0.1 Introduction

0.2 Characteristic Values

Definition 0.2.1: Characteristic Value and Vectors, Spaces

T : endo. on f.d.v.s V/F. A characteristic value of T is $c \in F$ s.t. $\exists \alpha \in V \backslash \{0\}$ s.t. $T\alpha = c\alpha$. This α is also called a characteristic vector of T associated to c. Also, $E_c := \{\alpha \in V \mid T\alpha = c\alpha\}$ is called the characteristic space of T associated to c.

Theorem 0.2.1

T : endo. on f.d.v.s. V/F. TFAE:

i) c is a characteristic value of T

ii) Operator $T - cI$ is singular (not invertible)

iii) $\det(T - cI) = 0$

Proof. ii) $\iff$ iii) is trivial. If i) holds, $\exists v \in V \backslash \{0\}$ $(Tv = cv) \Rightarrow (T - cI)v = 0$. Thus this is not injective, so singular. Thus i) $\iff$ ii). □

Definition 0.2.2: Characteristic Polynomials

$f(x) := \det(xI - A) \in F[x]$ is called characteristic polynomial of T. Then f is monic with $\deg(f) = n$ for $n \times n$ mat. A and $\forall c$ which is characteristic values, $f(c) = 0$.

Exercise 0.2.1

Check the choice of basis doesn't affect the char. poly. of T.

Proof. $B := P^{-1}AP$. $\det(xI - B) = \det(xI - P^{-1}AP) = \det(P^{-1}(xI - A)P) = \det(xI - A)$. □

Definition 0.2.3: Diagonalizable

T : endo. on f.d.v.s. V/F. If $\exists \mathfrak{B} = \{v_1, v_2, \ldots, v_n\}$ s.t. each v_i are char. vec. of T, we say T is diagonalizable.

Note:-

$[T]_{\mathfrak{B}} = \begin{bmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{bmatrix}$ with (may be) repititions. Then $[T]_{\mathfrak{B}}$ is diagonal mat. Furthermore, we can see $f(x) = \det(xI - [T]_{\mathfrak{B}})$ is decomposed complety into a product of linear factors.

Example 0.2.1

$A : n \times n$ mat. on f.d.v.s. $V/\mathbb{R}$. If char. poly. has no real sol., then it is not diagonalizable.

Lemma 0.2.1

T : endo. on f.d.v.s. V/F. Suppose $c_1, c_2, \dots, c_k$ are all possible distinct char. values of T and $W_i := \text{Null}(T - c_i I)$. Then $W := W_1 + \cdots + W_k \Rightarrow \dim(W) = \dim(W_1) + \cdots + \dim(W_k)$.

Proof. Trivially $\dim(W) \le \dim(W_1) + \cdots + \dim(W_k)$. Thus we have to check $\ge$ part. Suppose $\forall \beta_i \in W_i$ $(\beta_1 + \cdots + \beta_k = 0)$. We will show $\forall \beta_i = 0$. Suppose $\beta_1 + \beta_2 = 0$. Then $T\beta_1 + T\beta_2 = c_1\beta_1 + c_2\beta_2 = 0$. We can derive $(c_1 - c_2)\beta_2 = 0$. Since $c_1 \neq c_2$, $\beta_2 = 0$ thus $\beta_1 = 0$. Inductively, we can derive $\forall \beta_i = 0$. Thus $\dim(W) = \dim(W_1) + \cdots + \dim(W_k)$. $\square$

Theorem 0.2.2

T : endo. on n-d.v.s. V/F. $c_1, c_2, \dots, c_k$ are all possible distinct char. values of T and $W_i := \text{Null}(T - c_i I)$. TFAE:

i) T is diagonalizable

ii) Char. poly. $p(x) = \prod_{i=1}^{k}(x - c_i)^{d_i}$ where $d_i = \dim(W_i)$

iii) $d_1 + d_2 + \cdots + d_k = n = \dim(V)$

Proof. i)$\Rightarrow$ ii): $\exists \bigcup_{i=1}^{k} \mathfrak{B}_i$, basis of V where each $\mathfrak{B}_i$ are the part belonging to c_i. Then, $\text{span}(\mathfrak{B}_i) = W_i$, $\dim(W_i) = d_i \Rightarrow p(x) = \prod_{i=1}^{k}(x - c_i)^{d_i}$ where $d_i = \dim(W_i)$.

