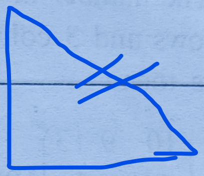
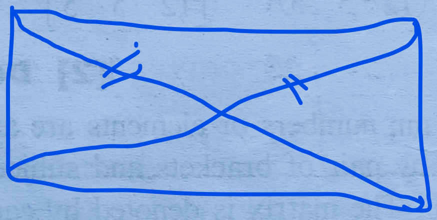
4 MATRIX

University Syllabus

❖ Introduction, Types of different – matrices (Row. Column , square, diagonal, transpose, Unit, null matrix) operation on matrices (Addition, Substraction, multiplication), Properties of transpose matrix, Adjoint of square matrix, Inverse of a square matrix Typical Examples.

An Outline of The Chapter





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[1] Introduction

The British Mathematician Arther Caley (1860) was the first person introduce the concept of matrix. The theory of matrix is not only useful in many branches of applied mathematics but is one of the powerful tools of mathematics in the study and understanding of several business, management, social and life sciences problems, input – output tables, demography, linear programming, design of experiments, game theory, decision theory, electrical engineering, network analysis etc. A matrix consists of a rectangular presentation of numerical elements arranged systematically in rows and columns describing various aspects. A most significant contribution of matrix algebra is its widely use in the solutions of a system of large number of simultaneous linear equations. We consider the following illustration.

In an elocution contest, a participant can speak either of three languages : Hindi, Gujarati and English. College – 1 sent 22 students of which 10 offered to speak in Hindi, 9 in English, 3 in Gujarati College – 2 sent 18 students of which 7 offered to speak Hindi, 8 in English and 3 in Gujarati College – 3 sent 22 students of which 12 offered to in Hindi 5 in English and 5 in Gujarati.

The information furnished in the above manner is cumbersome. The above information can be written in a more compact manner if we consider the following table form.

	Hindi	English	Gujarati
College–1	10	9	3
College–2	7	8	3
College–3	12	5	5

The number in the above data are known as a rectangular array. In the above data, there are 3 rows and 3 columns and hence $3 \times 3 = 9$ elements. If it is enclosed by a pair of square brackets then

$$\begin{pmatrix} 10 & 9 & 3 \\ 7 & 8 & 3 \\ 12 & 5 & 5 \end{pmatrix} \text{ or } \begin{bmatrix} 10 & 9 & 3 \\ 7 & 8 & 3 \\ 12 & 5 & 5 \end{bmatrix} \text{ is called a matrix.}$$

[2] Definition of Matrix

If mn numbers or elements are arranged in a rectangular array of m rows and n columns enclosed by pair of brackets and subject to certain rules of presentation is called a matrix of or $m \times n$. The matrix is denoted by capital bold face letters like A, B, C, D, ... X, Y, Z. The entry in the i^{th} row and j^{th} column is denoted a_{ij}. Different notations use for enclosing the elements or entries of a matrix are [ ], ( ), { } and ‖ ‖, but we shall use the pair of brackets notation [ ] throughout the chapter.

For example, in the illustration just as the colleges were numbered from 1 to 3 and languages were numbers 1 to 3. Then the matrix can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Where a_{11} = the number of students from college – 1 who offered language Hindi = 10, $a_{12} = 9$, $a_{13} = 3$, $a_{21} = 7$, $a_{22} = 8$, $a_{23} = 3$, $a_{31} = 12$, $a_{32} = 5$ and $a_{33} = 5$.

A general form of a matrix : A matrix of order $m \times n$ can be written as under.

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots & a_{1j} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2j} \ldots & a_{2n} \\ \vdots & \vdots & & \\ a_{i1} & a_{i2} \ldots & a_{ij} \ldots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} \ldots & a_{mj} \ldots & a_{mn} \end{bmatrix}$$

| row

— column

In this matrix, a_{21} represent an element lying in the second row and first column. In a_{ij} ($i = 1,2 ..m j = 1,2 ...n$) the first suffix indicates the order of the row and second suffix indicates the order of the column, where $a_{11}, a_{12} ...... a_{mn}$ stand for real numbers. The above matrix can also be written as

$A = [a_{ij}]_{m \times n}$ where $i = 1, 2 .....m$ and $j = 1,2 ...... n$

Note :
$$\begin{array}{c} \\ \text{Row - 1} \\ \text{Row - 2} \end{array} \begin{bmatrix} \text{Column - 1} & \text{Column - 2} & \text{Column - 3} \\ 4 & 7 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$ is a matrix of order 2×3

First row elements are 4, 7, 2 and First column elements are 4, 3.
Total number of elements $= 2 \times 3 = 6$.

[3] Types of Matrices

(1) **Square Matrix :** A matrix in which the number of rows is equal to the number of columns is called a square matrix.

e. g. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$ are square matrices.

2+2 3×3

In this case, A is square matrix of order 2 or 2 rowed matrix.

Remark : In square matrix all those elements a_{ij} for which $i = j$ i . e . a_{11} a_{22} a_{33} .... are called the diagonal elements.

(2) **Column Matrix :** A matrix having m rows and only one column is called a column matrix.

e. g. $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is a column matrix of order 3×1. In short, A matrix having a single column is called a column matrix.

r c

(3) **Row Matrix :** A matrix having single row and any number of column is called a row matrix. e. g. $R = [2, 3, 4, 5]$ is a row matrix of order 1×4.

(4) **Zero Matrix or Null Matrix :** If all the elements of a matrix are zero, then it is called a zero matrix or null matrix. It is denoted by the symbol O. e. g. $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(5) **Diagonal Matrix :** If all the elements except diagonal elements of a square matrix are zero, then it is called a diagonal matrix. e. g. $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3×3, It may be written as D = diag (2, 3, 4).

Remark : The square matrix A will be a diagonal matrix if all the elements a_{ij} for which $i \neq j$ are zero.

(6) **Scalar Matrix :** A diagonal matrix whose all the diagonal elements are equal is called a

Sq, dia

Scalar matrix. e. g. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \text{diag}\ (2, 2, 2)$

(7) **Unit Matrix or Identity Matrix :** If each element of scalar matrix is unity (or one) is called a unit matrix. Unit matrix of order $n \times n$ is written as I_n. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I_3,\ I_4,\ I_2$

$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are unit matrices of order 2, 3 and 4.

In general, in the square matrix whose elements are $a_{ij} = O$ if $i \neq j$ and $a_{ij} = 1$ if $i = j$.

(8) **Transpose of Matrix :** The matrix obtained from a given matrix A by interchanging rows and columns is called a transpose matrix of A and it is denoted by A' or A^T.

For example, If $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ then $A' = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 1 \\ 5 & 0 & 2 \end{bmatrix}$ ← Z

Obviously $(A')' = A$ or $(A^T)^T = A$. If order of matrix A is $m \times n$, then order of its transpose matrix is $n \times m$. Properties (i) $(A')' = A$ (ii) $(A + B)' = A' + B'$ (iii) $KA' = (KA)'$ (iv) $(AB)' = B'.A'$.

