0.1 Inner Products

Definition 0.1.1: Inner Product

An inner product $(-,-)$ on V is a function $(-,-) : V \times V \mapsto F$ satisfying:

1. $(-, \ )$ is linear functionoal with $(c\alpha+\beta,\gamma) = c(\alpha,\gamma)+(\beta,\gamma)$
2. $(\beta,\alpha) = \overline{(\alpha,\beta)}$
3. $\forall \alpha \in \mathbb{F}\backslash\{0\}\ ((\alpha,\alpha) > 0)$

Note:-

If $F = \mathbb{R}$, $(\beta,\alpha) = (\alpha,\beta)$. Thus 1. and 2. leads $(\ ,-)$ is also linear. Thus $(\ ,\ )$ is symmetric bilinear form.

But If $F = \mathbb{C}$, then $(\alpha, c\gamma) = \overline{c(\gamma,\alpha)}$. In this case, we call bbC is sesqui-linear. Also, $(\alpha,\alpha) = \overline{(\alpha,\alpha)}$, thus $(\alpha,\alpha) \in \mathbb{R}$.

Example 0.1.1 (Standard Inner Product)

$V := \mathbb{C}^n$, $[x_i],[y_i] \in \mathbb{C}^n$. Then $([x_i],[y_i]) = \sum_{i=1}^n x_i \bar{y}_i$ is called the standard inner product.

Example 0.1.2 (Positive Definite)

$F = \mathbb{R}^n$. $A : n \times n$ real mat. s.t. $\forall x \in \mathbb{R}^n$, $x^T A x > 0$. Then A is called positive definite. When A is symmetric pos. def., then $(x,y)_A := x^T A y$.

Exercise 0.1.1

Prove that $(x,y)_A$ is an inner product on $\mathbb{R}^n$.

Theorem 0.1.1

$F = \mathbb{R}$, $V = \mathbb{R}^n$. Let $(\ ,\ ) : V \times V \mapsto F$ be an arbitrary inn. prod. on V. Then $\exists$ a sym. pos. def. mat. A s.t. $(\ ,\ ) = (\ ,\ )_A$.

Proof. Choose a basis, the standard basis for convenient. $(e_i,e_j) =: g_{ij}$. Define $A := [g_{ij}]$. Let $x,y \in \mathbb{R}^n$. Then $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{j=1}^n y_j e_j$. $(x,y) = \sum_i \sum_j x_i y_j e_i e_j = \sum_i \sum_j x_i g_{ij} y_j = \sum_i x_i \sum_j g_{ij} y_j = [x^T]_{\mathfrak{B}} A [y]_{\mathfrak{B}}$. □

Definition 0.1.2: Hermitian Matrix

$n \times n$ mat. A is called Hermitian if $A^* = A = [a_{ij}]$ where $[A^*]_{ij} = [\overline{a_{ji}}] = \overline{A^T}$.

Theorem 0.1.2

$V = \mathbb{C}^n$. Let $(\ ,\ ) : V \times V \mapsto F$ be an inn. prod. on V. Then $(x,y) = x^* A y$ for some Hermitian pos. def. mat. A and vice versa.

Example 0.1.3

$V = C([a,b] \mapsto \mathbb{C})$: $\mathbb{C}$-v.s. of continuous functions on $[a,b]$. Define $f,g \in V$ as $(f,g := \int_a^b f(t)\overline{g(t)}dt)$. Then it is an inn. prod. on V of ∞-dim.

Definition 0.1.3: Quadratic Form

V : v.s. with inn. prod. Define the quadratic form $\alpha \mapsto ||\alpha||^2 = (\alpha,\alpha) \geq 0$ and $\alpha = 0 \iff ||\alpha||^2 = 0$.

Exercise 0.1.2 Polarization Identity

$||\alpha+\beta||^2 = ||\alpha||^2 + 2\Re(\alpha,\beta) + ||\beta||^2$. $F = \mathbb{R}$, $(\alpha,\beta) = \frac{1}{4}(||\alpha+\beta||^2 - ||\alpha-\beta||^2)$. $F = \mathbb{C}$, $(\alpha,\beta) = \frac{1}{4}(||\alpha+\beta||^2 - ||\alpha-\beta||^2 + i||\alpha+i\beta||^2 - i||\alpha-i\beta||^2)$.

