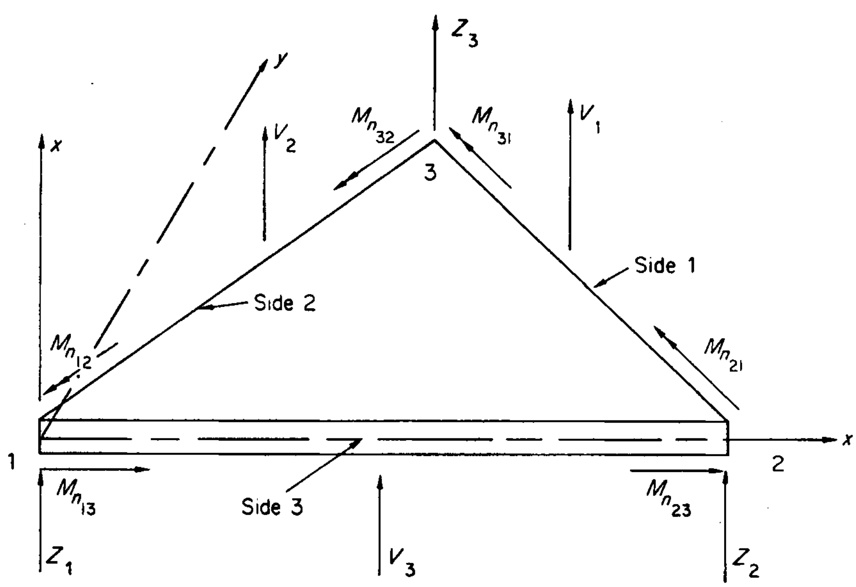

Figure 5. Generalized boundary forces of a triangular element

constant, the boundary forces can be represented by taking as generalized loads **Q**, the normal moments $M_{n_{ij}}$ and $M_{n_{ik}}$ at the two sides of the ith vertex, the corner force Z_i, and the equivalent and constant transverse shear V_i along the ith edge for i = 1, 2, 3. In this case, the total number of generalized forces **Q** is 12, while the total number of stress modes is 9. The difference is exactly the number of rigid body degrees of freedom. Hence, the element idealization corresponds to a kinematically stable system. It should be observed that if a quadrilateral is used, a constant moment distribution will not provide a sufficient number of β's to make the element kinematically stable.

The formulation of the equilibrium model by Fraeijs de Veubeke and Sander[30,31] is according to the section of the present paper dealing with the principle of minimum complementary energy-equilibrium model, with generalized displacements as unknown in the final matrix equations. Morley[32] has solved the plate bending problems by triangular equilibrium elements using the unknown stress resultants (values of stress functions) as unknowns. Morley's formulation is, in fact, based on the static and geometic analogy between the plate bending and the plate stretching problems, hence the plate bending problem by the equilibrium model can be set up in an analogous manner as the plane stress problem by the compatible model. By Morley's formulation the deflection of the plate can only be calculated by integrating the moment curvature relations. In view of the approximate character of the moments provided by the finite element analysis, the deflection obtained by integration is, in general, dependent upon the chosen integration path, hence, is not a unique solution. The dual finite element methods for stretching and bending of plates have also been discussed by Elias[36] and by Fraeijs de Veubeke and Zienkiewicz.[35]

Assumed Displacement Hybrid Model

In the initial investigation of the assumed displacement hybrid model Yamamoto[13] presented a simple example of bending of rectangular plate under uniformly distributed loading using rectangular element. The assumed modes contain the quadratic terms with eight independent parameters while the interelement forces are the transverse shear and the twisting and normal moments which are assumed to be constant along each edge. Yamamoto's paper contains the solution of plates of both clamped and simply supported boundaries using only one element for each quadrant of the plate. His solution for the centre deflection of a plate is shown later in this article.

Greene and others[33] have conducted a series of calculations for simply supported square plates under uniform loading using one square element for each quadrant of the plate but with different degrees of approximation for the interior displacements and the tractions* at the interelement boundary. Their results indicate that the degrees of approximation for the interior displacements and boundary tractions should be comparable, and solutions become ill conditioned when the number of degrees of freedom for the boundary tractions is over excessive. These conclusions are consistent with the conjunctures which have been made about this method.

