Mathematics II for Computer Science and Engineering stream

(Subject code: BMATS201)

Module 4: Numerical solution of ODE’s

Syllabus:

Introduction to various numerical techniques for handling Computer Science & Engineering applications.

Numerical Solution of Ordinary Differential Equations (ODE’s):

Numerical solution of ordinary differential equations of first order and first degree – Taylor’s series method, Modified Euler’s method, Runge-Kutta method of fourth order and Milne’s predictor-corrector formula (No derivations of formulae). Problems

4.1 Taylor's series method

Working rule:

- Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.
- Find $y'(x_0), y''(x_0), y'''(x_0), etc.$
- Substitute these values in Taylor's series $\boldsymbol{y(x) = y(x_0) + \frac{x-x_0}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \cdots}$

  (At least three non-zero terms must appear in the series.)

1. Find by Taylor's series method the value of y at $\mathbf{x = 0.1}$ and $\mathbf{x = 0.2}$ to 5 places of decimals from $\mathbf{\frac{dy}{dx} = x^2 y - 1, y(0) = 1}$. (July 2017)

By data, $x_0 = 0, y_0 = 1$

$y' = x^2 y - 1$	$y'(0) = -1$
$y'' = 2xy + x^2 y'$	$y''(0) = 0$
$y''' = 2y + 4xy' + x^2 y''$	$y'''(0) = 2$

By Taylor's series,

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(0) + \cdots$$

Put $x_0 = 0, y(0) = 1,\ y'(0) = -1, y''(0) = 0, y'''(0) = 2.$

$$y(x) = 1 + \frac{x}{1!}(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \cdots$$

Therefore, $y(x) = 1 - x + \frac{x^3}{3}$

$y(0.1) = 0.9003,\ \ y(0.2) = 0.8023.$

2. **Employ Taylor's series method to obtain approx. value of y at $\mathbf{x = 0.2}$ for the differential equation $\mathbf{\frac{dy}{dx} = 2y + 3e^x, y(0) = 0}$. Compare the numerical solution obtained with exact solution.** (Jan 2018)

By data, $x_0 = 0,\ y_0 = 0$

$y' = 2y + 3e^x$	$y'(0) = 3$
$y'' = 2y' + 3e^x$	$y''(0) = 9$
$y''' = 2y'' + 3e^x$	$y'''(0) = 21$

By Taylor's series,

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(0) + \cdots$$

Put $x_0 = 0, y(0) = 0,\ y'(0) = 3, y''(0) = 9, y'''(0) = 21.$

$$y(x) = 0 + \frac{x}{1!}(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \cdots$$

Therefore, $y(x) = 3x + 4.5x^2 + 3.5x^3$

$y(0.2) = 0.8110.$

3. **Solve by Taylor's series method the equation $\frac{dy}{dx} = \log xy$ for $y(1.1)$ and $y(1.2)$ given $y(1) = 2$.**

By data, $x_0 = 1,\ y_0 = 2$

$y' = \log x + \log y$	$y'(1) = 0.6932$
$y'' = \frac{1}{x} + \frac{1}{y}y'$	$y''(1) = 1.3466$
$y''' = -\frac{1}{x^2} + \frac{1}{y}y'' + \left(-\frac{1}{y^2}\right)(y')^2$	$y'''(1) = 0.4667$

By Taylor's series:

$$y(x) = y(x_0) + \frac{x - x_0}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(0) + \cdots$$

Put $x_0 = 1, y(1) = 2,\ y'(1) = 0.6932, y''(1) = 1.3466, y'''(1) = 0.4667.$

$$y(x) = 2 + \frac{x - x_0}{1!}(0.6932) + \frac{(x - x_0)^2}{2!}(1.3466) + \frac{(x - x_0)^3}{3!}(0.4667) + \cdots$$

Therefore, $y(x) = 2 + (x-1)0.6931 + (x-1)^2 0.6733 + (x-1)^3(0.4667)$

$y(1.1) = 2.0807, y(1.2) = 2.1692.$

4. **Find an approx. value of y when $x = 0.1$ if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$ using Taylor's series method.**

By data, $x_0 = 0,\ y_0 = 1$

$y' = x - y^2$	$y'(0) = -1$
$y'' = 1 - 2yy'$	$y''(0) = 3$
$y''' = -2yy'' - 2y'^2$	$y'''(0) = -8$
$y^{iv} = -2yy''' - 2y''y' - 4y'y''$	$y^{iv}(0) = 34$

Taylor's series:

$$y(x) = y(x_0) + \frac{x - x_0}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(0) + \cdots$$