0.2 Inner Product Spaces

Definition 0.2.1: Inner Product Space

$F = \mathbb{R}$ or $F = \mathbb{C}$. A vector space V/F with a specified inn. prod. called inner product space.

Note:-

$||\alpha|| = \sqrt{(\alpha,\alpha)}$.

Theorem 0.2.1

V : inn. prod. space. $\forall \alpha,\beta \in V\ \forall c \in F$, we have properties:

1. $||c\alpha|| = |c|||\alpha||$
2. $||\alpha|| > 0$ for $\alpha \neq 0$
3. $|(\alpha,\beta)| \leq ||\alpha||||\beta||$
4. $||\alpha+\beta|| \leq ||\alpha|| + ||\beta||$

Proof. 1 ane 2 are obvious from definition of inn. prod. For 3, just take $\alpha \neq 0$. Let $\beta^{\parallel} := \frac{(\beta,\alpha)}{||\alpha||^2}\alpha$, $\beta^{\perp} := \beta - \beta^{\parallel}$.

This is because: $(\beta^{\perp},\alpha) = 0 \iff (\beta,\alpha) = c(\alpha,\alpha)$, thus $c = \frac{(\beta,\alpha)}{(\alpha,\alpha)}$.

$0 \leq ||\beta^{\perp}||^2 = (\beta^{\perp},\beta^{\perp}) = (\beta,\beta) - |c|^2(\alpha,\alpha) = ||\beta||^2 - \frac{|(\alpha,\beta)|^2}{||\alpha||^2} \Rightarrow |(\alpha,\beta)| \leq ||\alpha||||\beta||$.

For 4, $||\alpha+\beta||^2 = (\alpha+\beta,\alpha+\beta) = (\alpha,\alpha)+(\alpha,\beta)+(\beta,\alpha)+(\beta,\beta) \leq (||\alpha||+||\beta||)^2$. Since both side are positive, we can just take off square. □

Note:-

The "angle" is defined as inner product. Using 3, we can derive $-1 \leq \frac{(\alpha,\beta)}{||\alpha||||\beta||} \leq 1$. Then for nonzero α,β, define angle θ as:

$$\cos\theta = \frac{(\alpha,\beta)}{||\alpha||||\beta||}$$

Definition 0.2.2: Orthogoanl and Orthogonal, Orthonormal Set

V : inn. prod. space. We say $\alpha, \beta \in V$ are orthogonal or perpendicular if their inn. prod. is 0. If $S \subset V$, S is orthogonal set if $\forall \alpha \neq \beta \in S$, $(\alpha, \beta) = 0$. If all element of S satisfies $||\alpha|| = 1$, we say S is orthonormal.

Theorem 0.2.2

Suppose $S \subset V$ be orthogonal set. Then S is lin. indep.

Proof. Take $\{\alpha_1, \alpha_2, \dots, \alpha_m\}$ as distinct vectors in S. Then $\beta = c_1\alpha_1 + \cdots + c_m\alpha_m$. So $(\beta, \alpha_k) = (\sum_j c_j\alpha_j, \alpha_k) = \sum_j c_j(\alpha_j, \alpha_k) = c_k(\alpha_k, \alpha_k)$. Since $(\alpha_k, \alpha_k) \neq 0$, $c_k = \frac{(\beta, \alpha_k)}{||\alpha_k||^2}$ for $1 \le k \le m$. When $\beta = 0$, this leads $\forall c_i = 0$, so S is lin. indep. set. □

Theorem 0.2.3 Gram-Schmidt

Let V be an inn. prod. space and let $\beta_1, \dots, \beta_n$ be any indep. vec. in V. Then we can construct orthogonal vectors $\alpha_1, \dots, \alpha_n$ in V s.t. for each $k \in [n]$ the set $\{\alpha_1, \dots, \alpha_k\}$ is a basis for the subspace spanned by $\beta_1, \dots, \beta_k$.

