0.1 Commutative Rings

Definition 0.1.1: Ring

R : a ring with two operation $+, \cdot$ s.t. $< R, + >$ form abelian group and $\cdot$ satisfies $a \cdot (b+c)$ and $(b+c) \cdot a$. A ring with unity is a ring with $1 \in R$ s.t. $\forall a \ (1 \cdot a = a \cdot 1 = a \in R)$.

0.2 Determinant Functions

Definition 0.2.1: n-Linear and Alternating

K : a ring. A function $D : K^{n \times n} \to K$. This is considered as a function on n rows and n columns.
i) We say D is n-linear if D is a linear function on the i-th row while fixing others. $D(ca_1 + a_1', a_2, \ldots, a_n) = cD(a_1', a_2, \ldots, a_n) + D(a_1, a_2, \ldots, a_n)$.
ii) An n-linear function $D : K^{n \times n} \to K$ is called alternating if $D(A) = 0$ when $\forall i \neq j \ (a_i = a_j)$.

Exercise 0.2.1

$D : K^{n \times n} \to K$: alternating n-linear function. $A \in K^{n \times n}$. $A' :=$ matrix obtained by interchanging i, j-th rows and fix others. Then $D(A') = -D(A)$.

Proof. Using given property. Exercise! □

Definition 0.2.2: Determinant Function

K : commu. ring with 1. $D : K^{n \times n} \to K$ be a function. We say D determinant function if D is n-linear, alternating, and $D(I_n) = 1$.

Theorem 0.2.1

$\exists!$ such D that we call the determinant function.

Theorem 0.2.2

Concrete description of D in terms of permutation.

Definition 0.2.3: Minor

K : commu. ring with 1, $n > 1$. Let $A \in K^{n \times n}$ and (i, j) for $1 \le i, j \le n$. $A(i|j)4$ is $(n-1) \times (n-1)$ mat. with i-th row and j-th col. removed. We call this (i, j)-minor.

Definition 0.2.4

$D(A(i|j)) = D_{ij}(A)$.

Theorem 0.2.3

$n > 1$, $D : K^{(n-1) \times (n-1)} \to K$, alternating $(n-1)$-linear function. Let $1 \le j \le n$. $A \in K^{n \times n}$. Define $E_j(A) := \sum_{i=1}^{n} (-1)^{i+j} A_{ij} D_{ij}(A)$. Then E_j is an alternating n-linear function on

$K^{n\times n}$. Also, if $D : K^{(n-1)\times(n-1)} \to K$ is a determinant function, so is E_j.

Proof. $A : n \times n$ mat. Note that $D_{ij}(A)$ is indep. of the entries of i-th row and j-th col. D is $(n-1)$-linear on $K^{(n-1)\times(n-1)}$, so $D_{ij}(A)$ is linear, further more $A_{ij}D_{ij}(A)$ is n-linear. Thus E_j is n-linear being a lin. comb. of n-linear functions. To prove alternating, suppose A has two equal rows at α_k, α_{k+1}. Take $i \neq k, k+1$. Then $D_{ij}(A) = 0$ because $A(i|j)$ has two identical rows and D is alternating. Then $E_j(A) = (-1)^{k+j}D_{kj}(A) + (-1)^{k+1+j}D_{k+1j}(A)$. Here, $A_{kj} = A_{k+1j}$, $D_{k+1j} = D_{kj}$, thus 0. This shows E_j is alternating n-linear. Also, since $I_n(i|j) = I_{n-1}$, we can see trivailly $E_j(I_n) = 1$. □

Corollary 0.2.1

For all $n \in \mathbb{N}$, $\exists$ det, function.

Proof. If $n = 1$, $D_1 = Id_k$ is a det. function. Suppose $n > 1$ and cor. holds for $1 \leq i < n$. Then D_{n-1} is a det. function, thus we can take $D_n = E_j$ written in terms of D_{n-1}. □

0.3 Permutations and the Uniqueness of Determinants

Definition 0.3.1: Permutation

A permutation σ of S is a bijective function $\sigma : S \to S$. We have $|S|!$ permutations.

Definition 0.3.2: Transposition

$\tau \in S_n$ is called transposition if it interchange just the values of 2 members.

Note:-

Every peprmutation can be written as a product of disjoint cycles. Also, every cycle is a product of non-disjoint transpositions.

Theorem 0.3.1

S_n be the permutations on n letters. $\sigma \in S_n$. For any permutation, the number of transpositions needed to express $\sigma \mod 2$ is an invariant of σ. Also, we define $\text{sgn}(\sigma)$ as 1 if mod is even, -1 if odd.

Corollary 0.3.1

$\sigma_1, \sigma_2 \in S_n$. Then $\text{sgn}(\sigma_1\sigma_2) = \text{sgn}(\sigma_1)\text{sgn}(\sigma_2)$.

0.4 Additional Properties of Determinants

0.5 Modules

This Chapter is Intentionally Skipped at Lectures

0.6 Multilinear Functions

This Chapter is Intentionally Skipped at Lectures

0.7 The Grassman Ring

This Chapter is Intentionally Skipped at Lectures