Mathematics II for Computer Science and Engineering stream

(Subject code: BMATS201)

Module 2: Vector Space and linear transformations

Module – 2: Vector Space and Linear Transformations	**(8 hours)**
Definition and examples of vector space, subspace, linear span, linear independent and dependent sets, Basis and dimension. Definition and examples of Linear transformations, Matrix of a linear transformation, rank-nullity theorem (without proof). Inner product space and orthogonality.	

2.1 Vector spaces

Definition:

A non-empty set of vectors V over the scalar field F under addition and scalar multiplication is said to be a vector space if it satisfies the following properties:

If $u, v, w \in V$ and $\alpha, \beta \in F$ then

Addition

(i) Closure: $u + v \in V$

(ii) Associative: $(u + v) + w = u + (v + w)$

(iii) Identity: There is $0 \in V$ such that $0 + u = u = u + 0$

(iv) Inverse: There is $-u \in V$ such that $u + (-u) = (-u) + u = 0$

(v) Commutative: $u + v = v + u$

Scalar multiplication

(vi) Closure: $\alpha u \in V$

(vii) Distributive 1: $\alpha (u + v) = \alpha u + \alpha v$

(viii) Distributive 2: $(\alpha + \beta) u = \alpha u + \beta u$

(ix) Associative: $\alpha(\beta u) = (\alpha\beta)u$

(x) Identity: $1u = u$

Problems:

1. **Let $V = \{(x, y) \in R^2 \,|\, x^2 + y^2 \leq 1\}$ be the unit disc in R^2. Is V a vector space?**

   $u = (1, 0) \in V$ and $\alpha = 2 \in F$. But $\alpha u = (2, 0) \notin V$
   The circle is not closed under scalar multiplication.
   Therefore, the unit disc is not a vector space.

2. **Let $V = \{(x, y) \in R^2 \mid y = x^2\}$ be the graph of the quadratic function. Is V a vector space?**

$u = (1, 1) \in V$ and $v = (2, 4) \in V$. But $u + v = (3, 5) \notin V$.
Therefore, V is not closed under addition.
Therefore, the graph of the quadratic function is not a vector space.

3. **Let $V = \{(x, y) \in R^2 \mid y = 2x\}$ be the graph of the function $f(x) = 2x$. Is V a vector space?**

Let $u = (a, 2a), v = (b, 2b), w = (c, 2c)$ be points in V and $\alpha, \beta \in F$.

Addition

(i) Closure: $u + v = (a + b, 2a + 2b)$
$= \left(a + b, 2(a + b)\right) \in V$

(ii) Associative: $(u + v) + w = (a + b, 2a + 2b) + (c, 2c)$
$= (a + b + c, 2a + 2b + 2c)$
$= \left(a + (b + c), 2a + (2b + 2c)\right)$
$= (a, 2a) + (b + c, 2b + 2c)$
$= u + (v + w)$

(iii) Identity: There exists $0 = (0, 0) \in V$ such that
$u + 0 = (a, 2a) + (0, 0) = (a, 2a) = u$

(iv) Inverse: There exists $-u = (-a, -2a) \in V$ such that $u + (-u) = 0$

(v) Commutative: $u + v = (a + b, 2a + 2b) = (b + a, 2b + 2a) = v + u$

Scalar multiplication

(vi) Closure:
$\alpha u = \alpha (a, 2a) = (\alpha a, \alpha 2a) = (\alpha a, 2(\alpha a)) \in V$

(vii) Distributive 1:
$\alpha (u + v) = \alpha(a + b, 2a + 2b)$
$= (\alpha(a + b), \alpha(2a + 2b))$
$= (\alpha a + \alpha b, 2\alpha a + 2\alpha b)$
$= (\alpha a, 2\alpha a) + (\alpha b, 2\alpha b)$
$= \alpha(a, 2a) + \alpha(b, 2b)$
$= \alpha u + \alpha v$

(viii) Distributive 1:
$(\alpha + \beta)u = (\alpha + \beta)(a, 2a)$
$= \left((\alpha + \beta)a, \alpha + \beta)2a\right)$
$= (\alpha a + \beta a, 2\alpha a + 2\beta a)$
$= (\alpha a, 2\alpha a) + (\beta a, 2\beta a)$
$= \alpha(a, 2a) + \beta(a, 2a)$
$= \alpha u + \beta u$

(ix) Associative:

$\alpha(\beta u) = \alpha\left(\beta(a, 2a)\right)$

$= \alpha(\beta a, 2\beta a)$

$= (\alpha\beta a,\ 2\alpha\beta a)$

$= \alpha\beta(a, 2a)$

$= (\alpha\beta)u$

(x) Identity:

$1u = 1(a, 2a) = (a, 2a) = u$

All the axioms of vector space are satisfied.

Therefore, the graph of the function f(x) = 2x is a vector space.

4. Let $V = P_n[t] = \{a_0 + a_1 t + a_n t^n \mid a_0, a_1, \ldots, a_n \in R\}$ be the set of all polynomials in the variable t and of degree $\leq n$. Is V a vector space?

Let $u = u_0 + u_1 t + \cdots + u_n t^n, v = v_0 + v_1 t + \cdots + v_n t^n$ and $w = w_{0_1} t + \cdots + w_n t^n$ be polynomials in V and $\alpha, \beta \in F$

Addition

(i) Closure:

$$u + v = (u_0 + u_1 t + \cdots + u_n t^n) + (v_0 + v_1 t + \cdots + v_n t^n)$$
$$= (u_0 + v_0) + (u_1 + v_1)t + \cdots + (u_n + v_n)t^n \in P_n[t]$$

(ii) Associative:

$$(u + v) + w = [(u_0 + v_0) + (u_1 + v_1)t + \cdots + (u_n + v_n)t^n] + [w_0 + w_1 t + \cdots + w_n t^n]$$
$$= (u_0 + v_0 + w_0) + (u_1 + v_1 + w_1)\,t + \cdots + (u_n + v_n + w_n)t^n$$
$$= [u_0 + u_1 t + \cdots + u_n t^n] + \begin{bmatrix}(v_0 + w_0) + (v_1 + w_1)t + \cdots \\ + (v_n + w_n)t^n\end{bmatrix}$$
$$= u + (v + w).$$

(iii) Identity:

There exists $0 \in P_n(t)$ such that

$$u + 0 = u_0 + u_1 t + \cdots + u_n t^n + 0 = u$$

(iv) Inverse:

There exists $-u = -u_0 - u_1 t - \cdots - u_n t^n \in V$ such that

$$u + (-u) = (u_0 + u_1 t + \cdots + u_n t^n) + (-u_0 - u_1 t - \cdots - u_n t^n) = 0.$$

(v) Commutative:

$$u + v = (u_0 + v_0) + (u_1 + v_1)t + \cdots + (u_n + v_n)tn$$
$$= (v_0 + u_0) + (v_1 + u_1)\,t + \cdots + (v_n + u_n)\,tn$$
$$= v + u$$

Scalar multiplication

(vi) Closure:

$$\alpha u = \alpha(u_0 + u_1 t + \cdots + u_n t^n)$$
$$= (\alpha u_0) + (\alpha u_1)t + \cdots + (\alpha u_n)t^n \in P_n[t]$$

(vii) Distributive 1:

$$\alpha\,(u + v) = \alpha[(u_0 + v_0) + (u_1 + v_1)\,t + \cdots + (u_n + v_n)t^n]$$
$$= \alpha(u_0 + v_0) + \alpha(u1 + v1)\,t + \cdots + \alpha(un + vn)t^n$$
$$= (\alpha u_0 + \alpha v_0) + (\alpha u_1 + \alpha v_1)\,t + \cdots + (\alpha u_n + \alpha v_n)t^n$$
$$= (\alpha u_0 + \alpha u_1\,t + \cdots + \alpha u_n\,t^n) + (\alpha v_0 + \alpha v_1\,t + \cdots + \alpha v_n t^n)$$

$$= \alpha(u_0 + u_1 t + \cdots + u_n t^n) + \alpha(v_0 + v_1 t + \cdots + v_n t^n)$$
$$= \alpha u + \alpha v$$

