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A Posteriori Error Estimators for the Stokes Equations*

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Summary. We present two a posteriori error estimators for the mini-element discretization of the Stokes equations. One is based on a suitable evaluation of the residual of the finite element solution. The other one is based on the solution of suitable local Stokes problems involving the residual of the finite element solution. Both estimators are globally upper and locally lower bounds for the error of the finite element discretization. Numerical examples show their efficiency both in estimating the error and in controlling an automatic, self-adaptive mesh-refinement process. The methods presented here can easily be generalized to the Navier-Stokes equations and to other discretization schemes.

Subject Classifications: AMS(MOS); 65N15, 65N30, 76D05; CR: G1.8.

1. Introduction

In the numerical approximation of partial differential equations, one often encounters the problem that the overall accuracy of the numerical solution is degraded by local singularities of the solution of the continuous problem such as, e.g., the singularity near a re-entrant corner. An obvious remedy is to refine the discretization in the critical regions, i.e., to place more gridpoints where the solution is less regular. The question is how to identify these regions automatically and how to determine a good balance of the number of gridpoints in the refined and unrefined regions such that the overall accuracy is optimal. This problem can be solved by trying to estimate locally the error of the numerical solution from the given data and the numerical solution itself.

In [4, 5, 7] automatic, self-adaptive mesh-refinement techniques based on a posteriori error estimators are investigated for finite element discretizations of 2nd order elliptic PDEs. The main idea is to solve local problems, which have the same structure as the original one and which involve the residual of the calculated numerical solution, using higher order finite elements. The

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error is then estimated by suitable norms of the solution of the local problems. Typically, one can prove that one obtains locally lower and globally upper bounds for the error of the numerical solution.

In this paper, we introduce two a posteriori error estimators for the so-called mini-element approximation (cf. Sect. 2) of the Stokes equations. The first one (cf. Sect. 3) simply computes suitable norms of the residual; the second one (cf. Sect. 4) requires the solution of local Stokes problems. For both estimators, we prove that they yield locally lower and globally upper bounds for the error of the numerical solution. In contrast to [7], we do not need a super-approximation assumption for the finite elements used for the estimator with respect to those used for the numerical solution. On the other hand, we have in our estimates additional higher order terms involving the right-hand side (cf. Theorems 3.1 and 4.2). Our estimators can easily be extended to the Navier-Stokes equations and to other approximation schemes (cf. Sect. 5). In Sect. 6 we present numerical examples which demonstrate the efficiency of the error estimators.

2. The Mini-Element Approximation of the Stokes Equations

We consider the Stokes equations

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \Gamma := \partial\Omega \end{aligned} \tag{2.1}$$

in a bounded, connected, polygonal domain Ω in $\mathfrak{R}^2$. The restriction to polygonal domains and to homogeneous boundary conditions is only made in order to avoid unnecessary technical difficulties in Sects. 3 and 4. The examples of Sect. 6 show that our methods can be applied equally well to domains with curved boundaries and to Stokes problems with non-homogeneous boundary conditions.

For any open subset S of Ω we denote by $H^k(S)$, $k \geqq 0$, and $L^2(S) = H^0(S)$ the usual Sobolev and Lebesgue spaces equipped with the semi-norm

$$|\varphi|_{k,S} := \{\sum_{|\alpha|=k} \int_S |D^\alpha \varphi|^2\}^{1/2}$$

and the norm

$$\|\varphi\|_{k,S} := \left\{\sum_{m=0}^{k} |\varphi|_{m,S}^2\right\}^{1/2}.$$

The scalar product of $L^2(S)$ is denoted by

$$(\varphi, \psi)_S := \int_S \varphi \psi.$$

Similarly, the norm and scalar product of $L^2(G)$ are defined for any measurable subset G of Γ. For simplicity we omit the index S if $S = \Omega$. Since no confusion can arise, we use the same notation for the corresponding norms and scalar products on $H^k(\Omega)^2$ and $L^2(\Omega)^2$, resp. Put

$$H_0^1(\Omega):=\{\varphi\in H^1(\Omega)\colon \varphi=0 \text{ on } \Gamma\},$$

$$L_0^2(\Omega):=\{\varphi\in L^2(\Omega)\colon (\varphi, 1)=0\},$$

and define

$$\mathcal{H}=H_0^1(\Omega)^2\times L_0^2(\Omega).$$

Note, that $|\cdot|_1$ is a norm on $H_0^1(\Omega)$ which is equivalent to $\|\cdot\|_1$.

We introduce a bilinear form $\mathcal{L}$ on $\mathcal{H}\times\mathcal{H}$ by

$$\mathcal{L}([\mathbf{u}, p]; [\mathbf{v}, q]):=(\nabla \mathbf{u}, \nabla \mathbf{v})-(p, \nabla\cdot\mathbf{v})-(q, \nabla\cdot\mathbf{u}) \qquad \forall [\mathbf{v}, q]\in\mathcal{H}. \tag{2.2}$$

Then the standard variational formulation of (2.1) is given by:

$$\text{Find } [\mathbf{u}, p]\in\mathcal{H} \text{ such that}$$

$$\mathcal{L}([\mathbf{u}, p]; [\mathbf{v}, q])=(\mathbf{f}, \mathbf{v}) \qquad \forall [\mathbf{v}, q]\in\mathcal{H}. \tag{2.3}$$

It is well-known (cf. [11]) that $\mathcal{L}$ satisfies the following inf-sup condition

$$\inf_{[\mathbf{u},p]\in\mathcal{H}} \sup_{[\mathbf{v},q]\in\mathcal{H}} \frac{\mathcal{L}([\mathbf{u}, p]; [\mathbf{v}, q])}{\{|\mathbf{u}|_1+\|p\|_0\}\{|\mathbf{v}|_1+\|q\|_0\}}\geqq\beta_c>0 \tag{2.4}$$

which implies that problem (2.3) has a unique solution.

