0.1 Cyclic Subspaces and Annihilaters

Definition 0.1.1: T-Cyclic Subspaces

T : endo. on f.d.v.s. V/F. Take $\alpha \in V$. Then the T-cyclic subspace generated by α is denoted as $Z(\alpha; T) := \{g(T)\alpha \in V \mid g(x) \in F[x]\}$. Just in case $Z(\alpha; T) = V$, we say V is cyclically generated by α and T, and α is a cyclic vector for T.

Note:-

$Z(\alpha; T)$ is always T-invariant. Also, $Z(\alpha; T)$ is very sensitive to choice of α. If $\alpha = 0$, nothing no show. If α is a char. vec., then $T\alpha = c\alpha$, so $Z(\alpha; T) = \text{span}\{\alpha\}$, which implies 1-dimensional. Also note that converse holds.

Definition 0.1.2: T-Annihilaters

The T-annihilater, denoted as $M(\alpha; T) := \{g(x) \in F[x] \mid g(T)\alpha = 0\}$.

Note:-

Note that annihilator is just a special case of conductor, which takes $W = 0$. We can also see that monic generator of annihilater divides minimal poly.

Theorem 0.1.1

T : endo. on f.d.v.s. V/F. p_α : T-annihilator of α. Then

i) $\deg(p_\alpha) = \dim(Z(\alpha; T))$

ii) If $\deg(p_\alpha) = k$, then $\{\alpha, T\alpha, \ldots, T^{k-1}\alpha\}$ forms a basis of $Z(\alpha; T)$

iii) Let $U := T|_{Z(\alpha;T)} : Z \mapsto Z$. Then min. poly. of U is $p_\alpha(x)$.

Proof. Take $g(x) = p_\alpha q(x) + r(x)$ by Euclidean algorithm for $\deg(p_\alpha) = k$. Note that $(p_\alpha) = M(\alpha; T)$. Thus $p_\alpha q \in M(\alpha; T) \Rightarrow g(T)\alpha = p_\alpha(T)q(T)\alpha + r(T)\alpha = r(T)\alpha \Rightarrow Z(\alpha; T) = \text{span}\{\alpha, T\alpha, \ldots, T^{k-1}\alpha\}$. Thus $\dim(Z(\alpha; T)) \le k$.

Claim 0.1.1

$\{\alpha, T\alpha, \ldots, T^{k-1}\alpha\}$ is a linearly independent set.

Proof. Suppose not. Then there is nonzero coefficients satisfying $\sum c_i T^i \alpha = 0$. Clearly $g(x) = \sum c_i x^i$ has deg $< k$. But p_α is the nonzero poly. of min. deg. in $M(\alpha; T)$, while $g(x) \in M(\alpha; T)$ with degree less than $p_\alpha(x)$. This is contradiction, so this set is linearly independent. $\square$

Thus by Claim 0.1.1, $Z(\alpha; T)$ is k-dimensional with $\deg(p_\alpha) = k$. i) and ii) done.
For iii), need to check $p_\alpha(U) = 0$ and it really is poly. with min. deg.
An arbitrary element of $Z(\alpha; T)$ is of the form $g(T)\alpha$ for some $g(x) \in F[x]$. Thus $p_\alpha(U)g(T)\alpha = p_\alpha(T)g(T)\alpha = g(T)p_\alpha(T)\alpha = 0$. Our first condition holds. Second condition is immediate from the minimality of the degree of p_α in $M(\alpha; T)$. $\square$

Lemma 0.1.1 Companion Matrices

T : endo. on f.d.v.s. V/F. $W = Z(\alpha; U) \subset V$ where $U := T|_{Z(\alpha;T)}$. Then w.r.t. the basis

$$\{\alpha, T\alpha, \dots, T^{k-1}\alpha\} = \mathfrak{B} \text{ of } Z,\ [U]_{\mathfrak{B}} = \begin{bmatrix} 0 & & & -c_0 \\ 1 & \ddots & & -c_1 \\ & \ddots & \ddots & \vdots \\ & & 1 & -c_{k-1} \end{bmatrix} \text{ where } p_\alpha = x^k + \sum_{i=0}^{k-1} c_i x^i.$$

This matrix is called companion matrix.