(viii) Distributive 2:

$$(\alpha + \beta)\, u = (\alpha + \beta)\,(u_0 + u_1 t + \cdots + u_n t^n)$$
$$= (\alpha + \beta)u_0 + (\alpha + \beta)u_1 t + \cdots + (\alpha + \beta)u_n t^n$$
$$= (\alpha u_0 + \alpha u_1\, t + \cdots + \alpha u_n\, t^n) + (\beta u_0 + \beta u_1\, t + \cdots + \beta u_n\, t^n)$$
$$= \alpha\,(u_0 + u_{1_n} t^n) + \beta\,(u_0 + u_1 t + \cdots + u_n t^n)$$
$$= \alpha u + \beta u$$

(ix) Associative:

$$\alpha(\beta u) = \alpha\,(\beta(u_0 + u_1 t + \cdots + u_n t^n))$$
$$= \alpha\,(\beta u_0 + \beta u_1 t + \cdots + \beta u_n\, t^n)$$
$$= \alpha\beta u_0 + \alpha\beta u_1 t + \cdots + \alpha\beta u_n t^n$$
$$= \alpha\beta(u_0 + u_1 t + \cdots + u_n t^n)$$
$$= (\alpha\beta)u$$

(x) Identity:

$$1u = 1(u_0 + u_1 t + \cdots + u_n t^n)$$
$$= 1.u_0 + 1.u_1 t + \cdots + 1.u_n t^n$$
$$= u_0 + u_1 t + \cdots + u_n t^n$$
$$= u$$

All the axioms of vector space are satisfied.

Therefore, $V = P_n[t]$ is a vector space.

5. Show that the set of all 2 × 2 matrices is a vector space over the field of reals R under usual addition and multiplication.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}, = \begin{pmatrix} i & j \\ k & l \end{pmatrix} \in V$ and $\alpha, \beta \in F$

Addition:

(i) Closed

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \in V$$

(ii) Associative

$$(A + B) + C = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] + \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$
$$= \begin{pmatrix} (a+e)+i & (b+f)+j \\ (c+g)+k & (d+h)+l \end{pmatrix}$$
$$= \begin{pmatrix} a+(e+i) & b+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right]$$
$$= A + (B + C)$$

(iii) Identity

There exists $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in V$ such that

$$0A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

(iv) Inverse

There exists $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in V$ such that

$$A + (-A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = 0$$

(v) Commutative

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a & f+b \\ g+c & h+d \end{pmatrix} = B + A$$

Scalar multiplication

(vi) Closed:

$$\alpha A = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \in V$$

(vii) Distributive 1:

$$\alpha(A + B) = \alpha \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] = \alpha \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a + \alpha e & \alpha b + \alpha f \\ \alpha c + \alpha g & \alpha d + \alpha h \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \alpha e & \alpha f \\ \alpha g & \alpha h \end{pmatrix} = \alpha A + \alpha B$$

(viii) Distributive 2:

$$(\alpha + \beta)A = (\alpha + \beta) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a + \beta a & \alpha b + \beta b \\ \alpha c + \beta c & \alpha d + \beta d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix}$$

$$= \alpha A + \beta B$$

(ix) Associative:

$$\alpha(\beta A) = \alpha \left[\beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right]$$

$$= \alpha \begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha(\beta a) & \alpha(\beta b) \\ \alpha(\beta c) & \alpha(\beta d) \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha\beta)a & (\alpha\beta)b \\ (\alpha\beta)c & (\alpha\beta)d \end{pmatrix}$$

$$= (\alpha\beta) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= (\alpha\beta)A$$

(x) Identity:

$$1A = 1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

All the axioms of vector space are satisfied.

Therefore, V is a vector space.

i.e., the set of all 2×2 matrices is a vector space.

6. **Let $V = M_{m\times n}$ be the set of all $m \times n$ matrices. Under the usual operations of addition of matrices and scalar multiplication, is $M_{m\times n}$ a vector space**?
It is proved that $M_{2\times 2}$ is a vector space in problem 5.
Similarly $M_{m\times n}$ is also a vector space.

7. Prove that the set of all real valued continuous (differentiable & integrable) functions of x defined in the interval [0, 1] is a vector space.
Let V be the set of all real valued continuous functions of x defined in [0, 1].
Let $f, g, h \in V$ and $\alpha, \beta \in F$
Define the function $f + g$ such that $(f + g)(x) = f(x) + g(x)$
and their scalar multiplication αf such that $(\alpha f)(x) = \alpha[f(x)]$.

Addition

(i) Closure:
If f and g are continuous functions, then their sum $f + g$ is also continuous.

(ii) Associative:

$$\begin{aligned}[(f + g) + h](x) &= (f + g)(x) + h(x)\\ &= [f(x) + g(x)] + h(x)\\ &= f(x) + [g(x) + h(x)]\\ &= f(x) + [g + h](x)\\ &= [f + (g + h)](x)\end{aligned}$$

Therefore, $(f + g) + h = f + (g + h)$.

(iii) Identity:
There exists $0(x) = 0 \in V$ such that
$(f + 0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$

(iv) Inverse:
There exists $-f(x) \in V$ such that
$[f + (-f)](x) = f(x) + (-f)(x) = f(x) - f(x) = 0.$

(v) Commutative:

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= g(x) + f(x)\\ &= (g + f)(x)\end{aligned}$$

Therefore, $f + g = g + f$

Scalar multiplication

(vi) Closure:

If f is continuous then αf is also continuous.

(vii) Distributive 1:

$$
\begin{aligned}
[\alpha\,(f + g)]\,(x) &= \alpha[(f + g)\,(x)] \\
&= \alpha[(f(x) + g(x)] \\
&= \alpha f(x) + \alpha g(x) \\
&= (\alpha f)\,(x) + (\alpha g)\,(x) \\
&= (\alpha f + \alpha g)\,(x)
\end{aligned}
$$

Therefore, $\alpha\,(f + g) = \alpha f + \alpha g$

(viii) Distributive 2:

$$
\begin{aligned}
(\alpha + \beta)\,f(x) &= \alpha f(x) + \beta f(x) \\
&= [\alpha f + \beta f]\,(x)
\end{aligned}
$$

Therefore, $(\alpha + \beta)\,f = \alpha f + \quad f$

(ix) Associative:

$$
\begin{aligned}
(\alpha\beta)\,f(x) &= [(\alpha\beta)\,f]\,(x) \\
&= [\alpha(\beta f)]\,(x) \\
&= \alpha(\beta f)(x)
\end{aligned}
$$

Therefore, $(\alpha\beta)f = \alpha(\beta f)$

(x) Identity:

$(1f)\,(x) = 1f(x) = f(x)$

Therefore, $1f = f$

All the axioms of vector space are satisfied. Therefore, V is a vector space.
i.e., set of all real valued cont. functions of X defined in the interval [0, 1] is a vector space.

Home work:

8. Show that the set $V = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in R\right\}$ is a vector space over the field of reals R under usual addition and multiplication.
9. Prove that the set of all convergent sequences of real numbers is a vector space over the field of real numbers.
10. Prove that the set of all ordered n-triples of complex numbers forms a vector space over the field of complex numbers.
11. Show that the set V of all ordered pairs of integers does not form a vector space over the field of real numbers R.
12. Show that the set of all pairs of real numbers over the field of reals defined as $(x_1, y_1) + (x_2, y_2) = (3y_1 + 3y_2, -x_1 - x_2)$ and $c(x_1, y_1) = (3cy_1 - cx_1)$ does not form a vector space.

2.2 Subspace

A subset W of a vector space V is called a subspace of V if it satisfies the following properties:

(1) The zero vector of V is also in W.

(2) If $u, v \in W$ then $u + v \in W$ (i.e., W is closed under addition),

(3) If $u \in W$ and k is a scalar then $ku \in W$ (i.e., W is closed under scalar multiplication).

Theorem: A non-empty subset W of a vector space V is a subspace of V iff $u, v \in W$ and α and β are scalars then $\alpha u + \beta v \in W$.

Remark: The intersection of two subspaces of a vector space V is a subspace of V but the union of two subspaces of V need not be a subspace of V.

1. Let $\boldsymbol{W = \{(x, y) \in R^2 \mid y = 2x\}}$ *be* the graph of the function $\boldsymbol{f(x) = 2x}$. Is W a subspace of $\boldsymbol{V = R^2}$?

