Module 3

Ordinary differential equation of first order

Introduction to first-order ordinary differential equations pertaining to the applications for Computer Science & Engineering.
Linear and Bernoulli's differential equations. Exact and reducible to exact differential equations - Integrating factors on $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ and $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$. Orthogonal trajectories, L-R & C-R circuits. Problems.
Non-linear differential equations: Introduction to general and singular solutions, Solvable for p only, Clairaut's equations,reducible to Clairaut's equations. Problems.

Self-Study: Applications of ODEs, Solvable for x and y.

Applications of ordinary differential equations: Rate of Growth or Decay, Conduction of heat.
(RBT Levels: L1, L2 and L3)

3.1 Linear and Bernoulli's differential equations

Introduction:

1. **Linear differential equation in y:**

   This is of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x alone.

   General solution is $y.IF = \int Q.IF\,dx + c$, where $IF = e^{\int P dx}$.

2. **Linear differential equation in x:**

   This is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y alone.

   General solution is $x.IF = \int Q.IF\,dy + c$, where $IF = e^{\int P dy}$.

3. **Bernoulli's differential equation in y:**

   This is of the form $\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x alone.

   Dividing this equation by y^n, $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ ---- (1)

   Put $y^{1-n} = t$, then $(1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$

   Equation (1) becomes $\frac{1}{1-n}\frac{dt}{dx} + Pt = Q$

   Reduced linear differential equation is $\frac{dt}{dx} + (1-n)Pt = (1-n)Q$

4. **Bernoulli's differential equation in x:**

   This is of the form $\frac{dy}{dx} + Px = Qx^n$, where P and Q are functions of y alone.

   Dividing this equation by x^n, $x^{-n}\frac{dy}{dx} + Px^{1-n} = Q$ ---- (1)

   Put $x^{1-n} = t$, then $(1-n)x^{-n}\frac{dx}{dy} = \frac{dt}{dy}$

   Equation (1) becomes $\frac{1}{1-n}\frac{dt}{dy} + Pt = Q$

   Reduced linear differential equation is $\frac{dt}{dy} + (1-n)Pt = (1-n)Q$

Problems:

1. **Solve $\frac{dy}{dx} + \boldsymbol{y\,cot\,x = cos\,x}$**

   This is a linear L.D.E in y with $P = \cot x\,, Q = \cos x$

   $IF = e^{\int P dx} = e^{\int \cot x dx} = \sin x$

   General solution is given by,

   $y.IF = \int Q.IF\, dx + c$

   $y.\sin x = \int \cos x\,.\sin x\, dx + c$

   $y \sin x = \frac{1}{2}\int \sin 2x\, dx + c$

   $y \sin x = -\frac{1}{4}\cos 2x + c$

2. **Solve $\frac{dy}{dx} + \boldsymbol{y\,tan\,x = y^3 sec\,x}$**

   Step 1: Reduce it to an L.D.E.

   Divide by y^3 on both sides,

   $\frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \sec x$ ---- (1)

   If $\frac{1}{y^2} = t$ then $-\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$ ---- (2)

   Multiply by 2 on both sides of (1)

   $\frac{-2}{y^3}\frac{dy}{dx} - \frac{2}{y^2}\tan x = -2\sec x$ ---- (3)

   Substitute (2) in (3)

   $\frac{dt}{dx} - 2t\tan x = -2\sec x$

   This is an L.D.E. in t with $P = -2\tan x\,, Q = -2\sec x$

   Step 2: Solve the reduced L.D.E.

   $IF = e^{\int -2\tan x dx} = e^{-2\log\sec x} = \cos^2 x$

   General solution is given by,

   $t.IF = \int Q.IF\, dx + c$

   $t\cos^2 x = \int -2\sec x \cos^2 x\, dx + c$

   $t\cos^2 x = -2\sin x + c$

   $\left(\frac{1}{y^2}\right)\cos^2 x = -2\sin x + c$

3. **Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$** **(May 22)**

Step 1: Reduce it to an L.D.E.

Divide by y^2 on both sides,

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\left(\frac{1}{y}\right) = x$ ---- (1)

If $\frac{1}{y} = t$ then $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ ---- (2)

Multiply by -1 on both sides of (1)

$-\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{x}\left(\frac{1}{y}\right) = -x$ ---- (3)

Substitute (2) in (3)

$$\frac{dt}{dx} - \frac{t}{x} = -x$$

This is an LDE in t with $P = -\frac{1}{x}, Q = -x$

Step 2: Solve reduced L.D.E.

$$IF = e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

General solution is given by,

$$t.IF = \int Q.IF\, dx + c$$

$$t\frac{1}{x} = \int -x\frac{1}{x} dx + c$$

$$t\frac{1}{x} = -x + c$$

$$\frac{1}{xy} = -x + c$$

4. $\boldsymbol{tan\, y \frac{dy}{dx} + tan\, x = cos\, y\, cos^2 x}$

Step 1: Reduce it to an L.D.E.

Divide by $\cos y$ on both sides,

$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$ ---- (1)

If $\sec y = t$ then $\sec y \tan y \frac{dy}{dx} = \frac{dt}{dx}$

Substitute in (1)

$\frac{dt}{dx} + t \tan x = \cos^2 x$

This is an LDE in t with $P = \tan x\,, Q = \cos^2 x$

Step 2: Solve reduced L.D.E.

$IF = e^{\int P\, dx} = e^{\int \tan x\, dx} = \sec x$

General solution is given by,

$t.IF = \int Q.IF\, dx + c$

$t \sec x = \int \cos^2 x \sec x\, dx + c$

$\sec y \sec x = \sin x + c$

5. Solve: $\boldsymbol{r \sin\theta - \cos\theta \frac{dr}{d\theta} = r^2}$

Step 1: Reduce it to an L.D.E.

Divide by r^2 on both sides,

$-\frac{\cos\theta}{r^2}\frac{dr}{d\theta} + \frac{\sin\theta}{r} = 1$ ---- (1)

If $\frac{1}{r} = t$ then $-\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dt}{d\theta}$

Substitute in (1)

$\cos\theta \frac{dt}{dx} + t\sin\theta = 1$

Divide by $\cos\theta$ on both sides,

$\frac{dt}{dx} + t\tan\theta = \sec\theta$

This is an LDE in t with $P = \tan\theta\,, Q = \sec\theta$

Step 2: Solve reduced L.D.E.

$IF = e^{\int P\,dx} = e^{\int \tan\theta\, d\theta} = \sec\theta$

General solution is given by,

$t.IF = \int Q.IF\, d\theta + c$

$t\sec\theta = \int \sec\theta\sec\theta\, d\theta + c$

$\frac{1}{r}\sec\theta = \tan\theta + c$

6. **Solve:** $\boldsymbol{\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2}$

Step 1: Reduce it to an L.D.E.

Divide by $z(\log z)^2$ on both sides,

$\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{\log z}\left(\frac{1}{x}\right)=\frac{1}{x^2}$ ---- (1)

If $\frac{1}{\log z}=t$ then $-\frac{1}{z(\log z)^2}\frac{dz}{dx}=\frac{dt}{dx}$ ---- (2)

Multiply by -1 on both sides of (1)

$-\frac{1}{z(\log z)^2}\frac{dz}{dx}-\frac{1}{\log z}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$ ---- (3)

Substitute (2) in (3)

$\frac{dt}{dx}-\frac{t}{x}=-\frac{1}{x^2}$

This is an LDE in t with $P=-\frac{1}{x}, Q=-\frac{1}{x^2}$

Step 2: Solve reduced L.D.E.

$IF=e^{\int -\frac{1}{x}dx}=e^{-\log x}=\frac{1}{x}$

General solution is given by,

$t.IF=\int Q.IF\,dx+c$

$t\frac{1}{x}=\int -\frac{1}{x^2}.\frac{1}{x}dx+c$

$t\frac{1}{x}=\frac{x^{-2}}{2}+c$

$\frac{1}{x\log z}=\frac{1}{2x^2}+c$

7. Solve: $x\frac{dy}{dx}+y=x^3y^6$

Step 1: Reduce it to an L.D.E.

Divide by x on both sides,

$$\frac{dy}{dx}+\left(\frac{1}{x}\right)y=x^2y^6$$

Divide by y^6 on both sides,

$$\frac{1}{y^6}\frac{dy}{dx}+\left(\frac{1}{x}\right)\frac{1}{y^5}=x^2 \qquad \text{---- (1)}$$

If $\frac{1}{y^5}=t$ then $-\frac{5}{y^6}\frac{dy}{dx}=\frac{dt}{dx}$ ---- (2)

Multiply by -5 on both sides of (1)

$$-\frac{5}{y^6}\frac{dy}{dx}-\left(\frac{5}{x}\right)\frac{1}{y^5}=-5x^2 \qquad \text{---- (3)}$$

Substitute (2) in (3)

$$\frac{dt}{dx}-5\frac{t}{x}=-5x^2$$

This is an LDE in t with $P=-\frac{5}{x}, Q=-5x^2$

Step 2: Solve reduced L.D.E.

$$IF=e^{\int -\frac{5}{x}dx}=e^{-5\log x}=\frac{1}{x^5}$$

General solution is given by,

$$t.IF=\int Q.IF\,dx+c$$

$$t\frac{1}{x^5}=\int -5x^2\frac{1}{x^5}dx+c$$

$$\frac{t}{x^5}=-5\int x^{-3}\,dx$$

$$\frac{1}{x^5y^5}=\frac{5}{2x^2}+c$$

8. Solve: $xy(1 + xy^2)\frac{dy}{dx} = 1$

Step 1: Reduce it to an L.D.E.

