0.1 Linear Transformations

Definition 0.1.1: Linear Transformation

$T : V_1 \to V_2$ for v.s. V_1, V_2 is funciton called linear transformation if this function satisfies $T(cx_1 + x_2) = cT(x_1) + T(x_2)$ for $x_i \in V_i$, $c \in F$.

Exercise 0.1.1

If T is a linear trans., then $T(0) = 0$.

Proof. $T(0) + T(0+0) = 2T(0)$. □

Exercise 0.1.2 If T is a linear trans., then $T(-x) = -T(x)$.

Theorem 0.1.1

V, W : f.d.v.s./F, $\{\alpha_1, \dots, \alpha_n\}$ be basis of V and $\{\beta_1, \dots, \beta_m\}$ be any given subset of W. Then $\exists! T : V \to W$ s,t, $T(\alpha_i) = \beta_i$.

Proof. Define $T_0(x_1\alpha_1 + \cdots + x_n\alpha_n) := \sum_{i=1}^{n} x_i\beta_i$. This is lin. trans. Thus existence is proven.

For uniqueness, if there if another U s.t. $U(\alpha_i) = \beta_i$, then $U(\sum x_i\alpha_i) = \sum x_i U(\alpha_i) = \sum x_i\beta_i = T_0(\sum x_i\alpha_i)$. Thus $U = T_0$. □

Definition 0.1.2: Null Space and Range

$T : V \to W$: lin. trans. of v.s./F. $N(T) \subset V$, $R(T) \subset W$ where $N(T) := \{v \in V \mid Tv =\}$ and $R(T) := \{w \in W \mid \exists v \in V \ (w = T(v))\}$.

Definition 0.1.3

$\mathrm{nullity}(T) := \dim_F(N(T))$, $\mathrm{rank}(T) := \dim_F(R(T))$.

Theorem 0.1.2

V : f.d.v.s./F, $T : V \to W$: lin. trans. Then $\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim(V)$.

Proof. Begin with $N(T)$. Choose basis $\{v_1, \dots, v_k\}$ of $N(T)$ and choose $v_{k+1}, \dots, v_n \in V$ s.t. $\{v_1, \dots, v_n\}$ is a basis of V.

Claim 0.1.1

$T(v_{k+1}), \dots, T(v_n)$ is a basis of $R(T)$.

Proof. For linear independence, suppose $c_{k+1}T(v_{k+1}) + \cdots + c_nT(v_n) = 0$. Then $T(c_{k+1}v_{k+1} + \cdots + c_nv_n) = 0$, so $c_{k+1}v_{k+1} + \cdots + c_nv_n \in N(T)$. Since $\{v_1, \dots, v_k\}$ is a basis of $N(T)$, $c_{k+1}v_{k+1} + \cdots + c_nv_n = a_1v_1 + \cdots + a_kv_k$. Since $\{v_1, \dots, v_n\}$ is basis, those are lin. indep. Thus all coefficients are 0, thus $T(v_{k+1}), \dots, T(v_n)$ are indep. □

Claim 0.1.2

$\mathrm{span}\{T(v_{k+1}), \dots, T(v_n)\} = R(T)$

Proof. Exercise! ☐

Thus $\dim(R(T)) = n - k$. ☐

Theorem 0.1.3

For $m \times n$ mat. A, row rank is equal to column rank.

Proof. $V := F^n$ and $W := F^m$. $T : V \to W$ is lin. trans. Then col. rank = dim. of spans of col. = $\dim(R(T))$ = $\text{rank}(T)$. Also, $\text{nullity}(T) = \dim(N(T)) = n - \text{rank}(T)$ = number of rows with leading 1's in RREF = number of cols. with leading 1's in RREF = dim. of col. space of A. Thus row rank is equal to col. rank. ☐

0.2 The Algebra of Linear Transformations

Definition 0.2.1: $L(V,W)$

$L(V,W)$ is set of all lin. trans. from V to W.

Theorem 0.2.1

V, W : F-v.s. Then $L(V,W)$ is itself vec. space over F.

Proof. Let $T, U \in L(V,W)$. Define $T + U : V \to W$ by $(T + U)(v) = T(v) + U(v)$.

