DS : Curs 1. 01.03.2019.

- will use Maple
- can obtain 1,5p from the lab.
- math.ubbcluj.ro/~abuica/dynsys.htm
- 2 tests : S3, S6 → 2p.

- we study processes that are dynamic = change in time.

I Continuous dynamical systems (time is a continuous variable $t \in \mathbb{R}$ $\underset{0\ 1}{\longrightarrow}$)
II Discrete dynamical systems (time is a discrete variable $t \in \mathbb{Z}$ $\underset{\dots -1\ 1\ 2 \dots}{\longrightarrow}$)

Differential equation

$x : I \to \mathbb{R}$, $I \subset \mathbb{R}$ open, nonempty interval
$t \to x(t) \in \mathbb{R}$

1. First-order scalar linear differential equations.

(1) $\underbrace{x' + \underset{\text{coeficient}}{a(t)}\, x}_{\text{the homogeneous part}} = \underbrace{f(t)}_{\text{non-homogeneous part, or the force.}}$, where $a, f \in C(I)$, $I \subset \mathbb{R}$, I nonempty, open interval.

Let $t_0 \in I$, $\eta \in \mathbb{R}$ an Initial Value Problem (IVP)

(2) $\begin{cases} x' + a(t)x = f(t) \\ x(t_0) = \eta \end{cases}$

Definition:
A function $\varphi : I \to \mathbb{R}$ is said to be a solution of (1) if $\varphi \in C'(I)$ s.t. $\varphi'(t) + a(t) = f(t)$, $\forall t \in I$.

Notation:
$C(I) = \{\varphi : I \to \mathbb{R} \text{ continuous}\}$
$C'(I) = \varphi : I \to \mathbb{R}$, s.t. $\exists\, \varphi'$ and both φ and φ' are continuous$\}$.

We will multiply the DE (1) with a function $\mu(t)$ (called integrating factor) such that we will be able to integrate afterwards. This method is called the intergrating factor method.

Notation: $A(t) = \int_{t_0}^{t} a(s)\, ds$ - is in the primitive of a s.t. $A(t_0) = 0$.

$A'(t) = a(t)$

Proposition 1 $\mu(t) = e^{A(t)}$ is an integrating factor of (1)

$$x' + a(t)x = f(t) \quad / \cdot e^{A(t)}$$

$$\underbrace{x' e^{A(t)} + x a(t) \cdot e^{A(t)}}_{(x e^{A(t)})'} = f(t) \cdot e^{A(t)}$$

$$(x e^{A(t)})' = f(t) e^{A(t)} \quad / \int_{t_0}^{t}$$

$$\int_{t_0}^{t} [x(s) \cdot e^{A(s)}]' ds = \int_{t_0}^{t} f(s) e^{A(s)} ds$$

$$x(s) e^{A(s)} \Big|_{t_0}^{t} = \int_{t_0}^{t} f(s) e^{A(s)} ds$$

$$x(t) e^{A(t)} - x(t_0) e^{A(t_0)} = \int_{t_0}^{t} f(s) e^{A(s)} ds$$

$$x(t) e^{A(t)} = x(t_0) + \int_{t_0}^{t} f(s) e^{A(s)} ds \quad / \cdot (e^{-A(t)})$$

$$x(t) = x(t_0) e^{-A(t)} + e^{-A(t)} \cdot \int_{t_0}^{t} f(s) e^{A(s)} ds$$

the general solution:

$$x = c \cdot e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} f(s) e^{A(s)} ds, \quad c \in \mathbb{R}.$$

Theorem:

The IVP (2) has a unique solution,

$$\varphi(t) = \eta \cdot e^{-A(t)} + e^{-A(t)} \int_{t_0}^{t} f(s) e^{A(s)} ds.$$

1.1. First order linear (scalar) homogeneous d.e. (LHDEs)

(3) $x' + a(t)x = 0$

we know the general solution: $x = c \cdot e^{-A(t)}$, $c \in \mathbb{R}$.

Theorem:

(i) Let x_1 be a solution of (3). Then either $x_1(t) = 0$, $\forall t \in I$ or $x_1(t) \neq 0$, $\forall t \in I$.

(ii) Let x_1 be a non-null solution of (3), Then the general solution of (3) is $x = c \cdot x_1$, $c \in \mathbb{R}$.

Proof:

Proof:

i). x_1 sol. of (3) $\rightarrow \exists\, c_1 \in \mathbb{R}$ s.t. $x_1 = c_1 e^{-A(t)}$ we have that either $c_1 = 0$ or $c_1 \neq 0$. Then it is easy to see the conclusion.

ii). $x_1 = c_1 \cdot e^{-A(t)}$, $c_1 \neq 0$.

the general sol. is $x = k \cdot e^{-A(t)}$, $k \in \mathbb{R}$ arbitrary.

$$x = k \cdot e^{-A(t)} = \frac{k}{c_1} \cdot x_1 = c \cdot x_1,\ c \in \mathbb{R}$$

$$\left\{ \frac{k}{c_1} ; k \in \mathbb{R} \right\} = \mathbb{R}$$

$$\boxed{c \overset{not}{=} \frac{k}{c_1}}$$

The separation of variables method to solve (3).

$x'(t) = -a(t)\, x(t)$; $x = 0$ is a solution

now we want to find the non-null solution

$$\frac{x'(t)}{x(t)} = -a(t) \quad / \int_{t_0}^{t}$$

$$\int_{t_0}^{t} \frac{x'(s)}{x(s)}\, ds = -\int_{t_0}^{t} a(s)\, ds$$

$$\ln|x(s)| \Big|_{t_0}^{t} = -A(t)$$

$$\ln|x(t)| - \ln|x(t_0)| = -A(t)$$

$$\ln\left|\frac{x(t)}{x(t_0)}\right| = -A(t)$$

$$\ln \frac{x(t)}{x(t_0)} = -A(t) \quad \text{because } \frac{x(t)}{x(t_0)} > 0,\ \forall t \in I$$

$$\frac{x(t)}{x(t_0)} = e^{-A(t)} \qquad x(t) = \underbrace{+x(t_0)}_{c} \cdot e^{-A(t)}$$

$$x = c \cdot e^{-A(t)},\ c \in \mathbb{R}.$$

Short-cut of variables method:

$x' = -a(t)\, x$ $\qquad x' = \frac{dx}{dt}$

$x = 0$ - solution.

$$\frac{dx}{dt} = -a(t)\, x$$

$$\frac{dx}{x} = -a(t)\, dt \quad / \int$$

$$\int \frac{dx}{x} = -\int a(t)\, dt$$

$$\ln|x| = -A(t) + c,\ c \in \mathbb{R}.$$

$$|x| = e^{-A(t) + c}$$

$\begin{cases} x = \pm e^{c} \cdot e^{-A(t)}, & c \in \mathbb{R} \\ x = 0 & \text{sol.} \end{cases}$

$\{0, e^{c}, -e^{-c} : c \in \mathbb{R}\} = \mathbb{R}$

$\rightarrow x = k \cdot e^{-A(t)},\ k \in \mathbb{R}.$

Conclusion:

4 methods: for eq. (9).

1). "Guess" a non-null solution x_1, and write the general solution as $x = c \cdot x_1$, $c \in \mathbb{R}$.

2). the separation of variables ~~so~~ method

3). The integrated factor method

4). Memorize $x = c \cdot e^{-A(t)}$, when $A'(t) = a(t)$

Examples:

Find the general solution of:

a) $x' = \lambda x$, $\lambda \in \mathbb{R}^*$ param. $x_1 = e^{\lambda t}$ - a non-null sol.

$\Rightarrow$ gen. sol $\Rightarrow x = c \cdot e^{\lambda t}$, $c \in \mathbb{R}$.

b) $tx' + 2x = 0$.

method 1: the integrating factor method

$\mu(t) = e^{A(t)}$, $A'(t) = a(t)$

$x' + \frac{2}{t}x = 0$. $a(t) = \frac{2}{t}$, $t \neq 0$ I can be either $(-\infty, 0)$ or $(0, \infty)$.

$A(t) = \underbrace{2\ln|t|}_{= \ln t^2} \Rightarrow \mu(t) = e^{\ln t^2} = t^2$

$x' + \frac{2}{t}x = 0 \ / \cdot t^2$

$t^2 x' + 2tx = 0$

$(t^2 x)' = 0 \rightarrow t^2 x = c \Rightarrow \boxed{x = \frac{c}{t^2}, \ c \in \mathbb{R}}$

method 2: separation of variables.

$x' = -\frac{2}{t}x$, $x = 0$ sol.

$x \neq 0$, $\rightarrow \frac{dx}{dt} = -\frac{2}{t}x$

$\frac{dx}{x} = -\frac{2}{t}dt$

$\ln|x| = -2\ln|t| + c$

$\ln|x| = \ln\frac{1}{t^2} + c$

$|x| = e^{\ln\frac{1}{t^2} + c}$

$x = \pm e^c \cdot e^{\ln\frac{1}{t^2}}$

$\begin{cases} x = \pm e^c \cdot \frac{1}{t^2}, \ c \in \mathbb{R} \\ x = 0 \end{cases}$

$x = c \cdot \frac{1}{t^2}$, $c \in \mathbb{R}$.

method 3:

i). Check that $x_1 = \frac{1}{t^2}$ is a sol.

ii). Find the general sol.

1.2. First order scalar linear non-homogeneous ~~non-null~~ D.E. (LNDE)

(1) $x' + a(t)x = f(t)$

Theorem:

Let x_h denote the general solution of the LHDE associated, $x' + a(t)x = 0$.
Let x_p denote a particular solution of (1).
Then the general solution of (1) is

$$\boxed{x = x_h + x_p}$$

Proof:

$x_p' + a(t)x_p = f(t)$, $\forall t \in I$. $(*)$

φ is an arbitrary sol. of (1) $\iff \varphi' + a(t)\varphi = f(t) \overset{(*)}{\iff}$

$\iff \varphi' + a(t)\varphi - [x_p' + a(t)x_p] = 0$.

$\iff (\varphi - x_p)' + a(t)[\varphi - x_p] = 0 \iff \varphi - x_p$ is a sol. of $x' + a(t)x = 0$.

So, to solve (1):

Step 1: Write the LHDE associated $x' + a(t)x = 0$.

(*) Step 2: Find its general sol., denoted x_h

(*) Step 3: Find x_p.

Step 4: The gen. sol of (1) $x = x_h + x_p$.

The Lagrange method (the variation of constants) to find x_p.

Step 2: $x_h = c \cdot e^{-A(t)}$, $c \in \mathbb{R}$

Step 3: Find x_p.

$x_p = \varphi(t) \cdot e^{-A(t)}$ $\quad \varphi = ?$

$\varphi' \cdot e^{-A(t)} + \varphi[-a(t)] \cdot e^{-A(t)} + a(t) \cdot \varphi \cdot e^{-A(t)} = f(t)$, $/ e^{A(t)}$

$\varphi' = f(t) \cdot e^{A(t)}$

$\varphi(t) = \int_{t_0}^{t} f(s) e^{A(s)} \, ds \Rightarrow$ $\boxed{x_p = e^{-A(t)} \int_{t_0}^{t} f(s) e^{A(s)} \, ds}$

Rules to find x_p in some particular cases.

1). The ~~y~~ eq. $x' - \lambda x = \alpha$, $\lambda \in \mathbb{R}^*$, $\alpha \in \mathbb{R}$
has a constant solution.

2). The eq. $x' - \lambda x = \alpha \cdot e^{bt}$, $\lambda \in \mathbb{R}^*$, $b \neq \lambda$.
has a sol. of the form $x_p = a \cdot e^{bt}$, $a = ?$

3). The eq. $x' - \lambda x = \alpha \cdot e^{\lambda t}$
has a solution of the form $x_p = a t e^{\lambda t}$, $a = ?$

4). The eq. $x' - \lambda x = \alpha_1 t + \alpha_2$, $\lambda \in \mathbb{R}^*$
has a sol. of the form $x_p = a_1 t + a_2$, $a_1, a_2 = ?$

5). The eq. $x' - \lambda x = a_1 \sin b_1 t + a_2 \sin b_2 t$

DS: Curs 2 08.03.2019.

Scalar linear differential equations

Let $n \geq 1$ be an integer
The unknown $t \in \mathbb{R} \mapsto x(t) \in \mathbb{R}$

(1) $x^{(n)} + a_1(t) \cdot x^{(n-1)} + \dots + a_{n-1}(t)\, x' + a_n(t) \cdot x = f(t)$

- where $a_1, \dots, a_n, f$ - are continuous on $I \iff \in C(I)$, $I \in \mathbb{R}$, I - open, nonempty interval
- $a_1, \dots, a_n$ - the coeficients, f is the force (on the non-hom. part)
- When $a_1, \dots, a_n$ are constant functions, we say that eq (1) has constant coeficients.

IVP: $t_0 \in I$, $\eta_1, \dots, \eta_n \in \mathbb{R}$ fixed. (2) $\begin{cases} \text{eq. (1)} \\ x(t_0) = \eta_1 \\ x'(t_0) = \eta_2 \\ \quad \vdots \\ x^{(n-1)}(t_0) = \eta_n \end{cases}$

Definition:
A function $\varphi : I \to \mathbb{R}$ is a solution of eq (1) if $\varphi \in C^n(I)$ and
$\varphi^{(n)}(t) + a_1(t) \cdot \varphi^{(n-1)}(t) + \dots + a_n(t) \cdot \varphi(t) = f(t)$, $\forall t \in I$.

Theorem. (the existence and uniqueness theorem)
The IVP (2) has a unique $\varphi^* \in C^n(I)$

The fundamental theorems for (1):
we define a map $\mathcal{L} : C^n(I) \to C(I)$
For $x \in C^n(I)$ we define $\mathcal{L}(x) \in C(I)$

$$\mathcal{L}(x)(t) = x^{(n)}(t) + a_1(t) x^{(n-1)}(t) + \dots + a_n(t) x(t), \quad \forall t \in I.$$

Lemma:
a). $C(I)$ has a structure of linear vector space with usual operations.
b). $\mathcal{L}$ is a linear map.

Proof of b:

Let $x_1, x_2 \in C^n(I)$ and $c_1, c_2 \in \mathbb{R}$.
Let $t \in I$.

$$\mathcal{L}(c_1x_1+c_2x_2)(t) = (c_1x_1+c_2x_2)^{(n)}(t) + a_1(t)(c_1x_1+c_2x_2)^{(n-1)}(t) + \dots + \underbrace{a_n(t)(c_1x_1+c_2x_2)(t)}_{c_1a_n(t)x_1(t)+c_2a_n(t)x_2(t)} = c_1\mathcal{L}(x_1)(t)+c_2\mathcal{L}(x_2)(t)$$

Direct consequences of the linearity of $\mathcal{L}$.

1). eq. (1) $\Leftrightarrow$ $\mathcal{L}(x) = f$

eq (3): $x^{(n)} + \dots + a_n(t)x = 0 \Leftrightarrow \mathcal{L}(x) = 0 \Leftrightarrow x \in \operatorname{Ker}\mathcal{L}$

2). Let $x_1, \dots, x_n$ be solutions of the LHDE (3) and $c_1, \dots, c_n \in \mathbb{R}$. Then $c_1x_1 + \dots + c_nx_n$ is a sol of (3).

3). The Superposition Principle

Let $f = \alpha_1 f_1 + \alpha_2 f_2$, where $f_1, f_2 \in C(I)$, $\alpha_1, \alpha_2 \in \mathbb{R}$.
Let x_{p_1} be a particular sol. of $\mathcal{L}(x) = f_1$
x_{p_2} — " — — " — $\mathcal{L}(x) = f_2$
then $x_p = \alpha_1 x_{p_1} + \alpha_2 x_{p_2}$ is a particular sol. of $\mathcal{L}(x) = f$.

Theorem 2: (The fundamental theorem for LHDE's)
The set of all solutions of the LHDE (3) is a linear space of dimention n.
Thus, there exists $x_1, x_2, \dots, x_n$ n linearly independent solutions of (3) and the general sol. of (3) is:

$$x = c_1x_1 + \dots + c_nx_n, \quad c_1, \dots, c_n \in \mathbb{R}.$$

Proof:

$\mathcal{L}$ is linear $\Rightarrow$ $\operatorname{Ker}\mathcal{L}$ is a linear subspace of $C(I)$
$\mathbb{R}^n$ is a linear space of dimention n.

Define $T: \operatorname{Ker}\mathcal{L} \to \mathbb{R}^n$

$$\varphi \in \operatorname{Ker}\mathcal{L} \longmapsto \begin{pmatrix} \varphi(t_0) \\ \varphi'(t_0) \\ \dots \\ \varphi^{(n-1)}(t_0) \end{pmatrix} \in \mathbb{R}^n$$

We will prove that T is bijective and T is linear. $\Leftrightarrow$
$\Leftrightarrow$ T is an izomorphism of linear spaces.

HW: Prove that T is linear.

T is bijective $\iff \forall \eta \in \mathbb{R}^n\ \exists!\ \varphi^* \in \operatorname{Ker} \mathcal{L}$ s.t. $T(\varphi^*) = \eta$.

$\iff \forall \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} \in \mathbb{R}^n\ \exists!\ \varphi^* \in C(I)$ s.t. φ^* is a solution of eq.(3) and $\varphi^*(t_0) = \eta_1, \dots, \varphi^{*(n-1)}(t_0) = \eta_n \iff$

$\iff \forall \eta \in \mathbb{R}^n\ \exists!\ \varphi^* \in C^n(I)$, sol. of the IVP

$$\begin{cases} x^{(n)} + \dots + a_n(t)x = 0 \\ \varphi(t_0) = \eta_1 \\ \varphi^{(n-1)}(t_0) = \eta_n \end{cases}$$

This is valid by the $\underbrace{\exists!\ \text{Theorem}}_{\text{Existance and uniqueness Thm.}}$

So, T is an izomorphism between $\operatorname{Ker} \mathcal{L}$ and $\mathbb{R}^n \Rightarrow$

$\Rightarrow$ the dimention of $\operatorname{Ker} \mathcal{L}$ is n. $\square$ → end of ~~de~~ proof.

Theorem 3 (the fundamental theorem for LNDE's)

The general sol of the LNDE (1) is $x = x_h + x_p$ where x_h is the general sol. of the LHDE associated and x_p is a particularization of the LNDE (1).

Proof.

The set of all solutions of an eq. $\mathcal{L}(x) = f$ in $\operatorname{Ker} \mathcal{L} + \{x_p\}$, wher x_p is a particular sol. of $\mathcal{L}(x) = f$.

Definition:

Let $x_1, \dots, x_m \in C(I)$. We say that they are lin. indep. if the following implication holds

$$c_1 x_1 + \dots + c_m x_m = 0 \iff c_1 = c_2 = \dots = c_m = 0.$$

Remark: We present two ways to prove the lin. indep.

First, take $c_1, \dots, c_m \in \mathbb{R}$ s.t. $c_1x_1 + \dots + c_m x_m = 0 \iff c_1 x_1(t) + \dots + c_m x_m(t) = 0$ $\forall t \in I$.

Method 1. We fix $t_1, \dots, t_m \in I$ and form the lin. system with unknowns $c_1, \dots, c_m$

$$c_1 x_1(t_1) + \dots + c_m x_m(t_1) = 0$$
$$\vdots$$
$$c_1 x_1(t_m) + \dots + c_m x_m(t_m) = 0$$

if $\begin{vmatrix} x_1(t_1) & \dots & x_m(t_1) \\ \vdots & & \vdots \\ x_1(t_m) & \dots & x_m(t_m) \end{vmatrix} \neq 0$ then $c_1 = \dots = c_m = 0$.

Method 2: $\begin{cases} c_1 x_1(t) + \dots + c_m x_m(t) = 0 \\ c_1 x_1'(t) + \dots + c_m x_m'(t) = 0 \\ \vdots \\ c_1 x_1^{(n-1)}(t) + \dots + c_m x_m^{(n-1)}(t) = 0 \end{cases}$, $\forall t \in I$.

if $\exists t_0 \in I$ s.t. $\begin{vmatrix} x_1(t_0) & \dots & x_m(t_0) \\ x_1'(t_0) & \dots & x_m'(t_0) \\ \vdots & & \vdots \\ x_1^{(n-1)}(t_0) & \dots & x_m^{(n-1)}(t_0) \end{vmatrix} \neq 0 \Rightarrow$
$\Rightarrow c_1 = \dots = c_m = 0$.