ii)$\Rightarrow$ iii): Trivial.

iii)$\Rightarrow$ i): $W_1 + \cdots + W_k = W \Rightarrow d_1 + \cdots + d_k = n$. Thus $W = V$. Thus V has a basis consisting of char. vec., so diagonalizable. $\square$

0.3 Annihilating Polynomials

Theorem 0.3.1

T : endo. on n-d.v.s. V/F. $p(x)$ as char. poly. of T, and $m(x)$ as min. poly. of T. Ignoring multiplicities, $p(x)$ and $m(x)$ has same sol. in F.

Proof. $m(c) = 0 \Rightarrow m(x) = (x - c)q(x)$. m is minimal implies $q(T) \neq 0$. Thus $\exists \beta \in V$ s.t. $q(T)\beta \neq 0$. This leads $(T - cI)q(T)\beta = 0$ since $(T - cI)q(T)\beta = m(T)\beta = 0\beta$. Thus $q(T)\beta$ is char. vec., which leads c as a char. value of T, so $p(c) = 0$.
Now if $p(c) = 0$, $\exists \alpha \in V \backslash \{0\}$ s.t. $T\alpha = c\alpha$. Thus $T^n\alpha = c^n\alpha$. So for any poly. $f(x) \in F[x]$, $f(T)\alpha = f(c)\alpha$. In particular, $m(T)\alpha = m(c)\alpha \Rightarrow m(c)\alpha = 0\alpha \Rightarrow m(c) = 0$. $\square$

Corollary 0.3.1

$p(x) = \prod_{i=1}^{k}(x - c_i)^{d_i} \Rightarrow m(x) = \prod_{i=1}^{k}(x - c_i)^{r_i}$ where $1 \le r_i \le d_i$.

Theorem 0.3.2 Cayley-Hamilton

T : endo. on n-d.v.s. V/F. $p(x)$ as char. poly. of T. Then $p(T) = 0$. In particular, $m(x) | p(x)$.

Proof. $K := \{h(T) \mid h(x) \in F[x]\}$ be image of $ev_T : F[x] \to L(v, v)$. Let $\mathfrak{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis of V. $A := [T]_{\mathfrak{B}}$ so that $T\alpha_i = \sum_{j=1}^{n} A_{ji}\alpha_j$ $(i \in [n]) \Rightarrow \sum_{j=1}^{n}(\delta_{ij}T - A_{ji}I)\alpha_j =$

0. Then $B := [B_{ij}]$ where $B_{ij} := (\delta_{ij}T - A_{ji}I)$. We know $\operatorname{adj}(B)\cdot B = B\cdot\operatorname{adj}(B) = \det(B)I$. By construction, $\sum_{j=1}^{n} B_{ij}\alpha_j = 0 \Rightarrow \sum_{j=1}^{n}\operatorname{adj}(B)_{ki}B_{ij}\alpha_j = 0$. Taking sums over i leads $0 = \sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{adj}(B)_{ki}B_{ij}\alpha_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\operatorname{adj}(B)_{ki}B_{ij}\right)\alpha_j = \sum_{j=1}^{n}\delta_{kj}\det(B)\alpha_j = \det(B)\alpha_k$. Since $\{\alpha_1,\ldots,\alpha_n\}$ is basis, $\det(B) = 0$, which is char. poly. of T. ☐

0.4 Invariant Subspaces

Theorem 0.4.1

T : endo. on f.d.v.s. V/F. $c_1, c_2, \ldots, c_k$ are all possible distinct char. values of T. Then T is diagonalizable $\iff m(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$.

Proof. Only for ($\Rightarrow$) here: Let $f(x)$ be a char. poly. of T. Then $m(x) \mid f(x)$. Thus $m(x) = (x-c_1)^{e_1}(x-c_2)^{e_2}\cdots(x-c_k)^{e_k}$.