Proof. $\mathfrak{B} := \{\alpha, T\alpha, \dots, T^{k-1}\alpha\} = \{\alpha_1, \dots, \alpha_k\}$. Then $U\alpha_1 = T\alpha = \alpha_2$, $U\alpha_2 = T^2\alpha = \alpha_3$, and so on, $U\alpha_{k-1} = T^{k-1}\alpha = \alpha_k$. By our supposition of p_α, $p_\alpha(U)\alpha = U^k\alpha + \sum_{i=0}^{k-1} c_i U^i \alpha$. Thus we can derive companion matrix of above form. □

Theorem 0.1.2

U has a cyclic vec. $\iff$ there is some ordered basis s.t. U is represented by the companion mat. of the min. poly. for U.

Corollary 0.1.1

If A is the companion mat. of a monic poly. p, then p is both min. and char. poly. of A.

0.2 Cyclic Decompositions and the Rational Form

Definition 0.2.1: Complementary Subspaces

T : endo. on f.d.v.s. V/F. $W \subset V$ as T-inv. subspaces. If $\exists$ T-inv. subspace $W' \subset V$ s.t. $V = W \oplus W'$, then we say W' is a complementary T-inv. subspaces of W.

Definition 0.2.2: T-Admissible

T : endo. on f.d.v.s. V/F. A subspace is T-admissible if W is T-inv. and $\exists f(x) \in F[x]\ \exists \beta \in V\ \exists \gamma \in W\ (f(T)\beta \in W \Rightarrow f(T)\beta = f(T)\gamma)$.

Lemma 0.2.1

T : endo. on f.d.v.s. V/F. Suppose W is T-inv. If its complementary T-inv. subspace exists, then W is T-admissible.

Proof. W is trivially T-inv. Suppose $f(T)\beta \in W$ for $(f(x) \in F[x]) \wedge (\beta \in V)$. Since $V = W \oplus W'$, $\beta = \gamma + \gamma'$ for unique $(\gamma \in W) \wedge (\gamma' \in W')$. Then $f(T)\beta = f(T)\gamma + f(T)\gamma'$. Since $f(T)\beta$ and $f(T)\gamma$ are T-inv. and in W, $f(T)\gamma'$ should be in W. Independence of W and W' implies thus $f(T)\gamma' = 0$. Thus $f(T)\beta = f(T)\gamma$, so W is T-admissible. □

Theorem 0.2.1 Cyclic Decomposition Theorem

T : endo. on f.d.v.s. V/F. Let $W_0 \subset V$ be any proper T-admissible subspace. $\exists \alpha_1, \dots, \alpha_r \in V \backslash \{0\}$ with respective T-annihilators $p_1, \dots, p_r$ s.t.

i) $V = W_0 \oplus (\bigoplus_{i=1}^r Z(\alpha_i\,;\,T))$

ii) $p_k \mid p_{k-1}$

Furthermore, the integer r and p_i are uniquely determined by i), ii), and the fact that no α_k is 0.

Proof. We will divide our proof to 4 steps. During our proof, we intentionally denote $f(T)\beta$ as $f\beta$.

Before: Let $\beta \in V \backslash W$. Consider $S(\beta\,;W) := \{g(x) \in F[x] \mid g(T)\beta \in W\}$. Then $\exists$ monic poly. generator f s.t. $f(T)\beta \in W$. By T-admissibility, $\exists \gamma \in W$ s.t. $f(T)\beta = f(T)\gamma$. Let $\alpha := \beta - \gamma$, then $f(T)\alpha = 0$. Since $\gamma \in W$, we can see that $S(\alpha\,;W) = S(\beta\,;W)$ and f is also the T-conductor of α to W. Since $f(T)\alpha = 0$, $f \in M(\alpha\,;T)$. Thus $(f) = S(\alpha\,;W) \subset M(\alpha\,;T)$. Conversely, if $g \in M(\alpha\,;T)$, $g(T)\alpha = 0 \in W$ so $M(\alpha\,;T) \subset S(\alpha\,;T)$. Thus $S(\alpha\,;W) = M(\alpha\,;T)$ and f is also a T-annihilater.

Claim 0.2.1

$W \cap Z(\alpha\,;T) = 0$.

Proof. Suppose $g(T)\alpha \in W \cap Z(\alpha\,;T)$. Then $g \in S(\alpha\,;W) = M(\alpha\,;T)$ implies $g(T)\alpha = 0$. Thus $W \cap Z(\alpha\,;T) = 0$, so $W + Z(\alpha\,;T) \Rightarrow W \oplus Z(\alpha\,;T)$. □

Step 1: Let's make following observation: Let $W \subset V$ be a proper T-inv. subspace. Then $\max_{\alpha \in V} S(\alpha\,;W)$ is obtained by some $\beta \in V$, so that $\deg(S(\beta\,;W))$ is maximized.

