Department of Civil Engineering, IIT Madras
CV 5100 (Modeling, Uncertainty and Data for Engineers)

3 Oct, 2025 | Quiz-2 | 8:00–8:50 pm

Name: Roll:

Instructions:
1. Answer questions in the provided space.
2. Walk through your reasoning without being verbose.
3. Make assumptions wherever necessary, and *state* it.
4. Z-table is provided at the end.

1. **(Uncertainty propagation)** Suppose X_1 and X_2 are two random variables with means μ_1 and μ_2, respectively and standard deviations σ_1 and σ_2. If correlation coefficient between X_1 and X_2 is ρ_{12}. Derive an expression for variance of a random variable $Y = a_1X_1 + a_2X_2 + c$, where a_1, a_2, and c are deterministic constants. [2]

Solution

$\mu_Y = E[Y] = a_1\mu_1 + a_2\mu_2 + c$

$Var(Y) = E[(Y-\mu_Y)^2]$ 1

$= E[\{(a_1x_1 + a_2x_2 + c) - (a_1\mu_1 + a_2\mu_2 + c)\}^2]$

$= E[\{a_1(x_1-\mu_1) + a_2(x_2-\mu_2)\}^2]$

$= E[a_1^2(x_1-\mu_1)^2] + E[a_2^2(x_2-\mu_2)^2] + E[2a_1a_2(x_1-\mu_1)(x_2-\mu_2)]$

$\Rightarrow \boxed{Var(Y) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\rho_{12}\sigma_1\sigma_2}$ 1

2. **(Approximation)** Let X be a random variable with mean $\mu_X = 10$ and standard deviation $\sigma_X = 3$. If $Y = 3X^2 + 2X + \cos^2(2\pi X) - 2e^{-3X}$, find
(a) the second-order approximation of the mean of Y and
(b) first-order approximation of the variance of Y. [2+1]

Solution

(a) $Y = 3x^2 + 2x + \cos^2(2\pi x) - 2e^{-3x} = g(x)$ [Say]

$\Rightarrow Y \cong g(\mu_x) + \frac{\partial g}{\partial x}(x-\mu_x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x-\mu_x)^2$

$\Rightarrow \mu_Y \cong g(\mu_x) + 0 + \frac{1}{2}\left.\frac{\partial^2 g}{\partial x^2}\right|_{\mu_x} \cdot \sigma_x^2$ 1 ← SO approximation

At $x = \mu_x$, $g(x) = 3\cdot 10^2 + 2\cdot 10 + \cos^2(20\pi) - 2e^{-30} = 321$

Two derivatives are:

$\partial g/\partial x = 6x + 2 - 2\pi\sin(4\pi x) + 6e^{-3x}$

$\partial^2 g/\partial x^2 = 6 - 8\pi^2\cos(4\pi x) - 18e^{-3x}$

at $x = \mu_x$: $\partial g/\partial x = 62$, $\partial^2 g/\partial x^2 = -72.96$

Hence,

$$\mu_Y \cong 321 + \frac{1}{2} \times (-72.96) \times 3^2$$

$$\Rightarrow \boxed{\mu_Y \cong -7.32} \quad 1$$

(b) $\sigma_Y^2 \cong \left(\frac{dg}{dx}\right)^2\Big|_{\mu_X} \cdot \sigma_X^2 \quad \leftarrow$ FO approximation

$$= 62^2 \cdot 3^2$$

$$\Rightarrow \boxed{\sigma_Y^2 \cong 34596} \quad 1$$

3. **(Moments)** Name first four moments of a random distribution. How are each of them used? [1]

Solution.

#	Name	Usage
First	mean	centrality
second central	variance	dispersion
third central	skewness	skewness
fourth central	kurtosis	tailedness

1

4. **(Least-square)** For a linear model $y = \beta_0 + \beta_1 \sin x + \beta_2 \cos x$, if five observations for (x, y) are given as $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). Write the matrix form for least-square estimate of parameters. [2]

