CV 510$^{0}_{1}$
Modeling, Uncertainty, and Data for Engineers

(July – Nov 2025)

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Module Overview

B.1 Uncertainty & Estimation

B.1.1 **Probabilistic basics and RVs**
- Probability laws
- Random variables
- Discrete, continuous RVs
- PMF/PDF/CDF
- Expectation
- Variance/cov./corr.

Q1C5, Funda6-7-8

B.1.2 **Core distributions**
- Gaussian
- Uniform
- Exponential
- Lognormal
- Symmetric/skewness
- *Tails*: thick/light

Q1C5, Funda9-10

B.1.3 **Propagation laws & least-sq.**
- Uncertainty propagation
- Least-square errors
- Parameter estimation (MLE)

Q1C2-C3, Funda9-10

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Flow (so far and week ahead)

- So far:
  - Probability axioms, rules, PMF/PDF/CDF
  - Variance, covariance, correlation
  - Distributions: uniform, Gaussian, exponential, Gumbel, lognormal
  - Standard normal; reading z-table
  - Uncertainty propagation: linear, nonlinear function (FO and SO approximation)
  - Fitting a distribution Linear regression (estimation): least-square method
- This week:
  - Estimation: Method of moments (MoM), Maximum likelihood estimation (MLE)
  - Inference: Confidence interval (CI)
  - Inference: Hypothesis testing (HT)
  - Goodness of fit (GoF): χ^2, KS
- Check-in with teachbook
  - MLE: Q1C3.6 and Q1C5.4.2
  - CI: Q1C3.5
  - HT: Q1C3.9
  - GoF: Q1C5.4

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Parameter estimation: method of moments

- Obtain the moments (e.g., mean, variance) from two sources:
  - Observations:
  - Parameters:

$$\mathrm{E}[X] = \mu_X = \int_{-\infty}^{+\infty} x f_X(x) \mathrm{d}x$$

$$\mathrm{Var}[X] = \sigma_X^2 = \mathrm{E}[(X - \mu_X)^2] = \mathrm{E}[X^2] - \mu_X^2$$

Depends on the distribution type

- Equate them to solve for the parameters

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Distributions and parameters

Dist.	PDF	CDF	Mean & Variance
Normal/ Gaussian	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$	$F(x) = \Phi(x)$ Use z-table	$\mathrm{E}[X] = \mu$ $\mathrm{Var}[X] = \sigma^2$
Uniform	$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$	$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a,b] \\ 1 & \text{otherwise} \end{cases}$	$\mathrm{E}[X] = \frac{1}{2}(a+b)$ $\mathrm{Var}[X] = \frac{1}{12}(b-a)^2$
Exponential	$f(x) = \lambda \exp(-\lambda x)$	$F(x) = 1 - \exp(-\lambda x)$	$\mathrm{E}[X] = \frac{1}{\lambda}$ $\mathrm{Var}[X] = \frac{1}{\lambda^2}$
Gumbel	$f(x) = \frac{1}{\beta}\exp[-z + \exp(-z)]$, where $z = \frac{x-\alpha}{\beta}$	$F(x) = \exp[-\exp(-z)]$	$\mathrm{E}[X] = \alpha + \beta\gamma$ $\mathrm{Var}[X] = \frac{\pi^2}{6}\beta^2$ $\gamma = 0.577$
Lognormal	$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]$	$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$ Use z-table	$\mathrm{E}[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$ $\mathrm{Var}[X] = [\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2)$

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Example

- Assume that Earthquakes in Rome expressed as MCS index follow Gumbel type-I distribution. Earthquake intensity and number of earthquakes between 1000 CE – 1980 CE are:

MSC intensity, x_i	2	3	4	5	6	7
Number, f_i	113	132	56	22	4	2

$\sum x = \sum x_i f_i$	$\sum x^2 = \sum x_i^2 f_i$	$\sum f_i$
994	3328	329

Find the parameters of the distribution.

- Observations:

$\mathrm{E}[X] = \mu_X = 994/329 = 3.02$

$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mu_X^2 = 10.12 - 3.02^2 \approx 0.99$

$\mathrm{E}[X^2] = 3328/329 \approx 10.12$

- Parameters for Gumbel type-I:

$\mathrm{E}[X] = \alpha + \beta\gamma$

$\mathrm{Var}[X] = \frac{\pi^2}{6}\beta^2$

- Equate them to solve for the parameters

Solve to find, $\alpha \approx 2.57, \beta \approx 0.77$

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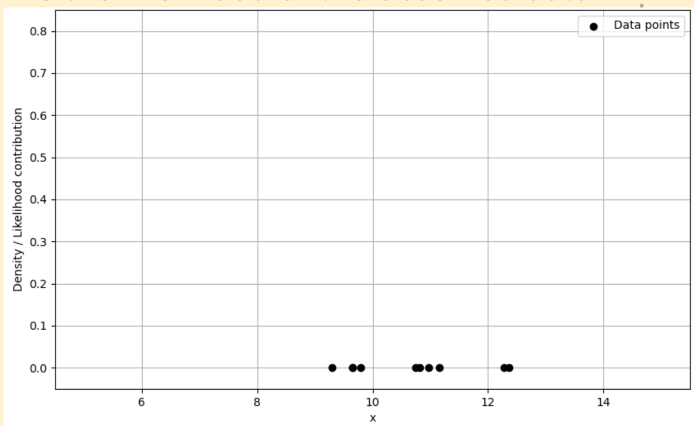
Maximum likelihood estimation

Basic idea

Objective of MLE:

To determine the parameters of a model
such that
they maximize the likelihood of the observed data.