Example:

a). Prove that $x = c_1 e^t + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$.
is the general solution of $x'' - x = 0$.

b) Prove that $x = c_1 \cosh(t) + c_2 \sinh(t)$, $c_1, c_2 \in \mathbb{R}$. is the general solution of $x'' - x = 0$.

(HW) c). $x = c_1 \cos(t) + c_2 \sin(t)$, $c_1, c_2 \in \mathbb{R}$ is sol. of $x'' + x = 0$.

$\boxed{\cosh(t) = \frac{e^t + e^{-t}}{2}}$; $\boxed{\sinh(t) = \frac{e^t - e^{-t}}{2}}$

$(\cosh(t))' = \sinh(t)$; $(\sinh(t))' = \cosh(t)$

Solutions:

$x'' - x = 0$ is a second order LHDE.

a) • e^t and e^{-t} are solutions of $x'' - x = 0$.
We check and we say that this is true.

• $\{e^t, e^{-t}\}$ are lin. indep.

$x_1(t) = e^t$ $\quad x_2(t) = e^{-t}$.

$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -e^t \cdot e^{-t} - e^t \cdot e^{-t} = -1 - 1 = -2 \neq 0$, $\forall t \in \mathbb{R} \Rightarrow$
$\Rightarrow \{e^t, e^{-t}\}$ are lin indep.

So, by the fundam. thm. for LHDE's we deduce that ...().

b). • $\cosh(t)$ and $\sinh(t)$ are sol $\overset{\text{check}}{\Rightarrow}$ TRUE

• they are lin indep. (HW)

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Other ideas:

Let $y: I \to \mathbb{C}$, $y(t) = u(t) + iv(t)$, wher $u, v: I \to \mathbb{R}$.

We say that $y: I \to \mathbb{C}$ verifies the eq. (1) if $y \in C^n(I, \mathbb{C})$ and

$$y^{(n)}(t) + \ldots + a_n(t) y(t) = f(t), \quad \forall t \in \mathbb{R}.$$

Lemma: If y verifies the LHDE $x^{(n)} + \ldots + a_n(t)x = 0$ then $u = \mathrm{Re}(y)$ and $v = \mathrm{Im}(y)$ are sol. of this eq.

Proof:

$$\mathcal{L}(y) = 0 \iff \mathcal{L}(u + iv) = 0 \underset{\text{"the linearity"}}{\iff} \underbrace{\mathcal{L}(u)}_{\in\mathbb{R}} + i \underbrace{\mathcal{L}(v)}_{\in\mathbb{R}} = 0 \Rightarrow$$

$$\Rightarrow \mathcal{L}(u) = 0 \text{ and } \mathcal{L}(v) = 0.$$

The complex exponetial function

For $z \in \mathbb{C}$ we consider the series

$$(5) \quad 1 + \frac{1}{1!} z + \ldots + \frac{1}{k!} z^k + \ldots$$

we know: for $z \in \mathbb{R}$, the series converges to e^z.

Theorem:

Series (5) converges $\forall z \in \mathbb{C}$ and its sum is denoted by e^z or $\exp(z)$.

Moreover, $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$, $\forall z_1, z_2 \in \mathbb{C}$

$$\frac{d}{dt}(e^{zt}) = z \cdot e^{zt}, \quad \forall t \in \mathbb{R}, \text{ for a fixed } z \in \mathbb{C}.$$

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22.03.2019

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The complex exponential

For $z \in \mathbb{C}$, $e^z := 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 \ldots + \frac{1}{k!}z^k + \ldots$

Let $\alpha, \beta \in \mathbb{R}$, $z = \alpha + i\beta \in \mathbb{C}$,

$\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma(t) = e^{(\alpha+i\beta)t}$, $\forall t \in \mathbb{R}$,

$\gamma(t) = e^{\alpha t + i\beta t} = e^{\alpha t} \cdot e^{i\beta t}$

Euler's formula: $e^{i\beta} = \cos\beta + i\sin\beta$

Proof: $z = i\beta$, $z^k = i^k \cdot \beta^k$

$i^{2p} = (-1)^p$, $i^{2p+1} = i(-1)^p$, $p \geq 0$

$$e^{i\beta} = \underbrace{\sum_{p=0}^{\infty} \frac{1}{(2p)!}(-1)^p \cdot \beta^{2p}}_{\text{the Taylor series of } \cos\beta} + i \underbrace{\sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \cdot (-1)^p \cdot \beta^{2p+1}}_{\text{the Taylor series of } \sin\beta} =$$

$= \cos\beta + i\sin\beta$

if $\beta = \pi \Rightarrow e^{i\pi} = \cos\pi + i\sin\pi \Rightarrow \boxed{e^{i\pi} + 1 = 0}$

continuare

$\gamma(t) = e^{(\alpha+i\beta)t} = e^{\alpha t}\cos(\beta t) + i\, e^{\alpha t}\sin(\beta t)$, $\forall t \in \mathbb{R}$

The derivative of γ: $\gamma'(t) = z \cdot e^{zt}$

Proof:

$\gamma'(t) = \alpha e^{\alpha t} \cdot \cos(\beta t) - \beta e^{\alpha t}\sin(\beta t) + i\alpha e^{\alpha t}\sin(\beta t) + i\beta e^{\alpha t}\cos(\beta t) =$

$= (\alpha + i\beta)\left(e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t)\right)$

$\tilde{\gamma} : (0, \infty) \to \mathbb{C}$, $\tilde{\gamma}(t) = t^z$ ($z \in \mathbb{C}$ fixed)

$\tilde{\gamma}(t) = (e^{\ln t})^z = e^{z\ln t} = e^{(\alpha+i\beta)\ln t} = e^{\alpha\ln t + i\beta\ln t} =$

$= \underbrace{e^{\alpha\ln t}}_{t^\alpha}\left(\cos(\beta\ln t) + i\sin(\beta\ln t)\right)$

$\frac{d}{dt}(t^z) = z \cdot t^{z-1}$.

2. LHDE with CC

(1) $x^{(n)} + a_1 x^{(n-1)} + \dots + a_n x = 0$, where $a_1, \dots, a_n \in \mathbb{R}$

Look for solutions of the form $x = e^{rt}$, $r \in \mathbb{C}$

$$x^{(k)} = r^{(k)} \cdot e^{rt} \overset{(1)}{\Longrightarrow} e^{rt} \underbrace{(r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n)}_{\ell(r)} = 0, \forall t \in \mathbb{R}$$

- $x = e^{rt}$ verifies (1) $\iff \ell(r) = 0$

Proposition:

(i) If $r = \alpha \in \mathbb{R}$ is a root of the characteristic polynomial then $e^{\alpha t}$ is a sol. of (1).

(ii) If $r = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ is a root of ℓ then $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are solutions of (1).

(iii) If $r = \alpha \in \mathbb{R}$ is a double root of ℓ then $e^{\alpha t}$ and $t e^{\alpha t}$ are solutions of (1).

Proof: (i) OK

(ii) $\gamma(t) = e^{(\alpha + i\beta)t}$ verifies (1) $\overset{\text{lecture 2}}{\Longrightarrow}$ $Re(\gamma(t))$ and $Im(\gamma(t))$ are sol of (1).

(iii) Let ~~(1) be~~ $x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = \mathcal{L}(x)$.

we need to prove that $\mathcal{L}(te^{\alpha t}) = 0$, $\forall t \in \mathbb{R}$.

$x = te^{\alpha t} \to x^k = te^{\alpha t} \cdot \alpha^k + e^{\alpha t} \cdot k \cdot \alpha^{k-1}$, $k \geq 1$

$$\mathcal{L}(te^{\alpha t}) = \sum_{k=0}^{n} a_{n-k} (te^{\alpha t})^{(k)} = \sum_{k=0}^{n} a_{n-k} \alpha^k \cdot te^{\alpha t} + \sum_{k=0}^{n} a_{n-k} k \alpha^{k-1} e^{\alpha t} =$$

$\boxed{a_0 = 1}$

$$= te^{\alpha t} \ell(k) + e^{\alpha t}(\ell'(k))$$

α double root of $\ell \iff \ell(\alpha) = \ell'(\alpha) = 0$

So $\mathcal{L}(te^{\alpha t}) = 0$.

The long-term behaviour of the solutions of a LHDE w/CC

Proposition

All solutions of eq. (1) go to 0 as $t \to \infty$ if and only if $Re(r) < 0$, for any r root of the charac. polynom.

Proof:

if $r = \alpha \in \mathbb{R}$ we have that $\lim_{t\to\infty} t^k e^{\alpha t} = 0$ iff $\alpha < 0$.

if $r = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, we have that

$\lim_{t\to\infty} t^k e^{\alpha t} \cos(\beta t) = 0$ iff $\alpha < 0$.

$-t^k e^{\alpha t} \leq t^k e^{\alpha t} \cos(\beta t) \leq t^k e^{\alpha t}$

Proposition

Let $a_1, a_2 \in \mathbb{R}$ and consider the DE

(3) $x'' + a_1 x' + a_2 x = 0$

we have that all solutions of (3) are periodic iff. $a_1 = 0$ and $a_2 > 0$. Moreover, when the solutions are periodic, they have the same main period.

Proof: $r^2 + a_1 r + a_2 = 0$

case I $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \neq \alpha_2$ root $\longmapsto e^{\alpha_1 t}, e^{\alpha_2 t}$

the gen. sol. is $x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$, $c_1, c_2 \in \mathbb{R}$.

case II $\alpha \in \mathbb{R}$ is a double root $\longmapsto e^{\alpha t}, t e^{\alpha t}$

the gen. sol. $x = c_1 e^{\alpha t} + c_2 t e^{\alpha t}$, $c_1, c_2 \in \mathbb{R}$.

case III $\alpha \pm i\beta$, $\beta \neq 0 \longmapsto e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$

$x = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$; $c_1, c_2 \in \mathbb{R}$

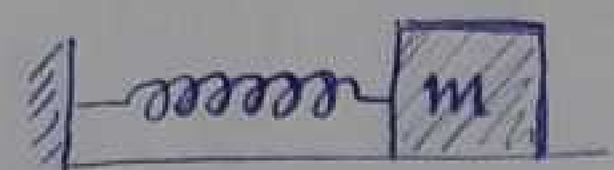
. any such function oscilates around 0.

the solutions are periodic iff . $\alpha = 0 \Leftrightarrow x = c_1\cos(\beta t) + c_2\sin(\beta t)$

Conclusion: all sol. of (3) are periodic iff $\pm i\beta$, $\beta \neq 0$ are roots of the char. eq. iff. the char. eq. is $(r+i\beta)(r-i\beta) = 0 \Leftrightarrow r^2 + \beta^2 = 0 \Leftrightarrow$

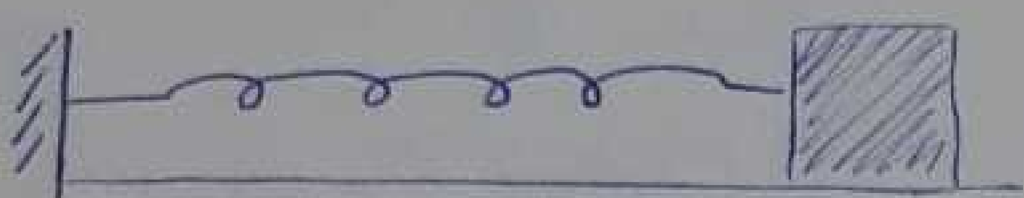
$\Leftrightarrow$ the diff. eq. is $x'' + \beta^2 x = 0 \Leftrightarrow a_1 = 0$ $a_2 = \beta^2 > 0$.

The spring-mass system



no motion $\to$ equilibrium state (or stationary state)

second Newton's law $F = ma$



$x(t)$ - the displacement from the equilibrium position.

$x'(t)$ - the instant velocity

$x''(t)$ - the acceleration

$F_r \approx k \cdot x(t)$ $\quad k>0$ the restoring force

$F_f \approx \gamma x(t)$ $\quad \gamma > 0$ the friction (damping) force

$F_e = f(t)$ - the extended force.

$$F = -F_r - F_f + F_e = -kx - \gamma x' + f(t)$$

$$ma = F \iff mx'' = -kx - \gamma x' + f(t)$$

$$\iff \boxed{x'' + \frac{\gamma}{m} x' + \frac{k}{m} x = f(t)} \quad \text{LNDE with CC.}$$

case I No damping, no external force.

$x'' + \frac{k}{m} x = 0 \qquad w_0 = \sqrt{\frac{k}{m}} \to$ the internal frequency of the spring.

$x = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$ - periodic, the main period $= \frac{2\pi}{w_0}$

case II With damping, without F_e.

$x'' + \frac{\gamma}{m} x' + \frac{k}{m} x = 0 \iff r^2 + \frac{\gamma}{m} r + \frac{k}{m} = 0.$

$\Delta = \frac{\gamma^2}{m^2} - 4\frac{k}{m} = \frac{\gamma^2 - 4km}{m^2}$

II.1 : $\underbrace{\gamma > \sqrt{4km}}_{\text{overdamping}}$ $(\iff \Delta > 0)$ no oscilation

$\left.\begin{array}{l} r_1 \cdot r_2 = \frac{k}{m} > 0 \\ r_1 + r_2 = -\frac{\gamma}{m} < 0 \end{array}\right\} \to \left\{\begin{array}{l} r_1 < 0 \\ r_2 < 0 \end{array}\right.$

$\Rightarrow$ any sol. goes to 0 as $t \to \infty$

II.2 $\gamma = \sqrt{4km}$ $\quad (\iff \Delta = 0)$

vertically dammped

II.3 $\gamma < \sqrt{4km}$ (underdamping)

$\Delta < 0 \quad r_{1,2} = \alpha \pm i\beta$ then are oscilations

$r_1 + r_2 = 2\alpha = -\frac{\gamma}{m} \Rightarrow \alpha = -\frac{\gamma}{2m} < 0 \Rightarrow$ any sol. goes to 0 as $t \to \infty$.

case III   No damping, with external force $F = A\cos(wt)$
where $A, w > 0$.

$$x'' + w_0^2 x = A\cos(wt)$$

we assume that $\boxed{w = w_0}$ Resonance

$$x_p = t\sin(wt)\cdot\frac{1}{2w_0}$$

$$x = c_1\cos(w_0 t) + c_2\sin(w_0 t) + \frac{1}{2w_0}t\cdot\sin(wt)$$

any solution oscilates with unbounded amplitude

Linear homogeneous systems with constant coeficients of 2 equations with 2 unknowns.

Reduction to a 2^{nd} order equation

Notation for the unknowns $x(t)$ and $y(t)$

$$(1)\begin{cases} x' = a_{11}x + a_{12}y \\ x'' = a_{21}x + a_{22}y \end{cases} \quad \text{where} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

A is set to be the matrix of the system

$$X \overset{not}{=} \begin{pmatrix} x \\ y \end{pmatrix} \quad (1) \Leftrightarrow \quad X' = AX$$

Remark:

Any second orde LHDE with CC can be written in the form (1) with the unknowns x ang $y = x'$.

$$(2) \quad x'' + \alpha_1 x' + \alpha_2 x = 0, \quad \alpha_1, \alpha_2 \in \mathbb{R}$$

$$(2) \Leftrightarrow \begin{cases} x' = y \\ y' = -\alpha_1 x - \alpha_2 y \end{cases} \quad \text{where matrix is} \quad A = \begin{pmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{pmatrix}$$

Remark

$\alpha \in \mathbb{C}$ is an eigenvalue of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ if and only if $\det(A - \alpha I_2) = 0$ if and only if $\alpha^2 - trA + \det A = 0$

Clarification:

The system $\begin{cases} x' = a_{11}x \\ y' = a_{22}y \end{cases}$ is called uncomplet system.

The general solution of this system is $\begin{cases} x = c_1 e^{a_{11}t} \\ y = c_2 \cdot e^{a_{22}t} \end{cases}, c_1, c_2 \in \mathbb{R}$

The other systems are called complet, that is any system (1) with $a_{12} \neq 0$ or $a_{21} \neq 0$ is complet.

Hypothesis: $\boxed{a_{12} \neq 0}$

Reduction method:

our aim: define a second order equation in X

Step 1: first eq $\Rightarrow y = \frac{1}{a_{12}}(x' - a_{11}x)$

Step 2: first eq $\Rightarrow x'' = a_{11}x' + a_{12}y'$

Step 3: second eq $\Rightarrow x'' = a_{11}x' + a_{12}(a_{21}x + a_{22}y)$

Step 4: Step 1 $\to x'' = a_{11}x' + a_{12}(a_{21}x + a_{22} \cdot \frac{1}{a_{12}}(x' - a_{11}x))$

$= a_{11}x' + a_{12} \cdot a_{21}x + a_{12} \cdot a_{22} \cdot \frac{1}{a_{12}}(x' - a_{11}x)$

$\Rightarrow (3) \quad x'' - (a_{11} + a_{22})x' + (a_{11} \cdot a_{22} - a_{12}a_{21})x = 0$

Step 5: We find the general solution of (3), then we find y using Step 1.

$r^2 - (a_{11} + a_{22})r + (a_{11} \cdot a_{22} - a_{12} \cdot a_{21}) = 0$

Exercises:

a). $\begin{cases} x' = 1x - y \\ y' = x + y \end{cases}$ b). $\begin{cases} x' = -y \\ y' = x \end{cases}$ c) $\begin{cases} x' = 2y \\ y' = x \end{cases}$ d) $\begin{cases} x' = -5x + 2y \\ y' = -3y \end{cases}$

Solutions:

a). $y = \frac{1}{1}(x' - x) \Rightarrow x'' = x' - y'$ $\qquad y = \frac{x' - a_{11}x}{a_{12}} =$

$x'' = x' - (x + y)$

$x'' = x' - (x - (x' - x))$

$x'' = x' - x + x' - x$

$x'' - 2x' + 2x = 0$ eq. charac. method.

$r^2 - 2r + 2 = 0 \quad \Delta = 4 - 8 = -4 = 4i^2 \quad r_{1,2} = 1 \pm i$

$r_1 = 1 + i \to e^t \cos t$

$r_2 = 1 - i \to e^t \sin t$ $\Rightarrow x = c_1 e^t \cos t + c_2 \sin t\, e^t$

$y = c_1 \cancel{e^t \cos t} + c_2 \cancel{e^t \sin t} + c_1 e^t \sin t \cancel{+ c_1 e^t \sin t} - c_2 e^t \cos t$
$- c_1 \cancel{e^t \cos t} - c_2 \cancel{e^t \sin t}$

$\begin{cases} y = c_1 e^t \sin t - c_2 e^t \cos t \\ x = c_1 e^t \cos t + c_2 e^t \sin t \end{cases}, \quad c_1, c_2 \in \mathbb{R}$

b). $\begin{cases} x' = -y \\ y' = x \end{cases}$ $a_{12} \neq 0$ $y = -\frac{1}{1}(x' - 0 \cdot x) = -x'$

$\Rightarrow y' = -x''$

$x'' = -x \Rightarrow x'' + x = 0$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i$

$x_1 = \sin t \qquad x_2 = \cos t$

$\begin{cases} x = c_1 \sin t + c_2 \cos t \\ y = c_1 \cos t + c_2 \sin t \end{cases}, c_1, c_2 \in \mathbb{R}$

d). $\begin{cases} x' = -5x + 2y \\ y' = -3y \end{cases}$

Step 1: $y = \frac{1}{a_{12}}(x' - a_{11}x) = \frac{1}{2}(x' + 5x)$

Step 2: $x'' = -5x' + 2y'$

Step 3: $y' = -3y \Rightarrow x'' = -5x' + 2(-3y) =$

$= -5x' - 6y$

Step 4: $y = \frac{1}{2}(x' + 5x) \Rightarrow$

$\Rightarrow x'' = -5x' - 6y = -5x' - \overset{3}{\cancel{6}} \cdot \frac{1}{\cancel{2}_1}(x' + 5x) =$

$= -5x' - 3x' - 15x = -8x' - 15x$

$\Rightarrow x'' + 8x' + 15x = 0.$

$x'' - \underbrace{(a_{11} + a_{22})}_{-5-3}x' + \underbrace{(a_{11} \cdot a_{22} - a_{12}a_{21})}_{(-5)\cdot(-3) - 2\cdot 0}x = 0.$

Step 5: $r^2 + 8r + 15 = 0 \qquad \Delta = 64 - 60 = 4 = 2^2$

$r_{1,2} = \frac{-8 \pm 2}{2} \Big< {-3 \atop -5} \mapsto x = c_1 e^{-3t} + c_2 e^{-5t}$

$2y = 2\frac{1}{2}(x' + 5x) = -3c_1 e^{-3t} \cancel{-5c_2 e^{-5t}} + 5c_1 e^{-3t} \cancel{+5c_2 e^{-5t}} =$

$= 2c_1 e^{-3t} \Rightarrow y = c_1 e^{-3t}.$

Sol: $\begin{cases} x = c_1 e^{-3t} + c_2 e^{-5t} \\ y = c_1 e^{-3t} \end{cases}$

Remark:

The roots of the charac. eq. of (3) are the eigenvalues of the system's matrix.

