CS 5001: Convex Optimization

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PROJECT REPORT

INTRODUCTION

Convex optimization is a subfield of mathematical optimization that studies the problem of minimizing convex functions over convex sets. Many classes of convex optimization problems admit polynomial-time algorithms, whereas mathematical optimization is in general NP-hard.

Convex optimization has applications in a wide range of disciplines, such as automatic control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, finance, statistics (optimal experimental design), and structural optimization, where the approximation concept has proven to be efficient. With recent advancements in computing and optimization algorithms, convex programming is nearly as straightforward as linear programming.

Definition

A mathematical optimization problem, or just optimization problem, has the form

minimize f_0 (x)
subject to $f_i(x) \le b_i$, i = 1,...,m.

Here the vector $x = (x_1,...,x_n)$ is the optimization variable of the problem, the function $f_0 : R^n \rightarrow R$ is the objective function, the functions $f_i : R^n \rightarrow R$, i = 1, . . . , m, are the (inequality) constraint functions, and the constants $b_1, . . . , b_m$ are the limits, or bounds, for the constraints. A vector $x^\star$ is called optimal, or a solution of the problem , if it has the smallest objective value among all vectors that satisfy the constraints: for any z with $f_1(z) \le b_1,...,f_m(z) \le b_m$, we have $f_0(z) \ge f_0(x^\star)$.

We generally consider families or classes of optimization problems, characterized by particular forms of the objective and constraint functions. As an important example, the optimization problem is called a linear program if the objective and constraint functions $f_0, . . . , f_m$ are linear, i.e., satisfy

$$f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)$$

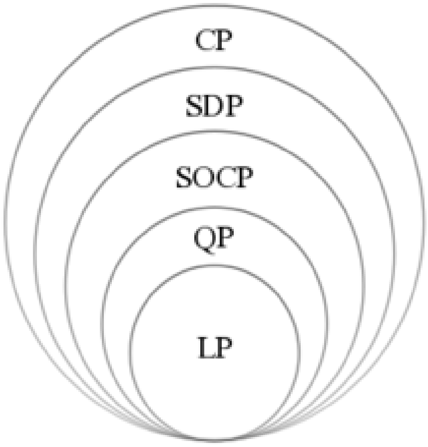
for all x, y $\in R^n$ and all $\alpha, \beta \in R$. If the optimization problem is not linear, it is called a nonlinear program.

This book is about a class of optimization problems called convex optimization problems. A convex optimization problem is one in which the objective and constraint functions are convex, which means they satisfy the inequality

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

for all x, $y \in R^n$ and all $\alpha, \beta \in R$ with $\alpha+\beta=1, \alpha \geq 0, \beta \geq 0$. Comparing and, we see that convexity is more general than linearity: inequality replaces the more restrictive equality, and the inequality must hold only for certain values of α and β. Since any linear program is therefore a convex optimization problem, we can consider convex optimization to be a generalization of linear programming.

SYSTEM MODEL



A hierarchy of convex optimization problems. (LP: linear program, QP: quadratic program, SOCP second-order cone program, SDP: semidefinite program, CP: cone program.)

Linear Programming

Optimization is the way of life. We all have finite resources and time and we want to make the most of them. From using your time productively to solving supply chain problems for your company – everything uses optimization. It's an especially interesting and relevant topic in data science.

It is also a very interesting topic – it starts with simple problems, but it can get very complex.

Linear programming (LP, also called linear optimization) is a method to achieve the best outcome (such as maximum profit or lowest cost) in a mathematical model whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming (also known as mathematical optimization).

More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints. It's feasible region is a convex polytope, which is a set defined as the intersection of finitely many half spaces, each of which is defined by a linear inequality. Its objective function is a real-valued affine (linear) function defined on this polyhedron. A linear programming algorithm finds a point in the polytope where this function has the smallest (or largest) value if such a point exists.

PROBLEM FORMULATION

A woman makes craft pens to sell at a seasonal craft show. She makes caps and barrels. Each cap takes her 1 hour to make and sells for a profit of 8.Thepairsofbarrelstake2hourstomake,butshegetsaprofitof20. She likes to have variety, so she wants to have at least as many caps as pairs of barrels. She also knows that she has approximately 40 hours for creating pens between now and the start of the show. She also knows that the craft show vender wants sellers to have more than 20 items on display at the beginning of the show. Assuming she sells all of her inventory, how many each of caps and barrel pairs should the woman make to maximize her profit?

Constraints

Let’s begin by identifying the constraints. To do this, we should first define some variables. Let x be the number of caps the woman makes, and let y be the number of pairs of barrels she makes.

We know that the woman has 40 hours to create the caps and barrels. Since they take 1 hour and 2 hours respectively, we can identify the constraint $x+2y\leq 40$.

