Results : (1) $A \cap A' = \phi$ (2) $A \cup A' = U$ (3) $\cup' = \phi$ (4) $\phi' = U$ (5) $(A')' = A$ (6) $A - B = A \cap B'$ (7) $B - A = B \cap A'$ (8) $A \subseteq B \Leftrightarrow B' \subseteq A'$ (9) $A - B = B' - A'$ (10) $(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$

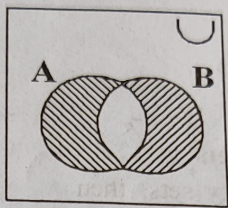
(E) **Symmetric Difference of Two Sets :**

If A and B are non empty sets, then the set $(A - B) \cup (B - A)$ is called a symmetric difference of A and B and it is denoted by $A \Delta B$ or $A \oplus B$.

$\therefore A \Delta B = \{x \mid x \in A \text{ and } x \notin B \text{ or } x \in B \text{ and } x \notin A\}$

$= \{x \mid x \notin A \cap B\}$.

In the following venn diagram, the shaded area represents $A \Delta B$.



For example,

Let $A = \{2, 3\}$ and $B = \{3, 4, 5\}$ then $A - B = \{2\}$ and $B - A = \{4, 5\}$

$\therefore A \Delta B = (A - B) \cup (B - A) = (2, 4, 5)$ or $A \Delta B = (A \cup B) - (A \cap B) = (2, 3, 4, 5\} - \{2\} = \{2\ 4, 5\}$

[7] Demorgan's Laws

1. **Complement of a union is the intersection of complements i.e.** $(A \cup B)' = A' \cap B'$

Proof : For proving the above law we have to prove the $(A \cup B)' \subseteq A' \cap B'$ and $(A \cap B)' \subseteq (A \cup B)'$

Let x is the any element of $(A \cup B)'$ i.e. $x \in (A \cup B)'$

$x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\therefore (A \cup B)' \subseteq A' \cap B'$ ....(i)

Let y is the any element of $A' \cap B'$ i.e. $y \in A' \cap B'$

$y \in A' \cap B' \Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin A \cup B$

$\Rightarrow y \in (A \cup B)'$

$\therefore (A' \cap B') \subseteq (A \cup B)'$......(ii)

From (i) and (ii) $\Rightarrow (A \cup B)' = A' \cap B'$

2. **Complement of an intersection is the union of the complements i.e.** $(A \cap B)' = A' \cup B'$ **Proof :** For proving $(A \cap B)' = A' \cup B'$, we must to prove the $(A \cap B)' \subseteq A' \cup B'$ and $A' \cup B' \subseteq (A \cap B)'$

Let x is any element of $(A \cap B)'$ i.e. $x \in (A \cap B)'$

$$
\begin{aligned}
x \in (A \cap B)' &\Rightarrow x \notin A \cap B \\
&\Rightarrow x \notin A \text{ or } x \notin B \\
&\Rightarrow x \in A' \text{ or } x \in B' \\
&\Rightarrow x \in A' \cup B'
\end{aligned}
$$

$\therefore$ $(A \cap B)' \subseteq A' \cup B'$....(i)

Let y is any element of $A' \cup B'$ i.e. $y \in A' \cup B'$

$$
\begin{aligned}
y \in A' \cup B' &\Rightarrow y \in A' \text{ and } y \in B' \\
&\Rightarrow y \notin A \text{ and } y \notin B \\
&\Rightarrow y \notin A \cap B \\
&\Rightarrow y \notin (A \cap B)'
\end{aligned}
$$

$\therefore$ $(A' \cup B') \subseteq (A \cap B)'$ .....(ii)

Now (i) and (ii) $\Rightarrow (A \cap B)' = A' \cup B'$

* **Demorgan's Law on Difference of sets :**

If A, B and C are any three non empty sets, then

(i) $A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A - (B \cap C) = (A - B) \cup (A - C)$

[8] Cartesian Product of Sets

Ordered pair : A pair of objects, listed in a specific order, is called an ordered pair, e.g. (a, b) is an ordered pair of two elements a and b, a is called the first element and be is called the second element.

