Module 5

Linear Algebra

Introduction of linear algebra related to Computer Science &Engineering.
Elementary row transformationofa matrix, Rank of a matrix. Consistency and Solution of system of linear equations - Gauss-elimination method, Gauss-Jordan method and approximate solution by Gauss-Seidel method. Eigenvalues and Eigenvectors, Rayleigh's power method to find the dominant Eigenvalue and Eigenvector.

Self-Study: Solution of system of equations by Gauss-Jacobi iterative method. Inverse of a square matrix by Cayley- Hamilton theorem.
Applications: Boolean matrix, Network Analysis, Markov Analysis, Critical point of a network system. Optimum solution.
(RBT Levels: L1, L2 and L3).

5.1 Rank of a matrix

Introduction:

- Elementary row operations are $R_i \to R_j$, $R_i \to kR_i$, $R_i \to k_1R_i + K_2R_j$
- A non-zero matrix is in Echelon form if
  (i) All the zero rows are below the non-zero rows.
  (ii) The first non-zero element in each non-zero row lies to the right of the non-zero element in any preceding row.
- In other words, For an Echelon form, all the elements below the first non-zero element of each row should be zero.
- Example:

$$\begin{pmatrix} 2 & 3 & 1 & 0 & 4 \\ 0 & -4 & -5 & 3 & 2 \\ 0 & 0 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -1 & 3 & 4 \\ 0 & 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

are in Echelon form.

$$\begin{pmatrix} 2 & 3 & 1 & 0 & 4 \\ 0 & -4 & -5 & 3 & 2 \\ 0 & 3 & 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -1 & 3 & 4 \\ 0 & 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are not in Echelon form.

- Apply elementary row transformation successively to get the echelon form of the given matrix.
- The rank of the matrix is the number of non-zero rows in the echelon form of the matrix. $Rank\ of\ A = \rho(A) = r$.
- The rank of the matrix is not altered by elementary transformations.

- A matrix and its transpose have the same rank.
- If A is a null matrix then $\rho(A) = 0$.
- If A is a non-singular $n \times n$ matrix then $\rho(A) = n$.

Find the rank of the following matrices by using elementary transformations:

1. $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{4} & \mathbf{2} \\ \mathbf{2} & \mathbf{6} & \mathbf{5} \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - 2R_1$

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix}$

$R_3 \to R_3 - R_1$

$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

This is in echelon form. The number of non-zero rows is 2.

Therefore, the rank of the given matrix is 2.

2. $\begin{pmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-6} & \mathbf{2} & \mathbf{4} \\ \mathbf{-3} & \mathbf{1} & \mathbf{-2} \end{pmatrix}$

Let $A = \begin{pmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & -2 \end{pmatrix}$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 + R_1$

$\sim \begin{pmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$

This is in echelon form. The number of non-zero rows is 2.

Therefore, the rank of the given matrix is 2.

3. $\begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{-1} \\ \mathbf{5} & \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{-4} & \mathbf{5} & \mathbf{12} & \mathbf{-1} \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{pmatrix}$

$R_2 \to 2R_2 - 5R_1, R_3 \to R_3 + 2R_1$

$\sim \begin{pmatrix} 2 & 3 & 4 & -1 \\ 0 & -11 & -20 & 3 \\ 0 & 11 & 20 & -3 \end{pmatrix}$

$R_3 \to R_3 + R_1$

$\sim \begin{pmatrix} 2 & 3 & 4 & -1 \\ 0 & -11 & -20 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

This is in echelon form. The number of non-zero rows is 2.

Therefore, the rank of the given matrix is 2.

4. $\begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{3} \\ \mathbf{3} & \mathbf{9} & \mathbf{12} & \mathbf{3} \\ \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{1} \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{pmatrix}$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - R_1$

$\sim \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -2 \end{pmatrix}$

$R_3 \to 3R_3 - R_2$

$\sim \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

This is in echelon form. The number of non-zero rows is 2.

Therefore, the rank of the given matrix is 2.

5. $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \mathbf{4} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{8} & \mathbf{4} & \mathbf{7} & \mathbf{13} \\ \mathbf{8} & \mathbf{4} & \mathbf{-3} & \mathbf{-1} \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{pmatrix}$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 4R_1, R_4 \to R_4 - 4R_1$

$$\sim \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{pmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 - 3R_1$

$$\sim \begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. The number of non-zero rows is 2.

Therefore, the rank of the given matrix is 2.

6. $\begin{pmatrix} \mathbf{-2} & \mathbf{-1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{-1} \end{pmatrix}$

Let $A = \begin{pmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$

$R_2 \to 2R_2 + R_1, R_3 \to 2R_3 + R_1$

$$\sim \begin{pmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$R_2 \to \frac{R_2}{3}$

$$\sim \begin{pmatrix} -2 & -1 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$R_3 \to R_3 + R_2, R_4 \to R_4 - R_2$

$$\sim \begin{pmatrix} -2 & -1 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{pmatrix} -2 & -1 & -3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. The number of non-zero rows is 3.

Therefore, the rank of the given matrix is 3.

7. $\begin{pmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-2} & \mathbf{-4} \\ \mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{6} & \mathbf{3} & \mathbf{0} & \mathbf{-7} \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$$

$R_3 \to 5R_3 - 4R_2, R_4 \to 5R_4 - 9R_2$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. The number of non-zero rows is 3.

Therefore, the rank of the given matrix is 3.

8. $\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{-3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} & \mathbf{0} \end{pmatrix}$ **(May 22)**

Let $A = \begin{pmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{pmatrix}$$

$R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{pmatrix}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. The number of non-zero rows is 2.

Therefore, the rank of the given matrix is 2.

9. $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} & \mathbf{-4} & \mathbf{6} \\ \mathbf{-1} & \mathbf{-3} & \mathbf{2} & \mathbf{-2} \\ \mathbf{2} & \mathbf{4} & \mathbf{-1} & \mathbf{6} \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{pmatrix}$

$R_2 \to R_2 - 2R_1\,, R_3 \to R_3 + R_1, R_4 \to R_4 - 2R_1$

$\sim \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$

$R_3 \to R_3 + R_2, R_4 \to R_4 - R_1$

$\sim \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}$

$R_3 \leftrightarrow R_4$

$\sim \begin{pmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

This is in echelon form. The number of non-zero rows is 4.

Therefore, the rank of the given matrix is 4.

10. $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{2} & \mathbf{4} & \mathbf{3} & \mathbf{2} \\ \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{6} & \mathbf{8} & \mathbf{7} & \mathbf{5} \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$

$R_2 \to R_2 - 2R_1\,, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{pmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. The number of non-zero rows is 3.

Therefore, the rank of the given matrix is 3.

