0.1 Vector Spaces

Definition 0.1.1: Vector Spaces

A vector space consists of the following:

1. field F of scalars
2. a set V of objects called vectors
3. $\forall\{\alpha,\beta,\gamma\}\subset V$, a rule called vector addition holds:
   - addition is commutative: $\alpha+\beta=\beta+\alpha$
   - addition is associative: $\alpha+(\beta+\gamma)=(\alpha+\beta)+\gamma$
   - $\exists!0\in V\ \ (\alpha+0=\alpha)$
   - $\exists!(-\alpha)\in V\ \ (\alpha+(-\alpha)=0)$
4. $\forall\{\alpha,\beta\}\subset V\ \ \forall\{c_1,c_2\}\subset F$, a rule called scalar multiplication holds:
   - $1\alpha=\alpha$
   - $(c_1c_2)\alpha=c_1(c_2\alpha)$
   - $c_1(\alpha+\beta)=c_1\alpha+c_1\beta$
   - $(c_1+c_2)\alpha=c_1\alpha+c_2\alpha$

Definition 0.1.2: Linear Combinations

$\alpha\in V$ is said to be linear combination of the vectors $\alpha_1,\dots,\alpha_n\in V$ if $\exists c_1,\dots,c_n\in F$ s.t.

$$\alpha=c_1\alpha_1+\cdots+c_n\alpha_n=\sum_{i=1}^{n}c_i\alpha_i$$

0.2 Subspaces

Definition 0.2.1: Subspaces

$W\subset V$ is called subspace if W satisfies vector space axioms.

Theorem 0.2.1

$((V$:f.d.v.s$/F)\wedge(\{0\}\subsetneq W\subset V))\Rightarrow(W$ is subspace $\iff\ \forall\{\alpha,\beta\}\in V\ \ \forall c\in F\ \ (c\alpha+\beta\in V))$.

Proof. We have to check: $W\neq\emptyset\Rightarrow\exists w\in W\Rightarrow 0\in W$. □

Theorem 0.2.2

$\{W_i\}$:= collection of subspaces of F-v.s. V. Let $W:=\cap W_i$. then W is also subspace.

Proof. All W_i has 0, thus $0\in\cap W_i$, which implies $W\neq\emptyset$. Let $v_1,v_2\in W$, $c\in F$. Then $\forall v_1,v_2\ (v_1,v_2\in W_i)$. Since W_i is subspace, $cv_1+v_2\in W_i$ for all i, thus also in W. □

Definition 0.2.2: Span

V : F-v.s. $S \subset V :=$ any nonempty subset. The span(S) is the intersetion of all subspaces of V that contains S.

Theorem 0.2.3

span(S) is set of All linear combination of S/F.

Proof. $W :=$ span(S) and let L be set of all lin. comb. of S/F. Then obviously, $L \subset W$ because $S \subset WW$ and W is subspace.

Conversely, note that $S \subset L$. If we prove L is subspace, then since $S \subset L$, $W =$ span$(S) \subset L$. Then L is apparently a subsapce. Thus $W = L$. $\square$

0.3 Bases and Dimensions

Definition 0.3.1: Linearly Independent

V : F-v.s., and take S as subset of V. We say S is linearly independent if $\exists \alpha_1, \ldots, \alpha_n \in S$ and $c_1, \ldots, c_n \in F$, not all zero, s.t. $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ has nontrivial solution. If S is not linaerly dependent, we say it is linearly independent.

Theorem 0.3.1

V : F-v.s. $\alpha_1, \ldots, \alpha_n$ are linearly independent $\iff$ $\forall i \in [n]\, \forall c_i \in F$ $((c_1\alpha_1 + \cdots + c_n\alpha_n = 0) \Rightarrow c_1 = c_2 = \cdots = c_n = 0$.

Proof. Exercise! $\square$

Definition 0.3.2: Basis

V : F-v.s. A basis of V is a subset $S \subset V$ s.t. S is lin. indep. and span$(S) = V$.

Definition 0.3.3: Finite Dimensional

If basis S has property $|S| < \infty$, we say V is finite dimensional vector space.

Theorem 0.3.2

V : F-v.s. that is spanned by $\{\beta_1, \ldots, \beta_n\} \subset V$. Then any lin. indep. set of vec. in V is finite and card. is no bigger than n.

Proof. E.T.S. that every subset S with more than n vec. are lin. dep. Suppose $S = \{\alpha_1, \ldots, \alpha_m\}$, for distinct vec. with $m \geq n$. Since $\{\beta_1, \ldots, \beta_n\}$ spans V, for each $1 \leq j \leq m$, $\alpha_j = \sum_{i=1}^{n} A_{ij}\beta_i$. Let $x_1, \ldots, x_m \in F$ be arbitrary chosen. Then $x_1\alpha_1 + \cdots + x_m\alpha_m = \sum_{j=1}^{m} x_j\alpha_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{ij}x_j\right)\beta_i$. Consider the system $[A_{ij}][\mathbf{x}^T] = 0$. This has at least 1 free variable, which leads system has nontrivial solution. $\square$

Corollary 0.3.1

V: F-v.s. that has finite spanning set. Then any two basis of V have same card.