Two ordered pairs (a, b) and (c, d) are equal if a = c and b = d.

i. e. $(a, b) = (c, d) \Rightarrow a = c$ and $b = d$.

Cartesian product of two sets : Let $A \neq \phi$ and $B \neq \phi$. The set of all ordered pairs (a, b) of elements $a \in A$ and $b \in B$ is called the Cartesian product of sets A and B. It is denoted by $A \times B$. Thus, $A \times B = \{(a, b) / a \in A, \ b \in B\}$

For example,

If $A = \{1, 2\}$ and $B = \{3, 4\}$ then

$$
\begin{aligned}
\text{(i)} \quad A \times B &= \{1, 2\} \times \{3, 4\} \\
&= \{(1, 3)\ (1, 4)\ (2, 3)\ (2, 4)\} \\
\text{(ii)} \quad B \times A &= \{3, 4\} \times \{1, 2\} \\
&= \{(3, 1)\ (3, 2)\ (4, 1), (4, 2)\} \\
\text{(iii)} \quad A \times A &= \{1, 2\} \times \{1, 2\} \\
&= \{(1, 1)\ (1, 2)\ (2, 1)\ (2, 2)\} \\
\text{(iv)} \quad B \times B &= \{3, 4\} \times \{3, 4\} \\
&= \{(3, 3)\ (3, 4)\ (4, 3)\ (4, 4)\}
\end{aligned}
$$

Note : (i) In general $A \times B \neq B \times A$

(ii) $n(A \times B) = n(A) \times n(B)$ if A and B are finite sets

(iii) $A \times B \times C = \{(a, b, c) / a \in A$ ...

$(A - B) \cap (A - C) = \{1, 8, 9\}$ ……(II)

From (I) and (II), it is proved that $A - (B \cup C) = (A - B) \cap (A - C)$

Example-5 If the universal set is $U = \{x \mid 1 \le x \le 12, x \in N\}$, $A = \{1, 9, 10\}$, $B = \{3, 4, 6, 11, 12\}$ and $C = \{2, 5, 6\}$ are subsets of U. Find the (i) $A \cup (B \cap C)$ and (ii) $(A \cup B) \cap (A \cup C)$

(Sau. Uni. B. C. A. (Sem.II) Oct–2009)

Ans. 5 Here, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A = \{1, 9, 10\}$

$B = \{3, 4, 6, 11, 12\}$

$C = \{2, 5, 6\}$

$B \cap C = \{6\}$ and $A \cup (B \cap C) = \{1, 6, 9, 10\}$…..(I)

$A \cup B = \{1, 3, 4, 6, 9, 10, 11, 12\}$

$A \cup C = \{1, 2, 5, 6, 9, 10\}$

$(A \cup B) \cap (A \cup C) = \{1, 6, 9, 10\}$ ……(II)

Now result (I) and (II) $\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example-6 Prove that $A \cap (B - C) = (A \cap B) - (A \cap C)$

(Sau. Uni. B. C. A. (Sem–II) Oct. 2009)

Ans. 6. (i) Let x is any element of $A \cap (B - C)$ i.e. $x \in A \cap (B - C)$

$x \in A \cap (B - C) \Rightarrow x \in A$ and $x \in B - C$

$\Rightarrow x \in A$ and ($x \in B$ but $x \notin (C)$

$x \in A \cap B$ but $x \notin (A \cap C)$

$\Rightarrow x \in \{(A \cap B) - (A \cap C)\}$

$\therefore A \cap (B - C) \subseteq (A \cap \boxed{\cancel{C}}\,B) - (A \cap C)$….(I)

y $\cancel{x} \in \{A \cap B) - (A \cap C) \Rightarrow y \in A$ and $y \in B$ but $y \notin A \cap C$

$\Rightarrow (y \in A)$ and ($y \in B$ but $y \notin A \cap C$)

$\Rightarrow y \in A$ and $y \in B$ but $y \notin C$

$\Rightarrow y \in A$ and $y \in B - C$

$\Rightarrow y \in A \cap (B - C)$

$\therefore (A \cap B) - (A \cap \cancel{B}\,C) \subseteq A \cap (B - C)$ …… *(II)*