Claim 0.4.1

$(T-c_1I)(T-c_2I)\cdots(T-c_kI) = 0$

Proof. Since T is diagonalizable, it has a basis $\{\alpha_1,\ldots,\alpha_n\}$ consisting of char. vec. Thus $T\alpha_j = c_{i(j)}\alpha_j$ where $c_{i(j)} \in \{c_1,\ldots,c_k\}$. This leads $(T - c_{i(j)}I)\alpha_j = 0$. Take $S := (T-c_1I)\cdots(T-c_kI)$. Then for each $j \in [n]$, $S(\alpha_j) = 0$. since each α_i form basis, $\forall v \in V\ (S(v) = 0)$. Thus Claim 0.4.1 holds. ☐

Oppisite of this proof is at Theorem 0.4.3. ☐

Corollary 0.4.1

T : endo. on n-d.v.s. V/F. Suppose T has n distinct char. values. If $f(x) = \prod_{i=1}^{n}(x-c_i)$ where distinct c_i, then $m(x) = f(x)$ thus it is diagonalizable.

Definition 0.4.1: T-Invariant Subspaces

T : endo. on n-d.v.s. V/F. Take subspace W. We say W is T-invariant or invariant under T if $T(W) \subset W$. If W is T-invariant, then T induces a endo. on W, denoted as $T|_W$.

$$\begin{array}{rccc} T: & V & \longrightarrow & V \\ & \updownarrow & & \updownarrow \\ T|_W: & W & \longrightarrow & W \end{array}$$

Example 0.4.1

$W = 0$ is trivailly T-invariant. Also, char. space E_c is T-invariant.

Lemma 0.4.1

Suppose W is T-invariant. $m(x)$ as min. poly. and $f(x)$ as char. poly. of T. Then $m_W(x) \mid m(x)$ and $f_W(x) \mid f(x)$ for each restriction to W.

Proof. Choose a basis $\mathfrak{B}' = \{\alpha_1, \dots, \alpha_k\}$ of W and extend it to $\mathfrak{B} = \{\alpha_1, \dots, \alpha_k, \alpha_{k+1} \dots, \alpha_n\}$ which is a basis of V. Since W is T-inv., $T\alpha_i \in \text{span}\{\mathfrak{B}'\}$. So $A = [T]_{\mathfrak{B}} = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ where $B = [T|_W]_{\mathfrak{B}'}$. Furthermore, $f(x) = \det(xI - A) = \det(xI - B) \cdot \det(xI - D)$. clearly, $f_W(x) \,|\, f(x)$.

Note that $A^r = \begin{bmatrix} B^r & C_r \\ 0 & D^r \end{bmatrix}$. Therefore, $\forall p(x) \in F[x]$ $(p(T) = 0)$, we can see $p_W(x) \,|\, p(x)$. Especially, $m_W(x) \,|\, m(x)$. □

Definition 0.4.2: T-Conductors

T : endo. on f.d.v.s. V/F. W be T-inv. subspaces. Suppose $\alpha \in V$. We define T-conductor as $S_T(\alpha\,; W) := \{g(x) \in F[x] \,|\, g(T)\alpha \in W\}$.

Lemma 0.4.2

$S_T(\alpha\,; W)$ is a nonzero ideal.

Proof. char. poly. $f(x)$ satisfies $f(T) = 0 \in W \Rightarrow f(x) \in S_T(\alpha\,; W)$. Trivially it is closed. Also, since polynomials are commutative and W is T-inv., it satisfies properties of ideals. □

Definition 0.4.3: T-Conductor as Generator

The unique monic poly. generator of $S_T(\alpha\,; W)$ is also often called the T-conductor of α to W.

Corollary 0.4.2

Min. poly. and char. poly. is in $S_T(\alpha\,; W)$, thus generator of that conductor divides both.

Definition 0.4.4: Triangulable

T : endo. on f.d.v.s. V/F. We say T is triangulable if V has a basis $\mathfrak{B}$ s.t. $[T]_{\mathfrak{B}}$ is an upper triangular mat.

Corollary 0.4.3

T is diagonalizable $\Rightarrow$ T is triangulable.

Lemma 0.4.3

T : endo. on f.d.v.s. V/F. Suppose $m(x) = \prod_{i=1}^{k}(x - c_i)^{r_i}$ where c_i are all distinct and $r_i \geq 1$. If W is T-inv. subspace, then $\exists \alpha \in V \backslash W$ $((T - cI)\alpha \in W)$ for some char. value $c = c_i$.