The existence and uniqueness:

Let $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2$ be fixed and $A \in \mathcal{M}_2(\mathbb{R})$ be fixed.
We have that the IVP

$$\begin{cases} \begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \\ x(0) = \gamma_1 \\ y(0) = \gamma_2 \end{cases}$$

has a unique solution

Definition:

- A matrix function $U : \mathbb{R} \to \mathcal{M}_2(\mathbb{R})$, $t \mapsto U(t)$ is said to be a matrix solution of system (1) if its collumns are solutions of system (1).
- A matrix solution of (1) is said to be fundamental matrix solution if its collumns are linearly independent functions.
- A matrix solution with $U(0) = I_2$ is called principal matrix solution.

Proposition:

i). U is a matrix solution iff. $U'(t) = A \cdot U(t)$, $\forall t \in \mathbb{R}$

ii). Let V be a matrix solution, we have that V is a fundamental matrix solution iff. $\det(V(0)) \neq 0$.

Exercises:

① Find the solutions of the IVP ② Find the principal matrix solution.

$$(1) \begin{cases} x' = -y \\ y' = x \\ x(0) = 1 \\ y(0) = 0 \end{cases} \quad (2) \begin{cases} x' = -y \\ y' = x \\ x(0) = 0 \\ y(0) = 1 \end{cases}$$

(1). $\begin{cases} x = C_1 \cos t + C_2 \sin t \\ y = C_1 \sin t - C_2 \cos t \end{cases}$, $\begin{matrix} x(0) = 1 \Leftrightarrow C_1 = 1 \\ y(0) = 0 \Leftrightarrow C_2 = 0 \end{matrix}$ $\Rightarrow \begin{cases} x = \cos t \\ y = \sin t \end{cases}$

(2) $\begin{matrix} x(0) = 0 \Leftrightarrow C_1 = 1 \\ y(0) = 0 \Leftrightarrow C_2 = -1 \end{matrix}$ $\Leftrightarrow \begin{cases} x = \sin t \\ y = \cos t \end{cases}$

b) ② (1) $U(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ $U(0) = I_2 \Rightarrow U$ is the principal matrix solution.

③ How many solutions have the following problems?

a). $\begin{cases} x''+t^2x=0 \\ x(0)=0 \end{cases}$ b). $\begin{cases} x''+t^2x=0 \\ x(0)=0 \\ x'(0)=0 \end{cases}$ c). $\begin{cases} x''+t^2x=0 \\ x(a)=0 \\ x'(a)=0 \\ x''(a)=1 \end{cases}$ a - param.

a). infinite nr. of solutions.

b). unique solution $x=0$

c). a only solution $x=0$ if $a=0$

$a\neq 0 \Rightarrow$ no solutions.

④ Let $w>0$ be a fixed parameter

Denote $\varphi(t,w)$ the unique sol. of the IVP $x''+x=\cos(wt)$

$x(0)=x'(0)=0$.

a). when $w\neq 1$, find $x_p = a\cos(wt)+b\sin(wt)$

b) when $w=1$, find $x_p = t(a\cos(wt)+b\sin(wt))$

c) find $\varphi(t,w)$

d) prove that $\lim\limits_{w\to 1} \varphi(t,w)=\varphi(t,1)$, $\forall t\in\mathbb{R}$.

Sol:

a). $x_p'' + x_p = \cos(wt)$

$x_p' = -aw\sin(wt)+bw\cos(wt)$

$x_p + x_p'' = -aw^2\cos(wt) - bw^2\sin(wt)+a\cos(at)+b\sin(at) = \cos(wt)$

$\Rightarrow \cos(wt)(-aw^2+a-1)+\sin(wt)(-bw^2+b)=0$, $\forall t\in\mathbb{R}$

$\begin{cases} -aw^2+a-1=0 \\ -bw^2+b=0 \end{cases} \Rightarrow b(1-w^2)=0 \Rightarrow \boxed{b=0}$

$a(-w^2+1)=1 \Rightarrow a=\frac{1}{-w^2+1} \Rightarrow x_p = \frac{1}{-w^2+1}\cos(wt)$

c). $w\neq 1$.

$x''+x=0 \Rightarrow r^2+1=0 \Rightarrow r_{1/2}=\pm i$

$x_h = c_1\sin t + c_2\cos t$

$x = c_1 \sin t + c_2\cos t + \frac{1}{-w^2+1}\cos(wt)$

$x(0)=0 \Rightarrow c_2 + \frac{1}{-w^2+1}=0 \Rightarrow c_2 = \frac{-1}{-w^2+1}$

$x'(t) = c_1\cos t - c_2\sin t - \frac{w}{-w^2+1}\sin(wt)$

$x'(0)=0 \Rightarrow c_1=0$

$\varphi(t,w) = \frac{-1}{-w^2+1}\cos t + \frac{1}{-w^2+1}\cos(wt)$

d). $\lim\limits_{w\to 1}\varphi(t,w) = \lim\limits_{w\to 1} \frac{-\cos t+\cos(wt)}{-w^2+1} = \frac{0}{0}$

$\overset{l'H}{=} \lim\limits_{w\to 1}\frac{-t\sin(wt)}{-2w} = \frac{-t\sin t}{-2} = \frac{t\sin t}{2}$.

5/5

DS : Curs 5 29.03.2019

Linear differential systems
(linear differential equations in $\mathbb{R}^n$)

Let $n \geq 1$ be a fixed integer. We consider:

$$(1)\begin{cases} x_1' = a_{11}(t)x_1 + \dots + a_{1n}(t)x_n + f_1(t) \\ x_2' = a_{21}(t)x_1 + \dots + a_{2n}(t)x_n + f_2(t) \\ \vdots \\ x_n' = a_{n1}(t)x_1 + \dots + a_{nn}(t)x_n + f_n(t) \end{cases} \iff (2)\ X' = A(t)X + F(t)$$

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & \dots & a_{2n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix} \quad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Hyp: $A \in C(I, M_n(\mathbb{R}))$, $F \in C(I, \mathbb{R}^n)$, $I \subset \mathbb{R}$ nonempty, open, interval.

Definition:
A function $\varphi : I \to \mathbb{R}^n$ is a solution of (2) if $\varphi \in C^1(I, \mathbb{R}^n)$ and $\varphi'(t) = A(t)\varphi(t) + F(t)$, $\forall t \in I$

Remark:
The system $X' = A(t)X$ is a linear homogeneous system.
When $F \not\equiv 0$ the system (2) is linear non-homogeneous.
When $A(t)$ is constant we say that system (2) has constant coeficients.

The fundamental theorems
The existence and uniqueness theorem.

Let $\eta \in \mathbb{R}^n$ be fixed. We have that the IVP

$$\begin{cases} X' = A(t)X + F(t) \\ X(t_0) = \eta \end{cases}$$

has a unique solution $\varphi^* : I \to \mathbb{R}^n$

Let $t_0 \in I$ be fixed.

The fundamental theorem for LHS's.

The set of solutions of system $X' = A(t)X$ is a linear space of dimention n.
Thus there exist $X_1, \dots, X_n$ n linearly independent solutions such that the general solution is $X = c_1 X_1 + \dots + c_n X_n$, where $c_1, \dots, c_n \in \mathbb{R}$.
Moreover, denoting $U(t) = (X_1(t) \;\; X_2(t) \;\; \dots \;\; X_n(t))$ we can write the general solution $X = U(t)C$, $C \in \mathbb{R}^n$.

Proof:

$\mathcal{L}(X)(t) = X'(t) - A(t)X(t)$, $\forall t \in I$.

$X \in C^1(I, \mathbb{R}^n) \longmapsto \mathcal{L}(X) \in C(I, \mathbb{R}^n)$

$C^1(I, \mathbb{R}^n)$ and $C(I_1, \mathbb{R}^n)$ are linear spaces.

$\mathcal{L}$ in a linear map i.e. $\mathcal{L}(\alpha X + \beta Y) = \alpha \mathcal{L}(X) + \beta \mathcal{L}(y)$, $\forall X, Y \in C^1(I, \mathbb{R}^n)$

$$\mathcal{L}(\alpha X + \beta Y)(t) = (\alpha X + \beta Y)'(t) - A(t)(\alpha X + \beta Y)(t) =$$
$$= \underline{\alpha X'(t)} + \beta Y'(t) - \alpha \underline{A(t)X(t)} - \beta A(t)Y(t) =$$
$$= \alpha \mathcal{L}(X)(t) + \beta \mathcal{L}(Y)(t), \; \forall t \in I$$

The set of solutions of the system $Y' = A(t)X$ is $\mathrm{Ker}\, \mathcal{L}$

$\mathcal{L}$ is a linear map $\Rightarrow$ $\mathrm{Ker}\, \mathcal{L}$ is a linear space.

We know that $\mathbb{R}^n$ is a linear space of dim n.

We consider $T : \mathrm{Ker}\, \mathcal{L} \longrightarrow \mathbb{R}^n$, when $t_0 \in I$ in fixed.
$T(X) = X(t_0)$

T is a linear map (H/W)

T is bijective $\Leftrightarrow$ $\forall \eta \in \mathbb{R}^n$, $\exists! X \in \mathrm{Ker}\, \mathcal{L}$. s.t. $T(X) = \eta$.

$\Leftrightarrow$ $\forall \eta \in \mathbb{R}^n$ $\exists!$ solution of $\begin{cases} X' = A(t)X \\ X(t_0) - \eta \end{cases}$ this is TRUE by the $\exists!$ theorem.

The fundamental theorem for LNS's

The general solution of the system $X' = A(t)X + F(t)$ is $X = X_h + X_p$, where X_h is the general solution of $X' = A(t)X$ and $X_p' = A(t)X_p + F(t)$ (X_p is a particular solution)

Proof: $\mathcal{L}(X) = F(t)$ $\quad \ker \mathcal{L} + \{X_p\}$.

The Lagrange method (or the variation of constants)

to find X_p : $X_p' - A(t)X_p = F(t)$.

Let $U(t)$ be s.t. the general solution of the LHS associated $X' = A(t)X$ is $X_h = U(t)\cdot C$, $C \in \mathbb{R}^n$.

Lagrange: $X_p = U(t)\varphi(t)$

$$U'(t)\varphi(t) + U(t)\varphi'(t) - A(t)U(t)\varphi(t) = F(t), \ \forall t \in I$$

$$U'(t) = (X_1'(t) \ \dots \ X_n'(t)) = (A(t)X_1(t) \ \dots \ A(t)X_n(t)) =$$

$$= A(t)\cdot(X_1(t) \ \dots \ X_n(t)) = A(t)\cdot U(t)$$

$\boxed{U'(t) = A(t)\cdot U(t)}$ $\Leftrightarrow$ U is a matrix solution

$$\varphi'(t) = U^{-1}(t)F(t), \ \forall t \in I$$

$$\varphi(t) = \int_{t_0}^{t} U^{-1}(s)F(s)\,ds, \ \forall t \in I$$

$$\Rightarrow \boxed{X_p(t) = U(t)\int_{t_0}^{t} U^{-1}(s)F(s)\,ds}$$

The columns of U are liniarly indep. $\Rightarrow$

$\Rightarrow \det U(t) \neq 0$, $\forall t \in I$.

LHS with CC

(3) $X' = AX$, where $A \in \mathcal{M}_n(\mathbb{R})$

Denote $E : \mathbb{R} \to \mathcal{M}_n(\mathbb{R})$ s.t. $E'(t) = AE(t)$, $\forall t \in \mathbb{R}$ and $E(0) = I_n$ (the identity matrix).

We have that $E(t)$ is said to be the principal matrix solution of $X' = AX$, and that the general solution of this system is $X = E(t)C$, $C \in \mathbb{R}^n$

The exponential matrix

Theorem: Let $A \in M_n(\mathbb{R})$ be fixed we have that the sense (of matrix)

$$I_n + \frac{1}{1!}A + \frac{1}{2!}\cdot A^2 + \dots + \frac{1}{k!}A^k + \dots$$

is convergent. Denote its sum by e^A or $\exp(A)$

Example:

1). $e^{O_n} = I_n$

2). $e^{I_n} = I_n + \frac{1}{1!}I_n + \frac{1}{2!}I_n + \dots + = \left(1 + \frac{1}{1!} + \dots + \frac{1}{k!} + \dots\right) I_n = e \cdot \frac{I}{1}$

3). $e^{I_n t} = e^t I_n$

4) $\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} t = \begin{pmatrix} e^{\lambda t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}$

5). $e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = I_2 + \frac{1}{1!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t - \frac{1}{2!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t^2 - \dots$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t$

$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} t^2 = -I_2 t^2$

$A^3 = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t^3$

$A^4 = I_2 t^4 \dots$

$$e^{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \dots & -\frac{1}{1!}t + \frac{1}{3!}t^3 + \dots \\ \frac{1}{1!}t - \frac{1}{3!}t^3 + \dots & 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 + \dots \end{pmatrix} =$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Theorem:

$$\frac{d}{dt}(e^{At}) = Ae^{At}$$

$$e^{At}\Big|_{t=0} = I_n$$

Therefore, $E(t) = e^{At}$, $\forall t \in \mathbb{R}$

Remark:

The general solution of $X' = AX$ is $X = e^{At}.C$, $C \in \mathbb{R}^n$

When A is diagonable

Definition:

Let $A, B \in \mathcal{M}_n(\mathbb{R})$.
We say that A and B are similar when there exists an inventible matrix $P \in \mathcal{M}_n(\mathbb{R})$ s.t. $A = P.B.P^{-1}$
This P is said to be a transitional matrix between A and B.

Definition:

Let $A \in \mathcal{M}_n(\mathbb{R})$. We say that A is diagonalizable when A is similar to a diagonal matrix.

Proposition:

Let A and B be similar. Then A and B have the same eigenvalues.

Proof:

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \iff \exists u \in \mathbb{R}^n$, $u = 0$, s.t.

$$\left.\begin{array}{l} Au = \lambda u \\ B = P^{-1}AP \end{array}\right\} \Rightarrow P^{-1}APu = \lambda P^{-1}Pu \iff Bu = \lambda u.$$

DS: Curs 6 05.04.2015

Linear systems with constant coeficients

(1) $x' = AX$, where $A \in \mathcal{M}_n(\mathbb{R})$, $n \in \mathbb{N}^*$

Recall from the last lecture:

- the F.T. if $X_1, \dots, X_n$ are n linearly indep. sol of (1), then the general solution of (1) is $X = c_1X_1 + \dots + c_nX_n$, $c_1, \dots, c_n \in \mathbb{R}$.
- we defined matrix exponential and we showed that e^{At} is the principal matrix of (1) and that the general solution of (1) can be written as $X = e^{At} \cdot c$, $c \in \mathbb{R}^n$

Notation

$\lambda_1, \dots, \lambda_n \in \mathbb{C}$ the eigenvalues of A (counted with their multiplies, thus it is not neccessary to be distinct)

$u_1, \dots, u_n \in \mathbb{C}^n$ a set of linearly indep. eigenvalues ($m \leq n$)

Similar matrices

Definition:

Let $A, B \in \mathcal{M}_n(\mathbb{R})$. We say that A is similar to B if there exists $P \in \mathcal{M}_n(\mathbb{R})$ invertible s.t. $A = P \cdot B \cdot P^{-1}$.

Property:

Let $\lambda \in \mathbb{C}$ and $u \in \mathbb{C}^n$ be such that u is an eigenvector of A corresponding to the eigenvalue λ. Then λ is an eigenvalue of B and $P^{-1}u$ is an eigenvector of B corresp. to λ.

The hypothesis is that A is similar to B.

Proof:

$Au = \lambda u \qquad u \neq 0$

$PBP^{-1}u = \lambda u \ / \cdot P^{-1}$ to the left.

$(P^{-1}P)BP^{-1}u = \underbrace{\lambda}_{\text{scalar}} \cdot P^{-1}u$

$\Rightarrow B\underbrace{[P^{-1}u]}_{\text{vector}} = \lambda \underbrace{[P^{-1}u]}_{\text{vector}}$ } $\Rightarrow P^{-1}u$ is an eigenvector of B corresp to the eigenvalue λ.

Since $u \neq 0$ we have $u \neq 0$

Property:

Let $D = diag(\lambda_1, \dots, \lambda_n)$ and P a matrix whose i^{th} column is the eigenvector u_i.

Then $A = PDP^{-1}$.

Consequences

$D^k = diag(\lambda_1^k, \dots, \lambda_n^k) \Rightarrow A^k = P diag(\lambda_1^k, \dots, \lambda_n^k) P^{-1}$.

$e^{Dt} \underset{\text{last lecture}}{=} diag(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \Rightarrow e^{At} = P diag(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) P^{-1}$ (1)

So, we gave a procedure to find to find the general sol of $X' = AX$ in the case that A is diagonalizable

Step 1: Find the eigenvalues and eigenvectors of A, if $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $\exists n$ lin. indep. eigenvectors then we include that A is diagonalizable.

Step 2: we find D and P.

Step 3: we find e^{At} using (1)

Step 4: $X = e^{At} C$, $C \in \mathbb{R}^n$

Proposition:

Let $A \in M_n(\mathbb{R})$, $\lambda \in \mathbb{R}$ be an eigenvalue of A, $u \in \mathbb{R}^n$ be an eigenvector of A corresp. to λ.

Then the function $\varphi : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = e^{\lambda t} u$ is a solution of $X' = AX$.

Proof:

Theorem:

The n functions are linearly indep. solutions of the system

Step 3: Write general sol $x = c_1 e^{\lambda_1 t} u_1 + \dots + c_n e^{\lambda_n t} u_n$; $c_1, \dots, c_n \in \mathbb{R}$

Example:

(i) Prove that the matrix $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ is diagonalizable.

(ii) Using the char. eq. method find the general sol of $x' = Ax$.

Solution:

(i) $\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(-1-\lambda) - 3 = 0$

$\iff \lambda^2 - 1 - 3 = 0 \iff \lambda^2 - 4 = 0 \iff \lambda_1 = -2$ and $\lambda_2 = 2$.

Find an eigenvector corresp to $\lambda_1 = -2$

$Au = -2u$, $u = ?$ $u \in \mathbb{R}^2$

$\iff (A + 2I_2)u = 0 \iff \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff a + b = 0$

choose $a = 1$
$\Rightarrow b = -1$.

$u = \begin{pmatrix} a \\ b \end{pmatrix}$ then $u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ an eigenvector corresp. to $\lambda_1 = -2$.

Find an eigenvector to $\lambda_2 = 2$

$Au = 2u \iff (A - 2I_2)u = 0 \iff \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff$

$\iff -a + 3b = 0$, we choose $b = 1 \Rightarrow a = 3$.

then $u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvect. corr. to $\lambda_2 = 2$.

$\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1 + 3 = 4 \neq 0 \Rightarrow u_1, u_2$ are lin indep $\Rightarrow$ A is ~~dig~~ diagon.

(ii) the gen. sol of $\begin{cases} x_1' = x_1 + 3x_2 \\ x_2' = x_1 - x_2 \end{cases}$ is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} x_1 = c_1 \cdot e^{-2t} + 3c_2 e^{2t} \\ x_2 = -c_1 e^{-2t} + c_2 e^{2t}, \end{cases} \quad c_1, c_2 \in \mathbb{R}$$

Henri Poincaré

The dynamical system associated to a differential equation

We consider the autonomous system in $\mathbb{R}^n$

(1) $\dot{x} = f(x)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$, is a C^1-functions.

The unknown is $x(t)$ and $\dot{x}$ is just the notation of Newton for x' used when t is the time.

Keywords: flow, initial state, equilibrium point, repeller, attractor, phase portrait.