(i) If $x = 0$ then $y = 2 \cdot 0 = 0$ and therefore $0 = (0, 0) \in W$.

(ii) Let $u = (a, 2a), v = (b, 2b) \in W$.

Then $u + v = (a, 2a) + (b, 2b)$

$$= (a + b, 2a + 2b)$$

$$= \big(a + b, 2\,(a + b)\big) \in W$$

Thus, W is closed under addition.

(iii) Let $u = (a, 2a) \in W$ and $\alpha \in F$

Then $\alpha u = (\alpha a, \alpha 2a) = \big(\alpha a, 2\,(\alpha a)\big) \in W$

W is closed under scalar multiplication.

All three conditions of a subspace are satisfied for W.

Therefore, W is a subspace of V.

2. Let $W = \{(x, y) \in R^2 | x \geq 0, y \geq 0\}$ be the first quadrant in R^2. Is W a subspace?

(i) The set W contains the zero vector $(0, 0)$.

(ii) Let $u = (x_1, y_1) \in W$ has components $x_1 \geq 0, y_1 \geq 0$

Let $v = (x_2, y_2) \in W$ has components $x_2 \geq 0, y_2 \geq 0$

Then $u + v = (x_1 + x_2, y_1 + y_2) \in W$

has components $x_1 + y_1 \geq 0, x_2 + y_2 \geq 0$.

Therefore, $u + v \in W$. Thus, W is closed under addition.

(iii) If $u = (1, 1)$ and $\alpha = -1$ then $\alpha u = (-1, -1) \notin W$.

Therefore, W is not closed under scalar multiplication.

Therefore, W is not a subspace of V.

3. Let $V = M_{n\times n}$ be the vector space of all $n \times n$ matrices. We define the trace of a matrix $A \in M_{n\times n}$ as the sum of its diagonal entries: $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$. Let W be the set of all $n \times n$ matrices whose trace is zero: $W = \{A \in M_{(n\times n)} \mid tr(A) = 0\}$. Is W a subspace of V?

(i) If 0 is the n × n zero matrix then clearly $tr(0) = 0$ and thus $0 \in M_{n\times n}$.

(ii) Suppose that A and B are in W. Then necessarily $tr(A) = 0$ and $tr(B) = 0$.

Consider the matrix $C = A + B$. Then

$$
\begin{aligned}
tr(C) = tr\,(A + B) &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn}) \\
&= (a_{11} + \cdots + a_{nn}) + (b_{11} + \cdots + b_{nn}) \\
&= tr(A) + tr(B) \\
&= 0
\end{aligned}
$$

Therefore, $tr(C) = 0$ and consequently $C = A + B \in W$,

Thus, W is closed under addition.

(iii) Now let α be a scalar and let $C = \alpha A$. Then

$tr(C) = tr(\alpha A) = \alpha a_{11} + \alpha a_{22} + \cdots + \alpha a_{nn} = \alpha tr(A) = 0.$

Thus, $tr(C) = 0$, that is, $C = \alpha A \in W$, and consequently W is closed under scalar multiplication. Therefore, the set W is a subspace of V.

4. **Let $V = P_n[t]$ and consider the subset W of V: $W = \{u \in P_n[t] \mid u'(1) = 0\}$. In other words, W consists of polynomials of degree n in the variable t whose derivative at t = 1 is zero. Is W a subspace of V?**

(i) The zero polynomial $0(t) = 0$ clearly has derivative at $t = 1$ equal to zero, that is, $0'(1) = 0$, and thus the zero polynomial is in W.

(ii) Now suppose that $u(t)$ and $v(t)$ are two polynomials in W. Then, $u'(1) = 0$ and also $v'(1) = 0$. To verify whether or not W is closed under addition, we must determine whether the sum polynomial $(u + v)(t)$ has a derivative at $t = 1$ equal to zero. From the rules of differentiation, we compute

$(u + v)'(1) = u'(1) + v'(1) = 0 + 0 = 0.$

Therefore, the polynomial $(u + v)$ is in W, and thus W is closed under addition.

(iii) Now let α be any scalar and let u(t) be a polynomial in W. Then u′ (1) = 0. To determine whether or not the scalar multiple αu(t) is in W we must determine if αu(t) has a derivative of zero at t = 1. Using the rules of differentiation, we compute that (αu) ′ (1) = αu′ (1) = α · 0 = 0. Therefore, the polynomial (αu) (t) is in W and thus W is closed under scalar multiplication.

All three properties of a subspace hold for W and therefore W is a subspace of $P_n[t]$.

5. **Let $V = P_n[t]$ and consider the subset W of V: $W = \{u \in P_n[t] \mid u(2) = -1\}$. In other words, W consists of polynomials of degree n in the variable t whose value $t = 2$ is −1. Is W a subspace of V?**

The zero polynomial $0(t) = 0$ clearly does not equal to −1 at $t = 2$.

Therefore, W does not contain the zero polynomial and, because all three conditions of a subspace must be satisfied for W to be a subspace, then W is not a subspace of $P_n[t]$.

6. **Let $V = R^3$ be a vector space and consider the subset W of V consisting of vectors of the form (a, a^2, b), where the first two components are the same. Is W a subspace of V?**

(i) The set W contains the zero vector $(0, 0, 0)$.

(ii) Let $u = (a, a, b)$ and $v = (c, c, d)$ then $u + v = (a + c, a + c, b + d) \in W$

Thus, W is closed under addition.

(iii) If we multiply (a, a, b) by a scalar k, we get $(ka, ka, kb) \in W$.

Thus, W is closed under scalar multiplication. Therefore, W is a subspace of V.

7. **Let $V = R^3$ be a vector space and consider the subset W of V consisting of vectors of the form (a, a^2, b), where the second component is the square of the first. Is W a subspace of V?**

(i) The set W contains the zero vector $(0, 0, 0)$.

(ii) Let $u = (a, a_2, b)$ and $v = (c, c_2, d)$, $u + v = (a + c, a_2 + c_2, b + d) \in W$

Because the second component of this vector is not the square of the first.

Thus, W is not closed under addition.

Therefore, W is not a subspace of V.

8. **Prove that the set W of $2 \in 2$ diagonal matrices is a subspace of the vector space M_{22} of 2×2 matrices.**

(i) The set W contains the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(ii) Let $u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $v = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ are in W then $u + v = \begin{pmatrix} a + c & o \\ 0 & b + d \end{pmatrix} \in W$.

Thus, W is closed under addition.

(iii) Let k be a scalar and $u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in W$ \, then $u = \begin{pmatrix} ka & 0 \\ 0 & kb \end{pmatrix} \in W$.

Thus, W is closed under scalar multiplication.

Therefore, W is a subspace of V.

9. **Let W be the set of vectors of the form $(a, a, a + 2)$. Show that W is not a subspace of $V = R^3$.**

The set W does not contain the zero vector $(0, 0, 0)$.

Therefore, W is not a subspace of V.

10. Prove that the subset $W = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$ of the vector space $V(R^3)$ is a subspace of $V(R^3)$.

(i) The set W contain the zero vector $(0, 0, 0)$ such that $0^2 + 0^2 + 0^2 \leq 1$.

(ii) Let u = (1, 0, 0) such that $1^2 + 0^2 + 0^2 \leq 1$ and

$v = (0, 0, 1)$ such that $0^2 + 0^2 + 1^2 \leq 1$ are in W

then $u + v = (1, 0,1)$ such that $1^2 + 0^2 + 1^2 = 2$

which is not less than or equal to 1.

i.e., $u + v = (1, 0,1) \notin W$

Therefore, W is not a subspace of V.

Home work:

11. Prove that the subset W = {(x, y, z) | x – 3y + 4z = 0} of the vector space R^3 is a subspace of R^3.
12. Prove that the subset W = {(x, y, z) | x + y + z = 0} of the vector space V_3 (R) is a subspace of V_3 (R).
13. Prove that the subset W = {(x, y, z) | x = y = z} of the vector space V_3 (R) is a subspace of V_3 (R).
14. Prove that the subset W = {(x, y, z) | 2x +3 y + z = 0} of the vector space V_3 (R) is a subspace of V_3 (R).
15. Verify the subset W = {(x, 2y, 3z) | x, y, z ∈ R} of the vector space V_3 (R) is a subspace of V_3 (R) or not.

