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ERROR ESTIMATES OF THE CLASSICAL AND IMPROVED TWO-GRID METHODS

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Abstract. In this paper, we obtain the first error estimate in L^2-norm for the classical two-grid method, then design an improved two-grid method by adding one more correction on coarse space to the classical two-gird method. Furthermore, we also present the error estimates both in L^2-norm and H^1-norm for the improved two-grid method. Especially, the L^2 error estimate of the improved two-grid method is one order higher than the L^2 error estimate of the classical two-grid. At last, we confirm and illustrate the theoretical result by numerical experiments.

AMS subject classifications: 65N30, 65B99

Key words: Two-grid methods, error estimate.

1 Introduction

The two-grid methods, firstly, introduced by Xu [13] for nonsymmetric or indefine linear elliptic partial differential equations, have been successfully applied to solve many problems in the last two decades, such as nonlinear elliptic problems [14–16,18], nonlinear parabolic equations [2,3], Navier-Stokes problems [5,8], Maxwell equations [20, 21], and eigenvalue problems [6, 9, 19, 22], etc. The main idea of the two-grid method is first to solve the original problem in a coarse mesh space with mesh size H, and then to solve an corresponding symmetric positive definite (SPD) problem in

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a fine mesh space with mesh size $h \ll H$. However, to our knowledge, there exists no work in the literature, which studies the L^2-norm for the two-grid methods.

Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. we consider the two-grid methods for solving the following nonsymmetric and/or indefinite linear partial differential equations

$$-\mathrm{div}(\alpha(x)\nabla u(x)) + \beta(x)\cdot\nabla u(x) + \gamma(x)u(x) = f(x), \qquad x \in \Omega, \tag{1.1}$$
$$u(x) = g(x), \qquad x \in \partial\Omega, \tag{1.2}$$

where $\alpha(x) \in \mathbb{R}^{2\times 2}$ is smooth functions on $\overline{\Omega}$ satisfying elliptic condition, namely, for some positive constant α_0,

$$\xi^T\alpha(x)\xi \geq \alpha_0|\xi|^2 \quad \forall\, \xi \in \overline{\Omega},$$

both $\beta(x) \in \mathbb{R}^2$ and $\gamma(x) \in \mathbb{R}^1$ are also smooth functions on $\overline{\Omega}$, $f(x)$ and $g(x)$ are given functions.

The classical two-grid methods for solving (1.1)-(1.2) (See Algorithm 3.1) are referred to [12, 13, 17], however, only the error estimates in H^1-norm existed in the previous literature. This paper provides an analysis of the first error estimate in L^2-norm for the classical two-grid method, and obtains H^1 error estimate by using a new proof. Furthermore, we design an improved two-grid method by adding one more correction on coarse space to the classical two-gird method, and provide the corresponding error estimates both in L^2-norm and H^1-norm for the improved two-grid method. More details, we has proved that the L^2 error estimate of the improved two-grid method is one order higher than the L^2 error estimate of the classical two-grid methods, although their H^1 error estimates are the same order. At last, we present some numerical results to assess and validate the theory developed for the proposed methods.

To avoid the repeated use of generic but unspecified constants, following [11], we use the notation $a \lesssim b$ meaning that there exists a positive constant C such that $a \leq Cb$, the above generic constants C are independent of the function under consideration, but they may depend on Ω and the shape-regularity of the meshes.

The remainder of the paper is organized as follows. In Section 2, we present the model problem and some preliminaries. In Section 3, we obtain the first error estimate in L^2-norm for the classical two-grid method. In Section 4, we design and analyze an improved two-grid method. Finally, we report some numerical experiments in support of the efficiency of the methods in Section 5.

2 Model Problem and Preliminaries

In this section, we will discuss the non-symmetric and/or indefinite linear partial differential equations and the corresponding finite element discretizations.