(9) **Sub Matrix :** A matrix obtained by deleting some rows or columns or both of a given matrix is called a sub matrix of a given matrix. e. g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3 \times 3}$ If we delete the first row and first column. The sub matrix of A is given by $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}_{2 \times 2}$

(10) **Triangular Matrix** : A square matrix $A = [a_{ij}]_{m \times n}$ is called upper triangular matrix if $a_{ij} = o$ for $i > j$ and is called lower triangular matrix if $a_{ij} = o$ for $I < j$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ and

upper

$B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ are upper and lower triangular matrices.

lower

(11) **Symmetric Matrix** : The square matrix A is said to be symmetric matrix if $A = A'$.

e. g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$. Therefore $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a symmetric matrix.

(12) **Skew Symmetric Matrix** : The square matrix A is said to be a skew – symmetric matrix if $A = -A'$ or $A' = -A$ or $A + A' = O$.

$A = -A'$

e. g. $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$ is a skew – symmetric matrix. Obviously all the diagonal elements are zero in skew – symmetric matrix.

(13) **Equal Matrices** : Two matrix A and B are said to be equal matrices if :

(i) they are of the same order.

(ii) the elements in the corresponding positions of two matrices are equal.

e. g. If $A = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$ then A and B are equal matrices. If two matrices A and B are equal, we write $A = B$, otherwise we write $A \neq B$.

[4] Basic Operations on Matrices

(i) **Multiplication of A Matrix By A Scalar :**

If A is a matrix and K is a scalar (real number), then the scalar Product Ka is the matrix obtained by multiplying every element of A by K is called the scalar multiple of A by K.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & 6 \\ 12 & 24 \end{bmatrix}$

Properties of scalar multiplications :

If A and B are the two matrices and k, l are scalars, then

(i) $K(A + B) = KA + KB$

(ii) $(K + 1) A = KA + 1A$

(iii) $(-K) A = -(KA) = K(-A)$

(iv) $K(1A) = (K1) A = 1(KA)$

(v) $1A = A$ and $(-1) A = -A$

(ii) Addition of Two Matrices :

Let $A = [a_{ij}]$, and $B = [b_{ij}]$ are two matrices of the same order $m \times n$, then their sum A + B is defined to be the matrix of order $m \times n$ such that $(A + B)_{ij} = a_{ij} + b_{ij}$ for all i and j.

2×2 . 3×3

For example, if

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \text{ then } A + B = \begin{bmatrix} 3+1 & 4+2 & 1+3 \\ 2+4 & 1+5 & 3+6 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 4 \\ 6 & 6 & 9 \end{bmatrix}$$

Remark : It should be noted that the addition is defined only for matrices of the same order.

(iii) Subtraction of Two Matrices :

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then their substraction A – B is defined to be the matrix of order $m \times n$ such that $(A - B)_{ij} = a_{ij} - b_{ij}$ forall i & j.

$$\text{For example, it } A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \text{ then } A - B = \begin{bmatrix} 3-2 & 4-3 \\ 5-4 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

* **Properties of Matrix Addition :**

(i) Matrix Addition is Commulative : If A and B are two $m \times n$ order matrices, then $A + B = B + A$.

1+2 = 3+1

(ii) Matrix Addition is Associative : It A, B and C are three $m \times n$ order matrices, then $(A + B) + C = A + (B + C)$.

(1+2)+5 = 1+(3+5)

(iii) Existence of Additive Identity : The null matrix is the identity for matrix addition, i. e, $A + O = O + A = A$ for every matrix A.

1+0=1

(iv) Existence of Additive Inverse : For every matrix $A = [a_{ij}]_{m \times n}$ there exists a matrix $[-a_{ij}]_{m \times n}$ denoted by – A such that $A + (-A) = O = (-A) + A$. The matrix – A is called the opposite matrix of A. A – B = A + (– B) Cancellation rules hold well in case of addition of matrices :

1+(-1)=0

If A, B and C are three $m \times n$ order matrices, then $A + B = A + C \Rightarrow B = C$ (Left Cancellation Law) and $(B + A = C + A \Rightarrow B = C$ (Right Cancellation Law).

(iv) Multiplication of Two matrices :

Two matrices A and B can be multiplied if and only if the number of columns of the first matrix A is same as number of rows of the second matrix B. If A is the $m \times n$ matrix and B is the $n \times p$ matrix, then AB is the $m \times p$ matrix. In the multiplication AB, A is known as prefector and B is known as post factor.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices, then their product AB is a matrix of order $m \times p$ and is defined as $(AB)_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1} b_{1j} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots + a_{in} b_{nj}$

$$= [\ a_{i1},\ a_{i2},\ \ldots\ldots\ \ a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}.$$

2×3 3×2

For example, if $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}$, then AB

$(AB)_{11}$ = (First row of A) (First column B)

$$= [2\ 1\ 1] \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2 \times 2 + 1 \times 1 + 1 \times 3 = 4 + 1 + 3 = 8$$

$$(AB)_{12} = [2\ 1\ 1] \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$ 2×0 + 1×2 + 1×1

$$= 0 + 2 + 1 = 3$$

$$(AB)_{21} = [1\ 0\ 2] \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$ 1×2 + 0×1 + 2+3

$$= 2 + 0 + 6 = 8$$

$$(AB)_{22} = [1\ 0\ 2] \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$ 1×0 + 0×2 + 2×1

$$= 0 + 0 + 2 = 2$$

Thus, $AB = \begin{bmatrix} 8 & 3 \\ 8 & 2 \end{bmatrix}$ 2×2

Remark : (i) $A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ and $B = [\ C_1\ C_2] \Rightarrow AB = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix}$

(ii) $AB = [a_1 \ a_2 \ \ldots . a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \ldots\ldots + a_n b_n$

Properties of Matrix Multiplication :

(i) Matrix multiplication is associative : $(AB)C = A(BC)$

(ii) Matrix multiplication in general not commutative : $AB \neq BA$

(iii) Matrix multiplication is distributive over matrix addition : $A(B + C) = AB + AC$

(iv) If A is a Square matrix, then $A^2 = AA$, $A^3 = A^2A = AA^2 = AAA$

$A^4 : A^3 A$

(v) $AB = O$ does not necessarily imply that $A = 0$ or $B = O$

For example,

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ then } AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(vi) If A is an $m \times n$ matrix, then $I_m A = A = A I_n$

(vii) Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$ is any polynomial in x.
Then $f(A) = a_0 I + a_1 A + a_2 A^2 + \ldots\ldots \ldots\ldots + a_n A^n$, where A is a square matrix
$f(A)$ is called a matrix polynomial.

(viii) $AO = O = OA$ and $A I_n = I_n A = A$, $I = I^2 = I^3 = \ldots = I^n$, $n \in N$

Example–1 If $A = \begin{bmatrix} 3 & 4 & 1 & -2 \\ -5 & 6 & 7 & 8 \end{bmatrix}$ and $B' = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$ then find $A + B$, $A - B$, $3A + 2B$, $5A - 2B$ and AB'.

