MAS 250 – MIDTERM

October 19, 2023 Time allowed: 165 minutes

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Honor Code: I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others.

Instructions: Show all your work on your solutions. You will NOT receive credit if you do not justify your answers.

Good luck!

1. True or False. If False, explain why it is false.

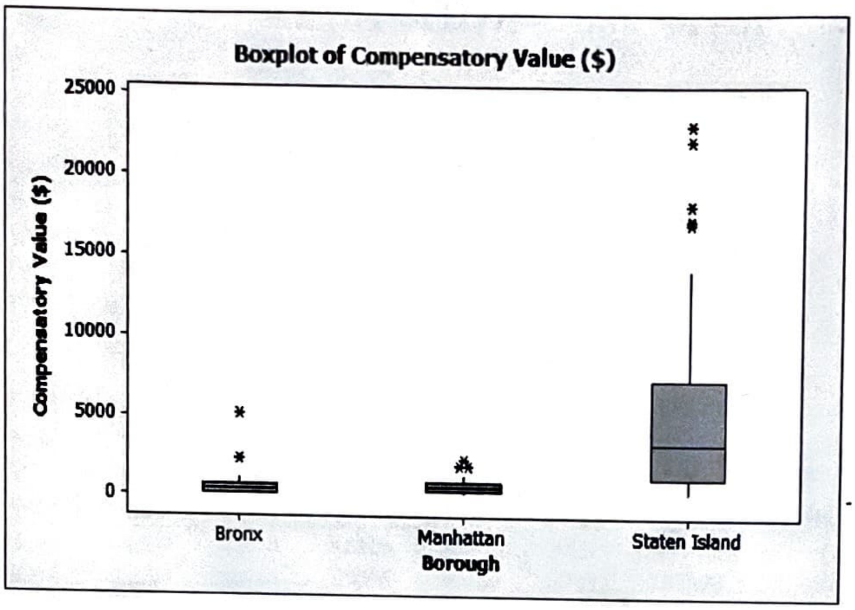
   (a) (3 points) It is claimed that the average life span of a new bulb is at least 1000 hours. Therefore, 100 new bulbs are selected and investigated. Then, the population is a collection of life span of the 100 selected new bulbs.

   (b) (3 points) To select 10,000 college students in Korea, a survey company randomly selected 100 universities first and then randomly selected 100 students from the 100 selected universities. This sampling method is called the stratified sampling.

   (c) (3 points) The empirical rule (e.g. approximately 95% of the data fall within sample mean $\pm$ $2\times$ sample standard deviation) assumes that the data distribution is bell-shaped.

   (d) (3 points) In the boxplots below, Staten Island has larger median value and variation, and more outliers compared to the other two regions.

   (e) (3 points) For two random variables X and Y, $Cov(X, Y) = 0$ implies that X and Y are independent.



2. Two boxes containing marbles are placed on a table. The boxes are labeled B_1 and B_2. Box B_1 contains 7 green marbles and 4 white marbles. Box B_2 contains 3 green marbles and 10 yellow marbles. The boxes are arranged so that the probability of selecting box B_1 is 1/3 and the probability of selecting box B_2 is 2/3. Liam is blindfolded and asked to select a marble. He will win a mobile phone if he selects a green marble.

   (a) (4 points) What is the probability that Liam will win the mobile phone?

   (b) (4 points) Given that Liam won the mobile phone, what is the probability that the green marble was selected from the first box?

3. (a) (4 points) When the moment generating function of X is $(.2 + .8e^t)^5$, calculate the exact probability of $P(\mu - \sigma < X < \mu + \sigma)$ using the distribution of X where $\mu = E(X)$ and $\sigma^2 = V(X)$.

   (b) (4 points) When the moment generating function of X is $e^{(e^t - 1)}$, calculate the exact probability of $P(\mu - \sigma < X < \mu + \sigma)$ using the distribution of X.

4. The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2, as measured on the Richter scale.

   (a) (4 points) Find the probability that an earthquake striking this region will exceed 5 on the Richter scale.

   (b) (4 points) Of the next ten earthquakes to strike this region, what is the probability that at least one will exceed 5 on the Richter scale?

5. (8 points) A couple decides to have children until they get a girl, but they agree to stop with a maximum of 3 children even if they have not gotten a girl. Assume that the probability of having a girl is 0.52. If X and Y denote the number of children and number of girls, respectively, then what are $\mathrm{E}(X)$ and $\mathrm{E}(Y)$?

6. Let $N(t)$ follows a Poisson process with $\lambda = 2$, and let $X_1, X_2, \ldots$ be the corresponding interarrival times.

   (a) (4 points) Find the probability that the first arrival occurs after $t = 0.5$?

   (b) (4 points) Given that we have had no arrivals before $t = 1$, find the probability that the first arrival occurs after $t = 3$.

7. The joint density function of X and Y is given by

$$f(x,y) = \begin{cases} kx, & 0 < y < 2x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) (3 points) Determine k.

   (b) (4 points) Derive the marginal density of X and Y.

   (c) (4 points) Find $P(Y \leq 1/3 | X = 1/4)$.

   (d) (4 points) Calculate $E(XY)$.

   (e) (3 points) Are X and Y independent? Why or why not?

8. The weekly amount of money spent on maintenance and repairs by a company is observed, over a long period of time, to be normal distributed with mean \$500 and standard deviation \$120. Assume that each week spending is independent and \$550 is budgeted for each week.

   (a) (3 points) What is the probability that the actual cost on a randomly selected week will exceed the budgeted amount?

   (b) (4 points) Find the probability that the mean weekly cost (of four weeks) of a randomly selected month exceeds the budgeted weekly amount.

   (c) (4 points) Assume that this budget plan will be sustained next three years (36 months). What is the probability that the mean weekly cost will exceed the budgeted weekly amount in more than 10 months during the next three years? Use the normal approximation to answer the question.

9. Suppose that $X_1, X_2, \ldots, X_{n+1}$ are independent and each $N(\mu, \sigma^2)$. Y and U are also independently sampled from a normal distribution with mean μ and variance σ^2. Identify the distribution of each question. Clearly justify your answer.

   (a) (4 points) Show that the distribution of

   $$X_{n+1} - \bar{X}_n \sim N\left(0, \frac{n+1}{n}\sigma^2\right)$$

   where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

   (b) (4 points) Identify the distribution of

   $$\frac{n}{n+1}\frac{(X_{n+1} - \bar{X}_n)^2}{\sigma^2}.$$

   (c) (4 points) Let

   $$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.$$

   Identify the distribution of

   $$\frac{X_{n+1} - \bar{X}_n}{\sqrt{\frac{n+1}{n}S_n^2}}$$

   You can use the fact that S_n^2 and $\bar{X}_n$ are independent, and

   $$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2.$$

   (d) (4 points) Identify the distribution of

   $$\frac{n}{n+1}\frac{(X_{n+1} - \bar{X}_n)^2}{S_n^2}.$$