2.1 Linear elliptic differential operators

We shall consider the following two elliptic differential operators

$$\mathcal{L}v = -\mathrm{div}(\alpha(x)\nabla v) \quad \text{and} \quad \hat{\mathcal{L}}\, v = \mathcal{L}v + \beta(x)\cdot\nabla v + \gamma(x)v. \tag{2.1}$$

It is easy to know that $\mathcal{L} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is an isomorphism, by using Lax-Milgram Theorem. Our basic assumption is that $\hat{\mathcal{L}} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is also an isomorphism. An application of the open mapping theorem yields

$$\|v\|_1 \lesssim \|\hat{\mathcal{L}}v\|_{-1} \quad \forall\, v \in H_0^1(\Omega). \tag{2.2}$$

Here, we use the standard Sobolev functional space and the corresponding norm (see [1]).

It is easy to see that if $\hat{\mathcal{L}}$ satisfies the above assumption, so does its formal adjoint:

$$\hat{\mathcal{L}}^* u = -\mathrm{div}(\alpha(x)\nabla u + \beta(x)u) + \gamma(x)u.$$

Namely $\hat{\mathcal{L}}^* : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ is also isomorphic and satisfies (2.2).

Corresponding to differential operators $\mathcal{L}$ and $\hat{\mathcal{L}}$, then for $u, v \in H_0^1(\Omega)$, we can define the following two bilinear forms

$$a(u,v) \;=\; < \mathcal{L}u, v > = \int_\Omega \alpha(x)\nabla u\cdot\nabla v \,\mathrm{d}\boldsymbol{x}, \tag{2.3}$$

$$\hat{a}(u,v) \;=\; < \hat{\mathcal{L}}u, v > = a(u,v) + \int_\Omega ((\beta\cdot\nabla u)v + \gamma(x)uv)\,\mathrm{d}\boldsymbol{x}. \tag{2.4}$$

We shall use the following well-known regularity result (see [4]).

Lemma 2.1. *If $u \in H_0^1(\Omega)$ and $\hat{\mathcal{L}}u \in \mathcal{L}^2(\Omega)$, then $u \in H^2(\Omega)$ and*

$$\|u\|_2 \lesssim \|\hat{\mathcal{L}}u\|_0,$$

where the constant depends on the coefficients of $\hat{\mathcal{L}}$ and the domain Ω.

2.2 Finite element discretizations and Galerkin projections

We assume that Ω is partitioned by a quasi-uniform triangulation $\mathcal{T}_h = \{\tau_i\}$. By this we mean that τ_i's are simplexes of size h with $h \in (0,1)$ and $\bar{\Omega} = \cup_i \bar{\tau}_i$ and there exist constants C_0 and C_1 not depending on h such that each element τ_i is contained in (contains) a ball of radius $C_1 h$ (respectively $C_0 h$).

For a given triangulation $\mathcal{T}_h$, a conforming finite element space $\mathbb{V}_h \subset \mathbb{V} \equiv H_0^1(\Omega)$ is defined by

$$\mathbb{V}_h = \{v_h \in C(\bar{\Omega}) : v_h|_\tau \in \mathcal{P}_k \quad \text{for all } \tau \in \mathcal{T}_h, v_h|_{\partial\Omega} = 0\},$$

where $\mathcal{P}_k$ is the set of polynomials of total degree at most k.

With the bilinear form $\hat{a}$ defined in (2.4), we obtain the the discrete variational problem (1.1)-(1.2): Find $u_h \in \mathbb{V}_h$ such that

$$\hat{a}(u_h,\chi) = (f,\chi) \quad \forall\, \chi \in \mathbb{V}_h. \tag{2.5}$$

For the nonsymmetric and/or indefinite problems, the following result (based on Schatz [10]) is of fundamental importance.