Ans. 1 $B' = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & -4 \\ 5 & 6 \end{bmatrix} \Rightarrow (B')' = B = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & -4 & 6 \end{bmatrix}$.

Transpose

Order of matrix, A' is 2×4 and order of matrix B is also 2×4. Hence $A + B$, $A - B$, $3A + 2B$ and $5A - 2B$ are possible.

(i) $A + B = \begin{bmatrix} 3 & 4 & 1 & -2 \\ -5 & 6 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 4 & 3 \\ -3 & 9 & 3 & 14 \end{bmatrix}$

(ii) $A - B = \begin{bmatrix} 3 & 4 & 1 & -2 \\ -5 & 6 & 7 & 8 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 3 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 & -7 \\ -7 & 3 & 11 & 2 \end{bmatrix}$

(iii) $3A + 2B = \begin{bmatrix} 9 & 12 & 3 & -6 \\ -15 & 18 & 21 & 24 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 & 10 \\ 4 & 6 & -8 & 12 \end{bmatrix} = \begin{bmatrix} 11 & 16 & 9 & 4 \\ -11 & 24 & 13 & 36 \end{bmatrix}$

(iv) $5A - 2B = \begin{bmatrix} 15 & 20 & 5 & -10 \\ -25 & 30 & 35 & 40 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 6 & 10 \\ 4 & 6 & -8 & 12 \end{bmatrix} = \begin{bmatrix} 13 & 16 & -1 & -20 \\ -29 & 24 & 43 & 28 \end{bmatrix}$

(v) $AB' = \begin{bmatrix} 3 & 4 & 1 & -2 \\ -5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & -4 \\ 5 & 6 \end{bmatrix}$

$= \begin{bmatrix} 3+8+3-10, & 6+12-4-12 \\ 5+12+21+40, & -10+18-28+48 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 68 & 28 \end{bmatrix}$

Example–2 If $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$, then find $A + B$, $A - B$, $A' + B'$, $A^2.B^2$ AB and BA

Ans. 2

(1) $A + B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$

(2) $A - B = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 4 & 1 \end{bmatrix}$

(3) $A' + B' = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}$

(4) $A^2 = AA = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4+5 & 10+15 \\ 2+3 & 5+9 \end{bmatrix} = \begin{bmatrix} 9 & 25 \\ 5 & 14 \end{bmatrix}$

$B^2 = BB = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1+3 & -1-2 \\ -3-6 & 3+4 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix}$

$$\cancel{A^2 + B^2 = \begin{bmatrix} 9 & 25 \\ 5 & 14 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix} = \begin{bmatrix} 13 & 22 \\ -4 & 21 \end{bmatrix}}$$

AB ≠ BA

$$A^2 - B^2 = \begin{bmatrix} 9 & 25 \\ 5 & 14 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -9 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 28 \\ 14 & 7 \end{bmatrix}$$

(5) $$AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2-15 & -2+10 \\ 1-9 & -1+6 \end{bmatrix} = \begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 5-3 \\ -6+2 & -15+6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -9 \end{bmatrix}$$

Example–3 If $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$, then find AB and BA if Possible.

Ans. 3 A is 3×3 matrix and B is 3×4 matrix. Here, No. of columns of A = No. of rows of B. hence AB is possible and is 3×4 matrix.

$$AB = C \Rightarrow \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{array}{c} C_1\,C_2\,C_3\,C_4 \\ \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix} \end{array} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \end{bmatrix}$$

$$C_{11} = R_1 C_1 = [2\ 1\ 0] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 + 2 + 0 = 4$$

$$C_{12} = R_1 C_2 = [2\ 1\ 0] \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 4 + 0 + 0 = 4$$

$$C_{13} = R_1 C_3 = [2\ 1\ 0] \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 6 + 1 + 0 = 7$$

$$C_{14} = R_1 C_4 = [2\ 1\ 0] \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = 8 + 2 + 0 = 10$$

$$C_{21} = R_2\, C_1 = [3\ 2\ 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3 + 4 + 3 = 10$$

$$C_{22} = R_2\, C_2 = [3\ 2\ 1] \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 6 + 0 + 1 = 7$$

$$C_{23} = R_2\, C_3 = [3\ 2\ 1] \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = 9 + 2 + 0 = 11$$

$$C_{24} = R_2\, C_4 = [3\ 2\ 1] \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = 12 + 4 + 5 = 21$$

$$C_{31} = R_3\, C_1 = [1\ 0\ 1] \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 0 + 3 = 4$$

$$C_{32} = R_3\, C_2 = [1\ 0\ 1] \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 + 0 + 1 = 3$$

$$C_{33} = R_3\, C_2 = [1\ 0\ 1] \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = 3 + 0 + 0 = 3$$

$$C_{34} = R_3\, C_4 = [1\ 0\ 1] \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = 4 + 0 + 5 = 9$$

Substituting these value in matrix C we get

$$C = AB = \begin{bmatrix} 4 & 4 & 7 & 10 \\ 10 & 7 & 11 & 21 \\ 4 & 3 & 3 & 9 \end{bmatrix}$$

Example–4 If $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, then prove that

(i) $(A + B)' = A' + B'$ (ii) $(AB)' = B'A'$ (iii) $(A - B)' = A' - B'$

Ans – 4 $A + B = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 4 & 2 \end{bmatrix} \Rightarrow (A + B)' = \begin{bmatrix} 5 & 4 \\ 7 & 2 \end{bmatrix}$ ....(I)

$A' + B' = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 7 & 2 \end{bmatrix}$ ....(II)

Now (I) and (II) $\Rightarrow (A + B)' = A' + B'$

$A - B = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \Rightarrow (A - B)' = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$ ....(III)

$A' - B' = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}$ .....(Iv)

Now (III) and (IV) $\Rightarrow (A - B)' = A' - B'$

$AB = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6+5 & 4+5 \\ 9+1 & 6+1 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ 10 & 7 \end{bmatrix}$

$\therefore (AB)' = \begin{bmatrix} 11 & 10 \\ 9 & 7 \end{bmatrix}$ ....(V)

$B'A' = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 6+5 & 9+1 \\ 4+5 & 6+1 \end{bmatrix} = \begin{bmatrix} 11 & 10 \\ 9 & 7 \end{bmatrix}$ ......(VI)

Now (V) and (VI) $\Rightarrow (AB)' = B'A'$

Example–5 If $f(x) = 2x^3 - 4x + 5$, then find f(A) Where $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

Ans - 5 $A^2 = AA = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1+8 & 2-6 \\ 4-12 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$

$A^3 = A^2 A = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9-16 & 18+12 \\ -8+68 & -16-51 \end{bmatrix} = \begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$

Let $f(x) = 2x^3 - 4x + 5$

$\therefore f(A) = 2A^3 - 4A + 5I_2$

$$= 2\begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 60 \\ 120 & -134 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -14-4+5 & 60-8+0 \\ +120-16+0 & -134+12+5 \end{bmatrix} = \begin{bmatrix} -13 & 52 \\ 104 & -117 \end{bmatrix}$$

Example - 6 If $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, then prove that $A^4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Ans. - 6 $A^2 = A\,A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A^3 = A^2\,A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^4 = A^3\,A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus any power of A is zero matrix. Such matrices whose integral power is zero matrix are said to be Nilpotent Matrices. A is Nilpotent Matrix and its index is 2.

