MAS250 Probability and Statistics

statical inference / generalization.

statistic $\Longrightarrow$ parameter

Chapter 3.

ELEMENTS OF PROBABILITY

Statistical Reasoning

- Suppose a new treatment for COVID has been developed. It is tested on sample 2000 patients and found 1200 were cured.
- Identify population, sample, parameter and 0.6. statistic.
- Does this suggest that the new treatment is effective or does it happen by chance?

3.2 Sample space and events

- Experiment: a process of observing a phenomenon that has variation in its outcomes.
  - Toss a coin, roll a die, clinical trials

S: {H. T} {1~6} {Cured. Not}

- The sample space S of an experiment
  - The set of all possible outcomes of the experiment

Toss a coin Twice

$S = \{HH. HT. TH. TT\}$

$A = \{\text{one } H\} = \{HT. TH\}$

$B = \{\text{all } T\} = \{TT\}$

- Events
  - Any subset E of the sample space is known as an event
  - If the outcome of the experiment is contained in E, then we say event E has occurred.

3.3 The algebra of events

For events E, F, and G,

- Commutative law

$$E \cup F = F \cup E$$

$$E \cap F = F \cap E$$

- Associative law

$$(E \cup F) \cup G = E \cup (F \cup G)$$

$$(E \cap F) \cap G = E \cap (F \cap G)$$

- Distributive law

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$$

$$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$$

- DeMorgan's laws

$A^c = \overline{A}$

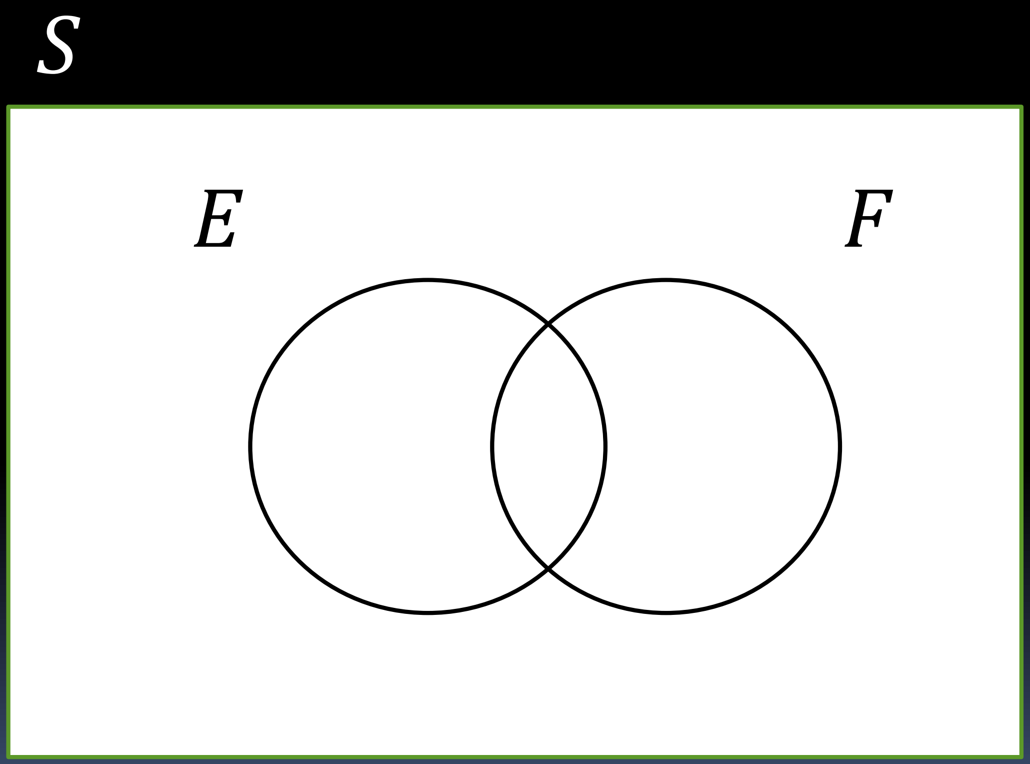
$$(E \cup F)^c = E^c \cap F^c$$

$$(E \cap F)^c = E^c \cup F^c$$

- Notation in the text book

$$EF := E \cap F$$

Venn diagram



3.4 Axioms of Probability

Let S be a sample space.

