Comput. Methods Appl. Mech. Engrg. 200 (2011) 1877–1891

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes–Darcy coupled problem

Gabriel N. Gatica [a,*], Ricardo Oyarzúa [b], Francisco-Javier Sayas [c]

[a] CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
[b] Departamento de Matemática, Facultad de Ciencias, Universidad del Bio-Bio, Casilla 5-C, Concepción, Chile
[c] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

ARTICLE INFO

Article history:
Received 16 June 2010
Received in revised form 26 November 2010
Accepted 8 February 2011
Available online 15 February 2011

Mathematics Subject Classifications (2000):
65N15
65N30
74F10
74S05

Keywords:
A posteriori error analysis
Efficiency
Reliability
Stokes
Darcy
Fully-mixed

ABSTRACT

In this paper we develop an a posteriori error analysis of a new fully mixed finite element method for the coupling of fluid flow with porous media flow in 2D. Flows are governed by the Stokes and Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law. We consider dual-mixed formulations in both media, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, and the traces of the porous media pressure and the fluid velocity on the interface, as the resulting unknowns. The set of feasible finite element subspaces includes Raviart–Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the traces. We derive a reliable and efficient residual-based a posteriori error estimator for the coupled problem. The proof of reliability makes use of the global inf–sup condition, Helmholtz decompositions in both media, and local approximation properties of the Clément interpolant and Raviart–Thomas operator. On the other hand, inverse inequalities, the localization technique based on element-bubble and edge-bubble functions, and known results from previous works, are the main tools for proving the efficiency of the estimator. Finally, some numerical results confirming the theoretical properties of this estimator, and illustrating the capability of the corresponding adaptive algorithm to localize the singularities of the solution, are reported.

1. Introduction

The derivation of new finite element methods for the Stokes–Darcy coupled problem, in which the respective interface conditions are given by mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law, has recently become a very active research area (see, e.g. [5,12,13,16,17,27,29,30,32,38,39,45,48,50–52,54,58] and the references therein). The above list includes porous media with cracks, nonlinear problems, and the incorporation of the Brinkman equation in the model (see [13,30,58]). In addition, most of the formulations employed are based on appropriate combinations of stable elements for the free fluid flow and for the porous medium flow: the first theoretical results in this direction go back to [29,45]. Indeed, an iterative subdomain method employing the primal variational formulation and standard finite element subspaces in both domains is proposed in [29], whereas the primal method in the fluid and the dual-mixed method in the porous medium are applied in [45]. In this way, the approach from [45] yields the velocity and the pressure in both domains, together with the trace of the porous medium pressure on the interface, as the main unknowns of the coupled problem. This trace unknown is motivated by the fact that one of the transmission conditions becomes essential. Then, new mixed finite element discretizations of the variational formulation from [45] have been introduced and analyzed in [38,39]. The stability of a specific Galerkin method is the main result in [38], and the resulting mixed finite element method is the first one that is conforming for the primal/dual-mixed formulation proposed in [45]. The results from [38] are improved in [39] where it is shown that the use of any pair of stable Stokes and Darcy elements implies the stability of the corresponding Stokes–Darcy Galerkin scheme. The analysis in [39] hinges on the fact that the operator defining the continuous variational formulation is given by a compact perturbation of an invertible mapping. Further techniques utilized in the literature include mortar finite element methods, discontinuous Galerkin (DG) schemes, and stabilized formulations (see, e.g. [5,16,17,27,

* Corresponding author. Tel.: +56 41 2203121; fax: +56 41 522055.
E-mail addresses: ggatica@ing-mat.udec.cl (G.N. Gatica), royarzua@ubiobio.cl (R. Oyarzúa), fjsayas@math.udel.edu (F.-J. Sayas).

0045-7825/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2011.02.009

28,32,48,50–52,54]). In particular, the main motivation for employing stabilized formulations either in both domains or in one of them, is the possibility of approximating the Stokes and Darcy flows with the same finite element subspaces. Certainly, different finite element subspaces in each flow region may lead to different approximation properties for each subproblem. On the contrary, using the same spaces guarantees the same accurateness along the entire domain and leads to simpler and more efficient computational codes.

Furthermore, the velocity–pressure-stress formulation for computational incompressible flows has gained considerable attention in recent years due to its natural applicability to non-Newtonian flows. Indeed, since in this case the constitutive equation is nonlinear, the stress can not be eliminated, and hence it becomes an unavoidable unknown in the corresponding solvability analysis. Actually, the main advantage of this formulation is that it allows for a unified analysis for linear and nonlinear flows. Nevertheless, the increase in the number of unknowns and the symmetry requirement for the stress tensor constitute the main drawbacks of this formulation. Moreover, the difficulty in deriving and using finite element subspaces of symmetric tensors in the Stokes and Lamé systems is already well known (see, e.g. [8,15]). In order to circumvent these disadvantages, at least partially, there are two main approaches. A first idea, which goes back to [7], consists of imposing the symmetry of the stress in a weak sense through the introduction of a suitable Lagrange multiplier (rotation in elasticity and vorticity in fluid mechanics). However, a more appealing idea nowadays is given by the use of the pseudostress instead of the stress in the corresponding setting of the Stokes equations. Indeed, this procedure has become very popular lately, thus yielding two new approaches for incompressible flows: the velocity–pressure-pseudostress and velocity-pseudostress formulations (see, e.g. [18–21,31,36]).

In the recent paper [40] we adopt the pseudostress based formulation mentioned above, and developed a new variational approach for the 2D Stokes–Darcy coupled problem, which allows, on the one hand, the introduction of further unknowns of physical interest, and on the other hand, the utilization of the same family of finite element subspaces in both media, without requiring any stabilization term. More precisely, in [40] we consider dual-mixed formulations in both domains, which yields the pseudostress and the velocity in the fluid, together with the velocity and the pressure in the porous medium, as the main unknowns. The pressure and the gradient of the velocity in the fluid can then be computed as a very simple postprocess of the above unknowns, in which no numerical differentiation is applied, and hence no further sources of error arise. In addition, since the transmission conditions become essential, we impose them weakly and introduce the traces of the porous media pressure and the fluid velocity, which are also variables of importance from a physical point of view, as the corresponding Lagrange multipliers. Then, we apply the well known Fredholm and Babuška–Brezzi theories to prove the unique solvability of the resulting continuous formulation and derive sufficient conditions on the finite element subspaces ensuring that the associated Galerkin scheme becomes well posed. Among the several different ways in which the equations and unknowns can be ordered, we choose the one yielding a doubly mixed structure for which the inf–sup conditions of the off-diagonal bilinear forms follow straightforwardly. In this way, the arguments of the continuous analysis can be easily adapted to the discrete case. In particular, a feasible choice of subspaces is given by Raviart–Thomas elements of lowest order and piecewise constants for the velocities and pressures, respectively, in both domains, together with continuous piecewise linear elements for the Lagrange multipliers.

On the other hand, it is well known that in order to guarantee a good convergence behaviour of most finite element solutions, specially under the eventual presence of singularities, one usually needs to apply an adaptive algorithm based on a posteriori error estimates. These are represented by global quantities $\boldsymbol{\eta}$ that are expressed in terms of local indicators η_T defined on each element T of a given triangulation $\mathcal{T}$. The estimator $\boldsymbol{\eta}$ is said to be efficient (resp. reliable) if there exists $C_{\mathrm{eff}} > 0$ (resp. $C_{\mathrm{rel}} > 0$), independent of the mesh sizes, such that

$$C_{\mathrm{eff}}\boldsymbol{\eta} + \text{h.o.t.} \leqslant \|\text{error}\| \leqslant C_{\mathrm{rel}}\boldsymbol{\eta} + \text{h.o.t.}, \tag{1.1}$$

where h.o.t. is a generic expression denoting one or several terms of higher order. In particular, the a posteriori error analysis of variational formulations with saddle-point structure has already been widely investigated by many authors (see, e.g. [2–4,14,22,24,35,43,46,47,49,55], and the references therein). These contributions refer mainly to reliable and efficient a posteriori error estimators based on local and global residuals, local problems, post-processing, and functional-type error estimates. In addition, the applications include Stokes and Oseen equations, Poisson problem, linear elasticity, and general elliptic partial differential equations of second order. However, up to our knowledge, the first a posteriori error analysis for the Stokes–Darcy coupled problem has been provided recently in [9], where a reliable and efficient residual-based a posteriori error estimator for the variational formulation analyzed in [38] is derived. Partially following known approaches, the proof of reliability makes use of suitable auxiliary problems, diverse continuous inf–sup conditions satisfied by the bilinear forms involved, and local approximation properties of the Clément interpolant and Raviart–Thomas operator. Similarly, Helmholtz decomposition, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions, are the main tools for proving the efficiency of the estimator.

Motivated by the discussion in the above paragraphs, our purpose now is to additionally contribute in the direction of [9] and provide the a posteriori error analysis of the fully-mixed variational approach introduced in [40]. According to this, the rest of this work is organized as follows. In Section 2 we recall from [40] the Stokes–Darcy coupled problem and its continuous and discrete fully-mixed variational formulations. The kernel of the present work is given by Section 3, where we develop the a posteriori error analysis. In Section 3.1 we employ the global continuous inf–sup condition, Helmholtz decompositions in both domains, and the local approximation properties of the Clément and Raviart–Thomas operators, to derive a reliable residual-based a posteriori error estimator. An interesting feature of our proof of reliability is the previous transformation of the global continuous inf–sup condition into an estimate involving global inf–sup conditions for each one of the components of the product space to which the vector of unknowns belongs. As a consequence, the a posteriori error analysis becomes much simpler than the one provided in [9], and it yields an approach that can be applied to several other boundary value problems (see, e.g. [37] and [41] for applications to nonlinear models). Then, in Section 3.2 we apply again Helmholtz decompositions, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions to prove the efficiency of the estimator. This proof benefits partially from the fact that some components of the a posteriori error estimator coincide with those obtained in [9] and the related work [22]. Finally, numerical results confirming the reliability and efficiency of the a posteriori error estimator and showing the good performance of the associated adaptive algorithm, are presented in Section 4.

We end this section with some notations to be used below. In particular, in what follows we utilize the standard terminology for Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, Γ is a closed Lipschitz curve, and $r \in \mathbb{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \quad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \quad \text{and} \quad \mathbf{H}^r(\Gamma) := [H^r(\Gamma)]^2.$$

However, for $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O}), \mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O}), \mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\Gamma)$, respectively. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O}), \mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$) and $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$ and $\mathbf{H}^r(\Gamma)$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div};\mathcal{O}) := \{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\,\mathbf{w} \in L^2(\mathcal{O})\},$$

is standard in the realm of mixed problems (see, e.g. [15] or [42]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathrm{div};\mathcal{O})$. The Hilbert norms of $\mathbf{H}(\mathrm{div};\mathcal{O})$ and $\mathbb{H}(\mathrm{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\mathrm{div};\mathcal{O}}$ and $\|\cdot\|_{\mathrm{div};\mathcal{O}}$, respectively. On the other hand, the symbol for the $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ inner products

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\lambda \quad \forall \xi, \lambda \in L^2(\Gamma), \qquad \langle \boldsymbol{\xi}, \boldsymbol{\lambda} \rangle_\Gamma := \int_\Gamma \boldsymbol{\xi}\cdot\boldsymbol{\lambda} \quad \forall \boldsymbol{\xi}, \boldsymbol{\lambda} \in \mathbf{L}^2(\Gamma)$$

will also be employed for their respective extensions as the duality products $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{1/2}(\Gamma)$. Finally, we employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to mean generic positive constants independent of the discretization parameters, which may take different values at different places.

2. The Stokes–Darcy coupled problem

In this section we follow very closely the presentation from [40] to introduce the model problem and the corresponding continuous and discrete mixed variational formulations.

2.1. The model problem

The Stokes–Darcy coupled problem models the interaction of an incompressible viscous fluid occupying a region Ω_S, which flows back and forth across the common interface into a porous medium living in another region Ω_D and saturated with the same fluid. Physically, we consider a simplified 2D model where Ω_D is surrounded by a bounded region Ω_S (see Fig. 2.1 below). Their common interface is supposed to be a Lipschitz curve Σ and we assume that $\partial\Omega_D = \Sigma$. The remaining part of the boundary of Ω_S is also assumed to be a Lipschitz curve Γ_S. For practical purposes, we can assume that both Γ_S and Σ are polygons. The unit normal vector field on the boundaries $\mathbf{n}$ is chosen pointing outwards from Ω_S (and therefore inwards to Ω_D when seen on Σ). On Σ we also consider a unit tangent vector field $\mathbf{t}$ in any fixed orientation of this closed curve.

The governing equations in Ω_S are those of the Stokes problem, which are written in the following non-standard velocity–pressure-pseudostress formulation:

$$\begin{aligned} &\boldsymbol{\sigma}_S = -p_S\mathbf{I} + \nu\nabla\mathbf{u}_S \quad \text{in } \Omega_S, \quad \mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S = \mathbf{0} \quad \text{in } \Omega_S,\\ &\mathrm{div}\,\mathbf{u}_S = 0 \quad \text{in } \Omega_S, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on } \Gamma_S, \end{aligned} \tag{2.1}$$

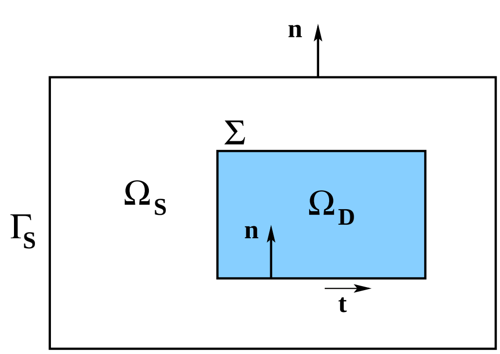

Fig. 2.1. Geometry of the problem.

where $\nu > 0$ is the viscosity of the fluid, $\mathbf{u}_S$ is the fluid velocity, p_S is the pressure, $\boldsymbol{\sigma}_S$ is the pseudostress tensor, $\mathbf{I}$ is the 2×2 identity matrix and $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ are known source terms. Here, div is the usual divergence operator acting on vector fields, and $\mathbf{div}$ denotes the action of div to the rows of each tensor. On the other hand, the flow equations in Ω_D are those of the linearized Darcy model:

$$\mathbf{u}_D = -\mathbf{K}\nabla p_D \quad \text{in } \Omega_D, \qquad \mathrm{div}\,\mathbf{u}_D = f_D \quad \text{in } \Omega_D, \tag{2.2}$$

where the unknowns are the pressure p_D and the flow $\mathbf{u}_D$, and the source term, given by $f_D \in L^2(\Omega_D)$, satisfies $\int_{\Omega_D} f_D = 0$. The matrix valued function $\mathbf{K}$, describing permeability of Ω_D divided by the viscosity ν, is symmetric, has $L^\infty(\Omega_D)$ components and is uniformly elliptic. Finally, the transmission conditions on Σ are given by

$$\begin{aligned} &\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n} \quad \text{on } \Sigma,\\ &\boldsymbol{\sigma}_S\mathbf{n} + \nu\kappa^{-1}(\mathbf{u}_S \cdot \mathbf{t})\mathbf{t} = -p_D\mathbf{n} \quad \text{on } \Sigma, \end{aligned} \tag{2.3}$$

where $\kappa := \alpha^{-1}\sqrt{(\nu\mathbf{K}\mathbf{t})\cdot\mathbf{t}}$ is the friction coefficient and α is a experimentally determined positive parameter. Throughout the rest of the paper we assume, without loss of generality, that κ is a positive constant.

The first equation in (2.3) corresponds to mass conservation on Σ, whereas the normal and tangential components of the second one constitute the balance of normal forces and the Beavers–Joseph–Saffman law, respectively. The latter establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidences, that $\mathbf{u}_D \cdot \mathbf{t}$ is negligible). We refer to [11] and [44] for further details on this interface condition. In connection with this issue, it is important to mention here that the use of the symmetric stress tensor is more physically accepted in fluids when dealing with Neumann or Robin-type boundary or transmission conditions. Moreover, it is known that the corresponding formulation is equivalent to the pseudostress-based formulation only if Dirichlet boundary conditions occur along the whole boundary. In the present case, the utilization of the latter approach is motivated, on one hand, by the fact that the first transmission condition in (2.3) deals precisely with Dirichlet data on the interface Σ, and on the other hand, by the inherent simplicity of not having to require any symmetry condition at all to the pseudostress. In addition, when writing the Stokes equation as we do in (2.1) with the pseudostress $\boldsymbol{\sigma}_S$, the corresponding integration by parts yields the appearing of $\boldsymbol{\sigma}_S\mathbf{n}$, and hence it makes sense, at least from a mathematical point of view, to incorporate this expresion, instead of the normal component of the symmetric stress, into the second equation in (2.3), thus yielding what we may call *a modified Beavers–Joseph–Saffman transmission condition*.

