An Optimal Poincaré Inequality for Convex Domains

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1. Introduction

Let G be a convex n-dimensional domain with boundary C. It is easily seen that the lowest eigenvalue of the free membrane problem

$$(1.1)\qquad \begin{aligned} \Delta v + \mu v &= 0 \quad \text{in } G \\ \partial v/\partial n &= 0 \quad \text{on } C \end{aligned}$$

is zero, the eigenfunction being any constant.

This corresponds to the fact that the solution of the interior Neumann problem

$$(1.2)\qquad \begin{aligned} \Delta \varphi &= 0 \qquad \text{in } G \\ \partial \varphi/\partial n & \quad \text{given on } C \end{aligned}$$

is only determined to within a constant. The latter is to be fixed by a normalization such as

$$(1.3)\qquad \int_G \varphi \, dG = 0.$$

The authors have previously introduced a method for bounding the pointwise value as well as the Dirichlet integral of a solution φ of the exterior Neumann problem in terms of a boundary integral of $(\partial\varphi/\partial n)^2$ [3]. In order to extend this method to the interior Neumann problem one needs a lower bound for the second eigenvalue μ_2 of (1). This eigenvalue is characterized by the minimum principle

$$(1.4)\qquad \mu_2 = \min_{\int_G u\,dG = 0} \frac{\int_G |\operatorname{grad} u|^2 \, dG}{\int_G u^2 \, dG}.$$

A lower bound for μ_2 can be used in the interior Neumann problem in the following manner (*cf.* [3]). Let $\vec{f}$ be a vector field which is piecewise continuously differentiable throughout G and points outward on C, so that

$$(1.5)\qquad \vec{f} \cdot \vec{n} \geqq k > 0 \quad \text{on } C.$$

For example, if G is star-shaped with respect to the origin, we may take $\vec{f}$ to be the radius vector. By the divergence theorem and the inequality $a^2 + b^2 \geqq 2ab$

we have, if φ is normalized by (1.3),

$$\begin{aligned}
\oint_C \varphi^2 \vec{f}\cdot\vec{n}\, dC &= \int_G [\varphi^2 \operatorname{div}\vec{f} + 2\varphi\vec{f}\cdot\operatorname{grad}\varphi]\, dG \\
&\leqq \int_G \varphi^2 [\operatorname{div}\vec{f} + |\vec{f}|^2]\, dG + \int_G |\operatorname{grad}\varphi|^2\, dG \\
&\leqq [1+\mu_2^{-1}\max(\operatorname{div}\vec{f}+|\vec{f}|^2)] \int_G |\operatorname{grad}\varphi|^2\, dG \\
&= [1+\mu_2^{-1}\max(\operatorname{div}\vec{f}+|\vec{f}|^2)] \oint_C \varphi\, \partial\varphi/\partial n\, dC.
\end{aligned} \tag{1.6}$$

Consequently by SCHWARZ's inequality

$$\oint_C \varphi^2 \vec{f}\cdot\vec{n}\, dC \leqq [1+\mu_2^{-1}\max(\operatorname{div}\vec{f}+|\vec{f}|^2)]^2 \oint_C (\vec{f}\cdot\vec{n})^{-1}(\partial\varphi/\partial n)^2\, dC. \tag{1.7}$$

If Γ is a fundamental solution of LAPLACE's equation with its singularity at the interior point P, we have

$$\begin{aligned}
|\varphi(P)| &= \left|\oint_c (\Gamma\,\partial\varphi/\partial n - \varphi\,\partial\Gamma/\partial n)\, dC\right| \\
&\leqq \left\{\oint_C (\vec{f}\cdot\vec{n})^{-1}(\partial\varphi/\partial n)^2\, dC\right\}^{\frac{1}{2}} \left(\left\{\oint_c f\cdot n\,\Gamma^2\, dC\right\}^{\frac{1}{2}} + \right. \\
&\quad \left. + [1+\mu_2^{-1}\max(\operatorname{div}\vec{f}+|\vec{f}|^2)] \left\{\oint_C (\vec{f}\cdot\vec{n})^{-1}(\partial\Gamma/\partial n)^2\, dC\right\}^{\frac{1}{2}}\right).
\end{aligned} \tag{1.8}$$

Thus, if a lower bound for μ_2 is known, $\varphi(P)$ may be explicitly bounded in terms of a the square integral of $\partial\varphi/\partial n$.