Lemma 2.2 (Lemma 8.2 of [17]). *If $h \ll 1$, then*

$$\|v_h\|_1 \lesssim \sup_{\varphi\in\mathbb{V}_h} \frac{\hat{a}(v_h,\varphi)}{\|\varphi\|_1} \quad \textit{and} \quad \|v_h\|_1 \lesssim \sup_{\varphi\in\mathbb{V}_h} \frac{\hat{a}(\varphi,v_h)}{\|\varphi\|_1} \quad \forall\, v_h \in \mathbb{V}_h. \tag{2.6}$$

Let $\hat{P}_h : \mathbb{V} \longrightarrow \mathbb{V}_h$ be the standard Galerkin projection defined by

$$\hat{a}(\hat{P}_h v,\chi) = \hat{a}(v,\chi) \quad \forall\, \chi \in \mathbb{V}_h. \tag{2.7}$$

The following lemma presents some basic error estimate for $\hat{P}_h$.

Lemma 2.3 (Lemmas 8.3 and 8.5 in [17]). *If $h \ll 1$, then $\hat{P}_h$ is well-defined and admits the following estimate*

$$\|u-\hat{P}_h u\|_0 + h\|u-\hat{P}_h u\|_1 \lesssim h^{r+1}\|u\|_{r+1} \quad \forall\, u \in H^{r+1}(\Omega).$$

3 Classical Two-Grid Method

In this section, we will provide an analysis of the first error estimate in L^2-norm for the classical two-grid method, and obtain H^1 error estimate by using a new proof.

We first describe the classical two-grid method in [13,15,17]. The basic mechanisms in our approach are two quasi-uniform triangulations of Ω, $\mathcal{T}_H$ and $\mathcal{T}_h$, with two different mesh sizes H and h ($H > h$), and the corresponding finite element spaces $\mathbb{V}_H$ and $\mathbb{V}_h$ ($\mathbb{V}_H \subset \mathbb{V}_h$) which will be called coarse and fine space respectively. In the applications given below, we shall always assume that

$$H = O(h^\lambda), \quad \text{for some} \quad 0 < \lambda < 1. \tag{3.1}$$

To discuss the algorithm and corresponding error estimates conveniently, we introduce the lower order terms of the operator $\hat{\mathcal{L}}$ (see (2.1)) by

$$\begin{aligned} N(v,\chi) &= \hat{a}(v,\chi) - a(v,\chi) \\ &= (\beta\cdot\nabla v,\chi) + (\gamma v,\chi). \end{aligned} \tag{3.2}$$

Now, we present the classical two-grid algorithm, seeing Algorithm 2.1 of [15] or Algorithm 8.1 of [17].

Algorithm 3.1 (Classical Two-Grid Method)**.**

1. Find $u_H \in \mathbb{V}_H$ such that

$$\hat{a}(u_H, \varphi) = (f, \varphi) \quad \forall\, \varphi \in \mathbb{V}_H. \tag{3.3}$$

2. Find $u^h \in \mathbb{V}_h$ such that

$$a(u^h, \chi) + N(u_H, \chi) = (f, \chi) \quad \forall\, \chi \in \mathbb{V}_h. \tag{3.4}$$

We note that the linear system in the second step of the above algorithm is SPD.

All of existed error analysis focus on H^1 error estimate, e.g., the following theoretical result can be found in [17]

Lemma 3.1 (Theorem 8.6 in [17])**.** *For $H \ll 1$, assume that $u_h \in \mathbb{V}_h$ satisfy* (2.5)*, and $u^h \in \mathbb{V}_h$ is the solution obtained by Algorithm 3.1, then*

$$\|u_h - u^h\|_1 \lesssim H^{r+1}\|u\|_{r+1} \quad \textit{and} \quad \|u - u^h\|_1 \lesssim (h^r + H^{r+1})\|u\|_{r+1}$$

provided that $u \in H^{r+1}(\Omega)$.

Next, we reestablish the H^1 error estimate of Algorithm 3.1 by using new method, our result is also slightly different from previous one (See Lemma 3.1).