The woman also has constraints on the number of products she will make. Specifically, her vender wants her to have more than 20 items. Thus, we know that $x+y>20$. Since, however, she cannot make part of a barrel on cap, we can adjust this inequality to $x+y\geq 21$.

Finally, the woman has her own constraints on her products. She wants to have at least as many caps as pairs of barrels. This means that $x\geq y$.

In addition, we have to remember that we cannot have negative numbers of products. Therefore, x and y are both positive too.

Thus, in summary, our constraints are:

$x+2y\leq 40$
$x+y\geq 21$
$x\geq y$
$x\geq 0$
$y\geq 0$

PROBLEM SOLUTION

The Objective Function

The woman wants to know how she can maximize her profits. We know that the caps give her a profit of 8,andbarrelsearnher20. Since she expects to sell all of the pens she makes, the woman will make a profit of P=8x+20y. We want to find the maximum of this function.

minimize $8x+20y$
subject to $x+2y \leq 40$
$x+y \geq 21$
$x \geq y$
$x \geq 0$
$y \geq 0$

SIMULATION AND RESULTS

CVX is a MATLAB-based modeling system for convex optimization. CVX turns MATLAB into a modeling language, allowing constraints and objectives to be specified using standard MATLAB expression syntax.

In this project ,we consider the convex optimization model discussed in problem formulation and run it in MATLAB using CVX.

Simulation results:

```
>> newproj

cvx_begin
   variable x      % caps
   variable y      %barrels

   maximize( 8* x + 20*y ) % max profit
   subject to
      x+2*y <=40;
      x+y >=21;
      x>=y;
      x>=0;
      y>=0;
cvx_end
```

```
Calling SDPT3 4.0: 5 variables, 2 equality constraints
   For improved efficiency, SDPT3 is solving the dual problem.
------------------------------------------------------------

 num. of constraints =  2
 dim. of linear var  =  5

*******************************************************************
   SDPT3: Infeasible path-following algorithms
*******************************************************************
 version  predcorr  gam  expon  scale_data
    NT      1      0.000   1        0
it pstep dstep pinfeas dinfeas  gap      prim-obj      dual-obj    cputime
-------------------------------------------------------------------
 0|0.000|0.000|1.5e+00|2.3e+00|3.0e+03| 2.447208e+02  0.000000e+00| 0:0:00| chol  1  1
 1|1.000|0.990|6.5e-07|2.7e-02|3.1e+02| 5.969890e+02  3.025510e+02| 0:0:00| chol  1  1
 2|0.807|1.000|2.8e-07|4.8e-04|7.3e+01| 3.955891e+02  3.229825e+02| 0:0:00| chol  1  1
 3|1.000|0.402|1.9e-06|3.1e-04|8.3e+01| 4.471843e+02  3.638785e+02| 0:0:00| chol  1  1
 4|0.973|0.909|9.8e-08|3.3e-05|3.4e+00| 3.757205e+02  3.722880e+02| 0:0:00| chol  1  1
 5|0.985|0.986|1.6e-09|9.7e-07|5.1e-02| 3.733685e+02  3.733180e+02| 0:0:00| chol  1  1
 6|0.989|0.989|1.1e-10|1.1e-08|5.6e-04| 3.733337e+02  3.733332e+02| 0:0:00| chol  1  1
 7|0.989|0.989|3.5e-12|1.4e-10|6.2e-06| 3.733333e+02  3.733333e+02| 0:0:00|
  stop: max(relative gap, infeasibilities) < 1.49e-08
-------------------------------------------------------------------
 number of iterations   =  7
 primal objective value =  3.73333338e+02
 dual   objective value =  3.73333331e+02
 gap := trace(XZ)       = 6.17e-06
 relative gap           = 8.25e-09
 actual relative gap    = 8.21e-09
 rel. primal infeas (scaled problem)   = 3.50e-12
 rel. dual     "        "       "      = 1.44e-10
 rel. primal infeas (unscaled problem) = 0.00e+00
 rel. dual     "        "       "      = 0.00e+00
 norm(X), norm(y), norm(Z) = 9.4e+00, 1.9e+01, 2.0e+01
 norm(A), norm(b), norm(C) = 4.3e+00, 2.3e+01, 4.6e+01
 Total CPU time (secs)  = 0.04
 CPU time per iteration = 0.01
 termination code       =  0
 DIMACS: 3.8e-12  0.0e+00  1.6e-10  0.0e+00  8.2e-09  8.3e-09
-------------------------------------------------------------------
```