Proof. Let $\beta \in V \backslash W$ and let $g(x)$ be the min. T-conducting poly. taking β to W. Then $g(x) \,|\, m(x)$. Since $\beta \notin W$, $\deg(g(x)) \geq 1$. Then $g(x) = \prod_{i=1}^{k}(x - c_i)^{e_i}$ for $e_i \leq r_i$. since $\deg(g) \geq 1$, $\exists j$ $(e_j \geq 1)$, so $(x - c_j) \,|\, g(x) \Rightarrow g(x) = (x - c_j)h(x)$. $\alpha := h(T)\beta$. This cannot be in W since $g(x)$ is the min. deg. fellow in $S_T(\beta\,; W)$. But $(T - c_j I)\alpha = g(T)\beta \in W$. Thus $(x - c_j) = S_T(\alpha\,; W)$. □

Theorem 0.4.2

T : endo. on n-d.v.s. V/F. T is triangulable $\iff$ $m(x) = \prod_{i=1}^{k}(x - c_i)^{r_i}$ for $r_i \geq 1$.

Proof. ($\Rightarrow$): Since T is triangulable, $\exists\mathfrak{B}$ s.t. $[T]_{\mathfrak{B}}$ is triangular. Thus char. poly. $f(x) = \prod_{i=1}^{k}(x-c_i)^{e_i}$ for $\sum e_i = n$, $e_i \geq 1$ and distinct c_i. Since $m(x)\,|\,f(x)$ our statement holds.

($\Leftarrow$): Suppose $m(x) = \prod_{i=1}^{k}(x-c_i)^{r_i}$. We use the Lemma 0.4.3 repeatedly over different choices of W. Take $W=0$ then $\exists\alpha_1 \in V\backslash W$ $((T-d_1)\alpha_1 = 0)$ for some d_1. Take $W_1 = \operatorname{span}\{\alpha_1\}$. Then $\exists\alpha_2 \in V\backslash W_1$ $((T-d_2)\alpha_2 = 0)$. Repeating this, we can derive $T\alpha_1 = d_1\alpha_1$, $T\alpha_2 = *\alpha_1 + d_2\alpha_2$, and so on, thus $[T]_{\{\alpha_1,\ldots,\alpha_n\}} = \begin{bmatrix} d_1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & d_k \end{bmatrix}$, which is upper triangular mat. $\square$

> **Theorem 0.4.3**
>
> T : endo. on n-d.v.s. V/F. T is diagonalizable $\iff$ $m(x) = (x-c_1)(x-c_2)\cdots(x-c_k)$.

Proof. Forward is at Theorem 0.4.1. ($\Leftarrow$): Let $W \subset V$ be subspace spanned by all char. vec. Suppose $W \subsetneq V$ toward contradiction. Since $T\alpha = c\alpha$ for each char. vec. α of T, W is T-inv. So by Lemma 0.4.3, $\exists\alpha \in V\backslash W$ $((T-c_jI)\alpha =: \beta \in W)$. Note that $\beta \in W\backslash\{0\}$. So we can write $\beta = \beta_1 + \cdots + \beta_k$ where $\beta_i \in E_{c_i}$. Here, $T\beta_i = c_i\beta_i$, and $T^k\beta_i = c_i^k\beta_i$. Thus $f(T)\beta = f(T)\beta_1 + \cdots + f(T)\beta_k$. $m(x) := (x-c_j)h(x)$ where $h(x) = \prod_{i\neq j}(x-c_i)$. Clearly $h(c_j) \neq 0$. Consider $h(x) - h(c_j) = (x-c_j)q(x) \Rightarrow h(T)\alpha - h(c_j)\alpha = q(T)(T-c_jI)\alpha = q(T)\beta \in W$. Also, $m(T)\alpha = (T-c_jI)h(T)\alpha = 0 \Rightarrow h(T)\alpha \in E_{c_j} \subset W$. Thus $(h(T)\alpha \in W) \wedge (q(T)\beta \in W)$ implies $h(c_j)\alpha \in W$, so $h(c_j) = 0$. This is contradiction to the fact that min. poly. has distinct roots, so $W = V$, which means V has basis consisting of char. vec., and T is diagonalizable. $\square$

> **Corollary 0.4.4**
>
> If F is algebraically closed, then T is always triangulable.

