Mathematics II for Computer Science and Engineering stream

(Subject code: BMATS201)

Module 1: Vertor Calculus

differential equations.

Module-1: Vector Calculus	(8 hours)
Scalar and vector fields. Gradient, Normal vector to the surface, Directional derivative, divergence and curl of vector fields – physical interpretation, solenoidal and irrotational vectors. Orthogonal Curvilinear coordinates: Scale factors, base vectors, transformation between cartesian and curvilinear systems, Cylindrical polar coordinates, Spherical polar coordinates.	

1.1 Gradient and directional derivative

Scalar and vector fields:

If every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a scalar function. Example: $\phi = x^2 + y^2 + z^2$, $\phi = xy^2z^3$

If every point (x, y, z) of a region R in space there corresponds vector $\vec{A}(x, y, z)$ then $\vec{A}$is called a vector point function. Example: $\vec{A} = x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$, $\vec{A} = xyz\,\hat{\imath} + yz\,\hat{\jmath} + z\,\hat{k}$

Gradient and directional derivative:

- The vector differential operator $\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \Sigma\hat{\imath}\frac{\partial}{\partial x}$
- The Gradient of the scalar point function ϕ is given by

  $grad\ \phi = \nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \sum\hat{\imath}\frac{\partial\phi}{\partial x}$
- Unit vector normal to the surface $\phi = c$ is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$
- Directional derivative of ϕ in the direction of $\vec{d}$ is given by $\nabla\phi.\hat{d}$
- Angle between the two surfaces $\phi_1 = C_1$ and $\phi_2 = C_2$ is given by

  $\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

Problems:

1. **Find $\nabla\phi$ if $\phi = \log\ (x^2 + y^2 + z^2)$**

$\phi = \log\ (x^2 + y^2 + z^2)$ $\frac{\partial \phi}{\partial x} = \frac{2x}{x^2+y^2+z^2}$ $\frac{\partial \phi}{\partial y} = \frac{2y}{x^2+y^2+z^2}$ $\frac{\partial \phi}{\partial z} = \frac{2z}{x^2+y^2+z^2}$	$\nabla\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ $= \hat{\imath}\frac{2x}{x^2+y^2+z^2} + \hat{\jmath}\frac{2y}{x^2+y^2+z^2} + \hat{k}\frac{2z}{x^2+y^2+z^2}$ $= \frac{2}{x^2+y^2+z^2}\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

2. **Find $\nabla\phi$ if $\phi = x^3 + y^3 + z^3 - 3xyz$ at the point $(1, -1, 2)$**

$\phi = x^3 + y^3 + z^3 - 3xyz$	At $(x, y, z) = (1, -1, 2)$
$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz$	$\frac{\partial \phi}{\partial x} = 3(1)^2 - 3(-1)(2) = 9$
$\frac{\partial \phi}{\partial y} = 3y^2 - 3xz$	$\frac{\partial \phi}{\partial y} = 3(-1)^2 - 3(1)(2) = -3$
$\frac{\partial \phi}{\partial z} = 3z^2 - 3xy$	$\frac{\partial \phi}{\partial z} = 3(2)^2 - 3(1)(-1) = 15$

$$\nabla\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = 9\hat{\imath} - 3\hat{\jmath} + 15\hat{k}$$

3. **Find $\nabla\phi$ if $\phi = 3x^2y - y^3z^2$ at the point (1,-2, -1)**

$\phi = 3x^2y - y^3z^2$	At $(x, y, z) = (1, -2, -1)$
$\frac{\partial \phi}{\partial x} = 6xy$	$\frac{\partial \phi}{\partial x} = 6(1)(-2) = -12$
$\frac{\partial \phi}{\partial y} = 3x^2 - 3y^2z^2$	$\frac{\partial \phi}{\partial y} = 3(1)^2 - 3(-2)^2(-1)^2 = -9$
$\frac{\partial \phi}{\partial z} = -2y^3z$	$\frac{\partial \phi}{\partial z} = -2(-2)^3(-1) = -16$

$$\nabla\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}$$

4. Find the unit vector normal to the surface $xy^3z^2 = 4$ at $(-1, -1, 2)$

$\phi = xy^3z^2 - 4$	At $(x, y, z) = (-1, -1, 2)$
$\frac{\partial \phi}{\partial x} = y^3z^2$	$\frac{\partial \phi}{\partial x} = (-1)(2)^2 = -4$
$\frac{\partial \phi}{\partial y} = 3xy^2z^2$	$\frac{\partial \phi}{\partial y} = 3(-1)(-1)^2(2)^2 = -12$
$\frac{\partial \phi}{\partial z} = 2xy^3z$	$\frac{\partial \phi}{\partial z} = 2(-1)(-1)^3(2) = 4$

$$\nabla\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = -4\hat{\imath} - 12\hat{\jmath} + 4\hat{k}$$

$$|\nabla\phi| = 4\left|-\hat{\imath} - 3\hat{\jmath} + \hat{k}\right| = 4\sqrt{(-1)^2 + (-3)^2 + (1)^2}$$

Unit vector normal to the surface is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-4\hat{\imath} - 12\hat{\jmath} + 4\hat{k}}{4\sqrt{11}} = \frac{-\hat{\imath} - 3\hat{\jmath} + \hat{k}}{\sqrt{11}}$$

5. Find the unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$

$\phi = x^3 + y^3 + 3xyz - 3$	At $(x, y, z) = (1, 2, -1)$
$\frac{\partial \phi}{\partial x} = 3x^2 + 3yz$	$\frac{\partial \phi}{\partial x} = 3(1)^2 + 3(2)(-1) = -3$
$\frac{\partial \phi}{\partial y} = 3y^2 + 3xz$	$\frac{\partial \phi}{\partial y} = 3(2)^2 + 3(1)(-1) = 9$
$\frac{\partial \phi}{\partial z} = 3xy$	$\frac{\partial \phi}{\partial z} = 3(1)(2) = 6$

$$\nabla\phi = \hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = -3\hat{\imath} + 9\hat{\jmath} + 6\hat{k} = 3\left(-\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right)$$

$$|\nabla\phi| = 3\sqrt{(-1)^2 + (3)^2 + (2)^2} = 3\sqrt{14}$$

Unit vector normal to the surface is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\left(-\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right)}{3\sqrt{14}} = \frac{-\hat{\imath} + 3\hat{\jmath} + 2\hat{k}}{\sqrt{14}}$$

6. **Find the unit vector normal to the surface $yz + zx + xy = c$ at (-1, 2, 3)**

$\phi = yz + zx + xy - c$	At $(x, y, z) = (1, 2, 3)$
$\frac{\partial\phi}{\partial x} = z + y$	$\frac{\partial\phi}{\partial x} = 3 + 2 = 5$
$\frac{\partial\phi}{\partial y} = z + x$	$\frac{\partial\phi}{\partial y} = 3 + 1 = 4$
$\frac{\partial\phi}{\partial z} = y + x$	$\frac{\partial\phi}{\partial z} = 2 + 1 = 3$

$$\nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$$

Unit vector normal to the surface is given by

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}}{\sqrt{25 + 16 + 9}} = \frac{5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}}{\sqrt{50}}$$

7. **Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at (2,-1, 2) along $2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}$. (MQP 1, Jan 2020)**

$\phi = 4xz^3 - 3x^2y^2z$	At $(x, y, z) = (2, -1, 2)$
$\frac{\partial\phi}{\partial x} = 4z^3 - 6xy^2z$	$\frac{\partial\phi}{\partial x} = 4(2)^3 - 6(2)(-1)^2(2) = 8$
$\frac{\partial\phi}{\partial y} = -6x^2yz$	$\frac{\partial\phi}{\partial y} = -6(2)^2(-1)(2) = 48$
$\frac{\partial\phi}{\partial z} = 12xz^2 - 3x^2y^2$	$\frac{\partial\phi}{\partial z} = 12(2)(2)^2 - 3(2)^2(-1)^2 = 84$

$$\nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 8\hat{\imath} + 48\hat{\jmath} + 84\hat{k}$$

$$\hat{d} = \frac{2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}{7}$$

Directional derivative of ϕ along $\vec{d}$ is

$$\nabla\phi.\hat{d} = \left(8\hat{\imath} + 48\hat{\jmath} + 84\hat{k}\right).\left(\frac{2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}{7}\right)$$

$$= \frac{16 - 144 + 504}{7} = \frac{376}{7} = 53.71$$

8. **Find the directional derivative of $\boldsymbol{\phi = x^2\, yz + 4xz^2}$ at (1, -2, -1) in the direction of the vector $\boldsymbol{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}$.**