Example - 7 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then prove that $A^2 - 4A - 5I_3 = O$

Ans. - 7 $A^2 = AA = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\text{L. H. S.} = A^2 - 4A - 5I_3$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= O =$ R. H. S.

Example – 8 Find AB and BA. Is AB = BA if $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $B \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$?

[Sau. Uni. B.C.A. Sem – II) 2005]

Ans. 8

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0-1 & 0-0+0 & -1+0-1 \\ 0+0+0 & 0-1+0 & 0+0+0 \\ -1+0-1 & 0-0+0 & -1+0-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix} \quad - \text{(i)}$$

$$BA = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0-1 & 0-0+0 & -1+0-1 \\ 0+0+0 & 0-1+0 & 0+0+0 \\ -1+0-1 & 0-0+0 & -1+0-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & 0 \\ -2 & 0 & -2 \end{bmatrix} \quad - \text{(ii)}$$

Now (i) and (ii) $\Rightarrow$ AB = BA

Example – 9 Find x, y, z, a, b, c and m if $\begin{bmatrix} x+2 & 5 & 3+y \\ 7 & 0 & z-2 \\ a+3 & b+5 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4x+10 & 5+m & 6y-10 \\ 14 & 2c & 0 \\ 0 & 2 & 4 \end{bmatrix}$

Ans. 9 By using definition of equal matrices

$x + 2 = \dfrac{4x+10}{2} = 2x + 5 \Rightarrow x = -3$

$5 = \dfrac{5+m}{2} \Rightarrow m = 10 - 5 = 5$

$3 + y = \dfrac{6y-10}{2} = 3y - 5 \Rightarrow y = 4$

$0 = 2C \Rightarrow = C = 0$

$z - 2 = 0 \Rightarrow z = 2$

$a + 3 = 0 \Rightarrow a = -3$

$b + 5 = \dfrac{2}{2} = 1 \Rightarrow b = -4$

Example – 10 If $A = \begin{bmatrix} 2 & 6 \\ 7 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 7 \\ 9 & 5 \end{bmatrix}$, then prove that A (BC) = (AB)C.

Ans. – 10

$$BC = \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 9 & 5 \end{bmatrix} = \begin{bmatrix} -12+45 & -21+25 \\ 0+72 & 0+40 \end{bmatrix} = \begin{bmatrix} 33 & 4 \\ 72 & 40 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -6+0 & 10+48 \\ -21+0 & 35+16 \end{bmatrix} = \begin{bmatrix} -6 & 58 \\ -21 & 51 \end{bmatrix}$$

$$\text{L. H. S.} = A(BC) = \begin{bmatrix} 2 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 33 & 4 \\ 72 & 40 \end{bmatrix} = \begin{bmatrix} 66+432 & 8+240 \\ 231+144 & 28+80 \end{bmatrix} = \begin{bmatrix} 498 & 248 \\ 375 & 108 \end{bmatrix}$$

$$\text{R. H. S.} = (AB)\,C = \begin{bmatrix} -6 & 58 \\ -21 & 51 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 9 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -24+522 & -42+290 \\ -84+259 & -147+255 \end{bmatrix} = \begin{bmatrix} 498 & 248 \\ 375 & 108 \end{bmatrix}$$

$\therefore$ L. H. S = R. H. S.

Example – 11 If A and B square matrices such that AB = A and BA = B. then prove that $A^2 = A$ and $B^2 = B$. **(Sau. Uni F.Y.B.B.A. – 2005)**

Ans. 11 Given that AB = A and BA = B

(i) $A^2 = AA$
$= (AB)\,A$
$= A\,(BA)$
$= AB$
$= A$

(ii) $B^2 = BB$
$= (BA)\,B$
$= B\,(AB)$
$= BA$
$= B$

Example – 12 For the following matrices A and B verify $(A + B)^2 = A^2 + 2AB + B^2$ and $A^2 - B^2 = (A - B)(A + B)$ if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

Ans. 12

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+2 \\ 2+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$B^2 = BB = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 2+6 & 0+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+4 & 0+6 \\ 2+2 & 0+3 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}$$

(i) $(A + B)^2 = (A + B)(A + B)$

$$= \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 4+8 & 4+8 \\ 8+16 & 8+16 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 24 & 24 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 10 & 12 \\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 16 \\ 12 & 11 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 16 & 16 \\ 20 & 20 \end{bmatrix}$$

$\therefore$ $(A + B)^2 \neq A^2 + 2AB + B^2$

(ii) $$A^2 - B^2 = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ -4 & -4 \end{bmatrix}$$

$$(A + B)(A - B) = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0+0 & 4-4 \\ 0+0 & 8-8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2 - B^2 \neq (A + B)(A - B)$

Example – 13 Mr. X is a trader, manufacturing tables and chairs. Each table requires 5 hours of labour and 6 units of material. Each chair requires. 3 hours of labour and 3 units of material. If Mr. X plans to produce 40 tables and 50 chairs in the next week. How many hours will be need to work and how much material will be require ?

Ans. 13

$$A = \begin{bmatrix} \textit{Tables} & \textit{Chairs} \\ 40 & 50 \end{bmatrix} \quad B = \begin{array}{cc} \text{Labour} & \text{Material} \\ \left[\begin{matrix} 5 \\ 3 \end{matrix}\right. & \left.\begin{matrix} 6 \\ 3 \end{matrix}\right] \end{array}$$

$$AB = \begin{bmatrix} 40 & 50 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 40 \times 5 + 50 \times 3 \\ 40 \times 6 + 50 \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 200 + 150 \\ 240 + 150 \end{bmatrix}$$

$$= \begin{bmatrix} 350 \\ 390 \end{bmatrix}$$

$\therefore$ The labours requirement is 350 hours & the material requirement is 390 units.