For the above β, $W + Z(\beta\,;T)$ is T-inv. and strictly larger than W. Applying this observation to the given $W_0 \subset V$: T-inv. proper subspaces. Then we obtain $\beta_1 \in V$ s.t. $\deg(S(\beta_1\,;T))$ is maximized among $\deg(S(\beta\,;W))$. Again, take $W_2 = W_1 + Z(\beta_2\,;T)$, which leads $W_0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_r = V$.

From this, we can derive at least $V = W_0 + \sum_{i=1}^{r} Z(\beta_i\,;T)$. Know Let's say $(p_k) := S(\beta_k\,;W_{k-1})$ has the maximum deg. among the conductors.

Step 2: Take W_i, β_i, p_i $i \in [r]$ as above. Fix $1 \le k \le r$ and let $\beta \in V$. Suppose $(f) = S(\beta\,;W_{k-1})$. Write $f\beta = \beta_0 + \sum_{i=1}^{k-1} g_i\beta_i$ for some $g_i \in F[x]$, $\beta_i \in W_i$.

Claim 0.2.2

$\beta_0 = f\gamma_0$ for some $\gamma_0 \in W_0$ and $f \mid g_i$.

Proof. If $k=1$, it means W_0 is T-admissible, so nothing to proof. Thus suppose $k>1$. By the Euclidean algorithm, $g_i = fh_i + r_i$. We want to prove all $r_i = 0$. Let $\gamma := \beta - \sum_{i=1}^{k-1} h_i\beta_i$. Then $\beta - \gamma = \sum_{i=1}^{k-1} h_i\beta_i \subset W_{k-1}$. This leads $S(\gamma\,;W_{k-1}) = S(\beta\,;W_{k-1})$. Also, $f\gamma = f\beta - \sum_{i=1}^{k-1} fh_i\beta_i = f\beta - \sum_{i=1}^{k-1}(g_i - r_i)\beta_i = \beta_0 + \sum_{i=1}^{k-1} g_i\beta_i - \sum_{i=1}^{k-1} g_i\beta_i + \sum_{i=1}^{k-1} r_i\beta_i$. Thus $f\gamma = \beta_0 + \sum_{i=1}^{k-1} r_i\beta_i \ \cdots$ (1). Toward contradiction, some $r_j \neq 0$ and say that j is the largest between such numbers.

$f\gamma = \beta_0 + \sum_{i=1}^{k-1} r_i\beta_i$ for nonzero r_i. Clearly $\dim(r_i) < \dim(f) \ \cdots$ (2). Consider conductor $(p) := S(\gamma\,;W_{j-1})$. With $W_{j-1} \subset W_{k-1}$, $S(\gamma\,;W_{j-1}) \subset S(\gamma\,;W_{k-1}) = (f)$. Thus $f \mid p$, i.e., $p = fq$ for some $q \in F[x]$. Applying g to (2) leads $p(\gamma) = g\beta_0 + \sum_{i=1}^{j-1} gr_i\beta_i + gr_j\beta_j$ where $p(\gamma) \in W_{j-1}$, $g\beta_0 \in W_0 \subset W_{j-1}$, $gr_i\beta_i \in W_i \subset W_{j-1}$. This eq. leads $gr_j\beta_j \in W_{j-1}$, and thus $\deg(gr_j) \ge \deg(S(\beta_j\,;W_{j-1})) = \deg(p_j)$ by definition, and $\deg(p_j) \ge \deg(S(\gamma\,;W_{j-1}))$ by mazimality condition of β_j, where $\deg(S(\gamma\,;W_{j-1})) = \deg(p) = \deg(fg)$. Consequently, $\deg(r_i) \ge \deg(f)$, which is contradiction. Thus all $r_i = 0$, and all $f \mid g_i$, and (1) says $f\gamma = \beta_0 \in W_0$. Since W_0 is T-admissible, $\exists \gamma_0 \in W_0$ s.t. $f\gamma = \beta_0 = f\gamma_0$. □

Step 3: Now we will find $\{\alpha_1, \dots, \alpha_r\}$ in V which satisfies i) and ii).