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - xy = x^2y^3$$

Divide by x^2 on both sides,

$$\frac{1}{x^2}\frac{dx}{dy} - y\left(\frac{1}{x}\right) = y^3 \qquad \text{---- (1)}$$

If $\frac{1}{x} = t$ then $-\frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$ ---- (2)

Multiply by -1 on both sides of (1)

$$-\frac{1}{x^2}\frac{dx}{dy} + y\left(\frac{1}{x}\right) = -y^3 \qquad \text{---- (3)}$$

Substitute (2) in (3)

$$\frac{dt}{dy} + yt = -y^3$$

This is an LDE in t with $P = y, Q = -y^3$

Step 2: Solve reduced L.D.E.

$$IF = e^{\int y\,dy} = e^{\frac{y^2}{2}}$$

General solution is given by,

$$t.IF = \int Q.IF\,dy + c$$

$$te^{\frac{y^2}{2}} = \int -y^3 e^{\frac{y^2}{2}} dy + c$$

Put $p = \frac{y^2}{2}$, $dp = y\,dy$

$$te^p = \int -2pe^p dp + c$$

$$te^p = -2(pe^p - e^p) + c$$

$$\frac{1}{x}e^{\frac{y^2}{2}} = -2\left(\frac{y^2}{2}e^{\frac{y^2}{2}} - e^{\frac{y^2}{2}}\right) + c$$

$$\frac{1}{x}e^{\frac{y^2}{2}} = (2 - y^2)e^{\frac{y^2}{2}} + c$$

9. **Solve:** $\boldsymbol{\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y}$

Step 1: Reduce it to an L.D.E.

Divide by $\cos^2 y$ on both sides,

$\sec^2 y \frac{dy}{dx} + 2x \sec^2 y \sin y \cos y = x^3$

$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$ ---- (1)

If $\tan y = t$ then $\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

Substitute in (1)

$\frac{dt}{dx} + 2xt = x^3$

This is an L.D.E. in t with $P = 2x, Q = x^3$

Step 2: Solve reduced L.D.E.

$IF = e^{\int P\, dx} = e^{\int 2x\, dx} = e^{x^2}$

General solution is given by,

$t.IF = \int Q.IF\, dx + c$

$te^{x^2} = \int x^3\, e^{x^2} dx + c$, Put $p = x^2, dp = 2xdx$

$te^p = \frac{1}{2} \int p\, e^p dp + c$

$te^p = \frac{1}{2}(p-1)e^p + c$

$(\tan y)e^{x^2} = \frac{1}{2}(x^2 - 1)e^{x^2} + c$

10. Solve: $\frac{dy}{dx} = \frac{y}{x-\sqrt{xy}}$

Step 1: Reduce it to an L.D.E.

$$\frac{x-\sqrt{xy}}{y} = \frac{dx}{dy}$$

$$\frac{dx}{dy} - \frac{x}{y} = -\sqrt{\frac{x}{y}}$$

Divide by $\sqrt{x}$ on both sides,

$$\frac{1}{\sqrt{x}}\frac{dx}{dy} - \sqrt{x}\left(\frac{1}{y}\right) = -\frac{1}{\sqrt{y}} \qquad \text{---- (1)}$$

If $\sqrt{x} = t$ then $\frac{1}{2\sqrt{x}}\frac{dx}{dy} = \frac{dt}{dy}$ ---- (2)

Divide by 2 on both sides of (1),

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy} - \frac{\sqrt{x}}{2}\left(\frac{1}{y}\right) = -\frac{1}{2\sqrt{y}} \qquad \text{---- (3)}$$

Substitute (2) in (3),

$$\frac{dt}{dy} - \frac{1}{2y}t = -\frac{1}{2\sqrt{y}}$$

This is an L.D.E. in t with $P = -\frac{1}{2y}, Q = -\frac{1}{2\sqrt{y}}$

Step 2: Solve reduced L.D.E.

$$IF = e^{\int P\,dy} = e^{\int -\frac{1}{2y}dy} = \frac{1}{\sqrt{y}}$$

General solution is given by,

$$t.IF = \int Q.IF\,dy + c$$

$$t\frac{1}{\sqrt{y}} = \int -\frac{1}{2\sqrt{y}}\,\frac{1}{\sqrt{y}}dy + c$$

$$\sqrt{\frac{x}{y}} = \int -\frac{1}{2y}\,dy + c$$

$$\sqrt{\frac{x}{y}} = -\frac{1}{2}\log y + c$$

3.2 Exact and reducible to exact differential equations

Exact differential equation:

- A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- General solution of an exact differential equation is

$$\int_{y-constant} M\,dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$$

Reducible to exact differential equation:

- If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, given differential equation is not exact.
- Reduce it to an exact differential equation by multiplying I.F on both sides.
- If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to N then $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$. Now $I.F = e^{\int f(x)\,dx}$
- If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is close to M then $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y)$. Now $I.F = e^{\int g(y)\,dy}$

Problems:

1. **Solve:** $(\boldsymbol{x^2+y^2+x})\boldsymbol{dx}+\boldsymbol{xy\,dy}=\boldsymbol{0}$ **(May 22)**

$(x^2+y^2+x)dx+xy\,dy=0$ ------ (1)

$M=x^2+y^2+x$	$N=xy$
$\frac{\partial M}{\partial y}=2y$	$\frac{\partial N}{\partial x}=y$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}=y$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{xy}(y)=\frac{1}{x}=f(x)\quad[say]$

$I.F=e^{\int f(x)\,dx}=e^{\int\frac{1}{x}dx}=x$

Multiply by x on both the sides of equation (1)

$(x^3+xy^2+x^2)dx+x^2y\,dy=0$

This is an exact D.E.

General solution is

$\int_{y-constant}M\,dx+\int(Terms\ of\ N\ not\ containing\ x)\ dy=c$

$\int_{y-constant}(x^3+xy^2+x^2)\,dx+\int(0)\ dy=c$

$\frac{x^4}{4}+\frac{x^2y^2}{2}+\frac{x^3}{3}=c$

2. Solve: $(4xy + 3y^2 - x)\,dx + x(x + 2y)dy = 0$

$(4xy + 3y^2 - x)\,dx + (x^2 + 2xy)dy = 0$ ------ (1)

$M = 4xy + 3y^2 - x$ $\frac{\partial M}{\partial y} = 4x + 6y$	$N = x^2 + 2xy$ $\frac{\partial N}{\partial x} = 2x + 2y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2+2xy} 2(x + 2y) = \frac{2}{x} = f(x) \quad [say]$

$I.F = e^{\int f(x)\,dx} = e^{\int \frac{2}{x}\,dx} = x^2$

Multiply by x^2 on both the sides of equation (1)

$(4x^3y + 3x^2y^2 - x^3)\,dx + (x^4 + 2x^3y)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant} (4x^3y + 3x^2y^2 - x^3)\,dx + \int (0)\ dy = c$

$x^4y + x^3y^2 - \frac{x^4}{4} = c$

3. Solve: $\boldsymbol{\left(xy^2 - e^{1/x^3}\right) dx - x^2 y\, dy = 0}$

$\left(xy^2 - e^{1/x^3}\right) dx + (-x^2 y) dy = 0$ ------ (1)

$M = xy^2 - e^{\frac{1}{x^3}}$ $\frac{\partial M}{\partial y} = 2xy$	$N = -x^2 y$ $\frac{\partial N}{\partial x} = -2xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4xy$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-x^2 y}(4xy) = -\frac{4}{x} = f(x) \quad [say]$

$I.F = e^{\int f(x)\, dx} = e^{-4\int \frac{1}{x} dx} = x^{-4}$

Multiply by x^{-4} on both the sides of equation (1)

$x^{-4}\left(xy^2 - e^{x^{-3}}\right) dx - x^{-2} y\, dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\, dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant} \left(x^{-3} y^2 - x^{-4} e^{x^{-3}}\right) dx + \int (0)\ dy = c$

$-\frac{1}{2} x^{-2} y^2 + \frac{1}{3} e^{x^{-3}} = c$

4. Solve: $(\boldsymbol{x^2 + y^3 + 6x})\, \boldsymbol{dx} + \boldsymbol{xy^2}\, \boldsymbol{dy} = \boldsymbol{0}$

$(x^2 + y^3 + 6x)\, dx + xy^2\, dy = 0$ ------ (1)

$M = x^2 + y^3 + 6x$ $\frac{\partial M}{\partial y} = 3y^2$	$N = xy^2$ $\frac{\partial N}{\partial x} = y^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y^2$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy^2}(2y^2) = \frac{2}{x} = f(x) \quad [say]$

$I.F = e^{\int f(x)\, dx} = e^{2\int \frac{1}{x} dx} = x^2$

Multiply by x^2 on both the sides of equation (1)

$\left(x^{-3}y^2 - x^{-4}e^{1/x^3}\right) dx - x^{-2}y\, dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\, dx + \int(Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant}\left(x^{-3}y^2 - x^{-4}e^{1/x^3}\right)dx + \int(0)\ dy = c$

$-\frac{y^2}{2x^2} + \frac{e^{\frac{1}{x^3}}}{3} = c$

5. Solve: $(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)\, dy = 0$

$(y^4 + 2y)\, dx + (xy^3 + 2y^4 - 4x)dy = 0$ ------ (1)

$M = y^4 + 2y$	$N = xy^3 + 2y^4 - 4x$
$\frac{\partial M}{\partial y} = 4y^3 + 2$	$\frac{\partial N}{\partial x} = y^3 - 4$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6 = 3(y^3 + 2)$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3}{y^4+2y}(y^3 + 2\,) = \frac{3}{y} = g(y) \quad [say]$

$I.F = e^{-\int g(y)\, dy} = e^{-3\int \frac{1}{y}\, dy} = y^{-3}$

Multiply by y^{-3} on both the sides of equation (1)

$y^{-3}(y^4 + 2y)\, dx + y^{-3}(xy^3 + 2y^4 - 4x)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\, dx + \int(Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant}(y + 2y^{-2})dx + \int(2y)\ dy = c$

$xy + \frac{2x}{y^2} + y^2 = c$

6. Solve: $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)dy = 0$ ------ (1)

$M = 3x^2y^4 + 2xy$	$N = 2x^3y^3 - x^2$
$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$	$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 6x^2y^3 + 4x = 2(3x^2y^3 + 2x)$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{3x^2y^4+2xy}(3x^2y^3 + 2x\ ) = \frac{2}{y} = g(y)\quad [say]$

$I.F = e^{-\int g(y)\,dy} = e^{-2\int\frac{1}{y}\,dy} = y^{-2}$

Multiply by y^{-2} on both the sides of equation (1)

$y^{-2}(3x^2y^4 + 2xy)\,dx + y^{-2}(2x^3y^3 - x^2)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int(Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant}\left(3x^2y^2 + \frac{2x}{y}\right)dx + \int(0)\ dy = c$

$x^3y^2 + \frac{x^2}{y} = c$

7. **Solve**: $(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)\, dy = 0$

$(xy^3 + y)\, dx + 2(x^2y^2 + x + y^4)dy = 0$ ------ (1)

$M = xy^3 + y$	$N = 2(x^2y^2 + x + y^4)$
$\frac{\partial M}{\partial y} = 3xy^2 + 1$	$\frac{\partial N}{\partial x} = 2(2xy^2 + 1) = 4xy^2 + 2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -xy^2 - 1 = -(xy^2 + 1)$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{xy^2+1}{xy^3+y} = -\frac{1}{y} = g(y) \quad [say]$

$I.F = e^{-\int g(y)\, dy} = e^{\int \frac{1}{y} dy} = y$

Multiply by y on both the sides of equation (1)

$y(xy^3 + y)\, dx + 2y(x^2y^2 + x + y^4)dy = 0$ This is an exact D.E.