We complete the description of our model problem by observing that the equations in the Stokes domain (cf. (2.1)) can be rewritten equivalently as

$$\begin{aligned} &\nu^{-1}\boldsymbol{\sigma}_S^d = \nabla\mathbf{u}_S \quad \text{in } \Omega_S, \qquad \mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S = \mathbf{0} \quad \text{in } \Omega_S,\\ &p_S = -\tfrac{1}{2}\mathrm{tr}\,\boldsymbol{\sigma}_S \quad \text{in } \Omega_S, \qquad \mathbf{u}_S = \mathbf{0} \quad \text{on } \Gamma_S, \end{aligned} \tag{2.4}$$

where $\mathrm{tr}\,\boldsymbol{\tau} := \tau_{11} + \tau_{22}$, and

$$\boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{2}(\mathrm{tr}\,\boldsymbol{\tau})\mathbf{I}$$

is the deviatoric part of the tensor $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$. Note that the third equation in (2.4) allows us to eliminate p_S from the system and compute it as a simple postprocess of the solution. Similarly, the first equation in (2.4) yields a straightforward postprocess formula for the gradient of the velocity in the fluid, and hence for the symmetric stress tensor as well.

Furthermore, we remark that, though the geometry described by Fig. 2.1 was chosen to simplify the presentation, the case of a fluid flowing only across a part of the boundary of the porous medium does not yield further complications for the a posteriori error analysis of the problem. We already discussed this issue in [40, Section 2.1], in connection with the respective a priori error analysis, and further details can be found in [32].

2.2. The fully-mixed variational formulation

We add two new unknowns to the system, namely $\boldsymbol{\varphi} := -\mathbf{u}_S|_\Sigma$ and $\lambda := p_D|_\Sigma$. The system will be written in terms of the unknowns $\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda)$ and $\underline{\mathbf{u}} := (\mathbf{u}_S, p_D)$. Then we recall from [40, Lemma 3.5] that the coupled problem given by (2.2), (2.3), and (2.4) has the one-dimensional kernel defined by

$$\{((\boldsymbol{\sigma}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda), (\mathbf{u}_S, p_D)) : \quad \boldsymbol{\sigma}_S = -c\mathbf{I}, \ \mathbf{u}_D = \mathbf{0},$$
$$\boldsymbol{\varphi} = \mathbf{0}, \ \lambda = c, \ \mathbf{u}_S = \mathbf{0}, \ p_D = c; \ c \in \mathbb{R}\}.$$

Hence, in order to solve this indetermination, we introduce

$$L_0^2(\Omega_D) := \left\{ q \in L^2(\Omega_D) : \int_{\Omega_D} q = 0 \right\},$$

and define the product spaces

$$\mathbb{X} := \mathbb{H}(\mathrm{div}; \Omega_S) \times \mathbf{H}(\mathrm{div}; \Omega_D) \times \mathbf{H}^{1/2}(\Sigma) \times H^{1/2}(\Sigma),$$
$$\mathbb{M} := \mathbf{L}^2(\Omega_S) \times L_0^2(\Omega_D),$$

endowed with the product norms

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbb{X}}^2 := \|\boldsymbol{\tau}_S\|_{\mathrm{div},\Omega_S}^2 + \|\mathbf{v}_D\|_{\mathrm{div};\Omega_D}^2 + \|\boldsymbol{\psi}\|_{1/2,\Sigma}^2 + \|\xi\|_{1/2,\Sigma}^2$$
$$\forall \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D, \boldsymbol{\psi}, \xi) \in \mathbb{X},$$

and

$$\|\underline{\mathbf{v}}\|_{\mathbb{M}}^2 := \|\mathbf{v}_S\|_{0,\Omega_S}^2 + \|q_D\|_{0,\Omega_D}^2 \quad \forall \underline{\mathbf{v}} := (\mathbf{v}_S, q_D) \in \mathbb{M}.$$

In this way, as explained in [40, Sections 2 and 3], it suffices to consider from now on the following modified variational formulation of (2.2), (2.3), and (2.4): Find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ such that

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}) &= \mathcal{F}(\underline{\boldsymbol{\tau}}) \quad \forall \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_D, \boldsymbol{\psi}, \xi) \in \mathbb{X}, \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{v}}) &= \mathcal{G}(\underline{\mathbf{v}}) \quad \forall \underline{\mathbf{v}} := (\mathbf{v}_S, q_D) \in \mathbb{M}, \end{aligned} \tag{2.5}$$

where

$$\mathcal{F}(\underline{\boldsymbol{\tau}}) := 0, \qquad \mathcal{G}(\underline{\mathbf{v}}) = \mathcal{G}((\mathbf{v}_S, q_D)) := -(\mathbf{f}_S, \mathbf{v}_S)_S - (f_D, q_D)_D, \tag{2.6}$$

and $\mathcal{A}$ and $\mathcal{B}$ are the bounded bilinear forms defined by

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}}) := {} & \mathbf{a}((\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\tau}_S, \mathbf{v}_D)) + \mathbf{b}((\boldsymbol{\tau}_S, \mathbf{v}_D), (\boldsymbol{\varphi}, \lambda)) \\ & + \mathbf{b}((\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\psi}, \xi)) - \mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi)), \end{aligned} \tag{2.7}$$

with

$$\mathbf{a}((\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\tau}_S, \mathbf{v}_D)) := \nu^{-1}(\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D)_D,$$
$$\mathbf{b}((\boldsymbol{\tau}_S, \mathbf{v}_D), (\boldsymbol{\psi}, \xi)) := \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma,$$
$$\mathbf{c}((\boldsymbol{\varphi}, \lambda), (\boldsymbol{\psi}, \xi)) := \nu\kappa^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}, \boldsymbol{\psi} \cdot \mathbf{t} \rangle_\Sigma + \langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma - \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma,$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := (\mathrm{div}\, \boldsymbol{\tau}_S, \mathbf{v}_S)_S - (\mathrm{div}\, \mathbf{v}_D, q_D)_D. \tag{2.8}$$

Hereafter we utilize, for each $\star \in \{S,D\}$, the following notations

$$(u, v)_\star := \int_{\Omega_\star} uv, \qquad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau},$$

for all $u, v \in L^2(\Omega_\star)$, $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega_\star)$, and $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$, where $\boldsymbol{\sigma} : \boldsymbol{\tau} := \mathrm{tr}(\boldsymbol{\sigma}^t \boldsymbol{\tau})$.

We find it important to remark that $\boldsymbol{\varphi}$ and λ can be interpreted as Lagrange multipliers associated to the transmission conditions (2.3). In addition, we notice that (2.5) is equivalent to the variational formulation defined in [40, Section 3.2, eq. (3.2)], in which $\boldsymbol{\sigma}_S$ is decomposed into $\boldsymbol{\sigma}_S = \boldsymbol{\sigma} + \mu$, with $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathrm{div}; \Omega_S)$ and $\mu \in \mathbb{R}$, where

$$\mathbb{H}_0(\mathrm{div}; \Omega_S) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathrm{div}; \Omega_S) : \int_{\Omega_S} \mathrm{tr}\, \boldsymbol{\tau} = 0 \right\}.$$

The following result taken from [40] establishes, in particular, the well-posedness of (2.5).

Theorem 2.1. *For each pair $(\mathcal{F}, \mathcal{G}) \in \mathbb{X}' \times \mathbb{M}'$ there exists a unique $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ solution to (2.5), and there exists a constant $C > 0$, independent of the solution, such that*

$$\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})\|_{\mathbb{X} \times \mathbb{M}} \leqslant C \left\{ \|\mathcal{F}\|_{\mathbb{X}'} + \|\mathcal{G}\|_{\mathbb{M}_0'} \right\}. \tag{2.9}$$

Proof. See [40, Theorem 3.9]. □

We now provide the converse of the derivation of (2.5). More precisely, the following theorem establishes that the unique solution of (2.5), with $\mathcal{F}$ and $\mathcal{G}$ given by (2.6), solves the original transmission problem described in Section 2.1. This result will be used later on in Section 3.2 to prove the efficiency of our a posteriori error estimator. We remark that no extra regularity assumptions on the data, but only $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ and $f_D \in L^2(\Omega_D)$, are required here.

Theorem 2.2. *Let $(\underline{\sigma}, \underline{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ be the unique solution of the variational formulation (2.5) with $\mathcal{F}$ and $\mathcal{G}$ given by (2.6). Then $\mathrm{div}\sigma_S = -\mathbf{f}_S$ in Ω_S, $\nu^{-1}\boldsymbol{\sigma}_S^d = \nabla \mathbf{u}_S$ in Ω_S, $\mathbf{u}_S \in \mathbf{H}^1(\Omega_S)$, $\mathrm{div}\,\mathbf{u}_D = f_D$ in Ω_D, $\mathbf{u}_D = -\mathbf{K}\nabla p_D$ in Ω_D, $p_D \in H^1(\Omega_D)$, $\mathbf{u}_D \cdot \mathbf{n} + \varphi \cdot \mathbf{n} = 0$ on Σ, $\sigma_S\mathbf{n} + \lambda\mathbf{n} - \nu\kappa^{-1} (\varphi \cdot \mathbf{t})\mathbf{t} = 0$ on Σ, $\lambda = p_D$ on Σ, $\varphi = -\mathbf{u}_S$ on Σ, and $\mathbf{u}_S = 0$ on Γ_S.*

Proof. It basically follows by applying integration by parts backwardly in (2.5) and using suitable test functions. We omit further details. □

We end this section by emphasizing, as already mentioned in the original paper [40], the main advantages of the present fully-mixed approach, namely: it provides either direct finite element approximations or very simple postprocess formulae for several additional quantities of physical interest; it yields, under a special ordering of the resulting equations and unknowns, a unified and straightforward analysis of the continuous and discrete formulations; it leads to independent but analogously structured stability assumptions on the finite element subspaces for the Stokes and Darcy regions; and it allows the utilization of the same kind of finite elements in both media, with the consequent simplification of the respective code.

2.3. A Galerkin method

Although the analysis in [40] provides general hypotheses for the well-posedness of a Galerkin scheme of (2.5), we will consider here the particular case described in [40, Section 5]. Let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be respective triangulations of the domains Ω_S and Ω_D formed by shape-regular triangles T of diameter h_T, and assume that $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ match in Σ, so that their union is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. Note that the shape-regularity of the meshes guarantees the boundedness of the quotients $h_T/h_{T'}$, for each $T, T' \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$, which may appear in our estimates below, for instance when applying an inverse inequality.

Then, for each $T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ we let $\mathrm{RT}_0(T)$ be the local lowest order Raviart–Thomas space,

$$\mathrm{RT}_0(T) := \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\}.$$

For each $\star \in \{S,D\}$ we define the global spaces

$$\mathbf{H}_h(\Omega_\star) := \left\{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div}; \Omega_\star) : \quad \mathbf{v}_h|_T \in \mathrm{RT}_0(T) \quad \forall T \in \mathcal{T}_h^\star \right\}, \tag{2.10}$$

and

$$L_h(\Omega_\star) := \{q_h : \Omega_\star \to \mathbb{R} : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall T \in \mathcal{T}_h^\star\}.$$

Hereafter, given a non-negative integer k and a subset S of $\mathbb{R}^2$, $\mathbb{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leqslant k$. Next, we let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{S}}$ (or $\mathcal{T}_h^{\mathrm{D}}$), and first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Note that because Σ_h is inherited from one of the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h}. If the cardinal of Σ_h is odd, we start by joining to adjacent elements and construct Σ_{2h} from this reduced partition.

Employing the above notations, we now introduce

$$\begin{aligned}
&\mathbb{H}_h(\Omega_{\mathrm{S}}) := \{\boldsymbol{\tau} : \Omega_{\mathrm{S}} \to \mathbb{R}^{2\times 2} : \quad \mathbf{c}^t\boldsymbol{\tau} \in \mathbf{H}_h(\Omega_{\mathrm{S}}) \quad \forall \mathbf{c} \in \mathbb{R}^2\},\\
&\mathbf{L}_h(\Omega_{\mathrm{S}}) := L_h(\Omega_{\mathrm{S}}) \times L_h(\Omega_{\mathrm{S}}),\\
&L_{h,0}(\Omega_{\mathrm{D}}) := L_h(\Omega_{\mathrm{D}}) \cap L_0^2(\Omega_{\mathrm{D}}),\\
&\Lambda_h(\Sigma) := \{\xi_h \in C(\Sigma) : \quad \xi_h|_e \in \mathbb{P}_1(e) \quad \forall e \text{ edge of } \Sigma_{2h}\},\\
&\boldsymbol{\Lambda}_h(\Sigma) := \Lambda_h(\Sigma) \times \Lambda_h(\Sigma),
\end{aligned}$$

and the product spaces

$$\mathbb{X}_h := \mathbb{H}_h(\Omega_{\mathrm{S}}) \times \mathbf{H}_h(\Omega_{\mathrm{D}}) \times \boldsymbol{\Lambda}_h(\Sigma) \times \Lambda_h(\Sigma) \quad \text{and}$$
$$\mathbb{M}_h := \mathbf{L}_h(\Omega_{\mathrm{S}}) \times L_{h,0}(\Omega_{\mathrm{D}}).$$

In this way, the Galerkin scheme of (2.5) becomes: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$ such that

$$\begin{aligned}
&\mathcal{A}(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}}_h) = \mathcal{F}(\underline{\boldsymbol{\tau}}) \forall \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi) \in \mathbb{X}_h,\\
&\mathcal{B}(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}) = \mathcal{G}(\underline{\mathbf{v}}) \forall \underline{\mathbf{v}} := (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}}) \in \mathbb{M}_h,
\end{aligned} \tag{2.11}$$

where $\underline{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h)$ and $\underline{\mathbf{u}}_h := (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})$.

The following theorem, also taken from [40], provide the well-posedness of (2.11), the associated Céa estimate, and the corresponding theoretical rate of convergence.

Theorem 2.3. *The Galerkin scheme (2.11) has a unique solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) \in \mathbb{X}_h \times \mathbb{M}_h$. Moreover, there exist $C_1, C_2 > 0$, independent of h, such that*

$$\|(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \leqslant C_1\left\{\|\mathcal{F}|_{\mathbb{X}_h}\|_{\mathbb{X}_h'} + \|\mathcal{G}|_{\mathbb{M}_h}\|_{\mathbb{M}_h'}\right\},$$

and

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \leqslant C_2\{\inf_{\underline{\boldsymbol{\tau}}_h \in \mathbb{X}_h} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}_h\|_{\mathbb{X}} + \inf_{\underline{\mathbf{v}}_h \in \mathbb{M}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|_{\mathbb{M}}\}.$$

If there exists $\delta \in (0,1]$ such that $\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbb{H}^\delta(\Omega_{\mathrm{S}}), \mathrm{div}\,\boldsymbol{\sigma}_{\mathrm{S}} \in \mathbf{H}^\delta(\Omega_{\mathrm{S}}), \mathbf{u}_{\mathrm{D}} \in \mathbf{H}^\delta(\Omega_{\mathrm{D}})$, and $\mathrm{div}\mathbf{u}_D \in H^\delta(\Omega_D)$, then, $\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^{1+\delta}(\Omega_S), p_D \in H^{1+\delta}(\Omega_D), \varphi \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in H^{1/2+\delta}(\Sigma)$, and there exists $C > 0$, independent of h, such that

$$\begin{aligned}
\|(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) - (\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \leqslant Ch^\delta\Big\{&\|\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathrm{div}\,\boldsymbol{\sigma}_{\mathrm{S}}\|_{\delta,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}}\\
&+\|\mathrm{div}\,\mathbf{u}_{\mathrm{D}}\|_{\delta,\Omega_{\mathrm{D}}} + \|\mathbf{u}_{\mathrm{S}}\|_{1+\delta,\Omega_{\mathrm{S}}} + \|p_{\mathrm{D}}\|_{1+\delta,\Omega_{\mathrm{D}}}\Big\}.
\end{aligned} \tag{2.12}$$

Proof. See [40, Theorems 5.3–5.5]. □

Note that the proofs of [40, Theorems 5.3–5.5] require $\mathcal{T}_h^{\mathrm{S}}$ and $\mathcal{T}_h^{\mathrm{D}}$ to be quasi-uniform in a neighborhood of Σ. Based on a recent result on stable discrete liftings of the normal trace of Raviart–Thomas elements in [53], the theorem can be easily generalized to any shape–regular triangulation.

