MATHEMATICS

SECTION A

January 27, 2024

1. If A is a square matrix of order 2 and $\left|A\right| = 4$,then find the value of $\left|2 \cdot AA'\right|$, where A' is the transpose of matrix A.
2. Find the angle between the line $\overrightarrow{r} = \left(2\hat{i} - \hat{j} + 3\hat{k}\right) + \lambda\left(3\hat{i} - \hat{j} + 2\hat{k}\right)$ and the plane $\overrightarrow{r}.\left(\hat{i} + \hat{j} + \hat{k}\right) = 3$.
3. Find the co-ordinates of the point,where the line $\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5}$ cuts the yz-plane.
4. Find the differential equation representing the family of curves $y = -Acos3x + Bsin3x$.
5. Find the differential of the function $\cos^{-1}\left(\sin 2x\right)$ w.r.t.x.
6. If an operation $*$ on the set of integers Z is defined by $a * b = 2a^2 + b$, then find
   (a) whether it is a binary or not, and
   (b) if a binary, then is it commutative or not.
7. Four cards are drawn one by one with replacement from a well-shuffled deck of playing cards. Find the probability that at least three cards are of diamonds.
8. The probability of two students A and B coming to school on time are $\frac{2}{7}$ and $\frac{4}{7}$,respectively. Assuming that the events 'A coming on time' and 'B coming on time' are independent, find the probability of only one of them coming to school on time.
9. Find the value of $(x - y)$ from the matrix equation $2\begin{pmatrix} x & 5 \\ 7 & y-3 \end{pmatrix} + \begin{pmatrix} -3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 15 & 14 \end{pmatrix}$
10. Solve the following differential equation :
$$\left(y + 3x^2\right)\frac{dx}{dy} = x$$
11. Using vectors, prove that the points $(2, -1, 3), (3, -5, 1)$ and $(-1, 11, 9)$ are collinear.
12. For any two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, prove that
$$\left(\overrightarrow{a} \times \overrightarrow{b}\right)^2 = \overrightarrow{a}^2\overrightarrow{b}^2 - \left(\overrightarrow{a} \cdot \overrightarrow{b}\right)^2$$
13. Find:
$$\int \frac{x-1}{(x-2)(x-3)}dx$$
14. Integrate:
$$\frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}$$
with respect to x.

15. Find:

$$\int e^x \left(\frac{2 + \sin 2x}{2\cos^2 x} \right) dx$$

16. If A and B are independent events with $P(A) = \frac{3}{7}$ and $P(B) = \frac{2}{5}$, then find $P(A' \cap B')$.

17. Find the equation of the plane passing through $(-1, 3, 2)$ and perpendicular to the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

18. Evaluate:

$$\int_1^5 (|x-1| + |x-2| + |x-4|)\, dx$$

19. If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & x^3 - 1 \\ y & y^2 & y^3 - 1 \\ z & z^2 & z^3 - 1 \end{vmatrix} = 0$,then using properties of determinants, show that $xyz = 1$.

20. Prove that :

$$\sin^{-1} \frac{4}{5} + \tan^{-1} \frac{5}{12} + \cos^{-1} \frac{63}{65} = \frac{\pi}{2}$$

21. Using vectors, find the value of x such that the four points A$(x, 5, -1)$,B$(3, 2, 1)$,C$(4, 5, 5)$ and D$(4, 2, -2)$ are coplanar.

22. Differentiate $\tan^{-1} \frac{3x - x^3}{1 - 3x^2}, |x| < \frac{1}{\sqrt{3}}$ w.r.t. $\tan^{-1} \frac{x}{\sqrt{1 - x^2}}$.

23. If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y), |x| < 1, |y| < 1$, show that $\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$.

24. Prove that the relation $\mathbf{R}$ in the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ given by $\mathbf{R} = \left\{(a, b) : \left|a - b\right| \text{ is even}\right\}$ is an equivalence relation.

25. Show that the function f in $A = R - \{\frac{2}{3}\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1}.

26. Find the particular solution of the differential equation: $\left(1 + e^{2x}\right) dy + \left(1 + y^2\right) e^x dx = 0$, given that $y(0) = 1$.

27. Find the particular solution of the differential equation: $x\frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0$,given that $y(1) = \frac{\pi}{2}$.

28. If $y = (\sin x^x) + \sin^{-1}\left(\sqrt{1 - x^2}\right)$, then find $\frac{dy}{dx}$

29. Find:

$$\int \cos 2x \cos 4x \cos 6x \, dx$$

30. Find the interval in which the function f given by $f(x) = \sin 2x + \cos 2x, 0 \le x \le \pi$ is strictly decreasing.

31. Using elementary row transformations,find the inverse of the matrix $\begin{pmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{pmatrix}$.

32. Using matrices, solve the following system of linear equations:

$$\begin{aligned} 2x + 3y + 10z &= 4 \\ 4x - 6y + 5z &= 1 \\ 6x + 9y - 20z &= 2 \end{aligned}$$

33. Using integration, find the area of the following region :

$$\{(x, y) : x^2 + y^2 \leq 16a^2 \textit{ and } y^2 \leq 6ax\}$$

34. Using integration, find the area of triangle ABC bounded by the lines

$$4x - y + 5 = 0, x + y - 5 = 0 \textit{ and } x - 4y + 5 = 0.$$

35. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit for type A souvenirs is ₹ 100 each and for type B souvenirs, profit is ₹ 120 each. How many souvenirs of each type should the company manufacture in order to maximise the profit ? Formulate the problem as a LPP and then solve it graphically.

36. Find the vector equation of the line passing through$(2, 1, -1)$ and parallel to the line $\overrightarrow{r} = \left(\hat{i} + \hat{j}\right) + \lambda\left(2\hat{i} - \hat{j} + \hat{k}\right)$.Also, find the distance between these two lines.

37. Find the coordinates of the foot Q of the perpendicular drawn from the point $P\,(1, 3, 4)$ to the plane $2x - y + z + 3 = 0$. Find the distance PQ and the image of P treating the planes as a mirror.

38. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack, two cards are drawn at random (without replacement) and both are found to be spades. Find the probability of the lost card being a spade.

39. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.