. Apply Theorem 0.3.2 to both side of two different basis. □

Lemma 0.3.1

$W \subsetneq V$ be finite dim. v.s. Then $\dim(W) < \dim(V)$.

Proof. Let S_0 be a basis of W. S_0 is lin. indep., so can enlarged it to get a basis of V. Since W is propersubset of V, $\exists v \in V \backslash W$. Take $S_1 = S_0 \cup \{v\}$, and repeat this. finite dimensional condition of V implies this algorithm terminates in finite times, and thus we can conclude $\dim(W) < \dim(V)$. □

Theorem 0.3.3

$W_1, W_2 \subset V$: finite v.s. Then $W_1 + W_2$ is a finitie dim. v.s. and $\dim(W_1) + \dim(W_2) = \dim(W_1 + W_2) - \dim(W_1 \cap W_2)$.

Proof. Choose $\{\alpha_1, \dots, \alpha_d\}$ a basis for $W_1 \cap W_2$. We can extend this into W_1 and W_2's basis. Take $\{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_a\}$ be basis for W_1 and $\{\alpha_1, \dots, \alpha_d, \gamma_{d+1}, \dots, \gamma_b\}$ be basis for W_2.

Claim 0.3.1

$\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_a, \gamma_{d+1}, \dots, \gamma_b$ is a basis for $W_1 + W_2$.

Proof. Suppose for arbitrary lin. indep. set B, $\text{span}(B) = W_1 + W_2$. Let $x \in W_1 + W_2$. Then $x = w_1 + w_2$ where $w_1 \in \text{span}\{\alpha, \beta\}$ and $w_2 \in \text{span}\{\alpha, \gamma\}$, thus $x \in \text{span}\{\alpha, \beta, \gamma\}$. On the other hand, each vec. in B is already in $W_1 + W_2$. Thus $\text{span}(B) = W_1 + W_2$. □

Claim 0.3.2

This B is lin. indep.

Proof. Suppose we have $\sum a_i \alpha_i + \sum b_j \beta_j + \sum c_k \gamma_k = 0$ for alal scalars are 0. Then $\sum a_i \alpha_i + \sum b_j \beta_j = -\sum c_k \gamma_k$. Thus $\sum c_k \gamma_k \in W_1 \cap W_2$ where $\{\alpha_1, \dots, \alpha_d\}$ is basis for $W_1 \cap W_2$ and γ are indep. with α. Thus $\forall k \in \mathbb{N}$ $(c_k = 0)$. Simarly, we can see that all scalars are 0. Thus B is indep. □

This two claim leads $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. □

0.4 Coordinates

Definition 0.4.1: Coordinates (Ordered Basis)

An ordered basis for F-v.s. V is a sequence of vec. that forms a basis.

Lemma 0.4.1

V : f.d.v.s./F. Suppose $B = \{v_1, \dots, v_n\}$ is an ordered basis of V. Then for each $x \in V$, $\exists!$ expression of the form $x = x_1 v_1 + \cdots + x_n v_n$ for some $x_i \in F$.

Proof. Existence of expression of form is trivial since B is basis of V.

For uniqueness, suppose we have two expression. Then indepence condition of each v_i leads these expression have exactly same coefficients. □

Definition 0.4.2: Coordinate Matrix

V : f.d.v.s./F, B be ordered basis. We define $[x]_B = [x_1\, x_2\, \cdots\, x_n]^T$ the coordinate matrix of x w.r.t. the basis B.

Theorem 0.4.1

V : f.d.v.s./F, B and B' be two different ordered basis of V. Then $\exists!$ invertible mat. P s.t. $\forall x \in B$, $[x]_B = P[x]_{B'}$, also $[x]_{B'} = P^{-1}[x]_B$.

Proof. Let $B := \{\alpha_1, \ldots, \alpha_n\}$ and $B' := \{\beta_1, \ldots, \beta_n\}$. For $\beta_j \in B'$, since B is a basis, $\beta_j = \sum_{i=1}^{n} P_{ij}\alpha_i$ and this P_{ij} are uniquely decided. Let $P := [P_{ij}]$. Let $x \in V$. Write $[x]_B = [x_1\, \ldots\, x_n]^T$, $[x]_{B'} = [x'_1\, \ldots\, x'_n]^T$. Then $x = \sum_i \left(\sum_j x'_j P_{ij}\right)\alpha$. By uniqueness, we can derive $[x]_B = P[x']_B$. Since B and B' are lin. indep., x=0 implies $[x]_B = [x]_{B'} = 0$. Thus P is invertible. $\square$

0.5 Summary of Row-Equivalence

This Chapter is Intentionally Skipped at Lectures

0.6 Computations Concerning Subspace

This Chapter is Intentionally Skipped at Lectures