# 40

# Domain decomposition for flow in porous media with fractures

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## Introduction

We are concerned with flow in a porous medium such as granite or sedimentary rock. Natural fractures occur in the rock and it is necessary to take them into account because they have a profound effect on the transport of the fluid. One can distinguish two types of fractures : numerous small fractures that can be treated with a double porosity model [Arb90], [KPL94], and more important fractures that one may wish to model individually using domain decomposition. The latter is the topic of this article.

The fractures that we are concerned with are filled with debris so we consider them as porous media. The permeability in the fracture is large in comparison with that in the surrounding rock, so the fluid circulates faster in the fracture. Thus we have a highly heterogeneous porous medium. One idea that has been used to take this into account is to treat the fracture as an interface and to assume that the fluid that flows into the fracture stays in the fracture. In fact, in many models the contrast in permeabilities is of such an order that the flow outside of the fracture is neglected. However, here we are concerned with the situation in which the exchange between the fracture and the rest of the domain is significant. To deal with this case we need to model both what happens in the fracture and what happens outside the fracture. One

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idea is to use domain decomposition treating each fracture as a separate sub-domain. This approach however, leads to well known difficulties due to the small width of a fracture in comparison with the size of the domain. The idea presented here is to use a domain decomposition model with the fractures as natural interfaces, and to model what happens in the interfaces. In this article, we shall consider only the simplest case with two sub-domains separated by one fracture.

This paper is organized as follows. In section 2, we indicate how the model is derived, and in section 3, we show that the corresponding problem has a unique solution. In section 4, we define the mixed finite element formulation and reduce the problem to an interface problem. Numerical results are given in Section 5.

## Model

Suppose that $\Omega$ is a connected and simply connected domain in $R^n$, $n = 2$ or 3, with boundary $\Gamma$. We assume that flow in $\Omega$ is governed by Darcy's equation and for simplicity we suppose that we have a Dirichlet boundary condition on all of $\Gamma$:

$$\begin{aligned} \operatorname{div} \mathbf{u} &= q && \text{in } \Omega \\ \mathbf{u} &= -K \nabla p && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma, \end{aligned} \tag{1}$$

where $p$ is the pressure, $\mathbf{u}$ the Darcy velocity, $K$ the permeability, $q$ an external source. We suppose that $K$ is positive, bounded above and away from 0:

$$0 < K_{min} \leq K(x) \leq K_{max} \qquad \text{almost everywhere in } \Omega.$$

Suppose that the fracture $\Omega_f$, a connected, simply connected sub-domain of $\Omega$, is the intersection of $\Omega$ with a hyperplane times an interval of length $d$ and that $\Omega_f$ separates $\Omega$ into two connected, simply connected sub-domains:

$$\Omega \backslash \bar{\Omega}_f = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \phi.$$

Denote by $\Gamma_i$ the part of the boundary of $\Omega_i$ in common with the boundary of $\Omega$, $i = 1, 2, f$ :

$$\Gamma_i = \partial\Omega_i \cap \Gamma, \quad i = 1, 2, f,$$

and by $\gamma_i$ the part of the boundary of $\Omega_i$ in common with the boundary of $\Omega_f$, $i = 1, 2$ :

$$\gamma_i = \partial\Omega_i \cap \partial\Omega_f \cap \Omega, \quad i = 1, 2.$$

Let $\mathbf{n}$ denote a unit vector in $R^n$ normal to the hyperplane of $\Omega_f$ and directed outward from $\Omega_1$ on $\gamma_1$ and $\eta$ an external, unit vector in $R^n$ normal to $\Gamma$ (see: figure 1).

If we denote by $p_i, \mathbf{u}_i, K_i$, and $q_i$ the restrictions of $p, \mathbf{u}, K$, and $q$ respectively to $\Omega_i$, $i = 1, 2, f$, we may write the transition problem

$$\begin{aligned} \operatorname{div} \mathbf{u}_i &= q_i && \text{in } \Omega_i, \quad i = 1, 2, f, \\ \mathbf{u}_i &= -K_i \nabla p_i && \text{in } \Omega_i, \quad i = 1, 2, f, \\ p_i &= 0 && \text{on } \Gamma_i, \quad i = 1, 2, f, \\ p_i &= p_f && \text{on } \gamma_i, \quad i = 1, 2, \\ \mathbf{u}_i \cdot \mathbf{n} &= -\mathbf{u}_f \cdot \mathbf{n} && \text{on } \gamma_i, \quad i = 1, 2, \end{aligned} \tag{2}$$