Let $\mathcal{T}_h$ be a family of triangulations of Ω such that any two triangles in $\mathcal{T}_h$ share at most a vertex or a whole edge and such that the smallest angle of all triangles is bounded from below by some constant $\alpha>0$ which is independent of h. Denote by $M_h^{k,-1}$, $k\geqq 0$, the set of all functions which are piecewise polynomials of degree $\leqq k$ and put $M_h^{k,s}:=M_h^{k,-1}\cap C^s(\Omega)$, $s\geqq 0$. Note, that $M_h^{k,s}$ is a subset of $H^{s+1}(\Omega)$, $s\geqq -1$. Denote for any $T\in\mathcal{T}_h$ by $\lambda_{T1}, \lambda_{T2}, \lambda_{T3}$ its barycentric co-ordinates and define

$$\mathcal{B}_h:=\operatorname{span}\{\lambda_{T1}\lambda_{T2}\lambda_{T3}\colon T\in\mathcal{T}_h\}.$$

The elements of $\mathcal{B}_h$ are often referred to as *bubble functions*. Note, that

$$(\nabla\varphi_h, \nabla\psi_h)=0 \qquad \forall \varphi_h\in M_h^{1,0},\ \psi_h\in\mathcal{B}_h. \tag{2.5}$$

Put

$$X_h:=\{M_h^{1,0}\cap H_0^1(\Omega)\oplus\mathcal{B}_h\}^2, \qquad Y_h:=M_h^{1,0}\cap L_0^2(\Omega)$$

and define

$$\mathcal{H}_h:=X_h\times Y_h.$$

The *mini-element* discretization [2] of (2.1) is then given by

$$\text{Find } [\mathbf{u}_h, p_h]\in\mathcal{H}_h \text{ such that}$$

$$\mathcal{L}([\mathbf{u}_h, p_h]; [\mathbf{v}_h, q_h])=(\mathbf{f}, \mathbf{v}_h) \qquad \forall [\mathbf{v}_h, q_h]\in\mathcal{H}_h. \tag{2.6}$$

From [2] we know that the mini-element satisfies the Babuska-Brezzi condition, which implies that the inequality

$$\inf_{[\mathbf{u}_h, p_h]\in\mathscr{H}_h} \sup_{[\mathbf{v}_h, q_h]\in\mathscr{H}_h} \frac{\mathscr{L}([\mathbf{u}_h, p_h]; [\mathbf{v}_h, q_h])}{\{|\mathbf{u}_h|_1 + \|p_h\|_0\}\{|\mathbf{v}_h|_1 + \|q_h\|_0\}} \geqq \beta_d > 0 \tag{2.7}$$

holds with a constant β_d which is independent of h. Because of ineq. (2.7), problem (2.6) has a unique solution and the error estimate

$$|\mathbf{u}-\mathbf{u}_h|_1 + \|p-p_h\|_0 \leqq ch\{|\mathbf{u}|_2 + |p|_1\}$$

holds provided $\mathbf{u}\in H^2(\Omega)^2$ and $p\in H^1(\Omega)$. Note, that in our analysis we will never need the H^2-regularity of the Stokes problem (2.1).

Each element $\mathbf{v}_h\in X_h$ can uniquely be written in the form $\mathbf{v}_h = \mathbf{v}_{h,l} + \mathbf{v}_{h,b}$ with a linear part $\mathbf{v}_{h,l}\in M_h^{1,0}$ and a bubble part $\mathbf{v}_{h,b}\in\mathscr{B}_h$. In all our computations we observed that the linear part $\mathbf{u}_{h,l}$ of the solution $\mathbf{u}_h$ of Eq. (2.6) is a better approximation to the solution $\mathbf{u}$ of Eq. (2.1) than $\mathbf{u}_h$ itself. This is underlined by experiences with industrial applications of the mini-element [J. Periaux private communication]. At first sight, this phenomenon might be surprising. But it can easily be explained as follows. In practical computations one does not compare $\mathbf{u}_h$ with $\mathbf{u}$ itself but instead with an easily computable interpolation $I_h\mathbf{u}\in X_h$ such as, e.g., the pointwise interpolation in the vertices and barycentres of the triangles. Equation (2.5), however, immediately implies that

$$|\mathbf{u}_{h,l} - (I_h\mathbf{u})_l|_1 \leqq |\mathbf{u}_h - I_h\mathbf{u}|_1 .$$

Moreover, the best approximation by elements of X_h has the same asymptotic order of convergence as the best approximation by elements of $\{M_h^{1,0}\cap H_0^1(\Omega)\}^2$. The bubble functions are only needed in order to stabilize the finite element approximation in the sense of [3, 8]. Due to these observations we will base our error estimators only on the linear part of the error. We therefore define

$$\mathbf{e}_h := \mathbf{u} - \mathbf{u}_{h,l}, \qquad \varepsilon_h := p - p_h . \tag{2.8}$$

Finally, denote by P_0 the L^2-projection onto $M_h^{0,-1}$ and by I_h the standard pointwise interpolation operator by elements of $M_h^{1,0}$.

3. An Error Estimator Based on the Residual

Our first error estimator is based on a suitable evaluation of the residual of the mini-element approximation inserted into Eq. (2.1). More precisely, we define for any $T\in\mathscr{T}_h$

$$\eta_{R,T} := \left\{|T|\,\|P_0\mathbf{f} - \nabla p_h\|_{0,T}^2 + \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} |E|\,\left\|\left[\frac{\partial\mathbf{u}_{h,l}}{\partial\mathbf{n}}\right]_J\right\|_{0,E}^2 + \|\nabla\cdot\mathbf{u}_{h,l}\|_{0,T}^2\right\}^{1/2} \tag{3.1}$$

where $|T|$ and $|E|$ are the area of T and the length of an edge E, resp., and where $\left[\frac{\partial\mathbf{u}_{h,l}}{\partial\mathbf{n}}\right]_J$ denotes the jump of $\frac{\partial\mathbf{u}_{h,l}}{\partial\mathbf{n}}$ across E.

Theorem 3.1. *There are two constants c_0, c_1, which only depend on Ω and on the smallest angle in the triangulation $\mathcal{T}_h$, such that the estimates*

$$|\mathbf{e}_h|_1 + \|\varepsilon_h\|_0 \leqq c_0 \{ \sum_{T\in\mathcal{T}_h} [\eta_{R,T}^2 + |T| \, \|\mathbf{f} - P_0\mathbf{f}\|_{0,T}^2] \}^{1/2} \tag{3.2}$$

and

$$\eta_{R,T} \leqq c_1 \{ |\mathbf{e}_h|_{1,T} + \|\varepsilon_h\|_{0,T} + |T|^{1/2} \, \|\mathbf{f} - P_0\mathbf{f}\|_{0,T} \} \qquad \forall T\in\mathcal{T}_h \tag{3.3}$$

hold.