5.2 Consistency of a system of linear equations

I. Homogeneous system of linear equations

Consistent	Solution exists
Inconsistent	Solution does not exist
Homogeneous system	$a_1x + b_1y + c_1z = 0$ $a_2x + b_2y + c_2z = 0$ $a_3x + b_3y + c_3z = 0$
Matrix form	$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ Or $AX = 0$
Coefficient matrix	$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$
$\rho(A) = 3$	Trivial solution (Unique) $x = 0, y = 0, z = 0$
$\rho(A) < 3$	Non-trivial solution (Infinite) $z = k, y = pk, x = qk$

Homogeneous system of equations is always consistent.

1. **Test for consistency and solve the following system of equations:**

   $\boldsymbol{x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0}$

   Coefficient matrix is

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{pmatrix}$$

   $$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 7R_1$$

   $$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{pmatrix}$$

   $$R_3 \to R_3 - 2R_2$$

   $$\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

   This is in echelon form. Number of non-zero rows is 3.

   $\rho(A) = 3$. Therefore, the given system of equations has trivial solution.

   Therefore, $x = 0, y = 0, z = 0$.

2. Test for consistency and solve the following system of equations:

$\boldsymbol{x+3y-2z=0, 2x-y+4z=0,\ x-11y+14z=0}$

Coefficient matrix is

$$A=\begin{pmatrix}1 & 3 & -2\\ 2 & -1 & 4\\ 1 & -11 & 14\end{pmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim\begin{pmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & -14 & 16\end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim\begin{pmatrix}1 & 3 & -2\\ 0 & -7 & 8\\ 0 & 0 & 0\end{pmatrix}$$

This is in echelon form. Number of non-zero rows is 2.

$\rho(A)=2$. Therefore, the given system of equations has non-trivial solutions.

Reduced system of equations is

$x+3y-2z=0$ ---- (1)

$-7y+8z=0$ ------- (2)

Choose $z=k$ then $y=\frac{8k}{7}$ and $x=-\frac{10}{7}k$

Therefore, $x=-\frac{10}{7}k$, $y=\frac{8k}{7}$, $z=k$

3. Test for consistency and solve the following system of equations:

$$\boldsymbol{x + 3y + 2z = 0, 2x - y + 3z = 0, 3x - 5y + 4z = 0, x + 17y + 4z = 0}$$

Coefficient matrix is

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$

$$\sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. Number of non-zero rows is 2.

$\rho(A) = 2$. Therefore, the given system of equations has non-trivial solutions.

Reduced system of equations is

$x + 3y + 2z = 0$ ---- (1)

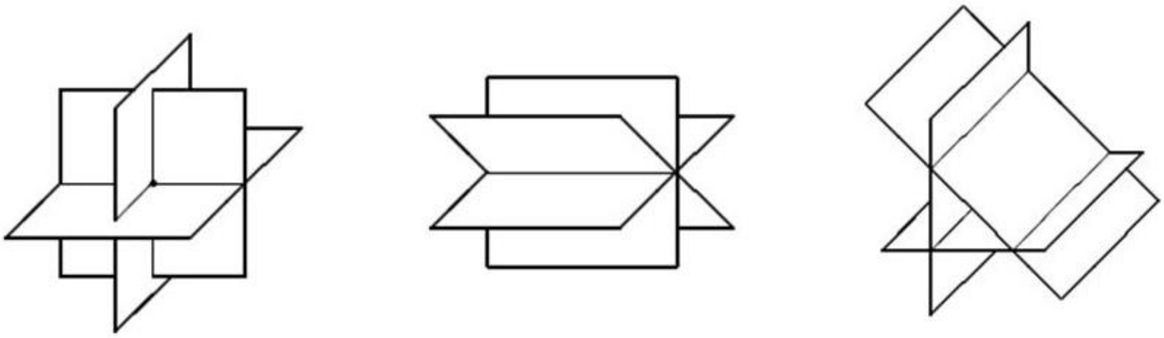
$-7y - z = 0$ ----- (2)

Choose $z = k$ then $y = \frac{-k}{7}$ and $x = -\frac{11}{7}k$

Therefore, $x = -\frac{11}{7}k$, $y = \frac{-k}{7}$, $z = k$

II. Non-homogeneous system of linear equations

<table>
<tr><td>Non-homogeneous system</td><td>$$a_1x + b_1y + c_1z = d_1$$ $$a_2x + b_2y + c_2z = d_2$$ $$a_3x + b_3y + c_3z = d_3$$</td></tr>
<tr><td>Matrix form</td><td>$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$ Or $AX = B$</td></tr>
<tr><td>Augmented matrix</td><td>$$(A, B) = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$$</td></tr>
<tr><td>$\rho(A) = \rho(A, B)$</td><td>Consistent<table><tr><td>$\rho(A) = \rho(A, B) = 3$</td><td>Unique solution</td></tr><tr><td>$\rho(A) = \rho(A, B) < 3$</td><td>Infinite number of solutions</td></tr></table></td></tr>
<tr><td>$\rho(A) \neq \rho(A, B)$</td><td>Inconsistent</td></tr>
</table>



Unique Solution　　Infinite Solutions　　No solutions

4. Test for consistency and solve the following system of equations:

$\boldsymbol{x+2y+3z=1},\ \boldsymbol{2x+3y+8z=2},\ \boldsymbol{x+y+z=3}$

Augmented matrix is

$$(A,B)=\begin{pmatrix}1 & 2 & 3 & 1\\ 2 & 3 & 8 & 2\\ 1 & 1 & 1 & 3\end{pmatrix}$$

$$R_2\to R_2-2R_1, R_3\to R_3-R_1$$

$$\sim\begin{pmatrix}1 & 2 & 3 & 1\\ 0 & -1 & 2 & 0\\ 0 & -1 & -2 & 2\end{pmatrix}$$

$$R_3\to R_3-R_2$$

$$\sim\begin{pmatrix}1 & 2 & 3 & 1\\ 0 & -1 & 2 & 0\\ 0 & 0 & -4 & 2\end{pmatrix}$$

This is in echelon form. Number of non-zero rows is 3.

$\rho(A)=\rho(A,B)=3.$

Therefore, the given system of equations is consistent and has a unique solution.

Reduced system of equations is

$x+2y+3z=1$ ---- (1)

$-y+2z=0$ ---- (2)

$-4z=2$ ---- (3)

Solving by back substitution,

$x=\frac{9}{2}, y=-1, z=-\frac{1}{2}$

5. Test for consistency and solve the following system of equations:

$\mathbf{5x + 3y + 7z = 4}$, $\mathbf{3x + 26y + 2z = 9}$, $\mathbf{7x + 2y + 10z = 5}$

Augmented matrix is

$$(A, B) = \begin{pmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{pmatrix}$$

$R_2 \to 5R_2 - 3R_1, R_3 \to 5R_3 - 7R_1$

$$\sim \begin{pmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{pmatrix}$$

$R_3 \to 11R_3 + R_2$

$$\sim \begin{pmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is in echelon form. Number of non-zero rows is 2.