Claim 0.2.1

$cT + U \in L(V,W)$

Proof. $(cT + U)(av_1 + v_2) = cT(av_1 + v_2) + U(av_1 + v_2)$ where both T and U is lin. trans. Thus trivially it is lin. trans. ☐

☐

Theorem 0.2.2

V : n-dim. v.s./F, W : m-dim. v.s./F. Then $\dim_F(L(V,W)) = nm$.

Proof. Suppose $B = \{\alpha_1, \ldots, \alpha_n\}$ is basis of V, $B' = \{\beta_1, \ldots, \beta_m\}$ is basis of W. For each (p,q) where $1 \leq p \leq m$ and $1 \leq q \leq r$, define $E^{p,q}(\alpha_i) = 0$ if $i \neq q$ and β_p if $i = q$. Then these are lin. indep. trans. $V \to W$ and they span $L(V,W)$. ☐

Lemma 0.2.1

$U \circ T$ is a lin. trans. in $L(V,Z)$ where $U : V \to W$ and $T : W \to Z$.

Proof. Exercise! ☐

Definition 0.2.2: Endomorphism (Linear Operator)

For the case $T : V \to V$, we say T is an endomorphism or linear operator.

Definition 0.2.3

$T : V \to W$ be lin. trans. Then

- one-to-one or injective if $T(v) = 0 \Rightarrow v = 0$. (nonsingular)
- onto or surjective if $T(V) = W$
- T is invertible if $\exists U : W \to V$ s.t. $U \circ T = T \circ U = Id$

Exercise 0.2.1

T is injective and surjective $\iff$ T is invertible.

Exercise 0.2.2

$T : V \to W$ is a nonsingular lin. trans. Then any lin. indep. subset S of V is sent to lin. indep. set $T(S)$.

Exercise 0.2.3

Suppose $T : V \to W$ is invertible. Then $\dim(V) = \dim(W)$ for f.d.v.s. V and W.

Theorem 0.2.3

Suppose V, W as f.d.v.s./F and $\dim(V) = \dim(W)$. Let $T : V \to W$ be a lin. trans. TFAE:

i) T is invertible

ii) T is nonsingular, i.e., T is injective

iii) T is onto, i.e., T is surjective

Proof. $\text{rank}(T) + \text{nullity}(T) = n$. T is nonsingular $\iff$ $\text{nullity}(T) = 0$ $\iff$ $\text{rank}(T) = n$ $\iff$ $R(T) = W$ $\iff$ T is onto. □

Definition 0.2.4: General linear Group

$G =$ invertible endo. on V. with inverse $\circ$. Then $G = GL(V)$ is the general linear group of V.

Definition 0.2.5: Group

If some algebraic structure is associative with identity, we say this algebraic structure is group.

0.3 Isomorphism

Definition 0.3.1: Isomorphism

$V, W : F$—v.s. We say a lin. trans. $T : V \to W$ is an isomorphism if T is an invertible lin. trans.

Theorem 0.3.1

V : n-d.v.s./F. Then V is isomorphic to F^n ($V \simeq F^n$).

Proof. $B := \{\alpha_1, \ldots, \alpha_n\}$ is basis of V. Define $T : V \to F^n$, i.e., $v \mapsto [v]_B$.

Claim 0.3.1

This is isomorphism $\iff$ T is injective.

Proof. Suppose $T(v) = 0$. Then $v = 0$. □

□

0.4 Representation of Transformation by Matrices

Theorem 0.4.1

V, W : F-v.s. and B, B' be basis, where $T : V \to W$ be lin. trans. Then $\exists! m \times n$ mat. A. s.t. $[Tv]_{B'} = A[v]_B$.

Theorem 0.4.2

V, W, Z : f.d.v.s./F, B, B', B'' be basis. Let $U \circ T : V \to Z$ be lin. trans. If $A_1 = [T]_{B,B'}$ and $A_2 = [T]_{B',B''}$, then $[U \circ T]_{B,B''} = A_2 \circ A_1$.

Theorem 0.4.3

T : endo. on f.d.v.s.V/F, where B_1, B_2 be two different basis of V. Let P be mat. s.t. $[v]_{B_1} = P[v]_{B_2}$. Then $[T]_{B_2} = P^{-1}[T]_{B_1}P$.

