MAS250 Probability and Statistics

$X_1 \sim X_n \overset{iid}{\sim} \cdot(\mu, \sigma^2)$

$\mu = E(X_i)$, $\sigma^2 = Var(X_i)$.

$E(\bar{X}) = \frac{1}{n}\Sigma E(X_i) = E(X) = \mu$.

$\bar{X}$: unbiased estimator of μ.

$Var(\bar{X}) = Var(\frac{1}{n}\Sigma X_i)$

$= \frac{1}{n^2}Var(\Sigma X_i)$ then if X_i are independent,

$= \frac{1}{n^2}\Sigma Var(X_i) = \frac{1}{n}Var(X_i) = \frac{\sigma^2}{n}$.

Mean Squared Error

(MSE) = $Bias^2 + Var$

CHAPTER 5.

SPECIAL RANDOM VARIABLES

'Bout Population

Random Variables

- Discrete random variables
  - Bernoulli random variable
  - Binomial random variable
  - Poisson random variable
  - Hypergeometric random variable
- Continuous random variables
  - Uniform random variable
  - Normal random variable
  - Exponential random variable
  - Gamma random variable

5.1 Bernoulli and Binomial RVs

mgf $\phi_X(t) = E[e^{tX}]$
$= e^0(1-p) + e^t p$
$\phi'_X(t) = pe^t = \phi''_X(t)$
$\| E(X)$ $\| E(X^2)$

- Bernoulli random variable
  - A r.v. X is said to be a Bernoulli random variable with parameter p $(0 \le p \le 1)$ if its pmf is given by

$$P\{X = 0\} = 1 - p, p\{X = 1\} = p$$

  - Binary outcome
  - p : the probability of a success
  - Notation: $X \sim Bernoulli(p)$ → Parameter
  - ean and variance

$= E(X^2) - E(X)^2$

$$E[X] = p, Var(X) = p(1-p)$$

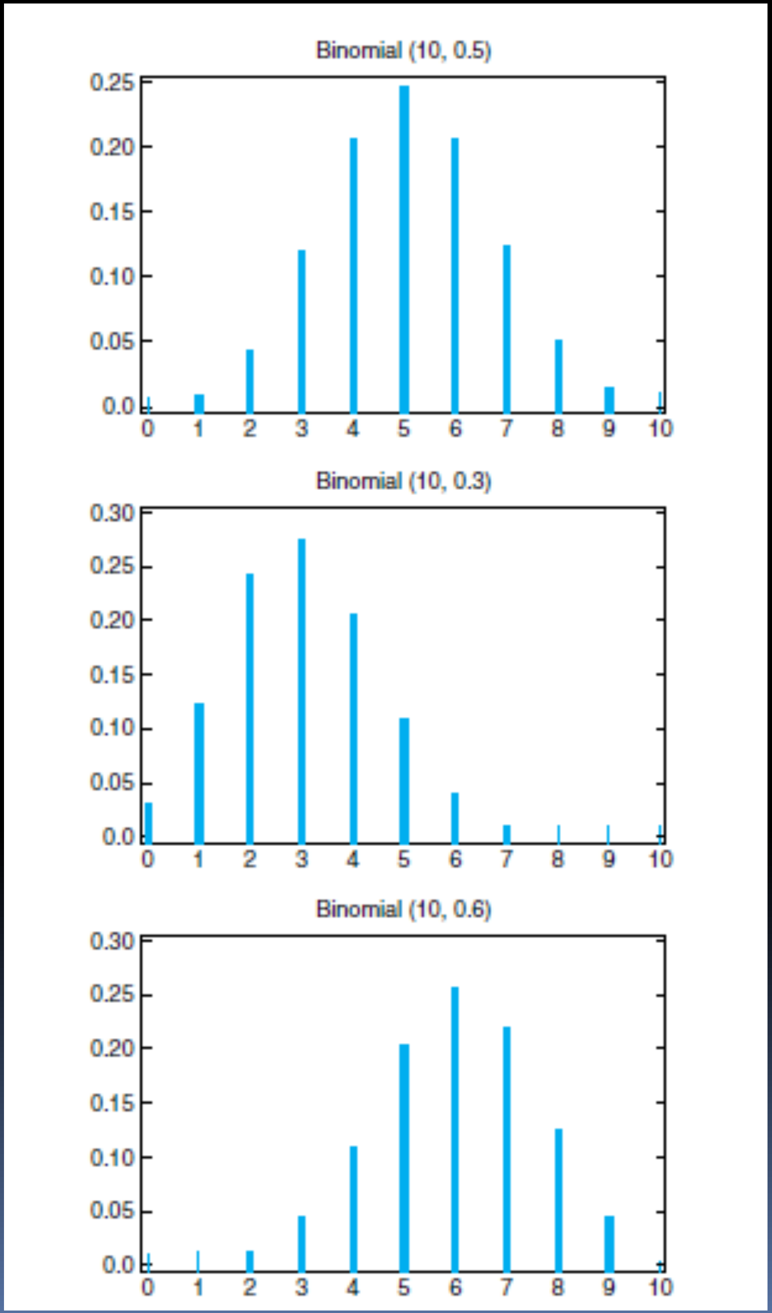
Tossing die 3 times
$X = \#$ of 6 showing up
$= 0.1.2.3$
$p = \frac{1}{6}$. $n = 3$.

$P(X=0) = (\frac{5}{6})^3$
$P(X=1) = 3 \cdot \frac{1}{6} \cdot (\frac{5}{6})^2$
$P(x=2) = 3 \cdot \frac{1}{6} \cdot (\frac{5}{6})^2$
$P(x=3) = (\frac{5}{6})^3$
$\Rightarrow P(x=i) = \binom{3}{i}(\frac{5}{6})^{3-i}(\frac{1}{6})^i$.

* Binomial expansion $(a+b)^n = \sum \binom{n}{i} a^i b^{n-i}$.

Condition of B.R.V. $\left\{ \begin{array}{l} \text{Binary outcome} \\ \text{Independent} \\ \text{Same probability} \end{array} \right.$

- Binomial random variable
  - X=# of successes in n independent Bernoulli trials
  - A r.v. X is said to be a Binomial random variable with parameters n and p $(0 \le p \le 1)$ if its pmf is given by
$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, i = 0,1,\cdots,n.$$
  - Notation: $X \sim B(n,p)$
  - For independent Bernoulli r.v.s X_i with parameter p,
$$\sum_{i=1}^{n} X_i \sim B(n,p)$$
→ # of success in n trial



Binomial (10, 0.5)
0.25
0.20
0.15
0.10
0.05
0.0
0 1 2 3 4 5 6 7 8 9 10
Binomial (10, 0.3)
0.30
0.25
0.20
0.15
0.10
0.05
0.0
0 1 2 3 4 5 6 7 8 9 10
Binomial (10, 0.6)
0.30
0.25
0.20
0.15
0.10
0.05
0.0
0 1 2 3 4 5 6 7 8 9 10

i) $X = \#$ of defective

$\sim B(10, 0.01)$

$\sum_{i=2}^{10} \binom{10}{i} \left(\frac{1}{100}\right)^i \left(\frac{99}{100}\right)^{10-i}$

$= 1 - \sum_{i=0}^{1} \binom{10}{i} \left(\frac{1}{100}\right)^i \left(\frac{99}{100}\right)^{10-i}$

≈ 0.005 ▪

ii) $Y = \#$ of returned package out of 3.

