Fuzzy Logic

- A linguistic variable is a fuzzy variable represented by a fuzzy set
- Fuzzy sets are expressed by linguistic values

Example: 'John is Tall'

Linguistic variable: John

Linguistic value(Fuzzy set): tall

- Linguistic variables and linguistic values are used in fuzzy rules to represent knowledge
  Examples:
  IF wind is strong THEN sailing is good
  IF speed is fast THEN stopping_distance is long

Linguistic Variables & values?

- A simple fuzzy rule has the following form:
  IF x is A THEN y is B
  Linguistic variables: x, y
  Linguistic values: A, B
  Antecedent: "x is A"
  Consequent: "y is B"

Example

IF speed is slow
THEN stopping_distance is short

IF speed is medium
THEN stopping_distance is average

IF speed is fast
THEN stopping_distance is long

Given speed = x km/h, what is the stopping_distance for 40km/h?

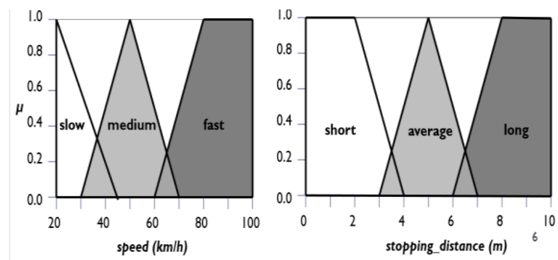
Fuzzy Inference

1. Fuzzification : define fuzzy sets & determine membership degrees of crisp inputs in appropriate fuzzy sets
2. Inference : calculate fuzzy output for each rule
3. Composition : aggregate rule outputs
4. Defuzzification : calculate the crisp output

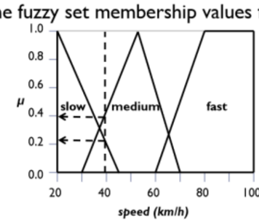
Rule 1: IF speed is slow
THEN stopping_distance is short

Rule 2: IF speed is medium
THEN stopping_distance is average

Rule 3: IF speed is fast
THEN stopping_distance is long

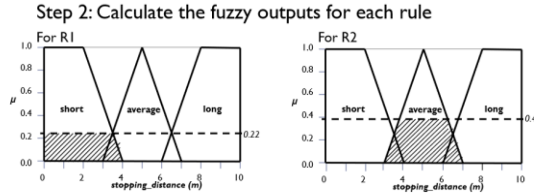


Step 1: Find the fuzzy set membership values for the input

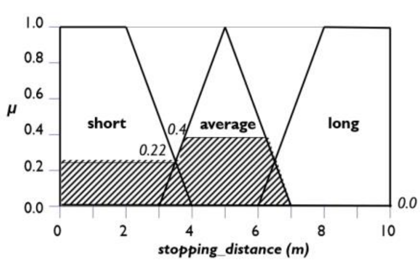


μ_{slow}(speed=40) = 0.22 , μ_{medium}(speed=40) = 0.4 , μ_{fast}(speed=40) = 0

Step 2: Calculate the fuzzy outputs for each rule



Step 3: Aggregate the rule outputs



Step 4: Defuzzification

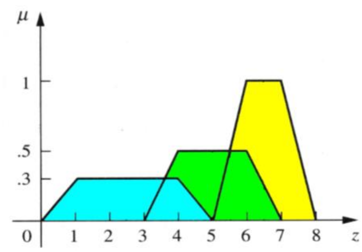
Calculate the centre of gravity (COG):

$$COG = \frac{\int_a^b \mu_A(x)x\,dx}{\int_a^b \mu_A(x)\,dx}$$

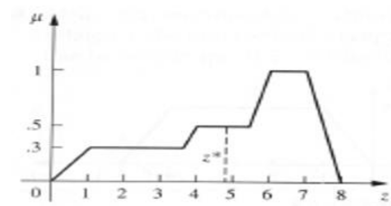
Use the estimate of the above:

$$COG = \frac{\sum_{x=a}^{b} \mu_A(x)x}{\sum_{x=a}^{b} \mu_A(x)} = \frac{(0+1+2+3)\times 0.22 + (4+5+6)\times 0.4 + 0.7\times 0}{0.22+0.22+0.22+0.22+0.4+0.4+0.4+0.4} = 2.95$$