General solution is

$\int_{y-constant} M\, dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant} (xy^4 + y^2)dx + \int (2y^5)\ dy = c$

$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$

8. Solve: $(y\ log\ y)\ dx + (x - log\ y)\ dy = 0$

$(y \log y)\ dx + (x - \log y)dy = 0$ ------ (1)

$M = y \log y$	$N = x - \log y$
$\frac{\partial M}{\partial y} = y\left(\frac{1}{y}\right) + \log y$	$\frac{\partial N}{\partial x} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + \log y - 1 = \log y$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{y \log y}(\log y) = \frac{1}{y} = g(y) \quad [say]$

$I.F = e^{-\int g(y)\,dy} = e^{-\int \frac{1}{y}\,dy} = \frac{1}{y}$

Multiply by $\frac{1}{y}$ on both the sides of equation (1)

$\frac{1}{y}(y \log y)\ dx + \frac{1}{y}(x - \log y)dy = 0$ This is an exact D.E.

General solution is

$\int_{y-constant} M\ dx + \int (Terms\ of\ N\ not\ containing\ x)\ dy = c$

$\int_{y-constant} \log y\ dx - \int \left(\frac{1}{y} \log y\right)\ dy = c$

$x \log y - \frac{(\log y)^2}{2} = c$

9. Solve: $\boldsymbol{y(x+y+1)\,dx + x(x+3y+2)\,dy = 0}$

$(xy + y^2 + y)\ \ dx + (x^2 + 3xy + 2x)dy = 0$ ------ (1)

$M = xy + y^2 + y$	$N = x^2 + 3xy + 2x$
$\frac{\partial M}{\partial y} = x + 2y + 1$	$\frac{\partial N}{\partial x} = 2x + 3y + 2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x - y - 1 = -(x + y + 1)$, close to M.

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{xy+y^2+y}(x + y + 1\ ) = -\frac{1}{y} = g(y)\quad [say]$

$I.F = e^{-\int g(y)\,dy} = e^{\int \frac{1}{y}\,dy} = y$

Multiply by y on both the sides of equation (1)

$y(xy + y^2 + y)\ \ dx + y(x^2 + 3xy + 2x)dy = 0$

This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int (Terms\ of\ N\ not\ containing\ x)\ \ dy = c$

$\int_{y-constant} (xy^2 + y^3 + y^2)dx + \int (0)\ \ dy = c$

$\frac{x^2y^2}{2} + xy^3 + xy^2 = c$

10. Solve: $\boldsymbol{2y\,dx + (2x\,log\,x - xy)\,dy = 0}$

$(2y)\,dx + (2x\log x - xy)dy = 0$ ------ (1)

$M = 2y$	$N = 2x\log x - xy$
$\frac{\partial M}{\partial y} = 2$	$\frac{\partial N}{\partial x} = 2x\left(\frac{1}{x}\right) + 2\log x - y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, this is not an exact D.E.

$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y = -(2\log x - y)$, close to N.

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x\log x - xy}(-2\log x + y) = -\frac{1}{x} = f(x) \quad [say]$

$I.F = e^{\int f(x)\,dx} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

Multiply by $\frac{1}{x}$ on both the sides of equation (1)

$\frac{1}{x}(2y)\;\; dx + \frac{1}{x}(2x\log x - xy)dy = 0$ This is an exact D.E.

General solution is

$\int_{y-constant} M\,dx + \int(Terms\ of\ N\ not\ containing\ x)\;\,dy = c$

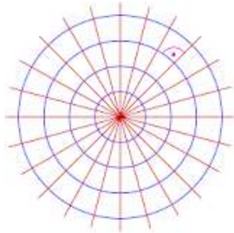
$\int_{y-constant} \frac{2y}{x}\;\,dx + \int(-y)\;\,dy = c$

$2y\log x - \frac{y^2}{2} = c$

3.3 Orthogonal Trajectory

Definition: Two families of curves such that every member of either family cuts each member of the other family at right angles are called orthogonal trajectories.

Example: Family of circles $x^2 + y^2 = a^2$ is the orthogonal trajectories to the family of straight lines $y = mx + c$. Where a and m are arbitrary constants.



Working rule to find the orthogonal trajectories of the family of curves $f(x, y, c) = 0$:

- Form the differential equation by eliminating arbitrary constant c.
- Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$. $[\tan(90 + \theta) = -\cot\theta]$
- Solve the modified differential equation.

Working rule to find the orthogonal trajectories of the family of curves $f(r, \theta, c) = 0$:

- Form the differential equation by eliminating arbitrary constant c.
- Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$. $[\tan(90 + \phi) = -\cot\phi]$
- Solve the modified differential equation.

Problems:

1. **Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.**

Consider $y^2 = 4ax$ ------ (1)

Differentiate w.r.to x,

$$2y\frac{dy}{dx} = 4a$$

Substitute in (1),

$$y^2 = 2xy\frac{dy}{dx}$$

$$y = 2x\frac{dy}{dx}$$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$$y = -2x\frac{dx}{dy}$$

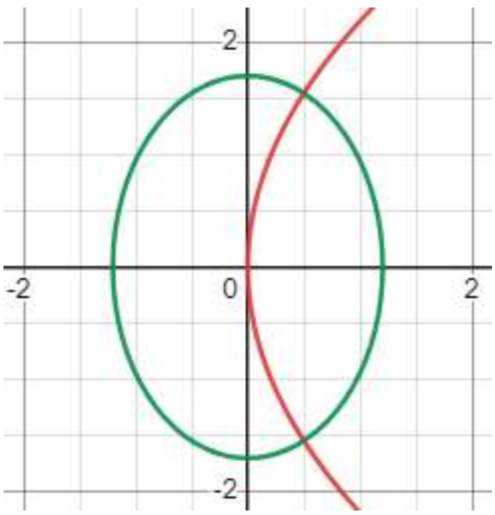
$$y\,dy = -2x\,dx$$

On integrating,

$$\frac{y^2}{2} = -x^2 + c$$

$$2x^2 + y^2 = k$$

This is the family of orthogonal trajectories of (1).



2. **Find the orthogonal trajectories of the family of circles $x^2 + y^2 = a^2$.**

Consider $x^2 + y^2 = a^2$ ------ (1)

Differentiate w.r.to x,

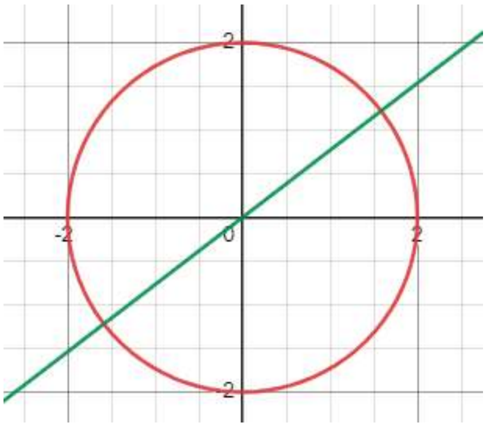
$2x + 2y\frac{dy}{dx} = 0$

$y\frac{dy}{dx} = -x$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$y\left(-\frac{dx}{dy}\right) = -x$

$\frac{1}{x}\, dx = \frac{1}{y}\, dy$



On integrating,

$\log x = \log y + \log c$

$x = yc$

This is the family of orthogonal trajectories of (1).

3. **Find the orthogonal trajectories of the family of curves $y^2 = c\, x^3$.**

Consider $y^2 = c\, x^3$ ------ (1)

Differentiate w.r.to x,

$2y\frac{dy}{dx} = 3c\, x^2$

$\times\, x \Rightarrow 2xy\frac{dy}{dx} = 3c\, x^3$

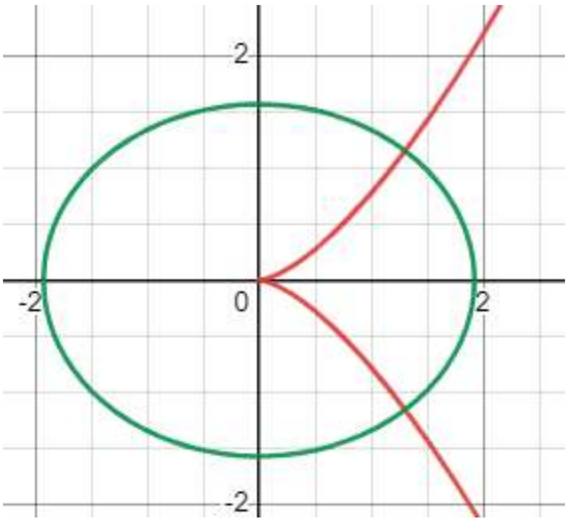
By (1), $2xy\frac{dy}{dx} = 3y^2$

$2x\frac{dy}{dx} = 3y$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$2x\left(-\frac{dx}{dy}\right) = 3y$

$-2x\, dx = 3y\, dy$



On integrating,

$-x^2 = \frac{3y^2}{2} + c$

$2x^2 + 3y^2 = k$

This is the family of orthogonal trajectories of (1).