Put $x_0 = 0, y(0) = 1,\ y'(0) = -1, y''(0) = 3, y'''(0) = -8.$

$$y(x) = 1 + \frac{x}{1!}(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \frac{x^4}{4!}(34) + \cdots$$

$$y(x) = 1 - x + \frac{3x^2}{2} - \frac{4}{3}x^3 + \frac{17}{6}x^4$$

$y(0.1) = 0.9139$

5. **Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method.**

By data, $x_0 = 1,\ y_0 = 0$

$y' = x + y$	$y'(1) = 1$
$y'' = 1 + y'$	$y''(1) = 2$
$y''' = y''$	$y'''(1) = 2$
$y^{iv} = y'''$	$y^{iv}(1) = 2$

By Taylor's series,

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(0) + \cdots$$

Put $x_0 = 1, y(1) = 0,\ y'(1) = 1, y''(1) = 2, y'''(1) = 2, y^{iv}(1) = 2$

$$y(x) = 0 + \frac{x - 1}{1!}(1) + \frac{(x - 1)^2}{2!}(2) + \frac{(x - 1)^3}{3!}(2) + \frac{(x - 1)^4}{4!}(2) + \cdots$$

$$y(x) = 1 + (x - 1) + (x - 1)^2 - \frac{1}{3}(x - 1)^3 + \frac{1}{12}(x - 1)^4$$

$y(1.1) = 1.1097, y(1.2) = 1.2375$

6. **Evaluate $y(0.1)$ correct to 6 places of decimal by Taylor's series method if $y(x)$ satisfies $y' = xy + 1, y(0) = 1$.**

By data, $x_0 = 0,\ y_0 = 1,$

$y' = xy + 1$	$y'(0) = 1$
$y'' = xy' + y$	$y''(0) = 1$
$y''' = xy'' + 2y'$	$y'''(0) = 2$

By Taylor's series,

$$y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(0) + \cdots$$

Put $x_0 = 0, y(0) = 1,\ y'(0) = 1, y''(0) = 1, y'''(0) = 2$

$$y(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2) + \cdots$$

$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$, $y(0.1) =\ 1.1053.$

7. Solve $y' = 3x + y^2, y(0) = 1$ using Taylor's series method and compute $y(0.1)$

By data, $x_0 = 0,\ y_0 = 1$

$y' = 3x + y^2$	$y'(0) = 1$
$y'' = 3 + 2yy'$	$y''(0) = 5$
$y''' = 2yy'' + 2y'^2$	$y'''(0) = 12$

By Taylor's series,

$$y(x) = y(x_0) + \frac{x - x_0}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(0) + \cdots$$

Put $x_0 = 0, y(0) = 1,\ y'(0) = 1, y''(0) = 5, y'''(0) = 12$

$y(x) = 1 + x + \frac{5x^2}{2} + 2x^3$

$y(0.1) = 1.1272.$

8. Using Taylor's series method find $y(0.1)$ to 3 decimal places given that $\frac{dy}{dx} = e^x - y^2$, $y(0) = 1$

By data, $x_0 = 0,\ y_0 = 1,$

$y' = e^x - y^2$	$y'(0) = 0$
$y'' = e^x - 2yy'$	$y''(0) = 1$
$y''' = e^x - 2y'^2 - 2yy''$	$y'''(0) = -1$

Taylor's series,

$$y(x) = y(x_0) + \frac{x - x_0}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(0) + \cdots$$

Put $x_0 = 0,\ y(0) = 1,\ y'(0) = 0, y''(0) = 1, y'''(0) = -1$

$y(x) = 1 + \frac{x^2}{2} - \frac{x^3}{6}$, $y(0.1) = 1.0048.$

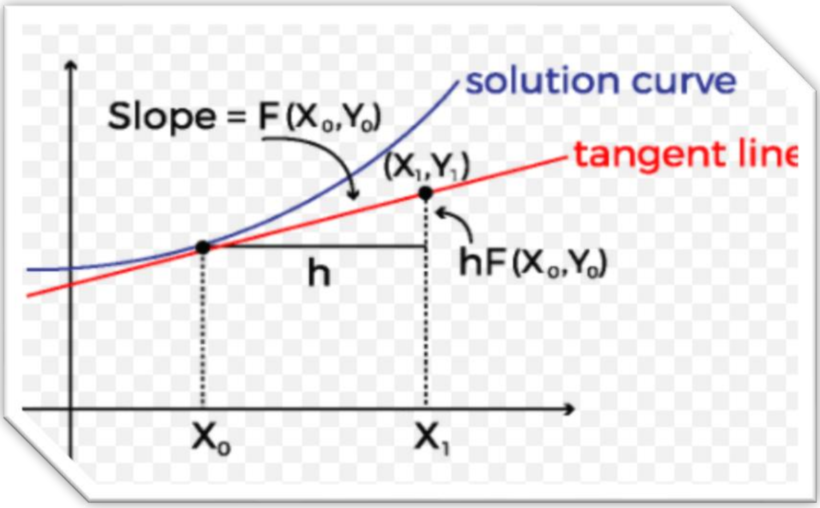
4.2 Modified Euler's method

- Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.
- Find the value of y_1 using Euler's formula $y_1 = y_0 + hf(x_0, y_0)$
- Improve the value of y_1 using Modified Euler's formula

$$\boldsymbol{y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)], y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]}$$