$\sim B(3, 0.005)$

$P(Y=1) = \binom{3}{1}(0.005)^1(0.995)^2$

≈ 0.015 ▪

- Example 5.1a
  - It is known that disks produced by a certain company will be defective with probability .01 independently each other. The company sells the disks in packages of 10 and offers a money-back guarantee if 2 or more of the 10 disks are defective. What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

i) $E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$

ii) $X = \sum_{i=1}^{n} X_i$. $X_i \sim \text{Bernoulli}(p)$

$E(X) = \sum_{i=1}^{n} (X_i) = np$.

$Var(X) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$.

iii) $\phi_X(t) = E[e^{tX}] = \sum_{n=0}^{\infty} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$

$= \sum \binom{n}{x} (e^t p)^x (1-p)^{n-x}$

$= (pe^t + 1 - p)^n$

$E[e^{tX}] = E[e^{t\sum X_i}] = E[\prod e^{tX_i}]$ if indep,

$= \prod E[e^{tX_i}]$ if iid

$= E[e^{tX_i}]^n = (1-p+pe^t)^n$

$E[X] = \phi_X'(0) = n(1-p+pe^t)^{n-1} \cdot pe^t \big|_{t=0}$

$= np$ ∎

Q. $X \sim B(n_1, p)$, $Y \sim B(n_2, p)$ } indep

$\Rightarrow X+Y \sim B(n_1+n_2, p)$

$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$

$= (1-p+pe^t)^{n_1} \cdot (1-p+pe^t)^{n_2}$

$= (1-p+pe^t)^{n_1+n_2}$

By uniqueness of mgf,

$X+Y \sim B(n_1+n_2, p)$ ∎

- More on the binomial distribution function
  - The mean and variance

$$E[X] = np, Var(X) = np(1-p)$$

  - The binomial distribution function

$$P\{X \leq i\} = \sum_{k=0}^{i} \binom{n}{k} p^k (1-p)^{n-k}, i = 0,1,\cdots,n$$

Calculate $\frac{P\{X=k+1\}}{P\{X=k\}}$.

  - The relation between $P\{X = k\}$ and $P\{X = k + 1\}$

$$P\{X = k + 1\} = \frac{p}{1-p} \frac{n-k}{k+1} P\{X = k\}$$

- More on the binomial distribution function
  - The moment generating function (mgf)
  - $\phi(t) = E(e^{tX}) = (1 - \mathrm{p} + \mathrm{p}e^{t})^{n}$

5.2 The Poisson RV

- Poisson random variable
  - A r.v. X is said to be a Poisson random variable with parameter λ $(\lambda > 0)$ if its pmf is given by

$$P\{X = i\} = e^{-\lambda} \boxed{\frac{\lambda^i}{i!}}, i = 0, 1, 2, \cdots.$$

$\Rightarrow \sum \frac{\lambda!}{i!} = e^{\lambda}.$

  - Notation: $X \sim Poisson(\lambda)$
  - The moment generating function (mgf)

$$\phi(t) = E[e^{tX}] = e^{\lambda(e^t - 1)}.$$

  - The mean and variance

$$E[X] = \lambda, Var(X) = \lambda$$

$X \sim \text{Poisson}(\lambda)$ $\quad * \ \text{Poisson}(e^t\lambda) = \frac{(\lambda e^t)^x e^{-\lambda e^t}}{x!}$

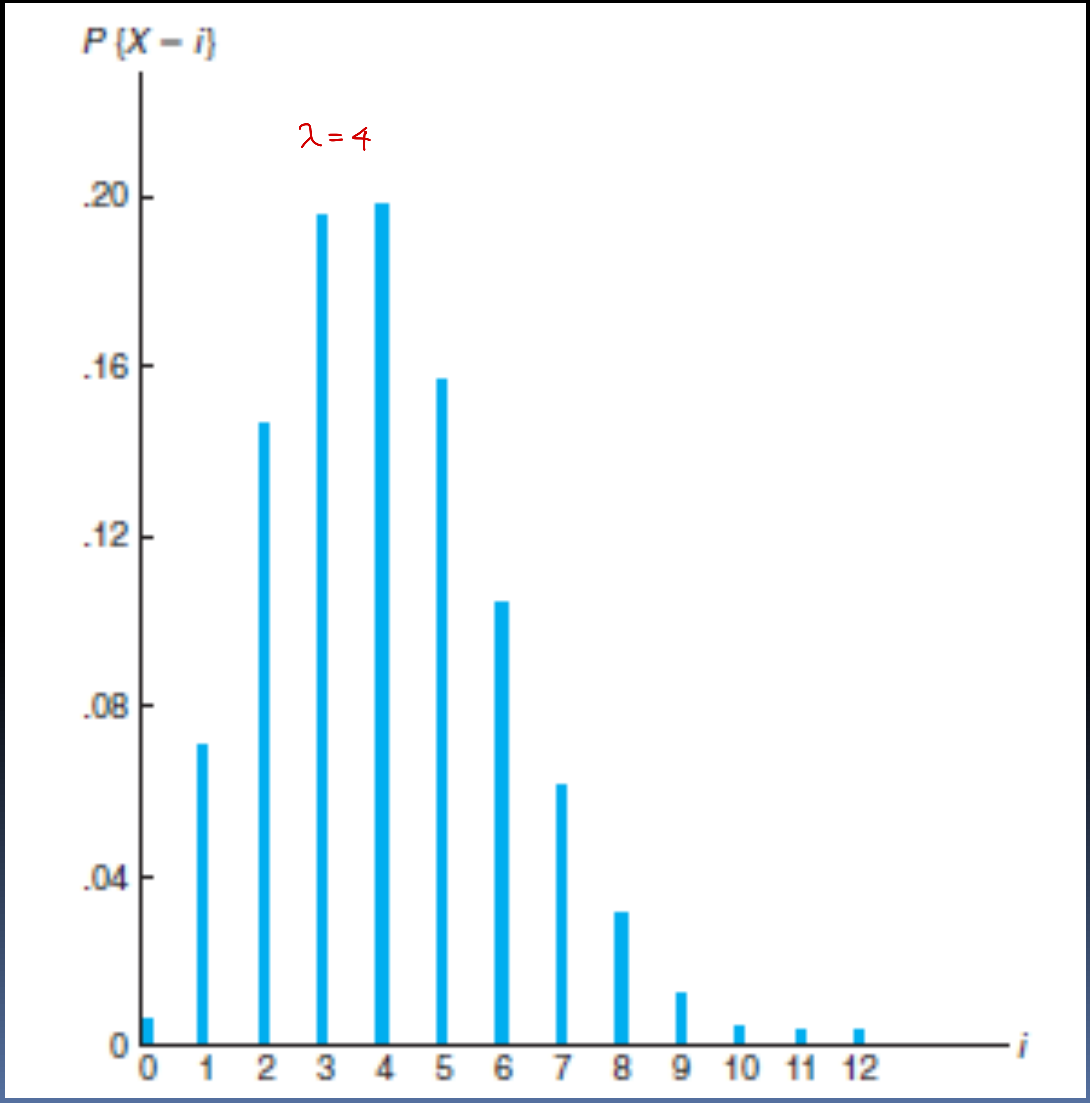
$$\phi_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(e^t\lambda)^x}{x!} = e^{-\lambda} \sum \frac{(e^t\lambda)^x (e^{-\lambda e^t}) \cdot e^{\lambda e^t}}{x!} = e^{\lambda(t-1)}.$$