Mixed Model by Reissner's Principle

Herrmann's plate bending analyses[12,20] have demonstrated the specific feature of the mixed model in finite element method. In both schemes Herrmann used triangular plate elements with linear distribution in w while in reference 12 the moments M_x, M_y and M_{xy} are assumed as linear

* In reference 33 Lagrangian multipliers are introduced to account only for discontinuities in w and $w_{,n}$ along the interelement boundary. These multipliers should be identified as boundary tractions.

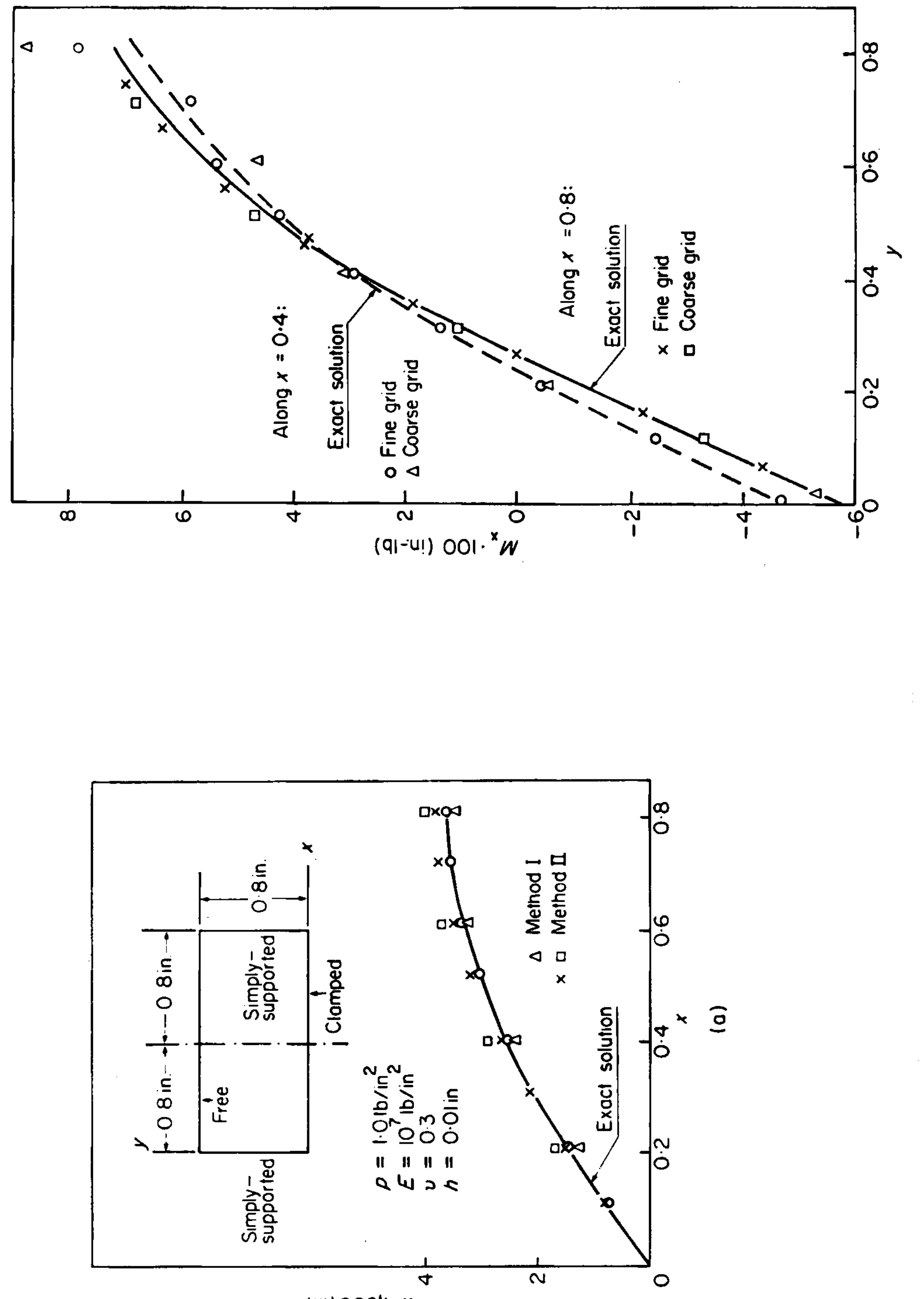

Figure 6. Finite element analyses of plate bending by Reissner's principle

and in reference 20 they are assumed as constant in each element. As has been pointed out before, the interelement boundary continuity conditions are more than sufficient in the first scheme, hence, the total number of unknowns in the first scheme is larger than that of the second. The results on a rectangular plate under uniform lateral loading are plotted in Figure 6. It is seen that the displacements obtained by the first formulation are more accurate while the stress results by the second formulation are superior.

RESULTS OF FINITE ELEMENT ANALYSIS OF BENDING OF SQUARE PLATES

The central deflection of a square plate with clamped edges under central loading has been analysed by different finite element methods using right-triangular elements. For comparison, the compatible model[21], the equilibrium model[30, 31] and the hybrid model[28] will be discussed. The results are plotted in Figure 7 in terms of the percentage error versus the number of meshes in half the plate width. The results clearly indicate the upper and lower bound solutions respectively by the equilibrium and compatible models and the intermediate values by the assumed