Theorem: (an eaisure and uniqueness theorem)

Let $\eta \in \mathbb{R}^n$. We have that the IVP

(2) $\begin{cases} x' = f(x) \\ x(0) = \eta \end{cases}$ has a unique solution, denoted by $\varphi(t, \eta)$ which is defined on a maximal interval $I_\eta = (\alpha, \beta_\eta)$

Moreover, we have that:

if $\varphi(\cdot, \eta)$ is bounded on $[0, \beta_\eta)$ then $\beta_\eta = +\infty$

if $\varphi(\cdot, \eta)$ is bounded on $(\alpha_\eta, 0]$ then $\alpha_\eta = -\infty$.

if $\varphi(\cdot, \eta)$ is bounded on I_η then $I_\eta = \mathbb{R} = (-\infty, +\infty)$

Definition:

The map $(t, \eta) \longmapsto \varphi(t, \eta)$ is called the flow of (1)

η is the initial state

$\varphi(t, \eta)$ → the state of the system at time t when it initiated at η.

The space $\mathbb{R}^n$ is called the state space.

Exercise:

DS: Curs 7. 12.04.2019

The dynamical system associated to a planar system

Let $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and we consider the planar autonomous system

$$(1)\ \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

· For any $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$ the IVP

$$(2)\ \begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

has a unique solution, denoted $t \longmapsto \varphi(t,\eta) \in \mathbb{R}^2,\ t \in I_\eta$

· The function $(t,\eta) \longmapsto \varphi(t,\eta)$ is said to be <u>the flow</u> of system (1). $\mathbb{R}^2$ is said to be the state space of (1).

· We say that $\eta^* \in \mathbb{R}^2$ is an equilibrium (or stationary state) of (2) if $\varphi(t,\eta^*) = \eta^*,\ \forall t \in \mathbb{R}$.

<u>Remark:</u>
η^* is an equilibrium point $\Leftrightarrow f(\eta^*) = 0$.
So, in order to find the equilibria of (1) we have to solve the system $\begin{cases} f_1(x,y) = 0 \\ f_2(x,y) = 0 \end{cases}$ for $(x,y) \in \mathbb{R}^2$

· For $\eta \in \mathbb{R}^2$ we define its <u>orbit</u> by $\gamma_\eta = \{\varphi(t,\eta) : t \in I_\eta\}$

<u>Remark:</u>
η^* is an equil. point $\Leftrightarrow \gamma_{\eta^*} = \{\eta^*\}$

<u>Definition:</u>
Let η^* be an equil. point.
We say that η^* is an <u>attractor</u> if $\exists$ a neighbourhood of η^*, denoted by V such that $\lim\limits_{t\to\infty} \varphi(t,\eta) = \eta^*,\ \forall \eta \in V$.
If η^* is an attractor s.t. $\lim\limits_{t\to\infty} \varphi(t,\eta) = \eta^*,\ \forall \eta \in \mathbb{R}^2$ we say that η^* is a <u>global attractor</u>.

- If we change ∞ to $-\infty$ in the attractor def. we say that η is a (repeller) repulsor (or even global repulsor)

- We say that γ_η is a periodic orbit (or closed orbit) when the solution $t \longmapsto \varphi(t,\eta)$ is a non-trivial periodic function.

Remark:
- A periodic orbit is a closed curve. ○
- If an orbit, is a closed curve then it is a periodic orbit.

- Let $U \subset \mathbb{R}^2$ be open, nonempty, and consider $H : U \to \mathbb{R}$ a C^1-function. We say that H is a <u>first integral in U</u> of (1) if (*) {H is not locally constant} and $H(\varphi(t,\eta)) = H(\eta)$, $\forall t$ s.t. $\varphi(t,\eta) \in U$, $\forall \eta \in U$.
  A first integral in $\mathbb{R}^2$ of (1) is called a global first integral.

- Let $U \subset \mathbb{R}^2$ be non-empty. We say that U is <u>invariant</u> for (1) if $\gamma_\eta \subset U$, $\forall \eta \in U$.

Definition
Let $H : U \to \mathbb{R}$ be a constant function.
Let $c \in \mathbb{R}$. The c-level curve of H is

$$\Gamma_c = \{ (x,y) \in U : H(x,y) = c \}$$

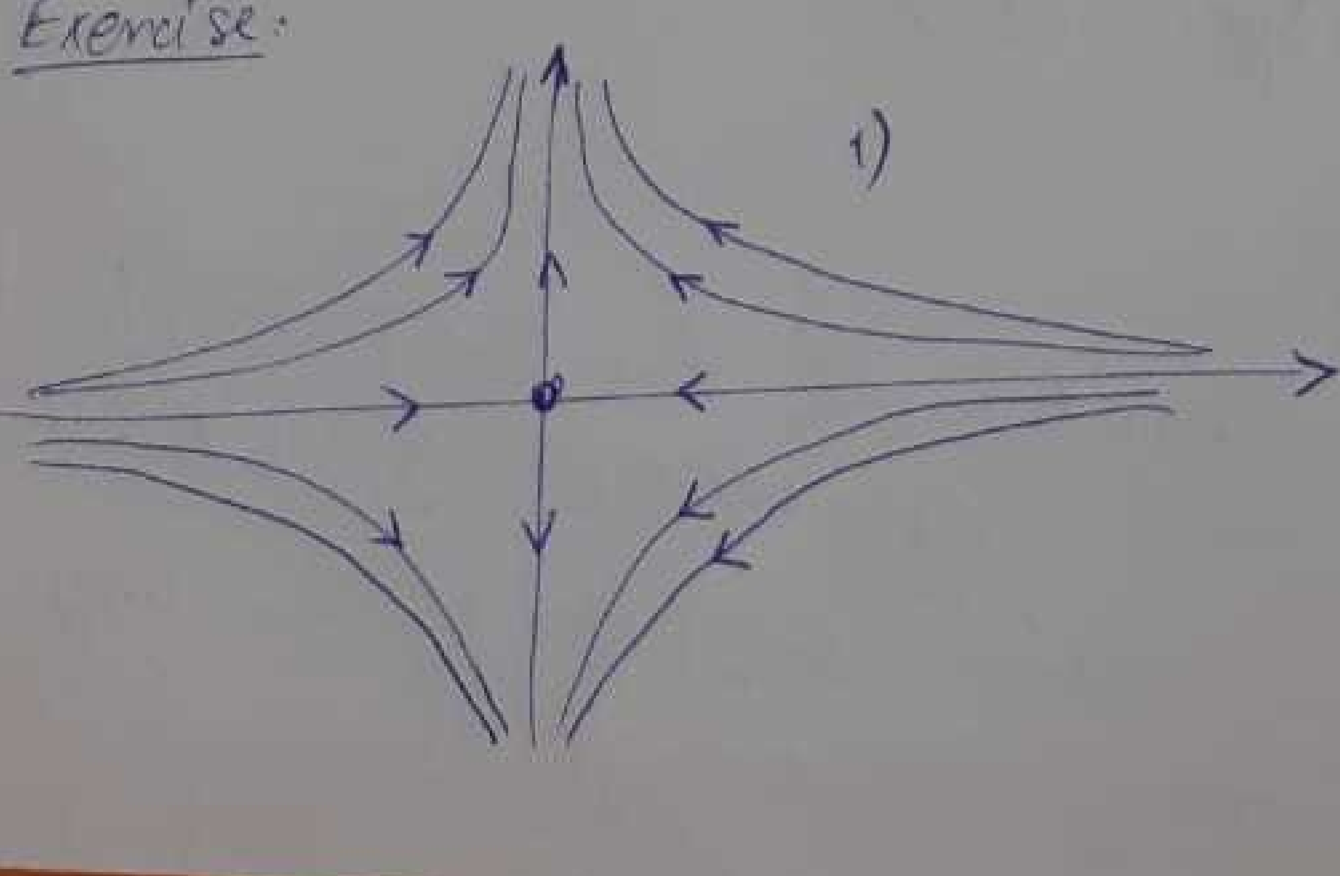
Remark:
Let H be a first integral in U of (1) and U ba an invariant set of system (1). Then $\forall \eta \in U$ we have

$$\gamma_\eta \subset \Gamma_{H(\eta)}$$

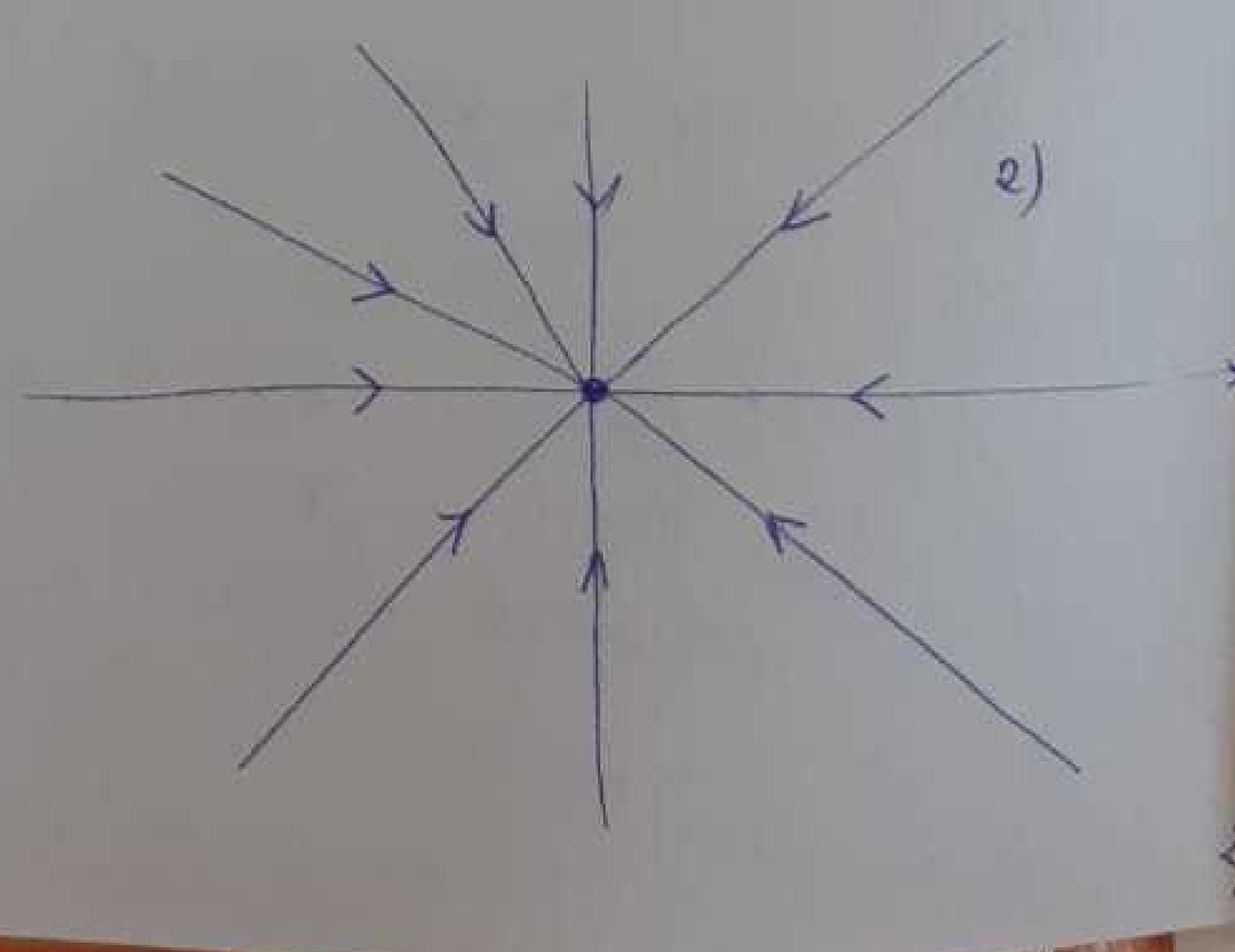
The <u>phase portait</u> of (1) is the representation <u>in the state space</u> $\mathbb{R}^2$ of ??? arrow on each orbit ??

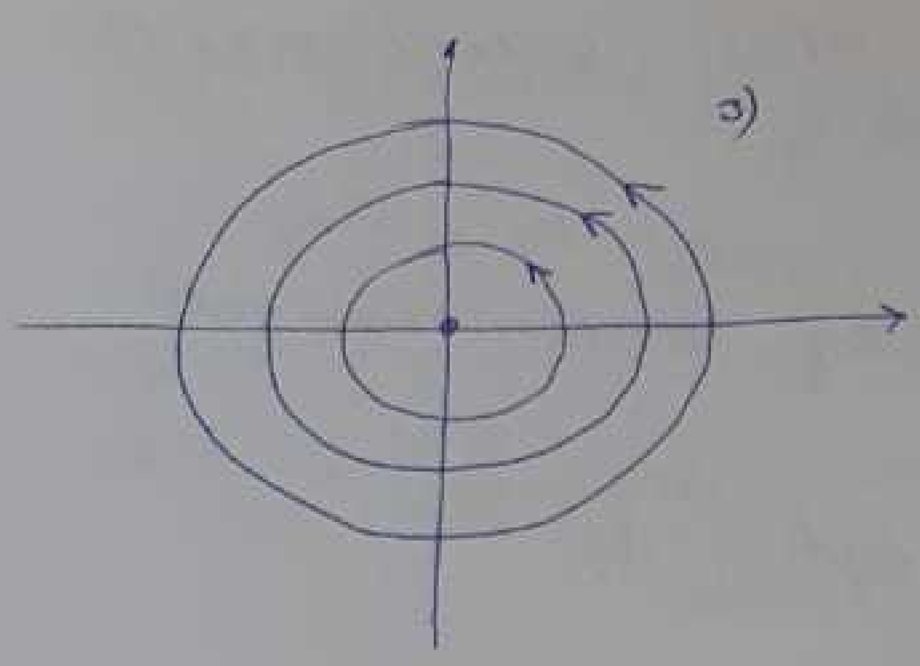
Exercise:

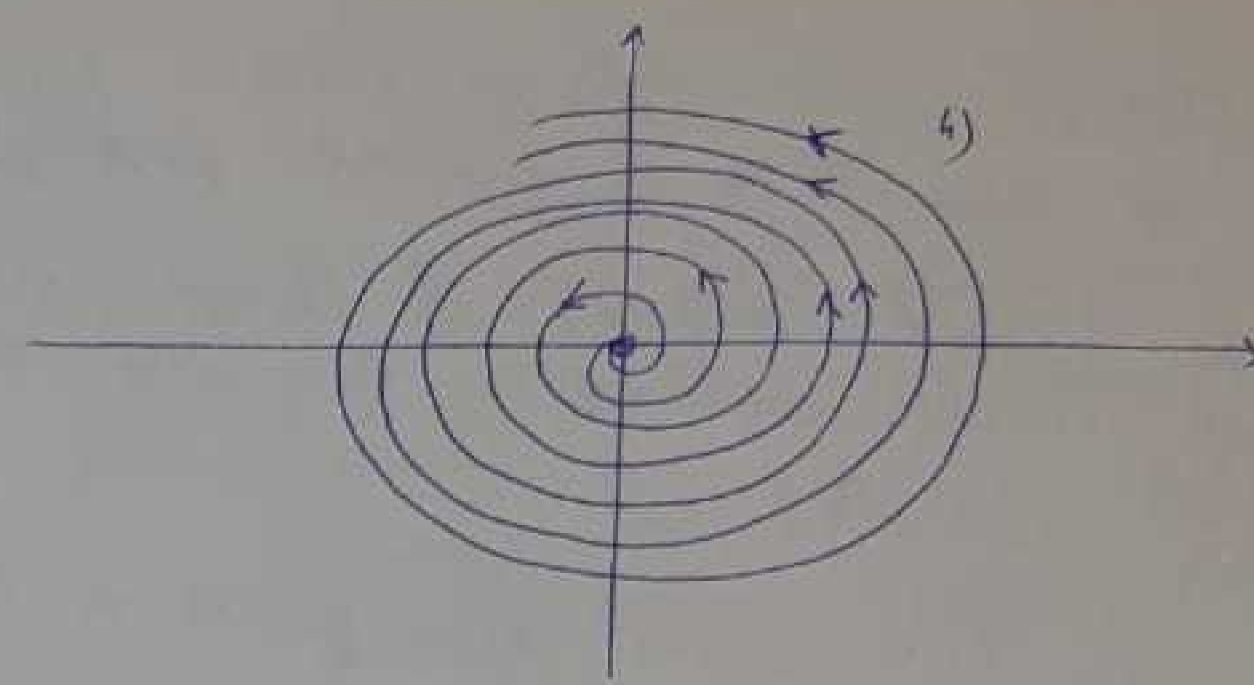
1)



2)







a). $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ b) $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ c) $\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases}$ d) $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases}$

Remarks based on given info for ex.

- systems a) – d) are linear systems
  $X' = AX, \quad A \in M_2(\mathbb{R})$
- the system $X' = AX$ has a unique equil. point $\eta^* = 0$
  $\Leftrightarrow \det A \neq 0$.

a). $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ Find equil. p., find the flow. Prove that it has a global first. int. Represent phase port.

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\det A = 1 \neq 0 \Rightarrow (0,0)$ – is only eq. point.

or, we solve $\begin{cases} -y = 0 \\ x = 0 \end{cases} \rightarrow$ – " –

let $\eta \in \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$, then IVP $\begin{cases} \dot{x} = -y \\ \dot{y} = x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$

$\ddot{x} = -\dot{y} = -x \Rightarrow \ddot{x} + x = 0$

$r^2 + 1 = 0 \Rightarrow r_{1/2} = \pm i \longmapsto \cos t, \sin t$

$x = c_1 \cos t + c_2 \sin t$
$y - x = c_1 \sin t - c_2 \cos t$ $\Rightarrow \begin{cases} x(0) = c_1 \\ y(0) = -c_2 \end{cases} \Rightarrow c_1 = \eta_1$ and $c_2 = \eta_2$

$\Rightarrow \varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 \cos t - \eta_2 \sin t \\ \eta_1 \sin t + \eta_2 \cos t \end{pmatrix}, \forall t \in \mathbb{R}, \forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$

We check $H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$ if it is a first integr.

$H(\varphi(t,\eta)) = (\eta_1 \cos t - \eta_2 \sin t)^2 + (\eta_1 \sin t + \eta_2 \cos t)^2 = \eta_1^2 \cos^2 t - 2\eta_1\eta_2 \cos t \sin t +$
$+ \eta_2^2 \sin^2 t + \eta_1^2 + 2\eta_1\eta_2 \sin t \cos t + \eta_2^2 \cos^2 t =$

5/5

$= (\eta_1^2+\eta_2^2)\cos^2 t + (\eta_1^2+\eta_2^2)\sin^2 t = \eta_1^2+\eta_2^2 = H(\eta)$, $\forall t\in\mathbb{R}, \forall \eta\in\mathbb{R}^2$

$\Rightarrow$ ~~capi~~ H is a global first integral.

The level curves of H are $x^2+y^2=c$, $c\in\mathbb{R}$, thus they are circles centered in O with arbitrary ~~solution~~ radious.

$x=-y$.

$y>0$
$x<0$ $\Rightarrow$ the arrow points to the left $\boxed{3}$
$x \downarrow$

b). $\begin{cases}\dot{x}=-x\\ \dot{y}=-y\end{cases}$ $\boxed{2}$

the equil. points $\begin{cases}-x=0\\-y=0\end{cases} \rightarrow (0,0)$ is the only equil. point.

Let $\eta=\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\in\mathbb{R}^2$ the IVP $\begin{cases}\dot{x}=-x\\ \dot{y}=-y\\ x(0)=\eta_1\\ y(0)=\eta_2\end{cases} \Rightarrow \varphi(t,\eta_1,\eta_2)=\begin{cases}\eta_1 e^{-t}\\ \eta_2 e^{-t}\end{cases}$

$\forall t\in\mathbb{R}$, $\forall \eta\in\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\in\mathbb{R}$

$H(x,y)=\frac{x}{y}$: $\begin{cases}U_1=\mathbb{R}\times(0,\infty)\\ U_2=\mathbb{R}\times(-\infty,0)\end{cases}$

$H(\varphi(t,\eta)) = \frac{\eta_1 e^{-t}}{\eta_2 e^{-t}} = \frac{\eta_1}{\eta_2}$, $\forall t\in\mathbb{R}$ $\Bigg\} \Rightarrow$ H is a first integral in U_1 & U_2. (not a global fr. int)

<u>Note that:</u>

$\lim \varphi(t,\eta) = \begin{pmatrix}0\\0\end{pmatrix}, \forall \eta\in\mathbb{R}^2$ (in other words $\begin{pmatrix}0\\0\end{pmatrix}$ is a global attract.)