3. A residual-based a posteriori error estimator

We first introduce some notations. For each $T \in \mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$ we let $\mathcal{E}(T)$ be the set of edges of T, and we denote by $\mathcal{E}_h$ the set of all edges of $\mathcal{T}_h^{\mathrm{S}} \cup \mathcal{T}_h^{\mathrm{D}}$, subdivided as follows:

$$\mathcal{E}_h = \mathcal{E}_h(\Gamma_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{S}}) \cup \mathcal{E}_h(\Omega_{\mathrm{D}}) \cup \mathcal{E}_h(\Sigma),$$

where $\mathcal{E}_h(\Gamma_{\mathrm{S}}) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_{\mathrm{S}}\}, \mathcal{E}_h(\Omega_\star) := \{e \in \mathcal{E}_h : e \subseteq \Omega_\star\}$ for each $\star \in \{\mathrm{S},\mathrm{D}\}$, and $\mathcal{E}_h(\Sigma) := \{e \in \mathcal{E}_h : e \subseteq \Sigma\}$. Note that $\mathcal{E}_h(\Sigma)$ is the set of edges defining the partition Σ_h. Analogously, we let $\mathcal{E}_{2h}(\Sigma)$ be the set of *double* edges defining the partition Σ_{2h}. In what follows, h_e stands for the diameter of a given edge $e \in \mathcal{E}_h \cup \mathcal{E}_{2h}(\Sigma)$. Now, let $\star \in \{\mathrm{D},\mathrm{S}\}$ and let $q \in [L^2(\Omega_\star)]^m$, with $m \in \{1,2\}$, such that $q|_T \in [C(T)]^m$ for each $T \in \mathcal{T}_h^\star$. Then, given $e \in \mathcal{E}_h(\Omega_\star)$, we denote by $[q]$ the jump of q across e, that is $[q] := (q|_{T'})|_e - (q|_{T''})|_e$, where T' and T'' are the triangles of $\mathcal{T}_h^\star$ having e as an edge. Also, we fix a unit normal vector $\mathbf{n}_e := (n_1,n_2)^{\mathrm{t}}$ to the edge e (its particular orientation is not relevant) and let $\mathbf{t}_e := (-n_2,n_1)^{\mathrm{t}}$ be the corresponding fixed unit tangential vector along e. Hence, given $\mathbf{v} \in \mathbf{L}^2(\Omega_\star)$ and $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega_\star)$ such that $\mathbf{v}|_T \in [C(T)]^2$ and $\boldsymbol{\tau}|_T \in [C(T)]^{2\times 2}$, respectively, for each $T \in \mathcal{T}_h^\star$, we let $[\mathbf{v}\cdot\mathbf{t}_e]$ and $[\boldsymbol{\tau}\mathbf{t}_e]$ be the tangential jumps of $\mathbf{v}$ and $\boldsymbol{\tau}$, across e, that is $[\mathbf{v}\cdot\mathbf{t}_e] := \{(\mathbf{v}|_{T'})|_e - (\mathbf{v}|_{T''})|_e\}\cdot\mathbf{t}_e$ and $[\boldsymbol{\tau}\mathbf{t}_e] := \{(\boldsymbol{\tau}|_{T'})|_e - (\boldsymbol{\tau}|_{T''})|_e\}\mathbf{t}_e$, respectively. From now on, when no confusion arises, we will simply write $\mathbf{t}$ and $\mathbf{n}$ instead of $\mathbf{t}_e$ and $\mathbf{n}_e$, respectively. Finally, for sufficiently smooth scalar, vector and tensors fields q, $\mathbf{v} := (v_1,v_2)^{\mathrm{t}}$ and $\boldsymbol{\tau} := (\tau_{ij})_{2\times 2}$, respectively, we let

$$\mathbf{curl}\mathbf{v} := \begin{pmatrix} \frac{\partial v_1}{\partial x_2} & -\frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} & -\frac{\partial v_2}{\partial x_1} \end{pmatrix}, \qquad \mathrm{curl}q := \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right)^{\mathrm{t}},$$

$$\mathrm{rot}\mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \quad \text{and} \quad \mathbf{rot}\boldsymbol{\tau} := \left(\frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2}, \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2}\right)^{\mathrm{t}}.$$

Next, let $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{X} \times \mathbb{M}$ and $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) := ((\boldsymbol{\sigma}_{\mathrm{S},h}, \mathbf{u}_{\mathrm{D},h}, \boldsymbol{\varphi}_h, \lambda_h), (\mathbf{u}_{\mathrm{S},h}, p_{\mathrm{D},h})) \in \mathbb{X}_h \times \mathbb{M}_h$ be the unique solutions of (2.5) and (2.11), respectively. Then, we introduce the global a posteriori error estimator:

$$\Theta := \left\{\sum_{T\in\mathcal{T}_h^{\mathrm{S}}} \Theta_{\mathrm{S},T}^2 + \sum_{T\in\mathcal{T}_h^{\mathrm{D}}} \Theta_{\mathrm{D},T}^2\right\}^{1/2}, \tag{3.1}$$

where, for each $T \in \mathcal{T}_h^{\mathrm{S}}$:

$$\begin{aligned}
\Theta_{\mathrm{S},T}^2 :=\ & \|\mathbf{f}_{\mathrm{S}} + \mathrm{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,T}^2 + h_T^2\|\mathbf{rot}\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2 + h_T^2\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\|_{0,T}^2\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{S}})} h_e\|\left[\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\right]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_{\mathrm{S}})} h_e\|\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t}\|_{0,e}^2\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\|\mathbf{u}_{\mathrm{S},h} + \boldsymbol{\varphi}_h\|_{0,e}^2\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \Big\{h_e\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|_{0,e}^2\\
&+h_e\|\nu^{-1}\boldsymbol{\sigma}_{\mathrm{S},h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\|_{0,e}^2\Big\},
\end{aligned}$$

and for each $T \in \mathcal{T}_h^{\mathrm{D}}$:

$$\begin{aligned}
\Theta_{\mathrm{D},T}^2 :=\ & \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2 + h_T^2\|\mathrm{rot}(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h})\|_{0,T}^2 + h_T^2\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\|_{0,T}^2\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_{\mathrm{D}})} h_e\|[\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t}]\|_{0,e}^2\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} h_e\|\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{t} + \lambda_h'\|_{0,e}^2\\
&+ \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)} \Big\{h_e\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{0,e}^2 + h_e\|p_{\mathrm{D},h} - \lambda_h\|_{0,e}^2\Big\}.
\end{aligned}$$

The derivatives $\boldsymbol{\varphi}_h'$ and λ_h' have to be understood as tangential derivatives in the direction imposed by the tangential vector field $\mathbf{t}$ on Σ.

3.1. Reliability of the a posteriori error estimator

The main result of this section is stated as follows.

Theorem 3.1. *There exists $C_{rel} > 0$, independent of h, such that*

$$\|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} \leqslant C_{\mathrm{rel}}\Theta. \tag{3.2}$$

We begin the derivation of (3.2) by recalling that the continuous dependence result given by (2.9) is equivalent to the global inf–sup condition for the continuous formulation (2.5). Then, applying this estimate to the error $(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) \in \mathbb{X} \times \mathbb{M}$, we obtain

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \leqslant C \sup_{\substack{(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \in \mathbb{X}\times\mathbb{M} \\ (\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \neq \mathbf{0}}} \frac{|R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}})|}{\|(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}})\|_{\mathbb{X}\times\mathbb{M}}}, \tag{3.3}$$

where $R : \mathbb{X}\times\mathbb{M} \to \mathbb{R}$ is the residual functional

$$R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := \mathcal{A}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h) + \mathcal{B}(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{v}}),\quad \forall (\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \in \mathbb{X}\times\mathbb{M}.$$

More precisely, according to (2.5) and the definitions of $\mathcal{A}$ and $\mathcal{B}$ (cf. 2.7, 2.8), we find that for any $(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := ((\boldsymbol{\tau}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}, \boldsymbol{\psi}, \xi), (\mathbf{v}_{\mathrm{S}}, q_{\mathrm{D}})) \in \mathbb{X}\times\mathbb{M}$ there holds

$$R(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) = R_1(\boldsymbol{\tau}_{\mathrm{S}}) + R_2(\mathbf{v}_{\mathrm{D}}) + R_3(\boldsymbol{\psi}) + R_4(\xi) + R_5(\mathbf{v}_{\mathrm{S}}) + R_6(q_{\mathrm{D}}),$$

where

$$R_1(\boldsymbol{\tau}_{\mathrm{S}}) := -\nu^{-1}\int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}^d_{\mathrm{S},h} : \boldsymbol{\tau}^d_{\mathrm{S}} - \int_{\Omega_{\mathrm{S}}} \mathbf{u}_{\mathrm{S},h}\cdot \mathbf{div}\,\boldsymbol{\tau}_{\mathrm{S}} - \langle \boldsymbol{\tau}_{\mathrm{S}}\mathbf{n}, \boldsymbol{\varphi}_h\rangle_\Sigma,$$

$$R_2(\mathbf{v}_{\mathrm{D}}) := -\int_{\Omega_{\mathrm{D}}} \mathbf{K}^{-1}\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{v}_{\mathrm{D}} + \int_{\Omega_{\mathrm{D}}} p_{\mathrm{D},h}\mathrm{div}\,\mathbf{v}_{\mathrm{D}} + \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma,$$

$$R_3(\boldsymbol{\psi}) := -\langle \boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda_h\rangle_\Sigma + \nu\kappa^{-1}\langle \boldsymbol{\psi}\cdot\mathbf{t}, \boldsymbol{\varphi}_h\cdot\mathbf{t}\rangle_\Sigma,$$

$$R_4(\xi) := \langle \mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\varphi}_h\cdot\mathbf{n}, \xi\rangle_\Sigma,$$

$$R_5(\mathbf{v}_{\mathrm{S}}) := -\int_{\Omega_{\mathrm{S}}} \mathbf{v}_{\mathrm{S}}\cdot(\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}),$$

and

$$R_6(q_{\mathrm{D}}) := -\int_{\Omega_{\mathrm{D}}} q_{\mathrm{D}}(f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}).$$

Hence, the supremum in (3.3) can be bounded in terms of $R_i, i \in \{1,\ldots,6\}$, which yields

$$\|(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}} - \underline{\mathbf{u}}_h)\|_{\mathbb{X}\times\mathbb{M}} \leqslant C\Big\{\|R_1\|_{\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})'} + \|R_2\|_{\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})'} + \|R_3\|_{\mathbf{H}^{1/2}(\Sigma)'} + \|R_4\|_{H^{1/2}(\Sigma)'} + \|R_5\|_{\mathbf{L}^2(\Omega_{\mathrm{S}})'} + \|R_6\|_{L^2_0(\Omega_{\mathrm{D}})'}\Big\}. \tag{3.4}$$

In this way, we have transformed (3.3) into an estimate involving global inf–sup conditions on each one of the spaces forming the product space $\mathbb{X}\times\mathbb{M}$. This splitting approach was recently extended in [37] and [41] to the derivation of residual-based a posteriori error estimators for certain nonlinear boundary value problems.

Throughout the rest of this section we provide suitable upper bounds for each one of the terms on the right hand side of (3.4). The following lemma, whose proof follows from straightforward applications of the Cauchy–Schwarz inequality, is stated first.

Lemma 3.1. *There hold*

$$\|R_5\|_{\mathbf{L}^2(\Omega_{\mathrm{S}})'} = \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|_{0,\Omega_{\mathrm{S}}} = \left\{\sum_{T\in\mathcal{T}^{\mathrm{S}}_h} \|\mathbf{f}_{\mathrm{S}} + \mathbf{div}\,\boldsymbol{\sigma}_{\mathrm{S},h}\|^2_{0,T}\right\}^{1/2}, \tag{3.5}$$

and

$$\|R_6\|_{L^2_0(\Omega_{\mathrm{D}})'} \leqslant \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|_{0,\Omega_{\mathrm{D}}} = \left\{\sum_{T\in\mathcal{T}^{\mathrm{D}}_h} \|f_{\mathrm{D}} - \mathrm{div}\,\mathbf{u}_{\mathrm{D},h}\|^2_{0,T}\right\}^{1/2}. \tag{3.6}$$

The next lemma estimates the supremum on the spaces defined in the interface Σ.

Lemma 3.2. *There exist $C_3, C_4 > 0$, independent of h, such that*

$$\|R_3\|_{\mathbf{H}^{1/2}(\Sigma)'} \leqslant C_3\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|^2_{0,e}\right\}^{1/2}, \tag{3.7}$$

and

$$\|R_4\|_{H^{1/2}(\Sigma)'} \leqslant C_4\left\{\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|^2_{0,e}\right\}^{1/2}. \tag{3.8}$$

Proof. It is clear from the definition of R_3 that

$$R_3(\boldsymbol{\psi}) = -\langle \boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}, \boldsymbol{\psi}\rangle_\Sigma \quad \forall \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma),$$

and hence

$$\|R_3\|_{\mathbf{H}^{1/2}(\Sigma)'} = \|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|_{-1/2,\Sigma}. \tag{3.9}$$

In order to estimate $\|\sigma_{S,h}\mathbf{n} + \lambda_h\,\mathbf{n} - \nu\kappa^{-1}(\varphi_h\cdot\mathbf{t})\mathbf{t}\|_{-1/2,\Sigma}$ in terms of local quantities we now apply a technical result from [23]. Taking $\tau_{\mathrm{S}} = \mathbf{0}$, $\mathbf{v}_{\mathrm{D}} = \mathbf{0}$ and $\xi = 0$ in the first equation of (2.11), we have

$$\langle \boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}, \boldsymbol{\psi}\rangle_\Sigma = 0 \quad \forall \boldsymbol{\psi} \in \boldsymbol{\Lambda}_h(\Sigma),$$

which says that $\sigma_{S,h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\varphi_h\cdot\mathbf{t})\mathbf{t}$ is $\mathbf{L}^2(\Sigma)$-orthogonal to $\boldsymbol{\Lambda}_h(\Sigma)$. Hence, applying [23, Theorem 2], and recalling that Σ_h and Σ_{2h} are of bounded variation, we deduce that

$$\begin{aligned}\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|^2_{-1/2,\Sigma} &\leqslant C\sum_{e\in\mathcal{E}_{2h}(\Sigma)} h_e\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|^2_{0,e}\\ &\leqslant C\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\boldsymbol{\sigma}_{\mathrm{S},h}\mathbf{n} + \lambda_h\mathbf{n} - \nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|^2_{0,e},\end{aligned}$$

which, together with (3.9), yields (3.7).

The proof of (3.8) proceeds analogously. Since

$$\|R_4\|_{H^{1/2}(\Sigma)'} = \|\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \boldsymbol{\varphi}_h\cdot\mathbf{n}\|_{-1/2,\Sigma},$$

and $\mathbf{u}_{\mathrm{D},h}\cdot\mathbf{n} + \varphi_h\cdot\mathbf{n}$ is $L^2(\Sigma)$-orthogonal to $\Lambda_h(\Sigma)$ (this is a consequence of the first equation of (2.11)), another straightforward application of [23, Theorem 2] yields the required estimate. □

Our next goal is to bound the remaining terms in right hand side of (3.4), for which we need some preliminary results. We begin with the following lemma showing the existence of stable Helmholtz decompositions for $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{D}})$ and $\mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$.

Lemma 3.3

(a) *There exists $C_D > 0$ such that every $\mathbf{v}_D \in \mathbf{H}(div;\Omega_D)$ can be decomposed as $\mathbf{v}_D = \mathbf{w} + curl\beta$, where $\mathbf{w} \in \mathbf{H}^1(\Omega_{\mathrm{D}}), \beta \in H^1(\Omega_{\mathrm{D}}), \int_{\Omega_{\mathrm{S}}}\beta = 0$ and*

$$\|\mathbf{w}\|_{1,\Omega_{\mathrm{D}}} + \|\beta\|_{1,\Omega_{\mathrm{D}}} \leqslant C_{\mathrm{D}}\|\mathbf{v}_{\mathrm{D}}\|_{\mathrm{div};\Omega_{\mathrm{D}}}.$$

(b) *There exists $C_S > 0$ such that every $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div};\Omega_{\mathrm{S}})$ can be decomposed as $\tau_S = \eta + \mathbf{curl}\chi$, where $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_{\mathrm{S}}), \boldsymbol{\chi} \in \mathbf{H}^1(\Omega_{\mathrm{S}})$ and*

$$\|\boldsymbol{\eta}\|_{1,\Omega_{\mathrm{S}}} + \|\boldsymbol{\chi}\|_{1,\Omega_{\mathrm{S}}} \leqslant C_{\mathrm{S}}\|\boldsymbol{\tau}_{\mathrm{S}}\|_{\mathbf{div};\Omega_{\mathrm{S}}}.$$

Proof. Let G be a convex domain with smooth boundary that contains Ω_D. Given $\mathbf{v}_D \in \mathbf{H}(\mathrm{div};\Omega_D)$, we take $z \in H_0^1(G) \cap H^2(G)$ to be the unique solution of

$$-\Delta z = \left\{ \begin{matrix} \mathrm{div}\,\mathbf{v}_D & \text{in} & \Omega_D \\ 0 & \text{in} & G \setminus \overline{\Omega}_D \end{matrix} \right\} \quad \text{in} \quad G, \quad z = 0 \quad \text{on} \quad \partial G.$$

It follows that

$$\|z\|_{2,G} \leqslant C\|\mathrm{div}\,\mathbf{v}_D\|_{0,\Omega_D} \leqslant C\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D},$$

and hence, defining $\mathbf{w} := -\nabla z$ in Ω_D, we find that $\mathrm{div}\mathbf{w} = \mathrm{div}\mathbf{v}_D$ in Ω_D and

$$\|\mathbf{w}\|_{1,\Omega_D} \leqslant \|z\|_{2,\Omega_D} \leqslant \|z\|_{2,G} \leqslant C\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D}.$$

In addition, since div $(\mathbf{v}_D - \mathbf{w}) = 0$ and Ω_D is connected, there exists $\beta \in H^1(\Omega_D)$, with $\int_{\Omega_D} \beta = 0$, such that $\mathbf{v}_D - \mathbf{w} = \mathrm{curl}\beta$ in Ω_D. In this way, using the generalized Poincaré inequality and the above estimate for $\mathbf{w}$, we deduce that

$$\|\beta\|_{1,\Omega_D} \leqslant C|\beta|_{1,\Omega_D} = C\|\mathrm{curl}\beta\|_{0,\Omega_D} = C\|\mathbf{v}_D - \mathbf{w}\|_{0,\Omega_D} \leqslant C\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D},$$

which completes the proof of (a).