$\phi = x^2\, yz + 4xz^2$	At $(x, y, z) = (1, -2, -1)$
$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2$	$\frac{\partial\phi}{\partial x} = 2(1)(-2)(-1) + 4(-1)^2 = 8$
$\frac{\partial\phi}{\partial y} = x^2 z$	$\frac{\partial\phi}{\partial y} = (1)^2(-1) = -1$
$\frac{\partial\phi}{\partial z} = x^2 y + 8xz$	$\frac{\partial\phi}{\partial z} = (1)^2(-2) + 8(1)(-1) = -10$

$$\nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 8\hat{\imath} - \hat{\jmath} - 10\hat{k}$$

$$\hat{d} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}$$

Directional derivative of ϕ along $\vec{d}$ is

$$\nabla\phi.\hat{d} = \left(8\hat{\imath} - \hat{\jmath} - 10\hat{k}\right).\left(\frac{2\hat{\imath} - \hat{\jmath} - 2\hat{k}}{3}\right)$$

$$= \frac{16+1+20}{3} = \frac{37}{3}$$

9. **Find the directional derivative of $\boldsymbol{\phi = x^2 + y^2 + 2z^2}$ at P (1, 2, 3) in the direction of the vector $\boldsymbol{\overrightarrow{PQ} = 4\hat{\imath} - 2\hat{\jmath} + \hat{k}}$.**

$\phi = x^2 + y^2 + 2z^2$	At $(x, y, z) = (1,2,3)$
$\frac{\partial\phi}{\partial x} = 2x$	$\frac{\partial\phi}{\partial x} = 2(1) = 2$
$\frac{\partial\phi}{\partial y} = 2y$	$\frac{\partial\phi}{\partial y} = 2(2) = 4$
$\frac{\partial\phi}{\partial z} = 4z$	$\frac{\partial\phi}{\partial z} = 4(3) = 12$

$$\nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 2\hat{\imath} + 4\hat{\jmath} + 12\hat{k}$$

$$\hat{d} = \frac{4\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{16+4+1}} = \frac{4\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{21}}$$

Directional derivative of ϕ along $\vec{d}$ is

$$\nabla\phi.\hat{d} = \left(2\hat{\imath} + 4\hat{\jmath} + 12\hat{k}\right).\left(\frac{4\hat{\imath} - 2\hat{\jmath} + \hat{k}}{\sqrt{21}}\right)$$

$$= \frac{8-8+12}{\sqrt{21}} = \frac{12}{\sqrt{21}}$$

10. Find the direction derivative of $\boldsymbol{\phi = xy^2 + yz^3}$ at $\boldsymbol{(2, -1, 1)}$ in the direction of the normal to the surface $\boldsymbol{x \log z - y^2 = -4}$ at $\boldsymbol{(-1, 2, 1)}$.

$\phi = xy^2 + yz^3$	At $(x, y, z) = (2, -1, 1)$
$\frac{\partial \phi}{\partial \mathrm{x}} = y^2$	$\frac{\partial \phi}{\partial \mathrm{x}} = (-1)^2 = 1$
$\frac{\partial \phi}{\partial \mathrm{y}} = 2xy + z^3$	$\frac{\partial \phi}{\partial \mathrm{y}} = 2(2)(-1) + 1^3 = -3$
$\frac{\partial \phi}{\partial \mathrm{z}} = 3yz^2$	$\frac{\partial \phi}{\partial \mathrm{z}} = 3(-1)(1)^2 = -3$

$$\nabla\phi = \hat{\mathrm{i}}\frac{\partial \phi}{\partial \mathrm{x}} + \hat{\mathrm{j}}\frac{\partial \phi}{\partial \mathrm{y}} + \hat{\mathrm{k}}\frac{\partial \phi}{\partial \mathrm{z}} = \hat{\imath} - 3\hat{\jmath} - 3\hat{k}$$

$\psi = xy^2 + yz^3$	At $(x, y, z) = (-1, 2,\ 1)$
$\frac{\partial \psi}{\partial \mathrm{x}} = \log z$	$\frac{\partial \psi}{\partial \mathrm{x}} = \log z = \log 1 = 0$
$\frac{\partial \psi}{\partial \mathrm{y}} = -2y$	$\frac{\partial \psi}{\partial \mathrm{y}} = -2(2) = -4$
$\frac{\partial \psi}{\partial \mathrm{z}} = \frac{x}{z}$	$\frac{\partial \psi}{\partial \mathrm{z}} = -\frac{1}{1} = -1$

$$\nabla\psi = \hat{\mathrm{i}}\frac{\partial \psi}{\partial \mathrm{x}} + \hat{\mathrm{j}}\frac{\partial \psi}{\partial \mathrm{y}} + \hat{\mathrm{k}}\frac{\partial \psi}{\partial \mathrm{z}} = -4\hat{\jmath} - \hat{k}$$

$$\hat{d} = \frac{\nabla\psi}{|\nabla\psi|} = \frac{-4\hat{\jmath} - \hat{k}}{\sqrt{16+1}}$$

Direction derivative of ϕ in the direction of the normal to the given surface is

$$\nabla\phi . \hat{d} = \left(\hat{\imath} - 3\hat{\jmath} - 3\hat{k}\right) . \left(\frac{-4\hat{\jmath} - \hat{k}}{\sqrt{17}}\right)$$

$$= \frac{0+12+3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$$

11. Find $\boldsymbol{a, b, c}$ so that the directional derivative of $\boldsymbol{\phi = axy^2 + byz + cz^2x^3}$ at $(\mathbf{1, 2, -1})$ has the maximum magnitude 64 in the direction parallel to the z axis.

$\phi = axy^2 + byz + cz^2x^3$	At $(x, y, z) = (1, 2, -1)$
$\frac{\partial\phi}{\partial x} = ay^2 + 3cx^2z^2$	$\frac{\partial\phi}{\partial x} = a(2)^2 + 3c(1)^2(-1)^2 = 4a + 3c$
$\frac{\partial\phi}{\partial y} = 2axy + bz$	$\frac{\partial\phi}{\partial y} = 2a(1)(2) + b(-1) = 4a - b$
$\frac{\partial\phi}{\partial z} = by + 2czx^3$	$\frac{\partial\phi}{\partial z} = b(2) + 2c(-1)(1)^3 = 2b - 2c$

$$\nabla\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (4a + 3c)\hat{\imath} + (4a - b)\hat{\jmath} + (2b - 2c)\hat{k}$$

By data, $\nabla\phi.\hat{k} = 64$ [$\because$ Direction parallel to the z axis is $\hat{k}$.]

$$\left[(4a + 3c)\hat{\imath} + (4a - b)\hat{\jmath} + (2b - 2c)\hat{k}\right].\hat{k} = 64$$

$$2b - 2c = 64$$

$$b - c = 32$$

Since $\nabla\phi$ is parallel to z axis,

$$4a + 3c = 0, 4a - b = 0$$

By solving $b - c = 32, 4a + 3c = 0, 4a - b = 0$

$$a = 6,\ b = 24,\ c = -8.$$

12. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at $(2, -1, 2)$ (July 2019)

$\phi_1 = x^2 + y^2 + z^2 - 9$	At $(x, y, z) = (2, -1, 2)$
$\frac{\partial \phi_1}{\partial \mathrm{x}} = 2x$	$\frac{\partial \phi_1}{\partial \mathrm{x}} = 2(2) = 4$
$\frac{\partial \phi_1}{\partial \mathrm{y}} = 2y$	$\frac{\partial \phi_1}{\partial \mathrm{y}} = 2(-1) = -2$
$\frac{\partial \phi_1}{\partial \mathrm{z}} = 2z$	$\frac{\partial \phi_1}{\partial \mathrm{z}} = 2(2) = 4$

$$\nabla \phi_1 = \hat{\mathrm{i}} \frac{\partial \phi_1}{\partial \mathrm{x}} + \hat{\mathrm{j}} \frac{\partial \phi_1}{\partial \mathrm{y}} + \hat{\mathrm{k}} \frac{\partial \phi_1}{\partial \mathrm{z}} = 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$$

$\phi_2 = z - x^2 - y^2 + 3$	At $(x, y, z) = (2, -1, 2)$
$\frac{\partial \phi_2}{\partial \mathrm{x}} = -2x$	$\frac{\partial \phi_2}{\partial \mathrm{x}} = -2(2) = -4$
$\frac{\partial \phi_2}{\partial \mathrm{y}} = -2y$	$\frac{\partial \phi_2}{\partial \mathrm{y}} = -2(-1) = 2$
$\frac{\partial \phi_2}{\partial \mathrm{z}} = 1$	$\frac{\partial \phi_2}{\partial \mathrm{z}} = 1$

$$\nabla \phi_2 = \hat{\mathrm{i}} \frac{\partial \phi_2}{\partial \mathrm{x}} + \hat{\mathrm{j}} \frac{\partial \phi_2}{\partial \mathrm{y}} + \hat{\mathrm{k}} \frac{\partial \phi_2}{\partial \mathrm{z}} = -4\hat{\imath} + 2\hat{\jmath} + \hat{k}$$

