Mathematics I for CSE Stream (BMATS101)

Module 1

Prerequisites:

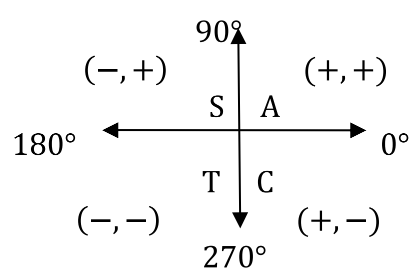
Trigonometry

Pythagorean identities	Reciprocal ratios
$\sin^2\theta+\cos^2\theta=1$ $\sec^2\theta-\tan^2\theta=1$ $cosec^2\theta-\cot^2\theta=1$	$\operatorname{cosec}\theta=\dfrac{1}{\sin\theta}$ $\sec\theta=\dfrac{1}{\cos\theta}$ $\cot\theta=\dfrac{1}{\tan\theta}$
Sum formulas	**Difference formulas**
$\sin(x+y)=\sin x\cos y+\cos x\sin y$ $\cos(x+y)=\cos x\cos y-\cos x\cos y$ $\tan(x+y)=\dfrac{\tan x+\tan y}{1-\tan x\tan y}$	$\sin(x-y)=\sin x\cos y-\cos x\sin y$ $\cos(x-y)=\cos x\cos y+\cos x\cos y$ $\tan(x-y)=\dfrac{\tan x-\tan y}{1+\tan x\tan y}$
Double angle formulas	**Triple angle formulas**
$\sin 2x=2\sin x\cos x$ $\cos 2x=\cos^2 x-\sin^2 x$ $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$	$\sin 3x=3\sin x-4\sin^3 x$ $\cos 3x=4\cos^3 x-3\cos x$ $\tan\ 3x=\dfrac{3\tan x-\tan^3 x}{1-3\tan^2 x}$
Half angle formulas	**Tangent formulas**
$\sin^2 x=\dfrac{1}{2}(1-\cos 2x)$ $\cos^2 x=\dfrac{1}{2}(1+\cos 2x)$ $\tan^2 x=\dfrac{1-\cos 2x}{1+\cos 2x}$	$\sin 2x=\dfrac{2\tan x}{1+\tan^2 x}$ $\cos 2x=\dfrac{1-\tan^2 x}{1+\tan^2 x}$ $\tan 2x=\dfrac{2\tan x}{1-\tan^2 x}$
Standard angle formulas	**ASTC Rule**

Standard angle formulas

θ	$o°$	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

ASTC Rule



Note:

$2\sin^2\frac{x}{2} = 1 - \cos x$	$\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2 = 1 + \sin x$	$\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$
$2\cos^2\frac{x}{2} = 1 + \cos x$	$\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 = 1 - \sin x$	$\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

Same ratio formulas:

$\sin(-\theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$	$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$	$\cos(2\pi - \theta) = \cos\theta$	$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$	$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$
$\cot(-\theta) = -\cot\theta$	$\cot(2\pi - \theta) = -\cot\theta$	$\cot(\pi - \theta) = -\cot\theta$	$\cot(\pi + \theta) = \cot\theta$
$\sec(-\theta) = \sec\theta$	$\sec(2\pi - \theta) = \sec\theta$	$\sec(\pi - \theta) = -\sec\theta$	$\sec(\pi + \theta) = -\sec\theta$
$\mathrm{cosec}(-\theta) = -\mathrm{cosec}\,\theta$	$\mathrm{cosec}(2\pi - \theta) = -\mathrm{cosec}\,\theta$	$\mathrm{cosec}(\pi - \theta) = \mathrm{cosec}\,\theta$	$\mathrm{cosec}(\pi + \theta) = -\mathrm{cosec}\,\theta$
(IV quadrant) Cos +ve	(IV quadrant) Cos +ve	(II quadrant) Sin +ve	(III quadrant) Tan +ve

Co ratio formulas:

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$	$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$	$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$	$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$	$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$
$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$	$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$	$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$
$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$	$\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$	$\cot\left(\frac{3\pi}{2} - \theta\right) = \tan\theta$	$\cot\left(\frac{3\pi}{2} + \theta\right) = -\tan\theta$
$\sec\left(\frac{\pi}{2} - \theta\right) = \mathrm{cosec}\,\theta$	$\sec\left(\frac{\pi}{2} + \theta\right) = -co\sec\theta$	$\sec\left(\frac{3\pi}{2} - \theta\right) = -\mathrm{cosec}\,\theta$	$\sec\left(\frac{3\pi}{2} + \theta\right) = co\sec\theta$
$\mathrm{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec\theta$	$\mathrm{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec\theta$	$\mathrm{cosec}\left(\frac{3\pi}{2} - \theta\right) = -\sec\theta$	$\mathrm{cosec}\left(\frac{3\pi}{2} + \theta\right) = -\sec\theta$
(I quadrant) All +ve	(II quadrant) Sin +ve	(III quadrant) Tan +ve	(IV quadrant) Cos +ve

Differentiation of some standard functions

Non Trigonometric functions	*Trigonometric functions*	*Hyperbolic functions*	*Inverse functions*
$(k)' = 0$	$(sin\ x)' = cos\ x$	$(sinh\ x)' = cosh\ x$	$(sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}$
$(x^n)' = n\ x^{n-1}$	$(cos\ x)' = -sin\ x$	$(cosh\ x)' = sinh\ x$	$(cos^{-1}x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$	$(tan\ x)' = sec^2x$	$(tanh\ x)' = sech^2x$	$(tan^{-1}x)' = \frac{1}{1+x^2}$
$(log\ x)' = \frac{1}{x}$	$(cot\ x)' = -\ cosec^2\ x$	$(coth\ x)' = -\ cosech^2\ x$	$(cot^{-1}x)' = -\frac{1}{1+x^2}$
$(e^x)' = e^x$	$(sec\ x)' = sec\ x. tan\ x$	$(sech\ x)' = -sech\ x. tanh\ x$	$(sec^{-1}x)' = \frac{1}{x\sqrt{x^2-1}}$
$(a^x)' = a^x log\ a$	$(cosec\ x)'$ $= -cosec\ x. cot\ x$	$(cosech\ x)'$ $= -cosech\ x. coth\ x$	$(cosec^{-1}x)' = -\frac{1}{x\sqrt{x^2-1}}$

Rules of differentiation

1. $(ku)' = ku'$ 2. $(u \pm v)' = u' \pm v'$	3. $(uv)' = uv' + vu'$ 4. $\left(\frac{u}{v}\right)' = \frac{vu'-uv\prime}{v^2}$

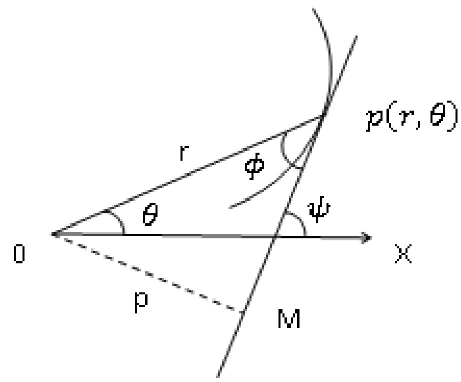
1.1 Polar curves

Introduction:

- Polar coordinates are $(x, y) = (r\cos\theta, r\sin\theta)$ where r - radial distance, θ- polar angle.
- Polar form of the equation of the curve $r = f(\theta)$ is called polar curve.
- $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$, $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$, $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$
- $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$, $\tan\left(\frac{\pi}{4} - \theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$
- Angle between radius vector and tangent is $tan\phi = r\frac{d\theta}{dr}$

Problems:

1. **Derive angle between radius vector and tangent. (May 22)**



Let $P(r, \theta)$ be any point on the polar curve $r = f(\theta)$.