We now let $\boldsymbol{\tau}_S \in \mathbb{H}(\mathrm{div};\Omega_S)$. Since Ω_S is not necessarily connected, we first perform a suitable extension of τ_S to the domain $\Omega := \Omega_S \cup \Sigma \cup \Omega_D$, and then apply (a) to each row of the resulting tensor. More precisely, let $\tau_{S,i} \in \mathbf{H}(\mathrm{div};\Omega_S)$ be the i-th row of τ_S, $i \in \{1,2\}$, and let $\phi_i \in H^1(\Omega_D)$ be the unique solution of the Neumann problem:

$$\Delta\phi_i = -\frac{\langle \boldsymbol{\tau}_{S,i} \cdot \mathbf{n}, 1 \rangle_\Sigma}{|\Omega_D|} \text{ in } \Omega_D, \quad \frac{\partial \phi_i}{\partial \mathbf{n}} = \boldsymbol{\tau}_{S,i} \cdot \mathbf{n} \text{ on } \Sigma, \quad \int_{\Omega_D} \phi_i = 0.$$

Then we define $\boldsymbol{\tau}_i^{ext} = \left\{ \begin{matrix} \boldsymbol{\tau}_{S,i} \text{ in } \Omega_S \\ \nabla\phi_i \text{ in } \Omega_D \end{matrix} \right.$, and notice that $\boldsymbol{\tau}_i^{ext} \in \mathbf{H}(\mathrm{div};\Omega)$ and

$$\begin{aligned}\|\boldsymbol{\tau}_i^{ext}\|_{\mathrm{div};\Omega} &\leqslant \|\boldsymbol{\tau}_{S,i}\|_{\mathrm{div};\Omega_S} + \|\nabla\phi_i\|_{\mathrm{div};\Omega_D} \leqslant \|\boldsymbol{\tau}_{S,i}\|_{\mathrm{div};\Omega_S} + C\|\boldsymbol{\tau}_{S,i} \cdot \mathbf{n}\|_{-1/2,\Sigma} \\ &\leqslant C\|\boldsymbol{\tau}_{S,i}\|_{\mathrm{div};\Omega_S}.\end{aligned}$$

Proceeding as in the proof of (a), but now for $\boldsymbol{\tau}_i^{ext} \in \mathbf{H}(\mathrm{div};\Omega)$, we deduce the existence of $\mathbf{w}_i \in \mathbf{H}^1(\Omega)$ and $\beta_i \in H^1(\Omega)$, with $\int_\Omega \beta_i = 0$, such that $\boldsymbol{\tau}_i^{ext} = \mathbf{w}_i + \mathrm{curl}\beta_i$ in Ω, and

$$\|\mathbf{w}_i\|_{1,\Omega} + \|\beta_i\|_{1,\Omega} \leqslant C\|\boldsymbol{\tau}_i^{ext}\|_{\mathrm{div};\Omega} \leqslant C\|\boldsymbol{\tau}_{S,i}\|_{\mathrm{div};\Omega_S}.$$

Hence, the proof of (b) follows by defining ith row of $\boldsymbol{\eta} := \mathbf{w}_i|_{\Omega_S}$ and $\boldsymbol{\chi} := (\beta_1|_{\Omega_S}, \beta_2|_{\Omega_S})$. □

We next recall two well–known approximation operators: the Raviart–Thomas interpolator (see [15] for example) and the Clément operator onto the space of continuous piecewise linear functions [22].

The Raviart–Thomas interpolation operator $\Pi_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{H}_h(\Omega_\star)$ (recall the discrete spaces in (2.10)), $\star \in \{S, D\}$, is given by the conditions

$$\Pi_h^\star \mathbf{v} \in \mathbf{H}_h(\Omega_\star) \quad \text{and} \quad \int_e \Pi_h^\star \mathbf{v} \cdot \mathbf{n} = \int_e \mathbf{v} \cdot \mathbf{n} \qquad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star. \tag{3.10}$$

As a consequence of (3.10), there holds

$$\mathrm{div}(\Pi_h^\star \mathbf{v}) = \mathcal{P}_h^\star(\mathrm{div}\mathbf{v}), \tag{3.11}$$

where $\mathcal{P}_h^\star, \star \in \{S, D\}$, is the $L^2(\Omega_\star)$-orthogonal projector onto the piecewise constant functions on $\Omega_\star$. A tensor version of $\Pi_h^\star$, say $\boldsymbol{\Pi}_h^\star : \mathbb{H}^1(\Omega_\star) \to \mathbb{H}_h(\Omega_\star)$, which is defined row-wise by $\Pi_h^\star$, and a vector version of $\mathcal{P}_h^\star$, say $\mathbf{P}_h^\star$, which is the $\mathbf{L}^2(\Omega_\star)$-orthogonal projector onto the piecewise constant vectors on $\Omega_\star$, might also be required. The local approximation properties of $\Pi_h^\star$ (and hence of $\boldsymbol{\Pi}_h^\star$) are stated as follows.

Lemma 3.4. *For each $\star \in \{S,D\}$ there exist constants c_1, $c_2 > 0$, independent of h, such that for all $\mathbf{v} \in \mathbf{H}^1(\Omega_\star)$ there hold*

$$\|\mathbf{v} - \Pi_h^\star \mathbf{v}\|_{0,T} \leqslant c_1 h_T \|\mathbf{v}\|_{1,T} \quad \forall T \in \mathcal{T}_h^\star,$$

and

$$\|\mathbf{v} \cdot \mathbf{n} - \Pi_h^\star \mathbf{v} \cdot \mathbf{n}\|_{0,e} \leqslant c_2 h_e^{1/2} \|\mathbf{v}\|_{1,T_e} \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^\star,$$

where T_e is a triangle of $\mathcal{T}_h^\star$ containing e on its boundary.

Proof. See [15]. □

The Clément operators $I_h^\star : H^1(\Omega_\star) \to X_{\star,h}$ approximate optimally non–smooth functions by continuous piecewise linear functions:

$$X_{\star,h} := \left\{ v \in C(\overline{\Omega}_\star) : \quad v|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h^\star \right\} \qquad \text{for each } \star \in \{S, D\}.$$

Of this operator, we will only use its approximation properties (see below). In addition, we will make use of a vector version of $I_h^\star$, say $\mathbf{I}_h^\star : \mathbf{H}^1(\Omega_\star) \to \mathbf{X}_{\star,h} := X_{\star,h} \times X_{\star,h}$, which is defined componentwise by $I_h^\star$. The following lemma establishes the local approximation properties of $I_h^\star$ (and hence of $\mathbf{I}_h^\star$).

Lemma 3.5. *For each $\star \in \{S,D\}$ there exist constants c_3,$c_4 > 0$, independent of h, such that for all $v \in H^1(\Omega_\star)$ there hold*

$$\|v - I_h^\star v\|_{0,T} \leqslant c_3 h_T \|v\|_{1,\Delta_\star(T)} \quad \forall T \in \mathcal{T}_h^\star,$$

and

$$\|v - I_h^\star v\|_{0,e} \leqslant c_4 h_e^{1/2} \|v\|_{1,\Delta_\star(e)} \quad \forall e \in \mathcal{E}_h,$$

where

$$\Delta_\star(T) := \cup\{T' \in \mathcal{T}_h^\star : \quad T' \cap T \neq \mathbf{0}\} \quad \text{and}$$
$$\Delta_\star(e) := \cup\{T' \in \mathcal{T}_h^\star : \quad T' \cap e \neq \mathbf{0}\}.$$

Proof. See [26]. □

Finally, we require the technical results given by the following two lemmas.

Lemma 3.6. *Let $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_S)$ and $\chi \in \mathbf{H}^1(\Omega_S)$. Then there hold*

$$\begin{aligned}|R_1(\boldsymbol{\eta} - \boldsymbol{\Pi}_h^S \boldsymbol{\eta})| \leqslant{}& c_1 \nu^{-1} \sum_{T \in \mathcal{T}_h^S} h_T \|\boldsymbol{\sigma}_{S,h}^d\|_{0,T} \|\boldsymbol{\eta}\|_{1,T} + c_2 \sum_{e \in \mathcal{E}_h(\Sigma)} h_e^{1/2} \|\mathbf{u}_{S,h} \\ &+ \boldsymbol{\varphi}_h\|_{0,e} \|\boldsymbol{\eta}\|_{1,T_e},\end{aligned}$$

and

$$\begin{aligned}|R_1(\mathbf{curl}(\boldsymbol{\chi} - \mathbf{I}_h^S \boldsymbol{\chi}))| \leqslant{}& c_3 \nu^{-1} \sum_{T \in \mathcal{T}_h^S} h_T \|\mathbf{rot}\boldsymbol{\sigma}_{S,h}^d\|_{0,T} \|\boldsymbol{\chi}\|_{1,\Delta_S(T)} \\ &+ c_4 \nu^{-1} \sum_{e \in \mathcal{E}_h(\Omega_S)} h_e^{1/2} \|[\boldsymbol{\sigma}_{S,h}^d \mathbf{t}]\|_{0,e} \|\boldsymbol{\chi}\|_{1,\Delta_S(e)} \\ &+ c_4 \nu^{-1} \sum_{e \in \mathcal{E}_h(\Gamma_S)} h_e^{1/2} \|\boldsymbol{\sigma}_{S,h}^d \mathbf{t}\|_{0,e} \|\boldsymbol{\chi}\|_{1,\Delta_S(e)} \\ &+ c_4 \sum_{e \in \mathcal{E}_h(\Sigma)} h_e^{1/2} \|\nu^{-1} \boldsymbol{\sigma}_{S,h}^d \mathbf{t} + \boldsymbol{\varphi}_h'\|_{0,e} \|\boldsymbol{\chi}\|_{1,\Delta_S(e)}.\end{aligned}$$

Proof. We first let $\zeta := \boldsymbol{\eta} - \boldsymbol{\Pi}_h^S \boldsymbol{\eta}$ and observe, according to (3.10) and (3.11), that

$$\int_e \mathbf{p} \cdot \zeta\mathbf{n} = 0 \quad \forall \mathbf{p} \in [\mathbb{P}_0(e)]^2, \quad \forall \text{ edge } e \text{ of } \mathcal{T}_h^S, \quad \text{and}$$
$$\mathrm{div}\,\zeta = \mathrm{div}\,\boldsymbol{\eta} - \mathbf{P}_h^S(\mathrm{div}\,\boldsymbol{\eta}).$$

Then, since $\sigma_{S,h}{}^d:\zeta^d = \sigma_{S,h}{}^d:\zeta$ and $\mathbf{u}_{S,h}$ is constant on each $T \in \mathcal{T}_h^S$, we deduce from the definition of R_1 and the above identities that

$$R_1(\zeta) = -\nu^{-1}\sum_{T\in\mathcal{T}_h^S}\int_T \boldsymbol{\sigma}_{S,h}^d : \zeta^d - \sum_{T\in\mathcal{T}_h^S}\int_T \mathbf{u}_{S,h}\cdot \operatorname{div}\zeta - \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \boldsymbol{\varphi}_h\cdot\zeta\mathbf{n}$$
$$= -\nu^{-1}\sum_{T\in\mathcal{T}_h^S}\int_T \boldsymbol{\sigma}_{S,h}^d : \zeta - \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \boldsymbol{\varphi}_h\cdot\zeta\mathbf{n} = -\nu^{-1}\sum_{T\in\mathcal{T}_h^S}\int_T \boldsymbol{\sigma}_{S,h}^d : \zeta - \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e (\mathbf{u}_{S,h}+\boldsymbol{\varphi}_h)\cdot\zeta\mathbf{n}.$$

We next let $\boldsymbol{\rho} := \boldsymbol{\chi} - \mathbf{I}_h^S\boldsymbol{\chi}$. Then, using that $\operatorname{div}(\mathbf{curl}\rho) = \mathbf{0}$, noting that $(\mathbf{curl}\rho)\mathbf{n} = (\nabla\rho)\mathbf{t}$ on Σ, integrating by parts on each $T \in \mathcal{T}_h^S$ and on Σ, and observing that $\boldsymbol{\varphi}_h' \in \mathbf{L}^2(\Sigma)$, we obtain

$$R_1(\mathbf{curl}\boldsymbol{\rho}) = -\nu^{-1}\int_{\Omega_S}\boldsymbol{\sigma}_{S,h}^d : \mathbf{curl}\boldsymbol{\rho} - \langle(\mathbf{curl}\boldsymbol{\rho})\mathbf{n},\boldsymbol{\varphi}_h\rangle_\Sigma$$
$$= \nu^{-1}\sum_{T\in\mathcal{T}_h^S}\left(-\int_T \boldsymbol{\rho}\cdot\mathbf{rot}\boldsymbol{\sigma}_{S,h}^d + \int_{\partial T}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\right) + \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e \boldsymbol{\rho}\cdot\boldsymbol{\varphi}_h' = -\sum_{T\in\mathcal{T}_h^S}\nu^{-1}\int_T\boldsymbol{\rho}\cdot\mathbf{rot}\boldsymbol{\sigma}_{S,h}^d$$
$$+ \sum_{e\in\mathcal{E}_h(\Omega_S)}\nu^{-1}\int_e\boldsymbol{\rho}\cdot[\boldsymbol{\sigma}_{S,h}^d\mathbf{t}] + \sum_{e\in\mathcal{E}_h(\Gamma_S)}\nu^{-1}\int_e\boldsymbol{\rho}\cdot\left(\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\right) + \sum_{e\in\mathcal{E}_h(\Sigma)}\int_e\boldsymbol{\rho}\cdot\left(\nu^{-1}\boldsymbol{\sigma}_{S,h}^d\mathbf{t}+\boldsymbol{\varphi}_h'\right).$$

Hence, straightforward applications of the Cauchy–Schwarz inequality to the above identities, together with the approximation properties of Lemmas 3.4 and 3.5, namely,

$$\|\boldsymbol{\eta}-\boldsymbol{\Pi}_h^S\boldsymbol{\eta}\|_{0,T} \leqslant c_1 h_T\|\boldsymbol{\eta}\|_{1,T}, \qquad \|\boldsymbol{\eta}\mathbf{n}-\boldsymbol{\Pi}_h^S\boldsymbol{\eta}\mathbf{n}\|_{0,e} \leqslant c_2 h_e^{1/2}\|\boldsymbol{\eta}\|_{1,T}$$
$$\|\boldsymbol{\chi}-\mathbf{I}_h^S\boldsymbol{\chi}\|_{0,T} \leqslant c_3 h_T\|\boldsymbol{\chi}\|_{1,\Delta_S(T)}, \qquad \|\boldsymbol{\chi}-\mathbf{I}_h^S\boldsymbol{\chi}\|_{0,e} \leqslant c_4 h_e^{1/2}\|\boldsymbol{\chi}\|_{1,\Delta_S(e)},$$

for each $T \in \mathcal{T}_h^S$ and for each $e \in \mathcal{E}(T)$, imply the required estimates and finish the proof. $\square$

Lemma 3.7. *Let $\mathbf{w} \in \mathbf{H}^1(\Omega_D)$ and $\beta \in H^1(\Omega_D)$. Then there hold*

$$|R_2(\mathbf{w}-\Pi_h^D\mathbf{w})| \leqslant c_1\sum_{T\in\mathcal{T}_h^D} h_T\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\|_{0,T}\|\mathbf{w}\|_{1,T} + c_2\sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\|p_{D,h}-\lambda_h\|_{0,e}\|\mathbf{w}\|_{1,T_e},$$

and

$$|R_2(\operatorname{curl}(\beta - I_h^D\beta))| \leqslant c_3\sum_{T\in\mathcal{T}_h^D} h_T\|\operatorname{rot}(\mathbf{K}^{-1}\mathbf{u}_{D,h})\|_{0,T}\|\beta\|_{1,\Delta_D(T)} + c_4 \times \sum_{e\in\mathcal{E}_h(\Omega_D)} h_e^{1/2}\|[\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}]\|_{0,e}\|\beta\|_{1,\Delta_D(e)} + c_4 \times \sum_{e\in\mathcal{E}_h(\Sigma)} h_e^{1/2}\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}+\lambda_h'\|_{0,e}\|\beta\|_{1,\Delta_D(e)}.$$

Proof. Since R_1 and R_2 have analogue structures, the proof proceeds similarly as for Lemma 3.6. $\square$

We are now in a position to bound the residual functionals R_1 and R_2.