Example – 14 If $2A - B = \begin{bmatrix} 8 & 11 \\ 8 & 7 \end{bmatrix}$ and $3A - B = \begin{bmatrix} 17 & 19 \\ 12 & 8 \end{bmatrix}$, then find 4A – 3B **[Sau. Uni.]**

Ans. 14

$$2A - B = \begin{bmatrix} 8 & 11 \\ 8 & 7 \end{bmatrix}$$

$$\therefore \; -(2A - B) = -\begin{bmatrix} 8 & 11 \\ 8 & 7 \end{bmatrix}$$

$$\therefore \; -2A + B = \begin{bmatrix} -8 & -11 \\ -8 & -7 \end{bmatrix} \quad \ldots\ldots\ldots \text{(I)}$$

$$3A - B = \begin{bmatrix} 17 & 19 \\ 12 & 8 \end{bmatrix} \quad \ldots\ldots \text{(II)}$$

Solving (I) and (II) we get

$$A = \begin{bmatrix} -8 & -11 \\ -8 & -7 \end{bmatrix} + \begin{bmatrix} 17 & 19 \\ 12 & 8 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 4 & 1 \end{bmatrix}$$

$$\therefore \; 2A = \begin{bmatrix} 18 & 16 \\ 8 & 2 \end{bmatrix}$$

$$2A - B = \begin{bmatrix} 8 & 11 \\ 8 & 7 \end{bmatrix} \Rightarrow B = 2A - \begin{bmatrix} 8 & 11 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 18 & 16 \\ 8 & 2 \end{bmatrix} - \begin{bmatrix} 8 & 11 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 5 \\ 0 & -5 \end{bmatrix}$$

$$4A - 3B = 4\begin{bmatrix} 9 & 8 \\ 4 & 1 \end{bmatrix} - 3\begin{bmatrix} 10 & 5 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 36 & 32 \\ 16 & 4 \end{bmatrix} - \begin{bmatrix} 30 & 15 \\ 0 & -15 \end{bmatrix} = \begin{bmatrix} 6 & 17 \\ 16 & 19 \end{bmatrix}$$

[Sau. Uni.]

Example - 15 If $A^2 = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$, find matrix A.

A^2 is a symmetric matrix and hence A will be symmetric matrix

Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$,

$$A^2 = AA = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ab+ba \\ ba+ab & b^2+a^2 \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$$

$\therefore \quad a^2 + b^2 = 13$ and $2ab = 12$

$\therefore \quad a^2 + b^2 + 2ab = 13 + 12 = 25 \Rightarrow (a + b)^2 = 25 \Rightarrow a + b = \pm 5$

$a^2 + b^2 - 2ab = 13 - 12 = 1 \Rightarrow (a - b)^2 = 1 \Rightarrow a - b = \pm 1$

$a + b = 5$ and $a - b = 1 \Rightarrow a = 3$ and $b = 2 \Rightarrow A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$a + b = 5$ and $a - b = -1 \Rightarrow a = 2$ and $b = 3 \Rightarrow A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$a + b = -5$ and $a - b = 1 \Rightarrow a = -2$ and $b = -3 \Rightarrow A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix}$

$a + b = -5$ and $a - b = -1 \Rightarrow a = -3$ and $b = -2 \Rightarrow A = \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix}$

Example - 16 Compute ABC if $A = [\ 2\ 1\ 3\ ]$ $B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & 5 \\ 5 & 2 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

Ans. 16 $ABC = (AB)\ C = A(BC)$

$$AB = [\,2\ 1\ 3\,]\begin{bmatrix}3 & 1 & 0\\ 2 & 0 & 5\\ 5 & 2 & 0\end{bmatrix} = [\,6+2+15,\ 2+0+6,\ 0+5+0\,]\ [23,\ 8,\ 5]$$

$$ABC = (AB)\,C = [23,\ 8,\ 5]\begin{bmatrix}3\\ 2\\ 5\end{bmatrix} = [69+16+25] = [110]$$

Example - 17 Prove that $A^3 = 4A$ if $A = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

Ans. 17 $A^2 = AA = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$$= \begin{bmatrix}1+1 & -1-1\\ -1-1 & 1+1\end{bmatrix} = \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} = 2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 2A$$

L. H. S $= A^3 = A^2 A = (2A)A = 2A^2 = 2(2A) = 4A =$ R. H. S.

Example - 18 If $P = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$ and $P^2 + 2P + Q = O$, then find Q

Ans. 18 $P^2 = P \times P = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix} = \begin{bmatrix}1+6 & 3+12\\ 2+8 & 6+16\end{bmatrix} = \begin{bmatrix}7 & 15\\ 10 & 22\end{bmatrix},\ 2P = \begin{bmatrix}2 & 6\\ 4 & 8\end{bmatrix}$

$$P^2 + 2P + Q = O \Rightarrow \begin{bmatrix}7 & 15\\ 10 & 22\end{bmatrix} + \begin{bmatrix}2 & 6\\ 4 & 8\end{bmatrix} + Q = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow\ Q = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}9 & 21\\ 14 & 30\end{bmatrix} = \begin{bmatrix}-9 & -21\\ -14 & -30\end{bmatrix}$$

(5) Determinant of the Square Matrix

Every square matrix can be associated to a unique number, which is called its determinant. The determinant of a square matrix A is denoted by $|A|$ or detA.

If $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then determinant of A is $|A| = \det A = \begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then determinant of A is $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a third order determinant.

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1\ (b_2c_3 - b_3c_2) - b_1\ (a_2c_3 - a_3c_2) + C_1\ (a_2b_3 - b_1a_2)$$

e. g. (i) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

$$= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1\ (45 - 48) - 2\ (36 - 42) + 3\ (32 - 25)$$
$$= 1\ (-3) - 2\ (-6) + 3\ (-3)$$
$$= -3 + 12 - 9 = 0$$

* **Singular Matrix :** Square matrix A is called a singular matrix if $|A| = 0$.

* **Non–Singular Matrix :** Square matrix A is called a non – singular matrix if $|A| \neq 0$.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a singular matrix and $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix.

[6] Adjoint of a Square Matrix

Let $A = [a_{ij}]$ is square matrix of order $n \times n$ and c_{ij} be the cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements in A is called the adjoint of A and is denoted by adj A. Thus, adj $A = (C_{ij})' = C_{ji}$ = cofactor of a_{ij} in A.

Properties of the Adjoint of the Matrix :

(i) If A is a non-singular matrix of order $n \times n$, then $A\ (\text{adj } A) = |A|\ I_n = (\text{adj } A)\ A$ and also $|\text{adj}A| = |A|^{n-1}$.

(2) If A is a square matrix of order $n \times n$, then adj $(A') = (\text{adj } A)'$
(3) adj (adj A) = A where A is a non-singular matrix.
(4) If A and B are two square matrices of the same order, then adj (AB) = (adj B) (adj A).
(5) Adjoint of a diagonal matrix is a diagonal matrix.

[7] Inverse of a Square Matrix

Let A is a square matrix of order $n \times n$ and $|A| \neq 0$ i. e. A is a non singular matrix. Then a matrix B is called inverse of A if $AB = BA = I_n$. Inverse of matrix A is denoted by A^{-1}.

Formula to find A^{-1} : $A^{-1} = \frac{1}{|A|}$ adjA where $|A| \neq 0$

Properties of Inverse Matrix

(1) $(AB)^{-1} = B^{-1} A^{-1}$
(2) $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$
(3) $(A)^{-1} = (A^{-1})$
(4) $(A^{-1})^{-1} = A$ i. e The inverse of the inverse is the original matrix itself.
(5) A is symmetric matrix $\Rightarrow A^{-1}$ is also symmetric matrix.
(6) $|A^{-1}| = |A|^{-1}$
(7) Square Matrix A is invertible if it is non-singular, $A^{-1} = \frac{1}{|A|}$ adj. A.

