Multi-Scale Finite-Volume Method for Elliptic Problems with Heterogeneous Coefficients and Source Terms

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Simulation of sub-surface flow in geologically complex formations is just one example in computational science, where efficient and accurate solutions of heterogeneous elliptic problems are of great interest. Often it is not feasible to resolve the whole range of relevant length scales associated with the spatial distribution of the highly varying coefficients. The MSFV method was originally developed for multi-phase flow in porous media. In the more general form presented here, it can be applied for solving large elliptic problems in various areas of computational science.

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1 The Multi-Scale Finite-Volume (MSFV) Method

We consider the elliptic problem

$$\frac{\partial}{\partial x_i}\left(\lambda_{ij}\frac{\partial u}{\partial x_j}\right) \;=\; r \quad \text{on}\,\Omega. \tag{1}$$

To compute the solution u may require very high spatial resolution, if the tensor $\lambda_{ij}(\mathbf{x})$ or the right-hand side $r(\mathbf{x})$ have complex fine-scale distributions. To overcome this resolution issue, various multi-scale methods have been developed [2, 1]. Here, the multi-scale finite-volume (MSFV) method [3, 4], which proofed to be very accurate and efficient for simulations of multi-phase flow in porous media, is presented in a general form. First, a coarse grid, which consist of M control volumes $\bar{\Omega}_k$ and a dual coarse grid, which consists of N volumes $\tilde{\Omega}^h$, are introduced. These grids can be much coarser than the fine grid representing λ_{ij} and r. Second, a set of basis functions is computed and used to construct and solve a coarse system. Third, the coarse-scale solution $\bar{u}_k$ at the locations $\mathbf{x}_k$ (inside $\bar{\Omega}_k$ and at the nodes of the dual grid) is used to reconstruct the fine-scale solution.

1.1 Localization and Basis Functions

The basic idea of the MSFV method is to employ the local fine-scale approximation

$$u^h(\mathbf{x}) \;=\; \hat{u}^h(\mathbf{x}) \;+\; \sum_{k=1}^{M}\hat{u}^h_k(\mathbf{x})\bar{u}_k \;\approx\; u(\mathbf{x}) \quad \text{for}\,\mathbf{x}\in\tilde{\Omega}^h, \tag{2}$$

where the basis functions $\hat{u}^h_k$ and the correction function $\hat{u}^h$ are numerical solutions of

$$\frac{\partial}{\partial x_i}\left(\lambda_{ij}\frac{\partial \hat{u}^h_k}{\partial x_j}\right) \;=\; 0 \quad \text{and} \quad \frac{\partial}{\partial x_i}\left(\lambda_{ij}\frac{\partial \hat{u}^h}{\partial x_j}\right) \;=\; r \quad \text{on}\,\tilde{\Omega}^h, \tag{3}$$

respectively and are zero for $\mathbf{x}\notin\tilde{\Omega}^h$. Note that the fine-scale distributions of λ_{ij} and r are used to compute these functions. The values of $\hat{u}^h_k$ and $\hat{u}^h$ at the corners of $\tilde{\Omega}^h$ are set, i.e. $\hat{u}^h_k(\mathbf{x}_l) = \delta_{kl}$ and $\hat{u}^h(\mathbf{x}_l) = 0$. At the faces of $\tilde{\Omega}^h$, the reduced problem boundary conditions

$$\frac{\partial}{\partial x_n}\left(\lambda_{ij}\frac{\partial \hat{u}^h_k}{\partial x_j}\tilde{\nu}^h_i\right)\tilde{\nu}^h_n \;=\; 0 \quad \text{and} \quad \frac{\partial}{\partial x_n}\left(\lambda_{ij}\frac{\partial \hat{u}^h}{\partial x_j}\tilde{\nu}^h_i\right)\tilde{\nu}^h_n \;=\; r \quad \text{at}\,\partial\tilde{\Omega}^h \tag{4}$$

are applied, where $\tilde{\nu}^h$ is the unit normal vector at $\partial\tilde{\Omega}^h$ pointing outwards.