Status: Solved
Optimal value (cvx_optval): +373.333

echo off
Done. (Check out the graph!)
>> x

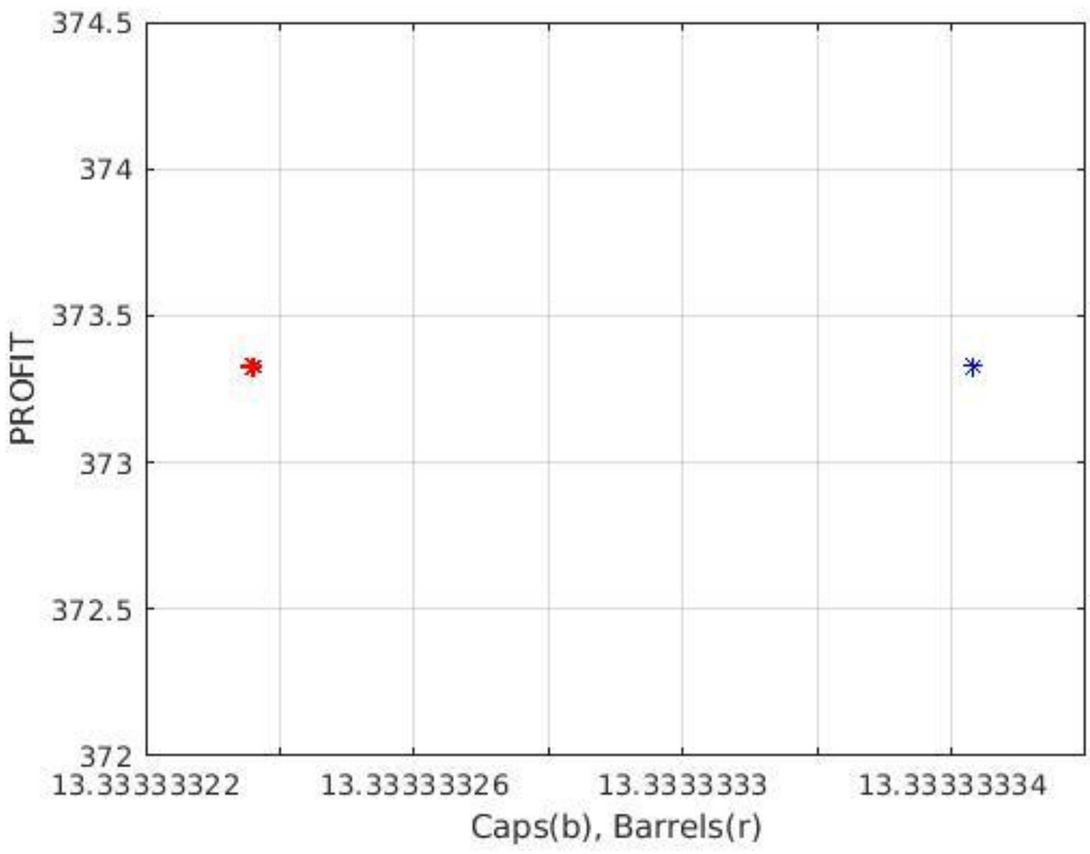
x =

   13.3333

>> y

y =

   13.3333



CONCLUSION

Therefore, x and y are about 13.33.

Now, we know from our function P=8x+20y is about 373.33

Now, the maximum in this case is the point (13.33,13.33). However, the woman cannot make 13.33 caps or 13.33 pairs of barrels. We can adjust by finding the nearest whole number coordinate that is inside the limit and testing it. In this case, we have (13, 13) or (14, 13). We will choose the latter since it will obviously yield a larger profit.

Then, we have:

P=14(8)+13(20)=372.

Thus, the woman should make 14 caps and 13 pairs of barrels for the greatest profit given her other constraints.

REFERENCES

1. https://en.wikipedia.org/wiki/Convex_optimization
2. https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf
3. https://www.analyticsvidhya.com/blog/2017/02/lintroductory-guide-on-linear-programming-explained-in-simple-english/
4. http://cvxr.com/cvx/
5. https://en.wikipedia.org/wiki/Linear_programming

APPENDIX

```
echo on

cvx_begin
   variable x        % caps
   variable y        %barrels

   maximize( 8* x + 20*y ) % max profit
   subject to
      x+2*y <=40;
      x+y >=21;
      x>=y;
      x>=0;
      y>=0;
cvx_end

echo off


figure(1)
ln=plot( x,  8* x + 20*y);
% xlabel( 'x' );
% ylabel( 'y' );bn.LineWidth = 2;
ln.Color = [0 0.5 0.5];
ln.Marker = '*';
ln.MarkerEdgeColor = 'b';
hold on;
bn=plot( y,  8* x + 20*y);
xlabel( 'Caps(b), Barrels(r)' );
ylabel( 'PROFIT' );
grid
disp( 'Done. (Check out the graph!)' );
bn.LineWidth = 2;
bn.Color = [0 0.5 0.5];
bn.Marker = '*';
bn.MarkerEdgeColor = 'r';

% cvx_quiet(s_quiet);
% cvx_pause(s_pause);
```