Angle between the surfaces is given by

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}).(-4\hat{\imath} + 2\hat{\jmath} + \hat{k})}{\sqrt{16+4+16}\sqrt{16+4+1}}$$

$$= \frac{-16-4+4}{6\sqrt{21}} = \frac{-16}{6\sqrt{21}} = \frac{-8}{3\sqrt{21}}$$

Therefore, $\theta = \cos^{1}\left(\frac{8}{3\sqrt{21}}\right)$

13. Find the angle between the surfaces $x^2 + y^2 - z^2 = 4$ and $z = x^2 + y^2 - 13$ at $(2, 1, 2)$ [MQP 2]

$\phi_1 = x^2 + y^2 - z^2 - 4$	At $(x, y, z) = (2, 1, 2)$
$\frac{\partial \phi_1}{\partial x} = 2x$	$\frac{\partial \phi_1}{\partial x} = 2(2) = 4$
$\frac{\partial \phi_1}{\partial y} = 2y$	$\frac{\partial \phi_1}{\partial y} = 2(1) = 2$
$\frac{\partial \phi_1}{\partial z} = -2z$	$\frac{\partial \phi_1}{\partial z} = -2(2) = -4$

$$\nabla\phi_1 = \hat{\imath}\frac{\partial \phi_1}{\partial x} + \hat{\jmath}\frac{\partial \phi_1}{\partial y} + \hat{k}\frac{\partial \phi_1}{\partial z} = 4\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$$

$\phi_2 = z - x^2 - y^2 + 13$	At $(x, y, z) = (2, 1, 2)$
$\frac{\partial \phi_2}{\partial x} = -2x$	$\frac{\partial \phi_2}{\partial x} = -2(2) = -4$
$\frac{\partial \phi_2}{\partial y} = -2y$	$\frac{\partial \phi_2}{\partial y} = -2(1) = -2$
$\frac{\partial \phi_2}{\partial z} = 1$	$\frac{\partial \phi_2}{\partial z} = 1$

$$\nabla\phi_2 = \hat{\imath}\frac{\partial \phi_2}{\partial x} + \hat{\jmath}\frac{\partial \phi_2}{\partial y} + \hat{k}\frac{\partial \phi_2}{\partial z} = -4\hat{\imath} - 2\hat{\jmath} + \hat{k}$$

Angle between the surfaces is given by

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\hat{\imath} + 2\hat{\jmath} - 4\hat{k}).(-4\hat{\imath} - 2\hat{\jmath} + \hat{k})}{\sqrt{16+4+16\sqrt{16+4+1}}}$$

$$= \frac{-16-4-4}{\sqrt{21}} = -\frac{24}{6\sqrt{21}} = -\frac{4}{\sqrt{21}}$$

Therefore, $\theta = \pi - \cos^1\left(\frac{4}{\sqrt{21}}\right)$

14. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at $(1, -2, 1)$.

$\phi_1 = xy^2z - 3x - z^2$	At $(x, y, z) = (1, -2, 1)$
$\frac{\partial\phi_1}{\partial x} = y^2z - 3$	$\frac{\partial\phi_1}{\partial x} = (-2)^2(1) - 3 = 1$
$\frac{\partial\phi_1}{\partial y} = 2xyz$	$\frac{\partial\phi_1}{\partial y} = 2(1)(-2)(1) = -4$
$\frac{\partial\phi_1}{\partial z} = xy^2 - 2z$	$\frac{\partial\phi_1}{\partial z} = (1)(2)^2 - 2(1) = 2$

$$\nabla\phi_1 = \hat{\imath}\frac{\partial\phi_1}{\partial x} + \hat{\jmath}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z} = \hat{\imath} - 4\hat{\jmath} + 2\hat{k}$$

$\phi_2 = 3x^2 - y^2 + 2z - 1$	At $(x, y, z) = (1, -2, 1)$
$\frac{\partial\phi_2}{\partial x} = 6x$	$\frac{\partial\phi_2}{\partial x} = 6(1) = 6$
$\frac{\partial\phi_2}{\partial y} = -2y$	$\frac{\partial\phi_2}{\partial y} = -2(-2) = 4$
$\frac{\partial\phi_2}{\partial z} = 2$	$\frac{\partial\phi_2}{\partial z} = 2$

$$\nabla\phi_2 = \hat{\imath}\frac{\partial\phi_2}{\partial x} + \hat{\jmath}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z} = 6\hat{\imath} + 4\hat{\jmath} + 2\hat{k}$$

Angle between two surfaces is given by

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(\hat{\imath} - 4\hat{\jmath} + 2\hat{k}).(6\hat{\imath} + 4\hat{\jmath} + 2\hat{k})}{\sqrt{1+16+4}\sqrt{36+16+4}}$$

$$= \frac{6-16+4}{\sqrt{21}\sqrt{56}} = -\frac{6}{\sqrt{21}\sqrt{56}}$$

Therefore, $\theta = \pi - \cos^1\left(\frac{3}{7\sqrt{6}}\right)$

15. Show that the surfaces $4x^2y + z^4 = 12$ and $6x^2 - yz = 9x$ intersect orthogonally at the point (1,-1,2)

$\phi_1 = 4x^2y + z^4 - 12$	At $(x, y, z) = (1, -1, 2)$
$\frac{\partial \phi_1}{\partial x} = 8xy$	$\frac{\partial \phi_1}{\partial x} = 8(1)(-1) = -8$
$\frac{\partial \phi_1}{\partial y} = 4x^2$	$\frac{\partial \phi_1}{\partial y} = 4(1)^2 = 4$
$\frac{\partial \phi_1}{\partial z} = 4z^3$	$\frac{\partial \phi_1}{\partial z} = 4(2)^3 = 32$

$$\nabla\phi_1 = \hat{\imath}\frac{\partial \phi_1}{\partial x} + \hat{\jmath}\frac{\partial \phi_1}{\partial y} + \hat{k}\frac{\partial \phi_1}{\partial z} = -8\hat{\imath} + 4\hat{\jmath} + 32\hat{k}$$

$\phi_2 = 6x^2 - yz - 9x$	At $(x, y, z) = (1, -1, 2)$
$\frac{\partial \phi_2}{\partial x} = 12x - 9$	$\frac{\partial \phi_2}{\partial x} = 12(1) - 9 = 3$
$\frac{\partial \phi_2}{\partial y} = -z$	$\frac{\partial \phi_2}{\partial y} = -2$
$\frac{\partial \phi_2}{\partial z} = -y$	$\frac{\partial \phi_2}{\partial z} = 1$

$$\nabla\phi_2 = \hat{\imath}\frac{\partial \phi_2}{\partial x} + \hat{\jmath}\frac{\partial \phi_2}{\partial y} + \hat{k}\frac{\partial \phi_2}{\partial z} = 3\hat{\imath} - 2\hat{\jmath} + \hat{k}$$

$$\nabla\phi_1 . \nabla\phi_2 = \left(-8\hat{\imath} + 4\hat{\jmath} + 32\hat{k}\right).\left(3\hat{\imath} - 2\hat{\jmath} + \hat{k}\right)$$

$$= -24 - 8 + 32 = 0$$

Therefore, given surfaces intersect orthogonally.

16. Find the values of a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ are orthogonal at the point $(1,-1,2)$. (MQP 2)

$\phi_1 = ax^2 - byz - (a+2)x$	At $(x,y,z) = (1,-1,2)$
$\frac{\partial \phi_1}{\partial x} = 2ax - a - 2$	$\frac{\partial \phi_1}{\partial x} = 2a(1) - a - 2 = a - 2$
$\frac{\partial \phi_1}{\partial y} = -bz$	$\frac{\partial \phi_1}{\partial y} = -b(2) = -2b$
$\frac{\partial \phi_1}{\partial z} = -by$	$\frac{\partial \phi_1}{\partial z} = -b(-1) = b$

$$\nabla\phi_1 = \hat{\imath}\frac{\partial \phi_1}{\partial x} + \hat{\jmath}\frac{\partial \phi_1}{\partial y} + \hat{k}\frac{\partial \phi_1}{\partial z} = (a-2)\hat{\imath} - 2b\hat{\jmath} + b\hat{k}$$

$\phi_2 = 4x^2y + z^3 - 4$	At $(x,y,z) = (1,-1,2)$
$\frac{\partial \phi_2}{\partial x} = 8xy$	$\frac{\partial \phi_2}{\partial x} = 8(1)(-1) = -8$
$\frac{\partial \phi_2}{\partial y} = 4x^2$	$\frac{\partial \phi_2}{\partial y} = 4(1)^2 = 4$
$\frac{\partial \phi_2}{\partial z} = 3z^2$	$\frac{\partial \phi_2}{\partial z} = 3(2)^2 = 12$

$$\nabla\phi_2 = \hat{\imath}\frac{\partial \phi_2}{\partial x} + \hat{\jmath}\frac{\partial \phi_2}{\partial y} + \hat{k}\frac{\partial \phi_2}{\partial z} = -8\hat{\imath} + 4\hat{\jmath} + 12\hat{k}$$

By data, The surface $ax^2 - byz = (a+2)x$ passes through the point (1, -1, 2).