* **Idempotent Matrix :** Square matrix A is called an idempotent matrix if $A^2 = A$ or $A(A - I_n) = O$
* **Orthogonal Matrix :** Square matrix A of order $n \times n$ is said to be orthogonal matrix if $AA' = A'A = I_n$. Every orthogonal matrix is non-singular matrix and every orthogonal matrix is invertible .

Remark : If A is an orthogonal matrix, then A' and A^{-1} are also orthogonal.

(2) square matrix A is called an **involutory** matrix if $A^2 = I$.

[8] Solution of Simultaneous Linear Equations by Inverse Matrix Method

(i) **System of two linear equations :**

Consider the two simultaneous linear equations in two variables x and y are as follows.

$$a_1x + b_1y + c_1 = 0 \;.....(I)$$
$$a_2x + b_2y + c_2 = 0 \;....(II)$$

These can be written in matrix form as follows :

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \text{ or } AX = B$$

Where A is 2×2 matrix and is called coefficient matrix, X and B are 2×1 column matrices.

$$AX = B \therefore A^{-1} A x = A^{-1}B \Rightarrow I_2X = A^{-1} B \Rightarrow X = A^{-1}. B$$

Note :

(1) If $|A| \neq 0$, the system is consistent and has a unique solution.

(2) If $|A| = 0$, the system of linear equations has either no solution or an infinite number of solution. We can not use this method if $|A| = 0$

(ii) System of three simultaneous equations in three variables x y and z .

$a_1 x + b_1 y + c_1 z = d_1$ .....(I)

$a_2 x + b_2 y + c_2 z = d_2$ .....(II)

$a_3 x + b_3 y + c_3 z = d_3$ ......(III)

These can be written in matrix form as follows :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = D$$

$$\text{where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = D \Rightarrow X = A^{-1} D$$

(iii) System of homogeneous linear equation

A linear equstion is said to be homogeneous if the constant term is zero. The equations $a_1 x + b_1 y + c_1 z = 0$, $a_2 x + b_2 y + c_2 z = 0$, $a_3 x\ b_3 y + c_3 z = 0$ Constitute a system of homogeneous linear equations and it can be written in matrix form as $A X = 0$.

$$\text{where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\&\; 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

* **Minors And cofactors of Square Matrix :**

Minor : Let $A = (a_{ij})$ be a square matrix of order $n \times n$. Then the minor A_{ij} of a_{ij} is the determent of the submatrix obtained by leaving i^{th} row and j^{th} column of A.

e. g. if $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, then, A_{23} = minor of $a_{23} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$

A_{32} = minor of $a_{32} = \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = 2 - 4 = -2$

Cofactor : Let $A = [a_{ij}]_{n \times n}$ Then the cofactor c_{ij} of a_{ij} is equal to $(-1)^{i+j}$ times the minor A_{ij} of a_{ij} i. e. $c_{ij} = (-1)^{i+j} A_{ij}$

Example - 19 If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, then prove that $A\,(ad_j\,A) = |A|\,I_3$

Ans. 19 Transpos of $A = A^t = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & -3 & 3 \end{bmatrix}$

$$\text{adj}A = \begin{bmatrix} +\begin{vmatrix} 2 & -1 \\ -3 & 3 \end{vmatrix}, & -\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}, & +\begin{vmatrix} 1 & 2 \\ 1 & -3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ -3 & +3 \end{vmatrix}, & +\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}, & -\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}, & -\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}, & +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(6-3) & -(3+1) & +(-3-2) \\ -(+3+6) & +(3-2) & -(-3-1) \\ +(-1-4) & -(-1-2) & +(2-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$\text{L. H. S} = A\,\text{adj}A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-9-5 & -4+1+3 & -5+4+1 \\ 3-18+15 & -4+2-9 & -5+8-3 \\ 6+9-15 & -8-1+9 & -10-4+3 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} = -11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= |A|\ I_3$

$= R.\ H.\ S.$

$|A| =$ determinant of matrix A

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$= 1\ (6 - 3) - 1\ (3 + 6) + 1\ (-1 - 4)$

$= 1\ (3) - 1\ (9) + 1\ (-5)$

$= 3 - 9 - 5$

$= -11$

Example – 20 If $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and B $\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$, then verity adj (AB) = (adj B) (adj A)

Ans. 20

$$AB \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12+2 & 9+2 \\ 4+1 & 3+1 \end{bmatrix} = \begin{bmatrix} 14 & 11 \\ 5 & 4 \end{bmatrix}$$

$$\text{adj } AB = \begin{bmatrix} 4 & -5 \\ -11 & 4 \end{bmatrix}' = \begin{bmatrix} 4 & -11 \\ -5 & 14 \end{bmatrix} \ldots\ldots \text{(I)}$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

and adj $B = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$ because $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$(\text{adj } B)\ (\text{adj} A) = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & -2-9 \\ -1-4 & 2+12 \end{bmatrix} = \begin{bmatrix} 4 & -11 \\ -5 & 14 \end{bmatrix} \ldots\ldots \text{(II)}$$

Now (I) and (II) $\Rightarrow$ adj AB = (adj B) (adj A)

Example – 21 Find A^{-1} if $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and prove that $AA^{-1} = A^{-1}A = I_2$

Ans. 21

$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}\ 3 - 2 = 1 \neq 0$. Therefore A is a non-singular matrix and A^{-1} is exist.

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 3-2 & -6+6 \\ 1-1 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ .... (i)}$$

$$A^{-1}A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3-2 & 2-2 \\ 3-3 & -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \text{ .... (ii)}$$

Now (i) and (ii) $\Rightarrow AA^{-1}\ A^{-1}\ A = I_2$

Example – 22 Prove that $A^2 = I_3$ if $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Ans. 22

$$A^2 = AA = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6+1 & -8+10-2 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Remark : A is an involuntary matrix

Examples – 23 If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then find the value of $|\text{adj } A|$

Ans. 23

Cofactor matrix of $A = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

$\therefore$ adj A = [Cofactor matrix of A $]^1 = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix}$

$$\therefore |\text{adj } A| = \begin{vmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \end{vmatrix} = a^2 \cdot a^2 \cdot a^2 = a^6$$

Example – 24 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$, then find A^{-1} [Sau. Uni.]

Ans. 24 $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{vmatrix} = 1(18 - 16) - 2\ (12 - 12) + 3\ (8 - 9)$

$= 1\ (2) - 2\ (0) + 3\ (-1) = 2 - 3 = -1$

$\therefore$ A is a non-singular matrix

$$\text{adj } A = \begin{bmatrix} +(18-16) & -(12-12) & +(8-9) \\ -(12-12) & +(6-9) & -(4-6) \\ +(8-9) & -(4-6) & +(3-4) \end{bmatrix}^{1}$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}^{1} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{-1} \begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

Example – 25 Find the trace of A if $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 3 & 2 & 1 \\ 2 & 1 & 4 & 1 \\ 7 & 3 & 2 & 4 \end{bmatrix}$

Ans. 25 A is a square matrix of order 4×4. The elements 1, 3, 4, 4 constitute the principal diagonal of this matrix. The sum of the diagonal elements of a square matrix A is said to be trace of A and is denoted by trac (A). Here trac (A) $= 1 + 3 + 4 + 4 = 12$

Example – 26 If matrix $A = \begin{bmatrix} 1 & 3 & x+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$ is a singular matrix, then find value of x.