$$\phi'_X(0) = e^{\lambda(e^t-1)} \cdot \lambda e^t \Big|_{t=0} = \lambda = E(X)$$

$$\phi''_X(0) = e^{\lambda(e^t-1)} \cdot \lambda^2 e^{2t} + e^{\lambda(e^t-1)} \cdot \lambda e^t \Big|_{t=0} = \lambda^2 + \lambda = E(X^2)$$

$$Var(X) = \lambda.$$



- Example 5.2a

$X \sim \text{Poisson}(\lambda=3)$

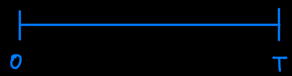
Suppose that the average number of accidents occurring weekly on a particular stretch of a highway equals 3. Calculate the probability that there is at least one accident this week.

$$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$$

$$= 1 - \frac{3^0 e^{-3}}{0!} = 1 - e^{-3}$$

- An approximation for $X \sim B(n, p)$
  - Let $\lambda = np$ be fixed. Then as $n \to \infty$,

$$P\{X = i\} = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \approx e^{-\lambda} \frac{\lambda^i}{i!}.$$



X = # of events in $[0, T]$

$\Rightarrow$ partition $[0, T]$ into $\frac{T}{n}$ that at most one event can occur in each interval

X_i = # of events at i_{th} partition

$$= \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$$

$$\Rightarrow X = \sum X_i \sim B(n, p)$$

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

$$= \underbrace{\boxed{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \cdots \cdot \frac{n-i+1}{n}}}_{\to 1} \cdot \frac{\lambda^i}{i!} \cdot \frac{\boxed{\left(1 - \frac{\lambda}{n}\right)^n} \to e^{-\lambda}}{\boxed{\left(1 - \frac{\lambda}{n}\right)^i} \to 1}$$

$$\approx e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$X_i = \begin{cases} 1 & i\text{th people selects own hat} \\ 0 & \text{else} \end{cases}$

$P(X_i=1) = \frac{1}{n}.$

$X = \sum X_i \approx B(n,p)$ for large n

$\approx \text{Poisson}(1)$

$np = \lambda = n \cdot \frac{1}{n} = 1.$

- Example 5.2e

  At a party n people put their hats in the center of a room, where the hats are mixed together. Each person then randomly chooses a hat. If X denotes the number of people who select their own hat, then, for large n, it can be shown that X has approximately a Poisson distribution with mean 1.

  - Why?

- When X_1 and X_2 are independent Poisson r.v.s with respective parameters λ_1 and λ_2 , $X_1 + X_2$ is a Poisson r.v. with parameter $\lambda_1 + \lambda_2$.
  - Why?

$$\phi_{X_1}(t) = e^{\lambda_1(t-1)}$$

$$\phi_{X_2}(t) = e^{\lambda_2(t-1)}$$

$$\phi_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}]$$

$$= E[e^{tX_1}]E[e^{tX_2}] = e^{\lambda_1(t-1)} \cdot e^{\lambda_2(t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(t-1)}$$

which is Poisson$(\lambda_1+\lambda_2)$ by uniqueness ■

$N_1 + N_2$
$\parallel$

- For $N \sim Poisson(\lambda)$, the number of events,
  - suppose that each of these event will independently be a type 1 event with probability p or a type 2 event with probability $1 - p$. Let N_1 and N_2 be, respectively, the numbers of events of type 1 and type 2.

(a) What is $P\{N_1 = n, N_2 = m\}$?

$$= P(N_1 = n, N_2 = m, N = n+m)$$
$$= P(N_1 = n, N_2 = m \mid N = n+m) \cdot P(N = n+m)$$
$$= \binom{n+m}{n} p^n (1-p)^m \cdot \frac{e^{-\lambda} \cdot \lambda^{n+m}}{(n+m)!}$$
$$= \frac{(n+m)!}{n!\,m!} \frac{(\lambda p)^n (\lambda(1-p))^m e^{-\lambda}}{(n+m)!}$$
$$= \frac{(\lambda p)^n}{n!} e^{-\lambda p} \cdot \frac{(\lambda(1-p))^m}{m!} \cdot e^{-\lambda(1-p)}$$
$$= \mathrm{Poisson}(\lambda p) \cdot \mathrm{Poisson}(\lambda(1-p))$$

(b) What is $P\{N_1 = n\}$?

(c) What is $P\{N_2 = m\}$?

$$P\{N_1 = n\} = \sum_{m=0}^{\infty} P(N_1 = n, N_2 = m) = \mathrm{Poisson}(\lambda)$$
$$P\{N_1 = m\} = \sum_{n=0}^{\infty} P(N_1 = n, N_2 = m) = \mathrm{Poisson}(\lambda(1-p))$$

- Property of a Poisson distribution

  If each of a Poisson number of events having mean λ is independently classified as being of one of the types $1, 2, 3, \dots, r$, with respective probabilities $p_1, p_2, \dots, p_r, \sum_{i=1}^{r} p_i = 1$, then the number of type $1, 2, 3, \dots, r$ events are independent Poisson random variables with respective means $\lambda p_1, \lambda p_2, \dots, \lambda p_r$.

- Computing the Poisson distribution function

$$\frac{P\{X = i + 1\}}{P\{X = i\}} = \frac{\lambda}{i + 1}.$$

Geometric (p)

X = # of trials until the 1st success

$P(X = i) = (1-p)^{i-1} p$

for $1 \le i < \infty$

⇝ Negative Binomial (r, p)

X = # of trials until the r-th successes.

5.3 The Hypergeometric RV

- Hypergeometric random variable
  - A r.v. X is said to be a hypergeometric random variable with parameters N, M, and n if its pmf is given by

$$P\{X = i\} = \frac{\binom{N}{i}\binom{M}{n-i}}{\binom{N+M}{n}}, i = 0,1,\cdots,\min(N,n).$$

  - c.f. a bin with N good items and M defective items and select n items from the bin. Then X is the number of good items selected.