Now (I) and (II) $\Rightarrow A \cap (B - C) = (A \cap B) - (A \cap C)$

Example-7 If $A = \{a, b, c, d\}$ $B = \{c, d, e\}$, $C = \{a, c, e\}$, then verify

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Ans. 7 (i) $B \cup C = \{a, c, d, e\}$

$A \cap (B \cup C) = \{a, c, d\}$….(I)

$A \cap B = \{c, d\}$

$A \cap C = \{a, c\}$

$(A \cap B) \cup (A \cap C) = \{a, c, d\}$ ……(II)

From result (I) and (II), $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $B \cap C = \{c, e\}$

(iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(v) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(vi) If $A \neq \phi$, $B \neq \phi$, then $A \times B = B \times A \Leftrightarrow A = B$

(vii) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

(viii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(ix) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$

(x) $n(A \times B) = n(B \times A) = n^2$ If $n(A) = n(B) = n$

[9] Results of Applications of Set Theory

1. (a) $n(A - B) = n(A) - n(A \cap B)$ where A is a finite set and n(A) is the number of elements in the set A

   (b) $n(B - A) = n(B) - n(A \cap B)$ where A and B are the finite sets and n(B) = No. of elements in the set B.

2. $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint sets $(= A \cap B = \phi)$
3. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ if A and B are not disjoint sets
4. $n(A \cup B \cup C) = n(A) + n(B) + n(C)$ if A, B and C are disjoint sets $(= A \cap B \cap C = \phi)$
5. $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$ if A, B and C are not disjoint sets
6. $n(A' \cap B') + n(A \cup B) = N \Rightarrow n(A' \cap B') = N - n(A \cup B)$ where $N = N(U)$
7. $n(A' \cap B' \cap C') = N - n(A \cup B \cup C)$ where $n(\cup) = N$
8. $n(A) + n(\bar{A}) = N(U) = N$
9. $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$

[10] Illustrative Examples

Example–1 If $A \{x / x^2 - 3x + 2 = 0, x \in R\}$ and $B = \{x \mid x^2 + 2x - 3 = 0, x \in R\}$, then find $A \cup B$ and $A \cap B$.

Ans. 1

$$\begin{aligned} A &= \{x \mid x^2 - 3x + 2 = 0\} \\ &= \{x \mid (x-2)(x-1) = 0\} \\ &= \{x \mid x - 2 = 0 \text{ or } x - 1 = 0\} \\ &= \{x \mid x = 2 \text{ or } x = 1\} \\ &= \{1, 2\} \\ B &= \{x \mid x^2 + 2x - 3 = 0\} \\ &= \{x \mid (x+3)(x-1) = 0\} \\ &= \{x \mid x + 3 = 0 \text{ or } x - 1 = 0\} \\ &= \{x \mid x = -3 \text{ or } x = 1\} \\ &= \{1, -3\} \end{aligned}$$

(i) $A \cup B = \{1, 2\} \cup \{1, -3\} = \{1, 2, -3\}$

(ii) $A \cap B = \{1, 2\} \cap \{1, -3\} = \{1\}$

Example–2 (a) ... $x \mid 3 \leq x \leq 13, x \in N\}$

$A = \{y \mid 2 < y < 7, y \in N\}$

$B = \{3, 5, 7, 9\}$

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By KhushDeep 🥰

$= \{(a, p)\ (a, q)\ (a, r)\ (b, p)\ (b, q)\ (b, r)\}$

(ii) $A \times B = \{a, b\} \times \{p, q\}$
$= \{(a, p), (a, q), (b, p)\ (b, q)\}$
$A \times C = \{a, b\} \times \{q, r\}$
$= \{(a, q)\ (a, r)\ (b, q)\ (b, r)\}$
$(A \times B) \cup (A \times C) = \{(a, p)\ (a, q)\ (a, r)\ (b, p), (b, q), (b, r)\}$
Obviously $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) $B \cap C = \{a\}$
$A \times (B \cap C) = \{a, b\} \times \{q\}$
$= \{(a, q)\ (b, q)\}$
$(A \times B) \cap (A \times C) = \{(a, q), (b, q)\}$
It is clear that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Example–13 If $A \times B = B \times A$ then prove that $A = B$