For an event E, there is a number $P\{E\}$ such that

- Axiom 1.

$$0 \leq P\{E\} \leq 1$$

- Axiom 2.

$$P\{S\} = 1$$

- Axiom 3. For any sequence of mutually exclusive events $\{E_i, i \geq 1\}$ (that is, $E_i \cap E_j = \emptyset$ when $i \neq j$)

$$P\{\cup_{i=1}^{n} E_i\} = \sum_{i=1}^{n} P\{E_i\}, n = 1,2, \ldots, \infty$$

- Proposition 3.4.1

$$P\{E^c\} = 1 - P\{E\}$$

- Proposition 3.4.2

$$P\{E \cup F\} = P\{E\} + P\{F\} - P\{E \cap F\}$$

- Probability of an event: a numerical value that represents the proportion of time the event is expected to occur when the experiment is repeated under identical conditions.

3.5 Equally likely outcomes

- *equally likely to occur* means, for a sample space $S = \{1, 2, \cdots, n\}$, we have

$$P\{\{1\}\} = P\{\{2\}\} = \cdots = P\{\{n\}\} = \frac{1}{n}$$

- Notation

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

: the number of cases when we select k elements among n items without considering the order

- Consider a set of n (different) objects. The number of permutations of the n objects (the number of ways of selecting the n objects) is $n! := n \cdot (n-1) \cdots 2 \cdot 1$.
  - If we select k objects from the n objects, then the number of different ways in the selection is $n(n-1) \cdots (n-k+1)$.
  - If we do not consider the order of the k objects in the selection, the number of different ways in the selection is $\binom{n}{k}$.

- A sample space (of points N) having equally likely outcomes

$$P\{E\} = \frac{the\ number\ of\ points\ in\ E}{N}$$

If they're indep.

- Basic principle of counting

Suppose that two experiments are to be performed. Then experiment 1 can result in any one of m possible outcomes and if, for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the two experiments.

- Generalized Basic Principle of Counting

If r experiments that are to be performed are such that the first one many result in any of n_1 possible outcomes, and if for each of these possible outcomes there are n_2 possible outcomes of the second experiment, and so on. Then there are a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

- Example 3.5.c
  - A class in probability theory consists of 6 men and 4 women. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same score,
  - (a) how many different rankings are possible?

    $10!$
  - (b) If all rankings are considered equally likely, what is the probability that women receive the top 4 scores? $\frac{4! \cdot 6!}{10!} = \frac{1}{210}$

- Example 3.5e
  - From a set of n items a random sample of k is to be selected. What is the probability a given item will be among the k selected?

$$\frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

- Example 3.5f
  - A basket ball team consists of 6 black and 6 white players. The players are to be paired in groups of two. If the pairings are done at random, what is the probability that none of the black players will have a white roommate?

If $2.2. \Rightarrow 3$ cases $= \dfrac{\binom{4}{2}\binom{2}{2}}{2!}$

$\Rightarrow$ For $6.6 \Rightarrow \dfrac{\left\{\dfrac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!}\right\}^2}{\dfrac{\binom{12}{2}\binom{10}{2}\cdots\binom{2}{2}}{6!}}$

The Birthday Problem

- Example 3.5g

  If n people are present in a room,

  - What is the probability that no two of them celebrate their birth day on the same day of the year?

    $\frac{{}_{365}P_n}{365^n}$

  - How large need n be so that this probability is less than $\frac{1}{2}$?