0.5 Simultaneous Triangulation; Simultaneous Diagonalization

> **Definition 0.5.1: Commuting Family**
>
> T_i : endo. on n-d.v.s. V/F. We say $\mathscr{F}$ is a commuting family of endo. if $\forall T_i, T_j \in \mathscr{F}$ $(T_iT_j = T_jT_i)$.

> **Definition 0.5.2: $\mathscr{F}$-Invariant**
>
> If $\forall T_i \in \mathscr{F}$ ($W$is T_i-invariant), then we say W is $\mathscr{F}$-inv.

> **Lemma 0.5.1**
>
> Suppose $\mathscr{F}$ is a commuting family of triangulable endo. Suppose $W \subsetneq V$, which is $\mathscr{F}$-inv. Then $\exists\alpha \in V\backslash W$ $(\forall T_i \in \mathscr{F}$ $((T_i - cI)\alpha \in \operatorname{span}\{W,\alpha\}))$.

Proof. We may assume $\{T_1,\ldots,T_r\}$, a maximal lin. indep. subset of $\mathscr{F}$. Applying Lemma 0.4.3 to T_1, $\exists\beta_1 \in V\backslash W$ $\exists c_1 \in F$ $((T_1-c_1I)\beta_1 \in W)$. Let $V_1 = \{\beta \in V \mid (T_1-c_1I)\beta \in W\}$. $\beta_1 \in V_1$, so it is nonempty and $W \subsetneq V_1 \subset V$. Here, by construction, V_1 is $\mathscr{F}$-inv. since $\forall T_i \in \mathscr{F}$ $((T_1-c_1I)T\beta = T(T_1-c_1I)\beta \in W)$.

Now, take $V_1 \subset V$ and let $U_2 := T_2|_{V_1}$. Applying Lemma 0.4.3 to $V_1 \backslash W$ and U_2, $\exists \beta_2 \in V_1 \backslash W$ $\exists c_2 \in F$ $((T_2 - c_2 I)\beta_2 \in W)$. So, $\beta_2 \notin W$, $(T_1 - c_1 I)\beta_2 \in W$, $(T_2 - c_2 I)\beta_2 \in W$. Take $V_2 = \{\beta \in V_1 \mid (T_2 - c_2 I)\beta \in W\}$. Then $(\beta_2 \notin W) \wedge (\beta_2 \in V_2)$. By repeating, we can get $W \subsetneq \cdots \subset V_1 \subset V$. Thus terminates in finite steps since $\dim(V) < \infty$. □

Corollary 0.5.1

V: f.d.v.s./F and $\mathscr{F}$ as comuuting family of triangulable endo. Then $\exists \mathfrak{B}$ s.t. $[T_i]_{\mathfrak{B}}$ are all upper triangular mat.

Proof. Exercise. Use our argument for a single operator and use Lemma 0.4.3 for commuting families. □

Corollary 0.5.2

V: f.d.v.s./F and $\mathscr{F}$ as comuuting family of diagonalizable endo. Then $\exists \mathfrak{B}$ s.t. $[T_i]_{\mathfrak{B}}$ are all diagonal mat.

Corollary 0.5.3

Suppoer F is algebraically closed and $\mathscr{F}$ as commuting family of endo. Then $\exists$ simultaneously triangulating basis.

0.6 Direct-Sum Decompositions

Definition 0.6.1: Independent

V: v.s./F. We say subspaces, just say W_i, are indep. if there common elements are just 0.

Definition 0.6.2: Internal Direct Sum

If $W = \sum_{i=1}^k W_i$ and each W_i are indep., then we say the sum is direct and we write it as $W = \bigoplus_{i=1}^k W_i$.

Exercise 0.6.1

If $W = \bigoplus_{i=1}^k$, then $\exists!$ expression of $w \in W$ w.r.t. each $w_i \in W_i$.