Lemma 3.2. *For $H \ll 1$, assume that $u_h \in \mathbb{V}_h$ satisfy* (2.5)*, and $u^h \in \mathbb{V}_h$ is the solution obtained by Algorithm 3.1, then*

$$\|u_h - u^h\|_1 \lesssim (h^{r+1} + H^{r+1})\|u\|_{r+1}$$

provided that $u \in H^{r+1}(\Omega)$.

Proof. By using (2.5), (3.4), (3.2) with Green formula and Cauchy-Schwarz inequality, then for any $\chi \in V_h$, we have

$$\begin{aligned} a(u_h - u^h, \chi) &= [(f,\chi) - N(u_h,\chi)] - [(f,\chi) - N(u_H,\chi)] \\ &= N(u_H - u_h, \chi) \\ &\lesssim \|u_H - u_h\|_0 \|\chi\|_1. \end{aligned}$$

Setting $\chi = u_h - u^h$, and making use of $a(\cdot,\cdot) \approx \|\cdot\|_1^2$, triangle inequality and standard finite element error estimate, we have

$$\begin{aligned} \|u_h - u^h\|_1 &\lesssim \|u_H - u_h\|_0 \\ &\leq \|u - u_H\|_0 + \|u - u_h\|_0 \\ &\lesssim (h^{r+1} + H^{r+1})\|u\|_{r+1}. \end{aligned}$$

The rest of the section is contributed to establish the first error estimate in L^2-norm for the classical two-grid method.

Lemma 3.3. *For $H \ll 1$, assume that $u_h \in \mathbb{V}_h$ satisfy* (2.5)*, and $u^h \in \mathbb{V}_h$ is the solution obtained by Algorithm 3.1, then*

$$\|u_h - u^h\|_0 \lesssim (h^r + H^{r+1})\|u\|_{r+1}$$

provided that $u \in H^{r+1}(\Omega)$.

Proof. By using triangle inequality and standard finite element error estimate, we have

$$\begin{aligned}\|u_h - u^h\|_0 &\leq \|u - u_h\|_0 + \|u - u^h\|_0 \\ &\leq \|u - u_h\|_0 + \|u - u^h\|_1 \\ &\lesssim h^{r+1}\|u\|_{r+1} + (h^r + H^{r+1})\|u\|_{r+1},\end{aligned}$$

which concludes the result by noting that $0 < h < H$.

Remark 3.1. *In view of Lemmas 3.3 and 3.1, for the classical two-grid method, we note that both the L^2 error estimate and H^1 error estimate are of the same order.*

In order to improve the order of L^2 error estimate, we will design and analyze an improved two-grid method in the next section.

4 Improved Two-Grid Method

In this section, we first present an improved two-grid method by just adding one coarse grid correction for Algorithm 3.1, then analyze the corresponding error estimate.

Algorithm 4.1 (Improved Two-Grid Method)**.**

1. Find $u_H \in \mathbb{V}_H$ such that

$$\hat{a}(u_H, \varphi) = (f, \varphi) \quad \forall\, \varphi \in \mathbb{V}_H. \tag{4.1}$$

2. Find $u^h \in \mathbb{V}_h$ such that

$$a(u^h, \chi) + N(u_H, \chi) = (f, \chi) \quad \forall\, \chi \in \mathbb{V}_h. \tag{4.2}$$

3. Find $e_H \in \mathbb{V}_H$ such that

$$\hat{a}(e_H, \varphi) = N(u_H - u^h, \varphi) \quad \forall\, \varphi \in \mathbb{V}_H. \tag{4.3}$$

4. Set

$$u^h_* = u^h + e_H. \tag{4.4}$$

Remark 4.1. *To solve problem* (4.3) *is equivalent to solve the following residual problem*

$$\hat{a}(e_H, \varphi) = (f, \varphi) - \hat{a}(u^h, \varphi) \quad \forall\, \varphi \in \mathbb{V}_H. \tag{4.5}$$