Therefore, $a(1)^2 - b(-1)(2) = (a+2)(1) \Rightarrow b = 1$.

By data, $\nabla\phi_1 . \nabla\phi_2 = 0$

$$\left[(a-2)\hat{\imath} - 2b\hat{\jmath} + b\hat{k}\right].\left[-8\hat{\imath} + 4\hat{\jmath} + 12\hat{k}\right] = 0$$

$$(a-2)(-8) - 2b(4) + 12b = 0$$

$$-8a + 4b + 16 = 0$$

$$-2a + b + 4 = 0$$

$$a = 2.5 \quad [\because\ b = 1]$$

Therefore, $a = 2.5$, $b = 1$.

17. Find the angle between the normals to the surface $xy = z^2$ at the points $(4, 1, 2)$ and $(3, 3, -3)$.

$\phi = xy - z^2$	At $(x, y, z) = (4, 1, 2)$	At $(x, y, z) = (3, 3, -3)$
$\frac{\partial\phi}{\partial x} = y$	$\frac{\partial\phi_1}{\partial x} = 1$	$\frac{\partial\phi_2}{\partial x} = 3$
$\frac{\partial\phi}{\partial y} = x$	$\frac{\partial\phi_1}{\partial y} = 4$	$\frac{\partial\phi_2}{\partial y} = 3$
$\frac{\partial\phi}{\partial z} = -2z$	$\frac{\partial\phi_1}{\partial z} = -2(2) = -4$	$\frac{\partial\phi_2}{\partial z} = -2(-3) = 6$

$$\nabla\phi_1 = \hat{\imath}\frac{\partial\phi_1}{\partial x} + \hat{\jmath}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z} = \hat{\imath} + 4\hat{\jmath} - 4\hat{k}$$

$$\nabla\phi_2 = \hat{\imath}\frac{\partial\phi_2}{\partial x} + \hat{\jmath}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z} = 3\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$$

Angle between two surfaces is given by

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(\hat{\imath}+4\hat{\jmath}-4\hat{k}).(3\hat{\imath}+3\hat{\jmath}+6\hat{k},)}{\sqrt{1+16+16}\sqrt{9+9+36}} = \frac{3+12-24}{9\sqrt{22}} = -\frac{1}{\sqrt{22}}$$

$$\theta = \pi - \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

18. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ prove that $grad\, r = \frac{\vec{r}}{r}$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$grad\ \phi = \nabla\phi$$

$$= \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}$$

$$= \frac{1}{r}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) = \frac{\vec{r}}{r}$$

19. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ prove that $\nabla r^n = nr^{n-2}\,\vec{r}$

$$\nabla r^n = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$$
$$= nr^{n-1}\left(\hat{\imath}\frac{\partial r}{\partial x} + \hat{\jmath}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$
$$= nr^{n-1}\left(\frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}\right)$$
$$= nr^{n-2}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$
$$= nr^{n-2}\vec{r}$$

20. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ prove that $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

$$\nabla\left(\frac{1}{r}\right) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$
$$= -\frac{1}{r^2}\left(\hat{\imath}\frac{\partial r}{\partial x} + \hat{\jmath}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$
$$= -\frac{1}{r^2}\left(\frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}\right)$$
$$= -\frac{1}{r^3}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$
$$= -\frac{\vec{r}}{r^3}$$

21. Find the angle between the surfaces $x\, log\, z = y^2 - 1$ and $x^2y = 2 - z$ at $(1, 1, 1)$.

Ans: $\theta = cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$

22. Find the angle between the directions of the normal to the surface $x^2yz = 1 = z^2$ at the points $(-1, 1, 1)$ and $(1, -1, -1)$.

Ans: $\theta = \pi$

1.2 Divergence and curl

Introduction:

- $div\,\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}\right)$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

- $Curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$
- Curl is analogous to rotation.
- Velocity is twice the angular velocity of rotation.

Problems:

1. **If $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ find $div\,\vec{F}$ and $curl\,\vec{F}$.** [July 2019, Jan 2020]

$\vec{F} = (x^3 + y^3 + z^3 - 3xyz)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= \hat{\imath}(3x^2 - 3yz) + \hat{\jmath}(3y^2 - 3xz) + \hat{k}(3z^2 - 3xy)$$

$div\,\vec{F} = \nabla.\vec{F}$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[\hat{\imath}(3x^2 - 3yz) + \hat{\jmath}(3y^2 - 3xz) + \hat{k}(3z^2 - 3xy)\right]$$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

$$= 6x + 6y + 6z$$

$$= 6(x + y + z)$$

$curl\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= \hat{\imath}(-3x + 3x) - \hat{\jmath}(-3y + 3y) + \hat{k}(-3z + 3z)$$

$$= \vec{0}$$

2. **Find $div\vec{F}$ and $curl\vec{F}$ if $\vec{F} = xyz^2\hat{\imath} + xy^2z\hat{\jmath} + x^2yz\hat{k}$.**

$$div\,\vec{F} = \nabla.\vec{F}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[xyz^2\hat{\imath} + xy^2z\hat{\jmath} + x^2yz\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\,(xyz^2) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(x^2yz)$$

$$= yz^2 + 2xyz + x^2y$$

$$curl\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^2 & xy^2z & x^2yz \end{vmatrix}$$

$$= \hat{\imath}(x^2z - xy^2) - \hat{\jmath}(2xyz - 2xyz) + \hat{k}(y^2z - xz^2)$$

3. **If $\vec{F} = \nabla(xy^3z^2)$ find $div\,\vec{F}$ and $curl\,\vec{F}$ at the point $(1, -1,\ 1)$.** [MQP 2]

$$\vec{F} = \nabla(xy^3z^2)$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)xy^3z^2$$

$$= y^3z^2\hat{\imath} + 3xy^2z^2\hat{\jmath} + 2xy^3z\,\hat{k}$$

At $(1, -1, 1)$,

$$\vec{F} = (-1)(1)\hat{\imath} + 3(1)(1)(1)\hat{\jmath} + 2(1)(-1)(1)\,\hat{k} = -\hat{\imath} + 3\hat{\jmath} - 2\,\hat{k}$$

$$div\,\vec{F} = \nabla.\vec{F}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(y^3z^2\hat{\imath} + 3xy^2z^2\hat{\jmath} + 2xy^3z\,\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\,(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$= 0 + 6xyz^2 + 2xy^3$$

At $(1, -1, 1)$,

$$div\,\vec{F} = 6(1)(-1)(1) + 2(1)(-1) = -6 - 2 = -8$$

$$curl\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= \hat{\imath}(6xy^2z - 6xy^2z) - \hat{\jmath}(2y^3z - 2y^3z) + \hat{k}(3y^2z^2 - 3y^2z^2)$$

$$= \vec{0}$$

4. If $\vec{F} = (x+y+1)\hat{\imath} + \hat{\jmath} - (x+y)\hat{k}$, then prove that $\vec{F}.curl\,\vec{F} = 0$.

$$curl\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ x+y+1 & 1 & -\mathrm{x}-\mathrm{y} \end{vmatrix}$$

$$= \hat{\imath}(-1-0) - \hat{\jmath}(-1-0) + \hat{k}(0-1)$$

$$= -\hat{\imath} + \hat{\jmath} - \hat{k}$$

$$\vec{F}.curl\,\vec{F} = \left[(x+y+1)\hat{\imath} + \hat{\jmath} - (x+y)\hat{k}\right].\left[-\hat{\imath} + \hat{\jmath} - \hat{k}\right]$$

$$= -x - y - 1 + 1 + x + y$$

$$= 0$$

5. If $\vec{v} = \vec{w} \times \vec{r}$, where w is a constant vector show that $\vec{w} = \frac{1}{2}(curl\,\vec{v})$.

Let $\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}$ and $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$.

Where $w_1,\ w_2,\ w_3$ are constant vectors.

$$\vec{v} = \vec{w} \times \vec{r}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & \mathrm{y} & \mathrm{z} \end{vmatrix}$$

$$= \hat{\imath}(w_2 z - w_3 y) - \hat{\jmath}(\mathrm{w}_1 z - w_3 x) + \hat{k}(\mathrm{w}_1 y - w_2 x)$$

$$curl\,\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ w_2 z - w_3 y & w_3 x - \mathrm{w}_1 z & \mathrm{w}_1 y - w_2 x \end{vmatrix}$$

$$= \hat{\imath}(w_1 - (-w_1)) - \hat{\jmath}(-\mathrm{w}_2 - w_2) + \hat{k}(\mathrm{w}_3 - (-w_3))$$

$$= 2\left(w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}\right)$$

$$= 2\vec{w}$$

Therefore, $\vec{w} = \frac{1}{2}(curl\,\vec{v})$

6. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ prove that (i) $\nabla.\vec{r} = 3$ (ii) $\nabla \times \vec{r} = 0$.