1.2 Coarse-Scale Solution and Reconstruction

Using Gauss' theorem and the approximation

$$u' \;=\; \sum_{h=1}^{N}u^h \;=\; \sum_{h=1}^{N}\hat{u}^h \;+\; \sum_{l=1}^{M}\bar{u}_l\sum_{h=1}^{N}\hat{u}^h_l \;\approx\; u \tag{5}$$

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one obtains for each coarse volume $\bar{\Omega}_k$

$$\int_{\partial\bar{\Omega}_k} \lambda_{ij} \frac{\partial u'}{\partial x_j} \bar{\nu}_i^k d\Gamma = \int_{\bar{\Omega}_k} r d\Omega \quad \text{or} \quad \sum_{l=1}^{M} \bar{u}_l \overbrace{\sum_{h=1}^{N} \int_{\partial\bar{\Omega}_k} \lambda_{ij} \frac{\partial \hat{u}_l^h}{\partial x_j} \bar{\nu}_i^k d\Gamma}^{A_{kl}} = \overbrace{\int_{\bar{\Omega}_k} r d\Omega - \sum_{h=1}^{N} \int_{\partial\bar{\Omega}_k} \lambda_{ij} \frac{\partial \hat{u}^h}{\partial x_j} \bar{\nu}_i^k d\Gamma}^{R_k}, \tag{6}$$

where $\bar{\nu}^k$ is the unit normal vector at $\partial\bar{\Omega}_k$ pointing outwards. Note that this results in a linear system, which can be solved for the coarse-scale values $\bar{u}_l$ at the locations $\mathbf{x}_l$. Reconstruction of a fine-scale solution can be obtained from Eq. (5), but the resulting flux, $\lambda_{ij}\partial u'/\partial x_j$, is in general discontinuous across dual volume boundaries. Therefore, a different reconstruction was devised, which employs $\bar{\nu}_i^k \lambda_{ij}\partial u'/\partial x_j$ as a flux boundary condition at $\partial\bar{\Omega}_k$ for the local problem $\partial(\lambda_{ij}\partial u''/\partial x_j)/\partial x_i = r$ on $\bar{\Omega}_k$. Note that this reconstruction does not affect the coarse-scale solution $\bar{u}_k$ and is only applied in the regions of Ω, where a fine-scale reconstruction is of interest, e.g. where transport is solved in subsurface flow simulations. Moreover, the basis and correction functions can be reused for subsequent time steps, even if the global boundary conditions change. Since most of the computational work is spent for small local problems (which are independent), the MSFV method is very well suited for massive parallel machines.

2 Numerical Results

Here, the MSFV method is demonstrated for incompressible flow in a highly heterogeneous isotropic medium, i.e. $\lambda_{ij} = \lambda\delta_{ij}$ in Eq. (1) has a complex fine-scale distribution ($\log\lambda$ is shown in Fig. 1a). The size of the domain is 12×6 and $r = 0$ everywhere. At the left and right boundaries constant values for u are applied (higher at the left side) and zero vertical gradient of u is enforced at the upper and lower boundaries. In addition to Eq. (1), the non-linear hyperbolic transport equation $\partial S/\partial t + V_i \partial[S^2/(S^2 + (1-S)^2)]/\partial x_i = 0$ with $V_i = -\lambda\partial u''/\partial x_i$ is solved on the fine grid for the phase saturation $S \in [0, 1]$. Initially, S is zero everywhere, whereas $S = 1$ for the fluid injected at the left side. Figs. 1b and 1c show the distribution of S at the same time, once computed with a fine-scale finite-voume method and once with the MSFV method, respectively. The fine grid has 120×60 and the coarse grid 12×6 cells. Note that the two solutions are almost identical, although a much coarser grid was used for the global problem in the MSFV method. Indirectly, this result illustrates the good agreement between the total fluxes V_i from the MSFV and fine-scale computations.

(a) (b) (c)

Fig. 1 (a): $\log\lambda$ (dark means low value); (b)&(c): S distributions from the MSFV and fine-scale simulations, respectively (dark means high value).

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