Proof. We can apply Gram-Schmidt orthogonalization process. $\alpha_1 := \beta_1$. $\beta_2 = \beta_2^{\perp\{\alpha_1\}} + \beta_2^{||\{\alpha_1\}}$. $\alpha_2 = \beta_2^{\perp\{\alpha_1\}} = \beta_2 - \beta_2^{||\{\alpha_1\}} = \beta_2 - \frac{(\beta_2, \alpha_1)}{||\alpha_1||^2}\alpha_1$. $\beta_3 = \beta_3^{\perp\{\alpha_1,\alpha_2\}} + \beta_3^{||\{\alpha_1,\alpha_2\}}$. $\alpha_3 = \beta_3^{\perp\{\alpha_1,\alpha_2\}} = \beta_3 - \frac{(\beta_3, \alpha_1)}{||\alpha_1||^2}\alpha_1 - \frac{(\beta_3, \alpha_2)}{||\alpha_2||^2}\alpha_2$, and so on. We can take orthogonal basis with this process. □

Corollary 0.2.1

All f.d. inn. prod. space has orthogonal basis.

Definition 0.2.3: Best Approximation

V : inn. prod. space. $W \subset V$, $\beta \in V \backslash W$. A best approximation of β to W is $\alpha \in W$ s.t. $\forall \gamma \in W \ (||\beta - \alpha|| \le ||\beta - \gamma||)$.

Theorem 0.2.4

Let W be a subspaces of an inn. prod. space V and let β be a vec. in V. Then

1. $\alpha \in W$ is a best approx. to β by vec. in $W \iff \beta - \alpha$ is orthogonal to every vec. in W
2. If a best approx. to β by vec. in W exists, it is unique
3. If W is f.d. and $\{\alpha_1, \dots, \alpha_n\}$ is any ortho. basis for W, then the vec. $\alpha = \sum_k \frac{(\beta, \alpha_k)}{||\alpha_k||^2}\alpha_k$ is the unique best approx. to β by vec. in W

Proof. Note that $\forall \gamma \in W$, $\beta - \gamma = (\beta - \alpha) + (\alpha - \gamma)$, and $||\beta - \gamma||^2 = ||\beta - \alpha||^2 + 2\Re(\beta - \alpha, \alpha - \gamma) + ||\alpha - \gamma||^2$. Now suppose $\beta - \alpha$ is ortho. to every vec. in W. Then since $(\alpha - \gamma) \in W$, we can see $||\beta - \gamma||^2 = ||\beta - \alpha||^2 + ||\alpha - \gamma||^2 \ge ||\beta - \alpha||^2$.

Conversely, suppose $\forall \gamma \in W \ (||\beta - \gamma|| \ge ||\beta - \alpha||)$. Then from above we can find that $\forall \gamma \in W \ (2\Re(\beta - \alpha, \alpha - \gamma) + ||\alpha - \gamma||^2 \ge 0)$. Since every vec. in W may be expressed in the form $\alpha - \gamma$ with $\gamma \in W$, we see that $2\Re(\beta - \alpha, \tau) + ||\tau||^2 \ge 0$. We may take $\tau = -\frac{(\beta - \alpha, \alpha - \gamma)}{||\alpha - \gamma||^2}(\alpha - \gamma)$.

Then the equality reduces to the statement $-\frac{|(\beta-\alpha,\alpha-\gamma)|^2}{\|\alpha-\gamma\|^2} \geq 0$, which holds iff $(\beta-\alpha,\alpha-\gamma)=0$. This completes the proof of 1. and ortho. condition is evidently satisfied by at most one vec. in W, thus proves 2.

Now suppose W is f.d. and let $\{\alpha_1,\dots,\alpha_n\}$ be ortho. basis for W. We know $\beta-\alpha$ is ortho. to each elements of basis, i.e., to every vec. in W, so α it is best approx. to β, which leads $\|\beta-\gamma\| \geq \|\beta-\alpha\|$. Therefore $\alpha \in W$ and it is best approx. to β. $\square$

Definition 0.2.4: Orthogonal Complement

$W^\perp := \{\beta \in V \mid \alpha \perp \beta \forall \alpha \in W\}$.

Exercise 0.2.1

V : f.d. inn. prod. space. Then $V = W \oplus W^\perp$

Proof. $\beta \in V$. Then $\mathrm{E}\beta$ is best approx. lies in W. It is easy to see that this is proj. Also, since $\alpha - E\alpha$ and $\beta - E\beta$ are each ortho. to W, $c(\alpha-E\alpha)+(\beta-E\beta) = (c\alpha+\beta)-(cE\alpha+E\beta) \in W^\perp$. Thus E is linear transformation by uniqueness of ortho. proj.