These results can be extended to general second order differential equations (*cf.* [3]).

In this paper we shall show that for a convex domain G in any number of dimensions

$$\mu_2 \geqq \pi^2 D^{-2} \tag{1.9}$$

where D is the diameter of G. This is the best bound that can be given in terms of the diameter alone in the sense that $\mu_2 D^2$ tends to π^2 for a parallepiped all but one of whose dimensions shrink to zero.

The inequality (1.9) is in general false for non-convex domains. In fact, for a sequence of domains which tends to two disjoint subdomains, μ_2 tends to zero. For the special class of domains G which are symmetric about all the coordinate planes of a rectangular coordinate system and have the property that the intersection of G with any line in a coordinate direction is simply connected, the authors have previously obtained an inequality of the form (1.9) with D replaced by the maximum length of intersection of G with a line in any coordinate direction [4].

A simple upper bound for μ_2 for any n-dimensional domain G in terms of its volume V is given by the isoperimetric inequality

$$\mu_2 \leqq p_n^2 K_n^{\frac{2}{n}} V^{-\frac{2}{n}}, \tag{1.10}$$

where K_n is the volume of the unit n-sphere and p_n is the lowest positive root of the equation

$$p\, J'_{\frac{1}{2}n}(p) - (\tfrac{1}{2}n - 1)\, J_{\frac{1}{2}n}(p) = 0. \tag{1.11}$$

Equality is attained when G is a sphere.

For $n=2$ this inequality was conjectured by KORNHAUSER & STAKGOLD [2] and proved by SZEGÖ [5]. For general n the proof was given by one of the authors [6].

The eigenvalue μ_2 itself is of interest in a variety of problems arising in mathematical physics. In two dimensions it is proportional to the square of the cutoff frequency of the lowest H-mode of a wave guide [2]. In three dimensions it is proportional to the lowest resonant frequency of an acoustic resonator with perfectly rigid walls. It is also inversely proportional to the relaxation time for diffusion in a body with perfectly reflecting boundary.

The proof of the lower bound (1.9) is based upon a lemma concerning a class of Sturm-Liouville systems. This lemma, which is of some interest in itself, is stated and proved in §2. The inequality (1.9) is proved for two dimensions in §3 and for higher dimensions in §4.

2. A one-dimensional lemma

In order to prove the lower bound (1.9) we require a somewhat stronger version of its one-dimensional analogue. It is the following lemma.

Lemma. *Let $p(y)$ be a non-negative convex function of y defined on the interval $0 \leqq y \leqq L$; then for any piecewise continuously differentiable function $u(y)$ which satisfies*

$$\int_0^L p(y)\, u(y)\, dy = 0 \tag{2.1}$$

it follows that

$$\int_0^L p(y)\, [u'(y)]^2\, dy \geqq \pi^2 L^{-2} \int_0^L p(y)\, [u(y)]^2\, dy. \tag{2.2}$$

Proof. We assume for the moment that p is strictly positive and twice differentiable. Then the function v which minimizes the quotient

$$\int_0^L p\, u'^2\, dy \Big/ \int_0^L p\, u^2\, dy \tag{2.3}$$

among functions u satisfying (2.1) must satisfy the Sturm-Liouville system [1, p. 348]

$$\begin{aligned} [p\, v']' + \lambda p\, v &= 0, \\ v'(0) = v'(L) &= 0, \end{aligned} \tag{2.4}$$

where λ is the minimum value of the quotient (2.3). We divide the equation (2.4) by p, differentiate with respect to y, and introduce the new variable

$$w = v'\, p^{\frac{1}{2}}. \tag{2.5}$$

The function w satisfies the Sturm-Liouville system

$$w'' + \left[\frac{1}{2}\frac{p''}{p} - \frac{3}{4}\frac{p'^2}{p^2}\right] w + \lambda w = 0, \qquad (2.6)$$
$$w(0) = w(L) = 0.$$

Because of the convexity of p the term in square brackets is non-positive. Hence, multiplying (2.6) by w and integrating by parts, we obtain

$$\lambda \geqq \frac{\int_0^L w'^2\, dy}{\int_0^L w^2\, dy}. \qquad (2.7)$$

Since $w(0) = w(L) = 0$ the quotient on the right of (2.7) is bounded below by the first eigenvalue of the vibrating string with fixed ends. Thus

$$\lambda \geqq \pi^2 L^{-2}. \qquad (2.8)$$

Since λ is the minimum of the quotient (2.3), (2.2) is proved when p is strictly positive and twice differentiable.