(i) $\nabla.\vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right) = 1 + 1 + 1 = 3.$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\imath}(0-0) + \hat{\jmath}(0-0) + \hat{k}(0-0) = \vec{0}$

7. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ prove that $\nabla.r^n\vec{r} = (n+3)r^n$

By data, $r^2 = x^2 + y^2 + z^2$

$2r\frac{\partial r}{\partial x} = 2x,\ 2r\frac{\partial r}{\partial y} = 2y,\ 2r\frac{\partial r}{\partial z} = 2z$

$\frac{\partial r}{\partial x} = \frac{x}{r}\ ,\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ,\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla.r^n\vec{r} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).r^n\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= r^n + nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z$

$= 3r^n + nr^{n-1}\left(\frac{x}{r}x + \frac{y}{r}y + \frac{z}{r}z\right)$

$= 3r^n + nr^{n-1}\left(\frac{x^2+y^2+z^2}{r}\right)$

$= 3r^n + nr^{n-1}\left(\frac{r^2}{r}\right)$

$= 3r^n + nr^n$

$= (n+3)r^n$

8. If $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ prove that $\nabla \times r^n\vec{r} = 0$.

$\nabla \times r^n\vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$

$= \hat{\imath}\left(\frac{\partial}{\partial y}r^n z - \frac{\partial}{\partial z}r^n y\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}r^n z - \frac{\partial}{\partial z}r^n x\right) + \hat{k}\left(\frac{\partial}{\partial y}r^n z - \frac{\partial}{\partial z}r^n y\right)$

$= \hat{\imath}\left(nr^{n-1}\frac{\partial r}{\partial y}z - nr^{n-1}\frac{\partial r}{\partial z}y\right) - \hat{\jmath}\left(nr^{n-1}\frac{\partial r}{\partial x}z - nr^{n-1}\frac{\partial r}{\partial z}x\right) +$

$\hat{k}\left(nr^{n-1}\frac{\partial r}{\partial y}z - nr^{n-1}\frac{\partial r}{\partial z}y\right)$

$= nr^{n-1}\left\{\hat{\imath}\left(\frac{\partial r}{\partial y}z - \frac{\partial r}{\partial z}y\right) - \hat{\jmath}\left(\frac{\partial r}{\partial x}z - \frac{\partial r}{\partial z}x\right) + \hat{k}\left(\frac{\partial r}{\partial y}z - \frac{\partial r}{\partial z}y\right)\right\}$

$= nr^{n-1}\left\{\hat{\imath}\left(\frac{y}{r}z - \frac{z}{r}y\right) - \hat{\jmath}\left(\frac{x}{r}z - \frac{z}{r}x\right) + \hat{k}\left(\frac{y}{r}z - \frac{z}{r}y\right)\right\}$

$= nr^{n-2}\{\hat{\imath}(yz - zy) - \hat{\jmath}(xz - zx) + \hat{k}(yz - zy)\} = \vec{0}$

1.3 Solenoidal and irrotational vectors

Introduction:

- $\nabla.\vec{F} = 0 \iff$ is a solenoidal vector.
- $\nabla \times \vec{F} = 0 \iff \vec{F}$ is irrotational.
- Irrotational vector is also known as curl free vector.
- If $\vec{F}$ is a conservative force field then $\vec{F} = \nabla\phi$.

Problems:

1. **Show that the vector $\vec{F} = (-x^2 + yz)\hat{\imath} + (4y - z^2x)\hat{\jmath} + (2xz - 4z)\hat{k}$ is solenoidal.**

$$\nabla.\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[(-x^2 + yz)\hat{\imath} + (4y - z^2x)\hat{\jmath} + (2xz - 4z)\hat{k}]$$

$$= \frac{\partial}{\partial x}(-x^2 + yz) + \frac{\partial}{\partial y}(4y - z^2x) + \frac{\partial}{\partial z}(2xz - 4z)$$

$$= -2x + 4 + 2x - 4$$

$$= 0$$

Therefore, the given vector is solenoidal.

2. **Show that the vector $\vec{V} = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}$ is solenoidal.**

$$\nabla.\vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} + 3x^2y^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0$$

$$= 0$$

Therefore, the given vector is solenoidal.

3. **Find the constant a so that the vector field $\vec{F} = (x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x - az)\hat{k}$ is solenoidal.**

By data, $\nabla.\vec{F} = 0$

$$\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[(x + 3y)\hat{\imath} + (y - 2z)\hat{\jmath} + (x - az)\hat{k}\right] = 0$$

$$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x - az) = 0$$

$$1 + 1 - a = 0$$

Therefore, $a = 2$.

4. **If $\vec{F} = (ax + 3y + 4z)\hat{\imath} + (x - 2y + 3z)\hat{\jmath} + (3x + 2y - z)\hat{k}$ is solenoidal, find $'a'$.**

By data, $\nabla.\vec{F} = 0$

$$\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[(ax + 3y + 4z)\hat{\imath} + (x - 2y + 3z)\hat{\jmath} + (3x + 2y - z)\hat{k}\right] = 0$$

$$\frac{\partial}{\partial x}(ax + 3y + 4z) + \frac{\partial}{\partial y}(x - 2y + 3z) + \frac{\partial}{\partial z}(3x + 2y - z) = 0$$

$a - 2 - 1 = 0$. Therefore, $a = 3$.

5. **Show that the vector $\vec{F} = (z + \sin y)\hat{\imath} + (x\cos y - z)\hat{\jmath} + (x - y)\hat{k}$ is irrotational.**

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z + \sin y & x\cos y - z & x - y \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x\cos y - z)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(x - y) - \frac{\partial}{\partial z}(z + \sin y)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(z + \sin y)\right]$$

$$= \hat{\imath}[-1 + 1] - \hat{\jmath}[1 - 1] + \hat{k}[\cos y - \cos y]$$

$$= \vec{0}$$

Therefore, the given vector is irrotational.

6. **Show that the vector $\vec{F} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$ is irrotational.**

Since $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k},\ r^2 = x^2 + y^2 + z^2$

$$2r\frac{\partial r}{\partial x} = 2x,\ 2r\frac{\partial r}{\partial y} = 2y,\ 2r\frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}\ ,\ \frac{\partial r}{\partial y} = \frac{y}{r}\ ,\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\vec{F} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}\hat{\imath} + \frac{y}{r}\hat{\jmath} + \frac{z}{r}\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix}$$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}\left(\frac{z}{r}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r}\right)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}\left(\frac{z}{r}\right) - \frac{\partial}{\partial z}\left(\frac{x}{r}\right)\right] + \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r}\right)\right]$$

$$= \hat{\imath}\left[-\frac{z}{r^2}\left(\frac{\partial r}{\partial y}\right) + \frac{y}{r^2}\left(\frac{\partial r}{\partial z}\right)\right] - \hat{\jmath}\left[-\frac{z}{r^2}\left(\frac{\partial r}{\partial x}\right) + \frac{x}{r^2}\left(\frac{\partial r}{\partial z}\right)\right] + \hat{k}\left[-\frac{y}{r^2}\left(\frac{\partial r}{\partial x}\right) + \frac{x}{r^2}\left(\frac{\partial r}{\partial y}\right)\right]$$

$$= \hat{\imath}\left[-\frac{z}{r^2}\left(\frac{y}{r}\right) + \frac{y}{r^2}\left(\frac{z}{r}\right)\right] - \hat{\jmath}\left[-\frac{z}{r^2}\left(\frac{x}{r}\right) + \frac{x}{r^2}\left(\frac{z}{r}\right)\right] + \hat{k}\left[-\frac{y}{r^2}\left(\frac{x}{r}\right) + \frac{x}{r^2}\left(\frac{y}{r}\right)\right]$$

$$= \frac{1}{r^3}\left\{\hat{\imath}(-zy + yz) - \hat{\jmath}(-zx + xz) + \hat{k}(-yx + xy)\right\} = \vec{0}$$

Therefore, the given vector is irrotational.

7. **Show that the vector $\vec{F} = \frac{x\hat{\imath}+y\hat{\jmath}}{x^2+y^2}$ is both solenoidal and irrotational. (MQP 1)**

$$\nabla.\vec{F} = \left(\hat{\imath}\frac{\partial}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial}{\partial \mathrm{y}} + \hat{\mathrm{k}}\frac{\partial}{\partial \mathrm{z}}\right).\left(\frac{x\hat{\imath}+y\hat{\jmath}}{x^2+y^2}\right)$$

$$= \frac{\partial}{\partial \mathrm{x}}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial \mathrm{y}}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{1}{(x^2+y^2)^2}(x^2+y^2-x.2x) + \frac{1}{(x^2+y^2)^2}(x^2+y^2-y.2y)$$

$$= \frac{1}{(x^2+y^2)^2}(y^2-x^2+x^2-y^2)$$

$$= 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{\imath}(0-0) - \hat{\jmath}(0-0) + \hat{k}\left[\frac{\partial}{\partial \mathrm{x}}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial \mathrm{y}}\left(\frac{x}{x^2+y^2}\right)\right]$$

$$= \hat{k}\left[\frac{-y.2x}{(x^2+y^2)^2} - \frac{-x.2y}{(x^2+y^2)^2}\right]$$

$$= \hat{k}\left[\frac{1}{(x^2+y^2)^2}(-2xy+2xy)\right]$$

$$= \vec{0}$$

Therefore, the given vector is both irrational and solenoidal.