Lemma 3.8. *There exists $C_1 > 0$, independent of h, such that*

$$\|R_1\|_{\mathbb{H}(\operatorname{div};\Omega_S)'} \leqslant C_1\left\{\sum_{T\in\mathcal{T}_h^S}\widehat{\Theta}_{S,T}^2\right\}^{1/2}, \tag{3.12}$$

where, for each $T \in \mathcal{T}_h^S$:

$$\widehat{\Theta}_{S,T}^2 := h_T^2\|\mathbf{rot}\boldsymbol{\sigma}_{S,h}^d\|_{0,T}^2 + h_T^2\|\boldsymbol{\sigma}_{S,h}^d\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_S)} h_e\left\|\left[\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\right]\right\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Gamma_S)} h_e\|\boldsymbol{\sigma}_{S,h}^d\mathbf{t}\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)}\left\{h_e\|\nu^{-1}\boldsymbol{\sigma}_{S,h}^d\mathbf{t}+\boldsymbol{\varphi}_h'\|_{0,e}^2 + h_e\|\mathbf{u}_{S,h}+\boldsymbol{\varphi}_h\|_{0,e}^2\right\}$$

Proof. Given $\tau_S \in \mathbb{H}(\operatorname{div};\Omega_S)$ we know from Lemma 3.3 that there exist $\boldsymbol{\eta} \in \mathbb{H}^1(\Omega_S)$ and $\chi \in \mathbf{H}^1(\Omega_S)$ such that $\tau_S = \eta + \mathbf{curl}\,\chi$ in Ω_S and

$$\|\boldsymbol{\eta}\|_{1,\Omega_S} + \|\boldsymbol{\chi}\|_{1,\Omega_S} \leqslant C\|\tau_S\|_{\operatorname{div};\Omega_S}. \tag{3.13}$$

Then, since $R_1(\boldsymbol{\tau}_{S,h}) = 0 \quad \forall\boldsymbol{\tau}_{S,h} \in \mathbb{H}_h(\Omega_S)$, which follows from the first equation of the Galerkin scheme (2.11) taking $(\mathbf{v}_D,\psi,\xi) = (\mathbf{0},\mathbf{0},\mathbf{0})$, we obtain

$$R_1(\boldsymbol{\tau}_S) = R_1(\boldsymbol{\tau}_S - \boldsymbol{\tau}_{S,h}) \quad \forall\boldsymbol{\tau}_{S,h} \in \mathbb{H}_h(\Omega_S). \tag{3.14}$$

In particular, we let $\tau_{S,h} := \boldsymbol{\Pi}_h^S\boldsymbol{\eta} + \mathbf{curl}(\mathbf{I}_h^S\boldsymbol{\chi})$, which can be seen as a discrete Helmholtz decomposition of $\tau_{S,h}$, and obtain

$$R_1(\boldsymbol{\tau}_S) = R_1(\boldsymbol{\eta}-\boldsymbol{\Pi}_h^S\boldsymbol{\eta}) + R_1(\mathbf{curl}(\boldsymbol{\chi}-\mathbf{I}_h^S\boldsymbol{\chi})). \tag{3.15}$$

Hence, applying Lemma 3.6 and noticing that the numbers of triangles in $\#\Delta_S(T)$ and $\#\Delta_S(e)$ are bounded, and finally using the estimate (3.13), we prove the upper bound (3.12). $\square$

Lemma 3.9. *There exists $C_2 > 0$, independent of h, such that*

$$\|R_2\|_{\mathbf{H}(\operatorname{div};\Omega_D)'} \leqslant C_2\left\{\sum_{T\in\mathcal{T}_h^D}\widehat{\Theta}_{D,T}^2\right\}^{1/2}, \tag{3.16}$$

where, for each $T \in \mathcal{T}_h^D$:

$$\widehat{\Theta}_{D,T}^2 := h_T^2\|\operatorname{rot}(\mathbf{K}^{-1}\mathbf{u}_{D,h})\|_{0,T}^2 + h_T^2\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\|_{0,T}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_D)} h_e\|[\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}]\|_{0,e}^2 + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_h(\Sigma)}\left\{h_e\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}+\lambda_h'\|_{0,e}^2 + h_e\|p_{D,h}-\lambda_h\|_{0,e}^2\right\}.$$

Proof. It follows basically the same lines of the proof of Lemma 3.8. In fact, given $\mathbf{v}_D \in \mathbf{H}(\operatorname{div};\Omega_D)$ we first apply Lemma 3.3 to deduce the existence of $\mathbf{w} \in \mathbf{H}^1(\Omega_D)$ and $\beta \in H^1(\Omega_D)$ such that $\mathbf{v}_D = \mathbf{w} + \operatorname{curl}\beta$ and

$$\|\mathbf{w}\|_{1,\Omega_D} + \|\beta\|_{1,\Omega_D} \leqslant C\|\mathbf{v}_D\|_{\operatorname{div};\Omega_D}. \tag{3.17}$$

Then, since $R_2(\mathbf{v}_{D,h}) = 0\,\forall\mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D)$, which corresponds to the first equation of the Galerkin scheme (2.11) with $(\tau_S,\psi,\xi) = (\mathbf{0},\mathbf{0},\mathbf{0})$, we obtain

$$R_2(\mathbf{v}_D) = R_2(\mathbf{v}_D - \mathbf{v}_{D,h}) \quad \forall\mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D). \tag{3.18}$$

Next, we choose $\mathbf{v}_{D,h} = \Pi_h^D\mathbf{w} + \operatorname{curl}\left(I_h^D\beta\right)$, notice that

$$R_2(\mathbf{v}_D) = R_2(\mathbf{w}-\Pi_h^D\mathbf{w}) + R_2\left(\operatorname{curl}(\beta - I_h^D\beta)\right),$$

and apply Lemma 3.7. Noticing again that the number of triangles in $\Delta_D(T)$ and $\Delta_D(e)$ are bounded, and employing now the upper bound (3.17), we conclude (3.16). $\square$

We end this section by observing that the reliability estimate (3.2) (cf. Theorem 3.1) is a direct consequence of Lemmas 3.1, 3.2, 3.8, and 3.9.

3.2. Efficiency of the a posteriori error estimator

The main result of this section is stated as follows.

Theorem 3.2. *There exists $C_{eff} > 0$, independent of h, such that*

$$C_{\text{eff}}\Theta \leqslant \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h\|_{\mathbb{X}} + \|\underline{\mathbf{u}} - \underline{\mathbf{u}}_h\|_{\mathbb{M}} + \text{h.o.t.}, \tag{3.19}$$

where h.o.t. stands, eventually, for one or several terms of higher order.

We remark in advance that the proof of (3.19) makes frequent use of the identities provided by Theorem 2.2. We begin with the estimates for the zero order terms appearing in the definition of $\Theta_{\text{S},T}^2$ and $\Theta_{\text{D},T}^2$.

Lemma 3.10. *There hold*

$$\|\mathbf{f}_\text{S} + \operatorname{div}\boldsymbol{\sigma}_{\text{S},h}\|_{0,T} \leqslant \|\boldsymbol{\sigma}_\text{S} - \boldsymbol{\sigma}_{\text{S},h}\|_{\text{div};T} \quad \forall T \in \mathcal{T}_{\text{S},h}$$

and

$$\|f_\text{D} - \operatorname{div}\mathbf{u}_{\text{D},h}\|_{0,T} \leqslant \|\mathbf{u}_\text{D} - \mathbf{u}_{\text{D},h}\|_{\text{div};T} \quad \forall T \in \mathcal{T}_{\text{D},h}.$$

Proof. It suffices to recall, as established by Theorem 2.2, that $\mathbf{f}_\text{S} = -\text{div}\sigma_\text{S}$ in Ω_S and $f_\text{D} = \text{div}\mathbf{u}_\text{D}$ in Ω_D. □

In order to derive the upper bounds for the remaining terms defining the global a posteriori error estimator Θ (cf. (3.1)), we proceed similarly as in [9], using results from [22,24] and [33], and apply Helmholtz decomposition, inverse inequalities, and the localization technique based on element-bubble and edge-bubble functions. To this end, we now introduce further notations and preliminary results. Given $T \in \mathcal{T}_h^\text{S} \cup \mathcal{T}_h^\text{D}$ and $e \in \mathcal{E}(T)$, we let ϕ_T and ϕ_e be the usual element-bubble and edge-bubble functions, respectively (see (1.5) and (1.6) in [57]). In particular, ϕ_T satisfies $\phi_T \in \mathbb{P}_3(T)$, $\operatorname{supp}\phi_T \subseteq T$, $\phi_T = 0$ on ∂T, and $0 \leqslant \phi_T \leqslant 1$ in T. Similarly, $\phi_e|_T \in \mathbb{P}_2(T)$, $\operatorname{supp}\phi_e \subseteq w_e := \cup\{T' \in \mathcal{T} : e \in \mathcal{E}(T')\}$, $\phi_e = 0$ on $\partial T\backslash e$, and $0 \leqslant \phi_e \leqslant 1$ in w_e. We also recall from [56] that, given $k \in \mathbb{N} \cup \{0\}$, there exists an extension operator $L : C(e) \to C(T)$ that satisfies $L(p) \in \mathbb{P}_k(T)$ and $L(p)|_e = p \forall p \in \mathbb{P}_k(e)$. A corresponding vector version of L, that is the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of ϕ_T, ϕ_e, and L are collected in the following lemma.

Lemma 3.11. *Given $k \in \mathbb{N} \cup \{0\}$, there exist positive constants c_1, c_2 and c_3, depending only on k and the shape regularity of the triangulations (minimum angle condition), such that for each triangle T and $e \in \mathcal{E}(T)$, there hold*

$$\|q\|_{0,T}^2 \leqslant c_1\|\phi_T^{1/2}q\|_{0,T}^2 \quad \forall q \in \mathbb{P}_k(T), \tag{3.20}$$

$$\|q\|_{0,e}^2 \leqslant c_2\|\phi_e^{1/2}q\|_{0,e}^2 \quad \forall q \in \mathbb{P}_k(e), \tag{3.21}$$

and

$$\|\phi_e^{1/2}L(q)\|_{0,T}^2 \leqslant c_3 h_e\|q\|_{0,e}^2 \quad \forall q \in \mathbb{P}_k(e). \tag{3.22}$$

Proof. See Lemma 1.3 in [56]. □

The following inverse estimate for polynomials will also be used.

Lemma 3.12. *Let k, l, $m \in \mathbb{N} \cup \{0\}$ such that $l \leqslant m$. Then, there exists $c > 0$, depending only on k, l, m and the shape regularity of the triangulations, such that for each triangle T there holds*

$$|q|_{m,T} \leqslant c h_T^{l-m}|q|_{l,T}, \forall q \in \mathbb{P}_k(T). \tag{3.23}$$

Proof. See Theorem 3.2.6 in [25]. □

In addition, we need to recall a discrete trace inequality, which establishes the existence of a positive constant c, depending only on the shape regularity of the triangulations, such that for each $T \in \mathcal{T}_h^\text{S} \cup \mathcal{T}_h^\text{D}$ and $e \in \mathcal{E}(T)$, there holds

$$\|v\|_{0,e}^2 \leqslant c\left\{h_e^{-1}\|v\|_{0,T}^2 + h_e|v|_{1,T}^2\right\} \quad \forall v \in H^1(T). \tag{3.24}$$

For a proof of inequality (3.24) we refer to Theorem 3.10 in [1] (see also eq. (2.4) in [6]).

The following lemma summarizes known efficiency estimates for ten terms defining $\Theta_{\text{S},T}^2$ and $\Theta_{\text{D},T}^2$. Their proofs, which apply the preliminary results described above, are already available in the literature (see, e.g. [9,10,22,33,34,36]). From now on we assume, without loss of generality, that $\mathbf{K}^{-1}\mathbf{u}_{\text{D},h}$ is polynomial on each $T \in \mathcal{T}_h^\text{D}$. Otherwise, additional higher order terms, given by the errors arising from suitable polynomial approximations, should appear in the corresponding bounds below, which explains the expression h.o.t. in (3.19).

Lemma 3.13. There exist positive constants $C_i, i \in \{1, \ldots, 10\}$, independent of h, such that.

(a) $h_T^2\|rot(\mathbf{K}^{-1}\mathbf{u}_{\text{D},h})\|_{0,T}^2 \leqslant C_1\|\mathbf{u}_\text{D} - \mathbf{u}_{\text{D},h}\|_{0,T}^2 \forall T \in \mathcal{T}_h^\text{D}$,
(b) $h_T^2\|\mathbf{rot}\boldsymbol{\sigma}_{\text{S},h}^d\|_{0,T}^2 \leqslant C_2\|\boldsymbol{\sigma}_\text{S} - \boldsymbol{\sigma}_{\text{S},h}\|_{0,T}^2 \forall T \in \mathcal{T}_h^\text{S}$,
(c) $h_e\|[\mathbf{K}^{-1}\mathbf{u}_{\text{D},h} \cdot \mathbf{t}]\|_{0,e}^2 \leqslant C_3\|\mathbf{u}_\text{D} - \mathbf{u}_{\text{D},h}\|_{0,w_e}^2 \forall e \in \mathcal{E}_h(\Omega_\text{D})$, *where the set w_e is given by $w_e := \cup\{T' \in \mathcal{T}_h^\text{D} : e \in \mathcal{E}(T')\}$,*
(d) $h_e\|[\boldsymbol{\sigma}_{\text{S},h}^d\mathbf{t}]\|_{0,e}^2 \leqslant C_4\|\boldsymbol{\sigma}_\text{S} - \boldsymbol{\sigma}_{\text{S},h}\|_{0,w_e}^2 \forall e \in \mathcal{E}_h(\Omega_\text{S})$, *where the set w_e is given by $w_e := \cup\{T' \in \mathcal{T}_h^\text{S} : e \in \mathcal{E}(T')\}$,*
(e) $h_e\|\boldsymbol{\sigma}_{\text{S},h}^d\mathbf{t}\|_{0,e}^2 \leqslant C_5\|\boldsymbol{\sigma}_\text{S} - \boldsymbol{\sigma}_{\text{S},h}\|_{0,T}^2 \forall e \in \mathcal{E}_h(\Gamma_\text{S})$, *where T is the triangle of $\mathcal{T}_h^\text{S}$ having e as an edge,*
(f) $h_T^2\|\mathbf{K}^{-1}\mathbf{u}_{\text{D},h}\|_{0,T}^2 \leqslant C_6\left\{\|p_\text{D} - p_{\text{D},h}\|_{0,T}^2 + h_T^2\|\mathbf{u}_\text{D} - \mathbf{u}_{\text{D},h}\|_{0,T}^2\right\} \forall T \in \mathcal{T}_h^\text{D}$,
(g) $h_T^2\|\boldsymbol{\sigma}_{\text{S},h}^d\|_{0,T}^2 \leqslant C_7\left\{\|\mathbf{u}_\text{S} - \mathbf{u}_{\text{S},h}\|_{0,T}^2 + h_T^2\|\boldsymbol{\sigma}_\text{S} - \boldsymbol{\sigma}_{\text{S},h}\|_{0,T}^2\right\} \forall T \in \mathcal{T}_h^\text{S}$,
(h) $h_e\|p_{\text{D},h} - \lambda_h\|_{0,e}^2 \leqslant C_8\left\{\|p_\text{D} - p_{\text{D},h}\|_{0,T}^2 + h_T^2\|\mathbf{u}_\text{D} - \mathbf{u}_{\text{D},h}\|_{0,T}^2 + h_e\|\lambda - \lambda_h\|_{0,e}^2\right\} \forall e \in \mathcal{E}_h(\Sigma)$, *where T is the triangle of $\mathcal{T}_h^\text{D}$ having e as an edge,*
(i) $\sum_{e \in \mathcal{E}_h(\Sigma)} h_e\|\mathbf{K}^{-1}\mathbf{u}_{\text{D},h} \cdot \mathbf{t} + \lambda_h'\|_{0,e}^2 \leqslant C_9\{\sum_{e \in \mathcal{E}_h(\Sigma)}\|\mathbf{u}_\text{D} - \mathbf{u}_{\text{D},h}\|_{0,T_e}^2 + \|\lambda - \lambda_h\|_{1/2,\Sigma}^2\}$, *where, given $e \in \mathcal{E}_h(\Sigma)$, T_e is the triangle of $\mathcal{T}_h^\text{D}$ having e as an edge, and*
(j) $\sum_{e \in \mathcal{E}_h(\Gamma_\text{S})} h_e\|\nu^{-1}\boldsymbol{\sigma}_{\text{S},h}^d\mathbf{t} + \boldsymbol{\varphi}_h'\|_{0,e}^2 \leqslant C_{10}\left\{\sum_{e \in \mathcal{E}_h(\Gamma_\text{S})}\|\boldsymbol{\sigma}_\text{S} - \boldsymbol{\sigma}_{\text{S},h}\|_{0,T_e}^2 + \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2\right\}$, *where, given $e \in \mathcal{E}_h(\Gamma_\text{S})$, T_e is the triangle of $\mathcal{T}_h^\text{S}$ having e as an edge.*

Proof. For (a) and (b) we refer to [22, Lemma 6.1]. Alternatively, (a) and (b) follow from straightforward applications of the technical result provided in [10, Lemma 4.3] (see also [36, Lemma 4.9]). Similarly, for (c), (d), and (e) we refer to [22, Lemma 6.2] or apply the technical result given by [10, Lemma 4.4] (see also [36, Lemma 4.10]). Then, for (f) and (g) we refer to [22, Lemma 6.3] (see also [36, Lemma 4.13] or [33, Lemma 5.5]). On the other hand, the estimate given by (h) corresponds to [9, Lemma 4.12]. In particular, its proof makes use of the discrete trace inequality (3.24). Finally, the proofs of (i) and (j) follow from very slight modifications of the proof of [33, Lemma 5.7]. Alternatively, an *elasticity version* of (i) and (j), which is provided in [34, Lemma 20], can also be adapted to our case. □

The estimates (i) and (j) in the previous lemma provide the only non-local bounds of the present efficiency analysis. However, under additional regularity assumptions on λ and $\boldsymbol{\varphi}$, we can give the following local bounds instead.