- The mean and variance

$np(1-p)\left(1-\frac{n-1}{N+M-1}\right)$

$$E[X]=\frac{nN}{N+M}, Var(X)=\frac{nNM}{(N+M)^2}\left(1-\frac{n-1}{N+M-1}\right)$$

- Example 5.3c For independent $X\sim B(n,p)$ and $Y\sim B(m,p)$, what is the conditional pmf of X, given that $X+Y=k$?

$$P(X=i \mid X+Y=k)=\frac{P(X=i, X+Y=k)}{P(X+Y=k)}=\frac{P(X=i, Y=k-i)}{P(X+Y=k)}=\frac{P(X=i)P(Y=k-i)}{P(X+Y=k)}$$

$$=\frac{\binom{n}{i}p^i(1-p)^{n-i}\binom{m}{k-i}p^{k-i}(1-p)^{m-k+i}}{\binom{n+m}{k}p^k(1-p)^{n+m-k}}$$

$$=\frac{\binom{n}{i}\binom{m}{k-i}}{\binom{n+m}{k}} \text{ : Hypergeometric} \blacksquare$$

5.4 The Uniform RV

- Uniform random variable over the interval $[\alpha, \beta]$
  - A r.v. X is said to be uniformly distributed over $[\alpha, \beta]$ if its pdf is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & if\ \alpha \leq x \leq \beta, \\ 0 & otherwise. \end{cases}$$

  - Notation : $X \sim U[\alpha, \beta]$
  - $P\{a < X \leq b\} = \frac{b-a}{\beta-\alpha}, \alpha \leq a \leq b \leq \beta.$
  - The mean and variance $E(X^2) = \int_\alpha^\beta x^2 \frac{1}{\beta-\alpha} dx$

$$\int_\alpha^\beta x \frac{1}{\beta-\alpha} dx = E[X] = \frac{\alpha+\beta}{2}, Var(X) = \frac{(\beta-\alpha)^2}{12}$$

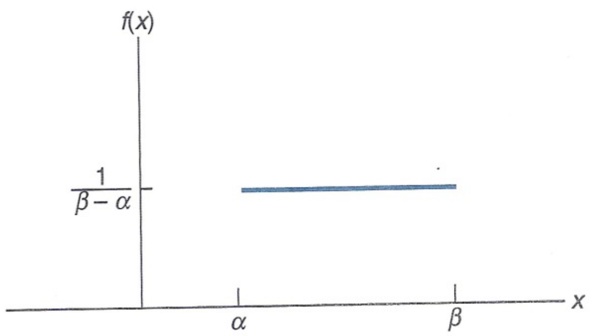


FIGURE 5.4 *Graph of $f(x)$ for a uniform* $[\alpha, \beta]$.

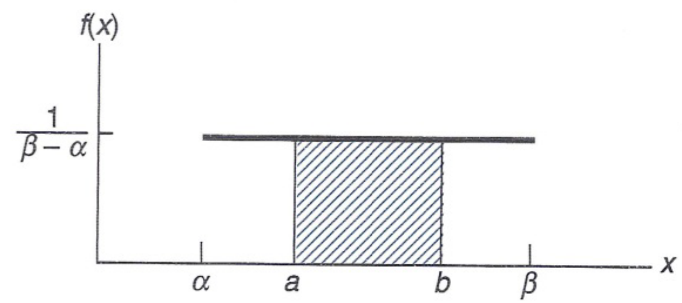


FIGURE 5.5 *Probabilities of a uniform random variable.*

5.5 Normal Random Variable

- Normal random variable
  - A r.v. X is said to be normally distributed with parameters μ and σ^2 if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

  - Notation: $X \sim N(\mu, \sigma^2)$ Variance * May be S.d.
  - The mgf

$$\phi(t) = E[e^{tX}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

$\phi_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$

$E(X) = \phi_X'(0) = (\mu + t\sigma^2)\phi_X(t) \Big|_{t=0}$
$= \mu.$

$E(X^2) = \phi_X''(0) = \sigma^2 \phi_X(t) + (\mu + t\sigma^2)^2 \phi_X(t) \Big|_{t=0}$
$= \sigma^2 + \mu^2$

$V(X) = \sigma^2$ ▪

- The mean and variance

$$E[X] = \mu, Var(X) = \sigma^2$$

  - Method 1: compute $E[X-\mu], E[(X-\mu)^2]$
  - Method 2: using MGF

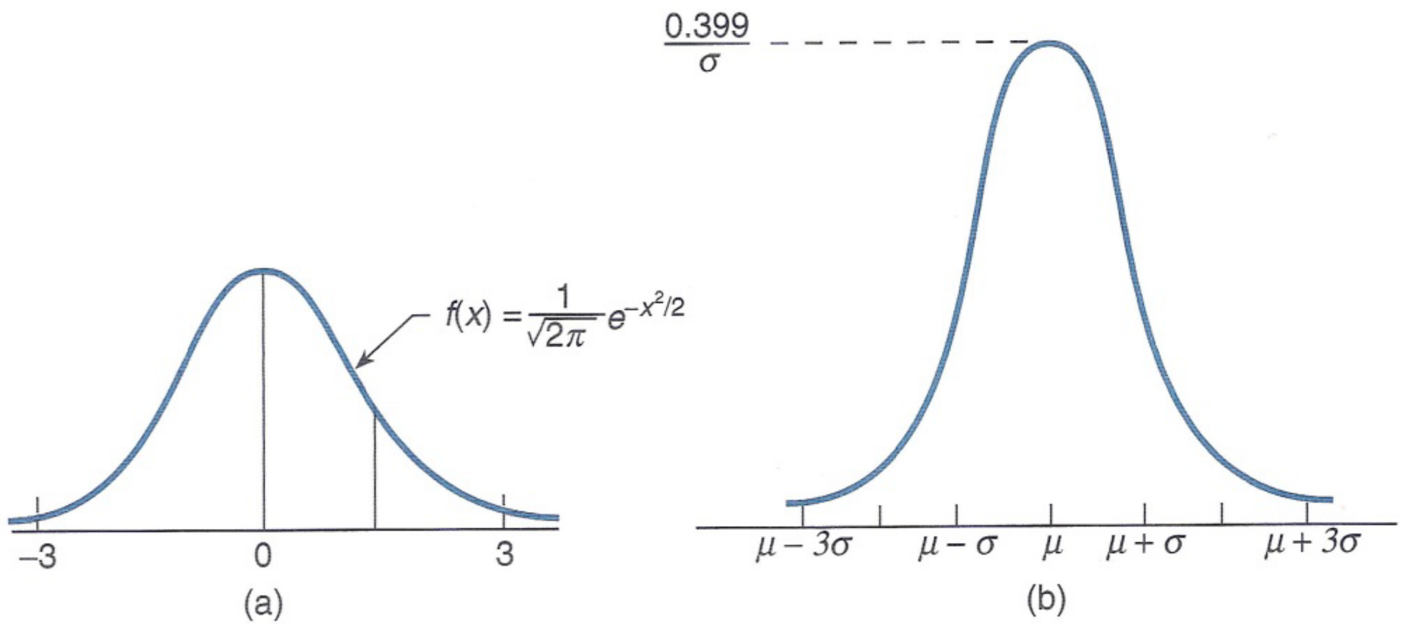


FIGURE 5.7 *The normal density function (a) with $\mu = 0, \sigma = 1$ and (b) with arbitrary μ and σ^2.*

$E[e^{t(aX+b)}]$

$= e^{(a\mu+b)t + \frac{t^2a^2\sigma^2}{2}}$

$= N(a\mu+b, a^2\sigma^2)$ by uniqueness of mgf.