8. **Find the constants $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\vec{\boldsymbol{F}} = (\boldsymbol{axy} + \boldsymbol{z^3})\hat{\boldsymbol{\imath}} + (\boldsymbol{3x^2} - \boldsymbol{z})\hat{\boldsymbol{\jmath}} + (\boldsymbol{bxz^2} - \boldsymbol{y})\hat{\boldsymbol{k}}$ is a conservative force field and find the scalar potential.** [Jan 2020]

$\vec{F}$ is a conservative force field.

$\therefore\ \nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = \vec{0}$$

$\hat{\imath}(-1+1) - \hat{\jmath}(bz^2 - 3z^2) + \hat{k}(6x - ax) = \vec{0}$

$-\hat{\jmath}(b-3)z^2 + \hat{k}(6-a)x = \vec{0}$

Equating components,

$b = 3$ and $a = 6$

To find: Scalar potential such that $\vec{\boldsymbol{F}} = \nabla\phi$

$(axy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (bxz^2 - y)\hat{k} = \hat{\imath}\frac{\partial\phi}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial\phi}{\partial \mathrm{y}} + \hat{k}\frac{\partial\phi}{\partial \mathrm{z}}$

$\frac{\partial\phi}{\partial \mathrm{x}} = 6xy + z^3$

On integrating,

$\phi = 6y\left(\frac{x^2}{2}\right) + xz^3 + f_1(y,z)$

$= 3x^2y + xz^3 + f_1(y,z)$ ------ (1)

$\frac{\partial\phi}{\partial \mathrm{y}} = 3x^2 - z$

On integrating,

$\phi = 3x^2y - yz + f_2(x,z)$ ------ (2)

$\frac{\partial\phi}{\partial \mathrm{z}} = 3xz^2 - y$

On integrating,

$\phi = 3x\left(\frac{z^3}{3}\right) - yz + f_3(x,y)$

$\phi = xz^3 - yz + f_3(x,y)$ ------ (3)

Combining (1), (2) and (3)

$\phi = 3x^2y + xz^3 - yz + c$

9. Find the values of $\boldsymbol{a, b, c}$ such that $\vec{\boldsymbol{F}} = (\boldsymbol{axy + bz^3})\hat{\boldsymbol{\imath}} + (\boldsymbol{3x^2 - cz})\hat{\boldsymbol{\jmath}} + (\boldsymbol{3xz^2 - y})\hat{\boldsymbol{k}}$ is irrotational, also find the scalar potential $\boldsymbol{\phi}$ such that $\vec{\boldsymbol{F}} = \boldsymbol{\nabla\phi}$.

[MQP 2, July 2019]

$\vec{F}$ is irrotational.

$\therefore\ \nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial \mathrm{x}} & \frac{\partial}{\partial \mathrm{y}} & \frac{\partial}{\partial \mathrm{z}} \\ axy + bz^3 & 3x^2 - cz & 3xz^2 - y \end{vmatrix} = \vec{0}$$

$\hat{\imath}(-1 + c) - \hat{\jmath}(3z^2 - 3bz^2) + \hat{k}(6x - ax) = \vec{0}$

$\hat{\imath}(-1 + c) - \hat{\jmath}(3 - 3b)z^2 + \hat{k}(6 - a)x = \vec{0}$

Equating components,

$c = 1,\ b = 1$ and $a = 6$

To find: Scalar potential

$\vec{\boldsymbol{F}} = \nabla\phi$

$(6xy + z^3)\hat{\imath} + (3x^2 - z)\hat{\jmath} + (3xz^2 - y)\hat{k} = \hat{\imath}\frac{\partial\phi}{\partial \mathrm{x}} + \hat{\jmath}\frac{\partial\phi}{\partial \mathrm{y}} + \hat{k}\frac{\partial\phi}{\partial \mathrm{z}}$

$\frac{\partial\phi}{\partial \mathrm{x}} = 6xy + z^3$

On integrating,

$\phi = 6y\left(\frac{x^2}{2}\right) + xz^3 + f_1(y, z)$

$\phi = 3x^2y + xz^3 + f_1(y, z)$ ------ (1)

$\frac{\partial\phi}{\partial \mathrm{y}} = 3x^2 - z$

On integrating,

$\phi = 3x^2y - yz + f_2(x, z)$ ------ (2)

$\frac{\partial\phi}{\partial \mathrm{z}} = 3xz^2 - y$

On integrating,

$\phi = 3x\left(\frac{z^3}{3}\right) - yz + f_3(x, y)$

$\phi = xz^3 - yz + f_3(x, y)$ ------ (3)

Combining (1), (2) and (3)

$\phi = 3x^2y + xz^3 - yz + c$

10. Show that $\vec{F} = (2xy^2 + yz)\hat{\imath} + (2x^2y + xz + 2yz^2)\hat{\jmath} + (2y^2z + xy)\hat{k}$ is a conservative force field and find the scalar potential.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + yz & 2x^2y + xz + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$= \hat{\imath}(4yz + x - x - 4yz) - \hat{\jmath}(y - y) + \hat{k}(4xy + z - 4xy - z)$$

$$= \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(0)$$

$$= \vec{0}$$

Therefore, $\vec{F}$ is a conservative force field.

To find: Scalar potential

$$\vec{\boldsymbol{F}} = \nabla\phi$$

$$(2xy^2 + yz)\hat{\imath} + (2x^2y + xz + 2yz^2)\hat{\jmath} + (2y^2z + xy)\hat{k} = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy^2 + yz$$

On integrating,

$\phi = x^2y^2 + xyz + f_1(y, z)$ ----- (1)

$$\frac{\partial\phi}{\partial y} = 2x^2y + xz + 2yz^2$$

On integrating,

$\phi = x^2y^2 + xyz + f_2(x, z)$ ----- (2)

$$\frac{\partial\phi}{\partial z} = 2y^2z + xy$$

On integrating,

$\phi = y^2z^2 + xyz + f_3(x, y)$ ------ (3)

Combining (1), (2) and (3)

$$\phi = x^2y^2 + y^2z^2 + xyz + c$$

11. **Show that $\vec{f} = 2xyz^3\,\hat{\imath} + x^2z^3\hat{\jmath} + 3x^2y\,z^2\hat{k}$ is irrotational and find ϕ such that $\vec{f} = \nabla\phi$.**

Ans: $curl\,\vec{f} = 0, \vec{f}$ is irrotational, $\phi = x^2yz^3$

12. Find the value of the constant $a$such that $\vec{F} = (axy - z^3)\hat{\imath} + (a - 2)x^2\hat{\jmath} + (1 - a)xz^2\,\hat{k}$ is irrotational and hence find a scalar potential ϕ such that $\vec{F} = \nabla\phi$.

Ans: $a = 4, \phi = 2x^2y - xz^3$

13. Show that $\vec{F} = (z + \sin y)\hat{\imath} + (x \cos y - z)x^2\hat{\jmath} + (x - y)\,\hat{k}$ is irrotational and hence find a scalar potential ϕ such that $\vec{F} = \nabla\phi$.

Ans: $curl\,\vec{f} = 0, \vec{f}$ is irrotational, $\phi = x\sin y - zy + xz$

14. Show that $\vec{F} = (y + z)\hat{\imath} + (z + x)\hat{\jmath} + (x + y)\,\hat{k}$ is irrotational and hence find a scalar potential ϕ such that $\vec{F} = \nabla\phi$.

Ans: $curl\,\vec{f} = 0, \vec{f}$ is irrotational, $\phi = xy + yz + zx$

1.4 Curvilinear coordinates

Introduction:

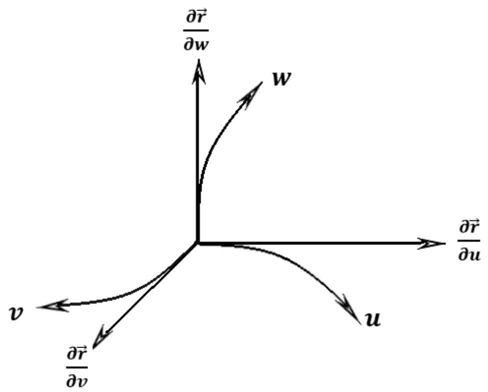
$P(x, y, z) -$ Coordinates of any point P in the Cartesian system.

$P(u, v, w) -$ Coordinates of any point P in the Curvilinear system.