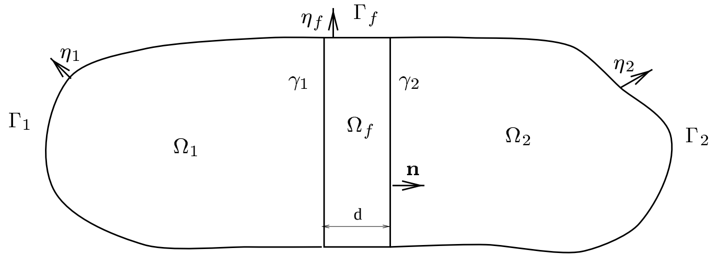


**Figure 1** The domain $\Omega$ with fracture $\Omega_f$

which is equivalent to (1).

To obtain a formulation in which the fracture is considered as an interface, we use an asymptotic expansion, cf:[Lio73], [GF96] . With an abuse of notation, we shall write $\Omega_2$ for $\Omega_{*,2} = \{x - d\mathbf{n} : x \in \Omega_2\}$, $\gamma$ for $\gamma_1 = \{x - d\mathbf{n} : x \in \gamma_2\}$, $\Omega$ for $\Omega_1 \cup \Omega_2 \cup \gamma$, $\Gamma_1$ for $\Gamma_{*,1} = \partial\Omega_1$, $\Gamma_2$ for $\Gamma_{*,2} = \partial\Omega_2$ and $\Gamma_f$ for $\Gamma_{*,f} = \partial\gamma$ (see: figure 2).

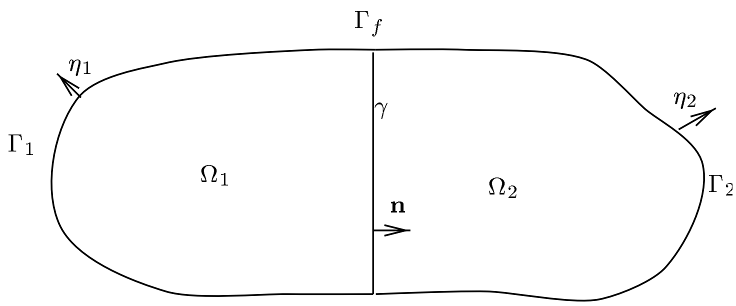


**Figure 2** The domain $\Omega$ with the fracture $\gamma$ as an interface

We introduce a real parameter $\epsilon$ $(0 < \epsilon < d)$ corresponding to the width of the fracture of the model and destined to tends to 0. We write the solution as an asymptotic expansion

$$u^\epsilon = \sum_{k=0}^{\infty} \epsilon^k u^k, \quad p^\epsilon = \sum_{k=0}^{\infty} \epsilon^k p^k.$$

The equations involving the first terms $p^0$ and $u^0$ lead to the problem

$$\begin{array}{rcll} \mathrm{div}_i\,\mathbf{u}_i &=& q_i & \text{on } \Omega_i \quad \text{for } i=1,2\\ \mathbf{u}_i &=& -K_i \nabla\, p_i & \text{on } \Omega_i \quad \text{for } i=1,2\\ p_i &=& 0 & \text{on } \Gamma_i \quad \text{for } i=1,2\\ p_i &=& p_f & \text{on } \gamma\\ \mathrm{div}_f\, d\,\mathbf{u}_f &=& d\,q_f + (\mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n}) & \text{on } \gamma\\ d\,\mathbf{u}_f &=& -d\,K_f \nabla_f\, p_f & \text{on } \gamma\\ p_f &=& 0 & \text{on } \partial\gamma. \end{array} \tag{3}$$

We define

$$\mathrm{div}_f \mathbf{v} \;=\; \mathrm{div}\,\mathbf{v} - \nabla(\mathbf{v}\cdot\mathbf{n})\cdot\mathbf{n} \qquad \text{and} \qquad \nabla_f\, r = \nabla r - \nabla r\cdot\mathbf{n}.$$