Ans. 26 A is singular matrix if $|A| = 0$

$$|A| = 0 \Rightarrow \begin{bmatrix} 1 & 3 & x+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$$

$= 0$

$\Rightarrow 1\,(40 - 40) - 3\,(20 - 24) + (x + 2)\,(10 - 12) = 0 \Rightarrow 0 + 32 - 2\,(x + 2) = 0$

$\therefore\ 12 - 2x - 4 = 0 \Rightarrow 2x = 8,\ x = 4$

Example - 27 Solve the following system of two linear equations in x and y by using inverse matrix method. **[Sau. Uni.]**

$$3x + 2y = 13$$
$$2x + 3y = 12$$

Cramer's method

Ans. 27 These can be written in matrix form as follows :

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \end{bmatrix}$$

$\therefore$ A X = D where $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $D = \begin{bmatrix} 13 \\ 12 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4$$

$= 5$

$\therefore\ |A| \neq 0$

$\therefore$ A is a non-singular matrix and hence A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\ \text{adj } A = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$AX = D \Rightarrow X = A^{-1} D$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 13 \\ 12 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 39-24 \\ -26+36 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$\therefore\ x = 3$ and $y = 2$ or $(x\ y) = (3, 2)$

Example - 28 Solve the following linear equations using inverse of matrix method. **[Sau. Uni.]**

$$x + y + z = 3$$
$$2x - y - z = 3$$
$$x - y + z = 9$$

Ans. 28 The above system of linear equations in x, y and z can be writer in the matrix from as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$\therefore$ A X = D $\Rightarrow$ X = A^{-1} D

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$

$$\det A = |A| = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = 1\,(-1-1) - 1\,(2+1) + 1\,(-2+1)$$

$$= 1\,(-2) - (3) - 1 = -2 - 3 - 1 = -6 \neq 0$$

$\therefore$ A is a non-singular matrix and hence A^{-1} exists.

$$A^1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} +(-1-1) & -(1+1) & +(-1+1) \\ -(2+1) & +(1-1) & -(-1-2) \\ +(-2+1) & -(-1-1) & +(-1-2) \end{bmatrix} = \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{-6} \begin{bmatrix} -2 & -2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & -3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 0 \\ 3 & 0 & -3 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|} \text{ adj } A. \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 0 \\ 3 & 0 & -3 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6+6+0 \\ 9+0-27 \\ 3-6+27 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -18 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$$

$\therefore\ x = 2, y = -3, z = 4$ or $(x\ y\ z) = (2, -3, 4)$ is the solution.

Example – 29 Show that the equations

$$2a + 6b + 11 = 0$$
$$6a + 20b - 6C - 3 = 0$$
$$6b - 18c + 1 = 0$$

are not consistent.

Ans. 29 The above symstem of equation can be written in the form of matrix as

$$\begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -11 \\ 3 \\ -1 \end{bmatrix}$$

$$\Rightarrow AX = D \quad \text{where } A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } D = \begin{bmatrix} -11 \\ 3 \\ -1 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} D$$

$$|A| = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix} = 2(-360 + 36) - 6(-108 - 0) + 0 = 648 + 648 = 0$$

$\therefore\ A^{-1}$ does not exist.

Hence the given system of linear equations are inconsistent.

Example – 30 Solve the following equations using inverse of matrix method

$$2x - 3y - 3 = 0 \text{ and } 4x - y - 11 = 0.$$

Ans. 30 The above equations can be written in the form of matrix as

$$\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$\Rightarrow AX = D \text{ where } A = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix} X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} D$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{|A|} \text{adj } A \times D$$

$$= \frac{1}{10}\begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix} = -2 + 12 = 10$$

$\therefore$ A is a non-singular matrix.

$$= \frac{1}{10}\begin{bmatrix} -3+33 \\ -12+22 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\therefore$ A^{-1} exist.

adj A= Cofactor matrix of A'

$$= \text{Cofactor matrix of } \begin{bmatrix} 2 & 4 \\ -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1) & -(-3) \\ +(4) & +(2) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix}$$

$\therefore$ Solution is $x = 3$ and $y = 1$

Example – 31 If $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, then prove that $(AB)^{-1} = B^{-1}A^{-1}$ [Sau. Uni.]

Ans. 31

$$AB = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12+3 & 8+3 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 15 & 11 \\ 4 & 3 \end{bmatrix}$$

$$|AB| = \begin{bmatrix} 15 & 11 \\ 4 & 3 \end{bmatrix} = 45 - 44 = 1$$

$$|A| = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} = 4 - 3 = 1$$

$$|B| = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = 3 - 2 = 1$$

$$\text{adj } AB = \begin{bmatrix} 3 & -11 \\ -4 & 15 \end{bmatrix} \quad \text{adj } A = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}, \text{ and adj } B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} \text{ adj } AB = \frac{1}{1}\begin{bmatrix} 3 & -11 \\ -4 & 15 \end{bmatrix} = \begin{bmatrix} 3 & -11 \\ -4 & 15 \end{bmatrix} \quad \text{.............(I)}$$

$$B^{-1} = \frac{1}{|B|} \text{ adj } B = \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{1}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

$$B^{1}A^{-1}=\begin{bmatrix}1&-2\\-1&3\end{bmatrix}\begin{bmatrix}1&-3\\-1&4\end{bmatrix}=\begin{bmatrix}1+2&-3-8\\-1-3&3+12\end{bmatrix}=\begin{bmatrix}3&-11\\-4&15\end{bmatrix}\quad\text{.......: (II)}$$

Now (I) and (II) $\Rightarrow (AB)^{-1}=B^{-1}A^{-1}$

Example – 32 If $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ and $AB=\begin{bmatrix}1&3\\0&2\end{bmatrix}$, then find matrix B.

Ans. 32

$$AB=\begin{bmatrix}1&3\\0&2\end{bmatrix}\Rightarrow B=A^{-1}\begin{bmatrix}1&3\\0&2\end{bmatrix}$$

$$=\frac{1}{|A|}\text{ adj }A\begin{bmatrix}1&3\\0&2\end{bmatrix}=\frac{1}{1-0}\begin{bmatrix}1&-2\\0&1\end{bmatrix}\begin{bmatrix}1&3\\0&2\end{bmatrix}=\frac{1}{1}\begin{bmatrix}1+0&3-4\\0+0&0+2\end{bmatrix}=\begin{bmatrix}1&-1\\0&2\end{bmatrix}$$

Example – 33 If $A=\begin{bmatrix}1&-1\\2&-1\end{bmatrix}$, $B=\begin{bmatrix}1&x\\4&y\end{bmatrix}$ and $(A+B)^2=A^2+B^2$, then find x and y .