Let χ be the angle from the X axis to the tangent.

Let p be the perpendicular distance from the origin to the tangent.

By diagram, $\chi = \theta + \phi$

$\tan\chi = \tan(\theta + \phi)$

$\tan\chi = \frac{\tan\theta+\tan\phi}{1-\tan\theta.\tan\phi}$ ------- (1)

But

$$\tan\chi = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$$

Divide by $\frac{dr}{d\theta}\cos\theta$ in numerator and denominator,

$$\tan\chi = \frac{\tan\theta+r\frac{d\theta}{dr}}{1-r\frac{d\theta}{dr}\tan\theta} \quad \text{--------} \ (2)$$

Equating components of (1) and (2),

$$\tan\phi = r\frac{d\theta}{dr}$$

2. Find the angle between radius vector and tangent to the following:
(i) $r^2\cos 2\theta = a^2$ (ii) $r = a(1+\cos\theta)$

(i) $r^2\cos 2\theta = a^2$ Take log on both sides, $\log(r^2\cos 2\theta) = \log a^2$ $\log r^2 + \log\cos 2\theta = 0$ $2\log r + \log\cos 2\theta = 0$ $2\log r = -\log\cos 2\theta$ Differentiate with respect to θ, $\frac{2}{r}\frac{dr}{d\theta} = \frac{2\sin 2\theta}{\cos 2\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$ $\cot\phi = \cot\left(\frac{\pi}{2} - 2\theta\right)$ $\phi = \frac{\pi}{2} - 2\theta$ This is the required angle between radius vector and tangent.	(ii) $r = a(1+\cos\theta)$ Take log on both sides, $\log r = \log a(1+\cos\theta)$ $\log r = \log a + \log(1+\cos\theta)$ Differentiate with respect to θ, $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{(1+\cos\theta)}$ $\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $\frac{1}{r}\frac{dr}{d\theta} = -\tan\frac{\theta}{2}$ $\cot\phi = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$ $\phi = \frac{\pi}{2} + \frac{\theta}{2}$ This is the required angle between radius vector and tangent.

3. **Find the angle between radius vector and tangent to the following:**
(i) $\boldsymbol{r^n = a^n \sec(n\theta + \alpha)}$ **(ii)** $\boldsymbol{r^m = a^m(\cos m\theta + \sin m\theta)}$

(iii) $r^n = a^n \sec(n\theta + \alpha)$

Take log on both sides,

$$\log r^n = \log a^n \sec(n\theta + \alpha)$$

$$\log r^n = \log a^n + \log \sec(n\theta + \alpha)$$

$$n \log r = n \log a + \log \sec(n\theta + \alpha)$$

Differentiate with respect to θ,

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + n\frac{\sec(n\theta + \alpha)\tan(n\theta + \alpha)}{\sec(n\theta + \alpha)}$$

$$\cot\phi = \tan(n\theta + \alpha)$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - (n\theta + \alpha)\right)$$

Angle between the radius vector and the tangent is $\phi = \frac{\pi}{2} - n\theta - \alpha$

(iv) $r^m = a^m(\cos m\theta + \sin m\theta)$

Take log on both sides,

$$\text{Log}\, r^m = \log a^m(\cos m\theta + \sin m\theta)$$

$$m \log r = m\ \log a^m + \log(\cos m\theta + \sin m\theta)$$

Differentiate with respect to θ,

$$\frac{\cancel{m}}{r}\frac{dr}{d\theta} = \frac{\cancel{m}(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

Divide by $\cos m\theta$ in Nr. And Dr.

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1 - \tan m\theta}{1 + \tan m\theta}$$

$$\cot\phi = \tan\left(\frac{\pi}{4} - m\theta\right)$$

$$\cot\phi = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right)$$

Angle between the radius vector and the tangent is $\phi = \frac{\pi}{4} + m\theta$

4. **Find the angle between radius vector and tangent to the following:**
$\boldsymbol{\frac{l}{r} = 1 + e\cos\theta}$ **.**

$\frac{l}{r} = 1 + e\cos\theta$

Take log on both sides,

$\log l - \log r = \log(1 + e\cos\theta)$

Differentiate with respect to θ,

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1 + e\cos\theta}$$

$$\cot\phi = \frac{e\sin\theta}{1 + e\cos\theta}$$

$$\tan\phi = \frac{1 + e\cos\theta}{e\sin\theta}$$

$$\phi = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)$$

5. **Show that the following pair of curves intersect orthogonally:**
$\boldsymbol{r = a(1 + \cos\theta)}$ **,** $\boldsymbol{r = b(1 - \cos\theta)}$ **(MQP 2)**

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$r = a(1 + \cos\theta)$

Take log on both sides,

$\log r = \log a\,(1 + \cos\theta)$

$\log r = \log a + \log(1 + \cos\theta)$

Differentiate w. r. to θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$\cot\phi_1 = -\tan\frac{\theta}{2}$

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$r = b(1 - \cos\theta)$

Take log on both sides,

$\log r = \log b\,(1 - \cos\theta)$

$\log r = \log b + \log(1 - \cos\theta)$

Differentiate w. r. to θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$\cot\phi_2 = \cot\frac{\theta}{2}$

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Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally.

6. Show that the following pair of curves intersect orthogonally:

$\boldsymbol{r^n = a^n \cos n\theta}$ **and** $\boldsymbol{r^n = b^n \sin n\theta}$ **(July 23)**

$r^n = a^n \cos n\theta$	$r^n = b^n \sin n\theta$
Take log on both sides,	Take log on both sides,
$\log r^n = \log a^n \cos n\theta$	$\log r^n = \log b^n \cos n\theta$
n $\log r = \log a^n + \log \cos n\theta$	$n \log r = \log b^n + \log \sin n\theta$
Differentiate w. r. to θ	Differentiate w. r. to θ
$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n \sin n\theta}{\cos n\theta}$	$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{n \cos n\theta}{\sin n\theta}$
$n\frac{1}{r}\frac{dr}{d\theta} = -n \tan n\theta$	$n\frac{1}{r}\frac{dr}{d\theta} = -n \cot n\theta$
$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$	$\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$
$\cot \phi_1 = -\tan n\theta$	$\cot \phi_2 = \cot n\theta$

Since $\cot \phi_1 . \cot \phi_2 = -1$, both intersect orthogonally.