Note 1: $y_1 = y_0 + p = y_0 + h\tan\theta = y_0 + h\left[\frac{dy}{dx}\right]_{(x_0, y_0)} = y_0 + h\, f(x_0, y_0)$

(Accuracy can be improved by using Modified Euler's method. By default, carryout two modifications.)



1. **Using Modified Euler's method, Find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$ by taking step length 0.1**

By data, $f(x, y) = x + y$, $h = 0.1$

Step 1: Find y_1

$x_0 = 0$	$x_1 = 0.1$
$y_0 = 1$	$y_1 = ?$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.1f(0, 1) = 1 + 0.1(0 + 1) = 1.1$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0.1}{2}[f(0, 1) + f(0.1, 1.1)]$$

$$= 1 + 0.05(0 + 1 + 0.1 + 1.1) = 1.115$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.1}{2}[f(0, 1) + f(0.1, 1.115)]$$

$$= 1 + 0.05[0 + 1 + 0.1 + 1.115] = 1.1105$$

$$\boxed{y_1 = y(0.1) = 1.1105}$$

Step 2: Find y_2

$x_1 = 0.1$	$x_2 = 0.2$
$y_1 = 1.1105$	$y_2 = ?$

By Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.1105 + 0.1f(0.1, 1.1105)$$

$$= 1.1105 + 0.1(0.1 + 1.1105) = 1.2316$$

By Modified Euler's formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.1105 + 0.05[0.1 + 1.1105 + 0.2 + 1.2316]$$

$$= 1.2426$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 1.1105 + 0.05[0.1 + 1.1105 + 0.2 + 1.2426]$$

$$= 1.2432$$

$$\boxed{y_2 = y(0.2) = 1.2432}$$

2. **Using the modified Euler's method, Find $\mathbf{y(0.1)}$ given that $\frac{dy}{dx} = x^2 + y$ and $\mathbf{y(0) = 1}$ take step $\boldsymbol{h = 0.05}$ and perform two modifications in each stage.**

By data, $f(x, y) = x^2 + y$, $h = 0.05$

Step 1: Find $\boldsymbol{y_1}$

$x_0 = 0$	$x_1 = 0.05$
$y_0 = 1$	$y_1 =?$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.05f(0,1) = 1 + 0.05(0 + 1) = 1.05$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.05, 1.05)]$$

$$= 1 + 0.025(0^2 + 1 + 0.05^2 + 1.1) = 1.0513$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.05}{2}[f(0,1) + f(0.1, 1.0513)]$$

$$= 1 + 0.05[0^2 + 1 + 0.05^2 + 1.0513] = 1.0514$$

$$\boxed{y_1 = y(0.1) = 1.0514}$$

Step 2: Find $\boldsymbol{y_2}$

$x_1 = 0.05$	$x_2 = 0.1$
$y_1 = 1.0514$	$y_2 =?$

By Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.0514 + 0.05f(0.05, 1.0514)$$

$$= 1.0514 + 0.05(0.05^2 + 1.0514) = 1.1041$$

By Modified Euler's formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.0514 + 0.025[0.05^2 + 1.0514 + 0.1^2 + 1.1041]$$

$$= 1.1056$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 1.0514 + 0.025[0.05^2 + 1.0514 + 0.1^2 + 1.1056]$$

$$= 1.1056$$

$$\boxed{y_2 = y(0.1) = 1.1056}$$

3. Using Modified Euler's method find $y(0.2)$ and $y(0.4)$ given that $y' = y + e^x, y(0) = 0$.

By data, $f(x,y) = y + e^x$, $h = 0.2$

Step 1: Find y_1

$x_0 = 0$	$x_1 = 0.2$
$y_0 = 0$	$y_1 = ?$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 0 + 0.2f(0,0) = 0 + 0.2(0+1) = 0.2$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1)]$$

$$= 0 + \frac{0.2}{2}[f(0,0) + f(0.2,0.2)]$$

$$= 0 + 0.1(0 + 1 + 0.2 + 1.2214) = 0.2421$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f\left(x_1,y_1^{(1)}\right)\right]$$

$$= 0 + \frac{0.2}{2}[f(0,0) + f(0.2,0.2421)]$$

$$= 0 + 0.1[0 + 1 + 0.2421 + 1.2214] = 0.2464$$

$$\boxed{y_1 = y(0.2) = 0.2464}$$

Step 2: Find y_2

$x_1 = 0.2$	$x_2 = 0.4$
$y_1 = 0.2464$	$y_2 = ?$

By Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 0.2464 + 0.2f(0.2, 1.2464)$$