Note that $(\beta \in W^\perp) \iff (E\beta = 0)$. The eq. $\beta = E\beta + (\beta - E\beta)$ shows $V = W + W^\perp$. Also, $W \cap W^\perp = \{0\}$, so $V = W \oplus W^\perp$. $\square$

0.3 Linear Functionals and Adjoints

Theorem 0.3.1

V : f.v.s./$\mathbb{R}$ or $\mathbb{C}$, V^* : dual vec. space. Let $f \in V^*$. Then $\exists! \beta \in V\ (f(-) = (-,\beta))$.

Proof. Choose ortho basis $\{\alpha_1,\dots,\alpha_n\}$ of V. For uniqueness, try $\beta = \sum_{i=1}^n c_i\alpha_i$. We can see $f(\alpha_j) = (\alpha_j, \sum_{i=1}^n c_i\alpha_i) = \sum_{i=1}^n \overline{c_i}(\alpha_j,\alpha_i) = \overline{c_j}(\alpha_j,\alpha_j)$. If such β exists, then it must be $\beta = \sum_{i=1}^n \overline{\frac{f(\alpha_i)}{\|\alpha_i\|^2}}\alpha_i$.

So take this as β. Now let's prove $f(-) = (-,\beta)$. We can see $(\alpha_j,\beta) = \sum_{i=1}^n \frac{f(\alpha_i)}{\|\alpha_i\|^2}(\alpha_i,\alpha_j) = \frac{f(\alpha_j)}{\|\alpha_j\|^2}(\alpha_j,\alpha_j) = f(\alpha_j)$. Thus such inn. prod. which corresponds to linear functional exists and unique. $\square$

Note:-

Usually V and V^* are not naturally related. But if V has inn. prod., then we can have an isomorphism.

Theorem 0.3.2

T : endo. on f.d.v.s.$V/\mathbb{R}$ or $\mathbb{C}$. Then $\exists! T : V \to T$ s.t. $(T\alpha,\beta) = (\alpha, T^*\beta)$ where T^* is a unique linear operator. If $F = \mathbb{R}$, T^* is transpose and if $F = \mathbb{C}$, T^* is conjugate transpose.

Proof. Fix $\beta \in V \Rightarrow (-,\beta) \in V^*$. Let's modify it a bit to get what we want. Theorem 0.3.1 says that $\exists! \beta' \in V\ \ ((T(-),\beta) = (-,\beta'))$. Define $T^* : V \to V : \beta \mapsto \beta'$. This mapping is well-defined. Also, easy to show that T^* is linear, and since for any $\beta \in V$, $T^*\beta$ is uniquely determined, thus uniqueness holds. $\square$

Theorem 0.3.3

T : endo. on f.d.v.s.$V/\mathbb{F}$ or $\mathbb{C}$, $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis. Let $A := [T]_{\mathfrak{B}} = [A]_{ij}$. Then $A_{ij} = (T\alpha_j, \alpha_i)$.

Proof. $\alpha \in V$. $\alpha = \sum_{i=1}^n (\alpha, \alpha_i)\alpha_i$. A is defined by A_{ij} s.t. $T(\alpha_j) = \sum_{i=1}^n A_{ij}\alpha_i$. Since $T\alpha_j = \sum_{i=1}^n (T\alpha_j, \alpha_i)\alpha_i$, $A_{ij} = (T\alpha_j, \alpha_i)$. □

Corollary 0.3.1

T : endo. on f.d.v.s.$V/\mathbb{F}$ or $\mathbb{C}$, $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$ be an orthonormal basis. Then $[T^*]_{\mathfrak{B}} = ([T]_{\mathfrak{B}})^*$ where L.H.S. is adjoint s.t. $(T\alpha, \beta) = (\alpha, T^*\beta)$ and R.H.S. is conjugate transpose.