7. Show that the following pair of curves intersect orthogonally:

$\boldsymbol{r = \frac{a}{1+\cos\theta}}$ **and** $\boldsymbol{r = \frac{b}{1-\cos\theta}}$ **. (MQP 2)**

$r = \dfrac{a}{1+\cos\theta}$ Take log on both sides, $\log r = \log\dfrac{a}{1+\cos\theta}$ $\log r = \log a - \log(1+\cos\theta)$ Differentiate w. r. to θ $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1+\cos\theta}$ $\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $\dfrac{1}{r}\dfrac{dr}{d\theta} = \tan\dfrac{\theta}{2}$ $\cot\phi_1 = \tan\dfrac{\theta}{2}$	$r = \dfrac{b}{1-\cos\theta}$ Take log on both sides, $\log r = \log\dfrac{b}{1-\cos\theta}$ $\log r = \log b - \log(1-\cos\theta)$ Differentiate w. r. to θ $\dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{\sin\theta}{1-\cos\theta}$ $\dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $\dfrac{1}{r}\dfrac{dr}{d\theta} = -\cot\dfrac{\theta}{2}$ $\cot\phi_2 = -\cot\dfrac{\theta}{2}$

Since $\cot\phi_1 . \cot\phi_2 = -1$, both intersect orthogonally

8. Show that the following pair of curves intersect orthogonally:

$\boldsymbol{r = a\theta}$ **and** $\boldsymbol{r = \frac{a}{\theta}}$

$r = a\theta$ Take log on both sides, $\log r = \log a\theta$ $\log r = \log a + \log\theta$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\theta}$ $\cot\phi_1 = \frac{1}{\theta}$	$r = \frac{a}{\theta}$ Take log on both sides, $\log r = \log\frac{a}{\theta}$ $\log r = \log a - \log\theta$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\theta}$ $\cot\phi_2 = -\frac{1}{\theta}$

By data, $a\theta = \frac{a}{\theta}$

Therefore, $\theta^2 = 1$.

$$\cot\phi_1 \,.\cot\phi_2 = \left(\frac{1}{\theta}\right)\left(-\frac{1}{\theta}\right)$$

$$= -\frac{1}{\theta^2}$$

$$= -1$$

Therefore, both intersect orthogonally.

9. Show that the following pair of curves intersect orthogonally:

$\boldsymbol{r = ae^{\theta}}$ **and** $\boldsymbol{re^{\theta} = b}$

$r = ae^{\theta}$ Take log on both sides, $\log r = \log a\theta$ $\log r = \log a + \theta$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 + 1$ $\cot \phi_1 = 1$	$re^{\theta} = a$ Take log on both sides, $\log re^{\theta} = \log a$ $\log r + \theta = \log a$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} + 1 = 0$ $\cot \phi_2 = -1$

Since $\cot \phi_1 . \cot \phi_2 = -1$, both intersect orthogonally.

10. Show that $\boldsymbol{r = 4\sec^2\frac{\theta}{2}}$ $\boldsymbol{and\ r = 9\ cosec^2\frac{\theta}{2}}$ **the pair of curves cut orthogonally.**

(May 22)

$r = 4\sec^2\frac{\theta}{2}$ Take log on both sides, $\log r = \log\left(4\ \sec^2\frac{\theta}{2}\right)$ $\log r = \log 4 + 2\log \sec\frac{\theta}{2}$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{2}{\sec\frac{\theta}{2}}\sec\frac{\theta}{2}\tan\frac{\theta}{2}$ $\cot \phi_1 = 2\tan\frac{\theta}{2}$	$r = 9\ cosec^2\frac{\theta}{2}$ Take log on both sides, $\log r = \log\left(9\ cosec^2\frac{\theta}{2}\right)$ $\log r = \log 9 + 2\log cosec\frac{\theta}{2}$ Differentiate with respect to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{2}{cosec\frac{\theta}{2}}cosec\frac{\theta}{2}\cot\frac{\theta}{2}$ $\cot \phi_2 = -2\cot\frac{\theta}{2}$

Since $\cot \phi_1 . \cot \phi_2 = -1$, both intersect orthogonally.

11. Find the angle of intersection of the following pair of curves:

$\boldsymbol{r = \sin\theta + \cos\theta}$ **and** $\boldsymbol{r = 2\sin\theta}$

$r = \sin\theta + \cos\theta$ Take log on both sides $\log r = \log\ (\sin\theta + \cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$ $\cot\phi_1 = \tan\left(\frac{\pi}{4} - \theta\right)$ $\cot\phi_1 = \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \theta\right)\right)$ $\cot\phi_1 = \cot\left(\frac{\pi}{4} + \theta\right)$ $\phi_1 = \frac{\pi}{4} + \theta$	$r = 2\sin\theta$ Take log on both sides $\log r = \log(2\sin\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{2\sin\theta}$ $\cot\phi_2 = \cot\theta$ $\phi_2 = \theta$

Therefore, $|\phi_1 - \phi_2| = \frac{\pi}{4}$

12. Find the angle of intersection of the following pair of curves:

$\boldsymbol{r = a(1 - \cos\theta)}$ **and** $\boldsymbol{r = 2a\cos\theta}$

$r = a(1 - \cos\theta)$ Take log on both sides $\log r = \log a(1 - \cos\theta)$ $\log r = \log a + \log\ (1 - \cos\theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1 - \cos\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $\cot\phi_1 = \cot\frac{\theta}{2}$ $\phi_1 = \frac{\theta}{2}$	$r = 2a\cos\theta$ Take log on both sides $\log r = \log\ (2a\cos\theta)$ $\log r = \log 2a + \log\cos\theta$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{\sin\theta}{\cos\theta}$ $= -\tan\theta$ $\cot\phi_2 = \cot\left(\frac{\pi}{2} + \theta\right)$ $\phi_2 = \frac{\pi}{2} + \theta$

By data, $1 - \cos\theta = 2\,cos\theta$

$$1 = 3\cos\theta$$

$$\theta = \cos^{-1}\frac{1}{3}$$

Angle of intersection of the given pair of curves is given by

$$|\phi_1 - \phi_2| = \frac{\pi}{2} + \frac{\theta}{2}$$

$$= \boldsymbol{\frac{\pi}{2} + \frac{1}{2}}\cos^{-1}\left(\frac{1}{3}\right)$$

13. Find the angle of intersection of the following pair of curves:

$\boldsymbol{r = a \log \theta}$ **and** $\boldsymbol{r = \frac{a}{\log \theta}}$ **(May 22)**