Assume, by contradiction that $\exists$ $\overline{H}:\mathbb{R}^2\to\mathbb{R}$ a global first integral.

$\Rightarrow \overline{H}(\varphi(t,\eta)) = \overline{H}(\eta)$, $\forall t\in\mathbb{R}$, $\forall \eta\in\mathbb{R}^2$

$\Rightarrow \lim_{t\to\infty} \overline{H}(\varphi(t,\eta)) = \overline{H}(\eta)$, $\forall \eta\in\mathbb{R}^2$

$\xRightarrow{H \text{ is cont}}$ $\overline{H}(0,0)=\overline{H}(\eta)$, $\forall\eta\in\mathbb{R}^2 \Rightarrow \overline{H}$ is a constant in $\mathbb{R}^2$

this means that $\overline{H}$ is not a f.i, <u>contradiction</u>

$H =$ is f.i $\Rightarrow$ the orbits lie on $\frac{x}{y}=c \Leftrightarrow y=\frac{1}{c}x$

these are linear through the origin $\Rightarrow$ 2

c). $\dot{x} = -x$
$\dot{y} = y$ d).

The equil. points. $\begin{cases} -x = 0 \\ y = 0 \end{cases} \Rightarrow (0,0)$ - is the only equil. point.

Let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$ and the IVP $\begin{cases} \dot{x} = -x \\ \dot{y} = y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases} \Rightarrow \varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{t} \end{pmatrix}$

Consider ~~$H(\varphi(t,\eta)) =$~~ $H(x,y) = x \cdot y$ def. on $\mathbb{R}^2$.
$H(\varphi(t,\eta)) = \eta_1 \cdot \eta_2 = H(\eta)$, $\forall t \in \mathbb{R}$, $\forall \eta \in \mathbb{R}^2 \Rightarrow H$ is a global first. int.

The phase portrait

DS : Curs 8

Planar dynamical systems

(1) $\dot{x} = f(x)$ where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

the flow : $(t, \eta) \longmapsto \varphi(t, \eta)$

the orbit for the initial state $\eta \in \mathbb{R}^2$ is

$$\gamma_\eta = \{ \varphi(t, \eta) : t \in I_\eta \}$$

Properties of the flow:

(i) $\varphi(0, \eta) = \eta$, $\forall \eta \in \mathbb{R}^2$

(ii) $\varphi(t, \varphi(s, \eta)) = \varphi(t+s, \eta)$, $\forall \eta \in \mathbb{R}^2$, $\forall t, s$

(iii) the flow is a continuous fundation in (t, η)

An important property of the orbits

Let $\eta, \tilde{\eta} \in \mathbb{R}^2$, $\eta \neq \tilde{\eta}$. Then either $\gamma_\eta = \gamma = \gamma_{\tilde{\eta}}$, or $\gamma_\eta \cap \gamma_{\tilde{\eta}} = \phi$

! through any point in $\mathbb{R}^2$, there exists at least an orbit passing.

First integrals for planar systems

$$(1) \Leftrightarrow \begin{cases} \dot{x} = f_1(x, y) \\ \dot{y} = f_2(x, y) \end{cases}$$

Proposition:

Let $U \in \mathbb{R}^2$ be open, connected, nonempty and $H \in C^1(U)$ be a non-locally constant function.

We have that H is a first integral in U of (1) iff

$$\frac{\partial H}{\partial x}(x, y) \cdot f_1(x, y) + \frac{\partial H}{\partial y}(x, y) \cdot f_2(x, y) = 0, \ \forall (x, y) \in U.$$

Proof:

H is a first integral in U of (1) $\Leftrightarrow$ $H(\varphi(t, \eta)) = H(\eta)$,

$\forall \eta \in U$, $\forall t$ s.t. $\varphi(t, \eta) \in U$ $\Leftrightarrow$ $\frac{d}{dt} H(\varphi(t, \eta)) = 0$,

$\forall \eta \in U$, $\forall t$ s.t. $\varphi(t, \eta) \in U$ $\Leftrightarrow$

$\frac{\partial H}{\partial x}(\varphi(t,\eta)) \cdot \dot{\varphi}_1(t,\eta) + \frac{\partial H}{\partial y}(\varphi(t,\eta)) \cdot \dot{\varphi}_2(t,\eta) = 0$, $\forall \eta \in$

$\forall t$ s.t. $\varphi(t,\eta) \in U$

$\iff \frac{\partial H}{\partial x}(\varphi(t,\eta)) \cdot f_1(\varphi(t,\eta)) + \frac{\partial H}{\partial y}(\varphi(t,\eta)) \cdot f_2(\varphi(t,\eta)) = 0$

$\forall \eta \in \mathbb{R}^2$, $\forall t$ s.t. $\varphi(t,\eta) \in U$

$\iff \frac{\partial H}{\partial x} f_1 + \frac{\partial H}{\partial y} f_2 = 0$ in U

A method to find the first integral in U for some systems (1)

Step 1: write $\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}$ (2)

Step 2: Integrate the above DE and put the general sol. as $H(x,y) = c$, $c \in \mathbb{R}$.

Step 3: Find a domain U for the function H found above and check (using either the definition or the characterization) that, indeed, H is a first integral in U.

A method to integrate eq. (2) in the case that it is separable, i.e. it has the form. $\frac{dy}{dx} = g_1(x) g_2(y)$

First, we separate the variables $\frac{dy}{g_2(y)} \cdot g_1(x) dx$

Then, we integrate $\int \frac{dy}{g_2(y)} \cdot \int g_1(x) dx$ and obtain $G_2(y) = G_1(x) + c$, $c \in \mathbb{R}$

Now, if it is possible, we simplify the previous, if not, we get $H(x,y) = G_2(y) - G_1(x)$

Examples:

Find a global first integral and represent a phase portrait of the following systems:

a) $\begin{cases} \dot{x} = -2y \\ \dot{y} = 3x \end{cases}$ b) $\begin{cases} \dot{x} = x \\ \dot{y} = -3y \end{cases}$

a). $\frac{dy}{dx} = \frac{3x}{-2y}$ this is separable $\Rightarrow 2y\,dy = -3x\,dx$

we intergrate $\int 2y\,dy = -3\int x\,dx \Rightarrow y^2 = -\frac{3}{2}x^2 + c$, $c \in \mathbb{R}$

$\Rightarrow H(x,y) = \frac{3}{2}x^2 + y^2$, $U = \mathbb{R}^2$

We write the pde for first integrals of (1) $(-2y)\frac{\partial H}{\partial x}(x,y) + 3x\frac{\partial H}{\partial y}(x,y) = 0$

We check $\frac{\partial H}{\partial x} = 3x$ $\quad \frac{\partial H}{\partial y} = 2y$

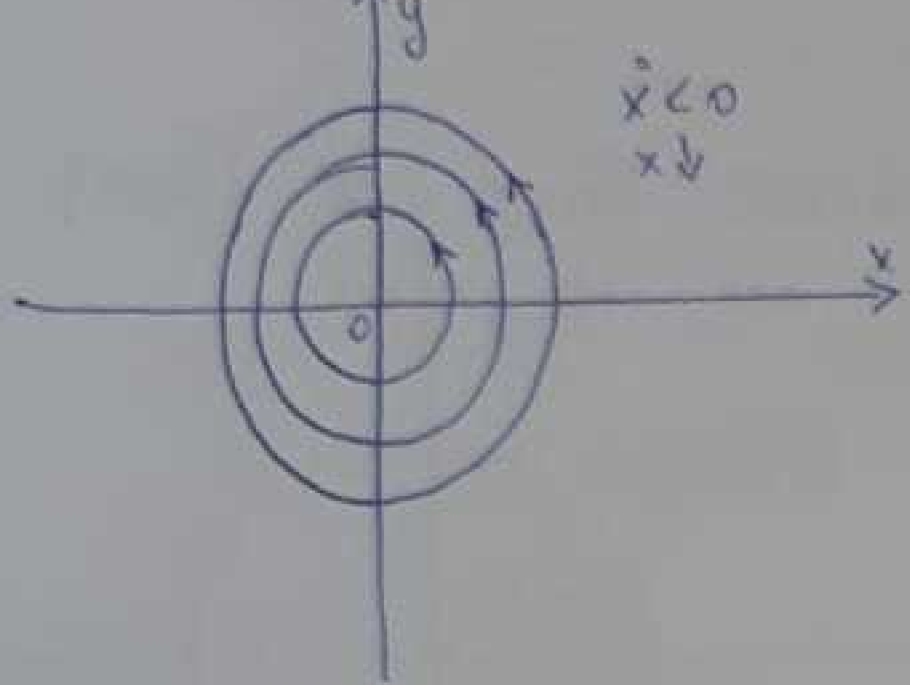
replace "$(-2y)\cdot 3x + 3x \cdot 2y = 0$, $\forall (x,y) \in \mathbb{R}^2$"

$\boxed{\text{TRUE}}$ $\Rightarrow$ the function $H : \mathbb{R}^2 \to \mathbb{R}$ $\quad H(x,y) = \frac{3}{2}x^2 + y^2$ is a global first integral.

Represent the level curves of H $\quad \frac{3}{2}x^2 + y^2 = c$, $c \in \mathbb{R}$

$c = 1$ $\quad \frac{3}{2}x^2 + y^2 = 1$

ellipses



the phase portrait of a).
Since the non-trivial orbits are closed, we deduce that any solution of a). is periodic in time.

b). $\begin{cases} \dot{x} = x \\ \dot{y} = -3y \end{cases}$ $\qquad$ the p.d.e for first integral

$x \cdot \frac{\partial H}{\partial x} - 3y\frac{\partial H}{\partial y} = 0$

$\frac{\partial y}{\partial x} = \frac{-3y}{x}$ is separable $\Rightarrow \frac{dy}{y} = \frac{-3dx}{x}$

$\int \frac{dy}{y} = -3\int \frac{dx}{x}$; $\ln|y| = -3\ln|x| + c$

$\ln|yx^3| = c$ $\qquad yx^3 = k$

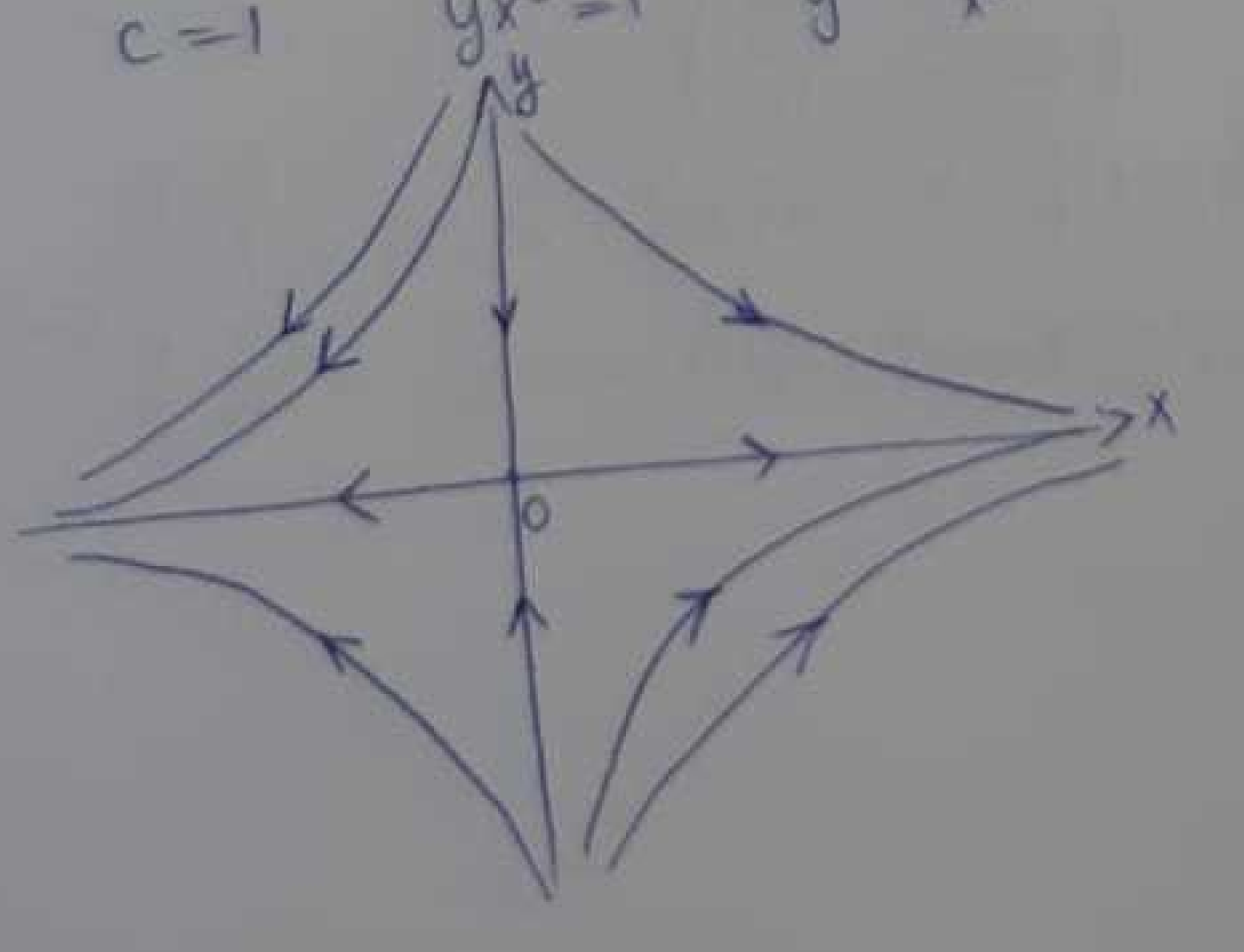
Take $H(x,y) = yx^3$, $u \in \mathbb{R}^3$

We check
"$x \cdot 3x^2 y - 3yx^3 = 0$, $\forall (x,y) \in \mathbb{R}^2$"
$\boxed{\text{TRUE}}$ $\Rightarrow$ H is a global first integral

the level curves of H

$yx^3 = c$

$c = 1$ $\quad yx^3 = 1$ $\quad y = \frac{1}{x^3}$



$f(x) = \frac{1}{x^3}$

$\lim_{x \to \infty} \frac{1}{x^3} = 0$

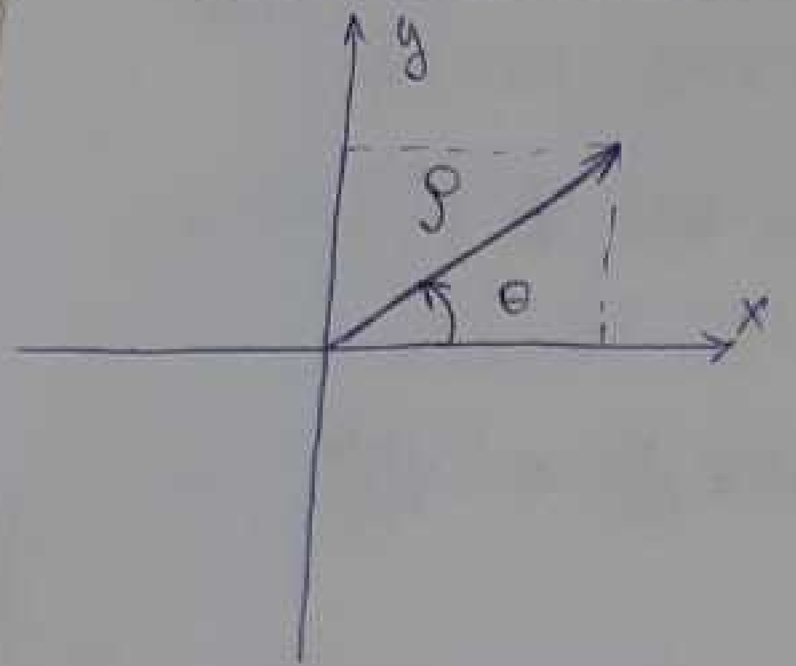
$\lim_{x \to 0_+} \frac{1}{x^3} = +\infty$

$f'(x) = -3\frac{1}{x^4} < 0$

$\begin{cases} \dot{x} = x \\ \dot{y} = -3y \end{cases}$

the only equilibrium is $(0,0)$

Polar coordinates in the plane



Property:

For any $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ there exists a unique pair (ρ,θ) in $(0,\infty) \times [0,2\pi]$ s.t.
$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

Definition:

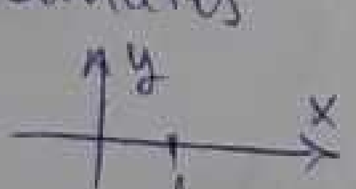
For a paint of charterian coordinates $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ we say that (ρ,θ) given by (1) are its polar coordinats

$$(1) \quad \begin{cases} \rho = \sqrt{x^2+y^2} \\ \cos\theta = \dfrac{x}{\rho} \\ \sin\theta = \dfrac{y}{\rho} \end{cases}$$

Examples:

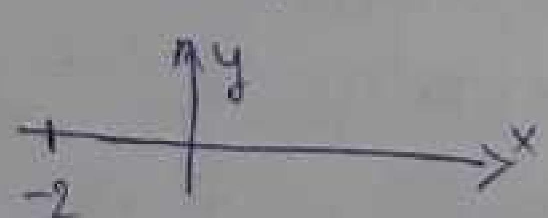
For the following points of charterian coord., find the polar coordinates

(a) $\begin{cases} x = 1 \\ y = 0 \end{cases}$



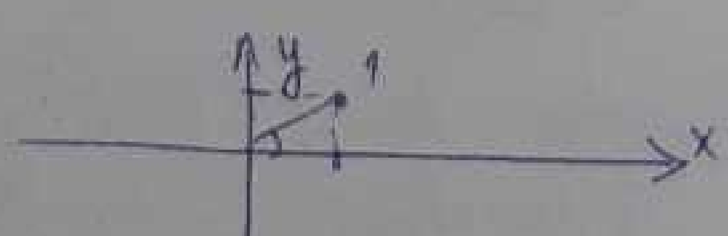
$(1,0)$

b) $\begin{cases} x = -2 \\ y = 0 \end{cases}$



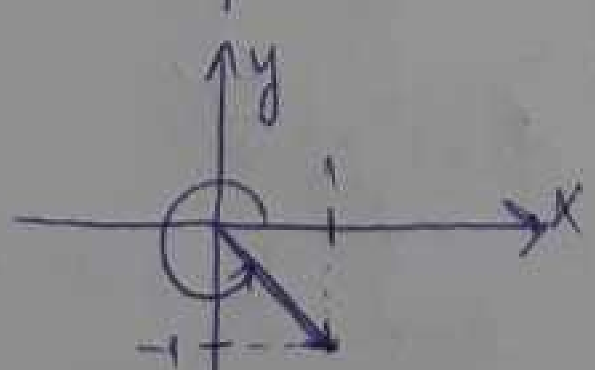
$(2,\pi)$

c) $\begin{cases} x = 1 \\ y = 1 \end{cases}$



$\left(\sqrt{2}, \frac{\pi}{4}\right)$

d). $\begin{cases} x = 1 \\ y = 1 \end{cases}$



$\left(\sqrt{2}, \frac{7\pi}{4}\right)$

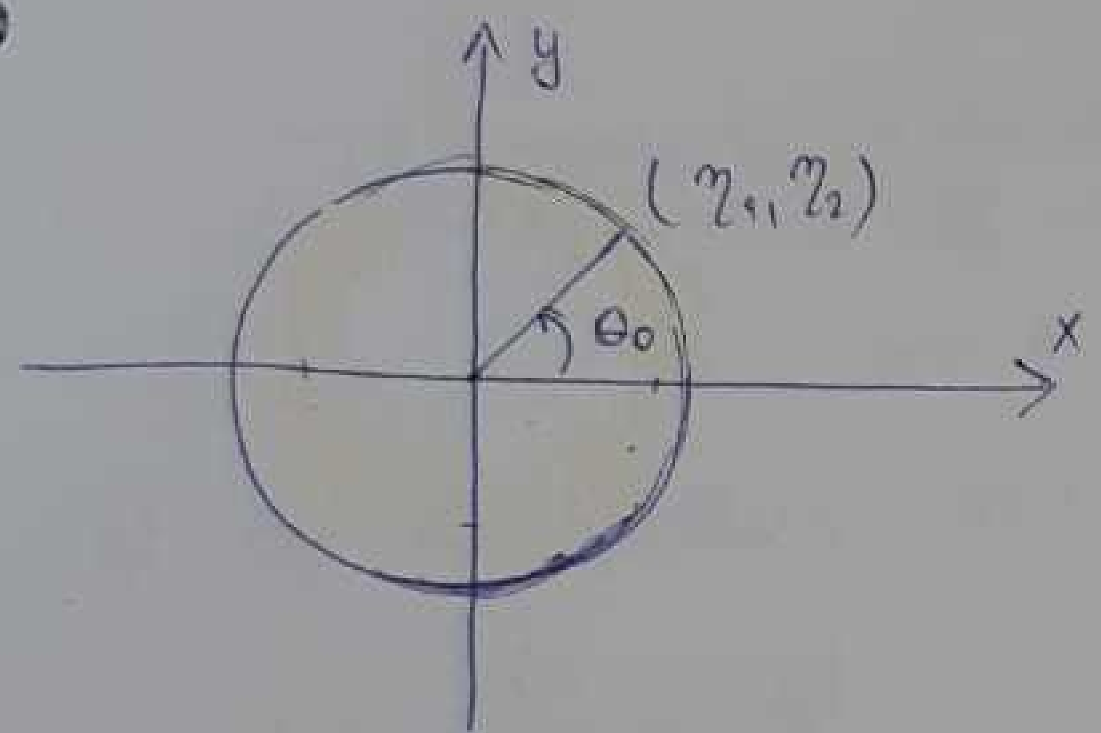
(2) $\begin{cases} x = \eta_1 \cos t + \eta_2 \sin t \\ y = \eta_1 \sin t + (-\eta_2 \cos t) \end{cases}, t \in \mathbb{R}$