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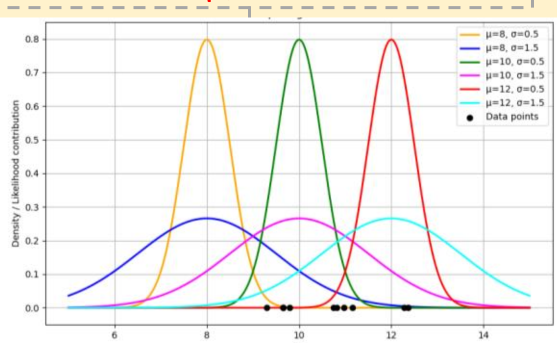
Maximum likelihood estimation

Find $\mathcal{L}(\mu, \sigma | x_i) = f(x_i | \mu, \sigma)$ for each dart, x_i.
Maximize the product of likelihood,

$$\mathcal{L}(\mu, \sigma | \mathbf{x}) = \prod_{i}^{n} f(x_i | \mu, \sigma)$$

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \mathcal{L}(\mu, \sigma | \mathbf{x})$$

$$\hat{\mu}, \hat{\sigma} = \arg\max_{\mu,\sigma} \ln[\mathcal{L}(\mu, \sigma | \mathbf{x})]$$



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Maximum likelihood estimation

Find $\mathcal{L}(\mu,\sigma|x_i) = f(x_i|\mu,\sigma)$ for each dart, x_i.

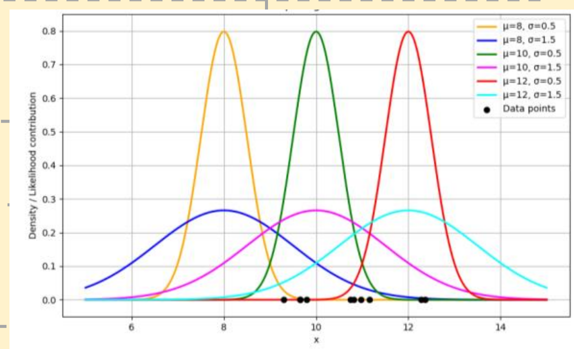
Maximize the product of likelihood,

$$\mathcal{L}(\mu,\sigma|\mathbf{x}) = \prod_{i}^{n} f(x_i|\mu,\sigma)$$

$$\hat{\mu},\hat{\sigma} = \arg\max_{\mu,\sigma} \mathcal{L}(\mu,\sigma|\mathbf{x})$$

$$\hat{\mu},\hat{\sigma} = \arg\max_{\mu,\sigma} \ln[\mathcal{L}(\mu,\sigma|\mathbf{x})]$$

Excel Demo!



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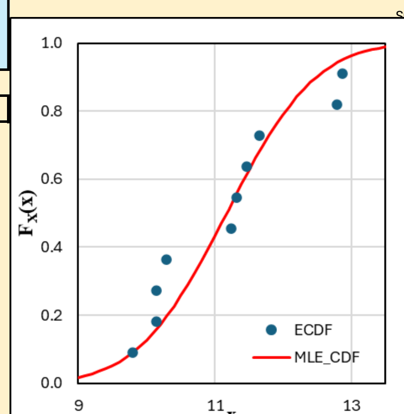
color	oran.	blue	green	mag.	red	turq.
μ_guess	**8**	**8**	**10**	**10**	**12**	**12**
σ_guess	**0.5**	**1.5**	**0.5**	**1.5**	**0.5**	**1.5**
x_i	Likelihood					
10.7	0.00	0.05	0.26	0.24	0.03	0.19
9.8	0.00	0.13	0.73	0.26	0.00	0.09
11.0	0.00	0.04	0.12	0.22	0.10	0.21
12.3	0.00	0.00	0.00	0.08	0.68	0.26
9.6	0.00	0.15	0.62	0.26	0.00	0.08
9.6	0.00	0.15	0.62	0.26	0.00	0.08
12.4	0.00	0.00	0.00	0.08	0.61	0.26
11.2	0.00	0.03	0.06	0.20	0.19	0.23
9.3	0.03	0.18	0.30	0.24	0.00	0.05
10.8	0.00	0.05	0.21	0.23	0.05	0.19
Sum-log-lik.	-166	-31	-32	-17	-59	-20

*Use "solver" to **maximize** Sum-log-lik. by changing μ, σ.*

μ_goal-seek	**10.67**
σ_goal-seek	**1.03**

Lik.	Log-lik.
0.39	-0.4
0.27	-0.6
0.37	-0.4
0.11	-0.9
0.24	-0.6
0.24	-0.6
0.10	-1.0
0.35	-0.5
0.16	-0.8
0.38	-0.4

Sum-log-lik. **-6.3**



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Questions, comments, or concerns?

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