- Properties of normal random variables
  - For $X \sim N(\mu, \sigma^2)$, two constants a and b,

$$Y = aX + b \sim N(a\mu + b, a^2\sigma^2)$$

  - It follows that $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$.

- The standard normal r.v. $Z \sim N(0,1)$
  - Its distribution function is denoted by $\Phi(x)$.
  - The pdf of Z is symmetric, i.e., for all x

$$\Phi(-x) = P\{Z < -x\} = P\{Z > x\} = 1 - \Phi(x)$$

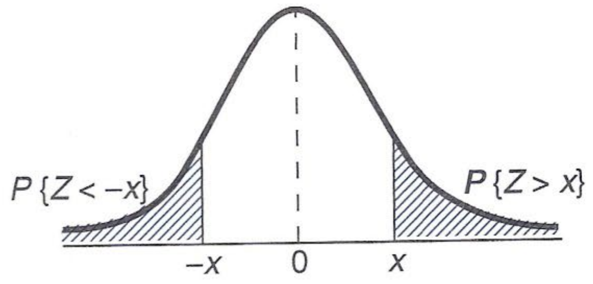


FIGURE 5.8 *Standard normal probabilities.*

- Example 5.5a: X~N(3,16)

$Z = \frac{X-3}{4}$

(a) P(X<11) $= P(Z < 2) \approx 0.97725$

(b) P(X>-1) $= P(Z > -1) = P(Z < 1) \approx 0.84$

(c) P(2<X<7) $= P(-0.25 < X < 1) = P(X<1) - P(X \leq -0.25)$

$\approx 0.84 - (1 - P(X \leq 0.25)) = 0.84 - (1 - 0.6)$

$= 0.44$

$$\phi_{\Sigma X_i}(t) = E(e^{t\Sigma X_i}) = E(\Pi\, e^{tX_i})$$
$$= \Pi\, E(e^{tX_i}) = \Pi\, e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}}$$
$$= e^{\Sigma \mu_i t + \frac{\Sigma \sigma_i^2 t^2}{2}}$$: mgf of $N(\Sigma \mu_i, \Sigma \sigma_i^2)$

by uniqueness ■

- More…
  - The sum of independent normal r.v.s is also a normal random variables. That is, if $X_i \sim N(\mu_i, \sigma_i^2)$, then $\sum_{i=1}^{n} X_i \sim N(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)$. Why?
  - For $\alpha \in (0,1)$, the quantity z_α is defined by
$$P\{Z > z_\alpha\} = \alpha.$$

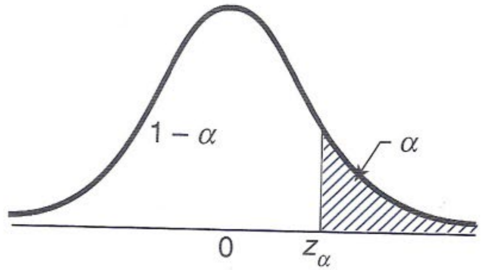


FIGURE 5.9 $P\{Z > z_\alpha\} = \alpha.$

(a) $X \sim N(12.08, 3.1^2)$

$P(X_1+X_2 > 25)$

$X_1+X_2 \sim N(24.16, \underbrace{2 \cdot 3.1^2}_{= 19.22})$

$Z = \frac{X-24.16}{\sqrt{19.22}}$

$\Rightarrow P(X_1+X_2 > 25) = P(Z > 0.19)$

$= 1 - P(Z \leq 0.19)$

$= 0.425$

(b) $P(X_1 > X_2 + 3) = P(X_1 - X_2 > 3)$

$X_1 - X_2 \sim N(0, 19.22)$

$Z = \frac{X}{\sqrt{19.22}}$

$\Rightarrow P(X_1 - X_2 > 3) = P(Z > 0.68)$

$= 1 - 0.752 = 0.248$

$Var(X_1 - X_2) = Var(X_1) + Var(-X_2) - 2\underbrace{Cov(X_1, X_2)}_{= 0}$

$= Var(X_1) + Var(X_2)$

since indep.

- Example 5.5d
  - Data from the National Oceanic and Atmospheric Administration indicate that the yearly precipitation in Los Angeles is a normal random variable with a mean of 12.08 inches and a standard deviation of 3.1 inches.
  - (a) Find the probability that the total precipitation during the next 2 years will exceed 25 inches.
  - (b) Find the probability that next year's precipitation will exceed that of the following year by more than 3 inches. Assume that the precipitation totals for the next 2 years are independent.

5.6 Exponential Random Variable

- Exponential random variable
  - A r.v. X is said to be exponentially distributed with parameter λ ($\lambda > 0$) if its pdf is given by

Scaling

$$f(x) = \begin{cases} \lambda e^{-\overset{\text{Rate}}{\lambda} x}, if\ x \geq 0, \\ 0, \quad otherwise. \end{cases}$$

* Some use $\beta = \frac{1}{\lambda}$

  - Notation: $X \sim Exp(\lambda)$
  - $P\{X \leq x\} = 1 - e^{-\lambda x}, x \geq 0.$ $P(X > x) = e^{-\lambda x}$: Survival Function
  - The mgf

$$\phi(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}, \lambda > t.$$

$$\phi_X(t) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \int_0^\infty \lambda e^{(t-\lambda)x}\,dx = -\frac{\lambda}{\lambda-t}e^{-(\lambda-t)x}\Big|_0^\infty = \frac{\lambda}{\lambda-t} \quad (\lambda > t)$$

$$\phi_X'(0) = \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{1}{\lambda}.$$

$$\phi_X''(0) = \frac{2\lambda(\lambda-t)}{(\lambda-t)^4}\Big|_{t=0} = \frac{2}{\lambda^2}.$$

$$E(X^2) - E(X)^2 = \frac{1}{\lambda^2}.$$

$$P(X > x) = e^{-\lambda x}.$$

$$P(X > s+t \mid X > t) = \frac{P(X > s+t)}{P(X > t)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s).$$

- Properties of exponential random variables
  - The mean and variance $E[X] = 1/\lambda, Var(X) = 1/\lambda^2$
  - The memoryless property. For all $s, t > 0$

$$P\{X > s + t \mid X > t\} = P\{X > s\}$$

- Proposition 5.6.1.