Proof. $A := [T]_{\mathfrak{B}} = [A_{ij}]$, $B := [T^*]_{\mathfrak{B}} = [B_{ij}]$. Then $A_{ij} = (T\alpha_j, \alpha_i)$ and $B_{ij} = (T^*\alpha_j, \alpha_i)$. Then $\overline{B_{ij}} = (\alpha_i, T^*\alpha_j) \Rightarrow \overline{B_{ji}} = (\alpha_j, T^*\alpha_i) = A_{ij}$. □

Exercise 0.3.1

$$(T_1 + T_2)^* = T_1^* + T_2^*,\ (cT)^* = \bar{c}T^*,\ (T_1T_2)^* = T_2^*T_1^*.$$

Definition 0.3.1: Hermitian

T : endo. on f.d.v.s.$V/\mathbb{R}$ or $\mathbb{C}$. We say T is Hermitian or self-adjoint if $T = T^*$.

0.4 Unitary Operators

Definition 0.4.1: Preserve

$T : V \to W$ on inn. prod. space V and W. Then we say T preserves the inn. prod. if $\forall \alpha, \beta \in V$ $(T\alpha, T\beta) = (\alpha, \beta)$. We say this is isometry.

Definition 0.4.2: Isomorphism of Inner Product Spaces

An isomorphism of inn. prod. space is a linear transf. s.t. it is an isomorphism of vec. spaces and preserves the inn. prod.

Theorem 0.4.1

$T : V \to W$ with same dim f.d. inn. prod. spaces. TFAE:

i) T preserves inn. prod.

ii) T is an isomorphism of inn. prod. spaces

iii) For arbitrary orthonormal basis $\mathfrak{B}$ of V, $T\mathfrak{B}$ is an orthonormal basis for W

iv) For some orthonormal basis mfB of V, $T\mathfrak{B}$ is an orthonormal basis for W

Proof. i) $\Rightarrow$ ii): Suppose $\exists \alpha \in N(T)$. Then $(T\alpha, T\alpha) = \|T\alpha\|^2 = \|\alpha\|^2 = 0$. Thus $\alpha = 0$. Since $\dim(V) = \dim(W)$, T is one-to-one, Thus T is an isomorphism.

ii) $\Rightarrow$ iii): Let $\mathfrak{B}$ an arbitrary orthonormal basis $\{\alpha_1, \dots, \alpha_n\}$. Then $(\alpha_i, \alpha_j) = \delta_{ij}$. Since T preserves, $(T\alpha_i, T\alpha_j) = \delta_{ij}$. Isomorphic condition of T implies then $T\mathfrak{B}$ is basis for W while $\{T\alpha_1, \dots, T\alpha_n\}$ is an orthonormal set.

iii) $\Rightarrow$ iv): Trivial.

iv) $\Rightarrow$ i): Let $\mathfrak{B}$ an orthonormal basis of V s.t. $T\mathfrak{B}$ is also an orthonormal basis.

Claim 0.4.1

$\forall \alpha, \beta \in V \ ((T\alpha, T\beta) = (\alpha, \beta))$.

Proof. $\alpha := \sum x_i\alpha_i$, $\beta := \sum y_i\alpha_i$. Then $T\alpha = \sum x_i T\alpha_i$ and $T\beta = \sum y_i T\alpha_i$. We can see $(T\alpha, T\beta) = (\sum x_i T\alpha_i, y_j T\alpha_j) = \sum_j \sum_i x_i \overline{y_j}(\alpha_i, \alpha_j)$

while $(\alpha, \beta) = (\sum x_i\alpha_i, \sum y_j\alpha_j) = \sum_j \sum_i x_i \overline{y_j}(\alpha_i, \alpha_j)$, and both are δ_{ij}. $\square$

$\square$

Theorem 0.4.2

$T : V \to W$ on inn. prod. space with preserving $\iff \|T\alpha\| = \|\alpha\|$.

Proof. $(\Rightarrow)$: Trivial since $\|T\alpha\|^2 = (T\alpha, T\alpha) = (\alpha, \alpha) = \|\alpha\|^2$.

$(\Leftarrow)$: By using polarization identity, we can easily derive this direction. $\square$

Definition 0.4.3: Unitary Operator

T is unitary operator if it is an isomorphism on inn. prod. space.

Theorem 0.4.3

$U : V \to V$ on inn. prod. space. Then U is unitary $\iff U^*$ exists and $UU^* = U^*U = I$.