Lemma 3.14. *Assume that $\lambda|_e \in H^1(e)$ for each $e \in \mathcal{E}_h(\Sigma)$, and that $\varphi|_e \in \mathbf{H}^1(e)$ for each $e \in \mathcal{E}_h(\Gamma_\text{S})$. Then there exist $\widetilde{C}_9, \widetilde{C}_{10} > 0$, such that*

$$h_e\|\mathbf{K}^{-1}\mathbf{u}_{D,h}\cdot\mathbf{t}+\lambda'_h\|^2_{0,e}\leqslant\widetilde{C}_9\left\{\|\mathbf{u}_D-\mathbf{u}_{D,h}\|^2_{0,T_e}+h_e\|\lambda'-\lambda'_h\|^2_{0,e}\right\}\quad\forall e\in\mathcal{E}_h(\Sigma),$$

and

$$h_e\|\nu^{-1}\boldsymbol{\sigma}^d_{S,h}\mathbf{t}+\boldsymbol{\varphi}'_h\|^2_{0,e}\leqslant\widetilde{C}_{10}\left\{\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|^2_{0,T_e}+h_e\|\boldsymbol{\varphi}'-\boldsymbol{\varphi}'_h\|^2_{0,e}\right\}\quad\forall e\in\mathcal{E}_h(\Gamma_S).$$

Proof. Similarly as for (i) and (j) from Lemma 3.13, it follows by adapting the corresponding *elasticity version* from [34]. We omit details here and refer to [34, Lemma 21]. □

It remains to provide the efficiency estimates for three residual terms defined on the edges of the interface Σ. They have to do with the transmission conditions and with the trace equation $\mathbf{u}_S+\boldsymbol{\varphi}=\mathbf{0}$ on Σ. More precisely, we have the following lemmas.

Lemma 3.15. *There exists $C>0$, independent of h, such that for each $e\in\mathcal{E}_h(\Sigma)$, there holds*

$$h_e\|\mathbf{u}_{D,h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n}\|^2_{0,e}\leqslant C\left\{\|\mathbf{u}_D-\mathbf{u}_{D,h}\|^2_{0,T}+h_T^2\|\mathrm{div}(\mathbf{u}_D-\mathbf{u}_{D,h})\|^2_{0,T}+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|^2_{0,e}\right\},$$

where T is the triangle of $\mathcal{T}_h^D$ having e as an edge.

Proof. We proceed similarly as in [9, Lemma 4.7]. Given $e\in\mathcal{E}_h(\Sigma)$, we let T be the triangle of $\mathcal{T}_h^D$having e as an edge, and define $v_e:=\mathbf{u}_{D,h}\cdot\mathbf{n}+\varphi_h\cdot\mathbf{n}$ on e. Then, applying (3.21), recalling that $\phi_e=0$ on $\partial T\backslash e$, extending $\phi_eL(v_e)$ by zero in $\Omega_D\backslash T$ so that the resulting function belongs to $H^1(\Omega_D)$, and using that $\mathbf{u}_D\cdot\mathbf{n}+\varphi\cdot\mathbf{n}=0$ on Σ, we get

$$\begin{aligned}\|v_e\|^2_{0,e}\leqslant c_2\|\phi_e^{1/2}v_e\|^2_{0,e}&=c_2\int_e\phi_e v_e(\mathbf{u}_{D,h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n})\\&=c_2\langle\mathbf{u}_{D,h}\cdot\mathbf{n}+\boldsymbol{\varphi}_h\cdot\mathbf{n},\phi_eL(v_e)\rangle_\Sigma\\&=c_2\langle\mathbf{u}_{D,h}\cdot\mathbf{n}-\mathbf{u}_D\cdot\mathbf{n},\phi_eL(v_e)\rangle_\Sigma\\&\quad+c_2\langle\boldsymbol{\varphi}_h\cdot\mathbf{n}-\boldsymbol{\varphi}\cdot\mathbf{n},\phi_eL(v_e)\rangle_\Sigma.\end{aligned}\tag{3.25}$$

Next, integrating by parts in Ω_D, and noting that $(\varphi_h\cdot\mathbf{n}-\varphi\cdot\mathbf{n})\in L^2(\Sigma)$, we find, respectively, that

$$\langle\mathbf{u}_{D,h}\cdot\mathbf{n}-\mathbf{u}_D\cdot\mathbf{n},\phi_eL(v_e)\rangle_\Sigma=\int_T\nabla(\phi_eL(v_e))\cdot(\mathbf{u}_{D,h}-\mathbf{u}_D)+\int_T\phi_eL(v_e)\mathrm{div}(\mathbf{u}_{D,h}-\mathbf{u}_D),$$

and

$$\langle\boldsymbol{\varphi}_h\cdot\mathbf{n}-\boldsymbol{\varphi}\cdot\mathbf{n},\phi_eL(v_e)\rangle_\Sigma=\int_e(\boldsymbol{\varphi}_h\cdot\mathbf{n}-\boldsymbol{\varphi}\cdot\mathbf{n})\phi_ev_e.$$

Thus, replacing the above expressions back into (3.25), applying the Cauchy–Schwarz inequality and the inverse estimate (3.23), and recalling that $0\leqslant\phi_e\leqslant1$, we obtain

$$\|v_e\|^2_{0,e}\leqslant C\left\{h_T^{-1}\|\mathbf{u}_D-\mathbf{u}_{D,h}\|_{0,T}+\|\mathrm{div}(\mathbf{u}_D-\mathbf{u}_{D,h})\|_{0,T}\right\}\|\phi_eL(v_e)\|_{0,T}+c\|v_e\|_{0,e}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}.$$

But, using again that $0\leqslant\phi_e\leqslant1$ and thanks to (3.22), we get

$$\|\phi_eL(v_e)\|_{0,T}\leqslant\|\phi_e^{1/2}L(v_e)\|_{0,T}\leqslant c_3^{1/2}h_e^{1/2}\|v_e\|_{0,e},\tag{3.26}$$

whence the previous inequality yields

$$\|v_e\|_{0,e}\leqslant Ch_e^{1/2}\left\{h_T^{-1}\|\mathbf{u}_D-\mathbf{u}_{D,h}\|_{0,T}+\|\mathrm{div}(\mathbf{u}_D-\mathbf{u}_{D,h})\|_{0,T}\right\}+c\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}.$$

Finally, it is easy to see that this estimate and the fact that $h_e\leqslant h_T$ imply the required upper bound for $h_e\|v_e\|^2_{0,e}$, which finishes the proof. □

Lemma 3.16. *There exists $C>0$, independent of h, such that for each $e\in\mathcal{E}_h(\Sigma)$, there holds*

$$h_e\|\boldsymbol{\sigma}_{S,h}\mathbf{n}+\lambda_h\mathbf{n}-\nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\|^2_{0,e}\leqslant C\left\{\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|^2_{0,T}+h_T^2\|\mathrm{div}(\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h})\|^2_{0,T}+h_e\|\lambda-\lambda_h\|^2_{0,e}+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|^2_{0,e}\right\},$$

where T is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. We proceed as in the previous lemma (see also [9, Lemma 4.6]). Indeed, given $e\in\mathcal{E}_h(\Sigma)$, we let T be the triangle of $\mathcal{T}_h^S$ having e as an edge, and define $\mathbf{v}_e:=\sigma_{S,h}\mathbf{n}+\lambda_h\mathbf{n}-\nu\,\kappa^{-1}(\varphi_h\cdot\mathbf{t})\mathbf{t}$ on e. Then, applying (3.21), recalling that $\phi_e=0$ on $\partial T\backslash e$, extending $\phi_e\mathbf{L}(\mathbf{v}_e)$ by zero in $\Omega_S\backslash T$ so that the resulting function belongs to $\mathbf{H}^1(\Omega_S)$, using that $\sigma_S\mathbf{n}+\lambda\mathbf{n}-\nu\kappa^{-1}(\varphi\cdot\mathbf{t})\mathbf{t}=0$ on Σ, and then integrating by parts in Ω_S, we arrive at

$$\begin{aligned}\|\mathbf{v}_e\|^2_{0,e}\leqslant c_2\|\phi_e^{1/2}\mathbf{v}_e\|^2_{0,e}&=c_2\int_e\phi_e\mathbf{v}_e\cdot\{\boldsymbol{\sigma}_{S,h}\mathbf{n}+\lambda_h\mathbf{n}-\nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t})\mathbf{t}\}\\&=c_2\int_T\nabla(\phi_e\mathbf{L}(\mathbf{v}_e)):(\boldsymbol{\sigma}_{S,h}-\boldsymbol{\sigma}_S)\\&\quad+c_2\int_T\phi_e\mathbf{L}(\mathbf{v}_e)\cdot\mathrm{div}(\boldsymbol{\sigma}_{S,h}-\boldsymbol{\sigma}_S)\\&\quad+c_2\int_e\phi_e\mathbf{v}_e\cdot\{(\lambda_h-\lambda)\mathbf{n}-\nu\kappa^{-1}(\boldsymbol{\varphi}_h\cdot\mathbf{t}-\boldsymbol{\varphi}\cdot\mathbf{t})\mathbf{t}\}.\end{aligned}$$

Next, applying the Cauchy–Schwarz inequality and the inverse estimate (3.23), recalling that $0\leqslant\phi_e\leqslant1$, and employing the vector version of (3.26), we deduce that

$$\|\mathbf{v}_e\|_{0,e}\leqslant Ch_e^{1/2}\left\{h_T^{-1}\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{0,T}+\|\mathrm{div}(\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h})\|_{0,T}\right\}+C\{\|\lambda-\lambda_h\|_{0,e}+\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}\},$$

which easily yields the required estimate, thus finishing the proof. □

Lemma 3.17. *There exists $C>0$, independent of h, such that for each $e\in\mathcal{E}_h(\Sigma)$, there holds*

$$h_e\|\mathbf{u}_{S,h}+\boldsymbol{\varphi}_h\|^2_{0,e}\leqslant C\left\{\|\mathbf{u}_S-\mathbf{u}_{S,h}\|^2_{0,T}+h_T^2\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|^2_{0,T}+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|^2_{0,e}\right\},$$

where T is the triangle of $\mathcal{T}_h^S$ having e as an edge.

Proof. Let $e\in\mathcal{E}_h(\Sigma)$ and let T be the triangle of $\mathcal{T}_h^S$ having e as an edge. We follow the proof of [9, Lemma 4.12] and obtain first an upper bound of $h_T^2|\mathbf{u}_S-\mathbf{u}_{S,h}|^2_{1,T}$. Indeed, using that $\nabla\mathbf{u}_S=\nu^{-1}\boldsymbol{\sigma}_S^d$ in Ω_S (cf. Theorem 2.2) and that $\mathbf{u}_{S,h}$ is constant in T, adding and substracting $\sigma_{S,h}{}^d$, and then applying the estimate (g) from Lemma 3.13, we deduce that

$$\begin{aligned}h_T^2|\mathbf{u}_S-\mathbf{u}_{S,h}|^2_{1,T}=\frac{h_T^2}{\nu^2}\|\boldsymbol{\sigma}_S^d\|^2_{0,T}&\leqslant Ch_T^2\left\{\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|^2_{0,T}+\|\boldsymbol{\sigma}^d_{S,h}\|^2_{0,T}\right\}\\&\leqslant C\left\{\|\mathbf{u}_S-\mathbf{u}_{S,h}\|^2_{0,T}+h_T^2\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|^2_{0,T}\right\}.\end{aligned}\tag{3.27}$$

Next, since $\varphi=-\mathbf{u}_S$ on Σ (cf. Theorem 2.2), we find that

$$h_e\|\mathbf{u}_{S,h}+\boldsymbol{\varphi}_h\|^2_{0,e}\leqslant2h_e\left\{\|\mathbf{u}_S-\mathbf{u}_{S,h}\|^2_{0,e}+\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|^2_{0,e}\right\},$$

which, employing the discrete trace inequality (3.24) and the estimate (3.27), yields

$$h_e\|\mathbf{u}_{S,h}+\boldsymbol{\varphi}_h\|_{0,e}^2 \leqslant C\Big\{\|\mathbf{u}_S-\mathbf{u}_{S,h}\|_{0,T}^2+h_T^2|\mathbf{u}_S-\mathbf{u}_{S,h}|_{1,T}^2+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2\Big\}$$
$$\leqslant C\Big\{\|\mathbf{u}_S-\mathbf{u}_{S,h}\|_{0,T}^2+h_T^2\|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{0,T}^2+h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2\Big\},$$

which completes the proof. □

We end this section by observing that the efficiency estimate (3.19) follows straightforwardly from Lemmas 3.10, 3.13, 3.15, 3.16, and 3.17. In particular, the terms $h_e\|\lambda-\lambda_h\|_{0,e}^2$ and $h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2$, which appear in Lemma 3.13 (item h), Lemmas 3.15, 3.16, and 3.17, are bounded as follows:

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\lambda-\lambda_h\|_{0,e}^2 \leqslant h\|\lambda-\lambda_h\|_{0,\Sigma}^2 \leqslant Ch\|\lambda-\lambda_h\|_{1/2,\Sigma}^2,$$

and

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2 \leqslant h\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,\Sigma}^2 \leqslant Ch\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2.$$

Note here that we have certainly overestimated $L^2(\Sigma)$ norms by $H^{1/2}$ norms, which, however, yields precisely what we need to complete the proof of efficiency of our a posteriori error estimator. Indeed, according to the left hand side of the inequality (1.1), which establishes the efficiency of the estimator, we just need to obtain

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\lambda-\lambda_h\|_{0,e}^2 \leqslant C\|\lambda-\lambda_h\|_{1/2,\Sigma}^2$$

and

$$\sum_{e\in\mathcal{E}_h(\Sigma)} h_e\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{0,e}^2 \leqslant C\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Sigma}^2,$$

which follow straightforwardly from the estimates above.

4. Numerical results

In [40, Section 5] we presented several numerical results illustrating the performance of the Galerkin scheme (2.11) with the subspaces $\mathbb{X}_h := \mathbb{H}_h(\Omega_S)\times\mathbf{H}_h(\Omega_D)\times\boldsymbol{\Lambda}_h(\Sigma)\times\Lambda_h(\Sigma)$ and $\mathbb{M}_h := \mathbf{L}_h(\Omega_S)\times L_{h,0}(\Omega_D)$ defined in Section 2.3. We now provide three examples confirming the reliability and efficiency of the respective a posteriori error estimator Θ derived in Section 3, and showing the behaviour of the associated adaptive algorithm.