$r = a \log \theta$ Take log on both sides $\log r = \log a + \log\ (\log \theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta \log \theta}$ $\cot \phi_1 = \frac{1}{\theta \log \theta}$ $\tan \phi_1 = \theta \log \theta$	$r = \frac{a}{\log \theta}$ Take log on both sides $\log r = \log a - \log\ (\log \theta)$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta \log \theta}$ $\cot \phi_2 = -\frac{1}{\theta \log \theta}$ $\tan \phi_2 = -\theta \log \theta$

By data, $a \log \theta = \frac{a}{\log \theta} \Rightarrow (\log \theta)^2 = 1$

$$\Rightarrow \theta = e$$

$\tan \phi_1 = \theta \log \theta = e \log e = e$

$\phi_1 = \tan^{-1} e$

$\tan \phi_2 = -\theta \log \theta = -e \log e = -e$

$\phi_2 = \tan^{-1}(-e) = -\tan^{-1} e$

Angle of intersection of the given pair of curves is given by

$|\phi_1 - \phi_2| = \tan^{-1} e + \tan^{-1} e = 2 \tan^{-1} e$

14. Find the angle of intersection of the following pair of curves:

$\boldsymbol{r = a \sin 2\theta}$ **and** $\boldsymbol{r = a \cos 2\theta}$

$r = a \sin 2\theta$	$r = a \cos 2\theta$
Take log on both sides	Take log on both sides
$\log r = \log(a \sin 2\theta)$	$\log r = \log(a \cos 2\theta)$
$\log r = \log a + \log\ (\sin 2\theta)$	$\log r = \log a + \log\ (\cos 2\theta)$
Differentiate w. r. to θ	Differentiate w. r. to θ
$\frac{1}{r}\frac{dr}{d\theta} = 2\cot 2\theta$	$\frac{1}{r}\frac{dr}{d\theta} = -2\tan 2\theta$
$\cot \phi_1 = 2\cot 2\theta$	$\cot \phi_2 = -2\tan 2\theta$
$\tan \phi_1 = \frac{1}{2}\tan 2\theta$	$\tan \phi_2 = -\frac{1}{2}\cot 2\theta$

By data, $a \sin 2\theta = a \cos 2\theta \Rightarrow \tan 2\theta = 1$

$$2\theta = \frac{\pi}{4}$$

$$\tan \phi_1 = \frac{1}{2}\tan 2\theta = \frac{1}{2}\tan\frac{\pi}{4} = \frac{1}{2} \Rightarrow \phi_1 = \tan^{-1}\frac{1}{2}$$

$$\tan \phi_2 = -\frac{1}{2}\cot 2\theta = -\frac{1}{2}\cot\frac{\pi}{4} = -\frac{1}{2} \Rightarrow \phi_2 = -\tan^{-1}\frac{1}{2}$$

Angle of intersection of the given pair of curves is given by

$$|\phi_1 - \phi_2| = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2} = 2\tan^{-1}\frac{1}{2}$$

Note: 1 $\tan|\phi_1 - \phi_2| = \left|\frac{\tan \phi_1 - tan\phi_2}{1+\tan \phi_1 . \tan \phi_2}\right| = \left|\frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{2}\cdot\frac{1}{2}}\right| = \frac{4}{3}$

Therefore, $|\phi_1 - \phi_2| = \tan^{-1}\frac{4}{3}$

Note: 2 $2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = \tan^{-1}\frac{4}{3}$

15. Find the angle of intersection of the following pair of curves:

$\boldsymbol{r = \frac{a\theta}{1+\theta}}$ **and** $\boldsymbol{r = \frac{a}{1+\theta^2}}$ **(Feb 23)**

$r = \frac{a\theta}{1+\theta}$	$r = \frac{a}{1+\theta^2}$
Take log on both sides	Take log on both sides
$\log r = \log\dfrac{a\theta}{1+\theta}$	$\log r = \log\dfrac{a}{1+\theta^2}$
$\log r = \log a\theta - \log\ (1+\theta)$	$\log r = \log a - \log\ (1+\theta^2)$
Differentiate w. r. to θ	Differentiate w. r. to θ
$\dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{1}{a\theta}(a) - \dfrac{1}{1+\theta}$	$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\theta}{1+\theta^2}$
$\cot\phi_1 = \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1}{\theta(1+\theta)}$	$\cot\phi_2 = -\frac{2\theta}{1+\theta^2}$
$\tan\phi_1 = \theta(1+\theta)$	$\tan\phi_2 = -\frac{1+\theta^2}{2\theta}$

By data, $\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$, $\theta + \theta^3 = 1 + \theta$, Therefore, $\theta = 1$.

$\tan\phi_1 = 1(1+1) = 2$, $\tan\phi_2 = -\frac{1+1}{2} = -1$

$\tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 + \tan\phi_2}{1 - \tan\phi_1 . \tan\phi_2} = \frac{2+1}{1-2} = -3$

$\phi_1 - \phi_2 = \tan^{-1}(-3) = -\tan^{-1} 3$

Angle of intersection of the given pair of curves is given by

$|\phi_1 - \phi_2| = \tan^{-1} 3$

16. Find the angle of intersection of the following pair of curves:

$\boldsymbol{r^2 \sin 2\theta = 4}$ **and** $\boldsymbol{r^2 = 16 \sin 2\theta}$

$r^2 \sin 2\theta = 4$	$r^2 = 16 \sin 2\theta$
Take log on both sides	Take log on both sides
$\log(r^2 \sin 2\theta) = \log 4$	$\log(r^2) = \log(16 \sin 2\theta)$
$\log r^2 + \log \sin 2\theta = \log 4$	$\log r^2 = \log 16 + \log \sin 2\theta$
Differentiate w. r. to θ	Differentiate w. r. to θ
$\frac{2}{r}\frac{dr}{d\theta} + \frac{2\cos 2\theta}{\sin 2\theta} = 0$	$\frac{2}{r}\frac{dr}{d\theta} = 0 + \frac{2\cos 2\theta}{\sin 2\theta}$
$\cot \phi_1 = -\cot 2\theta$	$\cot \phi_2 = \cot 2\theta$
$\cot \phi_1 = \cot(-2\theta)$	$\phi_2 = 2\theta$
$\phi_1 = -2\theta$	

By data, $16 \sin^2 2\theta = 4$, $\sin 2\theta = \frac{1}{2}$, $2\theta = \frac{\pi}{6}$

Therefore, $|\phi_1 - \phi_2| = 4\theta = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$

1.2 Pedal equations

Introduction:

If p is the perpendicular distance from the pole to the tangent of the polar curve, then the equation of the curve in terms of p and r is called pedal equation or $p - r$ equation.

$p - r$ equation is $p = r\,sin\phi$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$.