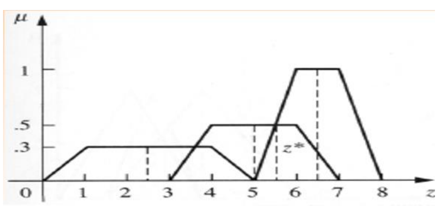
Result: The stopping_distance is 2.95 m



COG

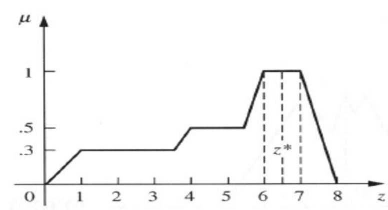


Weighted Average



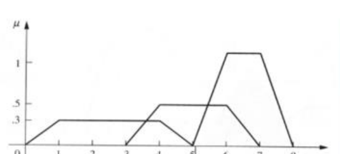
$$= \frac{(.3 \times 2.5) + (.5 \times 5) + (1 \times 6.5)}{.3 + .5 + 1} = 5.41 \text{ meters}$$

Mean Max



= (6+7)/2 = 6.5

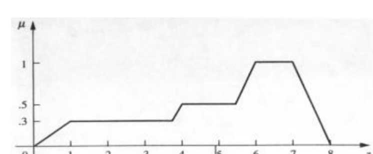
Centre of sums



½[(5-0)+ (4-1)]*0.3, ½ [4+2]*0.5, ½[4]*1

= (1.2*3+1.5*5+2*6.5)/1.2+1.5+2= 5.1

Centre of largest area



= [(.5+2.5+5+7)/4]=4.7

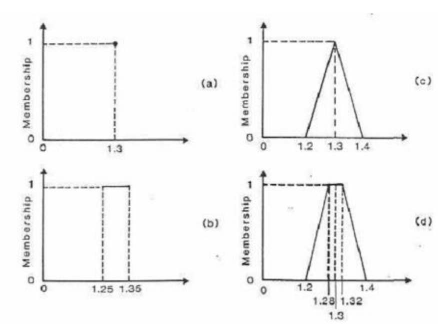
Definition

- Method of reasoning like humans
- Decision making - between digital values YES & No

Membership Function

- The membership function of a fuzzy number is of the form A : R-> [0,1]

$$A(x) = \begin{cases} f(x) & \text{for } x \in [a,b] \\ 1 & \text{for } x \in [b,c] \\ g(x) & \text{for } x \in [c,d] \\ 0 & \text{for } x < a \text{ and } x > d \end{cases}$$



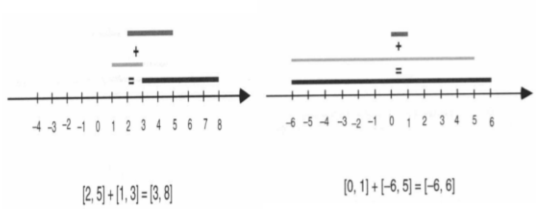
Fuzzy arithmetic on intervals

Based on two properties of Fuzzy numbers

1. Fuzzy numbers can be represented with α-cuts
2. Fuzzy number are closed intervals of real numbers for all α (0,1]

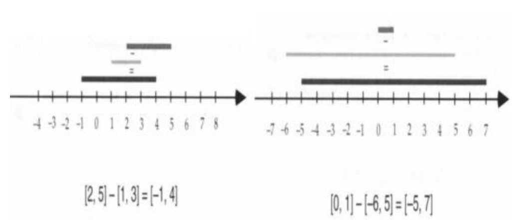
Addition

[a, b] + [c, d] = [a+c, b+d]



Fuzzy subtraction

[a, b] - [c, d] = [a-d, b-c]



Fuzzy Multiplication

$[a, b] \cdot [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$

$[-1,2].[-2,0.5] = [-4,2]$

Fuzzy Division

$[a, b] / [c, d] = [a, b] \cdot [1/d, 1/c]$

$=[\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$

$[-1, 1] / [-2, -0.5] = [-2, 2]$

Fuzzy arithmetic operations

Addition & Subtraction

Let $A = [a_1, a_2]$ and $B = [b_1, b_2]$ in $\Re$

If $x \in [a_1, a_2]$ and $y \in [b_1, b_2]$ than $x+y \in [a_1+b_1, a_2+b_2]$

Symbolically,we write

$$A(+)B = [a_1, a_2](+)[b_1, b_2] = [a_1+b_1, a_2+b_2]$$

subtraction:

$$A(-)B = [a_1, a_2](-) [b_1, b_2] = [a1-b_2, a_2-b_1]$$

Compute A(+)B and A(-)B,where

$$\mu_A(x)=\begin{cases} 0 & x \le -6 \\ (x+6)/4 & -6 < x \le -2 \\ (-x+3)/5 & -2 < x \le 3 \\ 0 & x > 3 \end{cases}, \quad \mu_B(x)=\begin{cases} 0 & x \le -1 \\ (x+1)/5 & -1 < x \le 4 \\ (-x+10)/6 & 4 < x \le 10 \\ 0 & x > 10 \end{cases}$$