Proof. Let $\mathbf{v}\in H^2(\Omega)^2 \cap H_0^1(\Omega)^2$ and $q\in L^2(\Omega)$. From standard approximation results [10] we conclude that the inequalities

$$\|\mathbf{v} - I_h\mathbf{v}\|_{0,T} \leqq c\,|T|^{1/2}\,|\mathbf{v}|_{1,T} \tag{3.4}$$

and

$$\|\mathbf{v} - I_h\mathbf{v}\|_{0,E} \leqq c\,|E|^{1/2}\,|\mathbf{v}|_{1,T} \tag{3.5}$$

hold for all $T\in\mathcal{T}_h$ and all edges E of T. Here and in the sequel, c denotes a generic constant which only depends on Ω and on the smallest angle in the triangulation $\mathcal{T}_h$. Subtracting Eq. (2.6) from Eq. (2.3), recalling Eq. (2.5), using partial integration, and applying ineqs. (3.4) and (3.5), we conclude that

$$\begin{aligned}
&\mathcal{L}([\mathbf{e}_h, \varepsilon_h]; [\mathbf{v}, q]) \\
&\quad = (\nabla \mathbf{e}_h, \nabla(\mathbf{v} - I_h\mathbf{v})) - (\varepsilon_h, \nabla\cdot(\mathbf{v} - I_h\mathbf{v})) - (q, \nabla\cdot\mathbf{e}_h) \\
&\quad = \sum_{T\in\mathcal{T}_h} \left\{ (-\Delta\mathbf{e}_h + \nabla\varepsilon_h, \mathbf{v} - I_h\mathbf{v})_T + (q, \nabla\cdot\mathbf{u}_{h,l}) + \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} \left(\left[\frac{\partial\mathbf{e}_h}{\partial\mathbf{n}}\right]_J, \mathbf{v} - I_h\mathbf{v} \right)_E \right\} \\
&\quad \leqq \sum_{T\in\mathcal{T}_h} \Big\{ \|P_0\mathbf{f} - \nabla p_h\|_{0,T}\,\|\mathbf{v} - I_h\mathbf{v}\|_{0,T} + \|\mathbf{f} - P_0\mathbf{f}\|_{0,T}\,\|\mathbf{v} - I_h\mathbf{v}\|_{0,T} \\
&\qquad + \|\nabla\cdot\mathbf{u}_{h,l}\|_{0,T}\,\|q\|_{0,T} + \frac{1}{2}\sum_{E\subset\partial T\cap\Omega} \left\| \left[\frac{\partial\mathbf{u}_{h,l}}{\partial\mathbf{n}}\right]_J \right\|_{0,E} \|\mathbf{v} - I_h\mathbf{v}\|_{0,E} \Big\} \\
&\quad \leqq \{ \sum_{T\in\mathcal{T}_h} [\eta_{R,T}^2 + |T|\,\|\mathbf{f} - P_0\mathbf{f}\|_{0,T}^2] \}^{1/2} \{ |\mathbf{v}|_{1,T} + \|q\|_{0,T} \}.
\end{aligned}$$

Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, this inequality together with ineq. (2.4) implies the estimate (3.2).

In order to prove ineq. (3.3), let $T\in\mathcal{T}_h$ and denote by E_{ij}, $1\leqq i<j\leqq 3$, the edge of T which has the vertices x_i and x_j as its endpoints. Put

$$\psi_T := \lambda_{T1}\lambda_{T2}\lambda_{T3} / \int_T \lambda_{T1}\lambda_{T2}\lambda_{T3}$$

and

$$\psi_{E_{ij}} := \lambda_{Ti}\lambda_{Tj} / \int_{E_{ij}} \lambda_{Ti}\lambda_{Tj}.$$

We define the functions $\mathbf{w}_T$ and q_T on T by

$$\mathbf{w}_T := \frac{1}{2} \sum_{E \subset \partial T \cap \Omega} |E|^2 \left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J \psi_E - \psi_T \left\{ |T|^2 [P_0 \mathbf{f} - \nabla p_h] + \frac{1}{2} \sum_{E \subset \partial T \cap \Omega} |E|^2 \left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J \int_T \psi_E \right\} \tag{3.6}$$

and

$$q_T := \nabla \cdot \mathbf{u}_{h,l}. \tag{3.7}$$

Note, that

$$\int_T \mathbf{w}_T = -|T|^2 \{P_0 \mathbf{f} - \nabla p_h\}, \qquad \int_E \mathbf{w}_T = \frac{1}{2} |E|^2 \left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J \quad \forall E \subset \partial T \cap \Omega.$$

We then obtain

$$\begin{aligned}
\eta_{R,T}^2 &= |T| \, \|P_0 \mathbf{f} - \nabla p_h\|_{0,T}^2 + \frac{1}{2} \sum_{E \subset \partial T \cap \Omega} |E| \left\| \left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J \right\|_{0,E}^2 + \|\nabla \cdot \mathbf{u}_{h,l}\|_{0,T}^2 \\
&= -(P_0 \mathbf{f} - \nabla p_h, \mathbf{w}_T)_T + \sum_{E \subset \partial T \cap \Omega} \left(\left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J, \mathbf{w}_T \right)_E + (\nabla \cdot \mathbf{u}_{h,l}, q_T)_T \\
&= -(P_0 \mathbf{f} - \nabla p_h, \mathbf{w}_T)_T - (\nabla \mathbf{e}_h, \nabla \mathbf{w}_T)_T - (\Delta \mathbf{e}_h, \mathbf{w}_T)_T - (\nabla \cdot \mathbf{e}_h, q_T)_T \\
&= -(\nabla \mathbf{e}_h, \nabla \mathbf{w}_T)_T - (\varepsilon_h, \nabla \cdot \mathbf{w}_T)_T - (q_T, \nabla \cdot \mathbf{e}_h) + (\mathbf{f} - P_0 \mathbf{f}, \mathbf{w}_T)_T \\
&\leqq c \{ |\mathbf{e}_h|_{1,T} + \|\varepsilon_h\|_{0,T} \} \{ |\mathbf{w}_T|_{1,T} + \|q_T\|_{0,T} \} \\
&\quad + \|\mathbf{f} - P_0 \mathbf{f}\|_{0,T} \, \|\mathbf{w}_T\|_{0,T}.
\end{aligned} \tag{3.8}$$