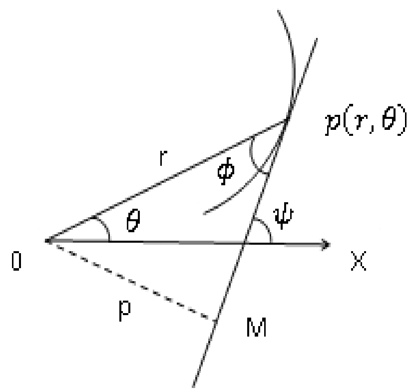
1. **With usual notations, prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ and hence deduce that $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$, where $u = \frac{1}{r}$.**

   $P(r,\theta)$ – Any point on the polar curve $r = f(\theta)$

   r – Radius vector

   p – perpendicular distance from the origin

   By diagram,



By diagram, $\frac{p}{r} = \sin\phi$ $p = r\sin\phi$ $\frac{1}{p^2} = \frac{1}{r^2\sin^2\theta}$ $\frac{1}{p^2} = \frac{1}{r^2}\,cosec^2\phi$ $= \frac{1}{r^2}(1+\cot^2\phi)$ $= \frac{1}{r^2}\left(1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)$	Therefore, $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ Put $u = \frac{1}{r}\ ,\frac{du}{d\theta} = -\frac{1}{r^2}\left(\frac{dr}{d\theta}\right).$ We ge $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

2. **Find the pedal equation of the curve $r^2 = a^2 \sin^2 \theta$**

To find: ϕ	**To find:** Pedal equation
$r^2 = a^2 \sin^2 \theta$	$p = r \sin \phi$
Take log on both sides,	$p = r \sin \theta$
$2 \log r = 2 \log a + 2 \log \sin \theta$	$p^2 = r^2 \sin^2 \theta$
$\log r = \log a + \log \sin \theta$	$p^2 = r^2 \left(\frac{r^2}{a^2}\right)$
$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos \theta}{\sin \theta}$	$a^2 p^2 = r^4$
$\cot \phi = \cot \theta$	$ap = r^2$
$\phi = \theta$	

3. **Find the pedal equation of the curve $r = 2(1 + \cos \theta)$**

To find: ϕ	**To find:** Pedal equation
$r = 2(1 + \cos \theta)$	$p = r \sin \phi$
Take log on both sides,	$p = r \sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$
$\log r = \log 2 + \log(1 + \cos \theta)$	$p^2 = r^2 \cos^2 \frac{\theta}{2}$
$\log r = \log 2 + \log(1 + \cos \theta)$	$p^2 = r^2 \left(\frac{r}{4}\right)$
$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin \theta}{1+\cos \theta} = \frac{-2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2 \cos^2\frac{\theta}{2}}$	$4p^2 = r^3$
$\cot \phi = -\tan \frac{\theta}{2} = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$	
$\phi = \frac{\pi}{2} + \frac{\theta}{2}$	

4. **Find the pedal equation of the curve $\boldsymbol{r^n = a^n \cos n\theta}$** **(May 22)**

To find: ϕ	**To find:** Pedal equation
$r^n = a^n \cos n\theta$ Take log on both sides, $n \log r = n \log a + \log \cos n\theta$ Differentiate w. r. to θ $\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{-n \sin n\theta}{\cos n\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$ $\cot \phi = \cot\left(\frac{\pi}{2} + n\theta\right)$ $\phi = \frac{\pi}{2} + n\theta$	$p = r \sin \phi$ $p = r \sin\left(\frac{\pi}{2} + n\theta\right)$ $p = r \cos n\theta$ $p = r\left(\frac{r^n}{a^n}\right)$ $a^n p = r^{n+1}$

5. **Find the pedal equation of the curve $\boldsymbol{r^m \cos m\theta = a^m}$**

To find: ϕ	**To find:** Pedal equation
$r^m \cos m\theta = a^m$ Take log on both sides, $m \log r + \log \cos m\theta = m \log a$ Differentiate w. r. to θ $\frac{m}{r}\frac{dr}{d\theta} + \frac{-m \sin m\theta}{\cos m\theta} = 0$ $\frac{1}{r}\frac{dr}{d\theta} = \tan m\theta$ $\cot \phi = \cot\left(\frac{\pi}{2} - m\theta\right)$ $\phi = \frac{\pi}{2} - m\theta$	$p = r \sin \phi$ $p = r \sin\left(\frac{\pi}{2} - m\theta\right)$ $p = r \cos m\theta$ $p = r\left(\frac{a^m}{r^m}\right)$ $r^{m-1} p = a^m$

6. **Find the pedal equation of the curve** $\boldsymbol{r^m = a^m(\cos m\theta + \sin m\theta)}$

To find: ϕ	**To find:** Pedal equation
$r^m = a^m(\cos m\theta + \sin m\theta)$ Take log on both sides, $m \log r = \log a^m + \log(\cos m\theta + \sin m\theta)$ Differentiate w. r. to θ $\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{-m\sin m\theta + m\cos m\theta}{\cos m\theta + \sin m\theta}$ $\frac{1}{r}\frac{dr}{d\theta} = \tan\left(\frac{\pi}{4} - m\theta\right)$ $\cot\phi = \cot\left(\frac{\pi}{2} - \frac{\pi}{4} + m\theta\right)$ $\phi = \frac{\pi}{4} + m\theta$	$p = r\sin\phi$ $p = r\sin\left(\frac{\pi}{4} + m\theta\right)$ $p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta)$ $p = \frac{r}{\sqrt{2}}\left(\frac{r^m}{a^m}\right)$ $\sqrt{2}a^m p = r^{m+1}$

7. **Find the Pedal equation of the curve** $\boldsymbol{r = ae^{m\theta}}$

To find: ϕ	**To find:** Pedal equation
$r = ae^{m\theta}$ Take log on both sides, $\log r = \log a + m\theta$ Differentiate w. r. to θ $\frac{1}{r}\frac{dr}{d\theta} = 0 + m$ $\cot\phi = m$	$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ $\frac{1}{p^2} = \frac{1}{r^2}(1 + m^2)$ $r^2 = p^2(1 + m^2)$

8. **Find the Pedal equation of the curve $\frac{l}{r} = 1 + e\cos\theta$**

To find: ϕ	**To find:** Pedal equation
$\frac{l}{r} = 1 + e\cos\theta$	$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$
Take log on both sides,	$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right)$
$\log\frac{l}{r} = \log(1 + e\cos\theta)$	$r^2(1 + e\cos\theta)^2 = p^2(1 + e^2 + 2e\cos\theta)$
$\log l - \log r = \log(1 + e\cos\theta)$	$l^2 = p^2\left(1 + e^2 + 2\left(\frac{l}{r} - 1\right)\right)$
Differentiate w. r. to θ	$l^2 = p^2\left(e^2 + \frac{2l}{r} - 1\right)$
$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$	
$\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$	

9. **Find the Pedal equation of the curve $\frac{2a}{r} = 1 - \cos\theta$**

To find: ϕ	**To find:** Pedal equation
$\frac{2a}{r} = 1 - \cos\theta$	$p = r\sin\phi$
Take log on both sides,	$p = r\sin\left(\pi - \frac{\theta}{2}\right) = r\sin\frac{\theta}{2}$
$\log 2a - \log r = \log(1 - \cos\theta)$	$\frac{p^2}{r^2} = \sin^2\frac{\theta}{2} = \frac{a}{r}$
Differentiate w. r. to θ	$p^2 = ar$
$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$	
$\cot\phi = -\cot\frac{\theta}{2} = \cot\left(\pi - \frac{\theta}{2}\right)$	
$\phi = \pi - \frac{\theta}{2}$	

1.3 Radius of curvature

Introduction:

- **Derivative of an arc length:**

Cartesian form	**Polar form**
$\frac{ds}{dx} = \sqrt{\mathbf{1} + \left(\frac{\boldsymbol{dy}}{\boldsymbol{dx}}\right)^{\mathbf{2}}}$	$\frac{ds}{d\theta} = \sqrt{\boldsymbol{r^2} + \boldsymbol{r_1^2}}$

- The rate of change of bending of a curve at p is called the curvature at p.