Take $\{\beta_1, \dots, \beta_r\}$ as ***Step 1***. Fix $1 \le k \le r$. Apply ***Step 2*** to the vec. $\beta = \beta_k$ and the T-conductor $f = p_k$. We obtain $p_k\beta_k = p_k\gamma_0 + \sum_{i=1}^{k-1} p_kh_i\beta_i$ for $\gamma_0 \in W_0$. Let $\alpha_k :=$

$\beta_k - \gamma_0 - \sum_{i=1}^{k-1} h_i\beta_i$. Since $\beta_k - \alpha_k \in W_{k-1}$, $S(\alpha_k\,;W_{k-1}) = S(\beta_k\,;W_{k-1}) = (p_k)$, and since $p_k\alpha_k = 0$, we have $W_{k-1} \cap Z(\alpha_k\,;T) = \{0\}$. Because each α_k satisfies this condition, $W_k = W_0 \oplus (\bigoplus_{i=1}^{k} Z(\alpha_i\,;T))$ and that p_k is the T-annihilater of α_k.

Since $p_i\alpha_i = 0$ for each i, we have the trivial relation $p_k\alpha_k = 0 + p_1\alpha_1 + \cdots + p_{k-1}\alpha_{k-1}$. Apply ***Step 2*** with β_i replaced by α_i and with $\beta = \alpha_k$, we can conclude p_k divides each p_i with $i < k$.

Step 4: We will show r and each poly. p_r are uniquely determined by the conditions.

Take γ_i, g_i $i \in [s]$ that satisfies conditions either. We will show $r = s$ and $p_i = g_i$.

The poly. g_1 is determined as the T-conductor of V into W_0. Let $S(V\,;W_0)$ be the collection of poly. f s.t. $\forall\beta \in V$ $(f\beta \in W_0)$, i.e., poly. f s.t. $R(f(T)) \subset W_0$. Then $S(V\,;W_0)$ is nonzero ideal. g_1 is the monic generator of this. Each $\beta \in V$ has the form $\beta = \beta_0 + f_1\gamma_1 + \cdots + f_s\gamma_s$ and so $g_1\beta = g_1\beta_0 + \sum_{i=1}^{s} g_1 f_i\gamma_i$. Since each g_i divides g_1, we have $g_1\gamma_i = 0$ for all i and $g_1\beta = g_1\beta_0 \in W_0$. Thus $g_1 \in S(V\,;W_0)$. Since g_1 is the monic poly. of least deg. which sends γ_1 into W_0, we see that g_1 is the monic poly. of least deg. in the ideal $S(V\,;W_0)$. By the same argu., p_1 also, so $p_1 = g_1$. Now note three facts:

1. $fZ(\alpha\,;T) = Z(f\alpha\,;T)$
2. If $V = \bigoplus_{i=1}^{k} V_i$, where each V_i is T-inv., $fV = fV_1 \oplus \cdots \oplus fV_k$.
3. If α and γ have the same T-annihilator, then $f\alpha$ and $f\gamma$ have the same T-annihilater and thus $\dim(Z(f\alpha\,;T)) = \dim(Z(f\gamma\,;T))$.

Now, proceed induction to show that $r = s$ and $p_i = g_i$. Suppose $r \geq 2$. Then $\dim(W_0) + \dim(Z(\alpha_1\,;T)) < \dim(V)$ Since $p_1 = g_1$, we know $\dim(Z(\alpha_1\,;T)) = \dim(Z(\gamma_1\,;T))$. Thus $\dim(W_0) + \dim(Z(\gamma_1\,;T)) < \dim(V)$. Then

$$p_2V = p_2W_0 \oplus Z(p_2\alpha_1\,;T)$$
$$p_2V = p_2W_0 \oplus Z(p_2\gamma_1\,;T) \oplus \cdots \oplus Z(p_2\gamma_s\,;T)$$

satisfies our desire. Furthermore, we conclude that $p_2\gamma_2 = 0$ and g_2 divides p_2. The argument can be reversed to show that p_2 divides g_2. Thus $g_2 = p_2$. □

Corollary 0.2.1

If, W is T-admissible, it has complementary T-inv. subspace. So with Lemma 0.2.1, if and only if condition holds.

Theorem 0.2.2

T : endo. There is $\alpha \in V$ s.t. T-annihilator of α is equal to min. poly.

Proof. With $W_0 = 0$, apply cyclic decomposition. Take $\alpha = \alpha_1$. T-conductor fo α_1 to W_0 is T-annihilater of α_1, which is the min. poly. □

Theorem 0.2.3

If T has cyclic vec., then char. poly. of T is equal to min. poly. of T.