Obviously, we have

$$\|q_T\|_{0,T} = \|\nabla \cdot \mathbf{u}_{h,l}\|_{0,T} \leqq \eta_{R,T}. \tag{3.9}$$

A simple homogeneity argument implies that

$$|\mathbf{w}_T|_{1,T} \leqq c \left\{ |T| \, |P_0 \mathbf{f} - \nabla p_h| + \sum_{E \subset \partial T \cap \Omega} |E| \left| \left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J \right| \right\} \leqq c \, \eta_{R,T} \tag{3.10}$$

and

$$\|\mathbf{w}_T\|_{0,T} \leqq c \, |T|^{1/2} \, |\mathbf{w}_T|_{1,T} \leqq c \, |T|^{1/2} \, \eta_{R,T}. \tag{3.11}$$

Combining ineqs. (3.8)–(3.11), we finally obtain the estimate (3.3). □

Note, that a similar error estimator was investigated in [1]. In contrast to $\eta_{R,T}$ it also incorporates the bubble part of the velocity and uses $\mathbf{f}$ itself instead of $P_0 \mathbf{f}$. However, only a global upper bound similar to ineq. (3.2) is established.

4. An Error Estimator Based on the Solution of Local Stokes Problems

Our second estimator is based on the solution of suitable local Stokes problems. For its description we introduce the spaces

$$\begin{aligned} X_T &:= [\operatorname{span}\{\lambda_{T1}\lambda_{T2}\lambda_{T3}, \lambda_{Ti}\lambda_{Tj}: 1 \leqq i < j \leqq 3, E_{ij} \subset \partial T \cap \Omega\}]^2, \\ Y_T &:= \operatorname{span}\{\lambda_{Ti}\lambda_{Tj}: 1 \leqq i < j \leqq 3\}, \qquad (4.1) \\ \mathscr{H}_T &:= X_T \times Y_T. \end{aligned}$$

Lemma 4.1. *There is a constant $\gamma > 0$ such that the inequality*

$$\inf_{[\mathbf{u},p]\in\mathscr{H}_T} \sup_{[\mathbf{v},q]\in\mathscr{H}_T} \frac{(\nabla\mathbf{u}, \nabla\mathbf{v})_T - (p, \nabla\cdot\mathbf{v})_T - (q, \nabla\cdot\mathbf{u})_T}{\{|\mathbf{u}|_{1,T} + \|p\|_{0,T}\}\{|\mathbf{v}|_{1,T} + \|q\|_{0,T}\}} \geqq \gamma > 0 \qquad (4.2)$$

holds for all $T \in \mathscr{T}_h$.

Proof. Let $T \in \mathscr{T}_h$ and $[\mathbf{u}, p] \in \mathscr{H}_T$ with $|\mathbf{u}|_{1,T} + \|p\|_{0,T} = 1$. Denote by $\hat{T}$ the reference triangle with vertices (0, 0), (1, 0), and (0, 1) and by $F_T(x) := B_T x + b_T$ an affine mapping from $\hat{T}$ onto T. Put

$$\hat{\mathbf{u}} := |\det B_T|^{1/2} B_T^{-t} \mathbf{u}(F_T), \qquad \hat{p} := |\det B_T|^{1/2} p(F_T).$$

A simple calculations shows that ineq. (4.2) holds for the reference triangle. Hence there is a $[\hat{\mathbf{v}}, \hat{q}] \in \mathscr{H}_{\hat{T}}$ with

$$|\hat{\mathbf{v}}|_{1,\hat{T}} + \|\hat{q}\|_{0,\hat{T}} = 1$$

and

$$(\nabla\hat{\mathbf{u}}, \nabla\hat{\mathbf{v}})_{\hat{T}} - (\hat{q}, \nabla\cdot\hat{\mathbf{u}})_{\hat{T}} - (\hat{p}, \nabla\cdot\hat{\mathbf{v}})_{\hat{T}} \geqq \gamma\{|\hat{\mathbf{u}}|_{1,\hat{T}} + \|\hat{p}\|_{0,\hat{T}}\} = \gamma.$$

Put

$$\mathbf{v} := |\det B_T|^{-1/2} B_T^t \hat{\mathbf{u}}(F_T^{-1}), \qquad q := |\det B_T|^{-1/2} \hat{q}(F_T^{-1}).$$

An easy calculation yields

$$|\mathbf{v}|_{1,T} + \|q\|_{0,T} = 1$$

and

$$(\nabla\mathbf{u}, \nabla\mathbf{v})_T - (p, \nabla\cdot\mathbf{v})_T - (q, \nabla\cdot\mathbf{u})_T \geqq \gamma.$$

The assertion now follows by a standard homogeneity argument. □

Because of Lemma 4.1 the local Stokes problem

Find $[\mathbf{u}_T, p_T] \in \mathscr{H}_T$ such that

$$(\nabla\mathbf{u}_T, \nabla\mathbf{v})_T - (q, \nabla\cdot\mathbf{u}_T)_T - (p_T, \nabla\cdot\mathbf{v})_T$$

$$= (P_0\mathbf{f} - \nabla p_h, \mathbf{v})_T + (q, \nabla\cdot\mathbf{u}_{h,l})_T + \sum_{E\subset\partial T\cap\Omega} \left(\left[\frac{\partial\mathbf{u}_{h,l}}{\partial\mathbf{n}}\right]_J, \mathbf{v}\right)_E \quad \forall[\mathbf{v}, q] \in \mathscr{H}_T \quad (4.3)$$

has a unique solution for all $T \in \mathscr{T}_h$. Put

$$\eta_{S,T} := |\mathbf{u}_T|_{1,T} + \|p_T\|_{0,T} \qquad \forall T \in \mathscr{T}_h. \tag{4.4}$$

Note, that the computation of $\eta_{S,T}$ requires for each $T \in \mathscr{T}_h$ the solution of a linear system of equations with 11 unknowns. The following theorem in combination with Theorem 3.1 shows that $\eta_{S,T}$ is globally an upper bound and locally a lower bound for the error of the mini-element approximation (2.6) of the Stokes problem (2.1).