$\rho = \sqrt{\eta_1^2 + \eta_2^2}$

Denote by $\rho_0 = \sqrt{\eta_1^2 + \eta_2^2}$ and $\theta_0 \in [0, 2\pi)$ s.t. $\begin{cases} \eta_1 = \rho_0 \cos\theta \\ \eta_2 = \rho_0 \sin\theta \end{cases}$

$\Rightarrow \begin{cases} x = \rho_0 \cos\theta \cdot \cos t + \rho_0 \cdot \sin\theta \cdot \sin t \\ y = \rho_0 \cdot \cos\theta \cdot \sin t - \rho_0 \cdot \sin\theta \cdot \cos t \end{cases}$

$\Rightarrow \begin{cases} x = \rho_0 \cos(t - \theta) \\ y = \rho_0 \sin(t - \theta) \end{cases} \Rightarrow \begin{cases} \rho(t) = \rho_0, \ \forall t \in \mathbb{R}^2 \\ \theta(t) = t - \theta_0 \end{cases}$



To transform a planar system $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$ in polar coordinates means to consider new unknowns $\rho(t)$ and $\theta(t)$ related by $\begin{cases} x(t) = \rho(t) \cdot \cos\theta(t) \\ y(t) = \rho(t) \cdot \sin\theta(t) \end{cases}$ and find a system in ρ and θ.

Practically, we have to use $\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2} \end{cases} \Leftrightarrow \begin{cases} \rho^2 = x^2 + y^2 \ \Big|' \text{w.r.t. } t \\ \tan\theta = \dfrac{y}{x} \end{cases}$

$\Rightarrow \begin{cases} \dot{\rho} = \cos\theta \cdot f_1(\rho\cos\theta, \rho\sin\theta) + \sin\theta \cdot f_2(\rho\cos\theta, \rho\sin\theta) \\ \dot{\theta} = \dfrac{1}{\rho} \cdot \cos\theta f_2(\rho\cos\theta, \rho\sin\theta) - \dfrac{1}{\rho} \cdot \sin\theta \cdot f_1(\rho\cos\theta, \rho\sin\theta) \end{cases}$

SD: Curs 10 (week 10) 10.05.2019

(15) $\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \end{cases}$

a). Find the equilibria and study their stability.
b). Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$
c). Represent in the phase plane the orbits corresponding to the initial value $(0, 2/3)$, $(4, 0)$

Solutions:

a). $-x + xy = 0$
$-2y + 3y^2 = 0 \Rightarrow y(-2+3y) = 0 \Rightarrow y_1 = 0$, $y_2 = 2/3$

$-x + x \cdot 0 = 0 \Rightarrow -x = 0 \Rightarrow x_1 = 0$
$-x + x\frac{2}{3} = 0 \Rightarrow \frac{-x}{3} = 0 \Rightarrow x_2 = 0$
$\Rightarrow A_1(0,0)$; $A_2(0, 2/3)$ equil. point.

b). By ~~the~~ definition $\varphi(t, 0, 2/3)$ is the unique sol. of the IVP.

i.V.P. $\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \\ x(0) = 0 \\ y(0) = 2/3 \end{cases}$ cannot solve "probleme neinițiale"

Since $(0, 2/3)$ is an equil. point, we have that $\varphi(t, 0, 2/3) = \begin{pmatrix} 0 \\ 2/3 \end{pmatrix}$
$\forall t \in \mathbb{R}$.
By definition $\varphi(t, 4, 0)$ is the unique sol. of the iVP.

$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \\ x(0) = 4 \\ y(0) = 0 \end{cases}$ First we solve the iVP: $\begin{cases} \dot{y} = -2y + 3y^2 \\ y(0) = 0 \end{cases} \Rightarrow \boxed{y = 0}$

i.V.P. $\begin{cases} \dot{x} = -x \\ x(0) = 4 \end{cases} \rightarrow \dot{x} + x = 0 \mapsto r + 1 = 0 \Rightarrow \boxed{r = -1}$

$\left.\begin{array}{l} x = e^{-t} \cdot c,\ c \in \mathbb{R} \\ x(0) = \boxed{c = 4} \end{array}\right\} \Rightarrow x = 4e^{-t}$

$\Rightarrow \varphi(t, 4, 0) = \begin{pmatrix} 4e^{-t} \\ 0 \end{pmatrix}$, $\forall t \in \mathbb{R}$

SD: Curs 10 (week 10) 10.05.2019

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$-x + x\frac{2}{3} = 0 \Rightarrow \frac{-x}{3} = 0 \Rightarrow x_2 = 0$
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i.V.P. $\begin{cases} \dot{x} = -x \\ x(0) = 4 \end{cases} \to \dot{x} + x = 0 \mapsto r + 1 = 0 \Rightarrow \boxed{r = -1}$

$x = e^{-t} \cdot c$, $c \in \mathbb{R}$
$x(0) = \boxed{c = 4}$ $\Rightarrow x = 4e^{-t}$

$\to \varphi(t, 4, 0) = \begin{pmatrix} 4e^{-t} \\ 0 \end{pmatrix}$, $\forall t \in \mathbb{R}$

$\cdot \varphi(t, 1, \frac{2}{3})$ is the unique sol of the IVP

$$\begin{cases} \dot{x} = -x + xy \\ \dot{y} = -2y + 3y^2 \\ x(0) = 1 \\ y(0) = 2/3 \end{cases}$$

We solve the IVP:

$$\begin{cases} \dot{x} = -2y + 3y^2 \\ y(0) = 2/3 \end{cases}$$

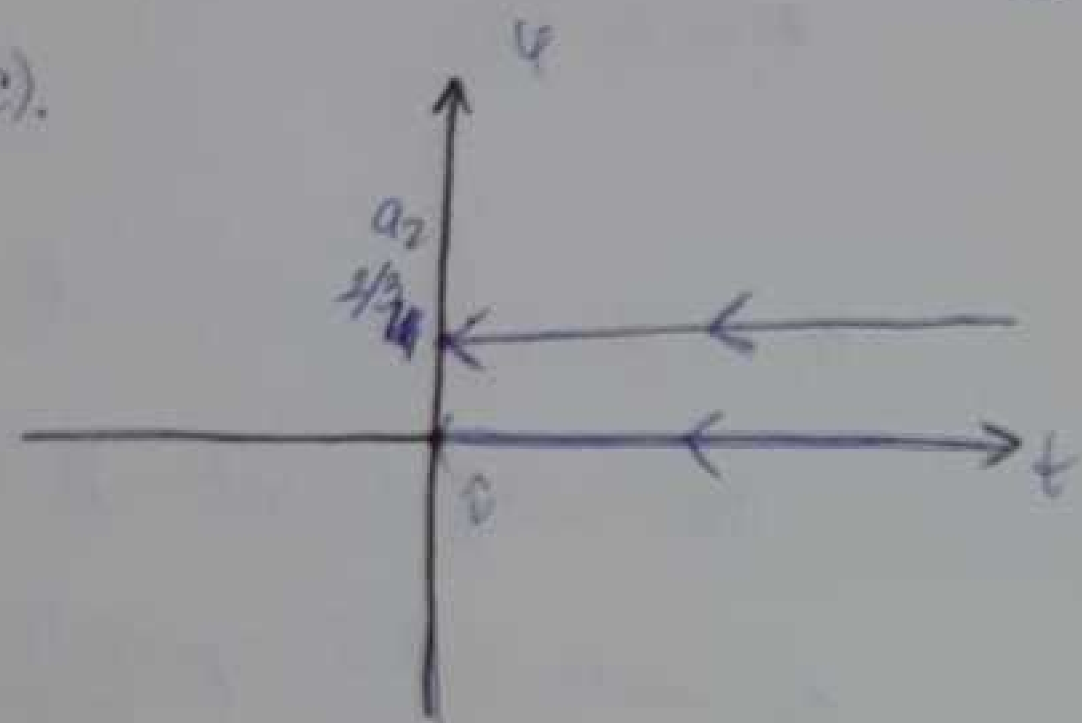
$\boxed{y = 2/3} \leftarrow$ unique sol. ct. fct.

$$\begin{cases} \dot{x} = -x + x\frac{2}{3} \\ x(0) = 1 \end{cases} \Rightarrow \dot{x} = -\frac{x}{3} \mapsto r + \frac{1}{3} = 0 \Rightarrow r = -\frac{1}{3}$$

$$\left.\begin{array}{l} x = e^{-\frac{1}{3}t} \cdot c \\ x(0) = 1 \end{array}\right\} \Rightarrow x(0) = \boxed{c = 1} \Rightarrow x = e^{-\frac{1}{3}t} \Rightarrow$$

$$\Rightarrow \varphi(t, 1, \tfrac{2}{3}) = \begin{pmatrix} e^{-1/3 t} \\ \frac{2}{3} \end{pmatrix}, \forall t \in \mathbb{R}$$

c).



(22) Consider the ~~flo~~ following planar systems

i). $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ linear

ii). $\begin{cases} \dot{x} = -y - x(x^2 + y^2) \\ \dot{y} = x - y(x^2 + y^2) \end{cases}$ non-linear

iii). $\begin{cases} \dot{x} = -y + xy \\ \dot{y} = x - x^2 \end{cases}$ non-linear

iv). $\begin{cases} \dot{x} = -y + x(x^2 + y^2) \\ \dot{y} = x + y(x^2 + y^2) \end{cases}$ non-linear

a) What is the type of the linear system i?

b) Show that the equil. $(0,0)$ of (ii), (iii), (iv) is not hyperbolic

c). Passing to polar coordinates, represent the phase portrait of these systems.

d). Study the type and stability of $(0,0)$ studying the phase portrait.

---

a). $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

We find the eigenvalues:

$\det(A - \lambda I_2) = 0 \Rightarrow \det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = 0 \Rightarrow$

$\Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 = -1 \Rightarrow \lambda_1 = i$

$\lambda_2 = -i$

$\Rightarrow$ the system is CENTER

b). For system a):

eq. point: $(0,0)$

$f(x,y) = \begin{pmatrix} -y - x^3 - xy^2 \\ x - yx^2 - y^3 \end{pmatrix}$

$Jf(x,y) = \begin{pmatrix} -x^2 - y^2 & -1 - 2xy \\ 1 - 2yx & -x^2 - 3y^2 \end{pmatrix}$

$Jf(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Homework: the Jacobian matrix of iii and iv is the same.

c) system i):

$(x,y) \longrightarrow (\rho, \theta)$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \longleftrightarrow \begin{matrix} \rho^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{matrix}$

NEED TO KNOW!

Coordinate polare.

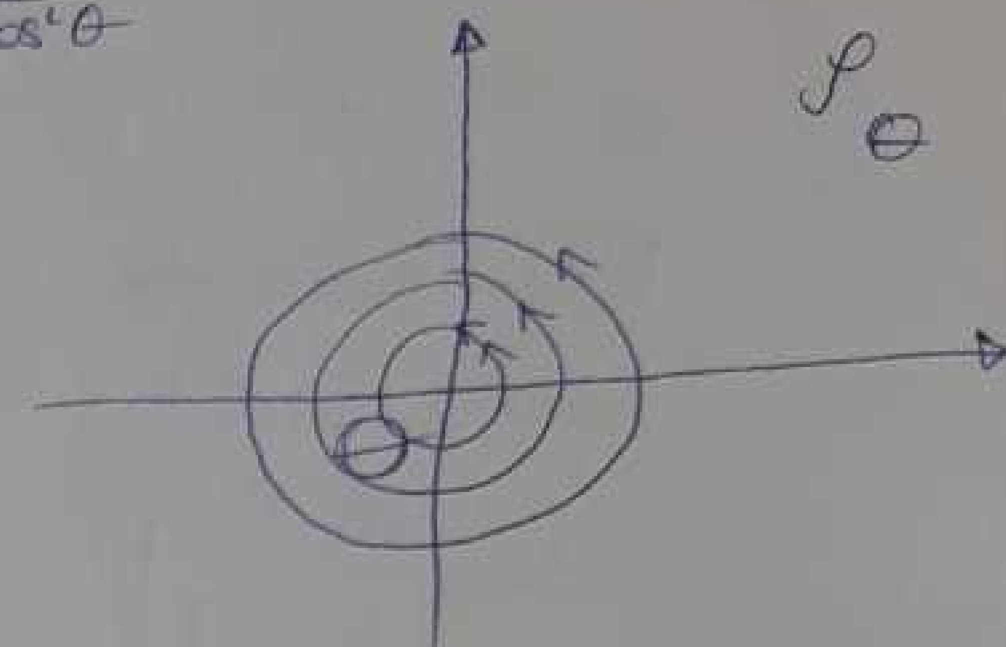
$\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$

$$\begin{cases} \rho\dot{\rho} = x\cdot(-y) + yx \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2+y^2}{x^2} \end{cases} \iff \begin{cases} \rho\cdot\dot\rho = 0 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\rho^2}{x} = \dfrac{1}{\cos^2\theta} \end{cases}$$

$\rho\ \theta$

$\dot\rho = 0 \Rightarrow \rho$ - ct.

$\dot\theta = 1 \Rightarrow \theta$ increases with time



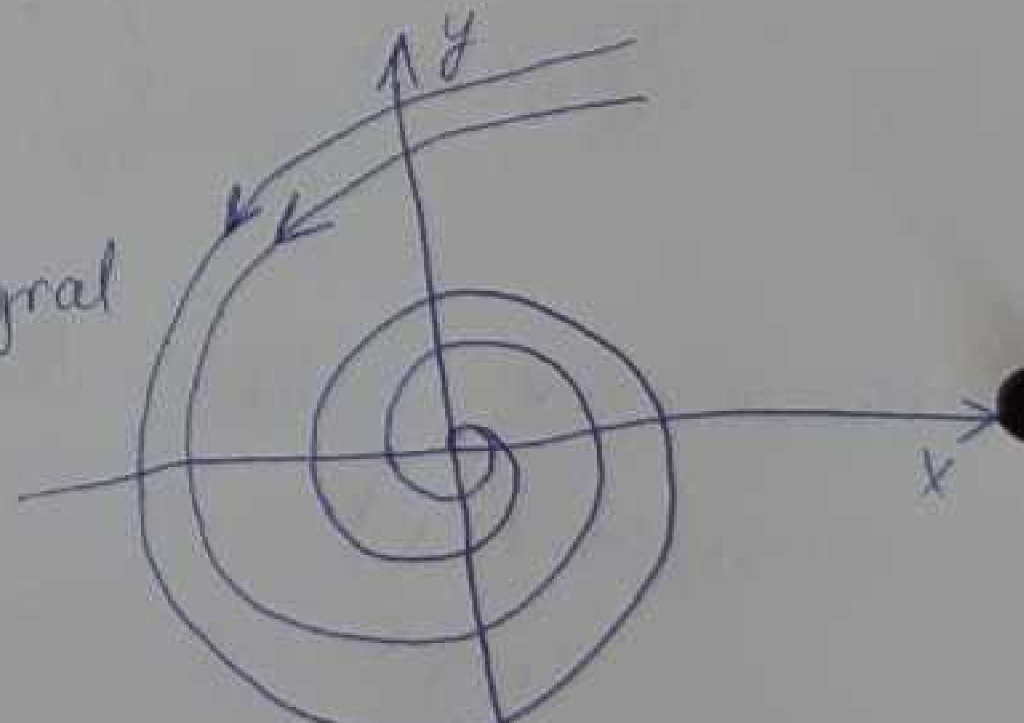
system ii):

$$\begin{cases} \rho\cdot\dot\rho = x\dot{x} + y\dot{y} \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\dot{y}x - \dot{x}y}{x^2} \end{cases} \iff \begin{cases} \rho\dot\rho = x[-y - x(x^2+y^2)] + y[x - y(x^2+y^2)] \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x[x - y(x^2+y^2)]}{x^2} \end{cases}$$

$$\iff \begin{cases} \rho\dot\rho = -xy - x^2(x^2+y^2) + xy - y^2(x^2+y^2) = -\rho^4 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2 - xy(x^2+y^2) + y^2 + xy(x^2+y^2)}{x^2} = \dfrac{\rho^2}{\rho^2\cdot\cos^2\theta} = \dfrac{1}{\cos^2\theta} \end{cases}$$