The model depends on the width $d$ of the fracture and the difference between fluxes from sub-domains is a source term for the fracture cf:[AJRS98]. To obtain a weak formulation of (3), we define the Hilbert spaces

$$\begin{array}{rcl} \mathbf{W} &=& \{\mathbf{u} = (\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_f) \in L^2(\Omega_1)^n \times L^2(\Omega_2)^n \times L^2(\gamma)^{n-1} : \ \ \mathrm{div}\ \mathbf{u}_i \in L^2(\Omega_i) \quad i=1,2\\ && \ \mathrm{div}_f\ d\,\mathbf{u}f - (\mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n}) \in L^2(\gamma)\}\\ M &=& \{p = (p_1,p_2,p_f) \in L^2(\Omega_1)\times L^2(\Omega_2)\times L^2(\gamma)\}. \end{array} \tag{4}$$

and their norms

$$\begin{array}{rcl} \|\mathbf{u}\|^2_{\mathbf{W}} &=& \sum_{i=1}^{2} \left(\|\mathbf{u}_i\|^2_{0,\Omega_i} + \|\mathrm{div}\ \mathbf{u}_i\|^2_{0,\Omega_i}\right) + \|\mathbf{u}_f\|^2_{0,\gamma} + \|\mathrm{div}_f\, d\,\mathbf{u}_f - (\mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n})\|^2_{0,\gamma}\\ \|p\|^2_M &=& \sum_{i=1}^{2} \|p_i\|^2_{0,\Omega_i} + \|p_f\|^2_{0,\gamma}. \end{array}$$

The weak formulation of (3) is given in terms of the bilinear forms

$$\alpha : \mathbf{W}\times\mathbf{W} \to R \qquad \text{and} \qquad \beta : \mathbf{W}\times M \to R$$

defined by

$$\begin{array}{rcl} \alpha(\mathbf{u},\mathbf{v}) &=& \sum_{i=1}^{2} \int_{\Omega_i} K_i^{-1}\mathbf{u}_i\cdot\mathbf{v}_i + \int_\gamma d\ K_f^{-1}\mathbf{u}_f\cdot\mathbf{v}_f\\ \beta(\mathbf{u},r) &=& \sum_{i=1}^{2} \int_{\Omega_i} \mathrm{div}\ \mathbf{u}_i\ r_i + \int_\gamma \left(\mathrm{div}_f(d\,\mathbf{u}_f) - (\mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n})\right) r_f \end{array}$$

and the linear forms $L_\sigma(q;\cdot) : M \to R$ and defined by

$$L_\sigma(q;r) = \sum_{i=1}^{2} \int_{\Omega_i} q_i\ r_i + \int_\gamma d\ q_f\ r_f,$$

The weak mixed formulation of problem (1) is Find $\mathbf{u}\in\mathbf{W}, p\in M$ such that

$$\begin{array}{rcll} \alpha(\mathbf{u},\mathbf{v}) - \beta(\mathbf{v},p) &=& 0 & \forall \mathbf{v}\in\mathbf{W}\\ \beta(\mathbf{u},r) &=& L_\sigma(q;r) & \forall r\in M. \end{array} \tag{5}$$

## Existence and uniqueness of the solution

We introduce the subspace $\tilde{\mathbf{W}}$ of $\mathbf{W}$ by $\tilde{\mathbf{W}} = \{\mathbf{v} \in \mathbf{W} : \beta(\mathbf{v}, r) = 0 \quad \forall r \in M\}$. To show the existence and uniqueness of the solution of (1), it is sufficient to show that $\alpha$ is $\tilde{\mathbf{W}}$-elliptic and that $\beta$ satisfies the inf-sup condition; cf:[BF91],[RT87]; that is there exist constants $C_\alpha$ and $C_\beta$ such that

$$\inf_{\mathbf{v}\in\tilde{\mathbf{W}}} \frac{\alpha(\mathbf{v},\mathbf{v})}{||\mathbf{v}||^2_{\tilde{\mathbf{W}}}} \geq C_\alpha$$

$$\inf_{r\in M} \sup_{\mathbf{v}\in\mathbf{W}} \frac{\beta(\mathbf{v},r)}{||r||_M \; ||\mathbf{v}||_{\mathbf{W}}} \geq C_\beta .$$