$$= 0.2464 + 0.2(0.2464 + 1.2214) = 0.5400$$

By Modified Euler's formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2,y_2)]$$

$$= 0.2464 + 0.1[0.2464 + 1.2214 + 0.5400 + 1.4918]$$

$$= 0.5968$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f\left(x_2,y_2^{(1)}\right)\right]$$

$$= 0.2464 + 0.1[0.2464 + 1.2214 + 0.5968 + 1.4918]$$

$$= 0.6025$$

$$\boxed{y_2 = y(0.2) = 0.6025}$$

4. Solve the following by Using Modified Euler's method, $y' = \log_{10}(x+y)$, $y(1) = 2$ at $x = 1.2$ and $x = 1.4$

By data, $f(x,y) = \log_{10}(x+y)$, $h = 0.2$

Step 1: Find y_1

$x_0 = 1$	$x_1 = 1.2$
$y_0 = 2$	$y_1 = ?$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + 0.2f(1,2) = 2.0954$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,y_0) + f(x_1,y_1)]$$

$$= 2 + \frac{0.2}{2}[f(1,2) + f(1.2, 2.0954)]$$

$$= 2.0995$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 2 + \frac{0.2}{2}[f(1,2) + f(0.2, 2.0995)]$$

$$= 2 + 0.1[0.3010 + 0.3551] = 2.0995$$

$$\boxed{y_1 = y(1.2) = 2.0995}$$

Step 2: Find y_2

$x_1 = 1.2$	$x_2 = 1.4$
$y_1 = 2.0995$	$y_2 = ?$

By Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 2.0995 + 0.2f(0.2, 2.0995)$$

$$= 2.2032$$

By Modified Euler's formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1,y_1) + f(x_2,y_2)]$$

$$= 2.0995 + 0.1[f(1.2, 2.0995) + f(1.4, 2.2032)]$$

$$= 2.2071$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1,y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 2.0995 + 0.1[f(1.2, 2.0995) + f(1.4, 2.2032)]$$

$$= 2.2071$$

$$\boxed{y_2 = y(0.4) = 2.2071}$$

5. **Solve the following by Using Modified Euler's method, $\frac{dy}{dx} = log\ (x + y), y(1) = 2$ at $x = 1.2$ and $x = 1.4$**

By data, $f(x, y) = log\ (x + y),\ h = 0.2$

Step 1: Find y_1

$x_0 = 1$	$x_1 = 1.2$
$y_0 = 2$	$y_1 = ?$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + 0.2f(1, 2) = 2 + 0.2(1.0986) = 2.2197$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 2 + \frac{0.2}{2}[1.0986 + f(1.2, 2.2197)]$$

$$= 2 + 0.1(1.0986 + 1.2296) = 2.2328$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 2 + \frac{0.2}{2}[f(1, 2) + f(0.2, 2.2328)]$$

$$= 2 + 0.1[1.0986 + 0.8890] = 2.1988$$

$$\boxed{y_1 = y(1.2) = 2.1988}$$

Step 2: Find y_2

$x_1 = 1.2$	$x_2 = 1.4$
$y_1 = 2.1988$	$y_2 = ?$

By Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 2.1988 + 0.2f(0.2, 2.1988)$$

$$= 2.1988 + 0.2(0.8750) = 2.3738$$

By Modified Euler's formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 2.1988 + 0.1[0.8750 + 1.3280]$$

$$= 2.4191$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 2.1988 + 0.1[0.8750 + 1.34]$$

$$= 2.4195$$

$$\boxed{y_2 = y(1.4) = 2.4195}$$

6. Using Modified Euler's method find the solution of the differential equation $\frac{dy}{dx} = x + |\sqrt{y}|$ with initial conditions $y = 1$ at $x = 0$ for the range $0 \le x \le 0.4$ in steps of 0.2

Solution:

By data, $f(x, y) = x + |\sqrt{y}|$, $h = 0.2$

Step 1: Find y_1

$x_0 = 0$	$x_1 = 0.2$
$y_0 = 1$	$y_1 = ?$

By Euler's formula,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.2f(0, 1) = 1 + 0.2(1) = 1.2$$

By Modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$= 1 + \frac{0.2}{2}[1 + f(0.2, 1.2)]$$

$$= 1 + 0.1(1 + 1.2954) = 1.2295$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.2}{2}[1 + f(0.2, 1.2295)]$$

$$= 1 + 0.1[1 + 1.3088] = 1.2309$$

$$\boxed{y_1 = y(0.2) = 1.2309}$$

Step 2: Find y_2

$x_1 = 0.2$	$x_2 = 0.4$
$y_1 = 1.2309$	$y_2 = ?$

By Euler's formula,

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.2309 + 0.2f(0.2, 1.2309)$$

$$= 1.2309 + 0.2(1.3095) = 1.4928$$

By Modified Euler's formula,

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)]$$

$$= 1.2309 + 0.1[1.3095 + 1.6218]$$

$$= 1.5240$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 1.2309 + 0.1[1.3095 + 1.6345\ ]$$

$$= 1.5253$$

$$\boxed{y_2 = y(0.4) = 1.5253}$$

7. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with boundary conditions $y = 1$ when $x = 0$. Find approximately y for $x = 0.1$ by Modified Euler's method. Carryout three modifications.

By data, $f(x, y) = \frac{y-x}{y+x}\, h = 0.1$

Step 1: Find y_1

$x_0 = 0$	$x_1 = 0.1$
$y_0 = 1$	$y_1 = ?$

By Euler's formula,

$y_1 = y_0 + hf(x_0, y_0)$

$= 1 + 0.1f(0, 1) = 1 + 0.1(1) = 1.1$

By Modified Euler's formula,

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$= 1 + \frac{0.1}{2}[1 + f(0.1, 1.1)]$

$= 1 + 0.1\left(1 + \frac{1.1-0.1}{1.1+0.1}\right) = 1.0916$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$

$= 1 + \frac{0.1}{2}[1 + f(0.1, 1.0916)]$

$= 1 + 0.1\left[1 + \frac{1.0916-0.1}{1.0916+0.1}\right] = 1.0916$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$

$= 1 + \frac{0.1}{2}[1 + f(0.1, 1.0916)]$

$= 1 + 0.1\left[1 + \frac{1.0916-0.1}{1.0916+0.1}\right] = 1.0916$

$$\boxed{y_1 = y(0.1) = 1.0916}$$

Home work

8. Find an approximate value of y when $x = 0.1$ using Modified Euler's method, given that $y' = 3x + \frac{y}{2}$ with $y(0) = 1, h = 0.1$
9. Find $y(20.2)$ and $y(20.4)$ using Modified Euler's method, given that $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right)$, $y(20) = 5$, taking $h = 0.2$

10. Using modified Euler's formula, compute $y(1.1)$ correct to five decimal places given that $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y = 1$ at $x = 1$. [Taking $h = 0.1$]
[Aug 21]

11. Solve $y'(x) = 3x + \frac{y}{2}, y(0) = 1$ then find $y(0.2)$ with $h = 0.2$ using modified Euler's method. [Sep 20]]

12. Given $\frac{dy}{dx} = x + \sin y, y(0) = 1$. Compute $y(0.4)$ with $h = 0.2$ using Euler's modified method. *[Jan 20]*
13. Using modified Euler's method to compute $y(0.2)$, given $\frac{dy}{dx} - xy^2 = 0$ under the initial condition $y(0) = 2$. Perform three iteration at each step, taking $h = 0.1$. [MQP 1, 18MAT]

14. Solve the differential equation $\frac{dy}{dx} = x\sqrt{y}$ under the initial condition $y(1) = 1$, by using modified Euler's method at the point $x = 1.4$. Perform three iterations at each step, taking h=0.2. [MQP 2, 18MAT]

4.3 Runge - Kutta method of fourth order

Working Rule:

- Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.
- $y_1 = y_0 + k = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

1. **Apply Runge - Kutta method of fourth order to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.**

By data, $f(x_0, y_0) = x_0 + y_0 = 1, h = 0.2$

$x_0 = 0$	$x_1 = 0.2$
$y_0 = 1$	$y_1 = ?$

$k_1 = hf(x_0, y_0)$

$= 0.2(1) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.2\, f(0.1,\ 1.1)$

$= 0.2(0.1 + 1.1) = 0.24$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.2f(0.1, 1.12)$

$= 0.2(0.1 + 1.12) = 0.244$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2f(0.2, 1.244)$

$= 0.2(0.2 + 1.244) = 0.2888$

$y_1 = y_0 + k$

$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.2 + 0.48 + 0.488 + 0.2888)$

$= 1.2428$

Conclusion: $y_1 = y(0.2) = 1.2428$

2. **Using Runge-Kutta method of fourth order solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$**

By data, $f(x_0, y_0) = f(0,1) = \frac{1-0}{1+0} = 1, h = 0.2$

$x_0 = 0$	$x_1 = 0.2$
$y_0 = 1$	$y_1 = ?$

$k_1 = hf(x_0\,, y_0)$

$= 0.2(1) = 0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right)$

$= 0.2\, f(0.1,\ 1.1)$

$= 0.2\left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right) = 0.1967$

$k_3 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right)$

$= 0.2f(0.1, 1.0984)$

$= 0.2\left(\frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2}\right) = 0.1967$