4. Find the orthogonal trajectories of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$.

Consider $x^{2/3} + y^{2/3} = a^{2/3}$ ------ (1)

Differentiate w.r.to x,

$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$

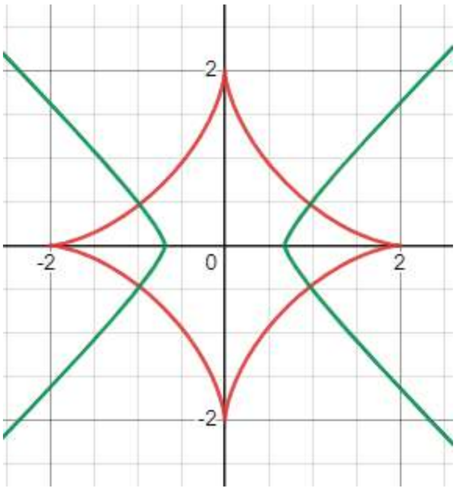
$\times\frac{3}{2} \Rightarrow x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\frac{dy}{dx} = 0$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$x^{-\frac{1}{3}} + y^{-\frac{1}{3}}\left(-\frac{dx}{dy}\right) = 0$

$y^{-\frac{1}{3}}\left(-\frac{dx}{dy}\right) = -x^{-\frac{1}{3}}$

$x^{\frac{1}{3}}\,dx = y^{\frac{1}{3}}\,dy$



On integrating,

$x^{4/3} = y^{4/3} + c$

$x^{4/3} - y^{4/3} = c$

This is the family of orthogonal trajectories of (1).

5. Show that the family of parabolas $y^2 = 4a(x+a)$ is self-orthogonal.

$y^2 = 4a(x+a)$

Diff. w. r. to x,

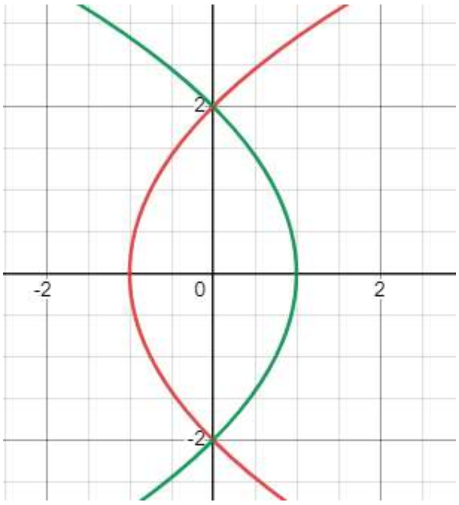
$2y\frac{dy}{dx} = 4a$

By substituting in (1),

$y^2 = 2y\frac{dy}{dx}\left(x + \frac{y}{2}\frac{dy}{dx}\right)$

$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$

$y = 2x\frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2$ ------- (1)



Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$y = 2x\left(-\frac{dx}{dy}\right) + y\left(-\frac{dx}{dy}\right)^2$

$y = -2x\left(\frac{dx}{dy}\right) + y\left(\frac{dx}{dy}\right)^2$

$y\left(\frac{dy}{dx}\right)^2 = -2x\left(\frac{dy}{dx}\right) + y$

$y = y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right)$ ------ (2)

Since (1) = (2),

The given family of parabolas is self-orthogonal.

6. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, where λ is the parameter. **(May 22)**

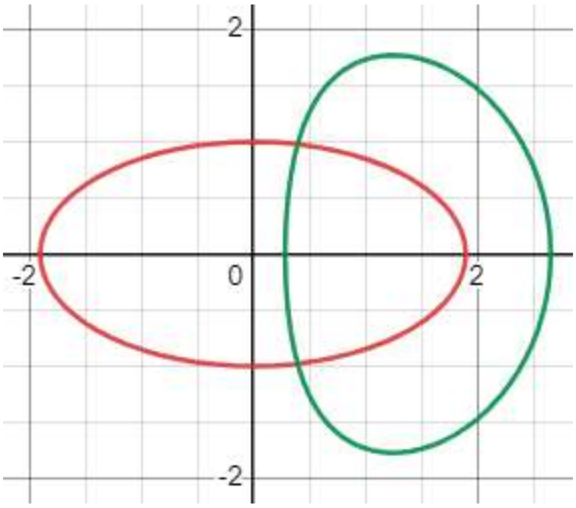
$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ ------ (1)

Diff. w.r.to x,

$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$

$\frac{y}{b^2+\lambda}\frac{dy}{dx}=-\frac{x}{a^2}$

$\frac{y}{b^2+\lambda}=-\frac{x}{a^2}\left(\frac{dx}{dy}\right)$



Substitute in (1),

$\frac{x^2}{a^2}-\frac{xy}{a^2}\left(\frac{dx}{dy}\right)=1$

$x^2-xy\left(\frac{dx}{dy}\right)=a^2$

$x^2-a^2=xy\left(\frac{dx}{dy}\right)$

$\frac{dy}{dx}=\frac{xy}{x^2-a^2}$

Replace $\frac{dy}{dx}=-\frac{dx}{dy}$

$-\frac{dx}{dy}=\frac{xy}{x^2-a^2}$

$\frac{a^2-x^2}{x}\,dx=y\,dy$

$\left(\frac{a^2}{x}-x\right)dx=y\,dy$

$a^2\log x-\frac{x^2}{2}=\frac{y^2}{2}+c$

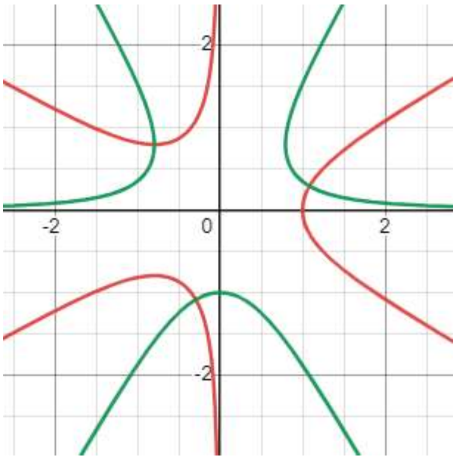
$x^2+y^2=2a^2\log x+k$

This is the family orthogonal trajectories of (1).

7. Find the orthogonal trajectories of the family of curves $x^3 - 3xy^2 = c$

$x^3 - 3xy^2 = c$------ (1)

Diff. w.r.to x,



$3x^2 - 3y^2 - 6xy\frac{dy}{dx} = 0$

$x^2 - y^2 = 2xy\frac{dy}{dx}$

Replace $\frac{dy}{dx} = -\frac{dx}{dy}$

$x^2 - y^2 = 2xy\left(-\frac{dx}{dy}\right)$

$\frac{dy}{dx} = \frac{2xy}{y^2-x^2}$

Pu $y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = \frac{2x(vx)}{(vx)^2-x^2} = \frac{2v}{v^2-1}$

$x\frac{dv}{dx} = \frac{2v}{v^2-1} - v = \frac{2v-v^3+v}{v^2-1} = \frac{3v-v^3}{v^2-1}$

$\frac{v^2-1}{3v-v^3}dv = \frac{1}{x}dx$

$-\frac{1}{3}\left(\frac{3-3v^2}{3v-v^3}\right)dv = \frac{1}{x}dx$

On integrating,

$-\frac{1}{3}\log(3v - v^3) = \log x + \log c$

$\log(3v - v^3) = -3\log x - 3\log c$

$x^3(3v - v^3) = k$

$x^3\left(3\left(\frac{y}{x}\right) - \frac{y^3}{x^3}\right) = k$

$3x^2y - y^3 = k.$ This is the family orthogonal trajectories of (1).

8. **Find the orthogonal trajectories of the family of curves $r^n = a^n cos\ n\theta$.**

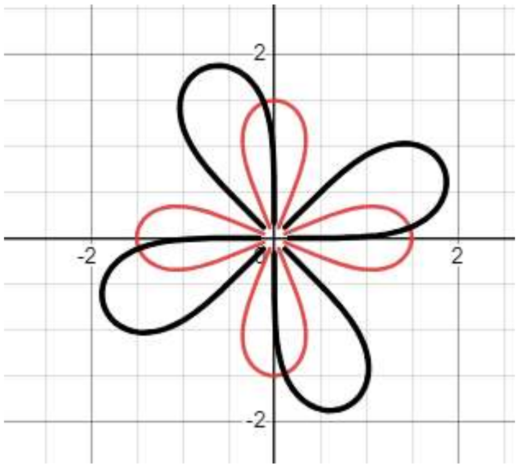
$r^n = a^n \cos n\theta$

$\log r^n = \log a^n \cos n\theta$

$n \log r = \log a^n + \log \cos n\theta$

$\frac{n}{r}\frac{dr}{d\theta} = -n\frac{\sin n\theta}{\cos n\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin n\theta}{\cos n\theta}$



Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = -\tan n\theta$

$\cot n\theta\, d\theta = \frac{1}{r}dr$

On integrating,

$\frac{1}{n}\log \sin n\theta = \log r + \log c$

$\log \sin n\theta = n \log rc$

$\sin n\theta = r^n c^n$

$r^n = k \sin n\theta$

This is the required O.T.

9. **Find the orthogonal trajectories of the family of curves $r^n cos\ n\theta = a^n$.**

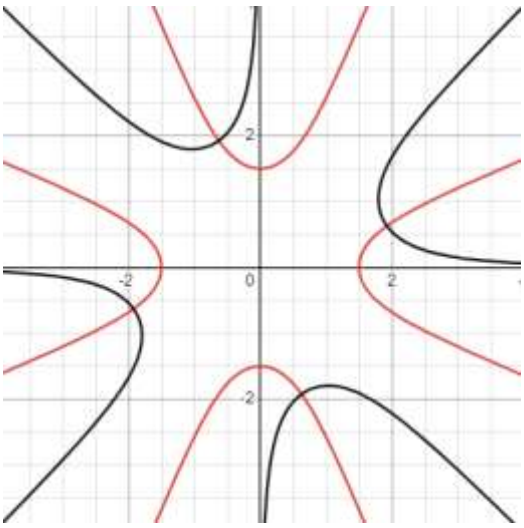
$r^n \cos n\theta = a^n$

$\log(r^n \cos n\theta) = \log a^n$

$n \log r + \log \cos n\theta = \log a^n$

$\frac{n}{r}\frac{dr}{d\theta} - n\frac{\sin n\theta}{\cos n\theta} = 0$

$\frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$



Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \tan n\theta$

$\cot n\theta\, d\theta = -\frac{1}{r}dr$

On integrating,

$\frac{1}{n}\log \sin n\theta = -\log r + \log c$

$\log \sin n\theta = n \log\frac{c}{r}$

$\sin n\theta = \frac{c^n}{r^n}$

$r^n \sin n\theta = k.$ This is the required O.T.