To see that $\alpha$ is $\tilde{\mathbf{W}}$-elliptic, we note that for $\mathbf{u} \in \tilde{\mathbf{W}}$, $||\mathbf{u}||^2_W = \sum_{i=1}^2 ||\mathbf{u}_i||^2_{0,\Omega_i} + ||\mathbf{u}_f||^2_{0,\gamma}$ so that

$$\begin{aligned} \alpha(\mathbf{u},\mathbf{u}) &= \sum_{i=1}^{2} \int_{\Omega_i} K_i^{-1}\mathbf{u}_i \cdot \mathbf{u}_i + \int_\gamma K_f^{-1}\mathbf{u}_f \cdot \mathbf{u}_f \\ &\geq K_{max}^{-1} \left( \sum_{i=1}^{2} ||\mathbf{u}_i||^2_{0,\Omega_i} + ||\mathbf{u}_f||^2_{0,\gamma} \right) \\ &= K_{max}^{-1} ||\mathbf{u}||^2_W . \end{aligned} \tag{6}$$

To see that $\beta$ satisfies the inf-sup condition, given $r \in M$, using the adjoint equation we construct a $\mathbf{v} \in \mathbf{W}$ such that $\beta(\mathbf{v}, r) = ||r||^2_M$ and $||\mathbf{v}||_{\mathbf{W}} \leq C||r||_M$, where $C$ is the constant of elliptic regularity for the adjoint problem.

For $r = (r_1, r_2, r_f) \in M$, let $(\varphi_1, \varphi_2, \varphi_f) \in H^2(\Omega_1) \times H^2(\Omega_2) \times H^2(\gamma)$ be the solution of

$$\begin{aligned} -\triangle\varphi &= \tilde{r} \quad \text{on } \Omega \\ \varphi &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $\tilde{r} \in L^2(\Omega)$ is given by $\tilde{r}_{|\Omega_i} = r_i$;and

$$\begin{aligned} -\triangle_f \varphi_f &= r_f \quad \text{on } \gamma \\ \varphi_f &= 0 \quad \partial\gamma. \end{aligned}$$

Pose $\mathbf{v}_i = -\nabla\varphi_{|\Omega_i}, i = 1, 2,$ and $\mathbf{v}_f = -\nabla_f \varphi_f$ and note that $\operatorname{div}\mathbf{v}_i = r_i \in L^2(\Omega_i), i = 1, 2,$ $\operatorname{div}_f \mathbf{v}_f = r_f \in L^2(\gamma)$ and $\mathbf{v}_1 \cdot \mathbf{n} - \mathbf{v}_2 \cdot \mathbf{n} = 0$. Thus $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_f) \in \mathbf{W}$ and it is easy to check that $\mathbf{v}$ has the desired properties.

## Domain decomposition for the interface problem

In this section we wish to formulate a domain decomposition problem based on mixed finite element methods; see [GW88, CMW95]. Toward this end, we introduce a quasi regular triangulation $\mathcal{T}_($ (of triangles and/or rectangles) of $\Omega$ compatible with the decomposition of $\Omega$ into the subdomains $\Omega_i$, $i = 1, 2$. Note that in this case a triangulation is induced on the interface $\gamma$. We let $M_h = M_{h,1} \times M_{h,2} \times M_{h,f}$ and

$\mathbf{W}_h = \mathbf{W}_{h,1} \times \mathbf{W}_{h,2} \times \mathbf{W}_{h,f}$ be finite dimensional subspaces of $L^2(\Omega_1) \times L^2(\Omega_2) \times L^2(\gamma)$ and $H(\mathrm{div};\Omega_1) \times H(\mathrm{div};\Omega_2) \times H(\mathrm{div}_f;\gamma)$ respectively, such that the pair $(M_{h,i},\ \mathbf{W}_{h,i})$ is a Raviart-Thomas space of order $k$ for $\Omega_i,\ i = 1,2$, subordinate to the triangulation $\mathcal{T}_{h,i}$ determined by $\mathcal{T}_h$; and the pair $(M_{h,f},\ \mathbf{W}_{h,f})$ is a Raviart-Thomas space of order $k$ for $\gamma$ associated with the triangulation $\mathcal{T}_{h,f}$ on $\gamma$ induced by $\mathcal{T}_($. The mixed formulation in the subdomain $\Omega_i$ $i = 1,2$, is

$$\begin{array}{rcll} \mathrm{div}\,\mathbf{u}_i &=& q_i & \text{in } \Omega_i, \\ \mathbf{u}_i &=& -K_i \nabla p_i & \text{in } \Omega_i, \\ p_i &=& 0 & \text{on } \Gamma_i, \\ p_i &=& p_f & \text{on } \gamma \end{array} \tag{7}$$