Definition 0.6.3: Projection

V: f.d.v.s./F. Supopose we have endo. $E : V \to V$ s.t. $E^2 = E$. Then we say E is a projection.

Example 0.6.1

$V := V_1 \oplus V_2$. $P_1 : V \mapsto V_1$ and $P_2 : V \mapsto V_2$. Then those classical 'projection' is actually a projection we defined above.

Lemma 0.6.1

Let E be a projection. Then for $V := V_1 \oplus V_2$ and $P_1 : V \mapsto V_1$, E really is a classical 'projection', i.e., $E = P_1 : V \mapsto V_1$.

Proof. $V_1 := R(E)$, $V_2 := N(E)$.

Claim 0.6.1

$V = V_1 \oplus V_2$

Proof. Let $v \in V$. Then $v = E(v)+v-E(v)$. $E(v) \in R(E)$. Also, $E(v-E(v)) = E(v)-E^2(v) = 0$, so $(v-E(v)) \in N(E)$. Thus $V = R(E)+N(E)$. To show this is direct, suppose we have $v_1+v_2 = 0$ for $(v_1 \in R(E)) \wedge (v_2 \in N(E))$. Then $v_1 = -v_2 \in R(E)$ and $\exists \alpha \in V \ (v_1 = R(\alpha))$. $E(v_1) = -E(v_2) = 0$ and $E(v_1) = E^2(\alpha) = E(\alpha) = v_1$. Since $E(v_1) = 0$, $v_1 = 0$. Thus $v_2 = 0$, which leads sum is direct. □

Now if $v \in V_1 \oplus V_2$, write $v = v_1 + v_2$, then $E(v) = E(v_1) = v_1$. So $E = P_1$. □

Theorem 0.6.1

V: f.d.v.s./F and $V = \bigoplus_{i=1}^{k} W_i$. Then $\exists E_i : V \mapsto W_i$ s.t.

i) Each E_i are projection

ii) $\forall i \neq j \ (E_i E_j = 0)$

iii) $I = \sum E_i$

iv) The range of E_i is W_i

Converse also holds. Furthermore, only i), ii), and iii) leads our theorem.

Proof. i), ii), and iv) are trivial by definition. For iii), take $\alpha \in V$. $\alpha = \sum E_i \alpha \Rightarrow I = \sum E_i$. Conversely, suppose we have E_i $i \in [k]$ s.t. they satisfy those first three conditions. We can take W_i as $R(E_i)$. Then, $V = W_1 + \cdots + W_k$. We have to show this is direct. By iii), we have $\alpha = \sum E_i \alpha$. This expression is unique since if $\alpha = \alpha_1 + \cdots + \alpha_k$ for $\alpha_i \in W_i$, then using i) and ii), we can derive $E_j \alpha = \sum_{i=1}^{k} E_j \alpha_i = E_j^2 \beta_j = E_j \beta_j = \alpha_j$ if we take $\alpha_i = E_i \beta_i$. □

0.7 Invariant Direct Sum

Theorem 0.7.1

T : endo. on n-d.v.s. V/F. $V = \bigoplus_{i=1}^{k} W_i$. Let $E_i : V \mapsto V$ be projection to W_i. Then W_i are T-inv. $\iff$ T commutes with E_i.

Proof. ($\Leftarrow$): Suppose T commutes with all E_i. Let $\alpha_i \in W_i = R(E_i)$. N.T.S. $T\alpha_i \in W_i$. We can write $\alpha_i = E_i \beta$. So $T\alpha_i = TE_i\beta = E_i T\beta$, which leads $T\alpha_i \in R(E_i) = W_i$. Since α_i was arbitrary element is W_i, W_i is T-inv.