Ans. 13 $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$
Let $(x, y) \in A \times B \Rightarrow (x, y) \in B \times A\ (\because A \times B = B \times A)$
$\Rightarrow x \in B$ and $y \in A$
$\Rightarrow A \subseteq B$ and $B \subseteq A$
$\Rightarrow A = B$

Example–14 If $A\ \{3x \mid x \in N\}$ and $B = \{7x \mid x \in N\}$ then find the set $A \cap B$.

Ans. 14 $A = \{3, 6, 9, 12, 15, 18, 21, 24.....\} = \{x \mid x$ is a multiple of $3, x \in N\}$
$B = \{7, 14, 21, 28......\} = \{x \mid x$ is a multiple of $7, x \in N\}$
$A \cap B = \{3, 6, 9, 12, 15, 18, 21, 24.....\} \cap \{7, 14, 21, 28...\}$
$= \{21, 42, 63....\}$
$\{x \in N : x$ is a multiple of $21\}$

Example–15 In a class of 25 students, 12 students have taken mathematics, 8 have taken mathematics but not statistics. find the number of students who have taken mathematics and statistics and those who have taken statistics but not mathematics.

Ans. 15 N = Total number of student in the class $= 25 = n(A \cup B)$
$n(A)$ = No. of students taking subject mathematics $= 12$
$n(B)$ = No. of students taking subject statistics
$n(A \cap B') = 8$
$A = (A \cap B') \cup (A \cap B) \Rightarrow n(A) = n(A \cap B') + n(A \cap B)$
$\therefore\ 12 = 8 + n(A \cap B) \Rightarrow n(A \cap B) = 12 - 8 = 4$
Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow$
$\therefore\ 25 = 12 + n(B) - 4$
$\therefore\ n(B) = 25 - 8 = 17$
Again $n(B) = n(B \cap A') + n(A \cap B)$
[illegible] $(B \cap A') + 4$
$n(B \cap A') = 13$ = No. of students taking statistics but not mathematics

$= 40 + 36 - 24 = 52$

or $n(M \cup S) = n(M \cap \overline{S}) + n(M \cap S) + n(S \cap \overline{M}) = 16 + 24 + 12$

$= 52$

(iv) $n(\overline{M} \cap \overline{S})$ = No. of students do not like Maths and statistics

$n(M \cup S) + n(\overline{M} \cap \overline{S}) = n(U) = N$

$\therefore 52 + (\overline{M} \cap \overline{S}) = 60$

$n(\overline{M} \cap \overline{S}) = 60 - 52 = 8$

Example-19 If $N = N(U) = 100$, $n(F) = 54$, $n(H) = 32$, $n(C) = 36$ $n(F \cap H) = 16$, $n(H \cap C) = 6$, $n(F \cap C) = 14)$ and $n(H \cap F \cap C) = 6$, then find $n(H' \cap F' \cap C')$

Ans. 19

$$\begin{aligned} n(H \cup F \cup C) &= n(H) + n(F) + n(C) - n(H \cap F) - n(H \cap C) - n(F \cap C) + n(H \cap F \cap C) \\ &= 32 + 54 + 36 - 16 - 6 - 14 + 6 \\ &= 92 \end{aligned}$$

$\therefore n(H' \cap F' \cap C') = N(U) - n(H \cup F \cup C) = 100 - 92 = 8$

Example-20 If $U = \{x \in N / 1 \le x \le 8\}$ $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then verify the De Morgan's two rules.