Solution

$$Y_1 = \beta_0 + \beta_1 \sin x_1 + \beta_2 \cos x_1$$

$$\vdots \qquad \qquad \vdots$$

$$Y_5 = \beta_0 + \beta_1 \sin x_5 + \beta_2 \cos x_5$$

$$\Rightarrow \underbrace{\begin{Bmatrix} Y_1 \\ \vdots \\ Y_5 \end{Bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 & \sin x_1 & \cos x_1 \\ \vdots & \vdots & \vdots \\ 1 & \sin x_5 & \cos x_5 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{Bmatrix}}_{X} \quad 1 \quad \Rightarrow Y_{5\times1} = A_{5\times3} X_{3\times1}$$

$$\Rightarrow \boxed{X = (A^T A)^{-1} (A^T Y)} \quad 1$$

5. **(Central limit theorem)** Prove that standard error of the mean of a sample of size n is inversely proportional to $\sqrt{n}$. [2]

Solution

$$\bar{x} = (x_1 + x_2 + \cdots + x_n)/n$$

$$\Rightarrow \quad \sigma_{\bar{x}}^2 = \frac{1}{n^2}\sigma_{x_1}^2 + \frac{1}{n^2}\sigma_{x_2}^2 + \cdots + \frac{1}{n^2}\sigma_{x_n}^2 \quad (1)$$

Since $\sigma_{x_1} = \sigma_{x_2} = \cdots = \sigma_{x_n} = \sigma$

$$\sigma_{\bar{x}}^2 = n \cdot \frac{1}{n^2}\sigma^2 \quad \Rightarrow \quad \sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$$

$$\Rightarrow \quad \boxed{\sigma_{\bar{x}} \propto \frac{1}{\sqrt{n}}} \quad (1)$$

6. **(Hypothesis testing)** A researcher is testing whether adding a new chemical increases the mean yield strength of a steel alloy, known to have standard deviation $\sigma = 40$ MPa. The current process has mean yield strength 500 MPa. They will take a sample of n = 16 specimens. The researcher uses a one-tailed test at significance level $\alpha = 0.05$.

(a) State the two hypotheses.

(b) Find the critical value of $\overline{X}$ above which H_0 will be rejected. Interpret what a Type I error means in this context.

(c) Suppose the true mean with the chemical is 550 MPa. Compute the probability of a Type II error under this alternative. Interpret what this probability means for the researcher.

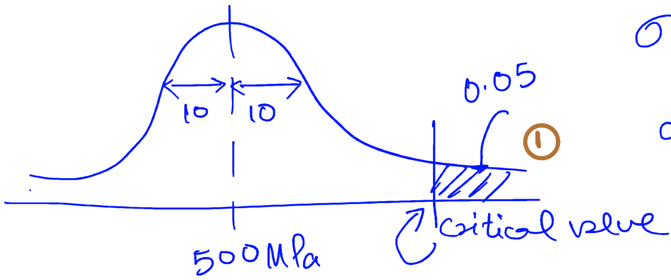
[1+2+2]

Solution

(a) $H_0: \mu = 500$ MPa

$H_1: \mu > 500$ MPa

(b)



$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = 10$ MPa

$\alpha = 0.05$

one-tailed $\Rightarrow$ $z_{critical} = \Phi^{-1}(1-0.05)$

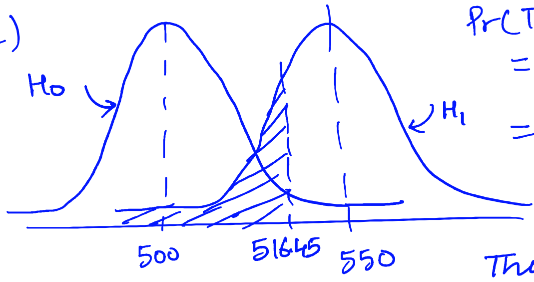
$= \Phi^{-1}(0.95) = 1.645$

Critical value $= 500 + 1.645 \times 10$

$\Rightarrow$ $\boxed{\text{Critical value} = 516.45 \text{ MPa}}$

Type I error would mean rejection of Ho, while the addition of the chemical does NOT increase the yield strength.

(c)



Pr(Type II error)

$= \Pr(H_0 \text{ is NOT rejected} \mid H_1 \text{ is true})$

$= \Pr(\bar{X} < 516.45 \mid \mu = 550, \sigma = 10)$

$= \Phi\left(\frac{516.45 - 550}{10}\right) = \Phi(-3.355)$

$= 0.0004$

Thus, there is a 0.04% probability that the researcher will incorrectly conclude that the chemical does NOT improve yield strength to 550MPa.