CV 510$^{0}_{1}$
Modeling, Uncertainty, and Data for Engineers
(July – Nov 2025)

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Module Overview

B.1 Uncertainty & Estimation

B.1.1 **Probabilistic basics and RVs**
- Probability laws
- Random variables
- Discrete, continuous RVs
- PMF/PDF/CDF
- Expectation
- Variance/cov./corr.

Q1C5, Funda6-7-8

B.1.2 **Core distributions**
- Gaussian
- Uniform
- Exponential
- Lognormal
- Symmetric/skewness
- *Tails*: thick/light

Q1C5, Funda9-10

B.1.3 **Propagation laws & least-sq.**
- Uncertainty propagation
- Least-square errors
- Parameter estimation

Q1C2-C3, Funda9-10

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Flow

- Least-square errors
- Parameter estimation

Linear models: intro

- Linear functional relationship

$$\mathrm{E}[Y \mid X = x] = Ax$$

A: deterministic constant

In other words,

$$Y = Ax + \epsilon$$

such that

$$\mathrm{E}[\epsilon] = 0$$

$\varepsilon \equiv$ random error: zero mean, finite variance

In a deterministic linear model, x is given:

$$\sigma_Y = \sigma_\epsilon$$

Linear models: example

- Linear function relationship

$$E[Y_i] = x_1 + x_2 t_i$$

t_i: observation time; epoch; given; deterministic

x_1: intercept
x_2: rate of change

Linear functional model is

$$E\left(\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_n \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

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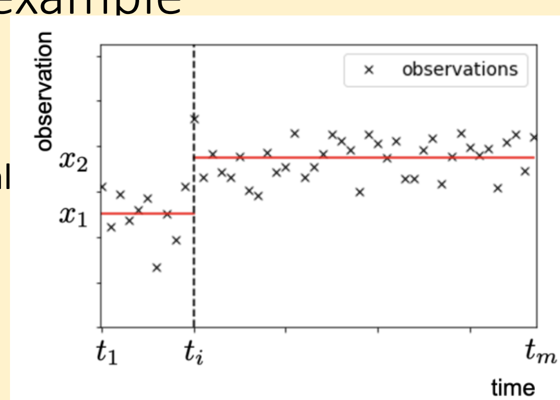
Linear models: example

- Is this a linear model



Write its linear functional relationship

$$E\left(\begin{bmatrix} Y_1 \\ \vdots \\ Y_{i-1} \\ Y_i \\ \vdots \\ Y_n \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

A linear model in x

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Linear models: example

- Is this a linear model

Step function, $s = x_2 - x_1$

Write its linear functional relationship

$$E\left(\begin{bmatrix} Y_1 \\ \vdots \\ Y_{i-1} \\ Y_i \\ \vdots \\ Y_n \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ s \end{bmatrix}}_{x}$$

A linear model in x

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Linear models: example

- Is this a linear model

$$E[Y_i] = x_1 + x_2 t_i + x_3 t_i^2$$

Write its linear functional relationship

$$E\left(\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

x_i are coefficients (yet to be determined) in this model.

A linear model in x

Misleading variable naming!

$$E[Y_i] = c_1 + c_2 t_i + c_3 t_i^2$$

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Linear models: example

- Is this a linear model

$$E[Y_i] = \beta_0 \cdot x_{1,i}^{\beta_1} \cdot x_{2,i}^{\beta_2}$$

Write its linear functional relationship

$$E\left(\begin{bmatrix}\ln Y_1\\ \ln Y_2\\ \vdots\\ \ln Y_n\end{bmatrix}\right) = \underbrace{\begin{bmatrix}1 & \ln x_{1,1} & \ln x_{2,1}\\ 1 & \ln x_{1,2} & \ln x_{2,2}\\ \vdots & \vdots & \vdots\\ 1 & \ln x_{1,n} & \ln x_{2},n\end{bmatrix}}_{A} \underbrace{\begin{bmatrix}\ln \beta_0\\ \beta_1\\ \beta_2\end{bmatrix}}_{x}$$

A linear model in x

x_i are coefficients (yet to be determined) in this model

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Elements of models

0. Eyeball the data.

Scatter, histogram, change scales (log-log, semilogX, semilogY, e^X, so on).

1. Estimation
2. Inference
3. Prediction
4. Explanation
5. Diagnosis

Model: $Y = Ax + \epsilon$ $\quad Y = x\beta + \epsilon$

~~Predicted~~/response: Y dependent var.