Theorem 4.2. *There are two constants c_2 and c_3, which only depend on Ω and on the smallest angle in the triangulation $\mathscr{T}_h$, such that the estimates*

$$\eta_{S,T} \leqq c_2\, \eta_{R,T} \tag{4.5}$$

and

$$\eta_{R,T} \leqq c_3\, \eta_{S,T} \tag{4.6}$$

hold for all $T \in \mathscr{T}_h$.

Proof. Observing that $\mathbf{v} = \mathbf{v} - I_h \mathbf{v}$ for all $\mathbf{v} \in X_T$ and recalling ineqs. (3.4) and (3.5), we conclude from Lemma 4.1 that

$$\begin{aligned} \eta_{S,T} &\leqq \frac{1}{\gamma} \sup_{[\mathbf{v},q] \in \mathscr{H}_T} \frac{1}{|\mathbf{v}|_{1,T} + \|q\|_{0,T}} \Big\{ \|P_0 \mathbf{f} - \nabla p_h\|_{0,T}\, \|\mathbf{v}\|_{0,T} \\ &\quad + \sum_{E \subset \partial T \cap \Omega} \left\| \left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J \right\|_{0,E} \|\mathbf{v}\|_{0,E} + \|\nabla \cdot \mathbf{u}_{h,l}\|_{0,T}\, \|q\|_{0,T} \Big\} \\ &\leqq c\, \eta_{R,T} \qquad \forall T \in \mathscr{T}_h. \end{aligned}$$

In order to prove ineq. (4.6), let $\mathbf{w}_T \in X_T$ be as in Eq. (3.6) and define $\tilde{q}_T \in Y_T$ by

$$\tilde{q}_T := |T|\, \nabla \cdot \mathbf{u}_{h,l} \sum_{\substack{1 \leqq i < j \leqq 3 \\ E_{ij} \subset \partial T \cap \Omega}} \lambda_{Ti} \lambda_{Tj} \Big/ \int_T \lambda_{Ti} \lambda_{Tj}.$$

From a simple homogeneity argument we conclude that

$$\|\tilde{q}_T\|_{0,T} \leqq c\, |T|^{1/2}\, |\nabla \cdot \mathbf{u}_{h,l}| = c\, \|\nabla \cdot \mathbf{u}_{h,l}\|_{0,T}.$$

Together with Eqs. (3.9), (3.10) and Lemma 4.1 this implies

$$\begin{aligned} \eta_{R,T}^2 &= (P_0 \mathbf{f} - \nabla p_h, \mathbf{w}_T)_T + \sum_{E \subset \partial T \cap \Omega} \left(\left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J, \mathbf{w}_T \right)_E + (\nabla \cdot \mathbf{u}_{h,l}, \tilde{q}_T)_T \\ &= (\nabla \mathbf{u}_T, \nabla \mathbf{w}_T)_T - (\tilde{q}_T, \nabla \cdot \mathbf{u}_T)_T - (p_T, \nabla \cdot \mathbf{w}_T)_T \\ &\leqq c\, \eta_{S,T} \{ |\mathbf{w}|_{1,T} + \|\tilde{q}_T\|_{0,T} \} \\ &\leqq c'\, \eta_{S,T}\, \eta_{R,T}. \end{aligned}$$

This completes the proof. □

Note, that in contrast to [4, 5, 7] we do not need a super-approximation assumption of the form

$$\inf_{\varphi_h \in M_h^{2,0}} \|\varphi - \varphi_h\|_0 = o(1) \inf_{\psi_h \in M_h^{1,0}} \|\varphi - \psi_h\|_0 \qquad \forall \varphi \in H^2(\Omega).$$

On the other hand, we have in Theorem 3.1 the additional higher order term $\sum_{T \in \mathcal{T}_h} |T|^{1/2} \|\mathbf{f} - P_0 \mathbf{f}\|_{0,T}$.

5. Generalization to Nonlinear Problems and to Other Discretization Schemes

The error estimators $\eta_{R,T}$ and $\eta_{S,T}$ can easily be generalized to the mini-element approximation

Find $[\mathbf{u}_h, p_h] \in \mathcal{H}_h$ such that

$$\mathcal{L}([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]) = (\mathbf{f}, \mathbf{v}_h) - \mathrm{Re}((\mathbf{u}_h \cdot \nabla) \mathbf{u}_h, \mathbf{v}_h) \qquad \forall [\mathbf{v}_h, q_h] \in \mathcal{H}_h$$

of the stationary, incompressible Navier-Stokes equations

$$\begin{aligned} -\Delta \mathbf{u} + \mathrm{Re}(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} \quad && \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= 0 && \text{on } \Gamma. \end{aligned}$$

One only has to replace the term $P_0 \mathbf{f} - \nabla p_h$ on the right-hand side of Eqs. (3.1) and (4.3) by $P_0[\mathbf{f} - \mathrm{Re}(\mathbf{u}_h \cdot \nabla) \mathbf{u}_h] - \nabla p_h$. Theorems 3.1 and 4.2 remain valid, provided Re $\|\mathbf{f}\|_0$ is sufficiently small.

The generalization of $\eta_{R,T}$ to the compressible Navier-Stokes equations is currently investigated. We will report on the numerical results in a forthcoming paper.

The results of Sects. 3 and 4 also carry over to other stable mixed finite element discretizations of the Stokes Eq. (2.1) such as, e.g., the Taylor-Hood element [11] which consists of continuous, piecewise quadratic velocities and continuous, piecewise linear pressures. One only has to replace the term $P_0 \mathbf{f} - \nabla p_h$ on the right-hand side of Eqs. (3.1) and (4.3) by $P_0 \mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h$. Similarly, one must replace in addition the term $\left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J$ by $\left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}} - p_h \mathbf{n}\right]_J$ when using discontinuous approximations of the pressure. The function $\mathbf{w}_T$ in the proof of Theorem 3.1 and the spaces X_T and Y_T in Sect. 4 must be chosen such that they can balance the terms $\left[\frac{\partial \mathbf{u}_{h,l}}{\partial \mathbf{n}}\right]_J$ and $P_0 \mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h$. This can always be achieved by constructing suitable bubble functions on the triangles and their edges. For the Taylor-Hood element, e.g., it is sufficient to take for $\mathbf{w}_T$ and X_T piecewise cubic functions, which vanish at the vertices in $\mathcal{T}_h$, and to choose Y_T as for the mini-element. The proofs of Sects. 3 and 4 then carry over immediately.