$\rho(A) = \rho(A, B) = 2.$

Therefore, the given system of equations is consistent

and has an infinite number of solutions.

Reduced system of equations is

$5x + 3y + 7z = 4$ ---- (1)

$121y - 11z = 33$ ---- (2)

Choose $z = k$ then $y = \frac{3+k}{11}$ and $x = \frac{7-16k}{11}$.

Therefore, $x = \frac{7-16k}{11}, y = \frac{3+k}{11}, z = k.$

6. Test for consistency and solve the following system of equations:

$\boldsymbol{x + y + z = -3, 3x + y - 2z = -2, 2x + 4y + 7z = 7}$

Augmented matrix is

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{pmatrix}$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{pmatrix}$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{pmatrix}$$

This is in echelon form. Number of non-zero rows is 3.

$\rho(A) = 2$ *and* $\rho(A, B) = 3.$ $\rho(A) \neq \rho(A, B).$

Therefore, the given system of equations is inconsistent

Therefore, the system has no solution.

7. Test for consistency and solve the following system of equations:

$\boldsymbol{x + 2y + 2z = 5, 2x + y + 3z = 6, 3x - y + 2z = 4, x + y + z = -1}$

Coefficient matrix is

$$(A, B) = \begin{pmatrix} 1 & 2 & 2 & 5 \\ 2 & 1 & 3 & 6 \\ 3 & -1 & 2 & 4 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & -7 & -4 & -11 \\ 0 & -1 & -1 & -6 \end{pmatrix}$$

$$R_3 \to 3R_3 - 7R_2, R_4 \to 3R_4 - R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & -2 & -14 \end{pmatrix}$$

$$R_4 \to 5R_4 - 2R_3$$

$$\sim \begin{pmatrix} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -60 \end{pmatrix}$$

This is in echelon form. Number of non-zero rows is 4.

$\rho(A) = 3$ and $\rho(A, B) = 4.$ $\rho(A) \neq \rho(A, B).$

Therefore, the given system of equations is inconsistent

Therefore, the system has no solution.

8. Find the values of λ and μ for which the system

$$\boldsymbol{x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu}$$

has (i) Unique solution (ii) Infinitely many solutions (iii) No solution. **(May 22)**

Augmented matrix is

$$(A, B) = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{pmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{pmatrix}$$

$$R_3 \to R_3 - R_2$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{pmatrix}$$

This is in echelon form.

(i) If $\lambda \neq 3$, then $\rho(A) = \rho(A, B) = 3$.

The given system of equations has a unique solution.

(ii) If $\lambda = 3, \mu = 10$ then $\rho(A) = \rho(A, B) = 2$.

The given system of equations has infinitely many solutions.

(iii) If $\lambda = 3, \mu \neq 10$ then $\rho(A) \neq \rho(A, B)$.

The given system of equations has no solution.

9. Find the values of λ and μ for which the system

$$\boldsymbol{2x+3y+5z=9,\ 7x+3y-2z=8,\ 2x+3y+\lambda z=\mu}$$

has (i) Unique solution (ii) Infinitely many solutions (iii) No solution.

Coefficient matrix is

$$(A,B)=\begin{pmatrix}2 & 3 & 5 & 9\\ 7 & 3 & -2 & 8\\ 2 & 3 & \lambda & \mu\end{pmatrix}$$

$$R_2 \to 2R_2-7R_1, R_3 \to R_3-R_1$$

$$\sim\begin{pmatrix}1 & 1 & 1 & 6\\ 0 & -15 & -39 & -47\\ 0 & 0 & \lambda-5 & \mu-9\end{pmatrix}$$

This is in echelon form.

(i) If $\lambda \neq 5$, then $\rho(A)=\rho(A,B)=3$.

The given system of equations has a unique solution.

(ii) If $\lambda=5, \mu=9$ then $\rho(A)=\rho(A,B)=2$.

The given system of equations has infinitely many solutions.

(iii) If $\lambda=5, \mu\neq 9$ then $\rho(A)\neq\rho(A,B)$.

The given system of equations has no solution.

5.3 Gauss elimination method

1. **Apply Gauss elimination method to solve the system of equations**

   $\boldsymbol{x - 2y + 3z = 2, 3x - y + 4z = 4, 2\,x + y - 2z = 5.}$

   Augmented matrix is

   $$(A{:}B) = \begin{pmatrix} 1 & -2 & 3 & 2 \\ 3 & -1 & 4 & 4 \\ 2 & 1 & -2 & 5 \end{pmatrix}$$

   $$R_2 \to R_2 - R_1\,,\ R_3 \to R_3 - R_1$$

   $$\sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 5 & -8 & 1 \end{pmatrix}$$

   $$R_3 \to R_3 - R_2$$

   $$\sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

   Reduced system of equations is

   $$x - 2y + 3z = 2$$
   $$5y - 5z = -2$$
   $$-3z = 3$$

   By back substitution,

   $-3z = 3$ gives $z = -1$

   $5y - 5(-1) = -2$ gives $y = -\frac{7}{5}$

   $x - 2\left(-\frac{7}{5}\right) + 3(-1) = 2$ gives $x = \frac{11}{5}$

   Therefore, the solution is

   $x = \frac{11}{5}\ ,\ \ y = -\frac{7}{5},\ z = -1.$

2. **Apply Gauss elimination method to solve the system of equations** $\boldsymbol{3x + y + 2z = 3, 2x - 3y - z = -3, x + 2y + z = 4.}$ (MQP 2)

Augmented matrix is

$$(A{:}B) = \begin{pmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{pmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{pmatrix}$$

$$R_3 \to 7R_2 - 5R_2$$

$$\sim \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & 0 & 8 & -8 \end{pmatrix}$$

Reduced system of equations is

$$x + 2y + z = 4$$

$$-7y - 3z = -11$$

$$8z = -8$$

By back substitution,

$8z = -8$ gives $z = -1$

$-7y - 3(-1) = -11$ gives $y = 2$

$x + 2(2) + (-1) = 4$ gives $x = 1$

Therefore, the solution is

$x = 1$, $y = 2$, $z = -1$.

3. Apply Gauss elimination method to solve the system of equations

$\boldsymbol{x + y + z = 9, x - 2y + 3z = 8, 2x + y - z = 3.}$

Augmented matrix is

$$(A:B) = \begin{pmatrix} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{pmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{pmatrix}$$

$R_3 \to 3R_3 - R_2$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{pmatrix}$$

Reduced system of equations is

$x + 2y + z = 4$

$-7y - 3z = -11$

$8z = -8$

By back substitution,

$8z = -8$ gives $z = -1$

$-7y - 3(-1) = -11$ gives $y = 2$

$x + 2(2) + (-1) = 4$ gives $x = 1$

Therefore, the solution is

$x = 1$, $y = 2$, $z = -1$.