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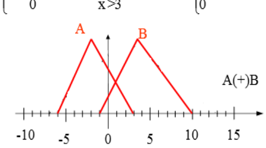

① Find α -cuts $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $[b_1^{(\alpha)}, b_2^{(\alpha)}]$

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$$\frac{a_1^{(\alpha)}+6}{4}=\alpha, \text{ and } \frac{-a_2^{(\alpha)}+3}{5}=\alpha$$

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Solving $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$, we obtain

$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [4\alpha - 6, -5\alpha + 3]$

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Similarly, $B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = [5\alpha - 1, -6\alpha + 10]$

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② $A_\alpha (+) B_\alpha = [9\alpha - 7, -11\alpha + 13] \triangleq [c_1^{(\alpha)}, c_2^{(\alpha)}] = (A(+)B)_\alpha$

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③ $x = 9\alpha - 7$, $\alpha = \frac{x+7}{9}$

$x = -11\alpha + 13$, $\alpha = \frac{-x+13}{11}$

①Find α -cuts $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $[b_1^{(\alpha)}, b_2^{(\alpha)}]$

$$\frac{a_1^{(\alpha)}+6}{4} = \alpha, \quad \text{and} \quad \frac{-a_2^{(\alpha)}+3}{5} = \alpha$$

Solving $a_1^{(\alpha)}$ and $a_2^{(\alpha)}$, we obtain

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② $A_\alpha (+) B_\alpha = [9\alpha - 7, -11\alpha + 13] \triangleq [c_1^{(\alpha)}, c_2^{(\alpha)}] = (A(+)B)_\alpha$

③ $x = 9\alpha - 7,\ \alpha = \frac{x+7}{9}$

$x = -11\alpha + 13,\ \alpha = \frac{-x+13}{11}$

$$\Rightarrow \mu_{A(+)B}(x) = \begin{cases} 0 & x \le -7 \\ (x+7)/9 & -7 < x \le 2 \\ (-x+13)/11 & 2 < x \le 13 \\ 0 & x > 13 \end{cases}$$

$A_\alpha (-) B_\alpha = [10\alpha - 16, -10\alpha + 4] = (A(-)B)_\alpha$

$10\alpha - 16 = x,\ \alpha = \frac{x+16}{10}$

$-10\alpha + 4 = x,\ \alpha = \frac{4-x}{10}$

$$\mu_{A(-)B}(x) = \begin{cases} 0 & x \le -16 \\ \frac{x+16}{10} & -16 < x \le -6 \\ \frac{-x+4}{10} & -6 < x \le 4 \\ 0 & x > 4 \end{cases}$$

To do

$$\mu_A(x) = \begin{cases} 0 & x \le 1 \\ \frac{x-1}{2} & 1 < x \le 3 \\ \frac{-x+6}{3} & 3 < x \le 6 \\ 0 & x > 6 \end{cases} \qquad \mu_B(x) = \begin{cases} 0 & x \le 2 \\ \frac{x-2}{2} & 2 < x \le 4 \\ \frac{-x+7}{3} & 4 < x \le 7 \\ 0 & x > 7 \end{cases}$$

Consider $A = [a_1, a_2]$ and $B = [b_1, b_2]$ in $\Re^+$(nonnegative real line)

$A(\bullet)B = [a_1, a_2](\bullet)[b_1, b_2] = [a_1 \bullet b_1, a_2 \bullet b_2]$