Definition 0.4.1: Similar

We say M and N are similar if $\exists$ invertible P s.t. $N = P^{-1}MP$.

0.5 Linear Functionals

Definition 0.5.1: Linear Functional

V : F-v.s. A lin. trans. $T : V \to F$ is called a linear functional.

Example 0.5.1

Definite integral and functions, especially constant function are linear functional.

Definition 0.5.2: Dual Vector Space

V : F-v.s. We normally write $V^* = L(V, F)$ the dual vector space of V.

Note:-

For finite dimensional V, $\dim(V^*) = \dim(V)$. But if V is infinite dimensional, $\dim(V^*)$ can be extremely large.

Lemma 0.5.1

V : n-d.v.s./F. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of V. Define $f \in V^*$ by declaring $f_i(\alpha_j) = \delta_{ij}$. Then $\{f_1, \ldots, f_n\}$ is basis of V^*.

Proof. Because $\dim(V^*) = \dim(V) = n$, E.T.S. that $f_1, \ldots, f_n$ are lin. indep. Suppose $\exists c_1 f_1 + \cdots + c_n f_n = 0$ for some $c_i \in F$ in V^*. Since $f_i(\alpha_j) = \delta_{ij}$, we can derive $c_j f_j(\alpha_j) = 0$. Thus $c_1 = \ldots = c_n = 0$, which implies $\{f_1, \ldots, f_n\}$ is basis. $\square$

Definition 0.5.3: The Dual Basis

$\{f_1, \ldots, f_n\} \subset V^*$ is called the dual basis of the basis $\{\alpha_1, \ldots, \alpha_n\}$ of V.

Lemma 0.5.2

V : n-d.v.s./F. $\{\alpha_1, \ldots, \alpha_n\}$ is basis of V. Let $\{f_1, \ldots, f_n\}$ is the dual basis. Then

i) For each $f \in V^*$ $f = \sum_{i=1}^n f(\alpha_i) f_i$

ii) For each $v \in V$ $v = \sum_{i=1}^n f_i(v) \alpha_i$

Proof. i): Since $f \in \operatorname{span}\{f_1, \ldots, f_n\}$, $\exists$ expression $f = \sum_{i=1}^n x_i f_i$ for some $x_i \in F$. Evaluate at $\alpha_j : f(\alpha_j) = x_j$.

ii): Since $v \in \operatorname{span}\{\alpha_1, \ldots, \alpha_n\}$, $\exists$ expression $v = \sum_{i=1}^n y_i \alpha_i$. Apply the dual basis. $\square$

Note:-

V : n-d.v.s./F. Let $f \in V^*$. Suppose $f \neq 0$ and $f : V \to F$ be surjective. $N_f := N(f)$. We know $\dim(N(f)) + \dim(R(f)) = \dim(V)$. Since $\dim(R(f)) = 1$, $\dim(N(f)) = n - 1$.

Definition 0.5.4: Hyperspace

V : f.d.v.s./F. subspace W which has property $\dim(W) = \dim(V) - 1$ is called hyperspace.

Definition 0.5.5: Annihilator

V : F-v.s. S be a nonempty subspace. The annihilator of S, $S^\circ = Ann(S)$ is defined to be $S^\circ := \{f \in V^* \mid \forall \alpha \in S\ (f(\alpha) = 0)\}$.

Exercise 0.5.1

$Ann(S)$ is subspace of V^*.

Example 0.5.2

If $S = \{0\}$, then $Ann(S) = V^*$.

Example 0.5.3

If $S = V$, then $Ann(S) = \{0\}$.

Theorem 0.5.1

V : n-d.v.s./F, and W be subspace. Then $\dim(W) + \dim(W^\circ) = \dim(V) = n$.

Proof. $k := \dim(W)$ with $\{\alpha_1, \ldots, \alpha_n\} \subset W$. Choose $\alpha_{k+1}, \ldots, \alpha_n \in V$ s.t. $\{\alpha_1, \ldots, \alpha_n\}$ is basis of V. Let $\{f_1, \ldots, f_k, f_{k+1}, \ldots, f_n\}$ be the dual basis.