Introducing for each $i,\ i = 1,2$, the bilinear forms

$$\alpha_i(\mathbf{u},\mathbf{v}) = \int_{\Omega_i} K_i{}^{-1}\,\mathbf{u}\cdot\mathbf{v}, \quad \mathbf{u},\mathbf{v} \in \mathbf{W}_{h,i}, \qquad \beta_i(\mathbf{v},r) = \int_{\Omega_i} \mathrm{div}\,\mathbf{v}\,r, \quad \mathbf{v} \in \mathbf{W}_{h,i},\ r \in M_{h,i};$$

and, for $q_i \in M_{h,i}$ and $p_f \in M_{h,f}$, the linear forms

$$L_{\sigma,i}(q_i;r) = \int_{\Omega_i} q_i\,r, \quad r \in M_{h,i}, \qquad L_{\gamma,i}(p_f;\mathbf{v}) = \int_\gamma p_f\,\mathbf{v}\cdot\mathbf{n}, \quad \mathbf{v} \in \mathbf{W}_{h,i},$$

we may write the weak form of (7):

$$\begin{array}{lcll} \mathbf{u}_i \in \mathbf{W}_{h,i}, \quad p_i \in M_{h,i} & & & \\ \alpha_i(\mathbf{u}_i,\mathbf{v}) - \beta_i(\mathbf{v},p_i) &=& (-1)^i L_{\gamma,i}(p_f;\mathbf{v}) & \forall \mathbf{v} \in \mathbf{W}_{h,i} \\ \beta_i(\mathbf{u}_i,r) &=& L_{\sigma,i}(q_i;r) & \forall r \in M_{h,i}. \end{array} \tag{8}$$

Following [CMW95] we decompose $\mathbf{u}_i$ and $p_i$ as follows:

$$\mathbf{u}_i = \hat{\mathbf{u}}_i + \mathbf{u}_i^0, \qquad p_i = \hat{p}_i + p_i^0,$$

with $\hat{\mathbf{u}}_i$, $\hat{p}_i$, $\mathbf{u}_i^0$ and $p_i^0$ determined by

$$\begin{array}{lcll} \hat{\mathbf{u}}_i \in \mathbf{W}_{h,i}, \quad \hat{p}_i \in M_{h,i} & & & \\ \alpha_i(\hat{\mathbf{u}}_i,\mathbf{v}) - \beta_i(\mathbf{v},\hat{p}_i) &=& (-1)^i L_{\gamma,i}(p_f;\mathbf{v}) & \forall \mathbf{v} \in \mathbf{W}_{h,i} \\ \beta_i(\hat{\mathbf{u}}_i,r) &=& 0 & \forall r \in M_{h,i}. \end{array} \tag{9}$$

and

$$\begin{array}{lcll} \mathbf{u}_i^0 \in \mathbf{W}_{h,i}, \quad p_i^0 \in M_{h,i} & & & \\ \alpha_i(\mathbf{u}_i^0,\mathbf{v}) - \beta_i(\mathbf{v},p_i^0) &=& 0 & \forall \mathbf{v} \in \mathbf{W}_{h,i} \\ \beta_i(\mathbf{u}_i^0,r) &=& L_{\sigma,i}(q_i;r) & \forall r \in M_{h,i}. \end{array} \tag{10}$$

In the fracture we have the problem

$$\begin{array}{lcll} \mathrm{div}\,d\,\mathbf{u}_f &=& d\,q_f\ +\ \mathbf{u}_1\cdot\mathbf{n} - \mathbf{u}_2\cdot\mathbf{n} & \text{in } \gamma, \\ \mathbf{u}_f &=& -K_f \nabla p_f & \text{in } \gamma, \\ p_f &=& 0 & \text{on } \Gamma_f. \end{array} \tag{11}$$