$k_4 = hf(x_0 + h\,, y_0 + k_3)$

$= 0.2f(0.2, 1.1967)$

$= 0.2\left(\frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2}\right) = 0.1891$

$y_1 = y_0 + k$

$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.2 + 0.3934 + 0.3934 + 0.1891)$

$= 1.1960$

Conclusion: $y_1 = y(0.2) = 1.1960$

3. Apply Runge-Kutta method of fourth order to find an approximate value of y for steps of 0.1 if $\frac{dy}{dx} = x + y^2$ given that $y = 1$, where $x = 0$ at $x = 0.1$

By data, $f(x_0, y_0) = f(0,1) = 0 + 1^2 = 1, h = 0.1$

$x_0 = 0$	$x_1 = 0.1$
$y_0 = 1$	$y_1 = ?$

$k_1 = hf(x_0\,, y_0)$

$\quad = 0.1(1) = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right)$

$\quad = 0.1\, f(0.05,\ 1.05)$

$\quad = 0.1(0.05 + 1.05^2) = 0.1153$

$k_3 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right)$

$\quad = 0.1\, f(0.05, 1.0576)$

$\quad = 0.1\ (0.05 + 1.0576^2) = 0.1168$

$k_4 = hf(x_0 + h\,, y_0 + k_3)$

$\quad = 0.1\, f(0.1, 1.1168)$

$\quad = 0.1\ (0.1 + 1.0576^2) = 0.1347$

$y_1 = y_0 + k$

$\quad = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\quad = 1 + \frac{1}{6}(0.1 + 0.2306 + 0.2332 + 0.1347)$

$\quad = 1.1164$

Conclusion: $y_1 = y(0.1) = 1.1164$

4. **Using Runge-Kutta method of fourth order find $\mathbf{y(0.1)}$ given that $\frac{dy}{dx} = 3x + \frac{y}{2}$, $\mathbf{y(0) = 1}$, taking $\mathbf{h = 0.1}$**

Solution: By data, $f(x_0, y_0) = f(0,1) = 3(0) + \frac{1}{2} = 0.5, h = 0.1$

$x_0 = 0$	$x_1 = 0.1$
$y_0 = 1$	$y_1 = ?$

$k_1 = hf(x_0, y_0)$

$= 0.1(0.5) = 0.05$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$= 0.1\, f(0.05,\ 1.025)$

$= 0.1(0.15 + 0.5125) = 0.0663$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= 0.1\, f(0.05, 1.0332)$

$= 0.1\ (0.15 + 0.5166) = 0.0667$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1\, f(0.1, 1.0667)$

$= 0.1\ (0.3 + 0.5334) = 0.0833$

$y_1 = y_0 + k$

$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}\{0.05 + 2(0.0663) + 2(0.0667) + 0.0833\}$

$= 1.0666$

Conclusion: $y_1 = y(0.1) = 1.0666$

5. **Using Runge-Kutta method of fourth order find $\mathbf{y(0.1)}$ given that $\mathbf{\frac{dy}{dx} = 3e^x + 2y}$, $\mathbf{y(0) = 0}$, taking $\mathbf{h = 0.1}$**

By data, $f(x_0, y_0) = f(0, 0) = 3(1) + 0 = 3, h = 0.1$

$x_0 = 0$	$x_1 = 0.1$
$y_0 = 0$	$y_1 = ?$

$k_1 = hf(x_0\,, y_0)$

$= 0.1(3) = 0.3$

$k_2 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right)$

$= 0.1\, f(0.05,\ 0.15)$

$= 0.1\big(3e^{0.05} + 2(0.15)\big) = 0.3454$

$k_3 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right)$

$= 0.1\, f(0.05, 0.1727)$

$= 0.1\,\big(3e^{0.05} + 2(0.1727)\big) = 0.3499$

$k_4 = hf(x_0 + h\,, y_0 + k_3)$

$= 0.1\, f(0.1, 0.3499)$

$= 0.1\,\big(e^{0.1} + 2(0.3499)\big) = 0.4015$

$y_1 = y_0 + k$

$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}\{0.3 + 2(0.3454) + 2(0.3499) + 0.4015\}$

$= 1.3487$

Conclusion: $y_1 = y(0.1) = 1.3487$

6. **Using Runge-Kutta method of fourth order find $\mathbf{y(0.2)}$ given that $\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{y-x}}{\mathbf{y+x}}$, $\mathbf{y(0.1) = 1.0912}$, taking $\mathbf{h = 0.1}$**

By data, $f(x_0, y_0) = f(0.1, 1.0912) = \frac{1.0912-0.1}{1.0912+0.1} = 0.8321, h = 0.1$