4. Apply Gauss elimination method to solve the system of equations

$\boldsymbol{x+y+z=9, 2x+y-z=0, 2x+5y+7z=52}$ **.**

Augmented matrix is

$$(A:B)=\begin{pmatrix}1 & 1 & 1 & 9\\ 2 & 1 & -1 & 0\\ 2 & 5 & 7 & 52\end{pmatrix}$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 2R_1$

$$\sim\begin{pmatrix}1 & 1 & 1 & 9\\ 0 & -1 & -3 & -18\\ 0 & 3 & 5 & 34\end{pmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\sim\begin{pmatrix}1 & 1 & 1 & 9\\ 0 & -1 & -3 & -18\\ 0 & 0 & -4 & -20\end{pmatrix}$$

Reduced system of equations is

$x+y+z=9$

$-y-3z=-18$

$-4z=-20$

By back substitution,

$-4z=-20$ gives $z=5$

$-y-3(5)=-18$ gives $y=3$

$x+(3)+(5)=9$ gives $x=1$

Therefore, the solution is

$x=1,\ y=3,\ z=5.$

5. Apply Gauss elimination method to solve the system of equations

$\boldsymbol{2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16}$ **.**

Augmented matrix is

$$(A:B) = \begin{pmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$

$$R_2 \to 2R_2 - 3R_1,\ \ R_3 \to 2R_3 - R_1$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 7 & 17 & 22 \end{pmatrix}$$

$$R_3 \to R_3 - 7R_2$$

$$\sim \begin{pmatrix} 2 & 1 & 1 & 10 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & -4 & -20 \end{pmatrix}$$

Reduced system of equations is

$2x + y + z = 10$

$y + 3z = 6$

$-4z = -20$

By back substitution,

$-4z = -20$ gives $z = 5$

$y + 3(5) = 6$ gives $y = -9$

$2x - 9 + 5 = 10$ gives $x = 7$

Therefore, the solution is

$x = 7,\ \ y = -9,\ \ z = 5.$

6. **Apply Gauss elimination method to solve the system of equations** $\boldsymbol{2x - 3y + z = -1, x + 4y + 5z = 25, 3x - 4y + z = 2}$ **.**

Augmented matrix is

$$(A\!:\!B) = \begin{pmatrix} 2 & -3 & 1 & -1 \\ 1 & 4 & 5 & 25 \\ 3 & -4 & 1 & 2 \end{pmatrix}$$

$$R_2 \to 2R_2 - R_1,\ \ R_3 \to 2R_3 - 3R_1$$

$$\sim \begin{pmatrix} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 1 & -1 & 7 \end{pmatrix}$$

$$R_3 \to 11R_3 - R_2$$

$$\sim \begin{pmatrix} 2 & -3 & 1 & -1 \\ 0 & 11 & 9 & 51 \\ 0 & 0 & -20 & 26 \end{pmatrix}$$

Reduced system of equations is

$$2x - 3y + z = -1$$

$$11y + 9z = 51$$

$$-20z = 26$$

By back substitution,

$-20z = 26$ gives $z = -1.3$

$11y + 9(-1.3) = 51$ gives $y = 5.7$

$2x - 3(5.7) - 1.3 = -1$ gives $x = 8.7$

Therefore, the solution is

$x = 8.7$, $y = 5.7$, $z = -1.3$

5.4 Gauss Jordan method

Introduction

In this method, using elementary row transformations, augmented matrix is reduced to the form $\begin{pmatrix} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \end{pmatrix}$.

1. **Apply Gauss Jordan method to solve the system of equations:**
   $\boldsymbol{x+3y+3z=16}$, $\boldsymbol{x+4y+3z=18}$, $\boldsymbol{x+3y+4z=19}$

   The augmented matrix associated to the given system of equations is

$$(A:B) = \begin{pmatrix} 1 & 3 & 3 & 16 \\ 1 & 4 & 3 & 18 \\ 1 & 3 & 4 & 19 \end{pmatrix}$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 3 & 3 & 16 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$R_1 \to R_1 - 3R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 10 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$R_1 \to R_1 - 3R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Therefore, $x = 1, y = 2$, $z = 3$.

2. Solve the system of equations by using the Gauss-Jordan method

$\boldsymbol{x+y+z=10, 2x-y+3z=19, x+2y+3z=22}$

The augmented matrix associated to the given system of equations is

$$(A:B)=\begin{pmatrix}1 & 1 & 1 & 10\\ 2 & -1 & 3 & 19\\ 1 & 2 & 3 & 22\end{pmatrix}$$

$$R_2 \to R_2-2R_1, R_3 \to R_3-R_1$$

$$\sim\begin{pmatrix}1 & 1 & 1 & 10\\ 0 & -3 & 1 & -1\\ 0 & 1 & 2 & 12\end{pmatrix}$$

$$R_2 \to \frac{R_2}{-3}$$

$$\sim\begin{pmatrix}1 & 1 & 1 & 10\\ 0 & 1 & -1/3 & 1/3\\ 0 & 1 & 2 & 12\end{pmatrix}$$

$$R_3 \to R_3-R_2, R_1 \to R_1-R_2$$

$$\sim\begin{pmatrix}1 & 0 & 4/3 & 29/3\\ 0 & 1 & -1/3 & 1/3\\ 0 & 0 & 7/3 & 35/3\end{pmatrix}$$

$$R_3 \to \frac{3}{7}R_3$$

$$\sim\begin{pmatrix}1 & 0 & 4/3 & 29/3\\ 0 & 1 & -1/3 & 1/3\\ 0 & 0 & 1 & 5\end{pmatrix}$$

$$R_2 \to R_2+\frac{1}{3}R_3\ , R_1 \to R_1-\frac{4}{3}R_3$$

$$\sim\begin{pmatrix}1 & 0 & 0 & 3\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & 5\end{pmatrix}$$

Therefore, $x=3,\ y=2\ ,\ z=5$.

3. Solve the system of equations by using Gauss-Jordan method

$\boldsymbol{x + y + z = 9,\ 2x + y - z = 0,\ 2x + 5y + 7z = 52}$

The augmented matrix associated to the given system of equations is

$$(A{:}B) = \begin{pmatrix} 1 & 1 & 1 & 9 \\ 2 & 1 & -1 & 0 \\ 2 & 5 & 7 & 52 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & 34 \end{pmatrix}$$

$R_2 \to -1 \times R_2$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 3 & 18 \\ 0 & 3 & 5 & 34 \end{pmatrix}$$

$R_3 \to R_3 - 3R_2, R_1 \to R_1 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & -2 & -9 \\ 0 & 1 & 3 & 18 \\ 0 & 0 & -4 & -20 \end{pmatrix}$$

$R_3 \to -\frac{1}{4} R_3$

$$\sim \begin{pmatrix} 1 & 0 & -2 & -9 \\ 0 & 1 & 3 & 18 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

$R_2 \to R_2 - 3R_3\ , R_1 \to R_1 + 2R_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

Therefore, $x = 1,\ y = 3\ ,\ z = 5$.