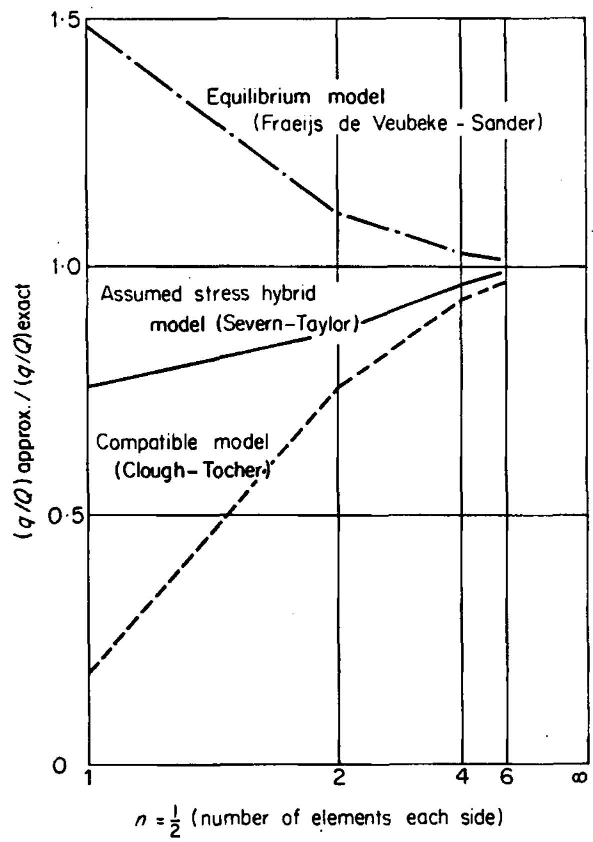

Figure 7. Centre deflection of clamped square plate under centre load (triangular elements)

stress hybrid model. It should be observed that the hybrid method provides a lower bound in this case and hence, provides more accurate solutions than the corresponding compatible model.

The second example, shown in Figure 8, is the central deflection of a simply supported square plate under central load, analysed by using 12 degrees of freedom rectangular elements. However, in this case, an equilibrium model will result in an element with kinematical deformation modes. Hence, only the results of the compatible and hybrid models are shown for comparison. The formulation of the hybrid method for rectangular plate elements[29] employed different moment approximations and the compatible model is the one used by Deák and Pian.[23] It is clearly indicated here that the hybrid method may yield either the upper or lower bound for the direct influence coefficient, and with the use of a sufficiently large number of stress modes it will yield a lower bound solution. The hybrid solutions include two different approaches: in one case the assumed stress distribution satisfies the condition of vanishing normal moments M_n along the simply supported edge; in the other the prescribed boundary stress condition is not considered as a restrained condition. As indicated in Figure 8, the first of these two approaches provides more accurate solutions, but the improvement becomes diminishingly small when a large number of elements is used in the analysis.

