MAS212A Midterm Exam
(13:00 - 15:45 (165 minutes), October 17th, 2023)

Direction: You should **justify all your answers properly, unless said otherwise.** Total score=200. (8 problems on 4 pages)

Problem 1. (16 points max) Read the questions and write one of (True, False, or Undecidable). No justifications needed. You will get 2 points for each correct answer, but (-1) point for each wrong answer.

(1) If V is an n-dimensional vector space over a field F, then any set of $n+1$ vectors in V is linearly dependent.

(2) For an n-dimensional vector space V, let $S \subset V$ be a subset of n vectors. If $0 \in S$, then S is linearly dependent.

(3) Let V and W be two n-dimensional vector spaces over F. Then they are isomorphic to each other.

(4) Let A be an $n \times n$ matrix. Suppose there is a nontrivial solution for the linear system $Ax = 0$. Then $\det(A) = 0$.

(5) For an $m \times n$ matrix A, it is row-equivalent to a unique reduced row echelon form.

(6) Every subset of a set of linearly independent vectors is linearly independent.

(7) When F is a field, the set F is not a vector space because F consists of just scalars.

(8) For a linear system $Ax = b$, the necessary and sufficient condition for the system to have a solution is $\text{rank}([A \mid b]) = \text{rank}(A)$.

Problem 2. (40 points - 10 each) Let F be a field. Answer the following questions on matrices whose entries are in F.

(1) Let A, B be $n \times n$ matrices. Prove that $\text{tr}(AB) = \text{tr}(BA)$, where $\text{tr}(-)$ is the trace.

(2) Let A, P be $n \times n$ matrices, and suppose that P is invertible. Deduce from (1) that $\text{tr}(P^{-1}AP) = \text{tr}(A)$.

(3) Prove that there does not exist a pair of $n \times n$ matrices A, B such that $AB - BA = I_n$.

(4) Let A be an $n \times n$ matrix. Suppose that for some linearly independent $n \times 1$ matrices $v_1, \cdots, v_n \in F^n$ and $\lambda_1, \cdots, \lambda_n \in F$, we have

$$Av_i = \lambda_i v_i, \quad 1 \leq i \leq n.$$

Prove that $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

More problems on the next page.

Problem 3. (40 points - 10 each) Let V be a finite dimensional F-vector space of dimension $n \geq 1$ and let $T : V \to V$ be a *nonzero* linear transformation.

Recall that the minimal polynomial $m(x) \in F[x]$ of T is the monic polynomial of the smallest degree such that $m(T) = 0$ as a linear transformation on V, if such a polynomial exists. Here $\deg m(x) \geq 1$ and it satisfies the property that for any other polynomial $g(x)$ such that $g(T) = 0$, we have $m(x) \mid g(x)$.

Answer the following questions.

(1) For some T, suppose that $m(x)$ is divisible by x. Prove that T is not invertible.

(2) Suppose T is an idempotent transformation, i.e. $T^2 = T$. Give concrete examples of $T : V \to V$ such that $m(x) = x, x - 1, x^2 - x$, respectively.

(3) Continue to suppose that T is an idempotent transformation. Prove that there exists a basis $\{\alpha_1, \cdots \alpha_r, \beta_{r+1}, \cdots, \beta_n\}$ of V such that $T(\alpha_i) = \alpha_i$ for $1 \leq i \leq r$, while $T(\beta_j) = 0$ for $r + 1 \leq j \leq n$. (Here $r = 0$ is allowed.)

(4) Suppose T is nilpotent, i.e. for some integer $N \geq 1$, we have $T^N = 0$. Let $r \geq 1$ be the smallest integer such that $T^r = 0$. Prove that $m(x) = x^r$.

Problem 4. (24 points) Answer the following questions.

(1) (15 points) Let V be an n-dimensional vector space over a field F. For any linear transform $T : V \to W$ to a vector space W, prove that

$$\text{rank}(T) + \text{nullity}(T) = n.$$

(2) (9 points) In addition to the above, suppose that W is also of dimension n.

Prove that the following three statements are equivalent:

(a) T is injective.

(b) T is surjective.

(c) T is an isomorphism.

More problems on the next page.

Problem 5. (20 points) Let V be an n-dimensional vector space over a field F, and let $W \subset V$ be a subspace. Recall that the annihilator $\mathrm{Ann}(W)$ of W (in the book, it is also denoted by W^0) is defined to be

$$\mathrm{Ann}(W) = \{f \in V^* \mid f(\alpha) = 0 \text{ for all } \alpha \in W\}.$$

Prove that $\dim W + \dim \mathrm{Ann}(W) = n$.

Problem 6. (20 points) Let V be an n-dimensional F-vector space. Let

$$F\langle -, - \rangle : V \times V \to F, \quad (v_1, v_2) \mapsto F\langle v_1, v_2 \rangle$$

be a 2-linear function, also called a *bilinear* function. In this problem, we will just write $\langle -, - \rangle$ instead of $F\langle -, - \rangle$.

We say that a bilinear function is *nondegenerate* if it has the following additional properties: (i) when $\langle v, w \rangle = 0$ for all $w \in V$, then $v = 0$ and (ii) when $\langle v, w \rangle = 0$ for all $v \in V$, then $w = 0$. Existence of such a function is not automatic, but in some good cases there exists.

Answer the following questions.

(1) (5 points) Suppose that there is a bilinear function $\langle -, - \rangle$. Let $v \in V$. Prove that $\langle v, - \rangle$ defines an element of the dual vector space V^*. (Denote it by $\hat{v}$.)

(2) (15 points) Suppose that there is a nondegenerate bilinear function $\langle -, - \rangle$. Prove that V has a basis $\{v_1, \cdots, v_n\}$ such that

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

(Such a basis is called an orthonormal basis of V with respect to $\langle -, - \rangle$.)

For an orthonormal basis $\{v_1, \cdots, v_n\}$ of V with respect to $\langle -, - \rangle$, deduce that $\{\hat{v}_1, \cdots, \hat{v}_n\}$ is the dual basis of V^* associated to $\{v_1, \cdots, v_n\}$, i.e. $\hat{v}_i(v_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$.

More problems on the next page.

Problem 7. (20 points) Let K be a commutative ring with identity. Let $D : K^{n \times n} \to K$ be a function that is n-linear on the rows, and alternating.
For each matrix $A \in K^{n \times n}$, in class we proved that D satisfies in general

$$D(A) = \det(A)D(I_n),$$

where

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)A_{1,\sigma(1)}A_{2,\sigma(2)} \cdots A_{n,\sigma(n)}.$$

Accepting the above, answer the following question: let $B \in K^{n \times n}$ be a fixed matrix. For each $X \in K^{n \times n}$, define the function

$$D(X) := \det(XB).$$

Prove that D is n-linear, alternating, and deduce that this proves $\det(AB) = \det(A)\det(B)$ for $A, B \in K^{n \times n}$.

Problem 8. (20 points - 10 each) Let $T : V \to V$ be a linear operator on an F-vector space V. Suppose that T is nilpotent, i.e. there exists an integer $N \geq 1$ such that $T^N = 0$. Let $I_V : V \to V$ be the identity function. We say that a linear operator $U : V \to V$ is *unipotent* if $U - I_V$ is nilpotent.

(1) Let T be a nilpotent linear operator. Prove that $I_V - T$ is an invertible linear transformation. Deduce that all unipotent linear transformations are invertible.

(2) Let $U : V \to V$ be a unipotent linear operator, and suppose V is finite dimensional. Write all possible minimal polynomials of U.

End of the Exam