4. Apply Gauss Jordan method to solve the system of equations:

$\boldsymbol{x - 2y + 3z = 2\,, 3x - y + 4z = 4\,, 2x + y - 2z = 5}$

The augmented matrix associated to the given system of equations is

$$(A{:}B) = \begin{pmatrix} 1 & -2 & 3 & 2 \\ 3 & -1 & 4 & 4 \\ 2 & 1 & -2 & 5 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 5 & -8 & 1 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

$R_2 \to \frac{R_2}{5}$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & -2/5 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

$R_1 \to R_1 + 2R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 6/5 \\ 0 & 1 & -1 & -2/5 \\ 0 & 0 & -3 & 3 \end{pmatrix}$$

$R_3 \to \frac{R_3}{-3}$

$$\sim \begin{pmatrix} 1 & 0 & 1 & 6/5 \\ 0 & 1 & -1 & -2/5 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$R_2 \to R_2 + R_3$, $R_1 \to R_1 - R_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 11/5 \\ 0 & 1 & 0 & -7/5 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Therefore, $x = \frac{11}{5}, y = -\frac{7}{5}, z = -1.$

5. Apply Gauss Jordan method to solve the system of equations:
$\boldsymbol{2x - 3y + z = -1\, , x + 4y + 5z = 25\, , 3x - 4y + z = 2}$

The augmented matrix associated to the given system of equations is

$$(A:B) = \begin{pmatrix} 2 & -3 & 1 & -1 \\ 1 & 4 & 5 & 25 \\ 3 & -4 & 1 & 2 \end{pmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{pmatrix} 1 & 4 & 5 & 25 \\ 2 & -3 & 1 & -1 \\ 3 & -4 & 1 & 2 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1\, ,\ R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & 4 & 5 & 25 \\ 0 & -11 & -9 & -51 \\ 0 & -16 & -14 & -73 \end{pmatrix}$$

$R_2 \to \frac{R_2}{-11}$

$$\sim \begin{pmatrix} 1 & 4 & 5 & 25 \\ 0 & 1 & 9/11 & 51/11 \\ 0 & -16 & -14 & -73 \end{pmatrix}$$

$R_1 \to R_1 - 4R_2\, , R_3 \to R_3 + 16R_2$

$$\sim \begin{pmatrix} 1 & 0 & 19/11 & 71/11 \\ 0 & 1 & 9/11 & 51/11 \\ 0 & 0 & -10/11 & 13/11 \end{pmatrix}$$

$R_3 \to -\frac{11}{10}R_3$

$$\sim \begin{pmatrix} 1 & 0 & 19/11 & 71/11 \\ 0 & 1 & 9/11 & 51/11 \\ 0 & 0 & 1 & -13/10 \end{pmatrix}$$

$R_1 \to R_1 - \frac{19}{11}R_3\, , R_2 \to R_2 - \frac{9}{11}R_3$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 87/10 \\ 0 & 1 & 0 & 57/10 \\ 0 & 0 & 1 & -13/10 \end{pmatrix}$$

Therefore, $x = \frac{87}{10}, y = \frac{57}{10}, z = -\frac{13}{10}$.

6. Apply Gauss Jordan method to solve the system of equations:
$\boldsymbol{2x + y + z = 10}$, $\boldsymbol{3x + 2y + 3z = 18}$, $\boldsymbol{x + 4y + 9z = 16}$ **(7, -9, 5)**

The augmented matrix associated to the given system of equations is

$$(A:B) = \begin{pmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{pmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{pmatrix}$$

$$R_2 \to R_2 - 3R_1 \,,\ \ R_3 \to R_3 - 2R_1$$

$$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{pmatrix}$$

$$R_2 \to \frac{R_2}{-10}$$

$$\sim \begin{pmatrix} 1 & 4 & 9 & 16 \\ 0 & 1 & 24/10 & 3 \\ 0 & -7 & -17 & -22 \end{pmatrix}$$

$$R_1 \to R_1 - 4R_2 \,, R_3 \to R_3 + 7R_2$$

$$\sim \begin{pmatrix} 1 & 0 & -6/10 & 4 \\ 0 & 1 & 24/10 & 3 \\ 0 & 0 & -2/10 & -1 \end{pmatrix}$$

$$R_3 \to -5R_3$$

$$\sim \begin{pmatrix} 1 & 0 & -6/10 & 4 \\ 0 & 1 & 24/10 & 3 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

$$R_1 \to R_1 + \frac{6}{10}R_3 \,, R_2 \to R_2 - \frac{24}{10}R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

Therefore, $x = 7, y = -9, z = 5$.

5.4 Gauss Seidel method

Introduction

Consider the system of equations
$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Rearrange this system of equations to a diagonally dominant system.

Rearrange the given system of equations in such a way that a_1, b_2, c_3 must be non zero and numerically largest comparing to other coefficients.

Rewrite this system of equations as
$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$
$$y = \frac{1}{b_2}(d_2 - c_2z - a_2x)$$
$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

Consider the initial approximations as $x_0 = 0, y_0 = 0, z_0 = 0$.

First iteration:
$$x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$
$$y_1 = \frac{1}{b_2}(d_2 - c_2z_0 - a_2x_1)$$
$$z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

Second iteration:
$$x_2 = \frac{1}{a_1}(d_1 - b_1y_1 - c_1z_1)$$
$$y_2 = \frac{1}{b_2}(d_2 - c_2z_1 - a_2x_2)$$
$$z_2 = \frac{1}{c_3}(d_3 - a_3x_2 - b_3y_2)$$

Third iteration:
$$x_3 = \frac{1}{a_1}(d_1 - b_1y_2 - c_1z_2)$$
$$y_3 = \frac{1}{b_2}(d_2 - c_2z_2 - a_2x_3)$$
$$z_3 = \frac{1}{c_3}(d_3 - a_3x_3 - b_3y_3)$$

Therefore, approximate solution of the given system of equations is

$x \cong x_3, y \cong y_3, z \cong z_3$

Accuracy can be improved by increasing number of iterations.

Initial approximation does not affect the solution, it may affect the number of iterations.

1. **Solve the following system of equations by Gauss-Seidel method**

$\mathbf{10x + y + z = 12, x + 10y + z = 12, x + y + 10z = 12}$

Rearrange this system of equations to a diagonally dominant system.

$$10x + y + z = 12$$
$$x + 10y + z = 12$$
$$x + y + 10z = 12$$

Rewrite this system of equations as

$$x = \frac{1}{10}(12 - y - z)$$
$$y = \frac{1}{10}(12 - z - x)$$
$$z = \frac{1}{10}(12 - x - y)$$

Consider the initial approximations as $x_0 = 0, y_0 = 0, z_0 = 0$.