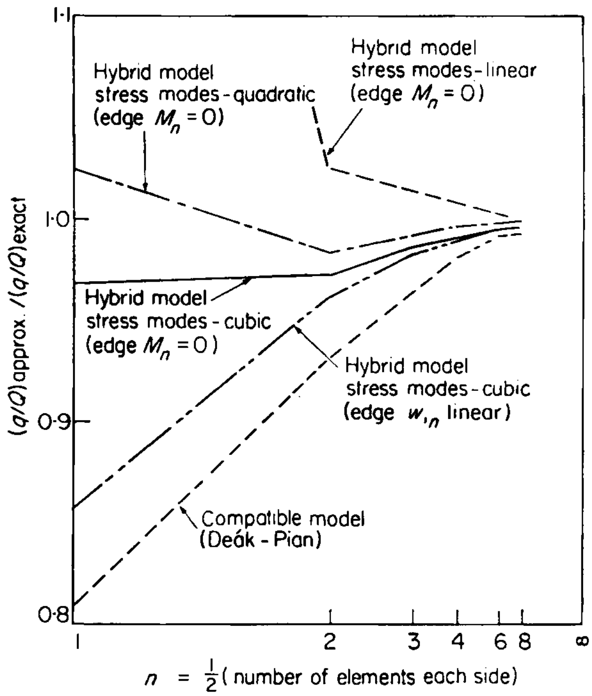

Figure 8. Centre deflection of simply supported square plate under centre load (rectangular elements)

The third example consists of a clamped-edge square plate under uniform loading. Figure 9 presents a comparison of the centre deflections obtained by the following methods:

1. Compatible model using triangular element by Clough and Tocher.[21]

2. Compatible model using triangular element by Bazeley and others.[24]

3. Compatible model using 16 degree of freedom rectangular elements by Bogner, Fox and Schmit.[22]

4. Assumed stress hybrid model using rectangular element.[29]

5. Equilibrium model by triangular elements using stresses as unknowns—deflections obtained by two integration paths by Morley.[32]

6. Assumed displacement hybrid model using rectangular elements by Yamamoto.[13]

Figure 10 presents a comparison of the stress distributions (M_x along the axes of symmetry $y = 0$) of a uniformly loaded square plate obtained by the various methods. It is seen that the assumed stress methods (equilibrium and hybrid models) are much more accurate methods in stress calculations.

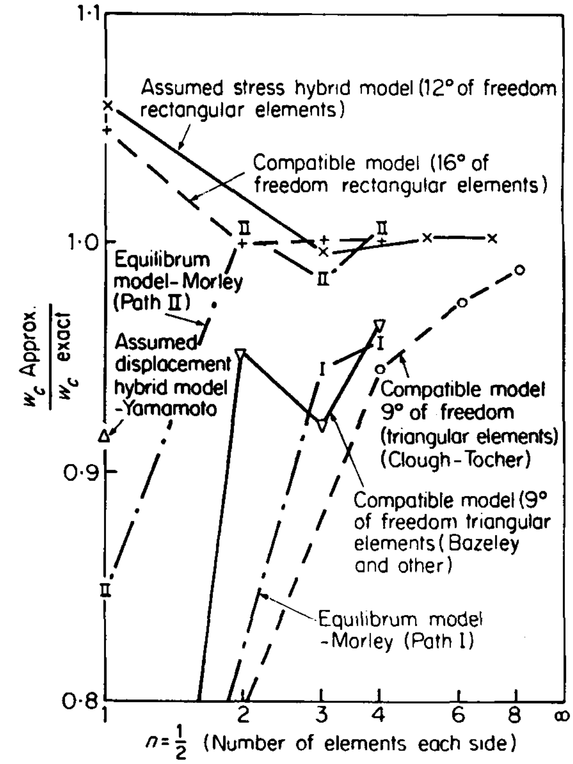

Figure 9. Centre deflection of clamped square plate under uniform loading

CONCLUSIONS

In addition to a brief discussion of the commonly used compatible displacement model in the finite element formulation in solid mechanics, the bases for the equilibrium model, the two hybrid models and the mixed model by Reissner's variational principle are presented. Descriptions of the features of the various formulations are summarized in Table I.*

Some of the existing solutions for plate bending finite element analyses are presented. From these isolated examples the following conclusions can be drawn:

1. The equilibrium and compatible models will provide, respectively, the upper and lower bounds for the strain energy and hence the direct flexibility influence coefficients.