$\vec{r}(x, y, z) -$ Position vector of P in the Cartesian system.

$\vec{r}(u, v, w) -$ Position vector of P in the Curvilinear system.

Orthogonal curvilinear coordinates:

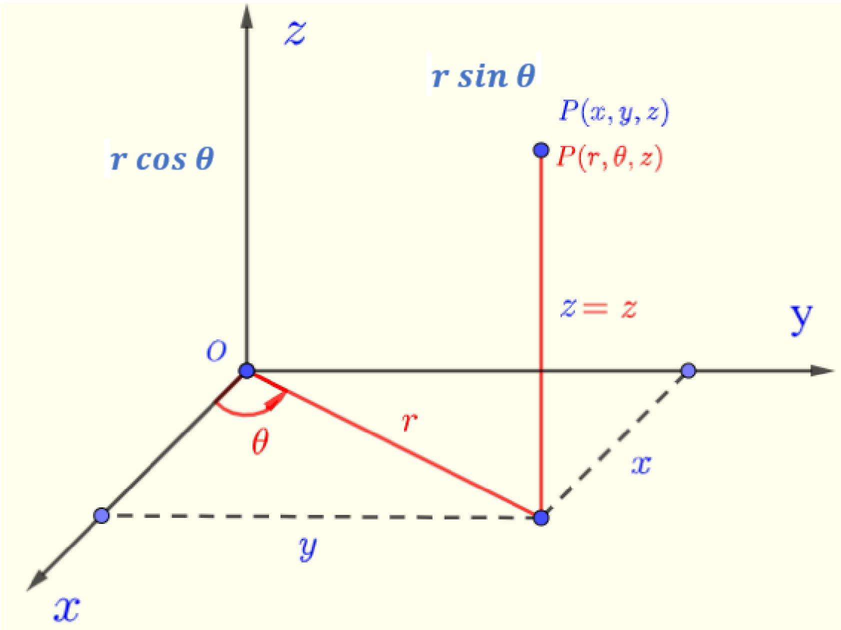


A system of curvilinear coordinates is said to be orthogonal if at each point the tangents to the coordinate curves are mutually perpendicular.

Tangent vectors	Scale factors	Basic vectors	For orthogonal coordinate system
$\frac{\partial \vec{r}}{\partial u}$	$h_1 = \left\lvert\frac{\partial \vec{r}}{\partial u_1}\right\rvert$	$\hat{e_1} = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}$,	$\hat{e_1}.\hat{e_2} = 0$
$\frac{\partial \vec{r}}{\partial u_2}$	$h_2 = \left\lvert\frac{\partial \vec{r}}{\partial u_2}\right\rvert$	$\hat{e_2} = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2}$	$\hat{e_2}.\hat{e_3} = 0$
$\frac{\partial \vec{r}}{\partial u_3}$	$h_3 = \left\lvert\frac{\partial \vec{r}}{\partial u_3}\right\rvert$	$\hat{e_2} = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial u_2}$	$\hat{e_3}.\hat{e_1} = 0$

Cylindrical coordinate system:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$



Tangent vectors	Scale factors	Basic vectors	For orthogonal coordinate system
$\frac{\partial \vec{r}}{\partial r}$	$h_1 = \left\lvert\frac{\partial \vec{r}}{\partial r}\right\rvert$	$\widehat{e_1} = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial r}$,	$\widehat{e_1}.\widehat{e_2} = 0$
$\frac{\partial \vec{r}}{\partial \theta}$	$h_2 = \left\lvert\frac{\partial \vec{r}}{\partial \theta}\right\rvert$	$\widehat{e_2} = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta}$	$\widehat{e_2}.\widehat{e_3} = 0$
$\frac{\partial \vec{r}}{\partial z}$	$h_3 = \left\lvert\frac{\partial \vec{r}}{\partial z}\right\rvert$	$\widehat{e_2} = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial z}$	$\widehat{e_3}.\widehat{e_1} = 0$

1. **Prove that the cylindrical coordinate system is orthogonal.**

In cylindrical polar coordinates system,

$x = r\cos\theta\,, y = r\sin\theta\,, z = z.$

Position vector: $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

$$\vec{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\vec{j} + z\hat{k}$$

Tangent vectors:

$$\frac{\partial\vec{r}}{\partial r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\frac{\partial\vec{r}}{\partial\theta} = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$$

$$\frac{\partial\vec{r}}{\partial z} = \hat{k}$$

Scale factors:

$$h_1 = \left|\frac{\partial\vec{r}}{\partial r}\right| = \left|\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}\right| = 1$$

$$h_2 = \left|\frac{\partial\vec{r}}{\partial\theta}\right| = \left|-r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\hat{k}\right| = r$$

$$h_3 = \left|\frac{\partial\vec{r}}{\partial z}\right| = \left|0\hat{\imath} + 0\hat{\jmath} + \hat{k}\right| = 1$$

Basic vectors:

$$\widehat{e_1} = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r} = \frac{1}{1}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}\right) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\widehat{e_2} = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta} = \frac{1}{r}\left(-r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\hat{k}\right) = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$

$$\widehat{e_3} = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial z} = \frac{1}{1}\left(0\hat{\imath} + 0\hat{\jmath} + \hat{k}\right) = \hat{k}$$

To prove: Cylindrical coordinate system is orthogonal

$$\widehat{e_1}.\widehat{e_2} = (\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}).(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath})$$

$$= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0$$

$$\widehat{e_2}.\widehat{e_3} = (-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}).\left(\hat{k}\right) = 0$$

$$\widehat{e_3}.\widehat{e_1} = \left(\hat{k}\right).(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) = 0$$

Therefore, cylindrical coordinate system is orthogonal.

2. **Express the vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical coordinates.**

In cylindrical polar coordinates system, $x = r\cos\theta\,, y = r\sin\theta\,, z = z$.

Position vector	**Tangent vectors**
$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ $\vec{r} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath} + z\hat{k}$	$\frac{\partial\vec{r}}{\partial r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $\frac{\partial\vec{r}}{\partial\theta} = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath}$ $\frac{\partial\vec{r}}{\partial z} = \hat{k}$
Scale factors	**Basic vectors**
$h_1 = \left\lvert\frac{\partial\vec{r}}{\partial r}\right\rvert$ $= \lvert\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}\rvert$ $= 1$ $h_2 = \left\lvert\frac{\partial\vec{r}}{\partial\theta}\right\rvert$ $= \lvert -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\hat{k}\rvert$ $= r$ $h_3 = \left\lvert\frac{\partial\vec{r}}{\partial z}\right\rvert$ $= \lvert 0\hat{\imath} + 0\hat{\jmath} + \hat{k}\rvert$ $= 1$	$\hat{e_1} = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r}$ $= \frac{1}{1}\left(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}\right)$ $= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ $\hat{e_2} = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta}$ $= \frac{1}{r}\left(-r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\hat{k}\right)$ $= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ $\hat{e_3} = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial z}$ $= \frac{1}{1}\left(0\hat{\imath} + 0\hat{\jmath} + \hat{k}\right)$ $= \hat{k}$

To express: $\vec{A}$ in cylindrical coordinates.

$$\vec{A} = z\hat{\imath} - 2x\hat{\jmath} + y\hat{k} = z\hat{\imath} - 2r\cos\theta\,\hat{\jmath} + r\sin\theta\,\hat{k}$$

$$A_1 = \vec{A}.\hat{e_1}$$

$$= \left(z\hat{\imath} - 2r\cos\theta\,\hat{\jmath} + r\sin\theta\,\hat{k}\right).(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$= z\cos\theta - 2r\sin\theta\cos\theta = z\cos\theta - r\sin 2\theta$$

$$A_2 = \vec{A}.\hat{e_2}$$

$$= \left(z\hat{\imath} - 2r\cos\theta\,\hat{\jmath} + r\sin\theta\,\hat{k}\right).(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}\,)$$

$$= -z\sin\theta - 2r\cos^2\theta$$

$$A_3 = \vec{A}.\hat{e_3}$$

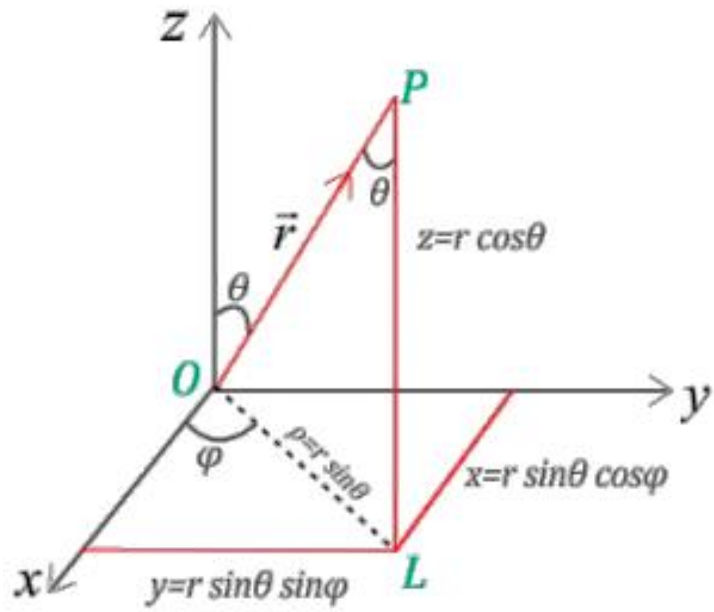
$$= \left(z\hat{\imath} - 2\rho\cos\phi\,\hat{\jmath} + \rho\sin\phi\,\hat{k}\right).\hat{k} = r\sin\theta$$