$x_0 = 0.1$	$x_1 = 0.2$
$y_0 = 1.0912$	$y_1 = ?$

$k_1 = hf(x_0\,, y_0)$

$= 0.1\ (0.8321) = 0.0832$

$k_2 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_1}{2}\right)$

$= 0.1\, f(0.15,\ 1.1328)$

$= 0.1 \left(\frac{1.1328-0.15}{1.1328+0.15}\right) = 0.0766$

$k_3 = hf\left(x_0 + \frac{h}{2}\,, y_0 + \frac{k_2}{2}\right)$

$= 0.1\, f(0.15, 1.1295)$

$= 0.1\ \left(\frac{1.1295-0.15}{1.1295+0.15}\right) = 0.0766$

$k_4 = hf(x_0 + h\,, y_0 + k_3)$

$= 0.1\, f(0.2, 1.1678\ )$

$= 0.1\ \left(\frac{1.1678-0.2}{1.1678+0.2}\right) = 0.0708$

$y_1 = y_0 + k$

$= y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1.0912 + \frac{1}{6}\{0{,}0832 + 2(0.0766) + 2(0.0766) + 0.0708\}$

$= 1.1679$

Conclusion: $y_1 = y(0.2) = 1.1679$

Home work:

7. Using Runge-Kutta method of fourth order find $\mathrm{y}(1.1)$ given that $y = 1.2$ when $x = 1$ and $\frac{\mathrm{dy}}{\mathrm{dx}} = 3\mathrm{x} + \mathrm{y}^2$.
8. Using Runge-Kutta method of fourth order find $\mathrm{y}(1.2)$ from $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$ taking $h = 0.1$

4.4 Milne's predictor - corrector method

Working Rule:

- Consider the ordinary differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2 \; and \; y(x_3) = y_3.$
- Find $y_4 = y(x_4)$ using Milne's predictor-corrector formula

$$\boldsymbol{y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3),\;\; y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)}$$

***Note*:**

- In Predictor corrector methods, four prior values are required to find y_4.
- Predictor formula is to predict the value of y_4.
- Corrector formula is to improve the value of y_4.
- Apply corrector formula twice to improve accuracy.

Problems:

1. **Given that $\frac{dy}{dx} = x - y^2$ and the data $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795,$ $y(0.6) = 0.1762$ compute y at $x = 0.8$ by applying Milne's method.**

By data,

x	y	$f = x - y^2$
$x_0 = 0$	$y_0 = 0$	$f_0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$f_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$f_2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$f_3 = 0.5689$

By Milne's predictor formula,

$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) = 0 + \frac{4(0.2)}{3}\big(2(0.1996) - 0.3932 + 2(0.5689)\big) = 0.3049$

$f_4 = x_4 - {y_4}^2 = 0.8 - 0.3049^2 = 0.7070$

By Milne's corrector formula,

$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) = 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7070) = 0.3046$

$f_4 = x_4 - {y_4}^2 = 0.8 - 0.3046^2 = 0.7072$

$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) = 0.0795 + \frac{0.2}{3}(0.3937 + 4(0.5689) + 0.7072) = 0.3046$

Conclusion: $\boldsymbol{y_4 = y(0.4) = 0.3046}$

2. Given $\frac{dy}{dx} = xy + y^2$, $y(0) = 1, y(0.1) = 1.1169$, $y(0.2) = 1.2773$, $y(0.3) = 1.5049$ compute $y(0.4)$ using Milne's method.

By data,

x	y	$f = xy + y^2$
$x_0 = 0$	$y_0 = 1$	$f_0 = 1$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7162$

By Milne's predictor formula,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 1 + \frac{4(0.1)}{3}(2(1.3591) - 1.8869 + 2(2.7162))$$

$$= 1.8352$$

$$f_4 = x_4y_4 + {y_4}^2$$

$$= (0.4)(1.8352) + 1.8352^2$$

$$= 4.1020$$

By Milne's corrector formula,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.2773 + \frac{0.1}{3}(1.8869 + 4(2.7162) + 4.1020)$$

$$= 1.8391$$

$$f_4 = x_4y_4 + {y_4}^2$$

$$= (0.4)(1.8352) + 1.8352^2$$

$$= 4.1179$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.2773 + \frac{0.1}{3}(1.8869 + 4(2.7162) + 4.1179)$$

$$= 1.8397$$

Conclusion: $y_4 = y(0.4) = 1.8397$

3. From the data given below find y at x = 0.4 using Milne's method. $\frac{dy}{dx} = x^2 + \frac{y}{2}$

x	1	1.1	1.2	1.3
y	2	2.2156	2.4549	2.7514

By data,

x	y	$f = x^2 + \frac{y}{2}$
$x_0 = 1$	$y_0 = 2$	$f_0 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$f_1 = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4549$	$f_2 = 2.6675$
$x_3 = 1.3$	$y_3 = 2.7514$	$f_3 = 3.0657$