10. Find the orthogonal trajectories of the family of curves $r = 2a\, cos\, \theta$.

$r = 2a\cos\theta$

$\log r = \log(2a\cos\theta)$

$\log r = \log 2a + \log\cos\theta$

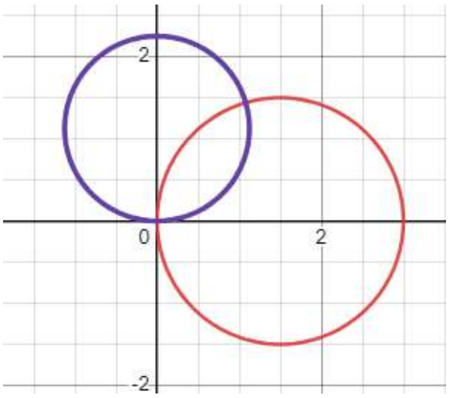
$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin\theta}{\cos\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = -\tan\theta$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = -\tan\theta$

$\cot\theta\, d\theta = \frac{1}{r}dr$



On integrating,

$\log\sin\theta = \log r + \log c$

$\log\sin\theta = \log cr$

$\sin\theta = cr$

$r = k\sin\theta$

This is the required O.T.

11. Find the orthogonal trajectories of the family of curves $r^n = a^n sin\, n\theta$.

$r^n = a^n\sin n\theta$

$\log r^n = \log a^n \sin n\theta$

$n\log r = \log a^n + \log\sin n\theta$

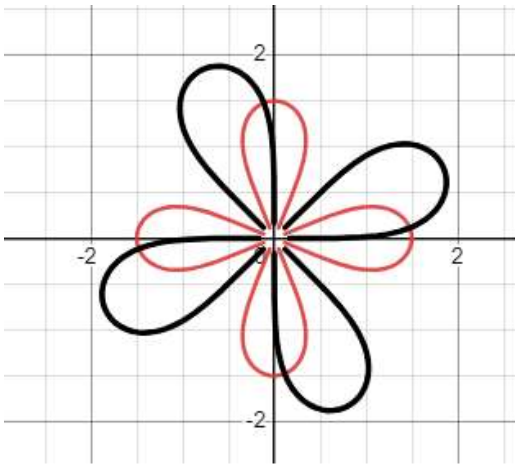
$\frac{n}{r}\frac{dr}{d\theta} = n\frac{\cos n\theta}{\sin n\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \cot n\theta$

$\tan n\theta\, d\theta = -\frac{1}{r}dr$



On integrating,

$\frac{1}{n}\log\sec n\theta = -\log r + \log c$

$\log\sec n\theta = n\log\frac{c}{r}$

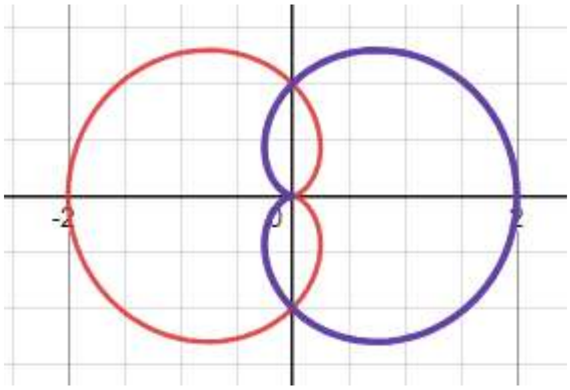
$\sec n\theta = \frac{c^n}{r^n}$

$r^n = k\cos n\theta$. This is the required O.T.

12. Find the orthogonal trajectories of the family of curves $r = a(1 - \cos\theta)$.

$r = a(1 - \cos\theta)$

$\log r = \log a + \log(1 - \cos\theta)$



$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$

$-\frac{1-\cos\theta}{\sin\theta}d\theta = \frac{1}{r}dr$

$-\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}d\theta = \frac{1}{r}dr$

$-\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}d\theta = \frac{1}{r}dr$

$-\frac{\sin\theta}{(1+\cos\theta)}d\theta = \frac{1}{r}dr$

On integrating,

$\log(1 + \cos\theta) = \log r + \log c$

$\log(1 + \cos\theta) = \log cr$

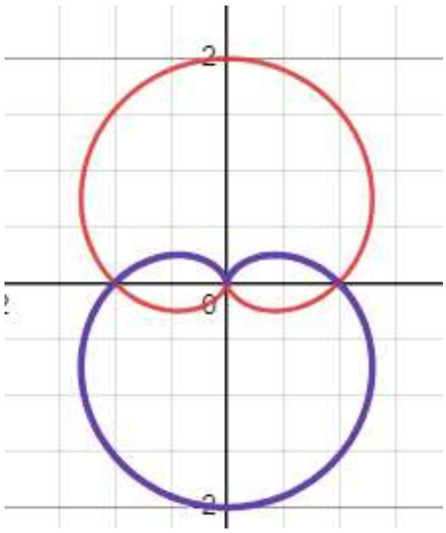
$1 + \cos\theta = cr$

$r = k(1 + \cos\theta)$. This is the required O.T.

13. Find the orthogonal trajectories of the family of curves $r = a(1 + \sin\theta)$.

$r = a(1 + \sin\theta)$

$\log r = \log a + \log(1 + \sin\theta)$



$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \frac{\cos\theta}{1+\sin\theta}$

$-\frac{1+\sin\theta}{\cos\theta}d\theta = \frac{1}{r}dr$

$-\frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}d\theta = \frac{1}{r}dr$

$-\frac{\cos\theta}{1-\sin\theta}d\theta = \frac{1}{r}dr$

On integrating,

$\log(1 - \sin\theta) = \log r + \log c$

$1 - \sin\theta = cr$

$r = k(1 - \sin\theta)$. This is the required O.T.

14. Find the orthogonal trajectories of the family of curves $\boldsymbol{r = 2a(\cos\theta + \sin\theta)}$.

$r = 2a(\cos\theta + \sin\theta)$

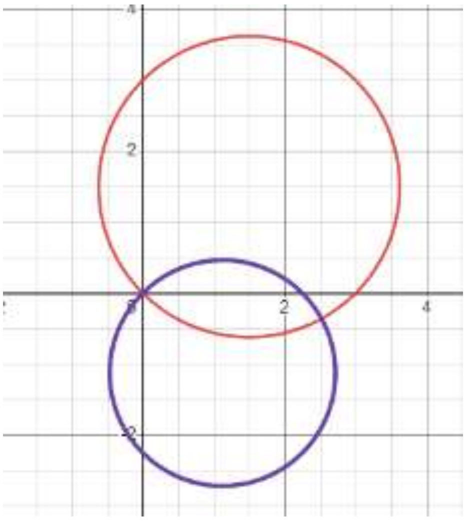
$\log r = \log 2a + \log(\cos\theta + \sin\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

$-\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}d\theta = \frac{1}{r}dr$



On integrating,

$\log(\cos\theta - \sin\theta) = \log r + \log c$

$\log(\cos\theta - \sin\theta) = \log cr$

$\cos\theta - \sin\theta = cr$

$r = k(\cos\theta - \sin\theta)$

This is the required O.T.

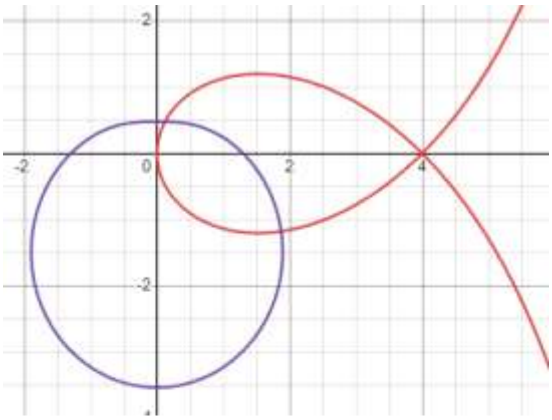
15. Find the orthogonal trajectories of the family of curves $\boldsymbol{r = 4a(\sec\theta + \tan\theta)}$.

$r = 4a(\sec\theta + \tan\theta)$

$\log r = \log 4a + \log(\sec\theta + \tan\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}$

$\frac{1}{r}\frac{dr}{d\theta} = \sec\theta$



Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

$-r\frac{d\theta}{dr} = \sec\theta$

$-\cos\theta\, d\theta = \frac{1}{r}dr$

On integrating,

$-\sin\theta = \log r + \log c$

$\log rc = -\sin\theta$

$rc = e^{-\sin\theta}$

$re^{\sin\theta} = k$

This is the required O.T.

16. Prove that the orthogonal trajectories of the family of curves $\frac{2a}{r} = 1 - \cos\theta$ is $\frac{2b}{r} = 1 + \cos\theta$.

$\frac{2a}{r} = 1 - \cos\theta$

$\log\frac{2a}{r} = \log(1 - \cos\theta)$

$\log 2a - \log r = \log(1 - \cos\theta)$

$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

Replace $\frac{1}{r}\frac{dr}{d\theta}$ by $-r\frac{d\theta}{dr}$

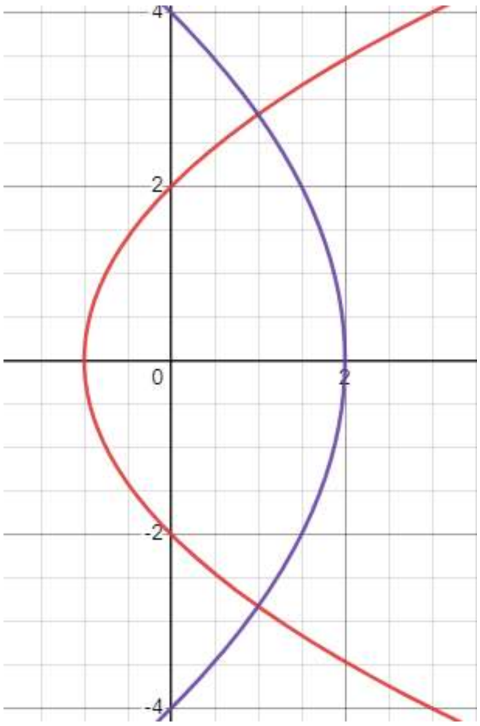
$r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$

$\frac{1-\cos\theta}{\sin\theta}d\theta = \frac{1}{r}dr$

$\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}d\theta = \frac{1}{r}dr$

$\frac{\sin^2\theta}{\sin\theta(1+\cos\theta)}d\theta = \frac{1}{r}dr$

$\frac{\sin\theta}{(1+\cos\theta)}d\theta = \frac{1}{r}dr$



On integrating,

$-\log(1 + \cos\theta) = \log r + \log c$

$\log 2b = \log r + \log(1 + \cos\theta)$

$\log\frac{2b}{r} = \log(1 + \cos\theta)$

$\frac{2b}{r} = 1 + \cos\theta$

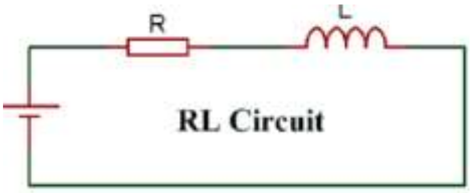
This is the required O.T.