Ans. 33

$$A=\begin{bmatrix}1&-1\\2&-1\end{bmatrix}\Rightarrow A^2=AA=\begin{bmatrix}1&-1\\2&-1\end{bmatrix}\begin{bmatrix}1&-1\\2&-1\end{bmatrix}$$

$$=\begin{bmatrix}1-2&-1+1\\2-2&-2+1\end{bmatrix}$$

$$=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$$

$$A+B=\begin{bmatrix}1&-1\\2&-1\end{bmatrix}+\begin{bmatrix}1&x\\4&y\end{bmatrix}$$

$$\Rightarrow (A+B)^2=(A+B)(A+B)$$

$$=\begin{bmatrix}2&x-1\\6&y-1\end{bmatrix}\begin{bmatrix}2&x-1\\6&y-1\end{bmatrix}$$

$$=\begin{bmatrix}4+6x-6&2x-2+(x-1)(y-1)\\12+6y-6&6x-6+(y-1)^2\end{bmatrix}$$

$$=\begin{bmatrix}-2+6x&2x-2+(x-1)(y-1)\\6+6y&6x-6+(y-1)^2\end{bmatrix}$$

$$B = \begin{bmatrix} 1 & x \\ 4 & y \end{bmatrix} \Rightarrow B^2 = B.B = \begin{bmatrix} 1 & x \\ 4 & y \end{bmatrix} \begin{bmatrix} 1 & x \\ 4 & y \end{bmatrix} = \begin{bmatrix} 1+4x & x+xy \\ 4+4y & 4x+y^2 \end{bmatrix}$$

$$A^2 + B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1+4x & x+xy \\ 4+4y & 4x+y^2 \end{bmatrix} = \begin{bmatrix} 4x & x+xy \\ 4+4y & 4x+y^2-1 \end{bmatrix}$$

$(A + B)^2 = A^2 + B^2$

$$\Rightarrow \begin{bmatrix} -2+6x & 2x-2+(x-1)(y-1) \\ 6+6y & 6x-6+(y-1)^2 \end{bmatrix} = \begin{bmatrix} 4x & x+xy \\ 4+4y & 4x+y^2-1 \end{bmatrix}$$

$\therefore -2 + 6x = 4x$ and $6 + 6y = 4 + 4y$

$\therefore 2x = 2$ and $2y = -2$

$\therefore \boxed{x = 1 \text{ and } y = -1}$

Example – 34 (i) If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $A X = B$, then find x and y.

(ii) If $A = [2 \ 3 \ 4]$ and $B = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, then Find AB, BA, A + 2B and A + 2B$'$,

Ans. 34

(i) $A X = B \Rightarrow X = A^{-1} B$

$$|A| = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$$

$$\text{adj } A = \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-5} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$$

$$\therefore \quad X = A^{-1} B = \frac{1}{-5} \begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} 9-4 \\ -12+2 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\therefore \quad x = -1, y = 2$

(ii) $$AB = [2\ 3\ 4] \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -2 + 6 + 4 = 8$$

$$BA = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [2, 3, 4] = \begin{bmatrix} -2 & -3 & -4 \\ 4 & 6 & 8 \\ 2 & 3 & 4 \end{bmatrix}$$

An order of matrix A is 1×3 and An order of matrix B is 3×1. Hence $A + 2B$ is not exist but $A + 2B'$ is exist.

$A + 2B^1 = [2\ 3\ 4] + 2\,[-1\ 2\ 1] = [2\ 3\ 4] + [-2\ 4\ 2] = [0\ 7\ 6]$

Example – 35 Find A if $A^{-1} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

Ans. 35 We know that $(A^{-1})^{-1} = A$

$$|A^{-1}| = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} = 5 - 4 = 1 \neq 0$$

$\therefore$ A^{-1} is a non-singular matrix and hence $(A^{-1})^{-1}$ is exist.

$$A = (A^{-1})^{-1} = \frac{1}{|A^{-1}|} \text{ adj } A^{-1}$$

$$= \frac{1}{1} \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$$

[Sau. Uni.]

Example – 36 Find $(AB)^{-1}$ if $A^{-1} = \begin{bmatrix} 7 & 6 \\ 1 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$

Ans. 36 We know that $(AB)^{-1} = B^{-1} A^{-1}$

$$\therefore (AB)^{-1} = B^{-1} A^{-1} = \frac{1}{|B|} \text{ adj } B. \frac{1}{|A|} \text{ adj } A$$

$$= \frac{1}{4-3} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \frac{1}{7-6} \begin{bmatrix} 1 & -6 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3 & -6+21 \\ -1-4 & 6+28 \end{bmatrix} = \begin{bmatrix} 4 & -27 \\ -5 & 34 \end{bmatrix}$$

Example - 37 If $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} A = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$, find A. [Sau. Uni.]

Ans. 37 $\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ is a matrix of order 3×1 and $\begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$ is a matrix of order 3×3

$\therefore$ A is a matrix of order 1×3

Let $A = [a \ b \ c]$

$$\therefore \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} [a \ b \ c] = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4a & 4b & 4c \\ a & b & c \\ 3a & 3b & 3c \end{bmatrix} = \begin{bmatrix} -4 & 8 & 4 \\ -1 & 2 & 1 \\ -3 & 6 & 3 \end{bmatrix} \Rightarrow a = -1,\ b = 2 \text{ and } c = 1$$

$\therefore A = [-1, 2, 1]$

Example - 38 Show that $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$, satisfies the equation $A^3 - 3A^2 - 9 + 9\, I_3 = 0$

where I_3 is the identity matrix of order 3×3 and 0 is the zero matrix of order 3×3. Hence find A^{-1}. [Sau. Uni.]

Ans. 38

$$A^2 = AA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & +1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & +1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & +1 \end{bmatrix} = \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix}$$

$$\text{L. H. S.} = A^3 - 3A^2 - A + 9\, I_3$$

$$= \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix} - \begin{bmatrix} 12 & 9 & 0 \\ -9 & 6 & -6 \\ 18 & 12 & 15 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & +1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^3 - 3\,A^2 - A + 9\,I_3 = 0$$

$$\Rightarrow \quad A^{-1}\,A^3 - 3\,A^{-1}\,A^2 + 9\,A^{-1}\,I_3 - A^{-1}\,A = A^{-1} \times 0$$

$$\Rightarrow \quad A^2 - 3A - I_3 + 9A^{-1} = 0$$

$$\Rightarrow \quad 9A^{-1} = 0 - A^2 + 3A + I_3$$

$$\Rightarrow \quad 9A^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 6 & 3 \\ 0 & 3 & -3 \\ 9 & -3 & +3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 9A^{-1} = \begin{bmatrix} 4 & 6 & 3 \\ 0 & 4 & -3 \\ 9 & -3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -1 \\ 3 & -7 & -1 \end{bmatrix}$$

[9] Exercise

[I] **Theoretical Questions :**

1. Explain the following terms.
Matrix, Square Matrix, Unit Matrix, Scalar Matrix.
[Sau. Uni. B. C. A. (Sem. II) April / May – 200

... Symmetric Matrix, Null Matrix, Square mat
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2. D...