2.3 Linear combination, Linear span

Linear combination:

Let V be a vector space over a field F and let $v_1, v_2, \dots, v_n \in V.$ Any vector of the form $a_1.v_1 + a_2.v_2 + \cdots + a_n.v_n \in V$, where $a_i \in F$ is called a linear combination of $v_1, v_2, \dots, v_n$.

Linear span:

Let S be a subset of a vector space over the field F. The set of all linear combination of vectors in S is called a linear span of S and is denoted by $L(S)$. If $S = \phi, then\ L(S) = 0$.

1. **Express the vector $\boldsymbol{v = (1, -2, 5)}$ as a linear combination of the vectors $\boldsymbol{v_1 = (1, 1, 1), v_2 = (1, 2, 3), v_3 = (2, -1, 1)}$ in the vector space $\boldsymbol{R^3(R)}$.**

   Let $v = a_1v_1 + a_2v_2 + a_3v_3$
   $(1, -2, 5) = a_1(1, 1, 1) + a_2(1, 2, 3) + a_3(2, -1, 1)$
   $(1, -2, 5) = (a_1 + a_2 + 2a_3,\ a_1 + 2a_2 - a_3,\ a_1 + 3a_2 + a_3)$
   Equating the corresponding elements,
   $a_1 + a_2 + 2a_3 = 1,\ a_1 + 2a_2 - a_3 = -2,\ a_1 + 3a_2 + a_3 = 5$
   Solving these equations, $a_1 = -6, a_2 = 3, a_3 = 2.$
   Hence, $(1, -2, 5) = -6(1, -1, 1) + 3(1, 2, 3) + 2(2, -1, 1)$.
2. **Express the vector $\boldsymbol{v = (2, -1, -8)}$ as a linear combination of the vectors $\boldsymbol{(1, 2, 1), (1, 1, -1)}$ and $\boldsymbol{(4, 5, -2)}$ in the vector space $\boldsymbol{R^3}$.**

   Let $v = a_1v_1 + a_2v_2 + a_3v_3$
   $(2, -1, -8) = a_1(1, 2, 1) + a_2(1, 1, -1) + a_3(4, 5, -2)$
   $(2, -1, -8) = (a_1 + a_2 + 4a_3,\ 2a_1 + a_2 + 5a_3,\ a_1 - a_2 - 2a_3)$
   Equating the corresponding elements,
   $a_1 + a_2 + 4a_3 = 2,\ 2a_1 + a_2 + 5a_3 = -1,\ a_1 - a_2 - 2a_3 = -8$
   Solving these equations, $a_1 = -4, a_2 = 2, a_3 = 1.$

Hence, $(2,-1,-8) = -4(1,2,1) + 2(1,1,-1) + (4,5,-2)$

3. **Express the vector $\boldsymbol{v} = (\mathbf{1}, \mathbf{3}, \mathbf{9})$ as a linear combination of the vectors $\boldsymbol{u_1} = (\mathbf{2}, \mathbf{1}, \mathbf{3}), \boldsymbol{u_2} = (\mathbf{1}, -\mathbf{1}, \mathbf{1}), \boldsymbol{u_3} = (\mathbf{3}, \mathbf{1}, \mathbf{5})$ in the vector space $\boldsymbol{R^3}(\boldsymbol{R})$.**

Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(1,3,9) = a_1(2,1,3) + a_2(1,-1,1) + a_3(3,1,5)$

$(1,3,9) = (2a_1 + a_2 + 3a_3,\ a_1 - a_2 + a_3,\ 3a_1 + 3a_2 + 5a_3)$

Equating the corresponding elements,

$2a_1 + a_2 + 3a_3 = 1,\ a_1 - a_2 + a_3 = 3,\ 3a_1 + 3a_2 + 5a_3 = 9$

Solving these equations, $a_1 = 10, a_2 = -5, a_3 = -12$.

Hence, $(1,3,9) = 10(2,1,3) - 5(1,-1,1) - 12(3,1,5)$

4. **For what value of $\boldsymbol{k}$ (if any) the vector $\boldsymbol{v} = (\mathbf{1}, -\mathbf{2}, \boldsymbol{k})$ can be expressed as a linear combination of vectors $\boldsymbol{v_1} = (\mathbf{3}, \mathbf{0}, -\mathbf{2})$ and $\boldsymbol{v_2} = (\mathbf{2}, -\mathbf{1}, -\mathbf{5})$ in $\boldsymbol{R^3}(\boldsymbol{R})$.**

Since vector $v = (1,-2,k)$ is a linear combination of vectors $v_1 = (3,0,-2)$ and $v_2 = (2,-1,-5)$, there exist scalars a and b such that $v = av_1 + bv_2$

$$(1,-2,k) = a(3,0,-2) + b(2,-1,-5)$$

$$(1,-2,k) = (3a + 2b, -b, -2a - 5b)$$

Equating the corresponding elements,

$$3a + 2b = 1, -b = -2, -2a - 5b = k$$

By solving these equations, $k = -8$.

5. **Find a condition on $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ so that $\boldsymbol{w} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is a linear combination of $\boldsymbol{u} = (\mathbf{1}, -\mathbf{3}, \mathbf{2})$ and $\boldsymbol{v} = (\mathbf{2}, -\mathbf{1}, \mathbf{1})$ in $\boldsymbol{R^3}$ so that $\boldsymbol{w} \in \boldsymbol{span}\ (\boldsymbol{u}, \boldsymbol{v})$.**

Since vector $w = (a,b,c)$ is a linear combination of vectors $u = (1,-3,2)$ and $v = (2,-1,1)$, there exist scalars x and y such that $w = xu + yv$.

That is, $(a,b,c) = x(1,-3,2) + y(2,-1,1)$

Equating the corresponding elements,

$$x + 2y = a, -3x - y = b, 2x + y = c.$$

Since $w \in span\,(u, v)$, $\begin{vmatrix} 1 & 2 & a \\ -3 & -1 & b \\ 2 & 1 & c \end{vmatrix} = 0$

We get, $a - 3b - 5c = 0$. This is the required condition.

6. **Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$**

Let $A = a_1 B + a_2 B + a_3 D, where\ a_1, , a_3 \in R.)$

$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ ------------ (1)

$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$

Equating the corresponding elements,

$a_1 + a_2 + a_3 = 3,\ a_1 + a_2 - a_3 = -1,\ -a_2 = 1,\ -a_1 = -2$

By solving these equations, $a_1 = 2, a_2 = -1, a_3 = 2$.

Linear combination of A is

(1) $\Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

7. **Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$**

Let $A = a_1 B + a_2 B + a_3 D, where\ a_1, , a_3 \in R.)$

$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1 \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$ ------------ (1)

$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 2a_3 & -3a_1 + 3a_3 \\ 2a_1 + 2a_2 & a_2 + 5a_3 \end{bmatrix}$

Equating the corresponding elements,

$2a_3 = 2,\ -3a_1 + 3a_3 = 0,\ 2a_1 + 2a_2 = 4,\ a_2 + 5a_3 = -5$

By solving these equations, $a_1 = 1, a_2 = 1, a_3 = 1$.

Linear combination of A is

(1) $\Rightarrow \begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$

8. **Determine whether the function $f(x) = 2x^2 + 6x + 7$ is a linear combination of $g(x) = x^2 - 1$ and $h(x) = 2x + 3$.**

   To check: $a_1 g(x) + a_2 h(x) = f(x)$ for some real values of a_1, a_2.

   $a_1(x^2 - 1) + a_2(2x + 3) = 2x^2 + 6x + 7$

Equating components, $a_1 = 2, a_2 = 3$. This system has the unique solution.

Therefore, $f(x)$ is a linear combination of $g(x)$ and $h(x)$ as

$$f(x) = 2g(x) + 3h(x).$$

9. **Let f (x) = $2x^2$ - 5 and g (x) = x +1. Show that the function h (x) = $4x^2$ + 3x - 7 lies in the subspace Span {f, g} of P_2.**

   It is enough to prove that that $h(x)$ is a linear combination of $f(x)$ and $g(x)$.