3.4 RL and RC circuits

Introduction:

Notation	Terminology	Unit
L	Inductance	Henry
C	Capacitance	Farad
R	Resistance	Ohms
E	Electro motive force (e.m.f.)	Volts
I	Current	Amperes
Q	Charge	Coloumb

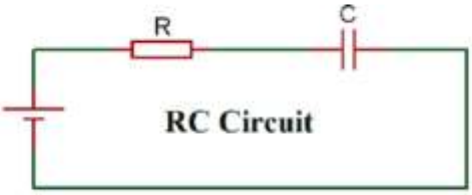
LR circuit:



- $I = \frac{dQ}{dt}$
- Voltage drop across resistance $R = RI$
- Voltage drop across inductance $L = L\frac{dI}{dt}$

By Kirchhoff's law, $L\frac{dI}{dt} + RI = E$ in LR circuit.

RC circuit:



- Voltage drop across capacitance $C = \frac{Q}{C}$

By Kirchhoff's law, $RI + \frac{Q}{C} = E$ in RC circuit.

1. **A resistance of $100\ \Omega$, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ secs. If $i = 0$ at $t = 0$.**

By data, $E = 20, R = 100, L = 0.5$

By Kirchhoff's law,

$L\frac{di}{dt} + Ri = E$

On substituting,

$0.5\frac{di}{dt} + 100i = 20$

$\frac{di}{dt} + 200i = 40$

By separating the variables,

$\frac{di}{40-200i} = dt$

On integrating,

$-0.005\log(40 - 200i) = t + c$

$\log(40 - 200i) = -200t + c'$

By taking anti log,

$40 - 200i = ke^{-200t}, k = e^{c'}$

By data, $i = 0$ at $t = 0$.

$40 = k$

Therefore, solution is

$40 - 200i = 40e^{-200t}$

Dividing by 40, $1 - 5i = e^{-200t}$

At $t = 0.5$

$1 - 5i = e^{-100}$

$i = \frac{1-e^{-100}}{5}$

2. **Find the current at any time $t > 0$, in a circuit having in series a constant electromotive force $40V$, a resistor 10Ω, an inductor $0.2H$ given that initial current is zero.**

By data, $E = 40, R = 10, L = 0.2$

By Kirchoff's law, $L\frac{dI}{dt} + RI = E$

On substituting, $0.2\frac{dI}{dt} + 10I = 40$

Therefore, $\frac{dI}{dt} + 50I = 200$

$200 - 50I = \frac{dI}{dt}$

$\frac{dI}{200-50I} = dt$

$-\frac{1}{50}\int \frac{(-50)dI}{200-50I} = \int dt$

$-\frac{1}{50}\log(200 - 50I) = t + c$

$\log(200 - 50I) = -50t - 50c$

$200 - 50I = ke^{-50t}$

By data, $I = 0\ at\ t = 0$.

Therefore, $k = 200$.

Solution is given by

$200 - 50I = 200e^{-50t}$

$50I = 200(1 - e^{-50t})$

$I = 4(1 - e^{-50t})$

3. **A generator having e.m.f. 100 volts is connected in series with a 10 ohms resistor and an inductor of 2 henries. If the switch is closed at a time $t = 0$, determine the current at time $t > 0$.**

By data, $E = 100, R = 10, L = 2$

By Kirchoff's law, $L\frac{dI}{dt} + RI = E$

On substituting, $2\frac{dI}{dt} + 10I = 100$

Therefore, $\frac{dI}{dt} + 5I = 50$

By separating the variables,

$\frac{dI}{50-5I} = dt$

On integrating,

$-\frac{1}{5}\log(50 - 5I) = t + c$

$\log(50 - 5I) = -5t - 5c$

By taking anti log,

$50 - 5I = e^{-5t}e^{-5c}$

$I = 10 - ke^{-5t}, k = \frac{e^{-5c}}{5}$

By data, at $t = 0, I = 0$.

$0 = 10 - k,\ k = 10$

Therefore, $I = 10 - 10e^{-5t}$

$= 10(1 - e^{-5t})$

4. **A decaying e.m.f. $E = 200e^{-5t}$ is connected in series with a $20\ ohm$ resistor and $0.01\ farad$ capacitor. Find the charge and current at any time assuming $Q = 0$ at $t = 0$. Find when the charge reaches the maximum. Calculate the maximum charge.**

To find: Q

By data, $E = 200\ e^{-5t}, R = 20, C = 0.01$

By Kirchoff's law, $RI + \frac{Q}{C} = E$

On substituting, $20I + \frac{Q}{0.01} = 200e^{-5t}$

Therefore, $\frac{dQ}{dt} + 5Q = 10e^{-5t}$

Solution is given by

$Q.e^{5t} = \int 10e^{-5t}\ e^{5t}dt + c$

$Q.e^{5t} = 10t + c$

By data, at $t = 0, Q = 0$.

$0 = 0 + c,\ \ c = 0$

Therefore, $Q.e^{5t} = 10t$

$Q = 10t\ e^{-5t}$

To find: When Q attains maximum.

$\frac{dQ}{dt} = 10(e^{-5t} - 5te^{-5t}) = 10(1 - 5t)e^{-5t}$

Q is maximum when $\frac{dQ}{dt} = 0$

$10(1 - 5t)e^{-5t} = 0$

$t = \frac{1}{5}$.

To find: Max Q

Maximum value of $Q = 10\left(\frac{1}{5}\right)e^{-1} = \frac{2}{e}$

5. **When a Resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\frac{di}{dt} + Ri = E$. If $E = 10\sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.**

$L\frac{dI}{dt} + RI = E$, Put $E = 10\sin t\,, i = 0, t = 0$

$L\frac{dI}{dt} + RI = 10\sin t$

$\frac{dI}{dt} + \frac{R}{L}I = 10\sin t$

$I.e^{\int\frac{R}{L}dt} = \int 10\sin t\,.e^{\int\frac{R}{L}dt}dt + c$

$I.e^{\frac{Rt}{L}} = \int 10\sin t\,.e^{\frac{Rt}{L}}dt + c$

$I.e^{\frac{Rt}{L}} = \frac{10e^{\frac{Rt}{L}}}{\frac{R^2}{L^2}+1}\left(\frac{R}{L}\sin t - \cos t\right) + c$

$I = \frac{10L}{R^2+L^2}(R\sin t - L\cos t) + ce^{-\frac{Rt}{L}}$

By data, at $t = 0, I = 0$.

$0 = \frac{10L}{L^2+R^2}(-L) + c$

Therefore, $c = \frac{10L^2}{L^2+R^2}$

$I = \frac{10L}{R^2+L^2}\left(R\sin t - L\cos t + Le^{-\frac{Rt}{L}}\right)$

6. **When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current I builds up at a rate given by $\boldsymbol{L\frac{di}{dt}+Ri=E}$. Find $\boldsymbol{i}$ as a function of $\boldsymbol{t}$. How long will it be , before the current has reached one-half its final value if $\boldsymbol{E=6}$ volts, $\boldsymbol{R=100}$ ohms and $\boldsymbol{L=0.1}$ henry?**

To find I:

$L\frac{dI}{dt}+RI=E$, Put $E=6, R=100, L=0.1$

$0.1\frac{dI}{dt}+100I=6$

$\frac{dI}{dt}+1000I=60$

$\frac{dI}{60-1000I}=dt$

$-0.001\log(60-1000I)=t+c$

$\log(60-1000I)=-1000t+c'$

$60-1000I=ke^{-1000t}$

When $t=0, i=0. so, k=60$

$60-1000I=60e^{-1000t}$

$1000I=60-60e^{-1000t}$

$I=0.06(1-e^{-1000t})$

To find t when I reaches max. value of I/2 :

When t is max, I is max. So, Max. value of $I=0.06$

$Max. value\ of\ \frac{I}{2}=0.03$

$0.03=0.06(1-e^{-1000t})$

$e^{1000t}=2$

$t=0.0006931\ sec$

3.5 Non-linear differential equations

Introduction: Product of variables and their first order derivatives are allowed in the non-linear differential equations.