$\dot\rho = -\rho^3 \Rightarrow \rho$ strictly decreases with time

$\dot\theta = 1 \Rightarrow \theta$ increases with time



iii). Phase portrait: → all the equil. → find a first integral

$$\begin{cases} \dot{x} = -y + xy = -y(1-x) \\ \dot{y} = x - x^2 \end{cases}$$

$$\begin{cases} -y + xy = 0 \\ x - x^2 = 0 \end{cases}$$

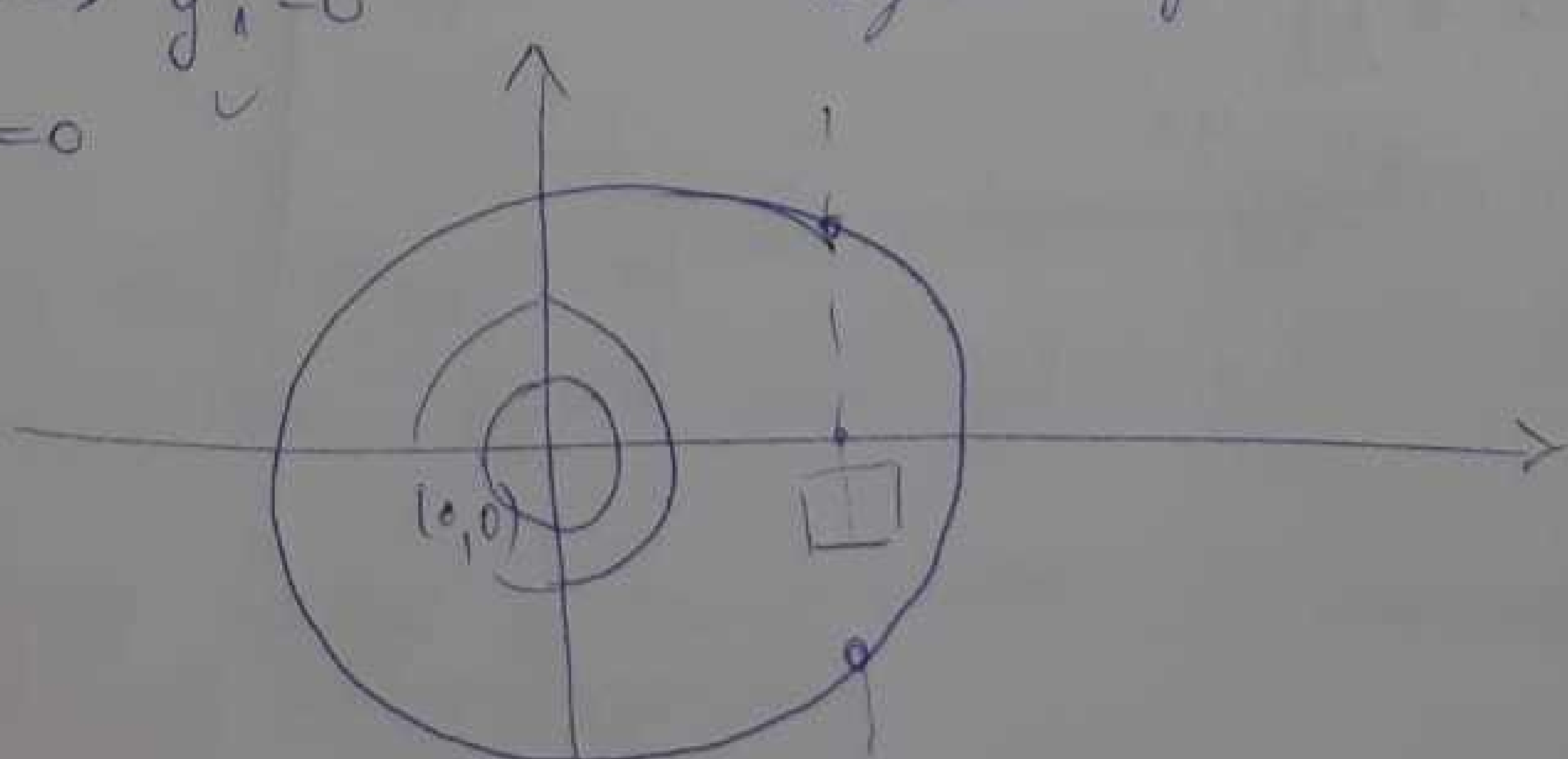
$x - x^2 = 0$

$x(1-x) = 0 \Rightarrow x_1 = 0,\ x_2 = 1$

$-y = 0 \to y_1 = 0$ ✓

$-y + y = 0$

equil. points: $(0,0)$, $(1,a)$, $a \in \mathbb{R}$



$$\frac{dy}{dx} = \frac{x-x^2}{-y+xy} \Rightarrow \frac{dy}{dx} = \frac{x(1-x)}{-y(1-x)} \Rightarrow \frac{dy}{dx} = \frac{x}{-y}$$

$$-y\,dy = x\,dx$$

$$-\int y\,dy = \int x\,dx$$

$$-\frac{y^2}{2} = \frac{x^2}{2} + c \;\Big|\cdot 2 \Rightarrow H(x,y) = x^2+y^2, \; \forall x,y \in \mathbb{R}^2$$

→ this might be a global first int.

$$\frac{\partial H}{\partial x}(x,y)\cdot f_1(x,y) + \frac{\partial H}{\partial y}(x,y)\cdot f_2(x,y) = 0$$

$$2x(-y+xy) + 2y(x-x^2) = 0$$

$$-2xy + 2x^2y + 2xy - 2yx^2 = 0$$

$$0 = 0 \quad (\text{True}, \forall (x,y) \in \mathbb{R}^2)$$

The direction field associated to a differential equation

(1) $y' = f(x,y)$

$y(x)$ is the notation for the unknown.

$f : \mathbb{R}^2 \to \mathbb{R}$ $\quad C^1$

A solution curve is represented in $\mathbb{R}^2$

(2) $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$

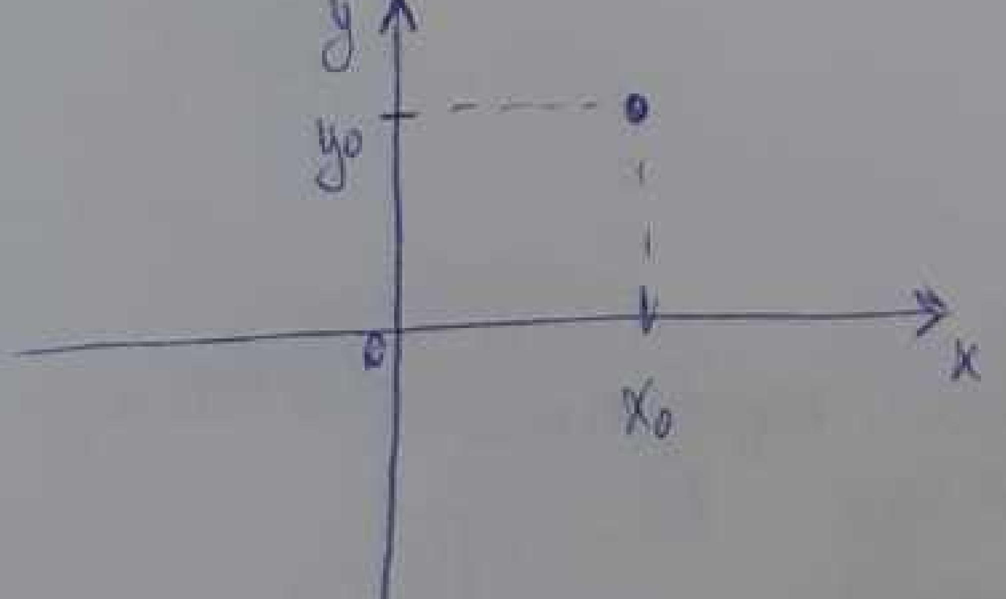
$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is the solution for the unknown

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2 \quad C^1$

An orbit is ~~p~~ represented in $\mathbb{R}^2$.

(1) $y' - f(x,y)$

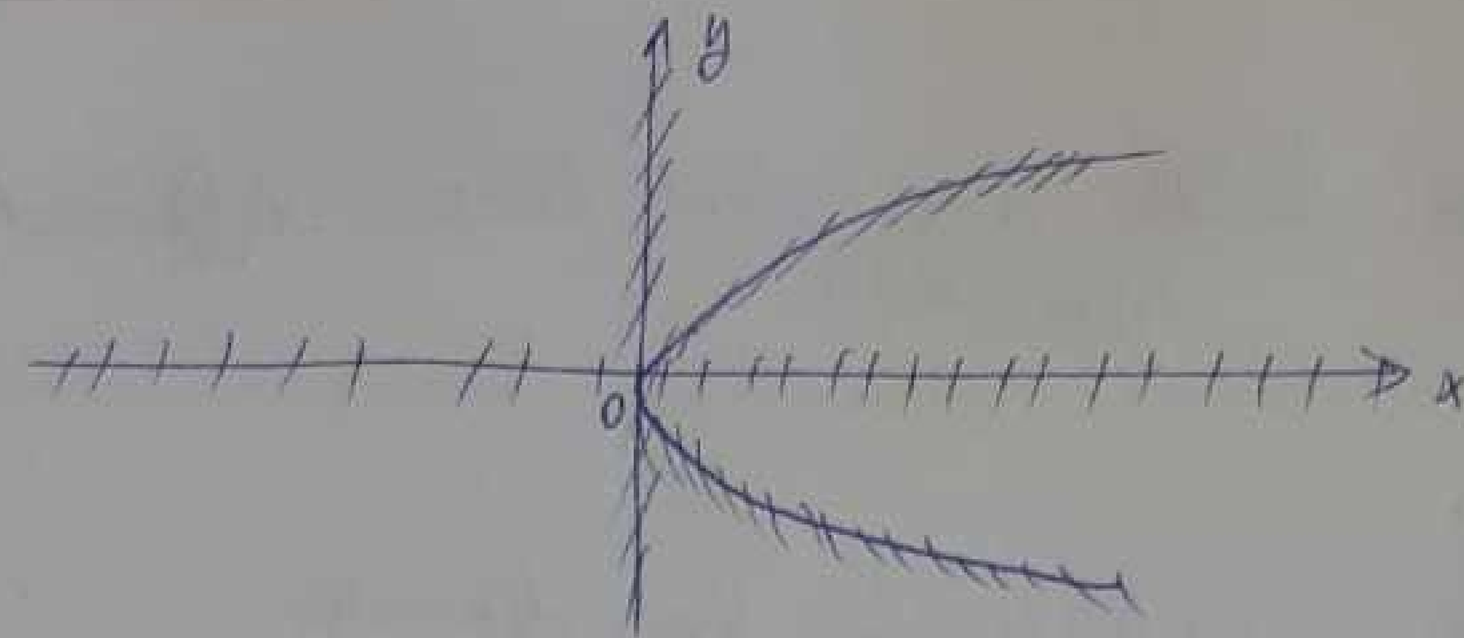
The notation field associated to (1) is a collection of vectors in $\mathbb{R}^2$



The vector through an arbitrary point $(x_0, y_0) \in \mathbb{R}^2$ has the slope $m = f(x_0, y_0)$

<u>Example:</u> $y' = 1 - \dfrac{x}{y^2}$

Let the vectors corresponding to $(1,1)$, $(0,1)$, $(1,0)$



Definition:

Let $m \in \mathbb{R} \cup \{\infty\}$ we define the m-isoclusive of the direction field of (1) as follows:

$$I_m = \{(x,y) \in \mathbb{R}^2 : f(x,y) = m\}$$

ex: Find the 0-izocline, 1-izocline of $y' = 1 - \frac{x}{y^2}$.

0-izocline: $1 - \frac{x}{y^2} = 0 \Leftrightarrow y^2 = x$ (a parabola)

1-izocline: $1 - \frac{x}{y^2} = 1 \Leftrightarrow x = 0$ (line).

Proposition:

Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary but fixed.
The slope of the direction field of (1) is (x_0, y_0) is equal to the slope of the solution curve of (1) that passes through (x_0, y_0).

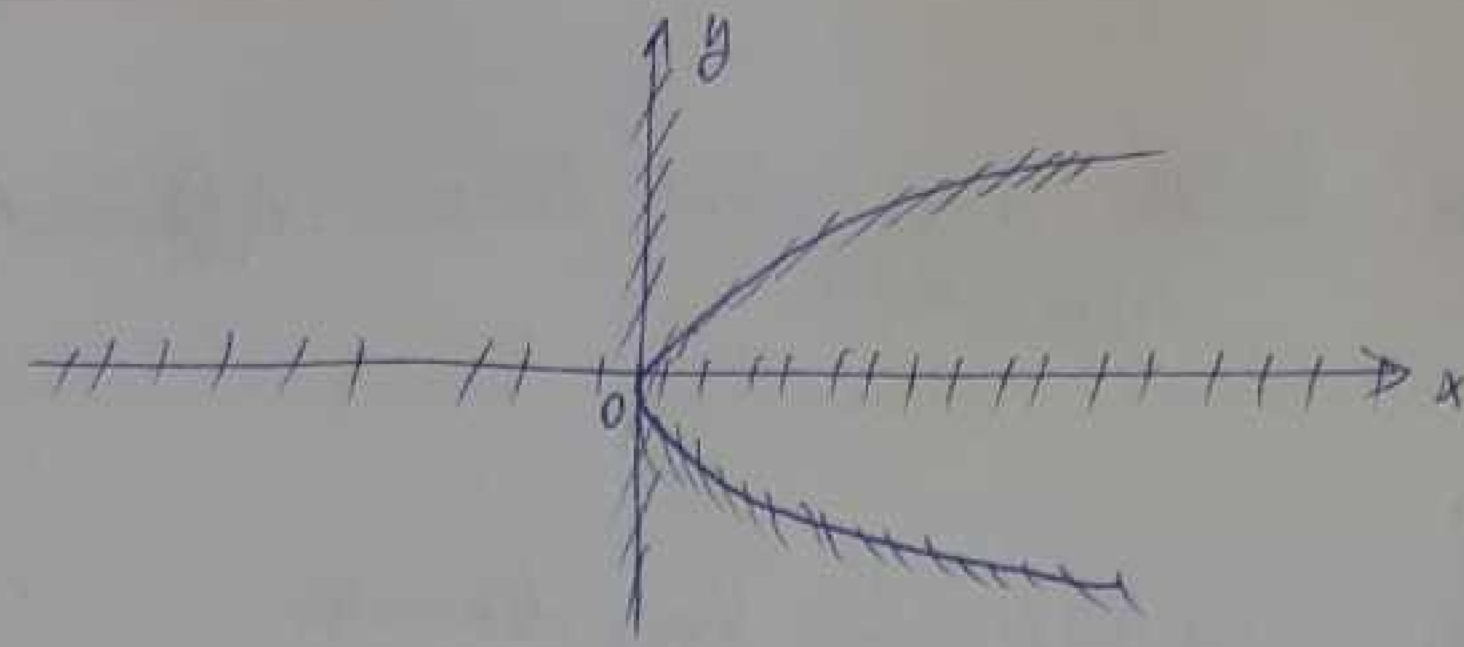
Proof: we consider the IVP:

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

It is known that this IVP has a unique solution, denoted by φ.

$$\Rightarrow \begin{cases} \varphi'(x) \overset{?}{=} f(x, \varphi(x)) \\ \varphi(x_0) = y_0 \end{cases} \Rightarrow$$ φ is the unique sol. curve of (1) that passes through (x_0, y_0).

$\Rightarrow \varphi'(x_0)$ is the slope of the solution curve that passes through (x_0, y_0); So we have to prove that
$\varphi'(x_0) = f(x_0, y_0)$



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ex: Find the 0-izocline, 1-izocline of $y' = 1 - \frac{x}{y^2}$.

0-izocline : $1 - \frac{x}{y^2} = 0 \iff y^2 = x$ (a parabola)

1-izocline : $1 - \frac{x}{y^2} = 1 \iff x = 0$ (line).

Proposition:

Let $(x_0, y_0) \in \mathbb{R}^2$ be arbitrary but fixed.
The slope of the direction field of (1) is (x_0, y_0) is equal to the slope of the solution curve of (1) that passes through (x_0, y_0)

Proof: we consider the IVP :

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

It is known that this IVP has a unique solution, denoted by φ.

$$\Rightarrow \begin{cases} \varphi'(x) = f(x, \varphi(x)) \\ \varphi(x_0) = y_0 \end{cases} \Rightarrow$$ φ is the unique sol. curve of (1) that passes through (x_0, y_0).

$\Rightarrow \varphi'(x_0)$ is the slope of the solution curve that passes through (x_0, y_0); So we have to prove that

$$\varphi'(x_0) = f(x_0, y_0)$$

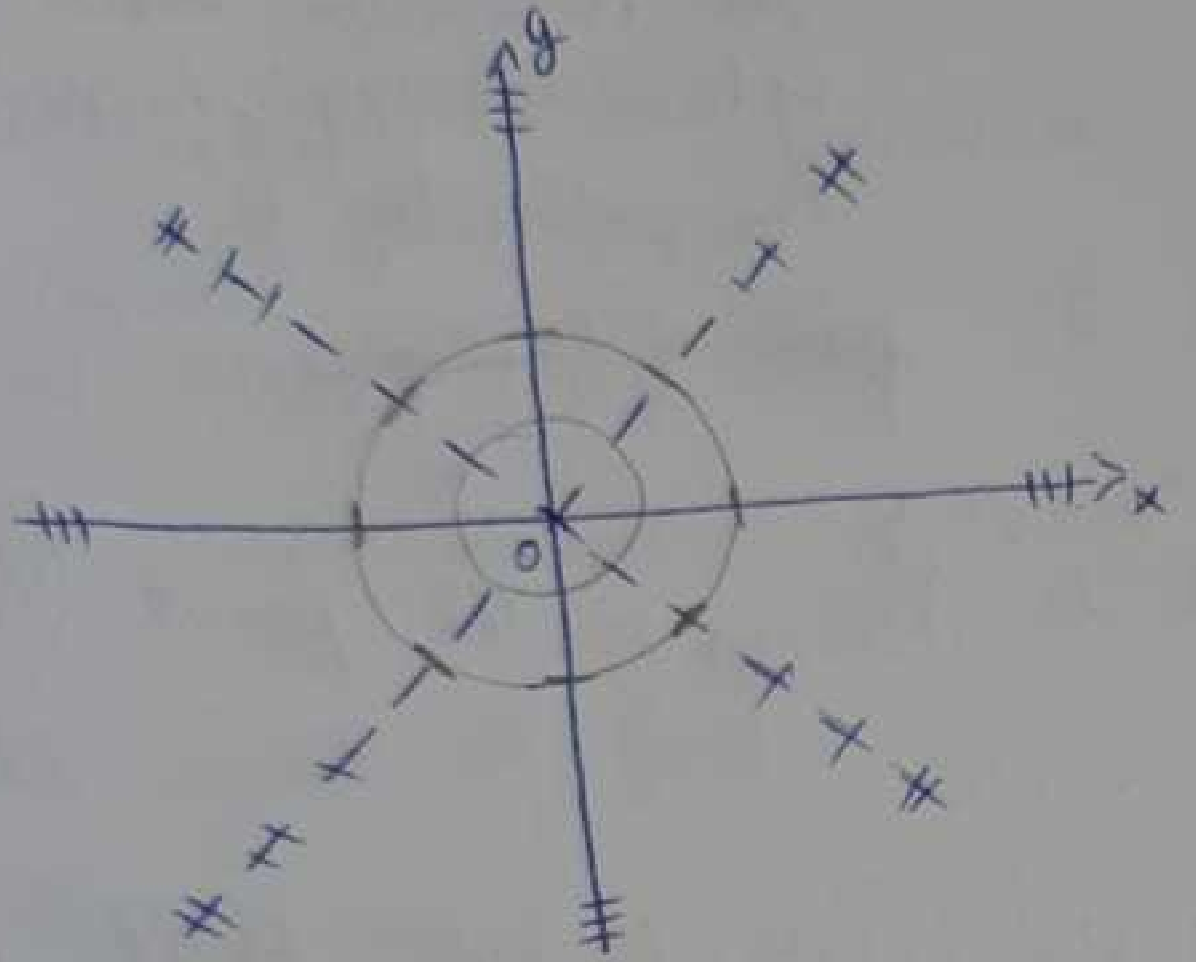
Example: $y' = -\frac{x}{y}$

$f(x,y) = -\frac{x}{y}$

0 - izocline $x = 0$

∞ - izocline $y = 0$

n - izocline $-\frac{x}{y} = n \Leftrightarrow y = \frac{1}{n} x$ (a line).



1 - izocline : $y = -x$

- Note that the solution curves lie on circles centered in the origin.

$$\boxed{\frac{dy}{dx} = -\frac{x}{y} \quad ydy = xdx \ ; \quad \frac{y^2}{2} = \frac{x^2}{2} + c \ ; \quad x^2 + y^2 = c, \ c \in \mathbb{R}.}$$

Definition:

The slope of the direction field associated to the planar system (2) is the point $(x_0, y_0) \in \mathbb{R}^2$ is

$$m = \frac{f_2(x_0, y_0)}{f_1(x_0, y_0)}$$

Proposition:

The slope of the direction field of (2) in $(x_0, y_0) \in \mathbb{R}^2$ is equal to the slope of the orbit of (2) that passes through (x_0, y_0).

Proof: we consider the IVP: $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$ and denote by $\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$ the unique sol

$$\Rightarrow \begin{cases} \dot{\varphi}_1(t) = f_1(\varphi_1(t), \varphi_2(t)) \\ \dot{\varphi}_2(t) = f_2(\varphi_1(t), \varphi_2(t)) \\ \varphi_1(0) = x_0 \\ \varphi_2(0) = y_0 \end{cases}$$

$\Rightarrow \gamma_{(x_0,y_0)} = \{(\varphi_1(t), \varphi_2(t)) \in \mathbb{R}^2\}$

It is known that the vector $(\varphi_1'(t), \varphi_2'(t))$ is tangent to $\gamma_{(x_0,y_0)}$ in the point $(\varphi_1(t), \varphi_2(t))$.

$\Rightarrow (\varphi_1'(0), \varphi_2'(0))$ is tangent to $\gamma_{(x_0,y_0)}$ is the point $(\varphi_1(0), \varphi_2(0)) = (x_0, y_0)$.