   $h(x) = a_1 f(x) + a_2 g(x)$

   $4x^2 + 3x - 7 = a_1(2x^2 - 5) + a_2(x + 1)$

   Equating coefficients, $a_1 = 2, a_2 = 3$. This system has the unique solution.

   Therefore, he function h is a linear combination of f and g.

   Therefore, h lies in Span {f, g}.

2.4 Linearly dependence, Basis and dimension

Linear dependence:

Let V be a vector over the field F. The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependant over F if there exist scalars $a_1, a_2, \dots, a_n \in F$ such that

$a_1.v_1 + a_2.v_2 + \cdots + a_n.v_n = 0, but\ not\ all\ a_i = 0., where\ i \in N.$

Linear independence:

Let V be a vector over the field F. The vectors $v_1, v_2, \dots, v_n$ are said to be linearly independent over F if there exist scalars $a_1, a_2, \dots, a_n \in F$ such that

$a_1.v_1 + a_2.v_2 + \cdots + a_n.v_n = 0 \Rightarrow$ All $a_i = 0., where\ i \in N.$

Basis of a vector space:

Let V be a vector space over the scalar field F. The set of vectors $\{v_1, v_2, \dots, v_n\}$is called a basis of V, if

(i) $v_1, v_2,\ \ v_n$ are linearly independent.

(ii) $v_1, v_2,\ \ v_n$span V.. That is, each vector V can be uniquely expressed as linear combination of $v_1, v_2,\ \ v_n$.

Example:

Consider the set {(1, 2, 3), (-2, 4, 1)} of vectors in R^3.

These vectors generate a subspace V of R^3 consisting of all vectors of the form

v = c_1(1, 2, 3) + c_2(-2, 4, 1)

Thus, the vectors (1, 2, 3) and (-2, 4, 1) span this subspace.

Furthermore, since the second vector is not a scalar multiple of the first vector the vectors are linearly independent.

Therefore {(1, 2, 3), (-2, 4, 1)} is a basis for V. Thus dim (V) = 2.

Dimension of a vector space V:

Number of elements in a basis of vector space V is called the dimension of V and is denoted by $\dim V.\ If\ V\ contains\ a\ basis\ with$ n elements then the $\dim V = n$.

Note:

(i) The vector space $\{0\}$ is defined to have dim 0, since empty set ϕ is independent and generates $\{0\}$.

That is , $\dim 0 = Number\ of\ elements\ in\ \phi = 0$ (Since no element is in ϕ).

(ii) When a vector is not of finite dimension, it is said to be of infinite dimension.

Properties:

1. For n vector of n dimensional vector space V to be a basis, it is sufficient that they span V or that they are linearly independent.
2. Let A be any m × n matrix which is equivalent to a row reduced echelon matrix E. Then the non-zero rows of E form a basis of the subspace spanned by the rows of A.

Theorem: Let V be an n dimensional vector space and let S be a set with n vectors. Then the following are equivalent.

1. S is a basis for V.
2. S is linearly independent.
3. S spans V.

Theorem: If S = $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V and T = $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is a linearly independent set of vectors in V, then $k \leq n$.

Remark: If S and T are both bases for V then k = n. This says that every basis has the same number of vectors. Hence the dimension is will defined.

Note:

(i) Since the dimension of a vector space is the number of elements in a basis, the number of non-zero rows in E is the dimension of the subspace spanned by the rows of A.

(ii) Since the rank of a matrix is the number of non-zero rows, the dimensional of the subspace spanned by the rows of A is equal to the rank of A.

(iii) To find the basis and the dimension of a subspace spanned the vectors, reduce the matrix whose rows are the given vectors to echelon form.

1. **Check whether the vectors $\boldsymbol{v_1 = (1,2,3), v_2 = (3,1,7)}$ and $\boldsymbol{v_3 = (2,5,8)}$ are linearly dependent or not.**

   $xv_1 + yv_2 + zv_3 = 0$

   $\Longrightarrow x(1,2,3) + y(3,1,7) + z(2,5,8) = (0,0,0)$

   $\Longrightarrow x + 3y + 2z = 0, 2x + y + 5z = 0, 3x + 7y + 8z = 0$

   $$\Longrightarrow \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{vmatrix} = 0$$

   This is impossible. Therefore, the given vectors are linearly independent.

2. **Check whether the vectors $\boldsymbol{v_1 = (1,4,9), v_2 = (3,1,4)}$ and $\boldsymbol{v_3 = (9,3,12)}$ are linearly dependent or not.**

   $xv_1 + yv_2 + zv_3 = 0$

   $\Longrightarrow x(1,4,9) + y(3,1,4) + z(9,3,12) = 0$

   $\Longrightarrow x + 3y + 9z = 0, 4x + y + 3z = 0, 9x + 4y + 12z = 0$

   $$\Longrightarrow \begin{vmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{vmatrix} = 0$$

   This is true. Therefore, the given vectors are linearly dependent.

3. **If $\boldsymbol{u, v, w}$ are linearly independent vectors in V(F), where F is the field of Complex numbers, then $\boldsymbol{\{u + v, v + w, w + u\}}$ is a linearly independent set of vectors.**

   Let $a(u + v) + b(v + w) + c(w + u) = 0, where\ a, b, c \in F$

   $(a + c)u + (a + b)v + (b + c)w = 0$

   Since u, v, w are linearly independent, $a + c = 0, a + b = 0, b + c = 0.$

   Therefore, $a = 0, b = 0, c = 0.$

   Therefore, $\{u + v, v + w, w + u\}$ is a linearly independent set of vectors.

4. **Let V be a vector space of all 2 × 3 matrices over R. Show that the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{2} & \mathbf{0} & \mathbf{5} \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} \mathbf{4} & \mathbf{-1} & \mathbf{2} \\ \mathbf{1} & \mathbf{-2} & \mathbf{3} \end{bmatrix}$ form a linearly independent set.**

Let $a\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b\begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 5 \end{bmatrix} + c\begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a-2c & 4a+5b+3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

By equating the corresponding elements,

$2a+b+4c=0,\ \ a+b-c=0, -a-3b+2c=0.$ ---- (1)

$3a-2b+c=0, -2a-2c=0,\ 4a+5b+3c=0.$ ----- (2)

On solving the system (1) of equations, the only solution is $a=0, b=0, c=0$.

This solution also satisfies the system (2) of equations.

Therefore, the given set of matrices is linearly independent.

5. **Determine whether or not each of the following forms a basis, $\boldsymbol{x_1} = (\mathbf{2}, \mathbf{2}, \mathbf{1}), \boldsymbol{x_2} = (\mathbf{1}, \mathbf{3}, \mathbf{7}), \boldsymbol{x_3} = (\mathbf{1}, \mathbf{2}, \mathbf{2})$in $\boldsymbol{R^3}$.**

Three vectors in R^3 form a basis if and only if they are linearly independent.

$x(2,2,1) + y(1,3,7) + z(1,2,2) = (0,0,0)$

$\Rightarrow 2x+y+z=0, 2x+3y+2z=0, x+7y+2z=0$

$\Rightarrow \begin{vmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{vmatrix} = 0$

This is impossible.

Therefore, Possible solution is $x=0, y=0, z=0$.

Therefore, the vectors x_1, x_2, x_3 are linearly independent and hence form a basis.

6. **Let W be the subspace of R^5 spanned by $x_1 = (1, 2, -1, 3, 4)$, $x_2 = (2, 4, -2, 6, 8)$, $x_3 = (1, 3, 2, 2, 6), x_4 = (1, 4, 5, 1, 8), x_5 = (2, 7, 3, 3, 9)$. Find a subset of vectors which forms a basis of W.**

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1, R_4 \to R_4 - R_1, R_5 \to R_5 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_5$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 3 & 5 & -3 & 1 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 3 & 5 & -3 & 1 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_2, R_4 \to R_4 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & -4 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows is 3.

Therefore, $\dim W = 3$ and $\{x_1, x_2, x_3\}$ forms a basis in W.