In fact, by using (4.2), $\mathbb{V}_H \subset \mathbb{V}_h$ *and* (3.2), *we have*

$$\begin{aligned} N(u_H - u^h, \varphi) &= N(u_H, \varphi) - N(u^h, \varphi) \\ &= \Big((f, \varphi) - a(u^h, \varphi) \Big) - N(u^h, \varphi) \\ &= (f, \varphi) - \hat{a}(u^h, \varphi) \end{aligned}$$

Lemma 4.1. *For $H \ll 1$, assume that u_h is the solution of* (2.5), *and u_*^h is given by Algorithm 4.1, then*

$$\|u_h - u_*^h\|_0 \lesssim H\|u_h - u_*^h\|_1. \tag{4.6}$$

Proof. Let us first prove the following Galerkin orthogonality

$$\hat{a}(u_h - u_*^h, \varphi) = 0, \quad \forall \varphi \in \mathbb{V}_H. \tag{4.7}$$

In fact, by using (2.5), (4.4) and (4.5), then for any $\varphi \in \mathbb{V}_H \subset \mathbb{V}_h$, we have

$$\begin{aligned} \hat{a}(u_h - u_*^h, \varphi) &= \hat{a}(u_h, \varphi) - \hat{a}(u_*^h, \varphi) \\ &= (f, \varphi) - \hat{a}(u^h + e_H, \varphi) \\ &= (f, \varphi) - (f, \varphi) = 0. \end{aligned}$$

Second, we prove (4.6).

To this end, we introduce the following auxiliary problem: Find $w \in H_0^1(\Omega) \cap H^2(\Omega)$, such that

$$\hat{a}(\chi, w) = (u_h - u_*^h, \chi) \quad \forall\, \chi \in \mathbb{V}_h. \tag{4.8}$$

Let $w_H \in \mathbb{V}_H$ be the L^2 projection of w, then

$$\|w - w_H\|_0 + H\|w - w_H\|_1 \lesssim H^2\|w\|_2 \lesssim H^2\|u_h - u_*^h\|_0, \tag{4.9}$$

In the last inequality, we apply Lemma 2.1 to operator $\mathcal{L}^*$.

Then using (4.7), (4.8), Cauchy-Schwarz inquality and(4.9), we have

$$\begin{aligned} \|u_h - u_*^h\|_0^2 &= \hat{a}(u_h - u_*^h, w) \\ &= \hat{a}(u_h - u_*^h, w - w_H) \\ &\lesssim \|u_h - u_*^h\|_1 \|w - w_H\|_1 \\ &\lesssim \|u_h - u_*^h\|_1 H \|u_h - u_*^h\|_0, \end{aligned}$$

which completes the proof of (4.6).

Lemma 4.2. *Under the hypotheses of Lemma 4.1, we have*

$$\|u_h - u_*^h\|_1 \lesssim \|u_h - u^h\|_1, \tag{4.10}$$

where u^h is given by (4.2).

Proof. Firstly, we prove the following equalities

$$e_H = u_H - \hat{P}_H u^h = \hat{P}_H(u_h - u^h). \tag{4.11}$$

In fact, by using (2.5), (4.5) and the definition of $\hat{P}_H$ (2.7), then for any $\varphi \in \mathbb{V}_H \subset \mathbb{V}_h$, we have

$$\begin{aligned} \hat{a}(u_H, \varphi) &= (f, \varphi) \\ &= \hat{a}(e_H + u^h, \varphi) \\ &= \hat{a}(e_H, \varphi) + \hat{a}(u^h, \varphi) \\ &= \hat{a}(e_H, \varphi) + \hat{a}(\hat{P}_H u^h, \varphi) \end{aligned}$$

which completes the proof of the first equality. The second equality follows by using the fact $u_H = \hat{P}_H u_h$.

Secondly, we prove (4.10).