First iteration:

$x_1 = \frac{1}{10}(12 - y_0 - z_0) = \frac{1}{10}(12 - 0 - 0) = 1.2$

$y_1 = \frac{1}{10}(12 - z_0 - x_1) = \frac{1}{10}(12 - 0 - 1.2) = 1.32$

$z_1 = \frac{1}{10}(12 - x_1 - y_1) = \frac{1}{10}(12 - 1.2 - 1.32) = 1.452$

Second iteration:

$x_2 = \frac{1}{10}(12 - y_1 - z_1) = \frac{1}{10}(12 - 1.32 - 1.452) = 1.2$

$y_2 = \frac{1}{10}(12 - z_1 - x_2) = \frac{1}{10}(12 - 1.452 - 1.2) = 0.9348$

$z_2 = \frac{1}{10}(12 - x_2 - y_2) = \frac{1}{10}(12 - 1.2 - 0.9348) = 0.9865$

Third iteration:

$x_3 = \frac{1}{10}(12 - y_2 - z_2) = \frac{1}{10}(12 - 0.9348 - 0.9865) = 1.0079$

$y_3 = \frac{1}{10}(12 - z_2 - x_3) = \frac{1}{10}(12 - 0.9865 - 1.0079) = 1.0005$

$z_3 = \frac{1}{10}(12 - x_3 - y_3) = \frac{1}{10}(12 - 1.0079 - 1.0005) = 0.9992$

Therefore, approximate solution of the given system of equations is

$x \cong 1, y \cong 1, z \cong 1$

2. Solve the following system of equation by Gauss-Seidel method:

$\mathbf{20x + y - 2z = 17}$, $\mathbf{3x + 20y - z = -18}$, $\mathbf{2x - 3y + 20z = 25}$

Rearrange this system of equations to a diagonally dominant system.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Rewrite this system of equations as

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 + z - 3x)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

Consider the initial approximations as $x_0 = 0, y_0 = 0, z_0 = 0$.

First iteration:

$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = \frac{1}{20}(17 - 0 - 0) = 0.85$

$y_1 = \frac{1}{20}(-18 + z_0 - 3x_1) = \frac{1}{20}(-18 + 0 - 3(0.85)) = -1.0285$

$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = \frac{1}{20}(25 - 2(0.85) + 3(-1.0285)) = 1.0107$

Second iteration:

$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = \frac{1}{20}\big(17 + 1.0285 + 2(1.0107)\big) = 1.0025$

$y_2 = \frac{1}{20}(-18 + z_1 - 3x_2) = \frac{1}{20}\big(-18 + 1.0107 - 3(1.0025)\big) = -0.9998$

$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = \frac{1}{20}(25 - 2(1.0025) - 3(0.9998)) = 0.9998$

Third iteration:

$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = \frac{1}{20}\big(17 + 0.9998 + 2(0.9998)\big) = 0.9999$

$y_3 = \frac{1}{20}(-18 + z_2 - 3x_3) = \frac{1}{20}(-18 + 0.9998 - 3(0.9999)) = -0.9999$

$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = \frac{1}{20}\big(25 - 2(0.9999) - 3(0.9999)\big) = 1$

Therefore, approximate solution of the given system of equations is

$x \cong 1, y \cong -1, z \cong 1$

3. Use the Gauss – Seidel iterative method to solve the system of equations

$\boldsymbol{27x + 6y - z = 85}$, $\boldsymbol{6x + 15y + 2z = 72}$, $\boldsymbol{x + y + 54z = 110}$

Carryout four iterations, taking the initial approximation to the solution as $(\mathbf{2}, \mathbf{3}, \mathbf{2})$.

Rearrange this system of equations to a diagonally dominant system.

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Rewrite this system of equations as

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 2z - 6x)$$

$$z = \frac{1}{54}(110 - x - y)$$

Consider the initial approximations as $x_0 = 2, y_0 = 3, z_0 = 2$.

First iteration:

$x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = \frac{1}{27}(85 - 6(3) + 2) = 2.5556$

$y_1 = \frac{1}{15}(72 - 2z_0 - 6x_1) = \frac{1}{15}(72 - 2(2) - 6(2.5556)) = 3.5111$

$z_1 = \frac{1}{54}(110 - x_1 - y_1) = \frac{1}{54}(110 - 2.5556 - 3.5111) = 1.9247$

Second iteration:

$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = \frac{1}{27}\big(85 - 6(3.5111) + (1.9247)\big) = 2.5133$

$y_2 = \frac{1}{15}(72 - 2z_1 - 6x_2) = \frac{1}{15}\big(72 - 2(1.9247) - 6(2.5133)\big) = 3.5381$

$z_2 = \frac{1}{54}(110 - x_2 - y_2) = \frac{1}{54}(110 - 2.5133 - 3.5381) = 1.9250$

Third iteration:

$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = \frac{1}{27}(85 - 6(3.5381\ ) + 1.9250) = 2.4332$

$y_3 = \frac{1}{15}(72 - 2z_2 - 6x_3) = \frac{1}{15}\big(72 - 2(1.9250\ ) - 6(2.4332)\big) = 3.5701$

$z_3 = \frac{1}{54}(110 - x_3 - y_3) = \frac{1}{54}(110 - 2.4332 - 3.5701) = 1.9259$

Therefore, approximate solution of the given system of equations is

$x = 2.4332,\ y = 3.5701,\ z = 1.9259$

4. Apply the Gauss –Seidel iterative method to solve the system of equations

$\mathbf{5x - y = 9, -x + 5y - z = 4, -y + 5z = -6}$

Rearrange this system of equations to a diagonally dominant system.

$$5x - y + 0z = 9$$

$$-x + 5y - z = 4$$

$$0x - y + 5z = -6$$

Rewrite this system of equations as

$$x = \frac{1}{5}(9 + y)$$

$$y = \frac{1}{5}(4 + x + z)$$

$$z = \frac{1}{5}(-6 + y)$$

Consider the initial approximations as $x_0 = 0, y_0 = 0, z_0 = 0$.

First iteration:

$x_1 = \frac{1}{5}(9 + y_0) = \frac{1}{5}(9 + 0) = 1.8$

$y_1 = \frac{1}{5}(4 + z_0 + x_1) = \frac{1}{5}(4 + 0 + 1.8) = 1.16$

$z_1 = \frac{1}{5}(-6 + y_1) = \frac{1}{5}(-6 + 1.16) = -0.968$

Second iteration:

$x_2 = \frac{1}{5}(9 + y_1) = \frac{1}{5}(9 + 1.16) = 2.032$

$y_2 = \frac{1}{5}(4 + z_1 + x_2) = \frac{1}{5}(4 - 0.968 + 2.032) = 1.0128$

$z_2 = \frac{1}{5}(-6 + y_2) = \frac{1}{5}(-6 + 1.0128) = -0.9974$

Third iteration:

$x_3 = \frac{1}{5}(9 + y_2) = \frac{1}{5}(9 + 1.0128) = 2.0026$

$y_3 = \frac{1}{5}(4 + z_2 + x_3) = \frac{1}{5}(4 - 0.9974 + 2.0026) = 1.001$

$z_3 = \frac{1}{5}(-6 + y_3) = \frac{1}{5}(-6 + 1.001) = -0.9998$

Therefore, approximate solution of the given system of equations is

$x \cong 2, y \cong 1, z \cong -1$

5. **Use the Gauss –Seidel iterative method to solve the system of equations**

$\boldsymbol{5x + 2y + z = 12, x + 4y + 2z = 15, x + 2y + 5z = 20}$

Rearrange this system of equations to a diagonally dominant system.