Problems:

1. **Solve:** $\boldsymbol{p^2 + p(x+y) + xy = 0}$

$(p+x)(p+y) = 0$

$p = -x$	$p = -y$
$\frac{dy}{dx} = -x$	$\frac{dy}{dx} = -y$
$dy = -x\,dx$	$\frac{1}{y}\,dy = -dx$
$\frac{x^2}{2} + y - c = 0$	$x + \log y - c = 0$

Therefore, the general solution is

$\left(\frac{x^2}{2} + y - c\right)(x + \log y - c) = 0$

2. **Solve:** $\boldsymbol{p^2 + 2p\,cosh\,x + 1 = 0}$

$p^2 + p\,(e^x + e^{-x}) + 1 = 0$

$p(p + e^x) + e^{-x}(p + e^x) = 0$

$(p + e^x)(p + e^{-x}) = 0$

$p = -e^x$	$p = -e^{-x}$
$\frac{dy}{dx} = -e^x$	$\frac{dy}{dx} = -e^{-x}$
$dy = -e^x\,dx$	$dy = -e^{-x}\,dx$
$e^x + y - c = 0$	$-e^{-x} + y - c = 0$

Therefore, general solution is

$(y + e^x - c)(y - e^{-x} - c) = 0$

3. Solve: $\boldsymbol{xyp^2 + p(3x^2 - 2y^2) - 6xy = 0}$

$xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

$xyp^2 + 3x^2p - 2y^2p - 6xy = 0$

$xp(yp + 3x) - 2y(yp + 3x) = 0$

$(xp - 2y)(yp + 3x) = 0$

$xp = 2y$ $x\frac{dy}{dx} = 2y$ $\frac{1}{y}dy = \frac{2}{x}\ dx$ $\log y = 2\log x + c$ $\log y = \log cx^2$ $y = cx^2$	$yp = -3x$ $y\frac{dy}{dx} = -3x$ $y\ dy = -3x\ dx$ $\frac{y^2}{2} + \frac{3x^2}{2} - c = 0$ $y^2 + 3x^2 - 2c = 0$

Therefore, general solution is

$(y - cx^2)(y^2 + 3x^2 - 2c) = 0$

4. Solve: $\boldsymbol{p(p + y) = x(x + y)}$

$p^2 + py - x^2 - xy = 0$

$(p^2 - x^2) + y(p - x) = 0$

$(p - x)(p + x + y) = 0$

$p = x$ $\frac{dy}{dx} = x$ $dy = x\ dx$ On integrating, $y = \frac{x^2}{2} + c$ $y - \frac{x^2}{2} - c = 0$	$p = -x - y$ $\frac{dy}{dx} = -x - y$ $\frac{dy}{dx} + y = -x$ This is an L.D.E. Solution is $ye^x = \int -x\,.\,e^x dx + c$ $ye^x = -(xe^x - e^x) + c$ $e^x(x + y - 1) - c = 0$

Therefore, general solution is

$\left[y - \frac{x^2}{2} - c\right][e^x(x + y - 1) - c] = 0$

5. Solve: $p^2 + 2pycot\ x = y^2$

$p^2 + 2py\cot x - y^2 = 0$

$(p + y\cot x)^2 - y^2 - y^2\cot^2 x = 0$

$(p + y\cot x)^2 - y^2 cosec^2 x = 0$

$(p + y\cot x + y\ cosec\ x)(p + y\cot x - y\ cosec\ x) = 0$

$p + y\cot x + y\ cosec\ x = 0$ $\frac{dy}{y} = (-cosec\ x - \cot x)dx$ $\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right)dx$ $\frac{dy}{y} = -\left(\frac{1-\cos^2 x}{\sin x(1-\cos x)}\right)dx$ $\frac{dy}{y} = -\left(\frac{\sin x}{1-\cos x}\right)dx$ On integrating, $\log y = -\log(1 - \cos x) + \log c$ $y(1 - \cos x) - c = 0$	$p + y\cot x - y\ cosec\ x = 0$ $\frac{dy}{y} = (cosec\ x - \cot x)dx$ $\frac{dy}{y} = \left(\frac{1-\cos x}{\sin x}\right)dx$ $\frac{dy}{y} = \left(\frac{1-\cos^2 x}{\sin x(1+\cos x)}\right)dx$ $\frac{dy}{y} = \left(\frac{\sin x}{1+\cos x}\right)dx$ On integrating, $\log y = -\log(1 + \cos x) + \log c$ $y(1 + \cos x) - c = 0$

Therefore, general solution is

$[y(1 - \cos x) - c][y(1 + \cos x) - c] = 0.$

6. Solve: $x^2\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{dy}{dx}\right) - 6y^2 = 0$

$x^2p^2 + xyp - 6y^2 = 0$

$(xp + 3y)(xp - 2y) = 0$

$xp + 3y = 0$ $x\frac{dy}{dx} = -3y$ $\frac{1}{y}dy = \frac{-3}{x}dx$ On integrating, $\log y = -3\log x + \log c$	$xp - 2y = 0$ $x\frac{dy}{dx} = 2y$ $\frac{1}{y}dy = \frac{2}{x}dx$ On integrating, $\log y = 2\log x + \log c$

$\log y + 3\log x = \log c$ $\log yx^3 = \log cx$ $yx^3 = c$ $x^3y - c = 0$	$\log y = \log x^2 + \log c$ $y = cx^2$ $y - cx^2 = 0$

Therefore, the general solution is

$(x^3y - c)(y - cx^2) = 0$

7. **Solve: $4y^2p^2 + 2pxy(3x + 1) + 3x^3 = 0$**

$(2yp)^2 + 2yp(3x^2 + x) + 3x^3 = 0$

$2yp(2yp + 3x^2) + x(2yp + 3x^2) = 0$

$(2yp + x)(2yp + 3x^2) = 0$

$2yp + x = 0$ $2y\frac{dy}{dx} = -x$ $2y\,dy = -x\,dx$ On integrating, $y^2 = -\frac{x^2}{2} + c$	$2yp + 3x^2 = 0$ $2y\frac{dy}{dx} = -3x^2$ $2y\,dy = -3x^2\,dx$ On integrating, $y^2 = -x^3 + c$

Therefore, the general solution is

$\left(y^2 + \frac{x^2}{2} - c\right)(x^3 + y^2 - c) = 0$

8. **Solve: $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$**

$p - \frac{1}{p} = \frac{x^2 - y^2}{xy}$

$\frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy}$

$xyp^2 - (x^2 - y^2)p - xy = 0$

$xp(yp - x) + y(yp - x) = 0$

$(xp + y)(yp - x) = 0$

$xp + y = 0$ $x\frac{dy}{dx} = -y$ $\frac{1}{y}\,dy = -\frac{1}{x}\,dx$ On integrating, $\log y = -\log x + \log c$	$yp - x = 0$ $y\frac{dy}{dx} = x$ $y\,dy = x\,dx$ On integrating, $\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$

$\log x + \log y = \log c$ $xy = c$	$y^2 - x^2 = c$ $y^2 - x^2 - c = 0$

Therefore, the general solution is

$(xy - c)(y^2 - x^2 - c) = 0$

9. Solve: $\boldsymbol{yp^2 + (x - y)p - x = 0}$

$yp(p - 1) + x(p - 1) = 0$

$(yp + x)(p - 1) = 0$

$yp + x = 0$ $y\frac{dy}{dx} = -x$ $y\,dy = -x\,dx$ On integrating, $\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c}{2}$ $y^2 + x^2 = c$ $x^2 + y^2 - c = 0$	$p - 1 = 0$ $\frac{dy}{dx} = 1$ $dy = dx$ On integrating, $y = x + c$ $y - x - c = 0$

Therefore, the general solution is

$(x^2 + y^2 - c)(y - x - c) = 0$

10. Solve: $\boldsymbol{x^2p^2 + xp - (y^2 + y) = 0}$

$(x^2p^2 - y^2) + (xp - y) = 0$

$(xp - y)(xp + y + 1) = 0$

$xp - y = 0$ $x\frac{dy}{dx} = y$ $\frac{1}{y}\,dy = \frac{1}{x}\,dx$ On integrating, $\log y = \log x + \log c$ $\log y = \log cx$ $y = cx$	$xp + y + 1 = 0$ $x\frac{dy}{dx} = -y - 1$ $\frac{1}{y+1}\,dy = -\frac{1}{x}\,dx$ On integrating, $\log(y + 1) = -\log x + \log c$ $\log x + \log(y + 1) = \log c$ $x(y + 1) = c$

Therefore, the general solution is

$$(y - cx)(xy + x - c) = 0$$

11. Solve: $\boldsymbol{xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\left(\frac{dy}{dx}\right) + xy = 0}$

$$xyp^2 - x^2p - y^2p + xy = 0$$

$$xp(yp - x) - y(yp - x) = 0$$

$$(yp - x)(xp - y) = 0$$

$yp - x = 0$ $y\frac{dy}{dx} = x$ $y\,dy = x\,dx$ On integrating, $\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$ $y^2 = x^2 + c$ $y^2 - x^2 - c = 0$	$xp - y = 0$ $x\frac{dy}{dx} = y$ $\frac{1}{y}\,dy = \frac{1}{x}\,dx$ On integrating, $\log y = \log x + \log c$ $\log y = \log cx$ $y = cx$

Therefore, the general solution is

$$(y^2 - x^2 - c)(y - cx) = 0$$

12. Solve: $\boldsymbol{y\left(\frac{dy}{dx}\right)^2 + (x - y)\left(\frac{dy}{dx}\right) - x = 0}$

$$yp^2 + xp - yp - x = 0$$

$$p(yp + x) - (yp + x) = 0$$

$$(p - 1)(yp + x) = 0$$

$p - 1 = 0$ $\frac{dy}{dx} = 1$ $dy = dx$ On integrating, $y = x + c$	$yp + x = 0$ $y\frac{dy}{dx} = -x$ $y\,dy = -x\,dx$ On integrating, $\frac{y^2}{2} = -\frac{x^2}{2} + \frac{c}{2}$

	$y^2 = -x^2 + c$

Therefore, the general solution is

$(y - x - c)(x^2 + y^2 - c) = 0$

3.6 Clairaut's equation and reducible to Clairaut's equation

Introduction:

This is of the form $y = px + f(p)$. General solution is $y = cx + f(c)$.

Working rule to find singular solution:

- Differentiate general solution partially w.r.to c.
- Substitute the value of c in the general solution.

Note:

$y = px + f(p)$ ---- (1)

Differentiate w.r.to x,

$\frac{dy}{dx} = p + p'x + f'(p)p'$

$p'x + f'(p)p' = 0$

$p'[x + f'(p)] = 0$

$p' = 0$

$p = c$

Substitute $p = c$ in (1),

$y = cx + f(c)$.

This is the general solution.

Problems:

1. Find the general solution and the singular solution of $\boldsymbol{p = sin\,(y - xp)}$.

$y - xp = \sin^{-1} p$

$y = xp + \sin^{-1} p$

This is in Clairaut's form.

General solution is $y = cx + \sin^{-1} c$ ------ (1)

Differentiate partially w.r.to c,

$0 = x + \frac{1}{\sqrt{1-c^2}}$	$1 - c^2 = \frac{1}{x^2}$
$\frac{1}{\sqrt{1-c^2}} = -x$	$c^2 = 1 - \frac{1}{x^2}$
$\sqrt{1-c^2} = -\frac{1}{x}$	$c = \frac{\sqrt{x^2-1}}{x}$

Substitute the value of c in (1).