If $\tilde{u}$ is any function defined on the interval $0 \leqq y \leqq L$, the function

$$u(y) \equiv \tilde{u}(y) - \left[\int_0^L p\, dy\right]^{-1} \int_0^L p \tilde{u}\, dy \qquad (2.9)$$

will satisfy (2.1). Hence (2.2) implies

$$\int_0^L p\, \tilde{u}'^2\, dy \geqq \pi^2 L^{-2} \left\{ \int_0^L p \tilde{u}^2\, dy - \left[\int_0^L p\, dy\right]^{-1} \left[\int_0^L p \tilde{u}\, dy\right]^2 \right\}. \qquad (2.10)$$

Clearly (2.10) is valid for the uniform limit of admissible functions p. In particular, then, p may be any non-negative convex function of y. Thus the lemma is proved.

Remarks. 1. The convexity of p was used only to show that the square bracket in (2.6) is non-positive. For this purpose it is sufficient to assume that $p^{-\frac{1}{2}}$ is a concave function of y. Therefore the lemma actually holds under this weaker condition.

2. By the minimax theorem [*1*, p. 352] we can show that if $p^{-\frac{1}{2}}$ is a concave function of y, the eigenvalues of the Sturm-Liouville system (2.4) satisfy the inequality

$$\lambda_k \geqq (k-1)^2 \pi^2 L^{-2}, \qquad k = 1, 2, \ldots. \qquad (2.11)$$

3. Equality in (2.11) is obtained if and only if $p^{-\frac{1}{2}}$ is linear in y. If p is assumed convex, it must then be constant.

3. The two-dimensional case

Let G be a convex plane domain with boundary C. Let μ_2 be defined as the infimum[1] of the quotient

$$\int_G |\operatorname{grad} u|^2\, dG \Big/ \int_G u^2\, dG \qquad (3.1)$$

[1] If the boundary C is smooth so that the problem (1.1) possesses eigenvalues, μ_2 is the second eigenvalue of (1.1).

among functions which have bounded second derivatives in G and satisfy

$$\int_G u\, dG = 0. \tag{3.2}$$

Let u be such a function. Consider the set of lines through the centroid of G. It follows from continuity that at least one such line divides G into two convex subdomains of equal area over each of which the integral of u vanishes. We now divide each of these subdomains into two more convex subdomains of equal area over each of which the integral of u vanishes.

Continuing this process, we arrive after a finite number of steps at a division of G into convex subdomains G_ν of arbitrarily small equal areas A_ν. Furthermore,

$$\int_{G_\nu} u\, dG = 0 \tag{3.3}$$

on each G_ν.

Let ϱ_ν be the radius of the largest circle contained in G_ν. Then clearly

$$A_\nu \geqq \pi \varrho_\nu^2. \tag{3.4}$$

Hence, if A_ν is sufficiently small, the width ϱ_ν of G is less than a preassigned ε:

$$\varrho_\nu \leqq \varepsilon. \tag{3.5}$$

This means that G_ν is contained between two parallel lines at distance ε. We introduce a rectangular coordinate system with the x_2-axis along one of these lines and the x_1-axis tangent to one end of G_ν. Let L_ν be the length of the projection of G_ν on the x_2 axis. Clearly, $L_\nu \leqq D$. Let $p(y)$ be the length of the intersection of G_ν with the line $x_2 = y$. Then $p(y) \leqq \varepsilon$. Because of the convexity of G_ν, $p(y)$ is convex in y.

Let M be a bound for the absolute values of u and its first and second derivatives. Then by the mean value theorem

$$\left| \int_{G_\nu} \left(\frac{\partial u}{\partial x_2}\right)^2 dG - \int_0^{L_\nu} p(y)\,[u(0,y)']^2\, dy \right| \leqq 2M^2 A_\nu\, \varepsilon, \tag{3.6}$$

$$\left| \int_{G_\nu} u^2\, dG - \int_0^{L_\nu} p(y)\,[u(0,y)]^2\, dy \right| \leqq 2M^2 A_\nu\, \varepsilon, \tag{3.7}$$

and

$$\left| \int_{G_\nu} u\, dG - \int_0^{L_\nu} p(y)\, u(0,y)\, dy \right| \leqq M A_\nu\, \varepsilon. \tag{3.8}$$