In what follows, N stands for the number of degrees of freedom defining $\mathbb{X}_h$ and $\mathbb{M}_h$. The solution of (2.5) and (2.11) are denoted

$$(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}) := ((\boldsymbol{\sigma}_S,\mathbf{u}_D,\boldsymbol{\varphi},\lambda),(\mathbf{u}_S,p_D))\in\mathbb{X}\times\mathbb{M}$$

and

$$(\underline{\boldsymbol{\sigma}}_h,\underline{\mathbf{u}}_h) := ((\boldsymbol{\sigma}_{S,h},\mathbf{u}_{D,h},\boldsymbol{\varphi}_h,\lambda_h),(\mathbf{u}_{S,h},p_{D,h}))\in\mathbb{X}_h\times\mathbb{M}_h.$$

The separate and total errors are defined by:

$$\mathbf{e}(\boldsymbol{\sigma}_S) := \|\boldsymbol{\sigma}_S-\boldsymbol{\sigma}_{S,h}\|_{\mathrm{div},\Omega_S},\qquad \mathbf{e}(\mathbf{u}_S) := \|\mathbf{u}_S-\mathbf{u}_{S,h}\|_{\mathrm{div};\Omega_S},$$
$$\mathbf{e}(\mathbf{u}_D) := \|\mathbf{u}_D-\mathbf{u}_{D,h}\|_{\mathrm{div};\Omega_D},\qquad \mathbf{e}(p_D) := \|p_D-p_{D,h}\|_{0,\Omega_D},$$
$$\mathbf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{1/2,\Sigma},\qquad \mathbf{e}(\lambda) := \|\lambda-\lambda_h\|_{1/2,\Sigma},$$

and

$$\mathbf{e}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}) := \Big\{(\mathbf{e}(\boldsymbol{\sigma}_S))^2+(\mathbf{e}(\mathbf{u}_S))^2+(\mathbf{e}(\mathbf{u}_D))^2+(\mathbf{e}(p_D))^2 +(\mathbf{e}(\boldsymbol{\varphi}))^2+(\mathbf{e}(\lambda))^2\Big\}^{1/2}.$$

The effectivity index with respect to Θ is given by

$$\mathrm{eff}(\Theta) := \mathbf{e}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})/\Theta.$$

Also, we let $r(\boldsymbol{\sigma}_S)$, $r(\mathbf{u}_S)$, $r(\mathbf{u}_D)$, $r(p_D)$, $r(\boldsymbol{\varphi})$, $r(\lambda)$, and $r(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})$ be the individual and global experimental rates of convergence given by

$$r(\%) := \frac{\log(\mathbf{e}(\%)/\mathbf{e}'(\%))}{\log(h/h')}\quad \text{for each } \%\in\{\boldsymbol{\sigma}_S,\mathbf{u}_S,\mathbf{u}_D,p_D,\boldsymbol{\varphi},\lambda\},$$

and

$$r(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}) := \frac{\log(\mathbf{e}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})/\mathbf{e}'(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}))}{\log(h/h')},$$

where h and h' denote two consecutive meshsizes with errors $\mathbf{e}$ and $\mathbf{e}'$. However, when the adaptive algorithm is applied (see details below), the expression $\log(h/h')$ appearing in the computation of the above rates is replaced by $-\frac{1}{2}\log(N/N')$, where N and N' denote the corresponding degrees of freedom of each triangulation.

The examples to be considered in this section are described next. In all of them we choose $\nu = 1$, $\mathbf{K} = \mathbf{I}$ and $\kappa = 1$. Example 1 is used to corroborate the reliability and efficiency of the a posteriori error estimator Θ. Examples 2 and 3 are utilized to illustrate the behavior of the associated adaptive algorithm, which applies the following procedure from [57]:

(1) Start with a coarse mesh $\mathcal{T}_h := \mathcal{T}_h^D\cup\mathcal{T}_h^S$.
(2) Solve the discrete problem (2.11) for the current mesh $\mathcal{T}_h$.
(3) Compute $\Theta_T := \Theta_{\star,T}$ for each triangle $T\in\mathcal{T}_h^\star, \star\in\{D,S\}$.
(4) Check the stopping criterion and decide whether to finish or go to next step.
(5) Use *blue-green* refinement on those $T'\in\mathcal{T}_h$ whose indicator $\Theta_{T'}$ satisfies
$$\Theta_{T'} \geqslant \frac{1}{2}\max_{T\in\mathcal{T}_h}\{\Theta_T:\quad T\in\mathcal{T}_h\}.$$
(6) Define resulting meshes as current meshes $\mathcal{T}_h^D$ and $\mathcal{T}_h^S$, and go to step 2.

Table 4.1
Example 1, quasi-uniform scheme.

N	h	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$r(p_D)$
321	0.5000	35.4015	–	0.6875	–	0.1996	–	0.0117	–
1201	0.2500	20.0107	0.8647	0.4266	0.7234	0.1121	0.8743	0.0057	1.0798
4641	0.1250	10.0700	1.0160	0.1615	1.4370	0.0531	1.1046	0.0023	1.3213
18241	0.0625	5.0492	1.0087	0.0801	1.0238	0.0259	1.0490	0.0011	1.0967
72321	0.0312	2.5268	1.0052	0.0401	1.0064	0.0129	1.0178	0.0005	1.0234
288001	0.0156	1.2637	1.0029	0.0200	1.0031	0.0064	1.0062	0.0003	1.0062
		$\mathbf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$	$r(\lambda)$	$\mathbf{e}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})$	$r(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})$	Θ	$\mathrm{eff}(\Theta)$
321	0.5000	4.2653	–	0.0981	–	35.6649	–	39.0015	0.9144
1201	0.2500	4.3919	–	0.0973	0.0124	20.4920	0.8399	22.6847	0.9033
4641	0.1250	1.7410	1.3690	0.0537	0.8781	10.2209	1.0292	11.1965	0.9129
18241	0.0625	0.8088	1.1202	0.0259	1.0670	5.1144	1.0117	5.5954	0.9140
72321	0.0312	0.3949	1.0408	0.0126	1.0516	2.5579	1.0060	2.7969	0.9145
288001	0.0156	0.1962	1.0123	0.0062	1.0266	1.2791	1.0031	1.3982	0.9148

Table 4.2
Example 2, quasi-uniform scheme.

N	h	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
608	0.3536	4.5187	0.1198	0.2649	0.0184	0.5760	0.1120
2332	0.1768	4.9963	0.0529	0.1520	0.0035	0.2653	0.0347
9140	0.0884	6.7481	0.0253	0.0778	0.0005	0.1485	0.0096
36196	0.0442	4.2857	0.0125	0.0392	0.0002	0.0771	0.0042
144068	0.0221	2.4834	0.0062	0.0196	0.0001	0.0348	0.0022

N	h	$\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$	$r(\boldsymbol{\sigma},\underline{\mathbf{u}})$	Θ	eff(Θ)
608	0.3536	4.5660	–	5.4033	0.8450
2332	0.1768	5.0060	–	5.2805	0.9480
9140	0.0884	6.7503	–	6.8230	0.9894
36196	0.0442	4.2866	0.6599	4.3158	0.9932
144068	0.0221	2.4837	0.7901	2.4958	0.9952

Table 4.3
Example 2, adaptive scheme.

N	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
608	4.5188	0.1199	0.2649	0.0184	0.5760	0.1121
1118	5.3792	0.0709	0.2262	0.0091	0.3185	0.0402
1391	7.2290	0.0661	0.2098	0.0082	0.2846	0.0215
1636	5.1151	0.0657	0.2094	0.0110	0.2591	0.0236
1884	3.9177	0.0657	0.2093	0.0108	0.2577	0.0229
3558	2.6519	0.0491	0.2020	0.0037	0.1626	0.0128
7164	1.8814	0.0320	0.1751	0.0067	0.1160	0.0171
13073	1.3945	0.0237	0.1591	0.0034	0.0742	0.0109
26227	0.9771	0.0165	0.1222	0.0030	0.0730	0.0103
35611	0.8163	0.0140	0.1089	0.0018	0.0384	0.0075
55318	0.6608	0.0114	0.0808	0.0005	0.0375	0.0039
70434	0.5825	0.0099	0.0747	0.0005	0.0357	0.0038
149402	0.4052	0.0070	0.0548	0.0003	0.0208	0.0023

N	$\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$	$r(\boldsymbol{\sigma},\underline{\mathbf{u}})$	Θ	eff(Θ)
608	4.5660	–	5.4033	0.8450
1118	5.3940	–	5.7977	0.9304
1391	7.2379	–	7.4956	0.9656
1636	5.1264	4.2524	5.4334	0.9435
1884	3.9324	3.7572	4.3145	0.9114
3558	2.6650	1.2238	2.9662	0.8985
7164	1.8934	0.9768	2.0913	0.9054
13073	1.4057	0.9902	1.5394	0.9132
26227	0.9876	1.0142	1.0951	0.9018
35611	0.8246	1.1796	0.9191	0.8972
55318	0.6669	0.9637	0.7388	0.9026
70434	0.5885	1.0359	0.6505	0.9046
149402	0.4095	0.9644	0.4550	0.8999

Table 4.4
Example 3, quasi-uniform scheme.

N	h	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
344	0.5000	16.8563	0.4452	0.7130	0.0674	1.8109	0.1615
1324	0.2500	11.3317	0.3329	0.3846	0.0130	2.5160	0.0826
5204	0.1250	7.0011	0.0849	0.1980	0.0038	0.8665	0.0458
20644	0.0625	4.4530	0.0412	0.0992	0.0018	0.3859	0.0203
82244	0.0312	2.8037	0.0206	0.0496	0.0009	0.1877	0.0097

N	h	$\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$	$r(\boldsymbol{\sigma},\underline{\mathbf{u}})$	Θ	eff(Θ)
344	0.5000	16.9751	–	18.8901	0.8986
1324	0.2500	11.6191	0.5626	13.1132	0.8861
5204	0.1250	7.0579	0.7284	7.8041	0.9044
20644	0.0625	4.4711	0.6626	5.0014	0.8940
82244	0.0312	2.8105	0.6717	3.1653	0.8879

In Example 1 we consider the regions $\Omega_D := (-0.5,0.5)^2$ and $\Omega_S := (-1,1)^2 \setminus \overline{\Omega}_D$, which yields a porous medium completely surrounded by a fluid, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by the smooth functions

Table 4.5
Example 3, adaptive scheme.

N	$\mathbf{e}(\boldsymbol{\sigma}_S)$	$\mathbf{e}(\mathbf{u}_S)$	$\mathbf{e}(\mathbf{u}_D)$	$\mathbf{e}(p_D)$	$\mathbf{e}(\boldsymbol{\varphi})$	$\mathbf{e}(\lambda)$
344	16.8564	0.4453	0.7131	0.0675	1.8109	0.1616
684	11.8048	0.3406	0.5828	0.0177	2.5165	0.0863
1367	10.6242	0.1330	0.4426	0.0099	0.8682	0.0530
1625	10.4486	0.1314	0.4426	0.0099	0.8682	0.0530
1863	10.3440	0.1278	0.4426	0.0098	0.8678	0.0530
2291	9.2480	0.1173	0.4427	0.0097	0.8672	0.0526
3109	7.5456	0.1013	0.4425	0.0098	0.8670	0.0522
11719	3.9053	0.0530	0.3296	0.0072	0.3875	0.0271
34611	2.2713	0.0202	0.2614	0.0058	0.1901	0.0092
60159	1.7281	0.0153	0.1723	0.0034	0.1759	0.0083
79482	1.5031	0.0111	0.1644	0.0032	0.1154	0.0072
115241	1.2620	0.0167	0.1498	0.0019	0.1101	0.0055
182014	0.9954	0.0130	0.1226	0.0012	0.0900	0.0027

N	$\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$	$r(\boldsymbol{\sigma},\underline{\mathbf{u}})$	Θ	eff(Θ)
344	16.9751	–	18.8901	0.8986
684	12.0893	0.9877	13.6112	0.8882
1367	10.6698	0.3608	11.3264	0.9420
1625	10.4949	0.1912	11.1221	0.9436
1863	10.3907	0.1460	10.8244	0.9599
2291	9.3000	1.0724	9.9113	0.9383
3109	7.6090	1.3146	8.2092	0.9269
11719	3.9388	1.0924	4.2413	0.9362
34611	2.2943	0.9981	2.4691	0.9292
60159	1.7456	0.9889	1.8902	0.9235
79482	1.5165	1.0102	1.5941	0.9513
115241	1.2757	0.9309	1.3418	0.9507
182014	1.0070	1.0350	1.0817	0.9309

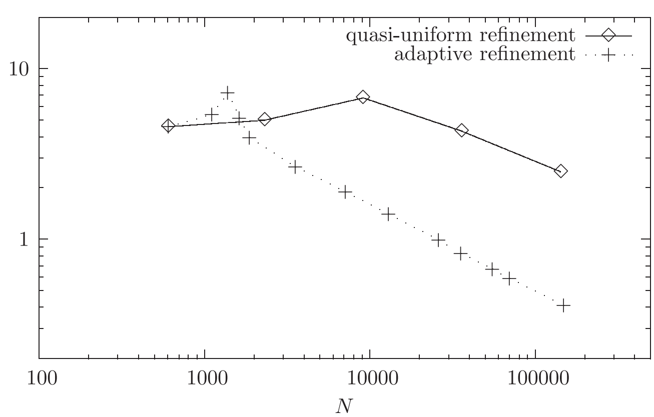

Fig. 4.1. Example 2, $\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$ vs. N for quasi-uniform/adaptive schemes.

$$\mathbf{u}_S(\mathbf{x}) = \begin{pmatrix} -2\sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ 2\sin(\pi x_1)\sin^2(\pi x_2)\cos(\pi x_1) \end{pmatrix} \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_S,$$
$$p_S(\mathbf{x}) = x_1^3 e^{x_2} \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_S,$$

and

$$p_D(\mathbf{x}) = x_1^3 \sin(x_2) \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_D.$$

In Example 2 we consider $\Omega_D := (-1,0)^2$ and let Ω_S be the L-shaped domain given by $(-1,1)^2 \setminus \overline{\Omega}_D$, which yields a porous medium partially surrounded by a fluid. Then we choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\mathbf{u}_S(\mathbf{x}) = \mathbf{curl}\Big(0.1\,(x_2^2-1)^2\sin^2(\pi x_1)\Big) \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_S,$$
$$p_S(\mathbf{x}) = \frac{1}{100(x_1^2+x_2^2)+0.1} \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_S,$$

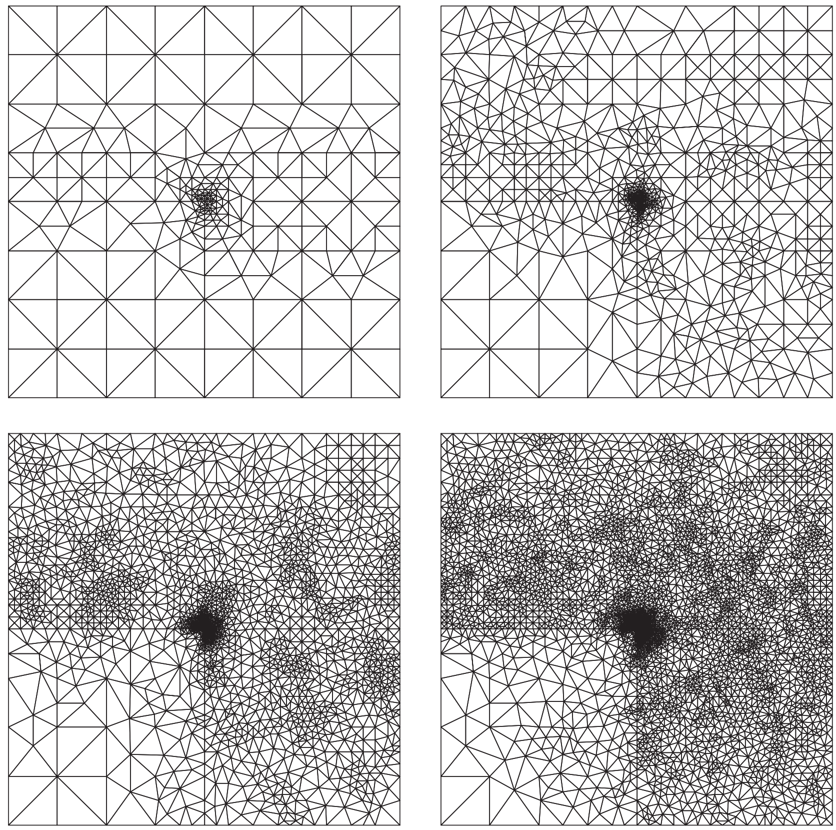

Fig. 4.2. Example 2, adapted meshes with 1884, 7164, 26227, and 55318 degrees of freedom.

and

$$p_D(\mathbf{x}) = \left(\frac{x_1+1}{10}\right)^2 \sin^3(2\pi(x_2+0.5)) \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_D.$$

Note that the fluid pressure p_S has high gradients around the origin.