Theorem 0.2.4 Generalized Cayley-Hamilton Theorem

T : endo. on f.d.v.s. V/F. m be min. poly. and p be char. poly. Then

i) $p\,|\,f$

ii) p and f have the same prime factors except for multiplicities

iii) If $p = f_1^{r_1} \cdots f_k^{r_k}$, then $f = f_1^{d_1} \cdots f_k^{d_k}$ where d_i is the nullity of $f_i(T)^{r_i}$ divided by the deg. of f_i.

Proof. $i)$: trivial from Cayley-Hamilton Theorem.

$ii)$: Cyclic decompose with W_0 says $\exists \alpha_1 \sim \alpha_r$ s.t. $V = \bigoplus_{i=1}^{r} Z(\alpha_i\,;\,T)$ with $m(x) = p_1(x)$ which is T-annihilater of α_1. $p_i \mid p_{i-1}$. Take $T_i := T|_{Z(\alpha_i\,;\,T)}$. Since $Z(\alpha_i\,;\,T)$ is a cyclic vec. space with cyclic vec. α_i, p_i is min. poly. for T_i is also char. poly. of T_i. Thus char. poly. $f(x) = \prod_{i=1}^{r} p_i$ and any prime factor of $m(x)$ divides $f(x)$ by i) while if a prime factor divides f, it divides one of p_i. Thus $p_i \mid p_{i-1} \mid \cdots \mid p_1 = m(x)$. Thus each prime factor of f also divides $m(x)$.

$iii)$: Apply primary decomposition: $W_i = N(f_i(T)^{r_i})$. Take $T_i := T|_{W_i}$. Then $f_i(x)^{r_i}$ is the min. poly. of T_i. Applying ii) to T_i its min. poly. Thus char. poly. of T_i is $f_i^{d_i}$ with $d_i \geq r_i$. Here, $\dim(W_i)$ is $d_i \cdot \deg(f_i)$. So $d_i = \frac{\dim(W_i)}{\deg(f_i)} = N(f_i(T)^{r_i}) / \deg(f_i)$. □

Corollary 0.2.2

T : nilpo. endo. on n-d.v.s. V/F. Then char. poly. of T is x^n.

Proof. T is nilpo. $\Rightarrow \exists N$ s.t. $T^N = 0 \Rightarrow$ min. poly. $m(x) \mid x^N \Rightarrow m(x) = x^r$. Thus $f(x) = x^n$. □

0.3 The Jordan Form

Note:-

How to find Jordan form?

Solution. ***Step 1:*** char. poly. $f(x) = \prod_{i=1}^{k}(x - c_i)^{d_i}$ for distinct c_i and $m(x) = \prod_{i=1}^{k}(x - c_i)^{r_i}$ for $1 \leq r_i \leq d_i$. Take $W_i = N((T - c_i I)^{r_i})$ as primary decomposition theorem. Then $V = \bigoplus_{i=1}^{k} W_i$. $T_i := T|_{W_i}$ where $m_i(x)$ of T_i is $(x - c_i)^{r_i}$.

Step 2: For each W_i, let $N_i := (T_i - c_i I) : W_i \to W_i$. Then N_i is nilpotent operator on W_i. Note that $T_i = N_i + c_i I$. Consider each W_i the cyclic decomposition of W_i w.r.t. N_i. So, $W_i = \bigoplus_{k=1}^{s_i} Z(\alpha_k\,;\,N_i)$. Take $\beta_j = \{\alpha_j, N_i\alpha_j, \ldots, N_i^{k_j - 1}\alpha_j\}$. Then

$$[N_i|_{Z(\alpha_j\,;\,N_i)}]_{\beta_j} = \begin{bmatrix} 0 & & & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad [T_i|_{Z(\alpha_j\,;\,N_i)}]_{\beta_j} = \begin{bmatrix} c_u & & & 0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ & & 1 & c_i \end{bmatrix}. \qquad (1)$$

Take $\mathfrak{B}^i = \cup \beta_j$. Then

$$[T_i|_{W_i}]_{\mathfrak{B}^i} = \begin{bmatrix} \square & & \\ & \ddots & \\ & & \square \end{bmatrix}$$

where each box is of the form at 1 R.H.S. Then finally take $B = \cup \mathfrak{B}^i$. This leads what we call Jordan form, where each small blocks are elementary Jordan blocks. □

0.4 Computation of Invariant Factors

This Chapter is Intentionally Skipped at Lectures.

0.5 Summary; Semi-Simple Operators

This Chapter is Intentionally Skipped at Lectures.