For the division of two intervals of confidence in $\Re_0^+$, we have

$A(:)B = [a_1, a_2](:)[b_1, b_2] = [a_1/b_2, a_2/b_1]$

Compute $A(\cdot)B$, $A(:)B$

$$\mu_A(x) = \begin{cases} 0 & x \le 1 \\ \frac{x-1}{2} & 1 < x \le 3 \\ \frac{-x+6}{3} & 3 < x \le 6 \\ 0 & x > 6 \end{cases} \qquad \mu_B(x) = \begin{cases} 0 & x \le 2 \\ \frac{x-2}{2} & 2 < x \le 4 \\ \frac{-x+7}{3} & 4 < x \le 7 \\ 0 & x > 7 \end{cases}$$

compute α-cuts

$$\alpha = \frac{a_1^{(\alpha)} - 1}{2} \qquad a_1^{(\alpha)} = 2\alpha + 1$$

$$\alpha = \frac{-a_2^{(\alpha)} + 6}{3} \qquad a_2^{(\alpha)} = -3\alpha + 6$$

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = [2\alpha + 1, -3\alpha + 6]$$

similarly, from $\mu_B(x)$, we have $B_\alpha = [2\alpha + 2, -3\alpha + 7]$

$A_\alpha(\cdot)B_\alpha = [(2\alpha + 1)(2\alpha + 2), (-3\alpha + 6)(-3\alpha + 7)]$

$= [4\alpha^2 + 6\alpha + 2, 9\alpha^2 - 39\alpha + 42] = (A(\cdot)B)_\alpha$

Solve $4\alpha^2 + 6\alpha + 2 = x,\ \alpha = (-3 \pm \sqrt{1+4x})/4$

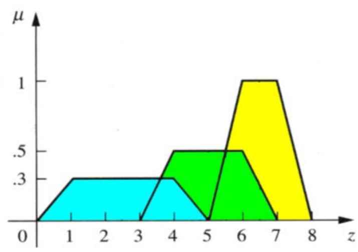
$9\alpha^2 - 39\alpha + 42 = x,\ \alpha = (13 \pm \sqrt{1+4x})/6$

$\because \alpha \in [0,1]$

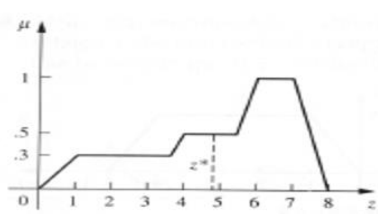
$$\therefore \mu_{A(\cdot)B}(x)=\begin{cases} 0 & x \le 2 \\ (-3+\sqrt{1+4x})/4 & 2<x \le 12 \\ (13-\sqrt{1+4x})/6 & 12<x \le 42 \\ 0 & x>42 \end{cases}$$

For $A_\alpha\,(:)B_\alpha = [\,\frac{2\alpha+1}{-3\alpha+7}\ ,\ \frac{-3\alpha+6}{2\alpha+2}\,]$

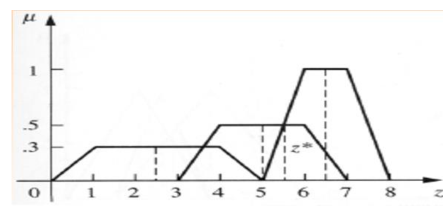
$$\Rightarrow \mu_{A(:)B}(x)=\begin{cases} 0 & x \le \frac{1}{7} \\ \frac{7x-1}{3x+2} & \frac{1}{7}<x \le \frac{3}{4} \\ \frac{-2x+6}{2x+3} & \frac{3}{4}<x \le 3 \\ 0 & x>3 \end{cases}$$



COG

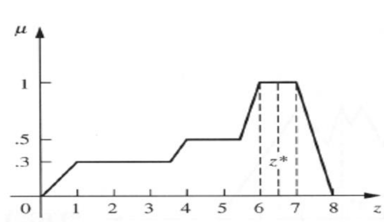


Weighted Average



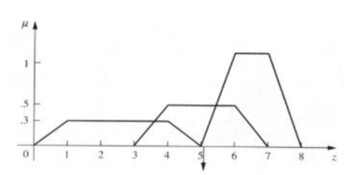
$$= \frac{(.3 \times 2.5) + (.5 \times 5) + (1 \times 6.5)}{.3 + .5 + 1} = 5.41 \text{ meters}$$

Mean Max



= (6+7)/2 = 6.5

Centre of sums



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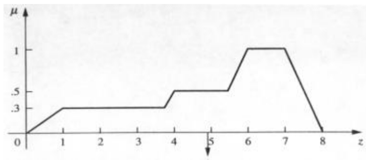
= (1.2*3+1.5*5+2*6.5)/1.2+1.5+2= 5.1

Centre of largest area



= [(.5+2.5+5+7)/4]=4.7