    $n \geq 23$

3.6 Conditional Probability

- For two events E and F, the conditional probability of E given that F has occurred, denoted by $P\{E|F\}$, is defined by

$$P\{E|F\} = \frac{P\{E \cap F\}}{P\{F\}}$$

- Example 3.6a
  - A bin contains 5 defective, 10 partially defective, and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail (i.e., not defective), what is the probability it is acceptable?

$=F$ $=E$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{25}{35/40} = \frac{5}{7}$$

3.7 Bayes' Formula

* $P(A \cap B) = P(A|B)P(B)$
$= P(B|A)P(A)$

$P(F_j|E) = \frac{P(F_j \cap E)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum P(E|F_i)P(F_i)}$

$\because P(E) = P(E \cap F_1) + \cdots + P(E \cap F_n) \Leftarrow$ Mutually Exclusive
$= \sum_{i=1}^{n} P(E \cap F_i) = \sum P(E|F_i)P(F_i).$

- S : Sample space

Suppose that $\{F_1, F_2, \cdots, F_n\}$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$.

$$P\{F_j|E\} = \frac{P\{E|F_j\}\,P\{F_j\}}{\sum_{i=1}^{n} P\{E|F_i\}\,P\{F_i\}}$$

$P(F) = 0.3$
$P(E|F) = 0.4$
$P(E|F^c) = 0.2$

$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$
$= 0.4 \cdot 0.3 + 0.2 \cdot 0.7$
$= 0.12 + 0.14 = 0.26$ ∎

- Example 3.7a
  - An insurance company believes that people can be divided into two classes – those that are accident prone and those that are not. Their statistics show that an accident-prone person (F) will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a non-accident-prone person (F^c). If we assume that 30 percent of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy (E)?

- Example 3.7c
  - Reconsider Example 3.7a and suppose that a new policy holder has an accident within a year of purchasing his policy. What is the probability that he is accident prone?

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)} = \frac{0.4 \cdot 0.3}{0.26} \approx 0.4615$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} =$$

- Example 3.7d
  - In answering a question on a multiple-choice test, a student either knows the answer or she guesses. Let p be the $\overset{F}{\text{probability}}$ that she knows the answer and $1 - p$ the $\overset{F^c}{\text{probability}}$ that she guesses. Assume that a student who guesses at the answer will be correct with probability $\frac{1}{m}$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that $\overset{E}{\text{she}}$ answered it correctly?

$P(F) = p$

$P(E|F^c) = \frac{1}{m}$

$P(E|F) = 1$

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)} = \frac{p}{p + \frac{1}{m} - \frac{p}{m}}.$$

If $m = 5$, $p = \frac{1}{2}$, $P(F|E) = \frac{5}{6}$ ▪

G : Suspect is guilty

C : Criminal Characteristic

$$P(G|C) = \frac{P(C|G)P(G)}{P(C|G)P(G) + P(C|G^c)P(G^c)}$$

$$= \frac{1 \cdot 0.6}{1 \cdot 0.6 + 0.2 \cdot 0.4} = \frac{0.6}{0.68} \approx 0.882$$

- Example 3.7f
  - At a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose now that a *new* piece of evidence that shows that the criminal has a certain characteristic is uncovered. If 20% of the (not guilty) population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect is among this group?

$$P(G|C) = \frac{P(C|G)P(G)}{P(C|G)P(G) + P(C|G^c)P(G^c)}$$

$$= \frac{0.9 \cdot 0.6}{0.9 \cdot 0.6 + 0.2 \cdot 0.4} = \frac{0.54}{0.62} \approx 0.871 \blacksquare$$

- Example 3.7f (continued)

  Let us now suppose that the new evidence is subjective to different possible interpretations, and in fact only shows that it is 90 percent likely that the criminal possesses this certain characteristic. In this case, how likely would it be that the suspect is guilty (assuming, as before, that he has this characteristic)?