Introducing the bilinear forms

$$\alpha_f(\mathbf{u},\mathbf{v}) = \int_\gamma d\,K_f{}^{-1}\,\mathbf{u}\,\mathbf{v}, \quad \mathbf{u},\mathbf{v} \in \mathbf{W}_{h,f}, \qquad \beta_f(\mathbf{v},r) = \int_\gamma \mathrm{div}_f\,d\,\mathbf{v}\,r, \quad \mathbf{v} \in \mathbf{W}_{h,f},\ r \in M_{h,f};$$

and, for $q_f \in M_{h,f}$ and $\mathbf{u}_i \in \mathbf{W}_{h,i}$, the linear forms

$$L_{\sigma,f}(q_f;r) = \int_\gamma d\, q_f\, r, \quad r \in M_{h,f}, \qquad L_{\Omega,i}(\mathbf{u}_i;r) = \int_\gamma \mathbf{u}_i \cdot \mathbf{n}\, r, \quad r \in M_{h,f}$$

we may write the weak form of (11):

$$\begin{array}{lcll} \mathbf{u}_f \in \mathbf{W}_{h,f}, \quad p_f \in M_{h,f} & & & \\ \alpha_f(\mathbf{u}_f, \mathbf{v}) - \beta_f(\mathbf{v}, p_f) & = & 0 & \forall \mathbf{v} \in \mathbf{W}_{h,f} \\ \beta_f(\mathbf{u}_f, r) & = & L_{\sigma,f}(q_f;r) + L_{\Omega,1}(\mathbf{u}_1;r) - L_{\Omega,2}(\mathbf{u}_2;r) & \forall r \in M_{h,f}. \end{array} \tag{12}$$

To obtain a problem defined on the interface $\gamma$, we also define, for $i = 1, 2$, the following bilinear form on $M_{h,f}$:

$$\mathcal{A}_i(s_f, r_f) = \alpha_i(\hat{\mathbf{u}}_i(s_f), \hat{\mathbf{u}}_i(r_f)),$$

where $(\hat{\mathbf{u}}_i(s_f), \hat{p}_i(s_f))$, respectively $(\hat{\mathbf{u}}_i(r_f), \hat{p}_i(r_f))$ is the solution of (9) for the data $L_{\gamma,i}(s_f;\,\cdot\,)$, respectively $L_{\gamma,i}(r_f;\,\cdot\,)$. Similarly we write $(\mathbf{u}_i(s_f), p_i(s_f))$, respectively $(\mathbf{u}_i(r_f), p_i(r_f))$ for the solution of (8) for the data $L_{\gamma,i}(s_f;\,\cdot\,)$, respectively $L_{\gamma,i}(r_f;\,\cdot\,)$, and we denote by $(u_i^0, p_i^0)$ the solution of (10). Using the symmetry of the forms $\alpha_i$, both equations of (9), and the decomposition of $\mathbf{u}_i$ into $\hat{\mathbf{u}}_i$ and $\mathbf{u}_i^0$ we have,

$$\begin{array}{rcl} \displaystyle\sum_{i=1}^{2} \mathcal{A}_i(s_f, r_f) & = & \displaystyle\sum_{i=1}^{2} \alpha_i(\hat{\mathbf{u}}_i(r_f), \hat{\mathbf{u}}_i(s_f)) \\ & = & \displaystyle\sum_{i=1}^{2} (-1)^i\, L_{\gamma,i}(r_f; \hat{\mathbf{u}}_i(s_f)) \\ & = & \displaystyle\sum_{i=1}^{2} (-1)^i\, L_{\gamma,i}(r_f; \mathbf{u}_i(s_f)) \;-\; \sum_{i=1}^{2} (-1)^i\, L_{\gamma,i}(r_f; \mathbf{u}_i^0) \end{array}$$

Then using the definitions of the forms $L_{\gamma,i}$ and $L_{\Omega,i}$ and both of the equations of (12) we have

$$\begin{array}{rcl} \displaystyle\sum_{i=1}^{2} (-1)^i\, L_{\gamma,i}(r_f; \mathbf{u}_i(s_f)) & = & \displaystyle\sum_{i=1}^{2} (-1)^i L_{\Omega,i}(\mathbf{u}_i(s_f); r_f) \\ & = & -\,\beta_f(\mathbf{u}_f(s_f), r_f) \;+\; L_{\sigma,f}(q_f; r_f) \\ & = & -\,\delta_f(s_f, r_f) \;+\; L_{\sigma,f}(q_f; r_f), \end{array}$$

where $\mathbf{u}_f(s_f)$ is the solution of (12) with $L_{\Omega,1}(\mathbf{u}_1(s_f);\cdot) - L_{\Omega,2}(\mathbf{u}_2(s_f);\cdot)$ as data and where the bilinear form $\delta_f(.,.)$ is

$$\delta_f(s_f, r_f) \;=\; < B_f A_f^{-1} B_f^T\, s_f, r_f >_{|\gamma}$$

$A_f$ and $B_f$ are the linear mapping respectively associated to the bilinear form $\alpha_f(.,)$ and $\beta_f(.,.)$.