7. **V is a vector space of polynomials over R. Find a basis and dimension of the subspace W of V, spanned by the polynomials, $x_1 = t^3 - 2t^2 + 4t + 1, x_2 = 2t^3 - 3t^2 + 9t - 1, x_3 = t^3 + 6t - 5, x_4 = 2t^3 - 5t^2 + 7t + 5$.**

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1, R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 1 & -1 & -1 & 3 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2, R_4 \to R_4 + R_2$

$$\sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non-zero rows of Echelon matrix form a basis.

Number of non-zero rows is 2.

Therefore, $\dim W = 2$ and $\{x_1, x_2\}$ forms a basis of W.

8. **Determine whether the set of vectors {(1, 2, 3), (-2, 1, 3), (3, 1, 0)} is a basis for R³ or not.**

Consider the determinant $\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 3 \\ 3 & 1 & 0 \end{vmatrix} = 1(0-3) - 2(0-9) + 3(-2-3) = 0$

Therefore, the vectors are linearly dependent.

Therefore, It is not a basis of R^3.

9. **Prove that the set {(1, 3, –1), (2, 1, 0), (4, 2, 1)} is a basis for R³.**

Consider the determinant $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{vmatrix} = -5 \neq 0$

Thus, the vectors are linearly independent.

The set {(l, 3, -1), (2, 1, 0), (4, 2, 1)} is therefore a basis for R^3.

10. **Find a basis for the subspace V of R^4 spanned by the vectors (1, 2, 3, 4), (–1, –1, –4, –2), (3, 4, 11, 8).**

We construct a matrix A having these vectors as row vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix}$$

Determine the reduced echelon form of A. We get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -1 & -4 & -2 \\ 3 & 4 & 11 & 8 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & -2 & 2 & -4 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero vectors of this reduced echelon form are $(1, 2, 3, 4)$, $(0, 1, -1, 2)$

These are the basis for the subspace V. The dimension of this subspace is two.

11. **Find the basis and dimension of the subspace spanned by the vectors {(1, –2, 3), (1, –3, 4), (–1, 1, –2)} in the vector space $V_3(R)$.**

Let S = {(1, –2, 3), (1, –3, 4), (–1, 1, –2)} be the given set in the vector space $V_3(R)$. Dim[$V_3(R)$] = 3.

Consider, $\begin{vmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{vmatrix} = 1(6-4) + 2(-2+4) + 3(1-3) = 0$

Given set S is linearly dependent.

Therefore, S is not a basis of $V_3(R)$.

To find the basis and dimension of the subspace of S

Consider, $\begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The final matrix has two non-zero rows.

Therefore, Subspace is S’ = {(1, –2, 3), (0, -1, 1)} and the dimension of subspace S’ is 2 i.e., dim (S') = 2.

12. Find the basis and dimension of the subspace spanned by the vectors {(1, 0, –1), (1, 2, 1), (0, –3, 2)} in the vector space $V_3(R)$.

Let S = {(1, 0, –1), (1, 2, 1), (0, –3, 2)} be the given set in the vector space $V_3(R)$.

$\mathrm{Dim}[V_3(R)] = 3$.

Consider, $\begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{vmatrix} = 1(4+3) - 1(-3-0) = 10 \neq 0$

Given set is linearly independent.

Therefore, S is a basis of $V_3(R)$ and dim(S) = 3.

13. Find the basis and dimension of the subspace spanned by the vectors {(2, 4, 2), (1, –1, 0), (1, 2, 1), (0, 3, 1)} in the vector space $V_3(R)$.

Let S = {(2, 4, 2), (1, –1, 0), (1, 2, 1), (0, 3, 1)} be the given set in the vector space

$V_3(R)$. $\mathrm{Dim}[V_3(R)] = 3$.

Any subset of $V_3(R)$ containing more than 3 vectors is linearly dependent.

Therefore, S is not a basis of $V_3(R)$.

To find the basis and dimension of the subspace of S

Consider, $\begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The final matrix has two non-zero rows.

Therefore, Subspace is S' = {(1, 2, 1), (0, –3, –1)} and the dimension of subspace S' is 2

i.e., dim (S') = 2.

14. Find the basis and dimension of the subspace spanned by the vectors {(2, -3, 1), (3, 0, 1), (0, 2, 1), (1, 1, 1)} in the vector space $V_3(R)$.

Let S = {(2, -3, 1), (3, 0, 1), (0, 2, 1), (1, 1, 1)} be the given set in the vector space $V_3(R)$. Dim[$V_3(R)$] = 3.

Any subset of $V_3(R)$ containing more than 3 vectors is linearly dependent.

Therefore, S is not a basis of $V_3(R)$.

To find the basis and dimension of the subspace of S

Consider, $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & -3 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 2 & 1 \\ 0 & -5 & -1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -7 \\ 0 & 0 & 7 \end{bmatrix}$

$$\approx \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

The final matrix has three non-zero rows.

Therefore, Subspace is S' = {(1, 1, 1), (0, –3, –2), (0, 0, –7)} and the dimension of subspace S' is 3. i.e., dim (S') = 3.

15. Show that the set S = $\left\{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\right\}$ form a basis of the vector space of the vector space $M_2(R)$ of 2 × 2 matrices and find its dimension.

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore, S spans $M_2(R)$ and $c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} =$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$

Given set is linearly independent.

Therefore, S is a basis of $M_2(R)$ and $\dim(S) = 4$.

16. Let $V = P_3$ (polynomials of degree 3) and let $S = \{1, t, t^2, t^3\}$. Show that S is a basis for V.

We must show both linear independence and span.

Linear Independence:

Let $c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3) = 0$

Then since a polynomial is zero if and only if its coefficients are all zero, we have

$$c_1 = c_2 = c_3 = c_4 = 0$$

Hence S is a linearly independent set of vectors in V.

A general vector in P_3 is given by $a + bt + ct^2 + dt^3$

We need to find constants c_1, c_2, c_3, c_4 such that

$c_1(1) + c_2(t) + c_3(t^2) + c_4(t^3) = a + bt + ct^2 + dt^3$

We just let $c_1 = a, c_2 = b, c_3 = c, c_4 = d$

Hence S spans V. We can conclude that S is a basis for V.

In general the basis $\{1, t, t^2, \dots, t^n\}$ is called the *standard basis* for P_n.

2.5 Linear transformations

Linear Transformation:

Let V and W be any two subspaces over the field F. A mapping T from V to W is called a linear transformation if

(i) $T(v_1 + v_2) = T(v_1) + T(v_2),\ for\ v_1, v_1 \in V$

(ii) $T(aV) = aT(v), \forall a \in F, \forall v \in V$

Algebra of linear transformations:

Let $T_1: Y \to V$ be two transformations where U and V are two vector spaces over the field F. We define the sum of T_1 and T_2 by $T_1 + T_2: U \to V$ such that $(T_1 + T_2)u = T_1(u) + T_2(u_2), u \in U$.

Scalar multiplication of linear transformations:

If $a \in F$, the function aT, that is, product of a linear transformation $T: U \to V$ with a, and defined by $aT: U \to V$ such that $(aT)u = a\big(T(u)\big)$ for all $u \in U$ is a linear transformation from U into V.

Composition of two linear transformations:

Let u, v, w be three vector spaces over a field F. We define T_2T_1, called the composition or product of T_2 and T_1, by $(T_2T_1)u = T_2(T_1(u))$. T_2T_1 is also denoted by $(T_2 \circ T_1)$.

Kernel of T (Nullity of T) and image of T:

Let T be a linear transformation from V to W.

$Ker\ T = \{v \in V: T(v) = 0, 0 \in W\}$

$Img(T) = \{w \in W: w = T(v) for\ some\ v \in V\}$

1. **Prove that the transformation T: $R^2 \to R^2$ defined by T (x, y) = (2x, x + y) is linear.**

Let x = (x_1, y_1) and y = (x_2, y_2) be elements of R^2 and let c be a scalar.

T (x + y) = T ((x_1, y_1) + (x_2, y_2))

= T ($x_1 + x_2$, $y_1 + y_2$)	by vector addition
= ($2x_1 + 2x_2$, $x_1 + x_2 + y_1 + y_2$)	by definition of T
= ($2x_1$, $x_1 + y_1$) + ($2x_2$, $x_2 + y_2$)	by vector addition
= T (x_1, y_1) + T (x_2, y_2)	by definition of T
= T (x) + T (y)	

Thus, T preserves vector addition.