Ans. 20 (i) $(A \cup B)' = A' \cap B'$ and (ii) $(A \cap B)' = A' \cup B'$

Here, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 2, 3\}$ & $B = \{2, 3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$ & $A \cap B = \{2, 3\}$

$A' = \{4, 5, 6, 7, 8\}$ & $B' = \{1, 5, 6, 7, 8\}$

$A' \cap B' = \{5, 6, 7, 8\}$ ......(I)

$(A \cup B)' = U - (A \cup B) = \{1, 2, 3, \ldots..8\} - \{1, 2, 3, 4,\} = \{5, 6, 7, 8\}$ ....(II)

Now (I) and (II) $\Rightarrow (A \cup B)' = A' \cap B'$

$\therefore (A \cap B)' = U - (A \cap B) = \{1, 2, 3, \ldots\ldots 8\} - \{2, 3\}$

$(A \cap B)' = \{1, 4, 5, 6, 7, 8\}$ .....(III)

$A' \cup B' = \{1, 4, 5, 6, 7, 8\}$ ......(IV)

From result (III) and (IV), $(A \cap B)' = A' \cup B'$

Example-21 Using Venn diagrams show that for any two sets A and B

(i) $(A \cap B)' = A' \cup B'$

(ii) $A \cup B = A \cup (B - A)$

(i) Let x is any element of $A \cup (B \cap C)$. Then $x \in A \cup (B \cap C)$

(i) $x \in A \cup (B \cap C) \Rightarrow x \in A$ or $x \in (B \cap C)$

$\Rightarrow x \in A$ or $[x \in B$ and $x \in C]$

$\Rightarrow [x \in A$ or $x \in B]$ and $[x \in A$ or $x \in C]$

$\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$......(I)

(ii) Let y is any element of $(A \cup B) \cap (A \cup C)$ $\therefore y \in (A \cup B) \cap (A \cup C)$

$y \in (A \cup B) \cap (A \cup C) \Rightarrow y \in (A \cup B)$ and $y \in (A \cup C)$

$\Rightarrow [y \in A$ or $y \in B]$ and $[y \in A$ or $y \in C]$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\therefore (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ ....(II)

From (I) and (II), we write the $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example-24 Find A, B and C sets if $A \cup B = \{1, 2, 3, 4\}$ $A \cup C = \{2, 3, 4, 5\}$ $A \cap B = \{2, 3\}$ and $A \cap C = \{2, 4\}$

Ans. 24 Since $A \cap B = \{2, 3\}$ and $A \cap C = \{2, 4\}$

$\therefore$ 2, 3, 4 $\in$ A, 2, 3, $\in$ B and 2, 4 $\in$ C.....(1)

Since $A \cup C = \{2, 3, 4, 5\}$ $1 \notin A$

Since $1 \notin A$ and $A \cup B = \{1, 2, 3, 4\}$, $1 \in B$ ......(2)

Since $A \cup B = \{1, 2, 3, 4\}$ $5 \notin A$

Since $5 \notin A$ and $A \cup C = \{2, 3, 4, 5\}$ $5 \in C$......(3)

Hence from (1), (2) and (3) we have

$A = \{2, 3, 4\}$

$B = \{1, 2, 3\}$

$C = \{2, 4, 5\}$

Example-25 Verify the result $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for the sets $A = \{1, 2, 3, 5\}$ $B = \{2, 3, 4, 6\}$ & $C = \{1, 2, 4, 5, 7\}$

Ans. 25 $A \cap B = \{2, 3\}$ $A \cap C$ $\{1, 2, 5\}$

$B \cup C = \{1, 2, 3, 4, 5, 6, 7\}$

$A \cap (B \cup C) = \{1, 2, 3, 5\} \cap \{1, 2, 3, 4, 5, 6, 7\} = \{1, 2, 3, 5\}$.....(I)

$(A \cap B) \cup (A \cap C) = \{2, 3\} \cup \{1, 2, 5\}$ $\{1, 2, 3, 5\}$.....(II)

From (I) and (II), we have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example-26 If $n(U) = 1000$, $n(A) = 658$, $n(B) = 372$, $n(C) = 590$, $n(A \cap B) = 166$, $n(B \cap C) = 126$, $n(A \cap C) = 434$ and $n(A \cap B \cap C) = 106$, find $n(A \cap \overline{B} \cap \overline{C})$