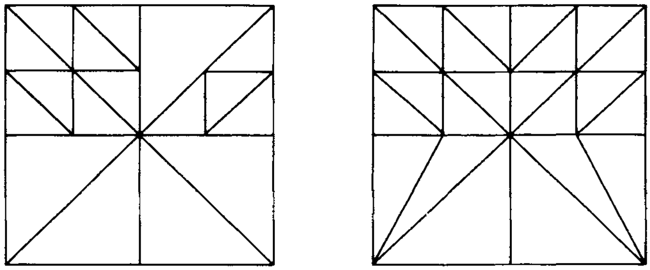

Fig. 1. Triangles divided due to error estimator and resulting triangulation

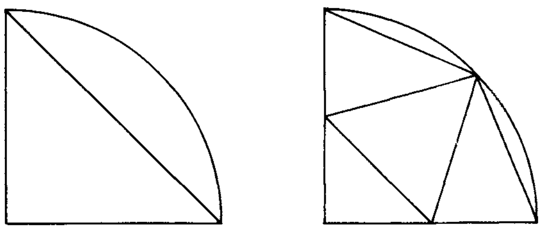

Fig. 2. Projection onto a curved boundary: original and refined triangulation

6. Numerical Examples

In this section we present four numerical examples which demonstrate the efficiency of the described error estimators, both for estimating the error of the finite element discretization and for controlling an automatic, self-adaptive mesh-refinement process. Before describing the examples in more detail, we want to comment on the refinement strategy and on the algorithm for the numerical solution of the discrete problem (2.6).

Starting from an initial triangulation $\mathcal{T}_{h_0}$ of a crude polygonal approximation of Ω, we construct a sequence of refined triangulations $\mathcal{T}_{h_k}$ as follows. Given $\mathcal{T}_{h_j}$, we compute the solution $\mathbf{u}_{h_j}$, p_{h_j} of problem (2.6) and determine

$$\eta_{R,j} := \max_{T \in \mathcal{T}_{h_j}} \eta_{R,T} \quad \text{or} \quad \eta_{S,j} := \max_{T \in \mathcal{T}_{h_j}} \eta_{S,T}.$$

Then all triangles $T \in \mathcal{T}_{h_j}$ with

$$\eta_{R,T} \geqq 0.5\,\eta_{R,j} \quad \text{or} \quad \eta_{S,T} \geqq 0.5\,\eta_{S,j}$$

are divided into four new triangles by joining the midpoints of the edges. In order to obtain a triangulation $\mathcal{T}_{h_{j+1}}$, which satisfies the conditions of Sect. 2, we must divide additional triangles into two or four new ones depending on whether they have one or more neighbours which have been cut into four smaller triangles. This process is illustrated in Fig. 1. In order to improve the approximation of curved boundaries during the refinement process, we project onto the curved boundary those midpoints of edges which belong to edges having their two endpoints on the curved boundary. This is illustrated in Fig. 2. Depending

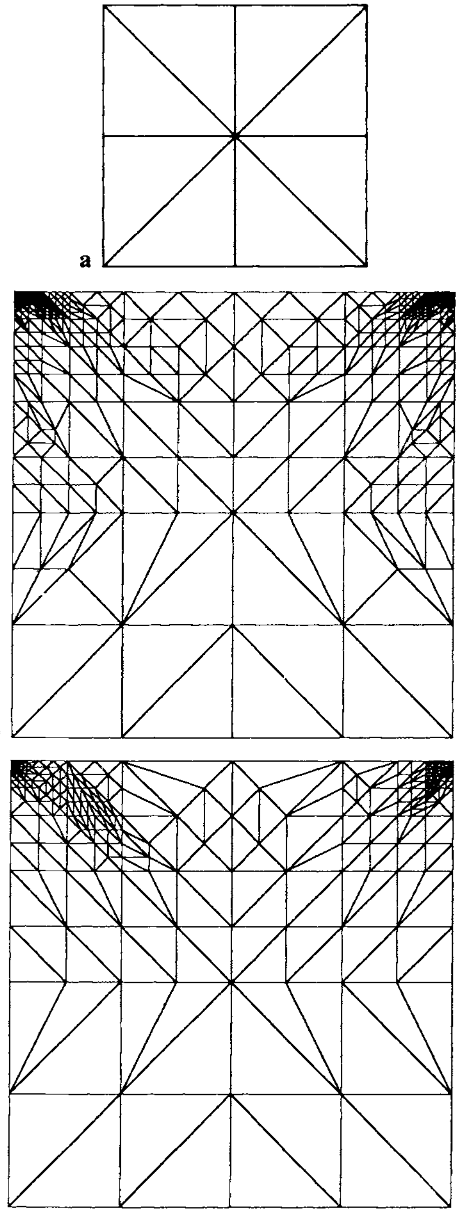

Fig. 3. a $\mathcal{T}_{h_0}$ for example 1 ($NT=8$, $NV=9$). **b** $\mathcal{T}_{h_{10}}$ using refinement strategy 1 for example 1 ($NT=1436$, $NV=773$). **c** $\mathcal{T}_{h_{10}}$ using refinement strategy 2 for example 1 ($NT=544$, $NV=307$)

on whether we use $\eta_{R,T}$ or $\eta_{S,T}$ in the above process, we will refer in the sequel to the first or to the second refinement strategy.

On a given triangulation, we solve the discrete problem (2.6) by a combined conjugate gradient – multi-grid algorithm as described in [12]. For completeness,

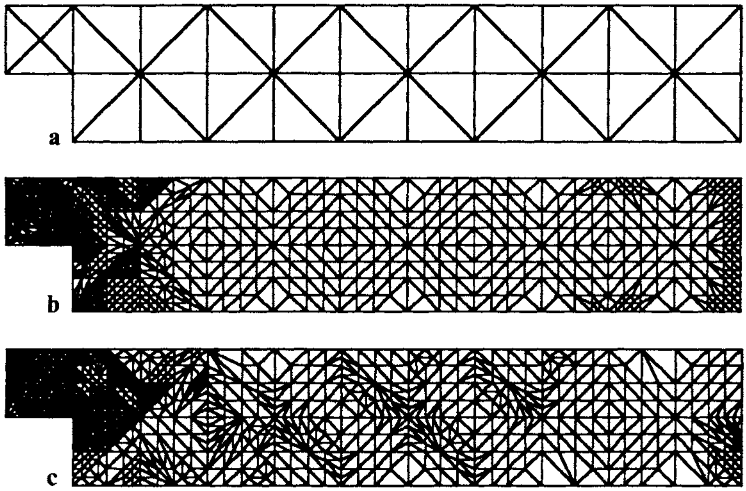

Fig. 4. **a** $\mathscr{T}_{h_0}$ for example 2 ($NT=44$, $NV=36$). **b** $\mathscr{T}_{h_{10}}$ using refinement strategy 1 for example 2 ($NT=1935$, $NV=1042$). **c** $\mathscr{T}_{h_{10}}$ using refinement strategy 2 for example 2 ($NT=1931$, $NV=1029$)

