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The Multiscale Finite Volume Method: A flexible tool to model physically complex flow in porous media

IVAN LUNATI and PATRICK JENNY
Institute of Fluid Dynamics, ETH Zurich, 8092 Zurich, Switzerland

Abstract

The Multiscale Finite-Volume (MSFV) method has been developed to solve homogeneous elliptic equation on large and highly heterogeneous domains efficiently and has been applied to multiphase flow problems. It employs an auxiliary coarse grid, together with its dual, to define and solve a coarse-scale pressure problem. A set of basis functions, which are local solutions on dual cells, is used to interpolate the coarse-grid pressure and obtain an approximate fine-scale pressure distribution. Then, by solving a set of local problems on coarse cells, a conservative flux field is constructed. The MSFV method has been modified to provide a tool that can model complex physical processes. In this case, the pressure equation might include a source term (which may arise due to gravity or capillary forces) and the basis functions are not good interpolators. An accurate fine-scale pressure can be computed by adding a correction function to the basis-function interpolated pressure. In this framework, the effects of physically complex processes are exclusively described by the correction function and are included in the conservative flux field by solving local problems including the full physics. In future we plan to apply a similar strategy to improve the ability of the MSFV method to solve nonlinear pressure equations as in the case of highly compressible flow.

1 Introduction

In the last decade, several multiscale techniques have been developed to model subsurface flow in heterogeneous reservoirs. Analogous to standard upscaling techniques the goal of these methods is to solve the flow problem efficiently by employing an auxiliary coarse grid. However, in contrast to upscaling multiscale methods target the full problem with the original resolution. The existing multiscale techniques can be subdivided in three major families: the multiscale finite-element methods [3]; the multiscale mixed finite-element methods [2, 1]; and the multiscale finite-volume (MSFV) method [4], which is considered in this paper.

The MSFV method was originally developed to solve elliptic homogeneous equations for single and multiphase flow problems with simplified physics, i.e. incompressible flow with negligible gravity and capillary effects [4, 5, 6], and has been recently extended to model compressible flow [8]. An auxiliary coarse grid is imposed and, together with its dual, used to define and solve a coarse-pressure problem. The MSFV method employs a set of basis functions, which are local solutions of the elliptic homogeneous equation, to relate the coarse-grid pressure to the fine-scale pressure distribution. If the equation is not homogeneous (i.e. the right hand side is not zero), the basis functions are not good interpolators and a correction function has to be introduced to obtain an accurate fine-scale pressure approximation. This was demonstrated for problems including wells [10] and density-driven flow [9]. In this paper, we describe how the ideas presented in [9] can be generalized to model physically complex systems.

2 MSFV method for linear elliptic operators

Here, we consider an equation of the form

$$\nabla \cdot (\mathbf{K}\nabla p - \boldsymbol{h}) = q, \tag{1}$$

where p is the (unknown) pressure; $\mathbf{K}$ a positive defined coefficient matrix, which will be assumed isotropic, i.e. $\mathbf{K} = K\mathbf{I}$, in the rest of the paper; $\boldsymbol{h}$ a vector that describes the effects of non-viscous forces, e.g. gravity or capillarity; and q a source term per unit volume. Note that $\mathbf{K}$ can describe the hydraulic conductivity for single phase flow and the total mobility for multiphase flow.

The MSFV method employs an auxiliary coarse grid, together with its dual, to solve Eq. (1). Given the domain Ω, a coarse grid with N cells is constructed, which defines a primary partition of the domain, $\bar{\Omega}_{i\in[1,N]}$. The dual coarse grid is constructed such that it has exactly one node in the interior of each coarse cell $\bar{\Omega}_i$, and it defines a dual partition, $\tilde{\Omega}^{e\in[1,M]}$. In the MSFV method the pressure field is approximated by composing local pressure solutions computed in the dual cells, i.e. $\tilde{p} = \cup_e \tilde{p}^e$. The flux field associate to this pressure field, $\tilde{\boldsymbol{u}} = -K\nabla\tilde{p} + \boldsymbol{h}$, is non-conservative (discontinuous) at the dual-cell boundaries, $\partial\tilde{\Omega}^e$, and cannot be successfully used to simulate transport. A conservative flux field approximation is obtained by composing solutions of Eq. (1) that are computed locally in the coarse cells $\bar{\Omega}_i$, i.e. $\bar{\boldsymbol{u}} = -K\nabla\bar{\psi} + \boldsymbol{h}$, where $\bar{\psi} = \cup_i \bar{\psi}_i$ and $\nabla \cdot (K\nabla\bar{\psi}_i - \boldsymbol{h}) = q$ in $\bar{\Omega}_i$ and $\nabla\bar{\psi}_i \cdot \boldsymbol{\nu} = \nabla\tilde{p} \cdot \boldsymbol{\nu}$ on $\partial\bar{\Omega}_i$ ($\boldsymbol{\nu}$ is the unit vector normal to $\partial\bar{\Omega}_i$ and pointing outwards). In the following we concentrate on the pressure approximation $\tilde{p}$ and we refer to the existing MSFV method literature for further details.