$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20$$

Rewrite this system of equations as

$$x = \frac{1}{5}(12 - 2y - z)$$
$$y = \frac{1}{4}(15 - 2z - x)$$
$$z = \frac{1}{5}(20 - x - 2y)$$

Consider the initial approximations as $x_0 = 0, y_0 = 0, z_0 = 0$.

First iteration:

$x_1 = \frac{1}{5}(12 - 2y_0 - z_0) = \frac{1}{5}(12 - 2(0) - 0) = 2.4$

$y_1 = \frac{1}{4}(15 - 2z_0 - x_1) = \frac{1}{4}(15 - 2(0) - 2.4) = 3.15$

$z_1 = \frac{1}{5}(20 - x_1 - 2y_1) = \frac{1}{5}(20 - 2.4 - 2(3.15)) = 2.26$

Second iteration:

$x_2 = \frac{1}{5}(12 - 2y_1 - z_1) = \frac{1}{5}(12 - 2(3.15) - 2.26) = 0.688$

$y_2 = \frac{1}{4}(15 - 2z_1 - x_2) = \frac{1}{4}(15 - 2(2.26) - 0.688) = 2.488$

$z_2 = \frac{1}{5}(20 - x_2 - 2y_2) = \frac{1}{5}(20 - 0.688 - 2(2.488)) = 2.8672$

Third iteration:

$x_3 = \frac{1}{5}(12 - 2y_2 - z_2) = \frac{1}{5}(12 - 2(2.488) - 2.8672) = 0.8314$

$y_3 = \frac{1}{4}(15 - 2z_2 - x_3) = \frac{1}{4}(15 - 2(2.8672) - 0.8314) = 2.1086$

$z_3 = \frac{1}{5}(20 - x_3 - 2y_3) = \frac{1}{5}\left(20 - 0.8314 - 2(2.1086)\right) = 2.9903$

Therefore, approximate solution of the given system of equations is

$x = 0.8314, y = 2.1086, z = 2.9903$

6. Solve the following system of equations by Gauss Seidel iterative method

$\boldsymbol{x+3y+10z=24,\ 2x+17y+4z=35,\ 28x+4y-z=32}$

Rearrange this system of equations to a diagonally dominant system.

$$28x+4y-z=32$$

$$2x+17y+4z=35$$

$$x+3y+10z=24$$

Rewrite this system of equations as

$$x=\frac{1}{28}(32-4y+z)$$

$$y=\frac{1}{17}(35-4z-2x)$$

$$z=\frac{1}{10}(24-x-3y)$$

Consider the initial approximations as $x_0=0, y_0=0, z_0=0$.

First iteration:

$x_1=\frac{1}{28}(32-4y_0+z_0)=\frac{1}{28}(32-4(0)+0)=1.1429$

$y_1=\frac{1}{17}(35-4z_0-2x_1)\ =\frac{1}{17}\big(35-4(0)-2(1.1429)\big)=1.9244$

$z_1=\frac{1}{10}(24-x_1-3y_1)=\frac{1}{10}\big(24-1.1429-3(1.9244)\big)=1.7084$

Second iteration:

$x_2=\frac{1}{28}(32-4y_1+z_1)=\frac{1}{28}(32-4(1.9244)+1.7084)=0.9290$

$y_2=\frac{1}{17}(35-4z_1-2x_2)\ =\frac{1}{17}\big(35-4(1.7084)-2(0.9290)\big)=1.5476$

$z_2=\frac{1}{10}(24-x_2-3y_2)=\frac{1}{10}\big(24-0.9290-3(1.5476)\big)=1.8428$

Third iteration:

$x_3=\frac{1}{28}(32-4y_2+z_2)=\frac{1}{28}(32-4(1.5476)+1.8428)=0.9876$

$y_3=\frac{1}{17}(35-4z_2-2x_3)\ =\frac{1}{17}\big(35-4(1.8428)-2(0.9876)\big)=1.5090$

$z_3=\frac{1}{10}(24-x_3-3y_3)=\frac{1}{10}\big(24-0.9876-3(1.5090)\big)=1.8485$

Therefore, approximate solution of the given system of equations is

$x=0.9876, y=1.5090, z=1.8485$

5.6 Rayleigh's power method

Introduction:

- This method is useful to find the largest Eigen value and its corresponding Eigen vector.
- If initial Eigen vector is not given, take $[1 \quad 0 \quad 0]^T$ as an initial Eigen vector.

Problems:

1. **Using Rayleigh's power method find the the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}$ by taking $[\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]^{\boldsymbol{T}}$ as initial eigen vector [carry out 6 iterations]. (MQP 1)**

$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0.5 \\ -0.5 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.5 \\ -0.5 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -4 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0.8 \\ -0.8 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.8 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 5.6 \\ 5.21 \\ -5.21 \end{pmatrix} = 5.6\begin{pmatrix} 1 \\ 0.93 \\ -0.93 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.93 \\ -0.93 \end{pmatrix} = \begin{pmatrix} 5.86 \\ 5.74 \\ -5.74 \end{pmatrix} = 5.86\begin{pmatrix} 1 \\ 0.98 \\ -0.98 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.98 \\ -0.98 \end{pmatrix} = \begin{pmatrix} 5.96 \\ 5.9 \\ -5.9 \end{pmatrix} = 5.96\begin{pmatrix} 1 \\ 0.99 \\ -0.99 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0.99 \\ -0.99 \end{pmatrix} = \begin{pmatrix} 5.98 \\ 5.98 \\ -5.98 \end{pmatrix} = 5.98\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

After 6 iterations, largest eigen value is 5.98, The corresponding eigen vector is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

2. Find the largest eigen value and the corresponding eigen vector of $\begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}$ **with the initial approximate eigen vector** $[\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]^{\boldsymbol{T}}$ **(MQP 2)**

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 0.33 \\ -0.33 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0.33 \\ -0.33 \end{pmatrix} = \begin{pmatrix} 4.68 \\ -0.68 \\ 0.68 \end{pmatrix} = 4.68\begin{pmatrix} 1 \\ -0.15 \\ 0.15 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.15 \\ 0.15 \end{pmatrix} = \begin{pmatrix} 6.6 \\ -2.6 \\ 2.6 \end{pmatrix} = 6.6\begin{pmatrix} 1 \\ -0.4 \\ 0.4 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.4 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 7.6 \\ -3.6 \\ 3.6 \end{pmatrix} = 7.6\begin{pmatrix} 1 \\ -0.47 \\ 0.47 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.47 \\ 0.47 \end{pmatrix} = \begin{pmatrix} 7.88 \\ -3.88 \\ 3.88 \end{pmatrix} = 7.88\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 4 \end{pmatrix} = 8\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$$

After 6 iterations, largest eigen value is 8

The corresponding eigen vector is $\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$.