Table 1. cpu time for mesh generation and for solution of discrete problems

	Tt		Ts	
	Strategy 1	Strategy 2	Strategy 1	Strategy 2
Ex. 1	0.77	1.43	10.77	2.77
Ex. 2	1.57	3.40	5.85	8.87

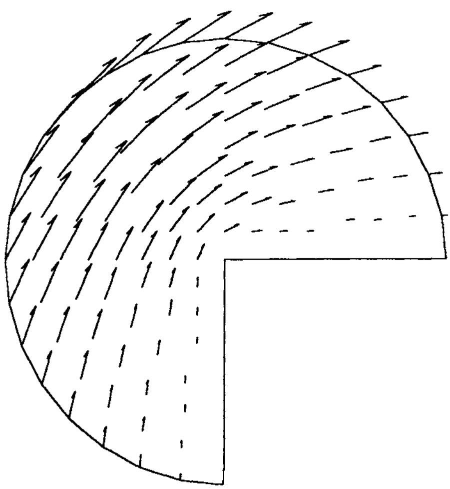

Fig. 5. Velocity field of example 3

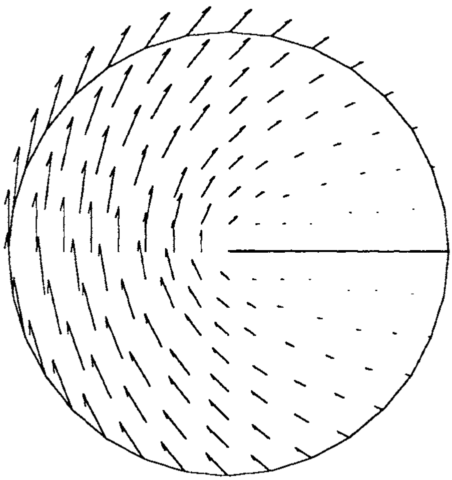

Fig. 6. Velocity field of example 4

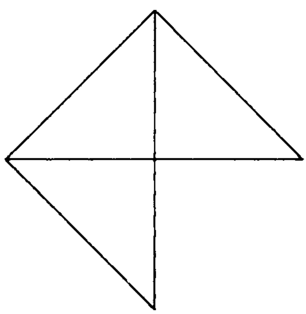

Fig. 7. $\mathcal{T}_{h_0}$ for example 3 ($NT=3$, $NV=5$)

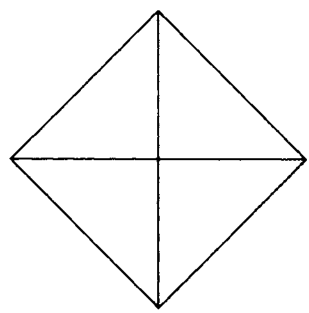

Fig. 8. $\mathcal{T}_{h_0}$ for example 4 ($NT=4$, $NV=6$)

we shortly describe this algorithm. Formally, we eliminate the velocity from the linear system (2.6). Thanks to the Brezzi condition (2.7) the resulting problem for the pressure is symmetric, positive definite, and has a condition number which is independent of h. To this problem we apply a standard conjugate gradient algorithm. In each iteration we then have to solve two discrete Poisson-type equations for the velocity. This is done by applying 2 to 4 iterations of a multi-grid algorithm, which was implemented by G. Wittum and which uses a V-cycle with two symmetric Gauß-Seidel relaxation steps per iteration for pre- and post-smoothing. The conjugate gradient algorithm is additionally preconditioned by a Quasi-Newton method with rank 2 updates as described in [9]. The resulting algorithm has a convergence rate between 0.3 and 0.5 even for highly anisotropic triangulations.

Our first two examples are standard test cases in computational fluid dynamics: (1) the driven cavity with unit tangential velocity at the top of the unit

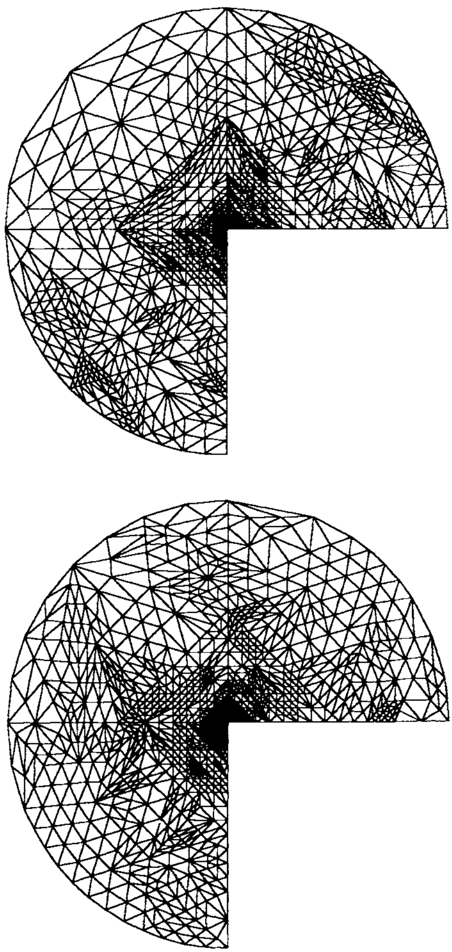

Fig. 9. a $\mathcal{T}_{h_{10}}$ using refinement strategy 1 for example 3 ($NT=2118$, $NV=1114$). **b** $\mathcal{T}_{h_{10}}$ using refinement strategy 2 for example 3 ($NT=2653$, $NV=1379$)

square and (2) the backward facing step with unit parabolic inflow and outflow and with channel width and channel length upstream/downstream of the step of 1/2 and 1/10, resp. In the first example, two singularities arise at the top corners of the square from the change of the boundary condition. In the second example, a singularity arises at the step from the re-entrant corner. Figures 3 and 4 show for both examples the initial triangulation and the triangulations after 10 refinement steps using the two refinement strategies.