2.1 Pressure approximation

The accuracy of the MSFV method depends on the quality of the localization assumption for the pressure approximation, that is on the ability of the boundary conditions assigned on $\partial\tilde{\Omega}^e$ to approximate the actual fine scale flow conditions. The standard implementation of the MSFV method requires that the flux derivative in the direction normal to $\partial\tilde{\Omega}^e$ is zero, i.e. $\frac{\partial u_\perp}{\partial x_\perp} = 0$. Generalizing this condition, we require that the divergence of the projection of the flux is equal to the source term and we write

$$\nabla \cdot [(\mathbf{I} - \mathbf{P}_\eta)\tilde{\boldsymbol{u}}] = [(\mathbf{I} - \mathbf{P}_\eta)\nabla] \cdot \tilde{\boldsymbol{u}} = \nabla_{\|} \cdot \tilde{\boldsymbol{u}} = q, \tag{2}$$

where $\mathbf{P}_\eta = \boldsymbol{\eta}\boldsymbol{\eta}^T$ is the projector operator in the direction $\boldsymbol{\eta}$ perpendicular to $\partial\tilde{\Omega}^e$, and we have defined $\nabla_\parallel = (\mathbf{I} - \mathbf{P}_\eta)\nabla$. Therefore, the approximate pressure is solution of

$$\begin{cases} \nabla \cdot K\nabla\tilde{p} = r & \text{in} \quad \tilde{\Omega}^e \\ \nabla_\parallel \cdot K\nabla\tilde{p} = r_\parallel & \text{on} \quad \partial\tilde{\Omega}^e \\ \tilde{p}(\boldsymbol{x}_j) = p_j \end{cases}, \tag{3}$$

where we have defined $r = q + \nabla \cdot \boldsymbol{h}$ and $r_\parallel = q + \nabla_\parallel \cdot \boldsymbol{h}$; and p_j is the pressure at the node, $\boldsymbol{x}_j$, of the dual grid, also called coarse-grid pressure. It is further assumed that the solution of Eq. (3) can be expressed as a linear combination of a set of basis functions, $\{\tilde{\varphi}^e_j\}$, plus a correction function, $\tilde{\varphi}^e_*$, i.e.

$$\tilde{p}^e = \sum_j \tilde{\varphi}^e_j p_j + \tilde{\varphi}^e_* \qquad \text{on} \quad \tilde{\Omega}^e, \tag{4}$$

and zero elsewhere. The linear combination of basis functions represents the solution of the localized homogeneous problem obtained by setting the right hand side to zero, whereas the correction function represents the effects due to the presence of a r.h.s. in Eq. (3). Since the homogeneous problem has to be satisfied in $\tilde{\Omega}^e$ for any value of the coarse pressure, each basis function has to be solution of

$$\begin{cases} \nabla \cdot K\nabla\tilde{\varphi}^e_j = 0 & \text{in} \quad \tilde{\Omega}^e \\ \nabla_\parallel \cdot K\nabla\tilde{\varphi}^e_j = 0 & \text{on} \quad \partial\tilde{\Omega}^e \\ \tilde{\varphi}^e_j(\boldsymbol{x}_i) = \delta_{ij} \end{cases}, \tag{5}$$

which is the original definition of the MSFV basis functions given in [4]. Each basis function $\tilde{\varphi}^e_j$ represents the contribution of a unit pressure signal at the node $\boldsymbol{x}_j$. For Eq. (4) to be solution of Eq. (3), the correction function must be defined by the local problem

$$\begin{cases} \nabla \cdot K\nabla\tilde{\varphi}^e_* = r & \text{in} \quad \tilde{\Omega}^e \\ \nabla \cdot K\nabla_\parallel\tilde{\varphi}^e_* = r_\parallel & \text{on} \quad \partial\tilde{\Omega}^e \\ \tilde{\varphi}^e_*(\boldsymbol{x}_i) = 0 \end{cases}. \tag{6}$$

Using the linearity of the operators it is straightforward to show that Eq. (4) is solution of Eq. (3), if basis and correction functions are defined by Eqs. (5) and (6), respectively. Moreover, given the localized problem to be satisfied and the definition of basis functions, the correction function is unique.