T (cx) = T (c (x_1, y_1))

= T (cx_1, cy1)	by scalar multiplication of a vector
= ($2cx_1$, cx_1+ cy_1)	by definition of T
= c ($2x_1$, $x_1 + y_1$)	by scalar multiplication of a vecto
= c T (x_1, y_1)	by definition of
= c T (x)	

Thus, T preserves scalar multiplication. Therefore, T is linear.

2. **Prove that the transformation $\boldsymbol{T}: \boldsymbol{R^2} \to \boldsymbol{R^2}$ defined by $\boldsymbol{T\,(x, y) = (3x, x + y)}$ is linear. Find the images of the vectors $\boldsymbol{(1, 3)}$ and $\boldsymbol{(-1, 2)}$ under this transformation.**

Let $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be arbitrary vectors in R^2 and c an arbitrary scalar. Then

$$
\begin{aligned}
T\,(x + y) &= T\,(x_1 + x_2,\, y_1 + y_2) \\
&= (3(x_1 + x_2\,), x_1 + x_2 + y_1 + y_2) \\
&= (3x_1, x_1 + y_1) + (3x_2,\, x_2 + y_2) \\
&= T(x) + T(y)
\end{aligned}
$$

Thus, T preserves vector addition.

We now show that T preserves scalar multiplication.

$$
\begin{aligned}
T\,(c\,x) &= T\,(cx_1, cy_1) \\
&= (3cx_1, cx_1 + cy_1) \\
&= c\,(3x_1, x_1 + y_1) \\
&= c\,T(x)
\end{aligned}
$$

Therefore, T is linear.

By definition of T, we have

$T(1,3) = (3.1, 1 + 3) = (3,4),$

$T(-1,2) = (3.(-1), -1 + 2) = (-3,1).$

3. **Prove that T (x, y) = (x + 2y, 3y, x - y) defines a linear transformation by T: $R^2 \rightarrow R^3$. Find the images of (0, 4) and (1, 1).**

Let $x = (x_1, y_1)$ and $y = (x_2, y_2)$ be arbitrary vectors in R^2 and c an arbitrary scalar.

Then

$$T(x + y) = T(x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2 + 2y_1 + 2y_2, 3y_1 + 3y_2, x_1 + x_2 - y_1 - y_2)$$
$$= (x_1 + 2y_1, 3y_1, x_1 - y_1) + (x_2 + 2y_2, 3y_2, x_2 - y_2)$$
$$= T(x) + T(y)$$

To prove that scalar multiplication is preserved under *T*,

$$T(cx) = T(cx_1, cy_1)$$
$$= T(cx_1 + 2cy_1, 3cy_1, cx_1 - cy_1)$$
$$= c(x_1 + 2y_1, 3y_1, x_1 - y_1)$$
$$= c\,T(x)$$

Therefore, *T* is linear.

By definition of *T*, we have

$T(0, 4) = (0+2\cdot4, 3\cdot4, 0 - 4) = (8, 12, -4)$

$T(1, 1) = (1+2\cdot1, 3\cdot1, 1 - 1) = (3, 3, 0).$

4. **Verify whether the transformation T: $R^1 \rightarrow R^3$ defined by T (x) = (x, $2x^2$, x^3) is linear or not.**

Let $x = (x_1)$ and $y = (x_2)$ be arbitrary vectors in R^1 and c be a scalar.

Then, $T(x + y) = T(x_1 + x_2)$

$$= (x_1 + x_2, 2(x_1 + x_2)^2, (y_1 + y_2)^3)$$

$$T(x) + T(y) = (x_1, 2x_1^2, y_1^3) + (x_2, 2x_2^2, y_2^3)$$
$$= (x_1 + x_2, 2x_1^2 + 2x_2^2, y_1^3 + y_2^3)$$

$T(x + y) \neq T(x) + T(y)$

Therefore, T is not linear.

5. Prove that the transformations T: $R^2 \to R$ defined by T (x, y) = y^2 is not linear. Justify your answer by counter example.

Let x = (x_1, y_1) and y = (x_2, y_2) be arbitrary vectors in R^2 and c be a scalar.

Then, T (x + y) = T $(x_1 + x_2,\ \ y_1 + y_2)$

$= (y_1 + y_2)^2$

And T(x) + T(y) = T $((x_1, y_1))$ + T $((x_2, y_2))$

$= y_1^{\ 2} + y_2^2$

T (x + y) ≠ T(x) + T(y)

Therefore, T is not linear.

For example, let x = (0, 1) and y = (0, 2)

T (x + y) = T ((0, 1) +(0, 2)) = T ((0,3)) = 9

T(x) + T(y) = T ((0,1)) + T ((0,2)) = 1 + 4 = 5

6. Show that the function $\boldsymbol{T}: \boldsymbol{R}^2 \to \boldsymbol{R}^3$ given by $\boldsymbol{T(x, y) = (x + y, x - y, y)}$ is a linear transformation.

$u = (x_1, y_1), v = (x_2, y_2) \in R^2$

$$T(u + v) = T(x_1 + x_2, y_1 + y_2)$$

$$= (x_1 + x_2 + y_1 + y_2,\ x_1 + x_2 - y_1 - y_2, y_1 + y_2)$$

$$= (x_1 + y_1 + x_2 + y_2,\ x_1 - y_1 + x_2 - y_2, y_1 + y_2)$$

$$= (x_1 + y_1,\ x_1 - y_1,\ y_1) + (x_2 + y_2,\ x_2 - y_2, y_2)$$

$$= T(u) + T(v)$$

Thus, T preserves vector addition.

Also for any scalar $a \in R$,

$$T(au) = T(ax_1, ay_1)$$

$$= (ax_1 + ay_1, ax_1 - ay_1, ay_1)$$

$$= a(x_1 + y_1, x_1 - y_1, y_1)$$

$$= aT(u)$$

Thus, T preserves scalar multiplication.

Therefore, T is a linear transformation.

7. **Check whether the function $\boldsymbol{T}: \boldsymbol{R}^3 \rightarrow \boldsymbol{R}^3$ defined by $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{y}, -\boldsymbol{x}, -\boldsymbol{z})$ is a linear transformation or not.**

Let $u = (x_1, y_1, z_1), v = (x_2, y_2, z_1) \in R^3$

$$T(u+v) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
$$= (y_1 + y_2, -x_1 - x_2, -z_1 - z_2)$$
$$= (y_1, -x_1, -z_1) + (y_2 - x_2 - z_2)$$
$$= T(u) + T(v)$$

Also for any scalar $a \in R$,

$$T(au) = T(ax_1, ay_1, az_1)$$
$$= (ay_1, -ax_1, -az_1)$$
$$= a(y_1, -x_1, -z_1)$$
$$= aT(u)$$

Therefore, T is a linear transformation.

8. **Check whether the function $\boldsymbol{T}: \boldsymbol{R}^3 \rightarrow \boldsymbol{R}^2$ defined by**

$\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = (\boldsymbol{2x - 3y}, \boldsymbol{7y + 2z})$ is a linear transformation or not .

Let $u = (x_1, y_1, z_1), v = (x_2, y_2, z_1) \in R^3$

$$T(u) = T(x_1, y_1, z_1) = (2x_1 - 3y_1, 7y_1 + 2z_1)$$

$$T(v) = T(x_2, y_2, z_2) = (2x_2 - 3y_2, 7y_2 + 2z_2)$$

$$T(u+v) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
$$= (2x_1 + 2x_2 - 3y_1 - 3y_2,\ 7y_1 + 7y_2 + 2z_1 + 2z_2)$$
$$= (2x_1 - 3y_1 + 2x_2 - 3y_2,\ 7y_1 + 2z_1 + 7y_2 + 2z_2)$$
$$= (2x_1 - 3y_1, 7y_1 + 2z_1) + (2x_2 - 3y_2, 7y_2 + 2z_2)$$
$$= T(u) + T(v)$$

Also for any scalar $a \in R$,

$$T(au) = T(ax_1, ay_1, az_1)$$
$$= (2ax_1 - 3ay_1,\ 7ay_1 + 2az_1)$$
$$= a(2x_1 - 3y_1,\ 7y_1 + 2z_1)$$
$$= aT(u)$$

Therefore, T is a linear transformation.

