MAS212A Final Exam
(13:00 - 15:45 (165 minutes), December 12th, 2023)

Direction: You should **justify all your answers properly, unless said otherwise**. Total score= 200. (8 problems on 3 pages)

Problem 1. (16 points max) Read the questions and write (True) if the given statement is always true. Otherwise write (False). No justifications needed. You will get 2 points for each correct answer, but (-1) point for each wrong answer.

(1) Let $T : V \to V$ be a linear operator on a finite dimensional vector space over a field F such that $0 \in F$ is a characteristic value of T. Then T is not surjective.

(2) Let $T : V \to V$ be a linear operator on a finite dimensional vector space over a field F such that its minimal polynomial is irreducible in $F[x]$. Then T is triangulable.

(3) Every real symmetric matrix over $\mathbb{R}$ is similar to a diagonal matrix with the entries in $\mathbb{R}$.

(4) If A is an $n \times n$ matrix satisfying $A^2 = I$ over a field F of characteristic 0, then A is diagonalizable.

(5) If A is an $n \times n$ matrix satisfying $A^4 = I$ over $F = \mathbb{R}$, but $A^2 \neq -I$, then A is diagonalizable.

(6) If A is an $n \times n$ matrix satisfying $A^3 = I$ over $F = \mathbb{C}$, then A is diagonalizable.

(7) If A is an $n \times n$ matrix satisfying $A^2 = A$ over a field F of characteristic 0, then considered as a linear transform $A : F^n \to F^n$, we can write $F^n = Null(A) \oplus Range(A)$.

(8) If A is an $n \times n$ matrix that is triangulable over $\mathbb{C}$, then it is also triangulable over $\mathbb{R}$.

Problem 2. (40 points - 10 each) Let V be a finite dimensional inner product space over $F = \mathbb{C}$, and let $T : V \to V$ be a self-adjoint (=hermitian) linear operator, i.e. $T = T^*$. Answer the following questions.

(1) If $c \in F$ is a characteristic value of T, then prove that c is necessarily a real number.

(2) Prove that $\det(T)$ is necessarily a real number.

(3) Let $f(x) \in F[x]$ be the characteristic polynomial of T. Prove that in fact all the coefficients of $f(x)$ are real, so that $f(x) \in \mathbb{R}[x]$.

(4) Prove that the minimal polynomial $m(x)$ of T is a product of distinct linear polynomials in $\mathbb{R}[x]$.

More problems on the next page.

Problem 3. (20 points) Let F be an algebraically closed field and let $V = F^{5}$. Let $T : V \to V$ be an F-linear operator. Let $f(x) = (x-2)^2(x-3)^3$ be the characteristic polynomial of T.

List all possible Jordan forms you can possibly obtain for T.

Problem 4. (24 points - 12 each) For $n \geq 1$, consider the unitary group $U(n) \subset \mathbb{C}^{n\times n}$, i.e. the $n \times n$ matrices A over $\mathbb{C}$ such that $AA^* = A^*A = I_n$. $U(n)$ can be seen also as a $\mathbb{R}$-subspace of $\mathbb{R}^{2n^2}$ via the identification $\mathbb{C} = \mathbb{R}^2$, sending $z = a + ib$ to (a, b).

For some $\epsilon > 0$, consider the open interval $J := (-\epsilon, \epsilon) \subset \mathbb{R}$. Suppose $\gamma : J \to \mathbb{R}^{2n^2}$ is a differentiable curve such that $\gamma(t) \in U(n)$ for all $t \in J$, while $\gamma(0) = I_n$. For such a curve, a tangent vector at $t = 0$ is given by $\gamma'(0) = \frac{d}{dt}\gamma(t)|_{t=0}$.
A tangent vector to $U(n)$ at I_n is defined to be an object of the form $\gamma'(0)$ for some such γ.

We denote by $u(n)$ the set of all tangent vectors to $U(n)$ at I_n.

(1) Prove that each vector in $u(n)$ is anti-self-adjoint (=anti-hermitian) matrices, i.e. $B^* = -B$ in $\mathbb{C}^{n\times n}$.

(2) It is known that $u(n)$ is precisely the set of all anti-self-adjoint matrices (No need to prove it). Compute the dimension of $u(n)$ as a real vector space in terms of n.

Problem 5. (20 points - 5 each) Let V be a finite dimensional vector space over a field F, and let $T : V \to V$ be a linear operator. Let $W \subset V$ be a proper nonzero T-invariant subspace. Answer the following questions.

(1) Suppose that there exists a T-invariant complementary subspace $W' \subset V$ such that $V = W \oplus W'$. Prove that W is T-admissible.

(2) Deduce the converse of (1) using the cyclic decomposition theorem.

(3) This time, suppose $F = \mathbb{R}$ or $\mathbb{C}$ and V is an inner product space. Let $W^\perp$ be the orthogonal complement of W in V. Prove that $W^\perp$ is an T^*-invariant subspace of V.

(4) Continue to suppose that V is an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. Suppose that T is self-adjoint. Prove that a nonzero proper T-invariant subspace $W \subset V$ always has a T-invariant complement.

More problems on the next page.

Problem 6. (30 points - 10 each) Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear operator, with the minimal polynomial

$$m(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0 \in F[x].$$

Suppose that there exists a nonzero $\alpha \in V$ such that V is the cyclic space $Z(\alpha, T)$. Answer the following questions.

(1) Prove that the characteristic polynomial $f(x)$ of T is also $m(x)$.

(2) What is the companion matrix (rational form) of T?

(3) Prove that $\mathrm{Tr}(T) = -a_{r-1}$, and $\det(T) = (-1)^r a_0$.

Problem 7. (30 points - 10 each) Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which can be considered as a 2×2 matrix over various different fields F. For a given field F, let $V = F^2$ and regard $A : V \to V$ as a F-linear transformation. Let $f(x)$ be its characteristic polynomial and $m(x)$ its minimal polynomial. Answer the following questions.

(1) Suppose $F = \mathbb{R}$. Compute $f(x)$ and $m(x)$, and dermine if A is (a) diagonalizable, (b) triangulable but not diagonalizable, or (c) not triangulable. In case of (a) / (b), find a matrix P such that $P^{-1}AP$ is diagonal / upper triangular.

(2) Suppose $F = \mathbb{C}$. Repeat the same questions in this case.

(3) Suppose $F = \mathbb{Z}/2$, the finite field with two elements $\{\bar{0}, \bar{1}\}$. Repeat the same questions in this case.

Problem 8. (20 points - 5 each) Let V be a finite dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$ and let $U : V \to V$ be a linear operator. Answer the following questions.

(1) Suppose U is a normal operator. Prove that $c \in F$ is a characteristic value of U with a characteristic vector $\alpha \in V$ if and only if $\bar{c} \in F$ is a characteristic value of U^* with α as a characteristic vector.

(2) Suppose U is a unitary operator. Prove that U is a normal operator, while prove that (i) all the characteristic values and (ii) $\det(U)$ are complex numbers z whose size $|z| = 1$.

(3) Suppose U is a self-adjoint unitary operator. Enumerate all possible characteristic values c of U.

(4) For $F = \mathbb{C}$, give concrete example of a normal operator that is neither self-adjoint, nor unitary.

End of the Exam.