Combining (4.4), (4.11), Cauchy-Schwarz inequality and Lemma 2.3, we obtain

$$\begin{aligned} \hat{a}(u_h - u_*^h, \chi) &= \hat{a}(u_h - u^h - e_H, \chi) \\ &= \hat{a}((I - \hat{P}_H)(u_h - u^h), \chi) \\ &\lesssim \|(I - \hat{P}_H)(u_h - u^h)\|_1 \|\chi\|_1 \\ &\lesssim \|u_h - u^h\|_1 \|\chi\|_1. \end{aligned}$$

The desired estimate then follows by using Lemma 2.2.

Using Lemmas 3.2, 4.2 and 4.1, we obtain the error estimate of Algorithm 4.1.

Theorem 4.1. *For $H \ll 1$, assume that u_h is the solution of* (2.5), *and u_*^h is given by Algorithm 4.1, then*

$$\|u_h - u_*^h\|_0 \lesssim H(h^{r+1} + H^{r+1})\|u\|_{r+1}, \quad \|u_h - u_*^h\|_1 \lesssim (H^{r+1} + h^{r+1})\|u\|_{r+1}$$

provided that $u \in H^{r+1}(\Omega)$.

Proof. Combining Lemma 4.2 with Lemma 3.2, we have

$$\|u_h - u_*^h\|_1 \lesssim \|u_h - u^h\|_1 \lesssim (h^{r+1} + H^{r+1})\|u\|_{r+1},$$

which implies the second inequality. The first inequality obviously follows from Lemma 4.1 and the second inequality.

5 Numerical Experiments

In this section, we will present some numerical experiments to demonstrate the efficiency of Algorithm 4.1. In all experiments, our computational domain is $\Omega = (0,1)^2$, the coefficients in partial differential equation (1.1) are as follows: $\alpha(x) = 1, \boldsymbol{\beta}(x) = (0,0)^t$ and different parameters $\gamma(x) = -1, -5, -10$, the true solution of (1.1)-(1.2) is $u(x) = \sin(\pi x)\cos(\pi y)$, then it is easy to obtain $f = (2\pi^2 + \gamma)\sin(\pi x)\cos(\pi y), g = \sin(\pi x)\cos(\pi y)$. We first partition the $x-$ and $y-$axes of the domain Ω into equally distributed subintervals, then divide one square into two triangles by using the line with slope -1. We choose piece linear conform finite element space. In particular, we use minimal residual (MINRES) method to solve the indefinite problems in (4.1) and (4.3), and use preconditioned conjugate gradient (PCG) method to solve SPD problem in (4.2), a relative reduction of the Euclidean norm of the residual vector by a factor of 10^{-10}, which is used as the termination criterion for both MINRES methods and PCG methods.

We present the numerical results for errors between the standard finite solution u_h and the improved two-grid solution u_*^h for a set of combination of h and H with different parameters $\gamma(x)$ in Tables 1 - 3 and Figures 1 - 3 . Here, we choose $H^2 = h$ with different H.

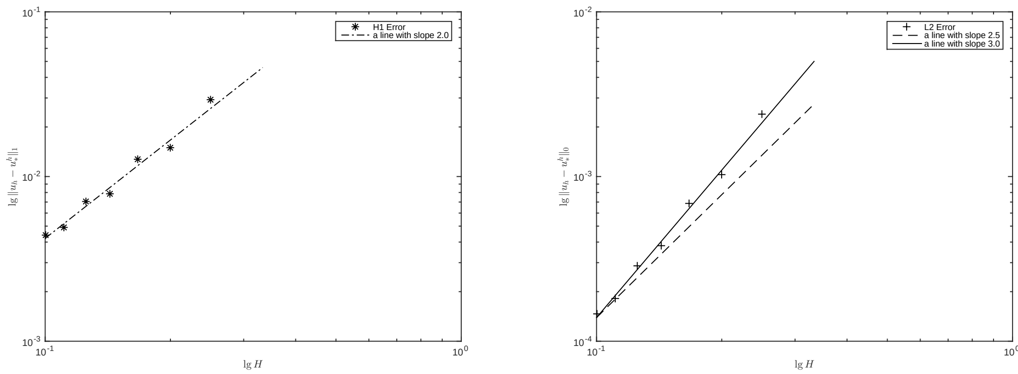
For $\gamma(x) = -1$, Table 1 shows that $\|u_h - u_*^h\|_0$ is three order and $\|u_h - u_*^h\|_1$ is second order with respect to coarse mesh size H, which is in support of the corresponding result in Theorem 4.1.