  If $X_1, X_2, \cdots, X_n$ are independent exponential random variables having respective parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$, then $\min_{1 \le i \le n} X_i$ is exponential with parameter $\sum_{i=1}^{n} \lambda_i$.

$P(\min X_i \leq x) = 1 - P(\min X_i > x)$

$\overset{\text{indep}}{=} 1 - P(X_1 > x) P(X_2 > x) \cdots P(X_n > x)$.

$= 1 - e^{-\lambda_1 x} \cdots e^{-\lambda_n x}$

$= 1 - e^{-\Sigma \lambda_i x}$: cdf of $Exp(\Sigma \lambda_i)$

$\Rightarrow f(x) = Exp(\Sigma \lambda_i)$

$P(\max X_i \leq x) \overset{\text{indep}}{=} P(X_1 \leq x) \cdots P(X_n \leq x)$. "Order Statistics"

- Example 5.6a
  - Suppose that a number of miles that a car run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000-mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery? What can be said when the distribution is not exponential?

$\lambda = \frac{1}{10000}$

$P(X > 5000) = e^{-\lambda x} = e^{-\frac{5000}{10000}} = e^{-\frac{1}{2}} \approx 0.6$

$(= P(X > t + 5000 \mid X > t))$

- Example 5.6b
  - A crew of workers has 3 interchangeable machines, of which 2 must be working for the crew to do its job. When in use, each machine will function for an exponentially distributed time having parameter λ before breaking down. The workers decide initially to use machine A and B and keep machine C in reserve to replace whichever of A or B breaks down first. They will then be able to continue working until one of the remaining machines breaks down. When the crew is forced to stop working because only one of the machines has not yet broken down, what is the probability that the still operable machine is machine C?

$\frac{1}{2}$

$$P(X > t) = P(\min X_i > t)$$
$$= e^{-\Sigma X_i t}$$

- Example 5.6c
  - A series system is one that needs all of its components to function in order for the system itself to be functional. For an n-component series system in which the component lifetimes are independent exponential random variables with respective parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$, what is the probability that the system survives for a time t?

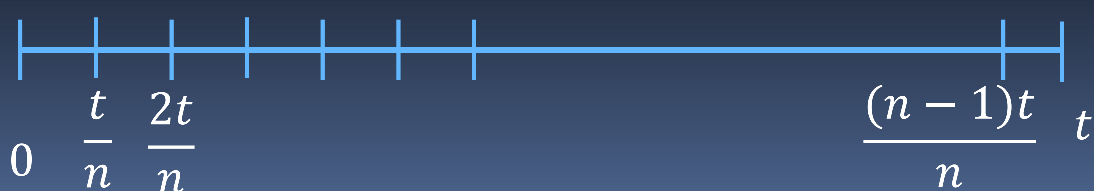
5.6.1 The Poisson process

time dependent

- Definition of the Poisson process $N(t)$ with rate λ
  - $N(0) = 0$
  - The number of events that occur in disjoint time intervals are independent (independent increments).
  - The distribution of the number of events that occur in a given interval depends only on the length of the interval and not on its location (stationary increments).
  - $\lim_{h\to 0} \frac{P\{N(h)=1\}}{h} = \lambda$ and $\lim_{h\to 0} \frac{P\{N(h)\geq 2\}}{h} = 0.$
  - c.f. $P\{N(h) = 1\} \approx \lambda h, P\{N(h) \geq 2\} \approx 0$ for sufficiently small h.

- To compute $P\{N(t) = k\}$, we divide the interval $[0, t]$ into n sub-intervals of equal size. Then we have two cases for $N(t) = k$.
  - Case 1: k of the n sub-intervals contain exactly 1 event and the other $n - k$ contain 0 events
  - ~~Case 2~~: at least 1 sub-interval contains 2 or more events
  - $P\{Case\ 1\} = \binom{n}{k} \boxed{\left(\frac{\lambda t}{n}\right)}^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \to e^{-\lambda t} \frac{(\lambda t)^k}{k!}$
  - $P\{Case\ 2\} \to 0$ as $n \to \infty$

    $P(N(h)) \approx \lambda h$ where $h = \frac{t}{n}$.
  - $P\{N(t) = k\} = P\{Case\ 1\} + P\{Case\ 2\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$



$P(X_i > t) = e^{-\lambda t}$: goal

$P(X_1 > t) = P(N(t) = 0)$

$= \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t} = Exp(\lambda)$

$P(X_2 > t \mid X_1 = s) = P(\text{No event in } [s, s+t] \mid X_1 = s)$: indep.

$= P(0 \text{ events in } [s, s+t])$: stationary.

$= P(0 \text{ events in } [0, t])$

$= e^{-\lambda t}$

- Proposition 5.6.2

  For a Poisson process $N(t)$ having rate λ

$$P\{N(t) = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, k \geq 0.$$

$= \text{Poisson}(\lambda t)$

- Proposition 5.6.3

  # of event occured in $(0, t)$

  $\Rightarrow N(t) \sim \text{Poisson}(\lambda t) \Rightarrow X_i \overset{iid}{\sim} Exp(\lambda)$

  - For a Poisson process with rate λ, the inter-arrival times $X_1, X_2, \cdots$ are independent exponential random variables, each with mean $1/\lambda$.

5.7 Gamma Random Variable

- Gamma random variable
  - A r.v. X is said to have a gamma distribution with parameters (α, λ) if its pdf is given by

$\alpha = 1$
$\Rightarrow Exp(\lambda)$

* Some use $\beta = \frac{1}{\lambda}$.

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, if\ x \geq 0, \\ 0, \quad otherwise. \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}\,dx = \int_0^\infty e^{-y} y^{\alpha-1}\,dy$$