Therefore,

$$\vec{A} = (z\cos\theta - r\sin 2\theta)\hat{e_1} - (z\sin\theta + 2r\cos^2\theta)\hat{e_2} + r\sin\theta\,\hat{e_3}$$

Spherical coordinate system

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$



Tangent vectors	Scale factors	Basic vectors	For orthogonal coordinate system
$\frac{\partial\vec{r}}{\partial r}$	$h_1 = \left\|\frac{\partial\vec{r}}{\partial r}\right\|$	$\hat{e_1} = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r}$,	$\hat{e_1}.\hat{e_2} = 0$
$\frac{\partial\vec{r}}{\partial\theta}$	$h_2 = \left\|\frac{\partial\vec{r}}{\partial\theta}\right\|$	$\hat{e_2} = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta}$	$\hat{e_2}.\hat{e_3} = 0$
$\frac{\partial\vec{r}}{\partial\phi}$	$h_3 = \left\|\frac{\partial\vec{r}}{\partial\phi}\right\|$	$\hat{e_2} = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\phi}$	$\hat{e_3}.\hat{e_1} = 0$

3. **Prove that the spherical system is orthogonal.**

For the spherical system,

$x = r\sin\theta\cos\phi\,,\;\; y = r\sin\theta\sin\phi\,, z = r\cos\theta\;.$

Position vector:

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} = r\sin\theta\cos\phi\,\vec{\imath} + r\sin\theta\sin\phi\,\vec{\jmath} + r\cos\theta\,\vec{k}$$

Tangent vectors:

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\vec{\imath} + \sin\theta\sin\phi\,\vec{\jmath} + \cos\theta\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\vec{\imath} + r\cos\theta\sin\phi\,\vec{\jmath} - r\sin\theta\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\vec{\imath} + r\sin\theta\cos\phi\,\vec{\jmath} + 0\vec{k}$$

Scalar factors:

$$h_1 = \left|\frac{\partial\vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\,\vec{\imath} + \sin\theta\sin\phi\,\vec{\jmath} + \cos\theta\,\vec{k}\right| = 1$$

$$h_2 = \left|\frac{\partial\vec{r}}{\partial\theta}\right| = \left|r\cos\theta\cos\phi\,\vec{\imath} + r\cos\theta\sin\phi\,\vec{\jmath} - r\sin\theta\,\vec{k}\right| = r$$

$$h_3 = \left|\frac{\partial\vec{r}}{\partial\phi}\right| = \left|-r\sin\theta\sin\phi\,\vec{\imath} + r\sin\theta\cos\phi\,\vec{\jmath} + 0\vec{k}\right| = r\sin\theta$$

Basic vectors:

$$\widehat{e_1} = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\widehat{e_2} = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta} = \frac{1}{r}\left(r\cos\theta\cos\phi\,\hat{\imath} + r\cos\theta\sin\phi\,\hat{\jmath} - r\sin\theta\,\hat{k}\right)$$

$$= \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

$$\widehat{e_3} = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial\phi} = \frac{1}{r\sin\theta}\left(-r\sin\theta\sin\phi\,\hat{\imath} + r\sin\theta\cos\phi\,\hat{\jmath} + 0\hat{k}\right)$$

$$= -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

To prove: Spherical system is orthogonal

$$\widehat{e_1}.\widehat{e_2} = \sin\theta\cos\theta - \sin\theta\cos\theta = 0$$

$$\widehat{e_2}.\widehat{e_3} = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\widehat{e_3}.\widehat{e_1} = -\sin\theta\cos\phi\sin\phi + \sin\theta\cos\phi\sin\phi = 0$$

Therefore, the spherical system is orthogonal.

4. **Represent $\vec{F} = y\hat{\imath} - z\hat{\jmath} + x\hat{k}$ in spherical polar coordinates.**

For the spherical system,

$x = r\sin\theta\cos\phi\,,\;\; y = r\sin\theta\sin\phi\,, z = r\cos\theta\;.$

Position vector:

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} = r\sin\theta\cos\phi\,\vec{\imath} + r\sin\theta\sin\phi\,\vec{\jmath} + r\cos\theta\,\vec{k}$$

Tangent vectors:

$$\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\vec{\imath} + \sin\theta\sin\phi\,\vec{\jmath} + \cos\theta\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\vec{\imath} + r\cos\theta\sin\phi\,\vec{\jmath} - r\sin\theta\,\vec{k}$$

$$\frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\vec{\imath} + r\sin\theta\cos\phi\,\vec{\jmath} + 0\vec{k}$$

Scalar factors:

$$h_1 = \left|\frac{\partial\vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\,\vec{\imath} + \sin\theta\sin\phi\,\vec{\jmath} + \cos\theta\,\vec{k}\right| = 1$$

$$h_2 = \left|\frac{\partial\vec{r}}{\partial\theta}\right| = \left|r\cos\theta\cos\phi\,\vec{\imath} + r\cos\theta\sin\phi\,\vec{\jmath} - r\sin\theta\,\vec{k}\right| = r$$

$$h_3 = \left|\frac{\partial\vec{r}}{\partial\phi}\right| = \left|-r\sin\theta\sin\phi\,\vec{\imath} + r\sin\theta\cos\phi\,\vec{\jmath} + 0\vec{k}\right| = r\sin\theta$$

Basic vectors:

$$\widehat{e_1} = \frac{1}{h_1}\frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\widehat{e_2} = \frac{1}{h_2}\frac{\partial\vec{r}}{\partial\theta} = \frac{1}{r}\left(r\cos\theta\cos\phi\,\hat{\imath} + r\cos\theta\sin\phi\,\hat{\jmath} - r\sin\theta\,\hat{k}\right)$$

$$= \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

$$\widehat{e_3} = \frac{1}{h_3}\frac{\partial\vec{r}}{\partial\phi} = \frac{1}{r\sin\theta}\left(-r\sin\theta\sin\phi\,\hat{\imath} + r\sin\theta\cos\phi\,\hat{\jmath} + 0\hat{k}\right)$$

$$= -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

To represent: $\vec{F} = y\hat{\imath} - z\hat{\jmath} + x\hat{k}$ in spherical coordinate system

$$\vec{F} = y\hat{\imath} - z\hat{\jmath} + x\hat{k}$$

$$= r\sin\theta\sin\phi\,\hat{\imath} - r\cos\theta\,\hat{\jmath} + r\sin\theta\cos\phi\,\hat{k}$$

$$F_1 = \vec{F}.\widehat{e_1}$$

$$= \left(r\sin\theta\sin\phi\,\hat{\imath} - r\cos\theta\,\hat{\jmath} + r\sin\theta\cos\phi\,\hat{k}\right).$$

$$\left(\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}\right)$$

$$= r\sin^2\theta\cos\theta\sin\phi\cos\phi - r\sin\theta\cos\theta\sin\phi - r\sin\theta\cos\theta\cos\phi$$

$$F_2 = \vec{F}.\widehat{e_2}$$

$$= \left(r\sin\theta\sin\phi\,\hat{\imath} - r\cos\theta\,\hat{\jmath} + r\sin\theta\cos\phi\,\hat{k}\right).\left(\cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}\right)$$

$$= r\sin\theta\cos\theta\sin\phi\cos\phi - rcos^2\theta\sin\phi - r\sin^2\theta\cos\phi$$

$$F_3 = \vec{F}.\widehat{e_3}$$

$$= \left(r\sin\theta\sin\phi\,\hat{\imath} - r\cos\theta\,\hat{\jmath} + r\sin\theta\cos\phi\,\hat{k}\right).(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$$

$$= -r\sin\theta\sin^2\phi - r\cos\theta\cos\phi$$

Therefore,

$$\vec{F} = r\sin\theta\cos\theta\,\{\sin\theta\sin\phi\cos\phi - \sin\phi - \cos\phi\}\widehat{e_1}$$
$$+\{r\sin\theta\cos\theta\sin\phi\cos\phi - rcos^2\theta\sin\phi - r\sin^2\theta\cos\phi\}\widehat{e_2}$$
$$-r\{\sin\theta\sin^2\phi + \cos\theta\cos\phi\}\widehat{e_3}$$