2. The assumed stress hybrid model will yield a structure which is more flexible than the compatible model of the same boundary displacement approximation and more rigid than the equilibrium model of the same interior stress approximation. Similarly the assumed displacement hybrid model will yield a structure more rigid than the equilibrium model of the same boundary traction and more flexible than the compatible model of the same interior displacement approximation. Thus, the hybrid models can most likely provide more accurate results than both the

* After the completion of the present paper one of the authors[34] has suggested another possible finite element formulation by using assumed displacement functions in each element and by independent assumption of displacement functions along the interelement boundary. The formulation also leads to the derivation of element stiffness matrices.

compatible and the equilibrium models. The assumed stress scheme is more convenient because its resulting matrix equations contain nodal displacements as unknowns.

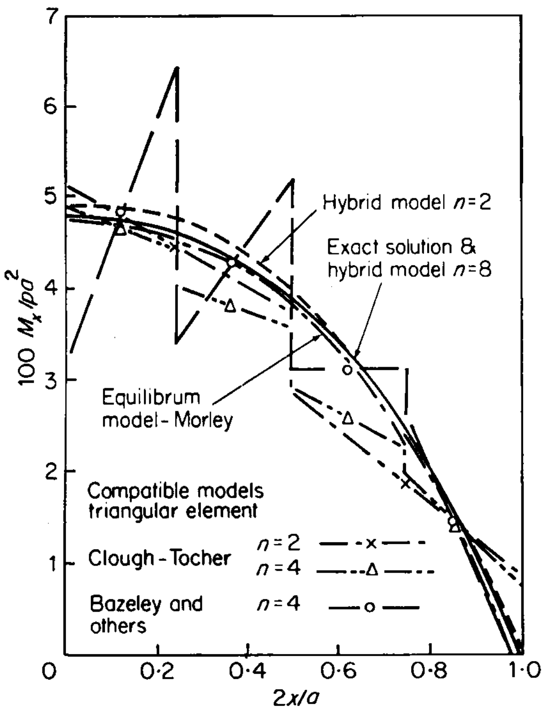

Figure 10. Bending moment along centre line of simply supported square plate under uniform loading

3. Finite element methods based on assumed stresses can provide more accurate stress estimation than the assumed displacement schemes.

4. The mixed-model by Reissner's principle has been demonstrated to be a very powerful tool for plate and shell analyses.

Table I. Classification of Finite Element Methods in Solid Mechanics

Model	Variational Principle	Assumed inside each element	Along Interelement Boundary	Unknowns in Final Equations
Compatible	Minimum Potential Energy	Smooth Displacement Distribution	Continuous Displacement	Nodal Displacements
Equilibrium	Minimum Complementary Energy	Smooth and Equilibrating Stress Distribution	Equilibrium of Boundary Tractions	(a) Generalized Nodal Displacements. (b) Stress Function Parameters
Hybrid (I)	Modified Complementary Energy	Smooth and Equilibrating Stress Distribution	Assumed Compatible Displacements	Nodal Displacements
Hybrid (II)	Modified Potential Energy	Smooth Displacement Distribution	Assumed Equilibrating Boundary Tractions	Boundary Redundant Forces
Mixed (for Plate Bending Problems)	Reissner's Variational Principle	Smooth Displacement and Stress Distributions	1. Continuous w, $w_{,s}$ and $w_{,n}$	Nodal Displacements
			2. Continuous w and M_n	Nodal values of w and M_n
			3. Continuous M_n, M_{ns} and $M_{s,n}$	Nodal values of M_n, M_{ns} and $M_{n,n}$
	Modified Reissner's Principle	Smooth Displacement and Stress Distribution	Assumed Distribution of interelement tractions	Displacement and stress parameters and boundary tractions

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