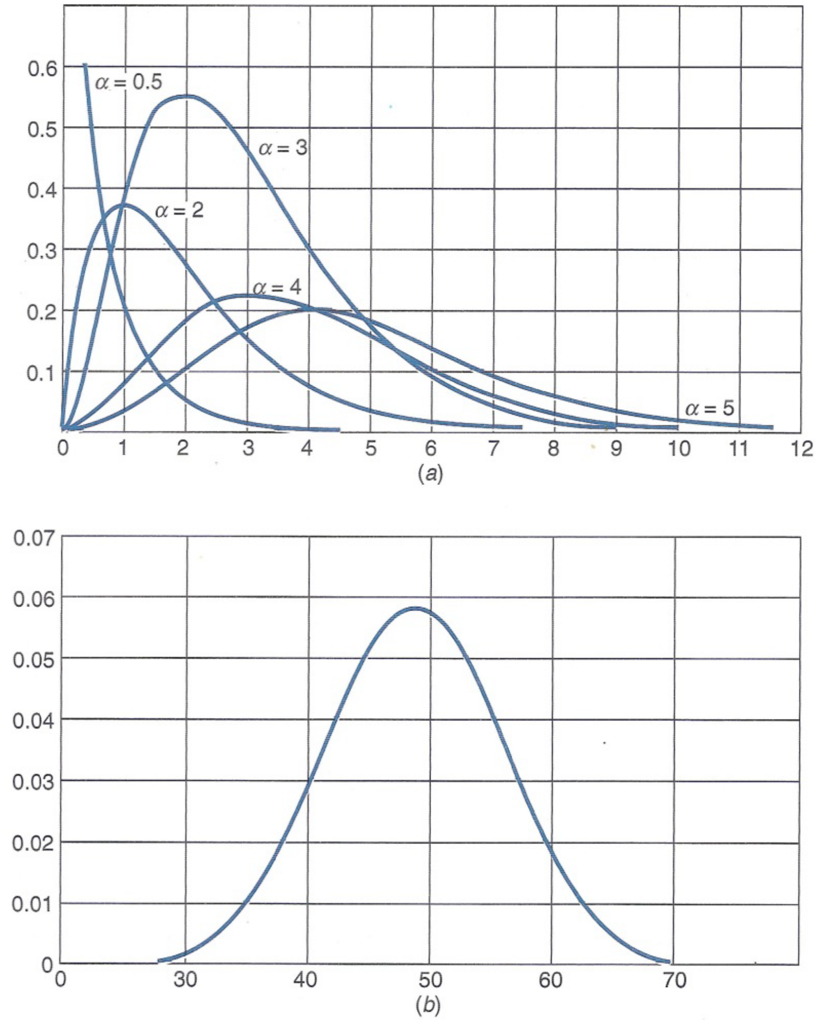


FIGURE 5.11 *Graphs of the gamma $(\alpha, 1)$ density for (a) $\alpha = .5, 2, 3, 4, 5$ and (b) $\alpha = 50$.*

- More…
  - $\Gamma(n) = (n-1)!$
  - $\phi(t) = E[e^{tX}] = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}, \lambda > t.$
  - The mean and variance $E[X] = \alpha/\lambda, Var(X) = \alpha/\lambda^2$
  - Notation: $X \sim gamma(\alpha, \lambda)$
  - $gamma(1, \lambda) = Exp(\lambda)$
  - For independent r.v.s $X_1 \sim gamma(\alpha_1, \lambda)$ and $X_2 \sim gamma(\alpha_2, \lambda)$,

$$X_1 + X_2 \sim gamma(\alpha_1 + \alpha_2, \lambda)$$

$\phi_{X_1+X_2}(t) = E(e^{t(X_1+X_2)}) \overset{\text{indep.}}{=} E(e^{tX_1})E(e^{tX_2}) = \left(\frac{\lambda}{\lambda-t}\right)^{X_1+X_2}$

- Proposition 5.7.1

  If $X_1, X_2, \cdots, X_n$ are independent gamma random variables having respective parameters (α_i, λ), then $\sum_{i=1}^{n} X_i$ is gamma with parameters $(\sum_{i=1}^{n} \alpha_i, \lambda)$.

- Corollary 5.7.2

  If $X_1, X_2, \cdots, X_n$ are independent exponential ($\alpha_i = 1$) random variables having common parameter λ, then $\sum_{i=1}^{n} X_i$ is gamma with parameters (n, λ).

5.8 Distributions Arising from The Normal

$Z \sim N(0.1)$

$Z^2 \sim \chi^2(1)$

- 5.8.1 The chi-square distribution

  If $Z_1, Z_2, \cdots, Z_n$ are independent standard normal r.v.s, then $X := Z_1^2 + \cdots + Z_n^2$ is said to have a chi-square distribution with n degrees of freedom.

  - Notation: $X \sim \chi_n^2$
  - For $\alpha \in (0,1)$, the quantity $\chi_{\alpha,n}^2$ is defined by

$$P\{X > \chi_{\alpha,n}^2\} = \alpha.$$

$\| \; \chi_\alpha^2(n)$

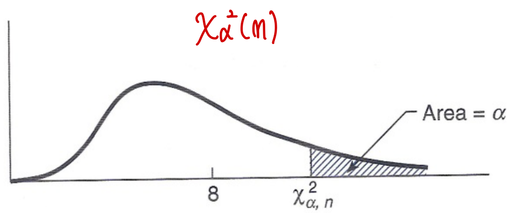


FIGURE 5.12 *The chi-square density function with 8 degrees of freedom.*

- The relation between chi-square and gamma.
  - The mgf of χ_1^2 $(= Z^2)$ $\quad E\left[e^{tZ^2}\right] = (1-2t)^{-\frac{1}{2}}$
  - The mgf of χ_n^2 $\quad E\left[e^{t\sum_{i=1}^{n} Z_i^2}\right] = (1-2t)^{-\frac{n}{2}}$
  - The mgf of $X \sim gamma(\alpha, \lambda)$

$$E\left[e^{tX}\right] = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$$

  - Hence, $\chi_n^2 \; =^{d} \; gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

- The pdf of χ_n^2

$$f(x) = \frac{\frac{1}{2} e^{-\frac{x}{2}} \left(\frac{x}{2}\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)}, \qquad x > 0$$

- The mean and variance

$$E[X] = n, Var(X) = 2n$$

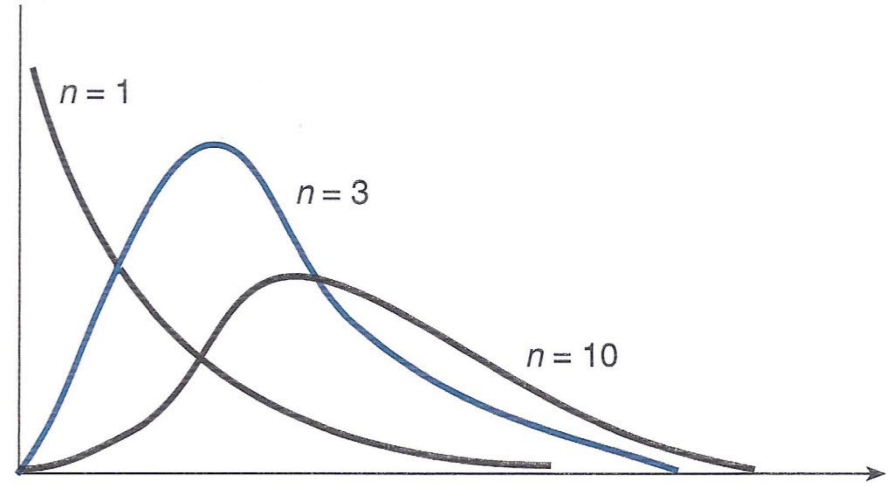


FIGURE 5.13 *The chi-square density function with n degrees of freedom.*

$P(D > 3)$

$\Rightarrow P(X_1^2 + X_2^2 > 9)$

where $X_i \sim N(0, 2^2)$.

$\Rightarrow Y_i = \frac{X_i}{2}$.