  It is denoted by $\mathrm{k} = \frac{\mathrm{d}\chi}{\mathrm{ds}}$.
- The reciprocal of the curvature of a curve at p is called the radius of curvature at p.

  It is defined by $\rho = \frac{ds}{d\psi}$.

Cartesian form	**Polar form**
$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$	$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-r\,r_2}$
Parametric form	**Pedal form**
$\rho = \frac{\left(\dot{x}^2+\dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$	$\rho = r\frac{dr}{dp}$

Note:

In Cartesian form, $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

In Polar form, $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

In Parametric form, $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$

1. **Derive radius of curvature for the Cartesian curve $\boldsymbol{y = f(x)}$.** **(May 22)**

$$tan\psi = \frac{dy}{dx}$$

$$tan\psi = y_1$$

$$\psi = tan^{-1}(y_1)$$

Differentiating w. r. to x,

$$\frac{d\psi}{dx} = \frac{1}{1 + y_1^2} . y_2$$

Therefore, radius of curvature is given by

$$\rho = \frac{ds}{d\psi}$$

$$= \frac{ds}{dx} . \frac{dx}{d\psi}$$

$$= \sqrt{1 + y_1^2} . \frac{1 + y_1^2}{y_2}$$

$$= \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

$Therefore,$

$$\boxed{\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

2. Derive radius of curvature for the parametric curve $\boldsymbol{x = f(t), y = g(t)}$.

$y_1 = \dfrac{dy}{dx}$ $= \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $= \dfrac{\dot{y}}{\dot{x}}$	$y_2 = \dfrac{d}{dx}\left(\dfrac{\dot{y}}{\dot{x}}\right)$ $= \dfrac{d}{dt}\left(\dfrac{\dot{y}}{\dot{x}}\right)\left(\dfrac{dt}{dx}\right)$ $= \left(\dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}\right)\left(\dfrac{1}{\dot{x}}\right)$

The radius of curvature is given by

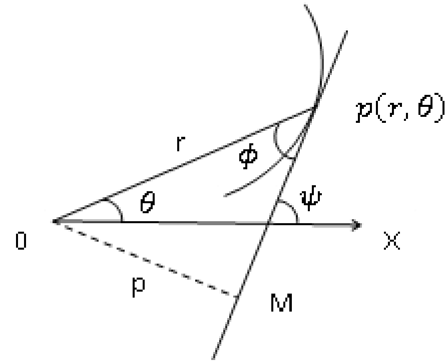
$$\rho = \frac{(1 + y_1^2)^{\frac{3}{2}}}{y_2}$$

$$= \frac{\left(1 + \frac{\dot{y}^2}{\dot{x}^2}\right)^{\frac{3}{2}}}{\left(\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}\right)}$$

Therefore,

$$\boxed{\rho = \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}}$$

3. **Derive radius of curvature for the polar curve $\boldsymbol{r = f(\theta)}$.**



By diagram, $\chi = \theta + \phi$

$$\frac{d\chi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta}.\frac{d\theta}{ds}$$

$$= \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$= \frac{1 + \frac{d\phi}{d\theta}}{\frac{ds}{d\theta}} \qquad \text{----------- (1)}$$

But

$$\tan\phi = r\frac{d\theta}{dr}$$

$$\phi = \tan^{-1}\left(\frac{r}{r_1}\right)$$

Differentiate with respect to θ,

$$\frac{d\phi}{d\theta} = \frac{1}{1 + \left(\frac{r}{r_1}\right)^2} . \frac{r_1.r_1 - rr_2}{r_1^2}$$

$$= \frac{r_1^2 - rr_2}{r^2 + r_1^2}$$

$$1 + \frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - rr_2}{r^2 + r_1^2}$$

$$= \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + r_1^2}$$

The radius of curvature is given by

$$\frac{1}{\rho} = \frac{d\chi}{ds}$$

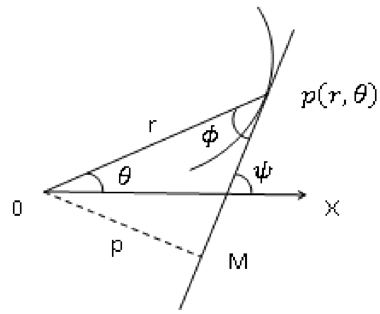
$$= \frac{1 + \frac{d\phi}{d\theta}}{\frac{ds}{d\theta}}$$

$$= \frac{1}{\sqrt{r^2 + r_1^2}} . \frac{r^2 + 2r_1^2 - rr_2}{r^2 + r_1^2}$$

Therefore,

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2}}$$

4. Derive radius of curvature for the pedal curve $p = f(r)$.



By diagram, $\chi = \theta + \phi$

Also,

$$p = r\sin\phi$$

Differentiate with respect to r,

$$\frac{dp}{dr} = \sin\phi + r\cos\phi\frac{d\phi}{dr}$$

$$= r\frac{d\theta}{ds} + r\frac{dr}{ds}\cdot\frac{d\phi}{dr}$$

$$= r\left(\frac{d\theta}{ds} + \frac{d\phi}{ds}\right)$$

$$= r\left(\frac{d}{ds}(\theta + \phi)\right)$$

$$= r\left(\frac{d\chi}{ds}\right)$$

$$\frac{ds}{d\chi} = r\frac{dr}{dp}$$

Therefore,

$$\boxed{\rho = r\frac{dr}{dp}}$$

5. Find the radius of curvature for $x^4 + y^4 = 2$ at $(1, 1)$ (July '16)

$$x^4 + y^4 = 2$$

Differentiate w. r. to x,

$$4x^3 + 4y^3 y' = 0$$

$$x^3 + y^3 y' = 0 \text{ -------- } (1)$$

Differentiate again w. r. to x,

$$3x^2 + 3y^2 (y')^2 + y^3 y" = 0 \text{ -------- } (2)$$

At $(1, 1)$,

$$(1) \Rightarrow 1 + y' = 0 \Rightarrow y' = -1$$

$$(2) \Rightarrow 3 + 3 - y" = 0 \Rightarrow y" = -6.$$

The radius of curvature is given by,

$$\rho = \frac{(1 + y'^2)^{3/2}}{y"}$$

$$= \frac{(1+1)^{\frac{3}{2}}}{-6}$$

$$= \frac{2\sqrt{2}}{-6}$$

$$= -\frac{\sqrt{2}}{3}$$

Here, negative sign indicates the direction of bending of the curve.
By ignoring sign,