Proof. $(\Rightarrow)$: If U is unitary, then, isomorphism, so $\exists U^{-1} : V \to V$ and $(U\alpha, \beta) = (U\alpha, I\beta) = (U\alpha, UU^{-1}\beta) = (\alpha, U^{-1}\beta)$. Thus $U^{-1} = U^*$.

$(\Leftarrow)$: Suppose $\exists U^* : V \to V$ s.t. $UU^* = U^*U = I$. Then U is invertible where $U^* = U^{-1}$. Then $(U\alpha, U\beta) = (\alpha, U^*U\beta) = (\alpha, \beta)$. $\square$

Definition 0.4.4: Unitary

$A : n \times n$ mat. on $\mathbb{R}$ or $\mathbb{C}$. We say A is unitary if $AA^* = A^*A = I$.

Theorem 0.4.4

$U : V \to V$ on inn. prod. space. Then U is unitary $\iff [U]_{\mathfrak{B}}$ for orthonormal basis $\mathfrak{B}$ is a unitary mat.

Proof. $[U]_{\mathfrak{B}}$ is unitary $\iff U$ is unitary. Then iff condition follows from Theorem 0.4.3. $\square$

Corollary 0.4.1

If U_1 and U_2 are unitary, then U_1U_2 also. Furthermore, U_1^{-1} is also unitary.

Definition 0.4.5: Unitary Group - Optional

For f.d.inn. prod. space, let $U(V)$ be a collection of all unitary op. on V. This is a group, i.e., closed under mat. multiplication.

Note:-

OPTIONAL.

When $V = \mathbb{C}^n$, $U(\mathbb{C}^n) = U(n)$: the n-th unitary group.

$V = \mathbb{R}^n, A : n \times n$ mat. on $\mathbb{R}$ s.t. $AA^t = A^tA = I$. Then $O(n)$ is the real orthogonal group.

$V = \mathbb{C}^n, A : n \times n$ mat. on $\mathbb{C}$ s.t. $AA^t = A^tA = I$. Then $O(n, \mathbb{C})$ is the complex orthogonal group.

$SU(n) = \{A \in U(n) \mid \det(A) = 1\}$ is special unitary group.

$SO(n) = \{A \in O(n) \mid \det(A) = 1\}$ is special orthogonal group. For example, $SO(2)$ is rotation and $SO(3)$, with $SO(3) \rtimes \mathbb{R}^3$ is rigid motion.

0.5 Normal Operators

Definition 0.5.1: Normal

T : endo on f.d.inn. prod. space. V/F. We say T is normal if $TT^* = T^*T$.

Note:-

Q. When do we have an orthonormal basis $\mathfrak{B}$ on V s.t. vec. in $\mathfrak{B}$ are also char. vec. of T?

Theorem 0.5.1

T : endo on f.d.inn. prod. space. V/F. Suppose T is normal. For char. vec. α of T, $c \in F$ is char. value $\iff$ $\bar{c}$ is char. value for T^* with char. vec. α.

Proof.

Claim 0.5.1

If U is normal, then $\|Uv\| = \|U^*v\|$.

Proof. $\|Uv\|^2 = (Uv, Uv) = (v, U^*Uv) = (v, UU^*v) = (U^*v, U^*v) = \|U^*v\|^2$. □

$\forall c \in F$, $U := T - cI$ is normal for normal T. Then $U^* = T^* - \bar{c}I$. $UU^* = U^*U$ is obvious. Thus $\|(T - cI)\alpha\| = \|(T^* - \bar{c}I)\alpha\|$ by Claim 0.5.1. Thus $(T - cI)\alpha = 0 \iff (T^* - \bar{c}I)\alpha = 0$. □

Theorem 0.5.2

T as Theorem 0.5.1 but not normal. Suppose $\exists$ orthonormal basis $\mathfrak{B}$ s.t. $[T]_{\mathfrak{B}}$ is upper triangular. Then T is normal $\iff$ $[T]_{\mathfrak{B}}$ is diagonal.

Proof. ($\Leftarrow$): Let $A := [T]_{\mathfrak{B}}$. $A^* = [T^*]_{\mathfrak{B}}$. A is diagonal, so A^* also. Trivially $AA^* = A^*A$, thus $TT^* = T^*T$, i.e., T is normal.