$\Rightarrow P(Y_1^2 + Y_2^2 > \frac{9}{4})$

$\Rightarrow P(\chi_2^2 > \frac{9}{4}) \approx 0.3247$

- Example 5.8d
  - When we attempt to locate a target in two-dimensional space, suppose that the coordinate errors are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceed 3.

- 5.8.2 The t-distribution

  If $Z \sim N(0,1)$ and χ_n^2 are independent, then the random variable T_n, defined by

$$T_n = \frac{Z \ {\scriptstyle \sim N(0,1)}}{\sqrt{\chi_n^2/n}}$$

  is said to have a t-distribution with n degrees of freedom.

  - When n is sufficiently large, $\frac{\chi_n^2}{n} \approx 1$ and $T_n \approx Z$.

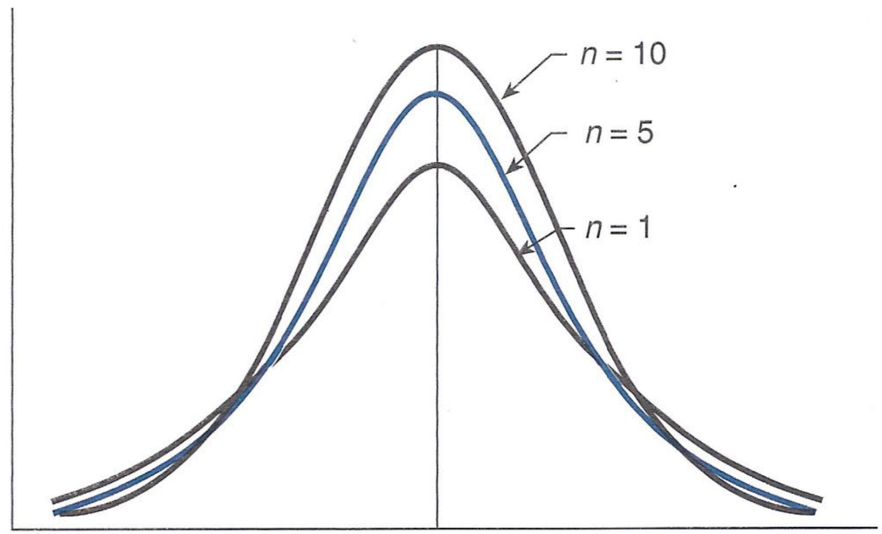


FIGURE 5.14 *Density function of* T_n.

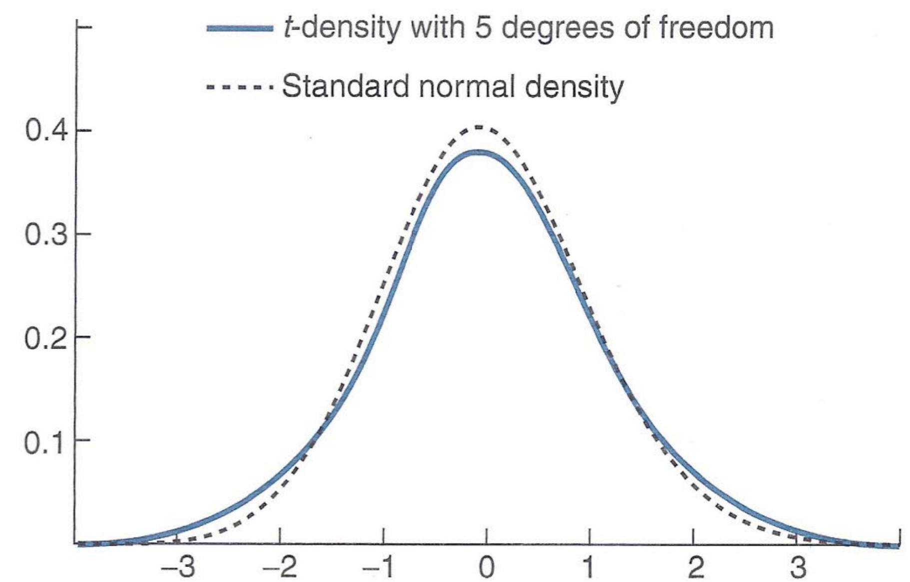


FIGURE 5.15 *Comparing standard normal density with the density of T_5.*

- More …
  - Notation: $X \sim T_n$
  - The mean and variance $E[X] = 0, Var(X) = \frac{n}{n-2}$ (n>2)
  - For $\alpha \in (0,1)$, the quantity $t_{\alpha,n}$ is defined by

    $$P\{X > t_{\alpha,n}\} = \alpha$$
  - The distribution function of T_n is symmetric.
  - From $\alpha = P\{X < -t_{\alpha,n}\} = 1 - P\{X \geq -t_{\alpha,n}\}$, we get $t_{1-\alpha,n} = -t_{\alpha,n}$.

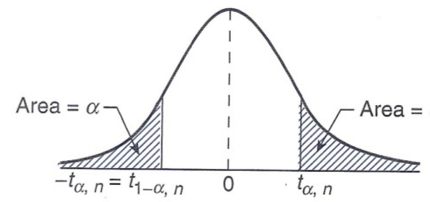


FIGURE 5.16 $t_{1-\alpha,n} = -t_{\alpha,n}$.

- 5.8.3 The F-distribution

  If χ_n^2 and χ_m^2 are independent, the random variable $F_{n,m}$, defined by

$$F_{n,m} := \frac{\chi_n^2/n}{\chi_m^2/m}$$

  is said to have an F-distribution with n and m degrees of freedom.

- More…
  - Notation: $X \sim F_{n,m}$
  - For $\alpha \in (0,1)$, the quantity $F_{\alpha,n,m}$ is defined by

$$P\{X > F_{\alpha,n,m}\} = \alpha$$

$$// \; 1-\frac{\chi_n^2/m}{\chi_m^2/m} < F_{\alpha,m,m}$$

  - $\alpha = P\left\{\frac{\chi_n^2/n}{\chi_m^2/m} > F_{\alpha,n,m}\right\} = 1 - P\left\{\frac{\chi_m^2/m}{\chi_n^2/n} > \frac{1}{F_{\alpha,n,m}}\right\}$
  - $F_{1-\alpha,m,n} = \frac{1}{F_{\alpha,n,m}}$

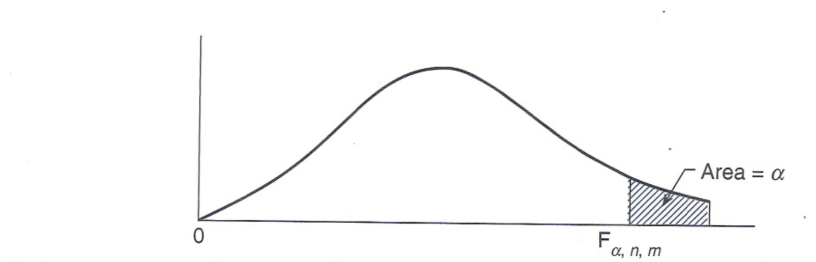


FIGURE 5.17 *Density function of $F_{n,m}$.*