Here, NT and NV denote the number of triangles and of vertices, resp. Figures 3 and 4 clearly show the ability of the two error estimators to identify the regions with a singular behaviour of the solution. For both examples we list in Table 1 the cpu time on the IBM 3090 of the Universität Heidelberg for generating the triangulation (Tt) and for solving the resulting discrete problems (Ts).

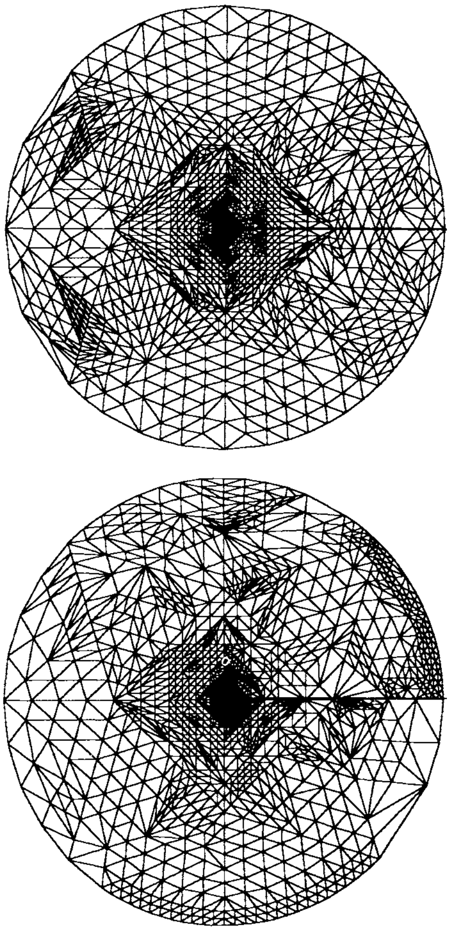

Fig. 10. a $\mathcal{T}_{h_{11}}$ using refinement strategy 1 for example 4 ($NT=3242$, $NV=1678$). **b** $\mathcal{T}_{h_{12}}$ using refinement strategy 2 for example 4 ($NT=3304$, $NV=1722$)

Table 2. Comparison of uniform refinement and of local refinement strategies

		NT	NV	NN	$\varepsilon_{\mathbf{u}}$	ε_p	q	Tt	Ts
Ex. 3	Uniform	12288	6305	42852	0.0189	0.0675		1.14	42.06
	Strategy 1	2118	1114	7365	0.0221	0.0572	0.3	1.40	13.30
	Strategy 2	2653	1379	9236	0.0242	0.0558	0.7	2.96	15.06
Ex. 4	Uniform	16384	8385	57156	0.0220	0.2025		1.53	40.60
	Strategy 1	3242	1678	11293	0.0326	0.2142	0.3	1.68	13.51
	Strategy 2	3304	1722	11497	0.0381	0.2210	0.8	7.27	14.78

In examples 3 and 4 we consider the Stokes equations in a circular segment with radius 1 and angle $3\pi/2$ and 2π, resp. and with homogeneous right-hand side, homogeneous Dirichlet boundary conditions on the straight parts of the boundary, and non-homogeneous Dirichlet boundary conditions on the curved part of the boundary. The exact solutions are given by

$$\mathbf{u}=(r^{\alpha}[(1+\alpha)\sin\varphi\,\psi(\varphi)+\cos\varphi\,\partial_{\varphi}\psi(\varphi)],$$

$$r^{\alpha}[\sin\varphi\,\partial_{\varphi}\psi(\varphi)-(1+\alpha)\cos\varphi\,\psi(\varphi)]),$$

$$p=-r^{\alpha-1}[(1+\alpha)^2\,\partial_{\varphi}\psi(\varphi)+\partial_{\varphi}^3\psi(\varphi)]/(1-\alpha),$$

with

$$\psi(\varphi)=\sin((1+\alpha)\varphi)\cos(\alpha\omega)/(1+\alpha)-\cos((1+\alpha)\varphi)$$
$$+\sin((\alpha-1)\varphi)\cos(\alpha\omega)/(1-\alpha)+\cos((\alpha-1)\varphi),$$

$$\alpha=856399/1572864, \qquad \omega=3\pi/2$$

for example 3 and

$$\psi(\varphi)=3\sin(0.5\,\varphi)-\sin(1.5\,\varphi),$$

$$\alpha=0.5, \qquad \omega=2\pi$$

for example 4. The velocity fields are shown in Figs. 5 and 6.

Starting from the initial triangulations shown in Figs. 7 and 8, we computed for both examples the discrete solutions on triangulations, which were obtained by uniformly refining the initial triangulation five times. Then, we applied the two refinement strategies until we obtained a discrete solution which has the same error when compared with the exact solution of the continuous problem as the one obtained by uniform refinement. The resulting triangulations are shown in Figs. 9 and 10.

In Table 2 we compare for both examples the two refinement strategies with the uniform refinement. There, NN, $\varepsilon_{\mathbf{u}}$, and ε_p denote the number of unknowns, the relative error of the velocity in the H^1-norm, and the relative error of the pressure in the L^2-norm, resp. The quantity q is the ratio of the actual error $|\mathbf{e}_{h_k}|_1+\|\varepsilon_{h_k}\|_0$ and of the estimated error. It is a measure for the quality of the error estimator. In contrast to positive definite problems [4, 5, 7], we cannot expect q to approach 1 since the constants in Theorems 3.1 and 4.2 depend on the inverse of the constants in the Brezzi conditions (2.4) and (2.7).

Figures 9 and 10, and Table 2 clearly show the efficiency of the two error estimators. The cpu-time for the solution process could still be improved by suitably adapting the techniques of [6] for multi-grid algorithms on locally refined grids.

Summarizing our numerical experiences with the error estimators presented here, we conclude that both are well suited for estimating the error of the finite element discretization and for controlling an automatic, self-adaptive mesh-refinement process. If one is primarily interested in the adaptive mesh-refinement, the estimator $\eta_{R,T}$ is preferable, since it is less expensive and yields reasonable triangulations. The estimator $\eta_{S,T}$ on the other hand, yields a slightly better quantitative information on the error of the discrete solution.

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