Finally, in Example 3 we take $\Omega_D := (-1,1)\times(-2,-1)$ and $\Omega_S := (-1,1)^2 \backslash [0,1]^2$, which yields a porous medium below a fluid, and choose the data $\mathbf{f}_S$ and f_D so that the exact solution is given by

$$\mathbf{u}_S(r,\theta) = \mathbf{curl}\left(0.1 r^{5/3}(r^2\cos^2(\theta)-1)^2(r\sin(\theta)-1)^2\sin^2\left(\frac{2\theta-\pi}{3}\right)\right)$$
$$\forall (r,\theta) \in \Omega_S,$$
$$p_S(\mathbf{x}) = 0.1 x_1 \sin(x_2) \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_S,$$

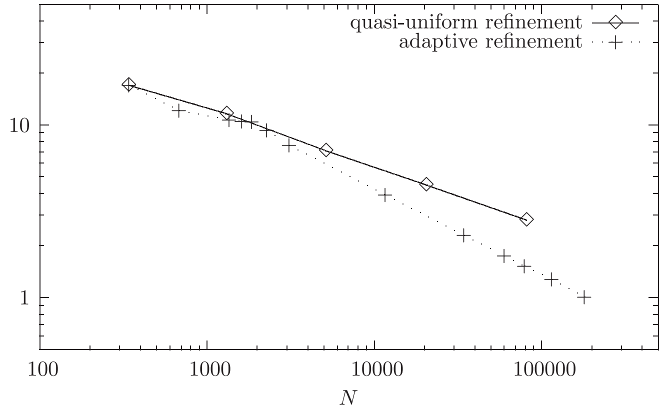

Fig. 4.3. Example 3, $\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$ vs. N for quasi-uniform/adaptive schemes.

and

$$p_D(\mathbf{x}) = 0.1(x_2+2)^2\sin^3(\pi x_1) \quad \forall \mathbf{x} := (x_1,x_2) \in \Omega_D.$$

Note that $\mathbf{u}_S$ is defined in polar coordinates and that its derivatives are singular at the origin.

The numerical results shown below were obtained using a MATLAB code. In Table 4.1 we summarize the convergence history of the mixed finite element method (2.11), as applied to Example 1, for a sequence of quasi-uniform triangulations of the domain. We observe there, looking at the corresponding experimental rates of convergence, that the $O(h)$ predicted by Theorem 2.3 (here $\delta = 1$) is attained in all the unknowns. In addition, we notice that the effectivity index eff(Θ) remains always in a neighborhood of 0.91, which illustrates the reliability and efficiency of Θ in the case of a regular solution.

Next, in Tables 4.2, 4.3, 4.4, 4.5 we provide the convergence history of the quasi-uniform and adaptive schemes, as applied to Examples 2 and 3. We observe that the errors of the adaptive procedures decrease faster than those obtained by the quasi-uniform ones, which is confirmed by the global experimental rates of convergence provided there. This fact is also illustrated in Figs. 4.1 and 4.3 where we display the total errors $\mathbf{e}(\boldsymbol{\sigma},\underline{\mathbf{u}})$ vs. the number of degrees of freedom N for both refinements. As shown by the values of $r(\boldsymbol{\sigma},\underline{\mathbf{u}})$, the adaptive method is able to keep the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain bounded from above and below, which confirms the reliability and efficiency of Θ in these cases of non-smooth solutions. Intermediate meshes obtained with the adaptive refinements are displayed in Figs. 4.2 and 4.4. Note that the method is able to recognize the region with high gradients in Example 2, and the singularity of the solution in Example 3.

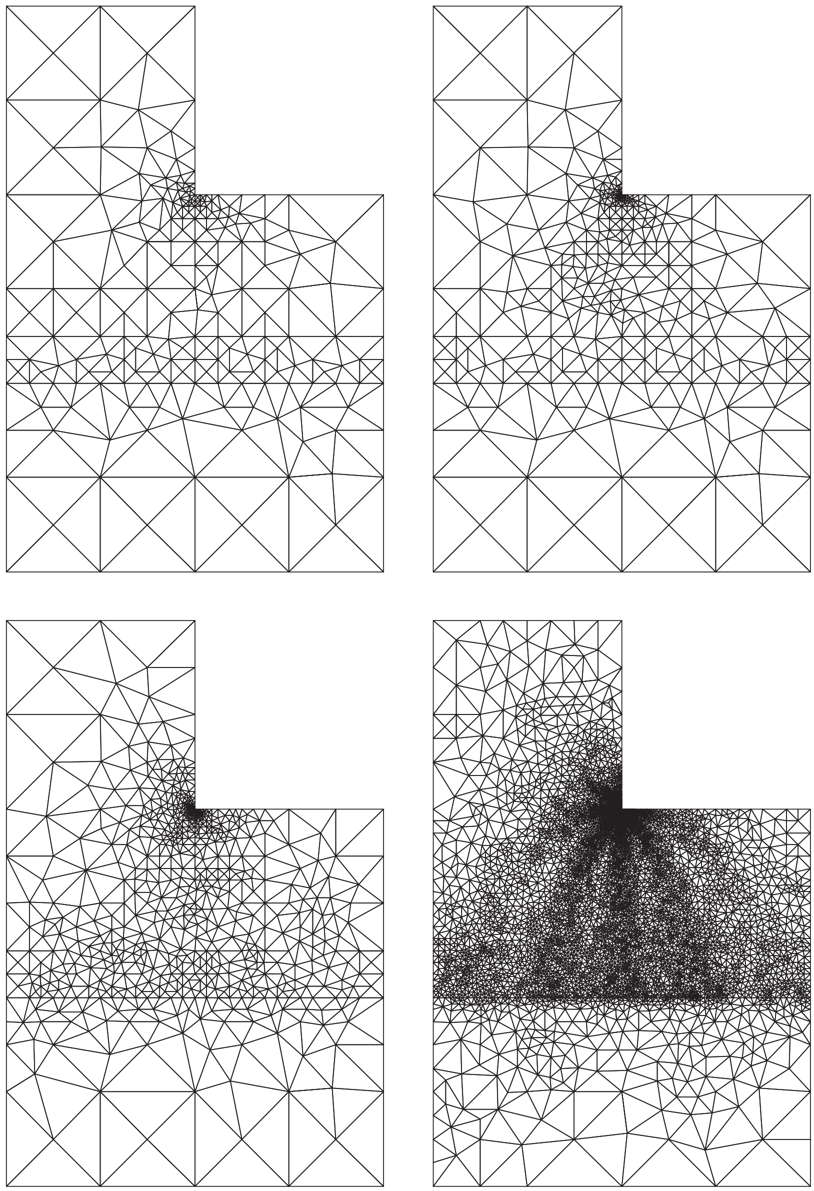

Fig. 4.4. Example 3, adapted meshes with 1863, 3109, 11719, and 60159 degrees of freedom.

References

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand, Princeton, New Jersey, 1965.

[2] M. Ainsworth, J.T. Oden, A unified approach to a posteriori error estimation based on element residual methods, Numerische Mathematik 65 (1993) 23–50.

[3] M. Ainsworth, J.T. Oden, A posteriori error estimators for the Stokes and Oseen equations, SIAM Journal on Numerical Analysisi 34 (1) (1997) 228–245.

[4] A. Alonso, Error estimators for a mixed method, Numerische Mathematik 74 (1996) 385–395.

[5] T. Arbogast, D.S. Brunson, A computational method for approximating a Darcy-Stokes system governing a vuggy porous medium, Computational Geosciences 11 (3) (2007) 207–218.

[6] D.N. Arnold, An interior penalty finite element method with discontinuous elements, SIAM Journal on Numerical Analysis 19 (4) (1982) 742–760.

[7] D.N. Arnold, F. Brezzi, J. Douglas, PEERS: A new mixed finite element method for plane elasticity, Japan Journal of Applied Mathematics 1 (1984) 347–367.

[8] D.N. Arnold, R. Winther, Mixed finite elements for elasticity, Numerische Mathematik 92 (3) (2002) 401–419.

[9] I. Babuška, G.N. Gatica, A residual-based a posteriori error estimator for the Stokes–Darcy coupled problem, SIAM Journal on Numerical Analysis 48 (2) (2010) 498–523.

[10] T.P. Barrios, G.N. Gatica, M. González, N. Heuer, A residual based a posteriori error estimator for an augmented mixed finite element method in linear elasticity, M2AN Mathematical Modelling and Numerical Analysis 40 (5) (2006) 843–869.

[11] G. Beavers, D. Joseph, Boundary conditions at a naturally impermeable wall, Journal of Fluid Mechanics 30 (1967) 197–207.

[12] C. Bernardi, T. Chacón Rebollo, F. Hecht, Z. Mghazli, Mortar finite element discretization of a model coupling Darcy and Stokes equations, M2AN Mathematical Modelling and Numerical Analysis 42 (3) (2008) 375–410.

[13] C. Bernardi, F. Hecht, O. Pironneau, Coupling Darcy and Stokes equations for porous media with cracks, M2AN Mathematical Modelling and Numerical Analysis 39 (1) (2005) 7–35.

[14] D. Braess, R. Verfürth, A posteriori error estimators for the Raviart-Thomas element, SIAM Journal on Numerical Analysis 33 (1996) 2431–2444.

[15] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer Verlag, 1991.

[16] E. Burman, P. Hansbo, Stabilized Crouzeix-Raviart elements for the Darcy-Stokes problem, Numerical Methods for Partial Differential Equations 21 (5) (2005) 986–997.

[17] E. Burman, P. Hansbo, A unified stabilized method for Stokes' and Darcy's equations, Journal of Computational and Applied Mathematics 198 (1) (2007) 35–51.

[18] Z. Cai, B. Lee, P. Wang, Least-squares methods for incompressible Newtonian fluid flow: Linear stationary problems, SIAM Journal on Numerical Analysis 42 (2) (2004) 843–859.

[19] Z. Cai, Ch. Tong, P.S. Vassilevski, Ch. Wang, Mixed finite element methods for incompressible flow: stationary Stokes equations, Numerical Methods for Partial Differential Equations 26 (4) (2010) 957–978.

[20] Z. Cai, Y. Wang, A multigrid method for the pseudostress formulation of Stokes problems, SIAM Journal on Scientific Computing 29 (5) (2007) 2078–2095.
[21] Z. Cai, Ch. Wang, S. Zhang, Mixed finite element methods for incompressible flow: stationary Navier-Stokes equations, SIAM Journal on Numerical Analysis 48 (1) (2010) 79–94.
[22] C. Carstensen, A posteriori error estimate for the mixed finite element method, Mathematics of Computation 66 (218) (1997) 465–476.
[23] C. Carstensen, An a posteriori error estimate for a first kind integral equation, Mathematics of Computation 66 (217) (1997) 139–155.
[24] C. Carstensen, G. Dolzmann, A posteriori error estimates for mixed FEM in elasticity, Numerische Mathematique 81 (1998) 187–209.
[25] P.G. Ciarlet, The finite Element Method for Elliptic Problems, North-Holland, Amsterdam, New York, Oxorfd, 1978.
[26] P. Clément, Approximation by finite element functions using local regularisation, RAIRO Modélisation Mathématique et Analyse Numérique 9 (1975) 77–84.
[27] M.R. Correa, Stabilized Finite Element Methods for Darcy and Coupled Stokes–Darcy Flows. D.Sc. Thesis, LNCC, Petrópolis, Rio de Janeiro, Brasil, 2006 (in Portuguese).
[28] M.R. Correa, A.F.D. Loula, A unified mixed formulation naturally coupling Stokes and Darcy flows, Computer Methods in Applied Mechanics and Engineering 198 (33–36) (2009) 2710–2722.
[29] M. Discacciati, E. Miglio, A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, Applied Numerical Mathematics 43 (2002) 57–74.
[30] V.J. Ervin, E.W. Jenkins, S. Sun, Coupled generalized nonlinear Stokes flow with flow through a porous medium, SIAM Journal on Numerical Analysis 47 (2) (2009) 929–952.
[31] L. Figueroa, G.N. Gatica, A. Márquez, Augmented mixed finite element methods for the stationary Stokes Equations, SIAM Journal on Scientific Computing 31 (2) (2008) 1082–1119.
[32] J. Galvis, M. Sarkis, Non-matching mortar discretization analysis for the coupling Stokes–Darcy equations, Electronic Transactions on Numerical Analysis 26 (2007) 350–384.
[33] G.N. Gatica, A note on the efficiency of residual-based a-posteriori error estimators for some mixed finite element methods, Electronic Transactions on Numerical Analysis 17 (2004) 218–233.
[34] G.N. Gatica, G.C. Hsiao, S. Meddahi, A residual-based a posteriori error estimator for a two-dimensional fluid-solid interaction problem, Numerische Mathematik 114 (1) (2009) 63–106.
[35] G.N. Gatica, M. Maischak, A posteriori error estimates for the mixed finite element method with Lagrange multipliers, Numerical Methods for Partial Differential Equations 21 (3) (2005) 421–450.
[36] G.N. Gatica, A. Márquez, M.A. Sánchez, Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations, Computer Methods in Applied Mechanics and Engineering 199 (17–20) (2010) 1064–1079.
[37] G.N. Gatica, A. Márquez, M.A. Sánchez, A priori and a posteriori error analyses of a velocity-pseudostress formulation for a class of quasi-Newtonian Stokes flows, Computer Methods in Applied Mechanics and Engineering 200, 17–20 (2011) 1619–1636.
[38] G.N. Gatica, S. Meddahi, R. Oyarzúa, A conforming mixed finite-element method for the coupling of fluid flow with porous media flow, IMA Journal of Numerical Analysis 29 (1) (2009) 86–108.
[39] G.N. Gatica, R. Oyarzúa, F.-J. Sayas, Convergence of a family of Galerkin discretizations for the Stokes–Darcy coupled problem, Numerical Methods for Partial Differential Equations DOI 10.1002/num, in press.
[40] G.N. Gatica, R. Oyarzúa, F.-J. Sayas, Analysis of fully-mixed finite element methods for the Stokes–Darcy coupled problem, Mathematics of Computation, in press, doi:10.1090/S0025-5718-2011-02466-X.
[41] G.N. Gatica, R. Oyarzúa, F.-J. Sayas, A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow, IMA Journal of Numerical Analysis, submitted for publication.
[42] V. Girault, P.A. Raviart, Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms, Springer Series in Computational Mathematics 5, Springer-Verlag, 1986.
[43] R.H.W. Hoppe, B.I. Wohlmuth, A comparison of a posteriori error estimators for mixed finite element discretizations by Raviart–Thomas elements, Mathematics of Computation 68 (228) (1999) 1347–1378.
[44] W. Jäger, M. Mikelic, On the interface boundary condition of Beavers, Joseph, and Saffman, SIAM Journal on Applied Mathematics 60 (2000) 1111–1127.
[45] W.J. Layton, F. Schieweck, I. Yotov, Coupling fluid flow with porous media flow, SIAM Journal on Numerical Analysis 40 (6) (2003) 2195–2218.
[46] M. Lonsing, R. Verfürth, A posteriori error estimators for mixed finite element methods in linear elasticity, Numerische Mathematik 97 (4) (2004) 757–778.
[47] C. Lovadina, R. Stenberg, Energy norm a posteriori error estimates for mixed finite element methods, Mathematics of Computation 75 (256) (2006) 1659–1674.
[48] A. Masud, A stabilized mixed finite element method for Darcy–Stokes flow, International Journal for Numerical Methods in Fluids 54 (6–8) (2008) 665–681.
[49] S. Repin, S. Sauter, A. Smolianski, Two-sided a posteriori error estimates for mixed formulations of elliptic problems, SIAM Journal on Numerical Analysis 45 (3) (2007) 928–945.
[50] B. Riviere, Analysis of a discontinuous finite element method for coupled Stokes and Darcy problems, Journal of Scientific Computing 22–23 (2005) 479–500.
[51] B. Riviere, I. Yotov, Locally conservative coupling of Stokes and Darcy flows, SIAM Journal on Numerical Analysis 42 (5) (2005) 1959–1977.
[52] H. Rui, R. Zhang, A unified stabilized mixed finite element method for coupling Stokes and Darcy flows, Computer Methods in Applied Mechanics and Engineering 198 (33–36) (2009) 2692–2699.
[53] D. Schötzau, F.-J. Sayas, in preparation.
[54] J.M. Urquiza, D. N'Dri, A. Garon, M.C. Delfour, Coupling Stokes and Darcy equations, Applied Numerical Mathematics 58 (5) (2008) 525–538.
[55] R. Verfürth, A posteriori error estimators for the Stokes problem, Numerische Mathematik 55 (1989) 309–325.
[56] R. Verfürth, A posteriori error estimation and adaptive mesh-refinement thecniques, Journal of Computational and Applied Mathematics 50 (1994) 67–83.
[57] R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley Teubner, Chichester, 1996.
[58] X. Xie, J. Xu, G. Xue, Uniformly stable finite element methods for Darcy–Stokes–Brinkman models, Journal of Computational Mathematics 26 (3) (2008) 437–455.