$$\boldsymbol{\rho = \frac{\sqrt{2}}{3}}$$

6. Find the radius of curvature of the Folium $x^3 + y^3 = 3axy$ at the point $(3a/2,\ 3a/2)$.

$$x^3 + y^3 = 3axy$$

Differentiate with respect to x

$$x^2 + y^2y' = a(xy' + y) \text{ ------ } (1)$$

Differentiate again with respect to x

$$2x + 2y(y')^2 + y^2y" = a\,(xy" + 2y') \text{ ------- } (2)$$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$(1) \Rightarrow \frac{9a^2}{4} + \frac{9a^2}{4}y' = a\left(\frac{3a}{2}y' + \frac{3a}{2}\right)$$

$$\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)\text{y}' = \frac{3a^2}{2} - \frac{9a^2}{4}$$

$$\boldsymbol{y' = -1}$$

$$(2) \Rightarrow 2\left(\frac{3a}{2}\right) + 2\left(\frac{3a}{2}\right) + \left(\frac{3a}{2}\right)^2 y" = a\left(\frac{3a}{2}y" - 2\right)$$

$$3a + 3a + \frac{9a^2}{4}y'' - \frac{3a^2}{2}y'' = -2a$$

$$\frac{3a^2}{4}y'' = -8a$$

$$\boldsymbol{y'' = \frac{-32}{3a}}$$

The radius of curvature is given by,

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''}$$

$$= \left(\frac{2^{\frac{3}{2}}}{-32}\right)3a$$

$$= -\frac{3a\sqrt{2}}{16}$$

Here, negative sign indicates the direction of bending of the curve.

By ignoring sign,

$$\boldsymbol{\rho = \frac{3a\sqrt{2}}{16}}$$

7. **Find the radius of curvature of the catenary $y = c\,cosh\frac{x}{c}$ at $(c, 0)$.**

$$y = c\,cosh\frac{x}{c}$$

Differentiate with respect to x,

$$y' = \sinh\frac{x}{c} \quad \text{-------- (1)}$$

Differentiate again with respect to x,

$$y'' = \frac{1}{c}\cosh\frac{x}{c} \quad \text{------- (2)}$$

$At\ (c, 0)$,

$$(1) \Longrightarrow y' = \sinh 1$$

$$(2) \Longrightarrow y'' = \frac{1}{c}\cosh 1$$

The radius of curvature is given by,

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''}$$

$$= \frac{(1 + \sinh^2 1)^{3/2}}{\frac{1}{c}\cosh 1}$$

$$= c\cosh^2 1$$

$$= \frac{y^2}{c}$$

Note:

$\sinh x = \frac{e^x - e^{-x}}{2}$	$\frac{d}{dx}(\sinh x) = \cosh x$	$\cosh^2 x - \sinh^2 x = 1$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\frac{d}{dx}(\cosh x) = \sinh x$	$\cosh^2 x = 1 + \sinh^2 x$

8. Find the radius of curvature of the parabola $\boldsymbol{y^2 = 4ax}$ at $\boldsymbol{(at^2, 2at)}$.

$$y^2 = 4ax$$

Differentiate w. r. to x,

$$yy' = 2a \text{ ---------- } (1)$$

Differentiate again w. r. to x,

$$y'^2 + yy'' = 0 \text{ ---------- } (2)$$

At $(at^2, 2at)$,

$$(1) \Longrightarrow 2aty' = 2a$$

$$\Longrightarrow y' = \frac{1}{t}$$

$$(2) \Longrightarrow \frac{1}{t^2} + 2aty'' = 0$$

$$\Longrightarrow y'' = -\frac{1}{2at^3}$$

The radius of curvature is given by,

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''}$$

$$= \frac{\left(1 + \frac{1}{t^2}\right)^{3/2}}{\left(-\frac{1}{2at^3}\right)}$$

$$= -2a(1 + t^2)^{\frac{3}{2}}$$

9. Find the radius of curvature of the curve $y = x^3(x - a)$ at $(a, 0)$.

$$y = x^3(x - a)$$

Differentiate w. r. to x,

$$y' = x^3 - (x - a)3x^2 \text{---------- (1)}$$

Differentiate again w. r. to x,

$$y'' = 6x^2 + 6x(x - a) \text{ -------- (2)}$$

$at\ (a, 0)$,

$$y' = a^3 - 0 = a^3$$

$$y'' = 6a^2 - 0 = 6a^2$$

The radius of curvature is given by,

$$\rho = \frac{(1 + y'^2)^{3/2}}{y''}$$

$$= \frac{(1 + a^6)^{\frac{3}{2}}}{6a^2}$$

10. Find the radius of curvature for $y^2 = \frac{a^2(a-x)}{x}$ where the curve meets the x-axis.

$$y^2 = \frac{a^2(a-x)}{x}$$

$$xy^2 = a^3 - a^2x$$

Differentiate w.r.to x ,

$$2xyy' + y^2 = -a^2$$

$$2xyy' = -\frac{a^3}{x}$$

$$\frac{dy}{dx} = -\frac{a^3}{2yx^2}$$

does not exist at $y = 0$.

$$\therefore \frac{dx}{dy} = -\frac{2x^2y}{a^3} \qquad \text{---- (1)}$$

Differentiate w.r.to y,

$$\frac{d^2x}{dy^2} = -\frac{2}{a^3}\left(x^2 + 2xy\frac{dx}{dy}\right) \qquad \text{------ (2)}$$

If $y = 0$ then $x = a$.

$$(1) \Longrightarrow \frac{dx}{dy} = 0$$

$$(2) \Longrightarrow \frac{d^2x}{dy^2} = -\frac{2}{a}$$

Therefore, radius of curvature is given by

$$\rho = \frac{\left(1 + \left(\frac{dx}{dy}\right)\right)^{3/2}}{\frac{d^2x}{dy^2}}$$

$$= -\frac{a}{2}$$

By ignoring the sign,

$$\boldsymbol{\rho = \frac{a}{2}}$$

.