By Milne's predictor formula,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 2 + \frac{4(0.1)}{3}\big(2(2.3178) - 2.6675 + 2(3.0657)\big)$$

$$= 3.0799$$

$$f_4 \;= x_4^2 + \frac{y_4}{2}$$

$$= 1.4^2 + \frac{3.0799}{2}$$

$$= 3.5$$

By Milne's corrector formula,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 2.4549 + \frac{0.1}{3}(2.6675 + 4(3.0657) + 3.5)$$

$$= 3.0692$$

$$f_4 \;= x_4^2 + \frac{y_4}{2}$$

$$= 1.4^2 + \frac{3.0692}{2}$$

$$= 3.4946$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 2.4549 + \frac{0.1}{3}(2.6675 + 4(3.0657) + 3.4946)$$

$$= 3.0690$$

Conclusion: $\boldsymbol{y_4 = y(1.4) = 3.0690}$

4. If $\frac{dy}{dx} = 2e^x - y,\ y(0) = 2, y(0.1) = 2.010, y(0.2) = 2.04,\ y(0.3) = 2.09$ find $y(0.4)$ using Milne's method.

By data,

x	y	$f = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$f_0 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$f_1 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.04$	$f_2 = 0.4028$
$x_3 = 0.3$	$y_3 = 2.09$	$f_3 = 0.6097$

By Milne's predictor formula,

$$y_4^{(p)} = y_0 + \tfrac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$$= 2 + \tfrac{0.4}{3}(2(0.2003) - 0.4028 + 2(0.6097))$$

$$= 2.1623$$

$$f_4 = 2e^{x_4} - y_4$$

$$= 2e^{0.4} - 2.1623$$

$$= 0.8213$$

By Milne's corrector formula,

$$y_4^{(c)} = y_2 + \tfrac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 2.04 + \tfrac{0.1}{3}(0.4028 + 4(0.6097) + 0.8213)$$

$$= 2.1621$$

$$f_4 = 2e^{x_4} - y_4$$

$$= 2e^{0.4} - 2.1621$$

$$= 0.8215$$

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 2.04 + \frac{0.1}{3}(0.4028 + 4(0.6097) + 0.8213)$$

$$= 2.1621$$

Conclusion: $\boldsymbol{y_4 = y(0.4) = 2.1621}$

5. **Find y if $2\frac{dy}{dx} = (1 + x^2)y^2$ at $x = 0.4$ and $y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12$, $y(0.3) = 1.21$ by Milne's predictor-corrector method.**

By data,

x	y	$f = (1 + x^2)y^2$
$x_0 = 0$	$y_0 = 1$	$f_0 = 0.5$
$x_1 = 0.1$	$y_1 = 1.06$	$f_1 = 0.5674$
$x_2 = 0.2$	$y_2 = 1.12$	$f_2 = 0.6523$
$x_3 = 0.3$	$y_3 = 1.21$	$f_3 = 0.7979$

By Milne's predictor formula,

$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) = 1.2807$

$f_4 = x_4 - {y_4}^2 = 0.9513$

By Milne's corrector formula,

$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) = 1.2798$

$f_4 = x_4 - {y_4}^2 = 0.95$

$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) = 1.2798$

$\boxed{y(1.4) = 1.2798}$

6. **Use Milne's predictor-corrector method to find $y(0.4)$ from $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1, y(0.1) = 1.1113$, $y(0.2) = 1.2507$, $y(0.3) = 1.426$. Apply the corrector formula twice. 1.6876**

Solution: By data,

$x_0 = 0$	$y_0 = 1$	$y_0' = {x_0}^2 + {y_0}^2 = 1$
$x_1 = 0.1$	$y_1 = 1.1113$	$y_1' = {x_1}^2 + {y_1}^2 = 1.2450$
$x_2 = 0.2$	$y_2 = 1.2507$	$y_2' = {x_2}^2 + {y_2}^2 = 1.6043$
$x_3 = 0.3$	$y_3 = 1.426$	$y_3' = {x_3}^2 + {y_3}^2 = 2.1235$

$y_4^{(p)} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') = 1 + \frac{2}{15}(5.1327) = 1.6844$

$y_4' = x_4^2 + {y_4}^2 = 2.9972$

$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') = 1.2507 + \frac{1}{30}(13.0955) = 1.6872$

$y_4' = x_4^2 + {y_4}^2 = 3.0066$

$y_4^{(c)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') = 1.2507 + \frac{1}{30}(13.1049) = 1.6875$

Therefore, $y(1.4) = 1.6875$