Combining the last two equations we obtain our interface problem:

$$\begin{array}{lcl} s_f \in M_{h,f} & & \\ \displaystyle\sum_{i=1}^{2} \mathcal{A}_i(s_f, r_f) \;+\; \delta_f(s_f, r_f) & = & \displaystyle\sum_{i=1}^{2} (-1)^{i+1} L_{\gamma,i}(r_f; \mathbf{u}_i^0) \;+\; L_{\sigma,f}(q_f; r_f), \quad r_f \in M_{h,f}. \end{array} \tag{13}$$

It is clear that the left hand side of the above equation determines a symmetric, positive definite form on $M_{h,f}$. Thus for a given source term $q = (q_1, q_2, q_f) \in M_{h,1} \times M_{h,2} \times M_{h,f}$ there is a unique solution $s_f \in M_{h,f}$. We remark that in the absense of a fracture, i.e. when the flux is continuous across the interface $\gamma$, $\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n}$, the second term of each side of (13) vanishes ($q_f$ is of course in this case equal to 0) and we obtain the standard interface problem given in [GW88, CMW95] for the case of two subdomains.

## Numerical Results

To illustrate the model we consider an ideal dimensionless problem. The domain is an horizontal rectangular slice of porous medium, of dimensions $2.1 \times 1$, with given pressure on the left and right boundaries and no flow conditions on the top and bottom boundaries. In the domain the permeability is equal to one. The domain is divided into two equally large sub-domains by a linear fracture parallel to the $x_2$ axis. The permeability in the fracture $\times$ the width of the fracture is equal to 2. For example the fracture could be of width 0.1 and could have a permeability equal to 20. Flow in the fracture is driven by a pressure drop of 10 between the two extremities of the fracture for the first example and a pressure drop of 5 for the second example.

Two cases are considered. A symmetric case where pressures on the left and on the right boundaries of the domain are equal. So the flow is driven only by the fracture and is symmetric. In the other case there is a pressure drop from the right boundary to the left one. Then the flow is a combination of the flow in the fracture and that going from left to right in the rest of the porous medium.

Numerical results are shown on figure 3. Arrows represent the flow field with length proportional to the magnitude of the velocity. The gray scale represents also the magnitude of the velocity with the lightest color corresponding to the largest velocity. These results were obtained by solving equation (13) with a direct method. We see that there is actual flow interaction between the fracture and the rest of the porous medium. In particular one can observe that some fluid is coming out of the fracture and then is coming back into it. In the nonsymmetric case we notice also that even though most of the flow is attracted into the fracture, there is still some flow on the left part of the domain pointing toward the left.

## Conclusion

A model for flow interaction between a fracture and the rest of the porous medium has been presented. In this model the fracture is an interface dividing the domain of calculation into sub-domains. Existence and uniqueness of the solution has been shown and the model has been reformulated as an interface problem. Simple numerical experiments show actual flow interaction between the fracture and the rest of the porous medium.

Extension to situations with several intersecting fractures is under way as well as the study of efficient preconditionners.

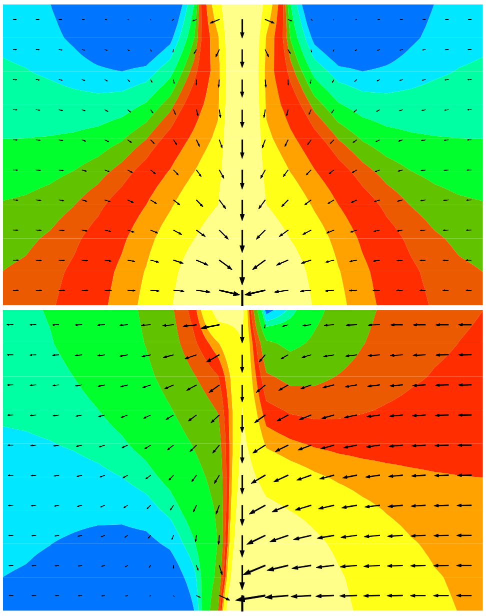

**Figure 3** Calculated Darcy's velocity for a symmetric and a nonsymmetric flow pattern

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