2.2 Coarse-scale pressure problem

The coarse-grid pressure coefficients, p_j, are determined by solving a set of coarse-scale mass conservation equations, which are obtained by integrating Eq. (1) over each coarse cell, $\bar{\Omega}_i$. Applying the Gauss' (or divergence) theorem, and using the approximate pressure, Eq. (4), we obtain the coarse-scale pressure equations,

$$\sum_j T_{ij} p_j = \sum_e \int_{\partial\bar{\Omega}_i \cap \tilde{\Omega}^e} K\nabla\tilde{\varphi}^e_* \cdot \boldsymbol{\eta} \, d\Gamma - \int_{\bar{\Omega}_i} r \, d\boldsymbol{x} \qquad i \in [1, N], \tag{7}$$

where

$$T_{ij} = -\sum_e \int_{\partial\bar{\Omega}_i \cap \tilde{\Omega}^e} K\nabla\tilde{\varphi}^e_j \cdot \boldsymbol{\eta} \, d\Gamma \tag{8}$$

is the coarse-scale transmissibility pertinent to the node $\boldsymbol{x}_j$. The first term on the r.h.s. of Eq. (7) can be regarded as a correction of the fluxes generated by the coarse-scale operator, T_{ij}. Indeed, T_{ij} does not include effects due to the presence of r and yields incorrect fluxes across $\partial\bar{\Omega}_i$ for a given pressure drop between the nodes; the first term on the r.h.s. represents a correction to these inaccurate fluxes. This correction is local, since it is obtained from a local solution of the flow problem, Eq. (6).

3 Application to multiphase flow and adaptivity

The flow of m immiscible phases through a porous medium is governed by m mass-balance equations of the form

$$\frac{\partial}{\partial t}\phi\rho_\alpha S_\alpha + \nabla\cdot\boldsymbol{u}_\alpha + \rho_\alpha q_\alpha = 0 \qquad \alpha\in[1,m], \tag{9}$$

where ϕ is the porosity of the medium; S_α the saturation; ρ_α the density; q_α the source term (positive when extracted);

$$\boldsymbol{u}_\alpha = -\rho_\alpha\lambda_\alpha k\left(\nabla p_\alpha - \rho_\alpha\boldsymbol{g}\right), \tag{10}$$

the mass flux per unit area of the α-phase. In Eq. (10), $\lambda_\alpha = k_{r\alpha}/\mu_\alpha$ is the α-phase relative mobility, i.e. the ratio of relative permeability, $k_{r\alpha}\in[0,1]$, to dynamic viscosity, μ_α; k is the intrinsic permeability, which is fluid independent; $\boldsymbol{g}$ the gravity acceleration vector; and p_α the pressure of the α-phase. Defining the macroscopic capillary pressure of the α-phase with respect to the pressure of a reference phase ω, $p_{c\alpha\omega} = p_\alpha - p_\omega$, and introducing the constitutive relationships, $k_{r\alpha}(S)$ and $p_{c\alpha\omega}(S)$, and the constraint $\sum_{\alpha=1}^{m} S_\alpha = 1$, the m mass-balance equations can be manipulated in order to obtain an equation for the pressure of the reference phase, $p = p_\omega$, and $m-1$ transport equations for the saturations.
In this framework, the main ideas of the MSFV method method apply to the solution of the pressure equation, which is elliptic and can be written in the form of Eq. (1) if the flow is incompressible. The phase-transport, Eqs. (9), can be solved by standard method, such as a Schwarz overlap method. Since the mobility depends on saturation, the basis functions have to be updated when saturation changes. To keep the MSFV method efficient, the basis functions are updated adaptively, i.e. only in regions where mobility changes exceed a specified threshold [5, 9]). This allows to reuse most basis functions for the successive step.