* Mutually exclusive: 'Can't' happen at same

3.8 Independent events

- Two events E and F are independent if
$$P\{E \cap F\} = P\{E\}\, P\{F\}.$$

- The events $\{E_1, E_2, \cdots, E_r\}$ are said to be independent if for every subset $\{E_{n_1}, E_{n_2}, \cdots, E_{n_k}\}$ $(\{n_1, n_2, \cdots, n_k\} \subseteq \{1,2, \cdots, r\})$ of these events

$$P\{E_{n_1} \cap E_{n_2} \cap \cdots \cap E_{n_k}\} = P\{E_{n_1}\}\, P\{E_{n_2}\} \cdots P\{E_{n_k}\}$$

- Proposition 3.8.1

  If E and F are independent, then E and F^c are also independent.

  $P(E) = P(E \cap F) + P(E \cap F^c)$
  $= P(E)P(F) + P(E \cap F^c)$

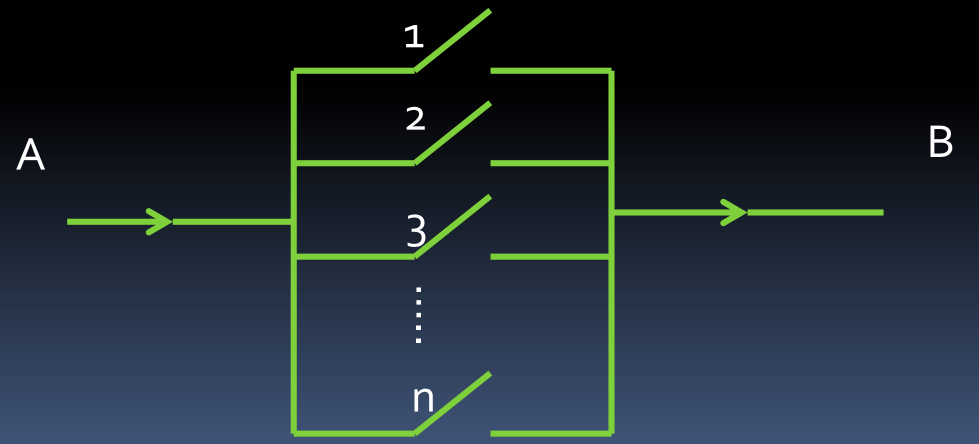
  $P(E \cap F^c) = P(E) - P(E)P(F)$
  $= P(E)(1 - P(F))$
  $= P(E)P(F^c)$ ▪

- If the events E, F and G are independent, then E will be independent of any event formed from F and G.

  $P(E \cap (F \cup G))$
  $= P(E)P(F \cup G)$

$P(\text{working}) = 1 - P(\text{no component working})$
$= 1 - \prod P(\text{component not working})$

- Consider the following system.
  - p_i: the probability that component i functions
  - Each component functions independently.
  - What is the probability that the system is working?



A_i : set contains type i.

$P(A_j \mid A_i) = \dfrac{P(A_j \cap A_i)}{P(A_i)}$.

$P(A_i) = 1 - P(A_i^c)$

$= 1 - P(\text{no type } i \text{ out of coupons})$

$= 1 - (1-p_i)^k$

$P(A_i \cap A_j) = 1 - P((A_i \cap A_j)^c) = 1 - P(A_i^c \cup A_j^c)$

$= 1 - \{P(A_i^c) + P(A_j^c) - P(A_i^c \cap A_j^c)\}$

$= 1 - \{(1-p_i)^k + (1-p_j)^k - (1-p_i-p_j)^k\}$

$\because P(A_i^c \cap A_j^c) = 1 - P(A_i \cup A_j)$

- Example 3.8e

  A set of k coupons, each of which is independently a type j coupon with probability $p_j, \sum_{j=1}^{n} p_j = 1$, is collected. Find the probability that the set contains a type j coupon given that it contains a type $i, i \neq j$.