3. Find the largest eigen value and the corresponding eigen vector for

$$\boldsymbol{A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}}$$ **with initial vector $\boldsymbol{(1 \quad 1 \quad 1)^T}$ [Carryout 5 iterations.]**

(July 2021 18MAT11)

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 0 \\ 0.67 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.67 \end{pmatrix} = \begin{pmatrix} 7.34 \\ -2.67 \\ 4.01 \end{pmatrix} = 7.34\begin{pmatrix} 1 \\ -0.36 \\ 0.55 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.36 \\ 0.55 \end{pmatrix} = \begin{pmatrix} 7.82 \\ -3.63 \\ 4.01 \end{pmatrix} = 7.82\begin{pmatrix} 1 \\ -0.46 \\ 0.51 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.46 \\ 0.51 \end{pmatrix} = \begin{pmatrix} 7.94 \\ -3.89 \\ 3.99 \end{pmatrix} = 7.94\begin{pmatrix} 1 \\ -0.49 \\ 0.5 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -0.49 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 7.98 \\ -3.97 \\ 3.99 \end{pmatrix} = 7.98\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$$

After 5 iterations, largest eigen value is 7.98

The corresponding eigen vector is $\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$.

4. Using Rayleigh's power method find the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ by taking $[\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]^T$ as initial eigen vector [carry out 5 iterations]. **(Jan 20, 18MAT11)**

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0 \\ 2 \end{pmatrix} = 2.5\begin{pmatrix} 1 \\ 0 \\ 0.8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 2.8 \\ 0 \\ 2.6 \end{pmatrix} = 2.8\begin{pmatrix} 1 \\ 0 \\ 0.93 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.93 \end{pmatrix} = \begin{pmatrix} 2.93 \\ 0 \\ 2.86 \end{pmatrix} = 2.93\begin{pmatrix} 1 \\ 0 \\ 0.98 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0.98 \end{pmatrix} = \begin{pmatrix} 2.98 \\ 0 \\ 2.96 \end{pmatrix} = 2.98\begin{pmatrix} 1 \\ 0 \\ 0.99 \end{pmatrix}$$

After 5 iterations, largest eigen value is 7.98

The corresponding eigen vector is $\begin{pmatrix} 1 \\ -0.5 \\ 0.5 \end{pmatrix}$.

5. Using Rayleigh's power method find the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ by taking $[\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]^{T}$ as initial eigen vector [carry out 7 iterations]. (MQP 1, 18MAT11)

$$\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25 \\ 1 \\ 2 \end{pmatrix} = 25\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.08 \end{pmatrix} = \begin{pmatrix} 25.2 \\ 1.12 \\ 1.68 \end{pmatrix} = 25.2\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix} = \begin{pmatrix} 25.18 \\ 1.12 \\ 1.72 \end{pmatrix} = 25.18\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix} = \begin{pmatrix} 25.18 \\ 1.12 \\ 1.72 \end{pmatrix} = 25.18\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix}$$

Two consecutive iterations give the same eigen value and eigen vector.

Largest eigen value is 25.18

The corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix}$.

6. Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$. Perform 5 iterations by taking the initial eigen vector as $[\mathbf{1} \quad \mathbf{1} \quad \mathbf{1}]^T$.

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = 4\begin{pmatrix} -0.75 \\ 0.25 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} -0.75 \\ 0.25 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 0.5 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1.5 \\ -1 \end{pmatrix} = 1.5\begin{pmatrix} 0.067 \\ 1 \\ 0.067 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} 0.067 \\ 1 \\ 0.067 \end{pmatrix} = \begin{pmatrix} 2.74 \\ -0.866 \\ 2.134 \end{pmatrix} = 2.74\begin{pmatrix} 1 \\ 0.316 \\ 0.779 \end{pmatrix}$$

After 5 iterations, largest eigen value is 25.18

The corresponding eigen vector is $\begin{pmatrix} 1 \\ 0.04 \\ 0.07 \end{pmatrix}$.

7. Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$ by Rayleigh's power method. (May 22)

Take $(1 \quad 0 \quad 0)^T$ as an initial eigen vector.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.5 \\ -2 \\ 0.5 \end{pmatrix} = 2.5\begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -0.8 \\ 0.2 \end{pmatrix} = \begin{pmatrix} 2.8 \\ -2.8 \\ 1.2 \end{pmatrix} = 2.8\begin{pmatrix} 1 \\ -1 \\ 0.43 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0.43 \end{pmatrix} = \begin{pmatrix} 3 \\ -3.43 \\ 1.86 \end{pmatrix} = 3.43\begin{pmatrix} 0.87 \\ -1 \\ 0.54 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 0.87 \\ -1 \\ 0.54 \end{pmatrix} = \begin{pmatrix} 2.74 \\ -3.41 \\ 2.08 \end{pmatrix} = 3.41\begin{pmatrix} 0.76 \\ -1 \\ 0.65 \end{pmatrix}$$

After 5 iterations, largest eigen value is 3.41

The corresponding eigen vector is $\begin{pmatrix} 0.76 \\ -1 \\ 0.65 \end{pmatrix}$.

8. Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{-3} & \mathbf{2} \\ \mathbf{4} & \mathbf{4} & \mathbf{-1} \\ \mathbf{6} & \mathbf{3} & \mathbf{5} \end{pmatrix}$ by Rayleigh's power method.

Take $(1 \quad 0 \quad 0)^T$ as an initial eigen vector.

$$\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 0.167 \\ 0.667 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.167 \\ 0.667 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.166 \\ 2.336 \\ 8.003 \end{pmatrix} = 8.003\begin{pmatrix} 0.021 \\ 0.292 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.021 \\ 0.292 \\ 1 \end{pmatrix} = \begin{pmatrix} 1.145 \\ 0.252 \\ 6.002 \end{pmatrix} = 6.002\begin{pmatrix} 0.191 \\ 0.042 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.191 \\ 0.042 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.065 \\ -0.068 \\ 6.272 \end{pmatrix} = 6.272\begin{pmatrix} 0.329 \\ -0.011 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}\begin{pmatrix} 0.329 \\ -0.011 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.326 \\ 0.272 \\ 6.941 \end{pmatrix} = 6.941\begin{pmatrix} 0.335 \\ 0.04 \\ 1 \end{pmatrix}$$

After 5 iterations, largest eigen value is 6.941

The corresponding eigen vector is $\begin{pmatrix} 0.335 \\ 0.04 \\ 1 \end{pmatrix}$.