3.1 Incompressible two-phase flow influenced by gravity and capillary forces

First, we consider the case of two incompressible fluids that move under the influence of gravity and capillary forces in a rigid porous medium. Here we consider a water-oil system and we observe the evolution of the saturation. In this case, the pressure equation is

$$\nabla\cdot\left[\lambda k\nabla p - \left(\rho_w\lambda_w + \rho_o\lambda_o\right)k\boldsymbol{g} + \lambda_w k\nabla p_c\right] = q \tag{11}$$

where w and o denote the water and the oil phase, respectively, $\lambda = \lambda_w + \lambda_o$ is the total mobility, and $p_c = p_w - p_o$ the capillary pressure. Defining

$$K = \lambda k, \qquad \boldsymbol{h} = \left(\rho_w\lambda_w + \rho_o\lambda_o\right)k\boldsymbol{g} - \lambda_w k\nabla p_c, \tag{12}$$

the application of the MSFV method is straightforward.
An adaptivity criterion based on the total-mobility changes can be introduced, such that basis and correction functions are selectively updated if

$$\frac{1}{1+\epsilon_\lambda} < \frac{\lambda'}{\lambda} < 1+\epsilon_\lambda, \tag{13}$$

is not fulfilled, where ϵ_λ is a predefined threshold and λ' the total mobility used to compute the current basis functions, i.e. the mobility at the last update. Analogously, a second criterion based on the r.h.s. changes is introduced, such that correction functions are updated if

$$\frac{1}{1+\epsilon_r} < \frac{r'}{r} < 1+\epsilon_r. \tag{14}$$

is not satisfied. If Eq. (13) is violated within a dual cell $\tilde{\Omega}^e$, one recomputes all the basis functions and the correction function pertinent to the cell and sets $\lambda' = \lambda$; if only Eq. (14) is violated, only the correction function is recomputed and r' is set to r.

4 MSFV method for nonlinear parabolic problems

The MSFV method has been applied to highly compressible multiphase flow in [8]. In this cases, the pressure is governed by a nonlinear parabolic equation of the form

$$C(p)\frac{\partial}{\partial t}p - \mathbf{R}_p p = 0, \tag{15}$$

where $\mathbf{R}_p$ is a nonlinear diffusion operator, and C a function of p, which represents the compressibility of the system. In the case of m compressible phases that flow in a rigid porous medium, for instance, we have

$$C(p) = \phi \sum_{\alpha=1}^{m} [c_\alpha(p) S_\alpha] \qquad \mathbf{R}_p = \sum_{\alpha=1}^{m} \frac{1}{\rho_\alpha} \nabla \cdot \rho_\alpha \lambda_\alpha k \nabla, \tag{16}$$

where we have neglected gravity and capillary forces. In Eq. (16), $c_\alpha = \frac{1}{\rho_\alpha}\frac{\partial \rho_\alpha}{\partial p}|_T$ is the isothermal compressibility coefficient of the α-phase. Nonlinear equations are solved numerically by iterative methods (e.g. Picard method or Newton-Raphson method) in which a time-discrete form of Eq. (15) is linearized. The MSFV method, which is based on the superimposition principle of the basis functions, can be applied at each iteration. If we use an implicit scheme for time discretization and define

$$\Delta_t^n p^\nu(\boldsymbol{x}, t) = \frac{p^\nu(\boldsymbol{x}, t^n) - p(\boldsymbol{x}, t^{n-1})}{t^n - t^{n-1}}, \tag{17}$$

we can write the time discrete equation

$$\mathbf{L}^{n,\nu-1} p^\nu = \left(-C^{\nu-1}\Delta_t^n + \mathbf{R}^{\nu-1}\right) p^\nu = 0, \tag{18}$$

where $\mathbf{L}^{n,\nu-1}$ is a linear parabolic operator (defined by the expression in brackets), and $C^{\nu-1} = C(p^{\nu-1})$ and $\mathbf{R}^{\nu-1} = \mathbf{R}_{p^{\nu-1}}$. In [8], a solution is sought that can be locally expressed as

$$\tilde{p}^{e,\nu}(\boldsymbol{x}, t) = \sum_j \tilde{\varphi}_j^e(\boldsymbol{x}) p_j^\nu(t) \qquad \text{on} \quad \tilde{\Omega}^e, \tag{19}$$

where the basis functions, $\tilde{\varphi}_j^e$, are again solutions of Eq. (5). The coarse-scale pressures, $\tilde{p}_j^\nu$, is obtained from the coarse-grid pressure problem

$$\sum_j V_{ij}^{\nu-1}\, \Delta_t^n p_j^\nu - \sum_j T_{ij}^{\mathbf{R},\nu-1} p_j^\nu = 0 \qquad i \in [1, N], \tag{20}$$

where we have defined the accumulation matrix

$$V_{ij}^{\nu-1} = \sum_e \int_{\bar{\Omega}_i \cap \tilde{\Omega}^e} C^{\nu-1} \tilde{\varphi}_j^e \, d\boldsymbol{x}, \tag{21}$$

and the transmissibilities

$$T_{ij}^{\mathbf{R},\nu-1} = -\sum_e \int_{\bar{\Omega}_i \cap \tilde{\Omega}^e} \mathbf{R}^{\nu-1} \tilde{\varphi}_j^e \, d\boldsymbol{x}. \tag{22}$$