Applying the inequality (2.10) of the lemma, we find, using (3.3) and $L_\nu \leqq D$, that

$$\begin{aligned} \int_{G_\nu} |\operatorname{grad} u|^2\, dG &\geqq \int_{G_\nu} \left(\frac{\partial u}{\partial x_2}\right)^2 dG \\ &\geqq \pi^2 D^{-2} \int_{G_\nu} u^2\, dG - 2M^2\left(1 + \pi^2 D^{-2}[1 + \tfrac{1}{2}\varepsilon]\right) A_\nu\, \varepsilon. \end{aligned} \tag{3.9}$$

We sum these inequalities over all the subdomains G_ν. The sum of the A_ν is the area of G. Since ε is arbitrarily small, we obtain the inequality

$$\int_G |\operatorname{grad} u|^2 \, dG \geqq \pi^2 D^{-2} \int_G u^2 \, dG. \tag{3.10}$$

Since u is any function with bounded second derivatives satisfying (3.2), we have, by definition,

$$\mu_2 \geqq \pi^2 D^{-2}. \tag{3.11}$$

4. The n-dimensional case

Let G be a convex n-dimensional domain with boundary C $(n \geqq 3)$. We again define μ_2 as the infimum of the Rayleigh quotient (3.1) amomg functions u having bounded second derivatives in G and satisfying the conditions (3.2)[1].

Let u be such a function. We consider the set of $n-1$-planes of the form $a x_{n-1} + b x_n = c$ passing through the centroid of G. By continuity we find that at least one of these planes divides G into two subdomains of equal n-volumes over each of which the integral of u vanishes. We divide these subdomains in the same way and continue the process until G is divided into subdomains G_ν of arbitrarily small n-volume $V_\nu^{[n]}$. If ϱ_ν is the radius of the largest inscribed n-sphere, we have

$$V_\nu^{[n]} \geqq K_n \varrho_\nu^n \tag{4.1}$$

where K_n is the volume of the unit n-sphere. Hence, by a sufficiently large number of subdivisions, we can make ϱ_ν less than a preassigned ε. This means that each G_ν is contained between two parallel $n-1$-planes at distance ε. In a particular G_ν we introduce new rectangular coordinates with the x_1-axis normal to these planes. We proceed to subdivide G_ν by means of planes of the form $a x_{n-1} + b x_n = c$ into subdomains on each of which the integral of u vanishes. We make these dividing planes pass through the centroids of the projections on $x_1 = 0$ of the domains being divided. After a finite number of such divisions we obtain subdomains G'_ν whose projections on $x_1 = 0$ have arbitrarily small $n-1$ volumes $V_\nu^{[n-1]}$. It follows as before that if $V_\nu^{[n-1]}$ is sufficiently small, the projection of G'_ν on $x_1 = 0$ lies between two parallel $n-2$-planes at distance at most ε. We keep the x_1-direction fixed and choose the x_2-direction of a new rectangular coordinate system in G'_ν perpendicular to these planes.

If $n > 3$ we divide each G'_ν further by means of planes $a x_{n-1} + b x_n = c$ passing through the centroids of the projections on $x_1 = x_2 = 0$ of G'_ν and of the succeeding domains.

In this way we eventually obtain a subdivision of G into a finite number of convex subdomains G''_ν over each of which the integral of u vanishes. Furthermore, each G''_ν is contained in a parallelepiped of the form

$$\begin{aligned} &0 \leqq x_i \leqq \varepsilon, \qquad i = 1, 2, \ldots, n-1 \\ &0 \leqq x_n \leqq L_\nu \end{aligned} \tag{4.2}$$

with respect to suitable rectangular coordinates.

[1] If the boundary C is smooth so that the problem (1.1) possesses eigenvalues, μ_2 is the second eigenvalue of (1.1).

Let $p(y)$ be the $n-1$ volume of the intersection of G_ν'' with $x_n = y$. Then $p(y)$ is convex because of the convexity of G_ν'', and $p(y) \leqq \varepsilon^{n-1}$ by (4.2).

The inequality

$$\mu_2 \geqq \pi^2 D^{-2} \tag{4.3}$$

is now derived exactly as in § 3.

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