11. Show that the radius of curvature at any point of the cycloid

$$\boldsymbol{x = a(\theta + \sin\theta),\ \ y = a(1 - \cos\theta)\ is\ 4a\cos\left(\frac{\theta}{2}\right)}$$

$$\frac{dx}{d\theta} = a(1 + \cos\theta),\ \frac{dy}{d\theta} = a\sin\theta$$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ $= \frac{a\sin\theta}{a(1+\cos\theta)}$ $= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$ $= \tan\frac{\theta}{2}$	$\frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\frac{\theta}{2}\times\frac{d\theta}{dx}$ $= \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{a(1+\cos\theta)}$ $= \frac{\frac{1}{2}\sec^2\frac{\theta}{2}}{a\left(2\cos^2\frac{\theta}{2}\right)}$ $= \frac{1}{4a}\sec^4\frac{\theta}{2}$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y'^2)^{\frac{3}{2}}}{y''}$$

$$= \frac{\sec^3\frac{\theta}{2}}{\frac{1}{4a}\sec^4\frac{\theta}{2}}$$

$$= 4a\cos\frac{\theta}{2}$$

12. Show that the radius of curvature at any point of the cycloid

$\boldsymbol{x = a(\theta - \sin\theta), \;\; y = a(1 - \cos\theta)}$ ***is*** $\boldsymbol{4a\sin\left(\frac{\theta}{2}\right)}$

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta$$

$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)}$ $= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $= \cot\frac{\theta}{2}$	$\frac{d^2y}{dx^2} = -\frac{1}{2}cosec^2\left(\frac{\theta}{2}\right) \times \frac{d\theta}{dx}$ $= -\frac{1}{2}cosec^2\left(\frac{\theta}{2}\right) \times \frac{1}{a(1-\cos\theta)}h$ $= -\frac{1}{4a}cosec^4\left(\frac{\theta}{2}\right)$

Therefore, radius of curvature is given by,

$$\rho = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1 + \cot^2\frac{\theta}{2}\right)^{\frac{3}{2}}}{-\frac{1}{4a}cosec^4\left(\frac{\theta}{2}\right)}$$

$$= -4a\sin\frac{\theta}{2}$$

By ignoring sign,

$$\boldsymbol{\rho = 4a\sin\frac{\theta}{2}}$$

13. Show that the radius of curvature at any point of the cycloid $\boldsymbol{x = a\,cos^3 t, y = a\,sin^3 t}$ **at** $\boldsymbol{t = \frac{\pi}{4}}$**.**

$\frac{dx}{d\theta} = -3a\cos^2 t \sin t\,,\ \frac{dy}{d\theta} = 3a\sin^2 t\cos t$

$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$	$y'' = \frac{d^2y}{dx^2} = -\sec^2 t \times \frac{dt}{dx} = \frac{\sec^2 t}{3a\cos^2 t\sin t} = \frac{1}{3a}\frac{\sec^4 t}{\sin t}$

At $t = \frac{\pi}{4}$,

$y' = -\tan\frac{\pi}{4} = -1$

$y'' = \frac{1}{3a}\frac{\sec^4\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1}{3a}4\sqrt{2}$

The radius of curvature is given by,

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

$$= \frac{(1+1)^{\frac{3}{2}}}{\frac{1}{3a}4\sqrt{2}}$$

$$= \frac{3a}{2}$$

14. Find the radius of curvature of the curve
$\boldsymbol{x = a(\cos t + t \sin t), y = a(\sin t - t \cos t)}$ **at any point** $\boldsymbol{t}$.

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t,$$

$$\frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $= \frac{at\sin t}{at\cos t}$ $= \tan t$	$\frac{d^2y}{dx^2} = \sec^2 t \times \frac{dt}{dx}$ $= \frac{\sec^2 t}{at\cos t}$ $= \frac{1}{at}\sec^3 t$

Therefore, radius of curvature is given by

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

$$= \frac{(1+\tan^2 t)^{\frac{3}{2}}}{\frac{1}{3a}4\sqrt{2}}$$

$$= \frac{\sec^3 t}{\frac{1}{at}\sec^3 t}$$

$$= at$$

15. For the cardioid $r = a(1-\cos\theta)$, show that $\frac{\rho^2}{r}$ is a constant.

Step 1: Find ϕ	**Step 3: Find radius of curvature.**
$r = a(1-\cos\theta)$ Take log on both sides, $\log r = \log a + \log(1-\cos\theta)$ Differentiate w. r. to x, $\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$ $\cot\phi = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$ $\cot\phi = \cot\frac{\theta}{2}$ $\phi = \frac{\theta}{2}$ **Step 2: Find $p-r$ equation.** $p = r\sin\phi = r\sin\frac{\theta}{2}$ $p^2 = r^2\sin^2\frac{\theta}{2}$ $p^2 = \frac{r^3}{2a}$ $r^3 = 2ap^2$	$r^3 = 2ap^2$ Differentiate w.r.to p, $3r^2\frac{dr}{dp} = 4ap$ $r\frac{dr}{dp} = \frac{4ap}{3r}$ $\rho = \frac{4ap}{3r}$ $\rho^2 = \frac{16a^2p^2}{9r^2}$ $= \frac{8ar}{9}$ $\frac{\rho^2}{r} = \frac{8a}{9}$ = Constant Therefore, $\frac{\rho^2}{r}$ is a constant.

16. If $\rho_1,\ \rho_2$ be the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + cos\theta)$ which passes through the pole, show that ${\rho_1}^2 + {\rho_2}^2 = \frac{16a^2}{9}$ May 22)

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Step 1: Find ϕ

$$r = a(1 + cos\theta)$$

$$\log r = \log a(1 + cos\theta)$$

$$\log r = \log a + \log(1 + \cos\theta)$$

Differentiate w.r.to r,

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{1 + \cos\theta}$$

$$\cot\phi = -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$= -\tan\frac{\theta}{2}$$

$$= \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

Therefore, $\phi = \frac{\pi}{2} + \frac{\theta}{2}$

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Step 2: Find $p - r$ equation.

$$p = r\sin\phi$$

$$= r\sin\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$p^2 = r^2\cos^2\frac{\theta}{2}$$

$$= \frac{r^2}{2}(1 + \cos\theta)$$

$$= \left(\frac{r^3}{2}\right)\left(\frac{r}{a}\right)$$

$$r^3 = 2ap^2$$

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Step 3: Find ρ

$$r^3 = 2ap^2$$

Differentiate w.r.to p

$$3r^2\frac{dr}{dp} = 4ap$$

$$r\frac{dr}{dp} = \frac{4ap}{3r}$$

$$\rho = \frac{4ap}{3r}$$

$$\rho^2 = \frac{16a^2p^2}{9r^2}$$

$$= \frac{8a\,(2ap^2)}{9r^2}$$

$$= \frac{8ar}{9}$$

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Step 4: To prove $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$

At $(r, \theta), r = a(1 + \cos\theta)$

$$\rho_1^2 = \frac{8ar}{9} = \frac{8a^2}{9}(1 + \cos\theta)$$

At $(r, \pi + \theta),\ r = a(1 + \cos(\pi + \theta))$

$$\rho_2^2 = \frac{8ar}{9} = \frac{8a^2}{9}(1 - \cos\theta)$$

Adding both.

$$\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$$

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