($\Rightarrow$): Let $\alpha \in V$. We can say $\alpha = v_1 + \cdots + v_k$ for each $v_i \in W_i$ uniquely. $W_i := R(E_i)$, so each $v_i = E_i(\alpha)$. So $\alpha = E_1(\alpha) \cdots + E_k(\alpha) \Rightarrow T\alpha = TE_1(\alpha) + \cdots TE_k(\alpha)$. Since $E_i(\alpha) \in W_i$ is T-inv., $T(E_i\alpha) = E_i(\beta_i) \in W_i \Rightarrow T\alpha = E_1(\beta_1) + \cdots + E_k(\beta_k)$. For $i \neq j$, $E_j T E_i \alpha = E_j E_i \beta_i = 0$. For $i = j$, $E_j T E_j \alpha = E_j \beta_j$. Thus $E_j T\alpha = E_j T E_1 \alpha + \cdots + E_j T E_k \alpha = E_j \beta_j = T E_j \alpha$. Thus $E_j T = T E_j$ since α is arbitrary. □

Theorem 0.7.2

T : endo. on n-d.v.s. V/F. If T is diagonalizable and if $c_1,\dots,c_k$ are the distinct char. values of T, then $\exists E_i$ on V s.t.

i) $T = c_1E_1 + \cdots + c_kE_k$

ii) $I = \sum E_i$

iii) $\forall i \neq j\ (E_iE_j = 0)$

iv) $E_i^2 = E_i$

v) The range of E_i is the char. space for T associated with c_i

Converse also holds. Furthermore, only i), ii), and iii) leads our theorem.

Proof. ($\Rightarrow$): Suppose diagonalizable with char. values c_i. $W_i := E_{c_i} = N(T - c_iI)$. Since T is diagonalizable, $V = \bigoplus_{i=1}^k W_i$. Thus ii)$\sim$v) are trivial. Now, $\alpha = \sum E_i\alpha \Rightarrow T\alpha = \sum TE_i\alpha = \sum T\alpha_i = \sum c_i\alpha_i = \sum c_iE_i\alpha$. Since α is arbitrary, $T = \sum c_iE_i$.

($\Leftarrow$): Using ii) and iii) to obtain iv). using i) and iv) to obtain $R(E_i) \subset N(T - c_iI)$. Since we assumed $E_i \neq 0$, c_i is char. value of T. Take $i) - c \times ii)$. Then $(T - cI) = (c_1 - c)E_1 + \cdots + (c_k - c)E_k$. so if $(T - cI)\alpha = 0$, we must have $(c_i - c)E_i\alpha = 0$. If $\alpha \neq 0$, then $E_i\alpha \neq 0$ for some i, so in this case, $c_i = c$. Certainly T is diagonalizable, since every nonzero vector in $R(E_i)$ is a char. vec. of T, and $I = \sum E_i$ shows these char. vec. span V. Now we have to show $N(T - c_iI) = R(E_i)$. This is clear since if $T\alpha = c_i\alpha$, then $\sum_{j=1}^k (c_j - c_i)E_j\alpha = 0$ hence $(c_j - c_i)E_j\alpha = 0$ for each j, and then $E_j\alpha = 0$ for $j \neq i$. Since $\alpha = \sum E_i\alpha$ and $E_j\alpha = 0$ for $j \neq i$, $\alpha = E_i\alpha$, which shows $\alpha \in R(E_i)$. □

0.8 The Primary Decomposition Theorem

Theorem 0.8.1 Primary Decomposition Theorem

T : endo. on f.d.v.s. V/F. $\exists$ a decomposition of V into $V = \bigoplus_{i=1}^k W_i$ s.t. $W_i = N(p_i(T)^{r_i})$ where $m(x) = \prod_{i=1}^k p_i(x)^{r_i}$ for $r_i \geq 1$ and irreducible, distinct p_i. Also, each W_i are T-inv., and $T_i := T|_{W_i}$ has min. poly. $p_i(T)^{r_i}$.

Proof. When $k = 1$, it is trivial. Suppose $k > 1$. Define $f_i(x) := \frac{m(x)}{p_i(x)^{r_i}} = \prod_{j\neq i} p_j(x)^{r_j}$. Then $\gcd(f, p_i^{r_i}) = 1$. Since each f_i are also relatively prime, $\exists g_1,\dots,g_k$ $(f_1g_1+\cdots+f_kg_k = 1)$. Define $h_i(x) := f_i(x)g_i(x)$. For $i \neq j$, $m \mid f_if_j$ thus $f_i(T)f_j(T) = 0$. Note that $\sum h_i(T) = I$. Define $E_i := h_i(T)$. Then $\sum E_i = I$ and $\forall i \neq j\ (E_iE_j = 0)$ since $E_iE_j = f_i(T)g_i(T)f_j(T)g_j(T) = 0$. Thus we can see E_i are projection. Thus $V = \bigoplus_{i=1}^k R(E_i)$ and each $R(E_i)$ are T-inv.