$y = \sqrt{x^2-1} + \sin^{-1}\frac{\sqrt{x^2-1}}{x}$

This is the required singular solution.

2. **Find the general solution and singular solution of $\boldsymbol{sin\, px\, cos\, y = cos\, px\, sin\, y + p}$**

$\sin(px - y) = p$

$px - y = \sin^{-1} p$

$y = px - \sin^{-1} p$

This is in Clairaut's form.

General solution is $y = cx - \sin^{-1} c$ ------ (1)

Differentiate partially w.r.to c,

$0 = x - \frac{1}{\sqrt{1-c^2}}$

$\frac{1}{\sqrt{1-c^2}} = x$

$\sqrt{1-c^2} = \frac{1}{x}$

$1 - c^2 = \frac{1}{x^2}$

$c^2 = 1 - \frac{1}{x^2}$

$c = \frac{\sqrt{x^2-1}}{x}$

Substitute the value of c in (1).

$y = \sqrt{x^2-1} - \sin^{-1}\frac{\sqrt{x^2-1}}{x}$

This is the required singular solution.

3. **Find the general solution and the singular solution of $\boldsymbol{p = log\ (px - y)}$.**

$p = \log(px - y)$

$e^p = xp - y$

$y = xp - e^p$

This is in Clairaut's form.

General solution is $y = cx - e^c$ ------ (1)

Differentiate partially w.r.to c,

$0 = x - e^c$

$x = e^c$

$c = \log x$

Substitute the value of c in (1).

$y = x\log x - x$

This is the required singular solution.

4. **Find the general solution and the singular solution of $(\boldsymbol{y - px})(\boldsymbol{p - 1}) = \boldsymbol{p}$.**

$y - xp = \frac{p}{p-1}$

$y = xp + \frac{p}{p-1}$

This is in Clairaut's form.

General solution is $y = cx + \frac{c}{c-1}$ ------ (1)

Differentiate partially w.r.to c,

$0 = x + \frac{(c-1)1 - c(1)}{(c-1)^2}$

$x = \frac{1}{(c-1)^2}$

$(c-1)^2 = \frac{1}{x}$

$c - 1 = \frac{1}{\sqrt{x}}$

$c = 1 + \frac{1}{\sqrt{x}}$

Substitute the value of c in (1).

$y = x + 2\sqrt{x} + 1$

$y = (\sqrt{x} + 1)^2$

This is the required singular solution.

5. **Find the general solution and the singular solution of $\boldsymbol{xp^2 - yp + a = 0}$.**

$yp = xp^2 + a$

$y = xp + \frac{a}{p}$

This is in Clairaut's form.

General solution is $y = cx + \frac{a}{c}$ ------ (1)

Differentiate partially w.r.to c,

$$0 = x - \frac{a}{c^2}$$

$$x = \frac{a}{c^2}$$

$$c = \sqrt{\frac{a}{x}}$$

Substitute the value of c in (1).

$$y = 2\sqrt{ax}$$

This is the required singular solution.

6. **Find the general solution and the singular solution of $xp^3 - yp^2 + 1 = 0$.**

$$yp^2 = xp^3 + 1$$

$$y = xp + \frac{1}{p^2}$$

This is in Clairaut's form.

General solution is $y = cx + \frac{1}{c^2}$ ------ (1)

Differentiate partially w.r.to c,

$$0 = x - \frac{2}{c^3}$$

$$x = \frac{2}{c^3}$$

$$c = \left(\frac{2}{x}\right)^{\frac{1}{3}}$$

Substitute the value of c in (1).

$$y = x\left(\frac{2}{x}\right)^{\frac{1}{3}} + \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

This is the required singular solution.

7. **Find the general solution and the singular solution of $y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\frac{dy}{dx}$.**

$$y = -2p^2 + (x+1)p$$

$$y = px + (p - 2p^2)$$

This is in Clairaut's form.

General solution is $y = cx + (c - 2c^2)$ ------ (1)

Differentiate partially w.r.to c,

$$0 = x + 1 - 4c$$

$$c = \frac{x+1}{4}$$

Substitute the value of c in (1).

$$y = x\left(\frac{x+1}{4}\right) + \left(\frac{x+1}{4} - \frac{(x+1)^2}{8}\right)$$

$8y = 2x(x+1) + 2(x+1) - (x+1)^2$

$8y = (x+1)(2x+2-x-1)$

$8y = (x+1)^2$

This is the required singular solution.

8. Solve $y^2(y - xp) = x^4p^2$ using substitutions $X = \frac{1}{x}$ and $Y = \frac{1}{y}$.

$y^2(y - xp) = x^4p^2$ ---- (1)

$P = \frac{dY}{dX} = \frac{d\left(\frac{1}{y}\right)}{d\left(\frac{1}{x}\right)} = \frac{-\frac{1}{y^2}dy}{-\frac{1}{x^2}dx} = \frac{x^2}{y^2}p$

Put $x = \frac{1}{X},\ y = \frac{1}{Y}\ , p = \frac{y^2}{x^2}P = \frac{X^2}{Y^2}\ P$ in (1)

$\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X}\ \frac{X^2}{Y^2}\ P\right) = \frac{1}{X^4}\left(\frac{X^2}{Y^2}\ P\ \right)^2$

$\frac{1}{Y^4}(Y - XP) = \frac{P^2}{Y^4}$

$Y - XP = P^2$

$Y = XP + P^2$

This is in Clairaut's form.

General solution is

$Y = cX + c^2\ \Rightarrow\ \frac{1}{y} = \frac{c}{x} + c^2$

9. Solve $(px - y)(py + x) = 2p$ by reducing into Clairaut's form, taking substitutions $X = x^2$ and $Y = y^2$.

$(px - y)(py + x) = 2p$ ---- (1)

$P = \frac{dY}{dX} = \frac{d(y^2)}{d(x^2)} = \frac{2y\ dy}{2x\ dx} = \frac{y}{x}p$

Put $x = \sqrt{X},\ y = \sqrt{Y}\ , p = \frac{x}{y}P = \frac{\sqrt{X}}{\sqrt{Y}}\ P$ in (1)

$\left(\frac{\sqrt{X}}{\sqrt{Y}}\ P\ \sqrt{X} - \sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}\ P\sqrt{Y} + \sqrt{X}\right) = 2\frac{\sqrt{X}}{\sqrt{Y}}\ P$

$\frac{1}{\sqrt{Y}}(P\ X - Y)\sqrt{X}\ (\ P + 1) = 2\frac{\sqrt{X}}{\sqrt{Y}}\ P$

$(\ P\ X - Y)(\ P + 1) = 2\ P$

$P\ X - Y = \frac{2P}{P+1}$

$Y = PX - \frac{2P}{P+1}$

This is in Clairaut's form.

General solution is

$Y = cX - \frac{2c}{c+1}$

Put $X = x^2$ and $Y = y^2$

$y^2 = cx^2 - \frac{2c}{c+1}$

10. Solve $\boldsymbol{x^2(y - px) = p^2 y}$ by reducing into Clairaut's form, using the substitutions $\boldsymbol{X = x^2}$ and $\boldsymbol{Y = y^2}$.

$x^2(y - px) = p^2 y$ ---- (1)

$P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}p$

Put $x = \sqrt{X},\ y = \sqrt{Y}\,, p = \frac{x}{y}P = \frac{\sqrt{X}}{\sqrt{Y}}\,P$ in (1)

$X\left(\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}}p\sqrt{X}\right) = \left(\frac{\sqrt{X}}{\sqrt{Y}}\,P\right)^2\sqrt{Y}$

$\frac{X}{\sqrt{Y}}(Y - pX) = \frac{X}{\sqrt{Y}}p^2$

$Y - pX = p^2$

$Y = pX + p^2$

This is in Clairaut's form.

General solution is

$Y = cX + c^2$

Put $Y = y^2,\ X = x^2$

$y^2 = cx^2 + c^2$

11. Solve $\boldsymbol{e^{4x}(p - 1) + e^{2y}p^2 = 0}$ by using substitutions $\boldsymbol{X = e^{2x}, Y = e^{2y}}$.

$e^{4x}(p - 1) + e^{2y}p^2 = 0$ ---- (1)

$P = \frac{dY}{dX} = \frac{2\,e^{2y}dy}{2\,e^{2x}dx} = \frac{e^{2y}}{e^{2x}}p$

Put $e^{2x} = X, e^{2y} = Y, p = \frac{e^{2x}}{e^{2y}}P = \frac{X}{Y}P$ in (1)

$X^2\left(\frac{X}{Y}P - 1\right) + Y\left(\frac{X}{Y}P\right)^2 = 0$

$\frac{X^2}{Y}(XP - Y) + \frac{X^2}{Y}P^2 = 0$

$XP - Y + P^2 = 0$

$Y = XP + P^2$

This is in Clairaut's form.

General solution is

$Y = cX + c^2$

Put $X = e^{2x}, Y = e^{2y}$

$e^{2y} = c\, e^{2x} + c^2$

12. Solve $(px + y)^2 = py^2$ by using the substitutions $X = y$ and $Y = xy$.

$(px + y)^2 = py^2$ ---- (1)

$P = \frac{dY}{dX} = \frac{d(xy)}{d(y)} = \frac{xdy+ydx}{dy} = \frac{xp+y}{p} = x + \frac{y}{p}$

$P - x = \frac{y}{p},\ p = \frac{y}{(P-x)}$

Put $x = \frac{Y}{y} = \frac{Y}{X},\ y = X, p = \frac{y}{(P-x)} = \frac{X}{\left(P-\frac{Y}{X}\right)} = \frac{X^2}{PX-Y}$ in (1).

$\left\{\left(\frac{X^2}{PX-Y}\right)\frac{Y}{X} + X\right\}^2 = \left(\frac{X^2}{PX-Y}\right)X^2$

$X^2\left\{\frac{Y}{PX-Y} + 1\right\}^2 = X^2\left(\frac{X^2}{PX-Y}\right)$

$\left\{\frac{PX}{PX-Y}\right\}^2 = \frac{X^2}{PX-Y}$

$P^2 = PX - Y$

$Y = PX - P^2$

This is in Clairaut's form.

General solution is

$Y = cX - c^2$

$xy = cy - c^2$