~~Predictor~~/feature: x ind./explanatory/covariate

Parameters: A or β

Estimating parameters aka "model building stage":

- Role of x is not to predict (y) as yet!
- It is to estimate A or β
  - Better call x at this stage feature/covariate/explanatory var.

Goal of estimation: Obtain parameter estimates ($\hat{A}$)

Elements of models

0. Eyeball the data.
   Scatter plot, histogram, change scales

1. Estimation   Goal: Obtain parameter estimates ($\hat{\beta}$)
   Concepts: least squares, maximum likelihood, fitting the model

2. Inference   Goal: Model comparison; uncertainty in parameter ($\hat{\beta}$)
   Concepts: Conf. interval for $\hat{\beta}$, hyp. testing, std. error, p-values

3. Prediction   Goal: Forecast new outcomes (x is now a predictor)
   Concepts: CI for $\hat{y}$ (prediction error), mean-squared error (MSE)

4. Explanation Goal: Interpret the fitted model, understand relationships
   Concepts: feature importance, causality

5. Diagnosis   Goal: Assess model assumptions and validity
   Concepts: error (constant Var.), unusual observations (outliers)

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Least-square errors

A linear model with a single feature has two parameters:

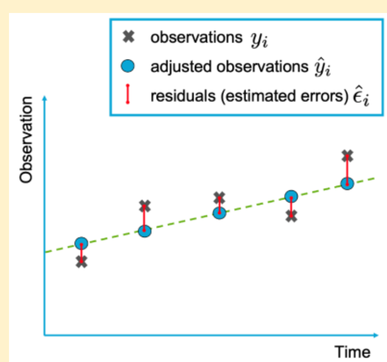
Intercept   slope

An example:

$$y_1 = mx_1 + c$$
$$y_2 = mx_2 + c$$
$$\dots$$
$$y_5 = mx_5 + c$$

5 equations; 2 unknowns (m, c)

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix}_{5\times1} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{5\times2} \begin{bmatrix} \\ \end{bmatrix}_{2\times1}$$



$$x = \begin{bmatrix} m \\ c \end{bmatrix}$$

$$A^{\mathrm{T}}y = A^{\mathrm{T}}Ax \ \Rightarrow x = \left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}}y$$

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Least-square errors

A linear model with a single feature has two parameters:

Intercept slope

An example:

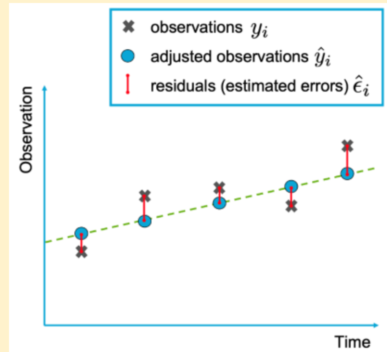
$y_1 = mx_1 + c$

...

$y_5 = mx_5 + c$

$A = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}; x = \begin{bmatrix} m \\ c \end{bmatrix}$

$A^{\mathrm{T}} y = A^{\mathrm{T}} A x \;\Rightarrow x = \left(A^{\mathrm{T}} A\right)^{-1} A^{\mathrm{T}} y$



~~Magic:~~ What does the calculated m and c mean?

Least-square estimates of m, c.

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Least-square errors

$$\mathrm{E}[Y \mid X = x] = Ax$$

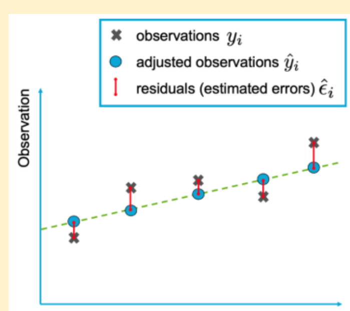
Observed	(x_i, y_i)	x and y
predicted	$\hat{y}_i = A\hat{x}_i$	$\hat{x}$ and $\hat{y}$

$$\hat{\epsilon}_i = y_i - \hat{y}_i = y_i - A\,\hat{x}_i$$

2-norm error,

$\|\epsilon\| = \sqrt{\epsilon_1^2 + \epsilon_2^2 + \cdots + \epsilon_n^2} = \sqrt{\epsilon^{\mathrm{T}} \epsilon}$

Minimizing $\|\epsilon\| \equiv$ minimizing $\|\epsilon\|^2$



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Parameter estimation

- Next week's class:
  - Estimation: Maximum likelihood estimation
  - Inference: Confidence interval
  - Inference: Hypothesis testing
  - Goodness of fit: χ^2, KS

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Questions, comments, or concerns?

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