Claim 0.8.1

$R(E_i) = W_i = N(p_i(T)^{r_i})$

Proof of Claim 0.8.1. Let $\alpha \in R(E_i)$. Then $\alpha = E_i\alpha \Rightarrow p_i(T)^{r_i}\alpha = p_i(T)^{r_i}f_i(T)g_i(T)\alpha = 0$ since $p_i(T)^{r_i}f_i(T)g_i(T) = m(T)g_i(T) = 0$. Thus $R(E_i) \subset N(p_i(T)^{r_i})$. Conversely, let $\alpha \in N(p_i(T)^{r_i})$. Note that if $i \neq j$, $p_i^{r_i} \mid f_j$ thus $p_i^{r_i} \mid f_jg_j = h_j$, thus $f_j(T)g_j(T)\alpha = h_j(T)\alpha = 0$.

In other words, $\forall i \neq j$, α is in V whose projection about E_j is 0. Thus α has only $R(E_i)$ component. Thus $N(p_i(T)^{r_i}) \subset R(E_i)$, consequently $R(E_i) = N(p_i(T)^{r_i})$. □

Now we have to show T_i has min. poly. as $p_i(x)^{r_i}$. Note that $W_i = N(p_i(T)^{r_i})$ implies $p_i(T)^{r_i}|_{W_i} = 0$. Thus $m_i(x) \mid p_i(x)^{r_i}$. So $m_i(x) = p_i^{s_i}$ for $1 \leq s_i \leq r_i$. E.T.S. $s_i = r_i$. Let $g(x)$ be poly. s.t. $g(T_i) = 0$.

Claim 0.8.2

$p_i(x)^{r_i} \mid g(x)$

Proof of Claim 0.8.2. $g(T_i) = 0 \iff g(T)f_i(T) = 0$. So min. poly. of T divides $g(x)f_i(x)$. Since $\gcd(p_i^{r_i}, f_i) = 1$, $m(x) \mid g(x)f_i(x)$ leads $p_i^{r_i} \mid g(x)$. In particular, $m_i(x)$ is divisible by $p_i^{r_i}$, thus $r_i = s_i$. □

□

Corollary 0.8.1

$E_1, \ldots, E_k$ be projection associated to primary decomposition of V w.r.t. T. Then each E_i is a poly. in T. In particular, if $U : V \mapsto V$ is another endo. commuting with T, then, U commutes with each E_i so W_i are U-inv.

Theorem 0.8.2

T : endo. on f.d.v.s. V/F. If T is triangulable, $\exists$ diagonalizable D and nilpotent N s.t. $T = D + N$ and $DN = ND$. Such D and N are uniquely determined by T.

Proof. $m(x) = \prod(x - c_i)^{r_i}$ for distinct c_i. Take $R(E_i) = W_i := N((T - c_i I)^{r_i})$ as like Theorem 0.8.1. Take $D := \sum c_i E_i$ and $N = T - D$.

Claim 0.8.3

N is nilpotent

Proof for Claim 0.8.3. $I = \sum E_i \Rightarrow T = \sum TE_i \Rightarrow N = T - D = \sum(T - c_i I)E_i$. Since each E_i are poly. in T and $E_i E_j = 0$, $N^r = \sum(T - c_i I)^r E_i$. By choosing $r = \max(r_1, \ldots, r_k)$, $N^r = 0$. □

D and N are commute since they are poly. in T. Thus existence is proven.

For uniqueness, suppose we have $T = D' + N' = D + N$. Then $D - D' = N' - N$. We know $D - D'$ is diagonalizable. Now suppose $N^r = N'^{r'} = 0$. Then $(N' - N)^A = \sum_{i=0}^{A} \binom{A}{i} N'^i N^{A-i}$. Taking $A > r + r'$ leads $(N' - N)^A = 0$. Take $\alpha := N' - N = D' - D$. Then α is diagonalizable and nilpotent, which leads $\alpha = 0$. Thus $D = D'$ and $N = N'$. □