9. Let $V(R) \to V_2(R)$ be a mapping $f(x) = (3x, 5x)$. Show that f is a linear transformation.

$$f(x_1 + x_2) = (3(x_1 + x_2), 5(x_1 + x_2))$$
$$= (3x_1 + 3x_2, 5x_1 + 5x_2)$$
$$= (3x_1, 5x_1) + (3x_2, 5x_2)$$
$$= f(x_1) + f(x_2)$$
$$f(ax) = \big(3(ax), 5(ax)\big)$$
$$= (3ax, 5ax)$$
$$= a(3x, 5x)$$
$$= af(x)$$

Hence f is a linear transformation.

10. Show that the transformation mapping $f: V_2(R) \to V_2(R)$ defined by $f(x, y) = (x + 6, y + 2)$ is not linear.

$$f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + 6, y_1 + y_2 + 2)$$
$$f(x_1, y_1) + f(x_2, y_2) = (x_1 + 6, y_1 + 2) + (x_2 + 6, y_2 + 2)$$
$$= (x_1 + x_2 + 12, y_1 + y_2 + 4)$$

Therefore, $f(x_1 + x_2, y_1 + y_2) \neq f(x_1, y_1) + f(x_2, y_2)$

Therefore, f is not a linear transformation.

11. Find a linear transformation T: $V_2(R) \to V_2(R)$ such that T(1, 2) = (3, 0) and T(2, 1) = (1, 2).

Let us express (1, 2) and (2, 1) as linear combination of the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$$(1, 2) = 1(1, 0) + 2(0, 1) = 1e_1 + 2e_2$$

$$(2, 1) = 2(1, 0) + 1(0, 1) = 2e_1 + 1e_2$$

Therefore, $T(e_1 + 2e_2) = T(1, 2)$ and $T(2e_1 + e_2) = T(2, 1)$

i.e., $T(e_1) + 2T(e_2) = (3, 0)$ and $2T(e_1) + T(e_2) = (1, 2)$

Solving these, we get

$T(e_1) = \left(\frac{-1}{3}, \frac{4}{3}\right)$ and $T(e_2) = \left(\frac{5}{3}, \frac{-2}{3}\right)$

Now, T (x, y) = T [x (1, 0) + y (0, 1)]

$= T(xe_1 + ye_2)$

$= x\,T(e_1) + y\,T(e_2)$

$= x\left(\frac{-1}{3}, \frac{4}{3}\right) + y\left(\frac{5}{3}, \frac{-2}{3}\right)$

$= \left(\frac{-x}{3} + \frac{5y}{3}, \frac{4x}{3} - \frac{2y}{3}\right)$

$= \left(\frac{-x+5y}{3}, \frac{4x-2y}{3}\right)$

i.e., T (x, y) = $\left(\frac{-x+5y}{3}, \frac{4x-2y}{3}\right)$ is the required liner transformation.

12. Find a linear transformation T: $V_2(R) \to V_3(R)$ such that T(-1, 1) = (-1, 0, 2) and T(2, 1) = (1, 2, 1).

Let us express (–1, 1) and (2, 1) as linear combination of the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

$(-1, 1) = -1(1, 0) + 1(0, 1) = -e_1 + e_2$

$(2, 1) = 2(1, 0) + 1(0, 1) = 2e_1 + e_2$

Therefore, $T(-e_1 + e_2) = T(-1, 1)$ and $T(2e_1 + e_2) = T(2, 1)$

i.e., $-T(e_1) + T(e_2) = (-1, 0, 2)$ and $2T(e_1) + T(e_2) = (1, 2, 1)$

Solving these, we get

$T(e_1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ and $T(e_2) = \left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right)$

Now, T (x, y) = T [x (1, 0) + y (0, 1)]

$= T(xe_1 + ye_2)$

$= x\,T(e_1) + y\,T(e_2)$

$= x\left(\frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\right) + y\left(\frac{-1}{3}, \frac{2}{3}, \frac{5}{3}\right)$

$= \left(\frac{2x-y}{3}, \frac{2x+2y}{3}, \frac{-x+5y}{3}\right)$

i.e., T (x, y) = $\left(\frac{2x-y}{3}, \frac{2x+2y}{3}, \frac{-x+5y}{3}\right)$ is the required liner transformation.

13. Find a linear transformation T: $V_3(R) \to V_2(R)$ such that T(1, 0, 0) = (-1, 0), T(0, 1, 0) = (1, 1) and T(0, 0, 1) = (0, -1).

$e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

$T(e_1) = (-1, 0)$, $T(e_2) = (1, 1)$ and $T(e_3) = (0, -1)$

Now, $T(x, y, z) = T[x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)]$

$= T(xe_1 + ye_2 + ze_3)$

$= x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$

$= x(-1, 0) + y(1, 1) + z(0, -1)$

$= (-x + y, y - z)$

i.e., $T(x, y, z) = (y - x, y - z)$ is the required liner transformation.

14. Find a linear transformation T: $V_3(R) \to V_3(R)$ such that T(1, 1, 1) = (1, 1, 1), T(1, 2, 3) = (-1, -2, -3) and T(1, 1, 2) = (2, 2, 4).

Let us express (1, 1, 1), (1, 2, 3) and (1, 1, 2) as linear combination of the

standard basis vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

$(1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1) = e_1 + e_2 + e_3$

$(1, 2, 3) = 1(1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1) = e_1 + 2e_2 + 3e_3$

$(1, 1, 2) = 1(1, 0, 0) + 1(0, 1, 0) + 2(0, 0, 1) = e_1 + e_2 + 2e_3$

Therefore, $T(1, 1, 1) = T(e_1 + e_2 + e_3)$

$T(1, 2, 3) = T(e_1 + 2e_2 + 3e_3)$

$T(1, 1, 2) = T(e_1 + e_2 + 2e_3)$

i.e., $T(e_1) + T(e_2) + T(e_3) = (1, 1, 1)$, $T(e_1) + 2T(e_2) + 3T(e_3) = (-1, -2, -3)$ and

$T(e_1) + T(e_2) + 2T(e_3) = (2, 2, 4)$

Solving these, we get

$T(e_1) = (4, 5, 8)$, $T(e_2) = (-4, -5, -10)$ and $T(e_3) = (1, 1, 3)$

Now, $T(x, y, z) = T[x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)]$

$= T(xe_1+ye_2+ze_3)$

= x T(e1) + y T(e2) + T(e3)

= x (4, 5, 8) + y (−4, −5, −10) + z (1, 1, 3)

= (4x – 4y + z, 5x – 5y + z, 8x – 10y + 3z)

i.e., T (x, y, z) = (4x – 4y + z, 5x – 5y + z, 8x – 10y + 3z) is the required liner transformation.

15. Let the transformations $T_1: R^3 \to R^2$ such that $T_1(x, y, z) = (4x, 3y - 2z)$ and $T_2: R^2 \to R^2$ such that $T_2(x, y) = (-2x, y)$. Compute T_1T_2 and T_2T_1.

Range of T_2 does not contain the domain of T_1.

Therefore, T_1T_2 is not defined.

Range of T_1 contains the domain of T_2.

Therefore, T_2T_1 is defined.

$T_2T_1(x, y, z) = T_2(T_1(x, y, z)) = T_2(4x, 3y - 2z) = (-8x, 3y - 2z)$

16. Ilustrate with an example that there exisits linear transformations $T_1: R^2 \to R^2$ and $T_2: R^2 \to R^2$ such that $T_2T_1 \neq 0$.

Let us define $T_1: R^2 \to R^2$ by $T_1(x, y) = (0, 4x)$

And $T_2: R^2 \to R^2$ by $T_2(x, y) = (x, 0)$.

Therefore, T_1T_2 and T_2T_1 both are defined.

Also, $(T_2T_1)(x, y) = T_2(T_1(x, y)) = T_2(0, 4x) = (0, 0) = 0(x, y)$

Therefore, $T_2T_1 = 0$.

Also, $T_1T_2(x, y) = T_1(T_2(x, y)) = T_1(x, 0) = (0, 4x) \neq 0(x, y)$

Therefore, $T_1T_2 \neq 0$.