Claim 0.5.1

$\{f_{k+1}, \ldots, f_n\}$ is a basis of W°

Proof. Let's see if $f_{k+1}, \ldots, f_n \in W^\circ$. Indeed, by the constructure of the dual basis, all f_i for $i \geq k+1$ vanishes on α_i for $1 \leq i \leq k$. Thus $f_{k+1}, \ldots, f_n \in W^\circ$.

Lin. indep. is obvious since this is part of basis of V^*. □

Claim 0.5.2

$\mathrm{span}\{f_{k+1}, \ldots, f_n\} = W^\circ$

Proof. $f \in W^\circ \subset V$. So $f = \sum_{i=1}^n f(\alpha_i) f_i$. Since $f \in W^\circ$, $f(\alpha_i) = 0$ for all $\alpha_i \in W$, $1 \leq i \leq k$. Thus $f = \sum_{i=1}^n f(\alpha_i) f_i$. □

□

Corollary 0.5.1

V : n-d.v.s./F. W be k-dim. subspace. Then W is intersection of $n-k$ hyperspaces in V of the form N_f for some $0 \neq f_i \in V^*$.

Proof. Basis of W can be extended to basis of V. Take $\{f_1, \ldots, f_n\} \subset V^*$ be the dual basis of $\{\alpha_1, \ldots, \alpha_n\}$. Then $W = \cap_{i=k+1}^n N_{f_i}$. □

Corollary 0.5.2

V : n-d.v.s./F. W be hyperspace. Then $W = N_f$ for some $0 \neq f \in V^*$.

Exercise 0.5.2

W_1, W_2 be subspaces. V : n-d.v.s./F. Then $W_1 = W_2 \iff W_1^\circ = W_2^\circ$.

0.6 The Double Dual

Definition 0.6.1: Double Dual

V : F-v.s. $V^{**} = L(V^*.F) = L(L(V, F), F)$.

Note:-

Dual is not natural in general, but double dual is natural. Define $L_\alpha \in V^{**}$ as: $L_\alpha : V^* \to F : f \mapsto f(\alpha)$.

Note:-

Define $\mathfrak{L} : V \to V^{**} : \alpha \mapsto L_\alpha$.

Claim 0.6.1

$\mathfrak{L}$ is a lin. trans.

Proof. Suppose $\alpha_1, \alpha_2 \in V, c \in F$. $\mathfrak{L}(c\alpha_1+\alpha_2) = L_{c\alpha_1+\alpha_2}(f) = f(c\alpha_1+\alpha_2) = cf(\alpha_1)+f(\alpha_2)$. □

Claim 0.6.2

$\mathfrak{L}$ is injective.

Proof. Suppose for some $\alpha \in V$, we have $\mathfrak{L}(\alpha)L_\alpha \in V^*$ is 0 $\iff$ $\forall f \in V^*$ $(L_\alpha(f) = 0)$ $\iff$ $\forall f \in V^*$ $(f(\alpha) = 0)$ $\iff$ $\alpha = 0$. Thus $\mathfrak{L}$ is injective. □

Note:-

Thus $\mathfrak{L}$ is not surjective in general for infinite dimensional V.

Theorem 0.6.1

V : f.d.v.s./F. Then $\mathfrak{L}$ is an iso. of vec. spaces.

Proof. V : n-d.v.s./F. Then $\dim(V^*) = \dim(V^* *) = n$. Thus $\mathfrak{L}$ is injective. lin. trans. from n-dim. to n-dim. is automatically surjective. □

Definition 0.6.2: Proper Subspace

V : v.s./F. Then $W \subset V$ is proper if it is not equal to V.

Definition 0.6.3: Maximal

V : v.s./F. A proper subspace $W \subsetneq V$ is said to be maximal if there is no intermediate subspace between W and V, i.e., if there is subspace $W \subset Z \subset V$, then either $W = Z$ or $V = Z$.

Note:-

If $\dim(V) = n$, then proper maximal subspace has dim. $n-1$.

Definition 0.6.4: Generalization of Hyperspace

V : v.s./F. A hyperspace of V is a proper maximal subspace of V.

Theorem 0.6.2

V : F-v.s. Suppose $f \in V^* \setminus \{0\}$,

0.7 The Transpose of a Linear Transformation