Three different compressibility models based on this approach have been considered [8]. They display an increasing level of complexity depending on whether the fine-scale pressure approximation, $\tilde{p}^{\nu-1}$, or the coarse-scale pressure, $p_j^{\nu-1}$, are used to compute $C_{ij}^{\nu-1}$ and $V_{ij}^{\nu-1}$. Numerical tests performed for a variety of highly compressible flow problems (gas flow, imbibition and drainage of a partially saturated reservoir, depletion of a gas-water reservoir and flooding of oil-gas reservoirs) demonstrated that this model provides accurate results in many cases, but exhibits some difficulties in modeling highly transient flow and very curved pressure profiles. This is due to the fact that the pressure interpolators, $\tilde{\varphi}_j^e$, are solution of elliptic (incompressible) problems, such that the approximate pressure does not converge to an exact solution of Eq. (18) in the dual cells, $\tilde{\Omega}^e$. By inserting Eq. (19) into Eq. (18), it is easy to show that the residual at the iteration ν is

$$res = \mathbf{L}^{n,\nu-1}\tilde{p}^{e,\nu} = -C^{\nu-1}\sum_j \tilde{\varphi}_j^e \Delta_t^n p_j^\nu + \sum_j p_j^\nu \left(\mathbf{R}^{\nu-1} - \nabla \cdot K\nabla\right)\tilde{\varphi}_j^e, \tag{23}$$

which may be important for highly transient flow or when $\mathbf{R}_p$ differs significantly from $\nabla \cdot K\nabla$.

4.1 Nonlinear parabolic problems with correction function

A possible way to overcome this problem is to employ a correction function and assume that the approximate pressure can be expressed as

$$\tilde{p}^{e,\nu}(\boldsymbol{x},t) = \sum_j \tilde{\varphi}_j^e(\boldsymbol{x}) p_j^\nu(t) + \tilde{\varphi}_*^{e,\nu}(\boldsymbol{x},t) \qquad \text{on} \quad \tilde{\Omega}^e, \tag{24}$$

such that the coarse-grid pressure problem reads

$$\sum_j V_{ij}^{\nu-1} \, \Delta_t^n p_j^\nu - \sum_j T_{ij}^{\mathbf{R},\nu-1} p_j^\nu = \sum_e \int_{\bar{\Omega}_i \cap \tilde{\Omega}^e} \mathbf{L}^{n,\nu-1} \tilde{\varphi}_*^{e\nu} \, d\boldsymbol{x} \qquad i \in [1,N]. \tag{25}$$

If $\tilde{\varphi}_*^{e,\nu}$ is defined as the local solution of

$$\mathbf{L}^{n,\nu-1} \left(\tilde{\varphi}_*^{e,\nu} - \tilde{\varphi}_*^{e,\nu-1} \right) = -\mathbf{L}^{n,\nu-1} \tilde{p}^{\nu-1} \qquad \text{in} \quad \tilde{\Omega}^e, \tag{26}$$

the residual in this case reads

$$res = \mathbf{L}^{n,\nu-1} \tilde{p}^{e,\nu} = \sum_j \mathbf{L}^{n,\nu-1} \left[\tilde{\varphi}_j^e \left(p_j^\nu - p_j^{\nu-1} \right) \right], \tag{27}$$

and the approximate pressure is solution of Eq. (15) if the coarse-scale pressure converges. This approach needs to be investigated and tested numerical with respect to stability.

5 Conclusions

The MSFV method has been recently extended to become a flexible method to model physically complex flow in porous media. In particular, the ideas presented in [8, 9] have been employed to develop a MSFV method for the black-oil equation [7]. We believe that the accuracy of the method can be improved by the general formulation presented in this paper, which extends the correction function introduced in [9]. This function describes all non-elliptic and nonlinear effects and, on a rigorous basis, allows for basis functions that are again solutions of linear elliptic problems. This formulation provides a general framework, which relies on a single approximation, i.e. the localization assumption, and represents a promising tool that can be applied to a broad range of problems.

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