H	h	$\|u_h - u_*^h\|_0$	$\|u_h - u_*^h\|_0 * H^{-3}$	$\|u_h - u_*^h\|_1$	$\|u_h - u_*^h\|_1 * H^{-2}$
$\frac{1}{4}$	$\frac{1}{16}$	2.4094e-3	0.1542	2.9469e-2	0.4715
$\frac{1}{5}$	$\frac{1}{25}$	1.0280e-3	0.1285	1.5060e-2	0.3765
$\frac{1}{6}$	$\frac{1}{36}$	6.8519e-4	0.1480	1.2819e-2	0.4615
$\frac{1}{7}$	$\frac{1}{49}$	3.8280e-4	0.1313	7.8878e-3	0.3865
$\frac{1}{8}$	$\frac{1}{64}$	2.8633e-4	0.1466	7.0547e-3	0.4515
$\frac{1}{9}$	$\frac{1}{81}$	1.8134e-4	0.1322	4.8951e-3	0.3965
$\frac{1}{10}$	$\frac{1}{100}$	1.4590e-4	0.1459	4.4150e-3	0.4415

Table 1: Errors for Algorithm 4.1 ($\gamma(x) = -1$)

Furthermore, in order to show clear convergence phenomena, we plot the $\lg - \lg$ figure according to Table 1, then the slopes of the approximating lines are the corresponding convergence orders.

For $\gamma(x) = -5$ and $\gamma(x) = -10$, we report the corresponding numerical experiments in Tables 2 - 3 and Figures 2 - 3. We find that $\|u_h - u_*^h\|_0$ is still three order and $\|u_h - u_*^h\|_1$ is still second order with respect to coarse mesh size H.

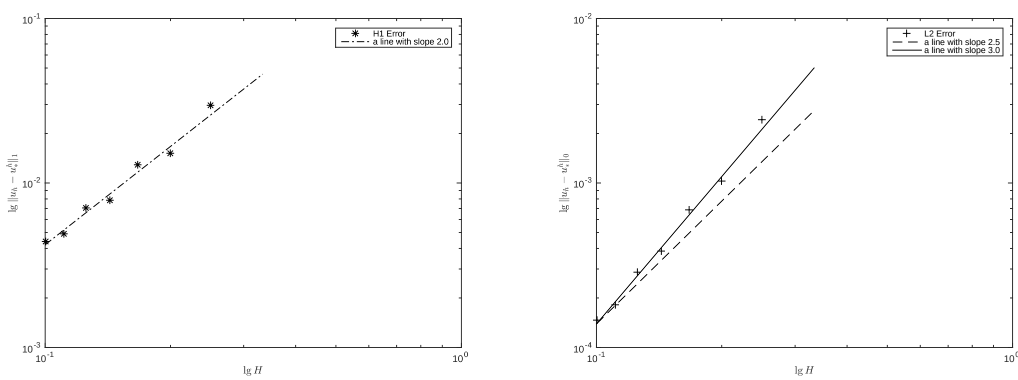


(a) H1 Error (b) L2 Error

Figure 1: $\lg - \lg$ plot of Errors for Algorithm 4.1 ($\gamma(x) = -1$)