($\Rightarrow$): Suppose T is normal. We are given that A is upper triangular. Let $\mathfrak{B} = \{\alpha_1, \ldots, \alpha_n\}$. Then

$$A = [T]_{\mathfrak{B}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}$$

where T is normal, and α_1 is char. vec., where a_{11} are char. value w.r.t. α_1. By Theorem 0.5.1, $T^*\alpha_1 = \overline{a_{11}}\alpha_1$. On the other hand, since $[T^*]_{\mathfrak{B}} = A^*$, $T^*\alpha_1 = \overline{a_{11}}\alpha_1 + \overline{a_{12}}\alpha_2 + \cdots + \overline{a_{1n}}\alpha_n$. Thus

$$A = [T]_{\mathfrak{B}} = \begin{bmatrix} a_{11} & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}.$$

Applying this algorithm to each α_i leads A is diagonal. □

Lemma 0.5.1

T : endo on f.d.inn. prod. space. $V/\mathbb{R}$ or $\mathbb{C}$. Let $W \subset V$ be T-inv. subspace. Then $W^\perp$ is automatically T^*-inv.

Proof. Let $\beta \in W^\perp$. N.T.S. $T^*\beta \in W^\perp$, i.e., $\forall \alpha \in W \ \ ((\alpha, T^*\beta) = (T\alpha, \beta) = 0)$. Since W is T-inv., this clearly holds. □

Theorem 0.5.3

T : endo on f.d.inn. prod. space. $V/\mathbb{C}$. Then $\exists$ orthonormal basis $\mathfrak{B}$ for V s.t. $[T]_{\mathfrak{B}}$ is upper triangular mat.

Proof. We prove it by induction on $n = \dim(V)$. If $n = 1$, it is obvious. So suppose $n > 1$ and assume Theorem 0.5.3 holds for any inn. prod. space with dim $< n$. Since $F = \mathbb{C}$, applying Fundamental Theorem of Algebra to T^*, $\exists$ char. value $c \in \mathbb{C}$, and a char. vec. α s.t. $T^*\alpha = c\alpha$. By replacing α to $\frac{\alpha}{\|\alpha\|}$, α itself has length 1. Define $W = \operatorname{span}\{\alpha\}^\perp$. Since $\operatorname{span}\{\alpha\}$ is T^*-inv, which leads $W = \operatorname{span}\{\alpha\}^\perp$ is T-inv. by the Lemma 0.5.1. Then we can see

$$\begin{array}{rcccl} T: & V & \longrightarrow & V & \dim(V) = n \\ & \updownarrow & & \updownarrow & \\ T|_W: & W & \longrightarrow & W & \dim(W) = n-1 \end{array}$$

By induction hypothesis, $\exists$ orthonormal basis $\mathfrak{B}' = \{\alpha_1, \dots, \alpha_{n-1}\}$ s.t. $[T|_W]_{\mathfrak{B}'}$ is upper triangular. Take $\alpha_n := \alpha$, and $\mathfrak{B} = \mathfrak{B}' \cup \{\alpha_n\}$. Then

$$[T]_{\mathfrak{B}} = \begin{bmatrix} [T|_W]_{\mathfrak{B}'} & * \\ 0 & * \end{bmatrix}.$$

Thus $[T]_{\mathfrak{B}}$ is upper triangular. □

Corollary 0.5.1

T : endo on f.d.inn. prod. space. $V/\mathbb{C}$ where T is normal. Then V has orthonormal basis consisting of char. vec. of T. In particular, T is diagonalizable.

Corollary 0.5.2

With Theorem 0.5.2 and Theorem 0.5.3, if $A \in M_{n\times n}(\mathbb{C})$, $\exists$ unitary mat. $P \in U(n)$ s.t. $P^{-1}AP$ is upper triangular. In case $AA^* = A^*A$, $P^{-1}AP$ is diagonal, i.e., A is normal implies A is unitary diagonalizable.

Example 0.5.1

T : endo on f.d.inn. prod. space. V/F. If T is hermitian, i.e., self-adjoint, then T is normal. Also, if T is unitary operator, it is normal.