H	h	$\|u_h - u_*^h\|_0$	$\|u_h - u_*^h\|_0 * H^{-3}$	$\|u_h - u_*^h\|_1$	$\|u_h - u_*^h\|_1 * H^{-2}$
$\frac{1}{4}$	$\frac{1}{16}$	2.4109e-3	0.1543	2.9475e-2	0.4716
$\frac{1}{5}$	$\frac{1}{25}$	1.0288e-3	0.1286	1.5064e-2	0.3766
$\frac{1}{6}$	$\frac{1}{36}$	6.8565e-4	0.1480	1.2822e-2	0.4615
$\frac{1}{7}$	$\frac{1}{49}$	3.8309e-4	0.1314	7.8898e-3	0.3866
$\frac{1}{8}$	$\frac{1}{64}$	2.8652e-4	0.1466	7.0562e-3	0.4515
$\frac{1}{9}$	$\frac{1}{81}$	1.8148e-4	0.1323	4.8963e-3	0.3966
$\frac{1}{10}$	$\frac{1}{100}$	1.4600e-4	0.1460	4.4159e-3	0.4416

Table 2: Errors for Algorithm 4.1 ($\gamma(x) = -5$)



(a) H1 Error (b) L2 Error

Figure 2: $\lg - \lg$ plot of Errors for Algorithm 4.1 ($\gamma(x) = -5$)

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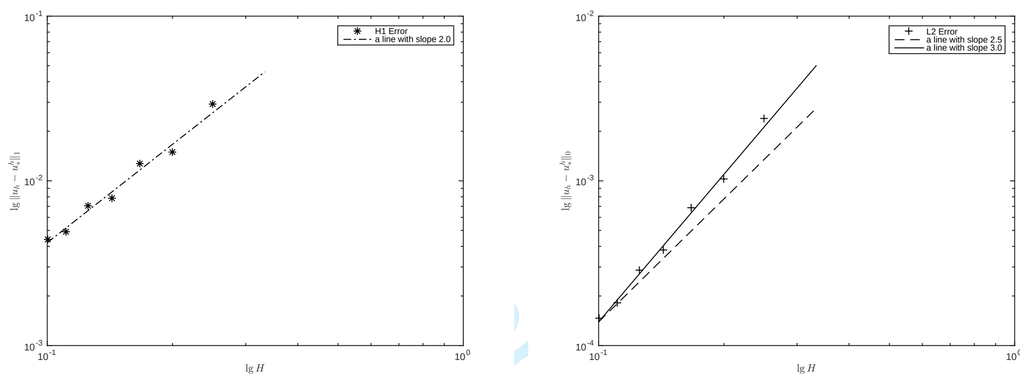
H	h	$\|u_h - u_*^h\|_0$	$\|u_h - u_*^h\|_0 * H^{-3}$	$\|u_h - u_*^h\|_1$	$\|u_h - u_*^h\|_1 * H^{-2}$
$\frac{1}{4}$	$\frac{1}{16}$	2.4078e-3	0.1541	2.9463e-2	0.4714
$\frac{1}{5}$	$\frac{1}{25}$	1.0272e-3	0.1284	1.5056e-2	0.3764
$\frac{1}{6}$	$\frac{1}{36}$	6.8472e-4	0.1479	1.2817e-2	0.4614
$\frac{1}{7}$	$\frac{1}{49}$	3.8251e-4	0.1313	7.8857e-3	0.3865
$\frac{1}{8}$	$\frac{1}{64}$	2.8613e-4	0.1465	7.0531e-3	0.4514
$\frac{1}{9}$	$\frac{1}{81}$	1.8121e-4	0.1321	4.8938e-3	0.3964
$\frac{1}{10}$	$\frac{1}{100}$	1.4579e-4	0.1459	4.4140e-3	0.4415

Table 3: Errors for Algorithm 4.1 ($\gamma(x) = -10$)



(a) H1 Error

(b) L2 Error

Figure 3: $\lg-\lg$ plot of Errors for Algorithm 4.1 ($\gamma(x) = -10$)

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