$\Rightarrow \dfrac{\varphi_2'(0)}{\varphi_1'(0)}$ is the slope of the orbit of (2) that passes through (x_0, y_0). Thus we have to prove that

$$\frac{f_2(x_0,y_0)}{f_1(x_0,y_0)} = \frac{\varphi_2'(0)}{\varphi_1'(0)}$$

we have $\dfrac{\varphi_2'(0)}{\varphi_1'(0)} = \dfrac{f_2(\varphi_1(0), \varphi_2(0))}{f_1(\varphi_1(0), \varphi_2(0))} = \dfrac{f_2(x_0,y_0)}{f_1(x_0,y_0)}$ q.e.d. $\Rightarrow \square$

ex:

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x. \end{aligned}$$

Numerical methods to find approximate solutions of differential equations.

(1) $y' = f(x, y)$

$f: \mathbb{R}^2 \to \mathbb{R}$ C^1

$(x_0, y_0) \in \mathbb{R}^2$

(IVP$_1$) $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ – has a unique sol. $\varphi: [x_0, x^*] \to \mathbb{R}$

we consider a partition of $[x_0, x^*]$.

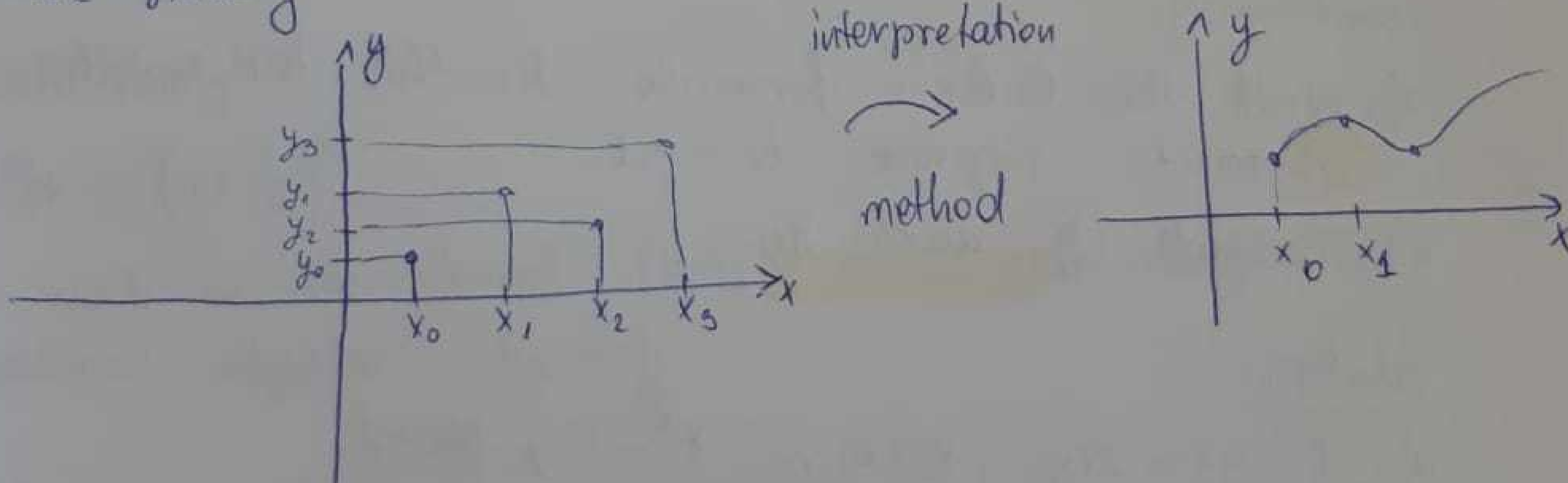
$x_0 < x_1 < x_2 < \dots < x_n = x^*$ (we cover in n steps the interval $[x, x^*]$)

We want to find $y_1, \dots, y_n$ "good" approximations for $\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)$.

$y_k \approx \varphi(x_k)$ $k = \overline{1,n}$

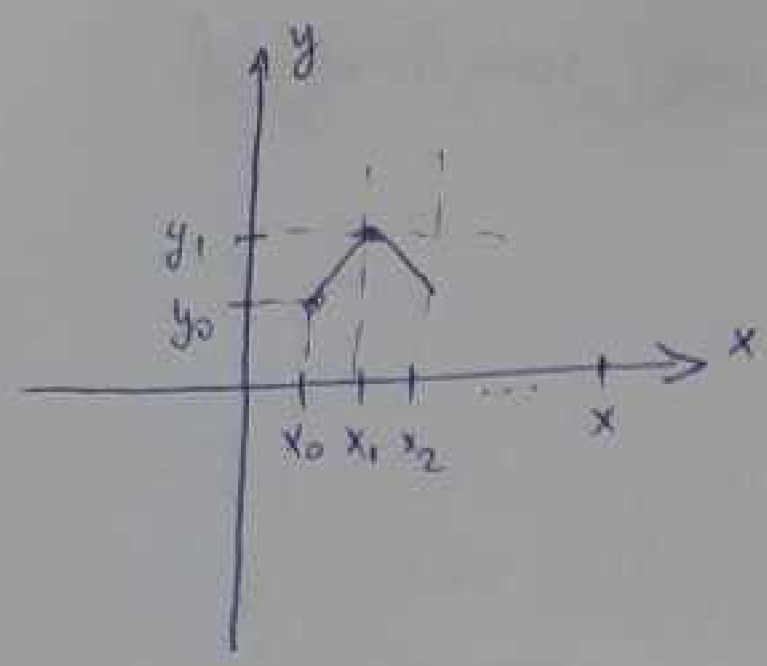
$y_k = ??$

we finally obtain (x_k, y_k), $k = \overline{1,n}$



The Euler's formula:

$$y_{k+1} = y_k + (x_{k+1} - x_k) f(x_k, y_k), \quad k = \overline{0, n-1}$$



(x_0, y_0)

$m_0 = f(x_0, y_0)$

$y - y_0 = m_0(x - x_0)$

$y_1 = y_0 + (x_1 - x_0) f(x_0, y_0)$

(x_1, y_1)

$m_1 = f(x_1, y_1)$

$y - y_1 = m_1(x - x_1)$

$y_2 = y_1 + (x_2 - x_1) f(x_1, y_1)$

...

Example (for EXAM):

we consider the IVP:

$$\begin{cases} y' = 2xy, & x \in [0,1] \\ y(0) = 1 \end{cases}$$

In the case that $x_{k+1} - x_k = h$, $\forall k$ we say that h is the stepsize (was written on another table)

continuing:

a). Write the Euler's formula for this IVP with constant stepsize $h = 0.1$.

b). Compute y_1 and y_2

Solution:

a). $f(x,y) = 2xy$, $x_0 = 0$, $x^* = 1$, $y_0 = 1$.

$$\begin{cases} y_{k+1} = y_k + 0.1 \cdot 2 \cdot x_k \cdot y_k, & \forall k = \overline{0,9} \\ x_{k+1} = x_k + h \end{cases}$$

$$\Leftrightarrow \begin{cases} y_{k+1} = y_k + 0.2\, x_k y_k, & k = \overline{0,9} \\ x_k = \frac{k}{10}, & k = \overline{1,10} \\ x_0 = 0, & y_0 = 1 \end{cases}$$

b). $x_1 = \frac{1}{10}$

$$y_1 = y_0 + \frac{2}{10} x_0 y_0 = 1 + \frac{2}{10} \cdot 0.1 = 1.$$

$$y_2 = y_1 + \frac{2}{10} x_1 y_1 = 1 + \frac{2}{10} \cdot \frac{1}{10} \cdot 1 = 1 + \frac{2}{100} = 1.02.$$

Improved Euler's formula:

$$y_{k+1} = y_k + \frac{h}{2} (x_k, y_k) + \frac{h}{2} f(x_k + h, y_k + h f(x_k, y_k))$$

Exercițiu:

we consider the IVP : $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$

Write the Euler's formula with stepsize $h_n = \frac{1}{n}$ $(n \geq 1)$ and obtain approximations for the number e.

Solution:

We have that $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = e^x$ is the unique sol. of this IVP. Note that $\varphi(1) = e$.

$f(x, y) = y$.

$$y_{k+1} = y_k + \frac{1}{n} \cdot y_k, \ \forall k \geq 0. \Leftrightarrow \begin{cases} y_{k+1} = (1 + \frac{1}{n}) \cdot y_k, \ \forall k \geq 0, \\ y_0 = 1 \end{cases}$$

Note that

$$y_k = \left(1 + \frac{1}{n}\right)^k$$

we work on the interval $[0, 1]$ with a partition with constant stepsize $h_n = \frac{1}{n}$

So $x_n = 1$

$y_n \approx \varphi(x_n) = \varphi(1) = e$ | $y_n = \left(1 + \frac{1}{n}\right)^n \approx e$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Discrete dynamical systems

We consider the non-linear difference equation in $\mathbb{R}^n$

(1) $x_{k+1} = f(x_k)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

Remark:

1) For an arbitrary $\eta \in \mathbb{R}^n$, the IVP $x_{k+1} = f(x_k)$, $x_0 = \eta$ has a unique solution $\eta, f(\eta), \underbrace{f(f(\eta))}_{\overset{\text{notation}}{=} f^2(\eta)}, \ldots, f^k(\eta), \ldots$

where $f^k = \underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}} \to$ (the k^{th} iterate of f)

2) the flow of (1) is $\varphi : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi(k, \eta) = f^k(\eta)$, $\forall k \geq 0$, $\forall \eta \in \mathbb{R}^n$.

Definition:

1). $\mathbb{R}^n$ is called the state space ; η is the initial state, $f^k(\eta)$ is the state at time k.

2). The positive orbit of η is $\gamma_\eta^+ = \{\eta, f(\eta), \ldots, f^k(\eta) \ldots\}$

In the case that f is invertible we define the orbit of η

$\gamma_\eta = \{\ldots f^{-k}(\eta), \ldots, f^{-2}(\eta), f^{-1}(\eta), \eta, \ldots\}$

where f^{-1} is the inverse of f (i.e. $f \circ f^{-1} = f^{-1} \circ f = id$)

and $f^{-k} = (f^{-1})^k$

3). We say that $\eta^* \in \mathbb{R}^n$ is a fixed point of f if $f(\eta^*) = \eta^*$

4). Let η^* be a fixed point of f.

- We say that η^* is stable where $\forall \varepsilon > 0 \ \exists \delta > 0$ such that whenever $\|\eta - \eta^*\| < \delta$ we have $\| f^k(\eta) - \eta^* \| < \varepsilon, \ \forall k \geq 0.$
- We say that η^* is an attractor when $\exists r > 0$ such that whenever $\|\eta - \eta^*\| < r$ we have that $\lim\limits_{k \to \infty} f^k(\eta) = \eta^*.$
- The basin of attraction of an attractor η^* is $A_{\eta^*} = \{ \eta \in \mathbb{R}^n : \lim\limits_{k \to \infty} f^k(\eta) = \eta^* \}$

Remark:

Let η^* be a fixed point of f.

- Then η^* is a fixed point of f^k, $\forall k \geq 1$.

Proof by induction:

η^* is a fixed point of $f \Leftrightarrow \boxed{f(\eta^*) = \eta^*} \Rightarrow$

$\Rightarrow f(f(\eta^*)) = f(\eta^*) \Rightarrow$

$\Rightarrow f^2(\eta^*) = \eta^* \Rightarrow$ by def. η^* is a fix. point of f^2.

$f^k(\eta^*) = \eta^* \overset{??}{\Rightarrow} f^{k+1}(\eta^*) = \eta^*$ (HW. prove!)

- The positive orbit of the f. p. η^* is $\gamma^+_{\eta^*} = \{\eta^*\}$
- The unique sol. of IVP $\begin{cases} x_{k+1} = f(x_k) \\ x_0 = \eta^* \end{cases}$

is the constant sequence $x_k = f^k(\eta^*) = \eta^*, \ \forall k \geq 0.$

Proposition:

If $(x_k)_{k\geq 1}$ is a solution of (1), which is a convergent sequence, then its limits is a fixed point of f.

Proof:

Let $\eta^* \in \mathbb{R}^n$; $\lim_{k\to\infty} x_k = \eta^*$.

By hyp. $x_{k+1} = f(x_k)$, $\forall k \geq 0$.

$\lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(x_k)$

$\eta^* = f(\eta^*)$ (f cont)

Definition:

Let $\eta^* \in \mathbb{R}^n$ and $p \in \mathbb{N}$, $p \geq 2$.
We say that η^* is a p-periodic point of f when $f^p(\eta^*) = \eta^*$ and η^* is not a fixed point of $f, f^2, \dots, f^{p-1}$

Remark:

Let η^* be a p-periodic point of f.

1) Then $\gamma^+_{\eta^*} = \{\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*)\}$ - called p-periodic cycle

The unique sol. of the IVP $x_{k+1} = f(x_k)$, $x_0 = \eta^*$ is $\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*), \eta^*, f(\eta^*), \dots$

2). η^* is a p-periodic point $\Rightarrow$ $f(\eta^*), \dots, f^{p-1}(\eta^*)$ are also p-periodic points.

Example:

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 1 - 2x^2$
Find its fixed points and its 2-periodic points.

---

fixed points: $f(x) = x$, $x = ?$ $x \in \mathbb{R}$.

$\Leftrightarrow 1 - 2x^2 = x \Rightarrow 2x^2 + x - 1 = 0 \Rightarrow x_{1/2} = \begin{cases} -1 \\ \frac{1}{2} \end{cases}$

2-periodic points : $f^2(x) = x$, $x \in \mathbb{R}$, $f(x) \neq x$.

$$f^2(x) = f(f(x)) = f(1-2x^2) = 1 - 2(1-2x^2)^2 =$$
$$= 1 - 2(1 - 4x^2 + 4x^4) =$$
$$= 1 - 2 + 8x^2 - 8x^4 = -8x^4 + 8x^2 - 1$$

$$f^2(x) = x \Rightarrow -8x^4 + 8x^2 - 1 = x \quad /(-1).$$
$$8x^4 - 8x^2 + 1 + x = 0$$

We know that -1 and $\frac{1}{2}$ are fixed points $\Rightarrow$ they are f. p. for $f^2(x)$ as well. !

$\Rightarrow$

	x^4	x^3	x^2	x^1	x^0
	8	0	-8	1	1
-1	8	-8	0	1	0
$\frac{1}{2}$	8	-4	-2	0	

$\Rightarrow 8x^3 - 8x^2 + 1 = 0$

$\Rightarrow 8x^2 - 4x - 2 = 0.$

$$\Leftrightarrow 4x^2 - 2x - 1 = 0. \Rightarrow x_{3/4} = \begin{cases} \frac{1-\sqrt{5}}{4} \\ \frac{1+\sqrt{5}}{4} \end{cases}$$

<u>Conclusion</u>: f has 2 fixed points : $\{-1, \frac{1}{2}\}$ and f has on 2-periodic orbit $\left\{\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right\}$

<u>Definition:</u>

Let η^* be a p-periodic point of f. We say that its orbit $\gamma^+_{\eta^*}$ is stable / attractor / unstable when η^* is a stable / attractor / unstable fixed point of f^p.

<u>Remark:</u>

Let η^* be a p-periodic point of f which is an attractor.
Then $\exists r > 0$ s.t. whenever $\|\eta - \eta^*\| < r$ we have
$$\lim_{k \to \infty} (f^p)^k(\eta) = \eta^*$$

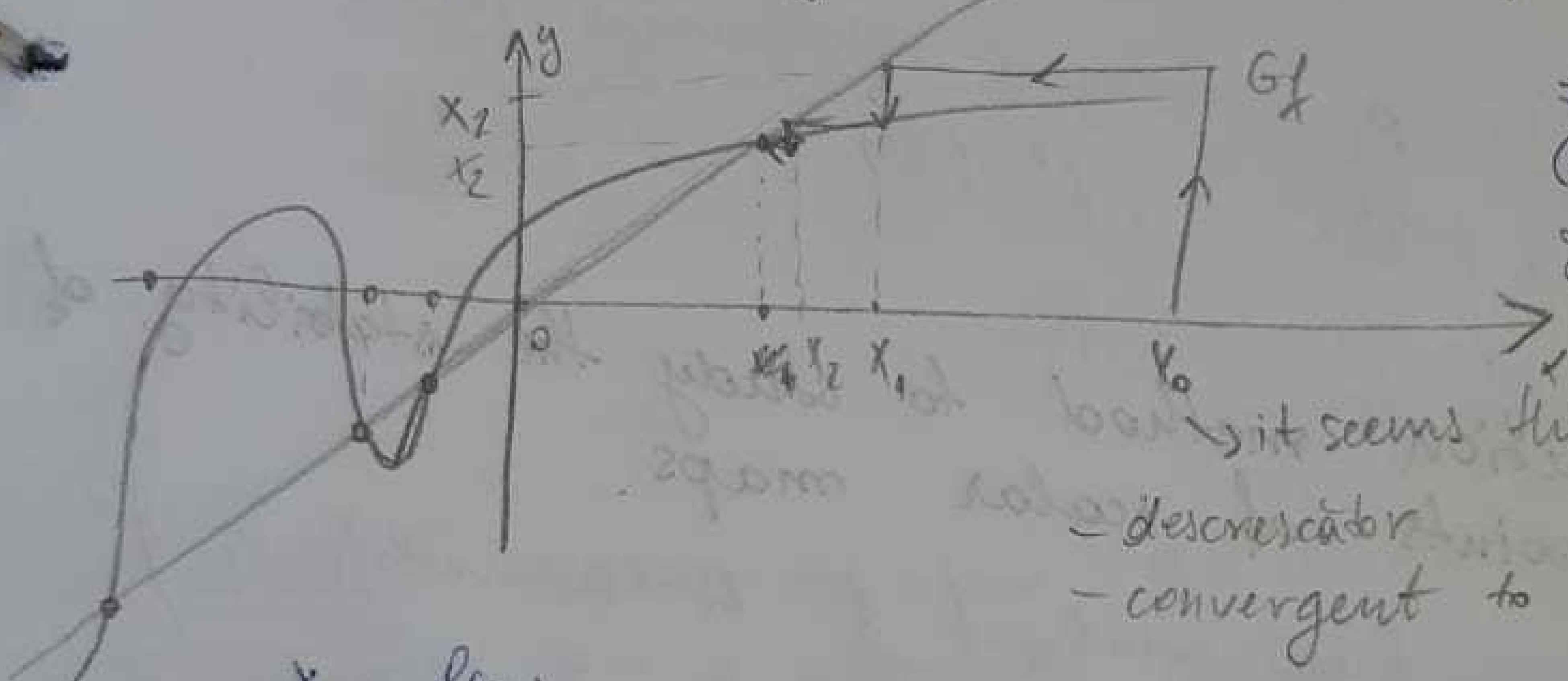
f has the 2-periodic orbit $\left\{ \frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4} \right\}$

$f^2(x) = -8x^4 + 8x^2 - 1$

$g = f^2$; $g'(x) = -32x^3 + 16x$

HW: $g'\left(\frac{1-\sqrt{5}}{4}\right) = \ldots$

The cob-web or stair-step diagram to study the dynamic of a scalar map f.



$f(x) = x$.
G_f
$y = x$.

↳ it seems that the sequence is:
- descrescător
- convergent to the fixed point nearby.

$x_1 = f(x_0)$
$x_2 = f(x_1)$

$\gamma^+_\eta = \{f^k(\eta) : \forall k \geq 0\} =$

$= \{\eta, f(\eta), \dots, f^{p-1}(\eta),$

$f^p(\eta), f^{p+1}(\eta), \dots, f^{2p-1}(\eta),$

$f^{2p}(\eta), f^{2p+1}(\eta), \dots, f^{3p-1}(\eta), \dots$

$\vdots$

$f^{kp}(\eta), f^{kp}(f(\eta)) \quad \dots \quad \}$

$\downarrow k \to \infty \quad \downarrow k \to \infty \quad \downarrow k \to \infty$

$\eta^* \quad f(\eta^*) \quad f^{p-1}(\eta^*)$

$\boxed{\eta = \text{eta}}$

Thus, $(f^k(\eta))_{k \geq 0}$ has p convergent subsequences with the limits $\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*)$.

The liniarization method to study the stability of the fixed points of scalar maps.

Let η^* be a fixed point of $f : \mathbb{R} \to \mathbb{R}$ which is C^1.

If $|f'(\eta^*)| < 1$ then η^* is an attractor.

If $|f'(\eta^*)| > 1$ then η^* is unstable.

Exercise:

Study the stability of the fixed points and of the 2-periodic orbit of $f(x) = 1 - 2x^2$.

f has the f.p. -1 and $\frac{1}{2}$.

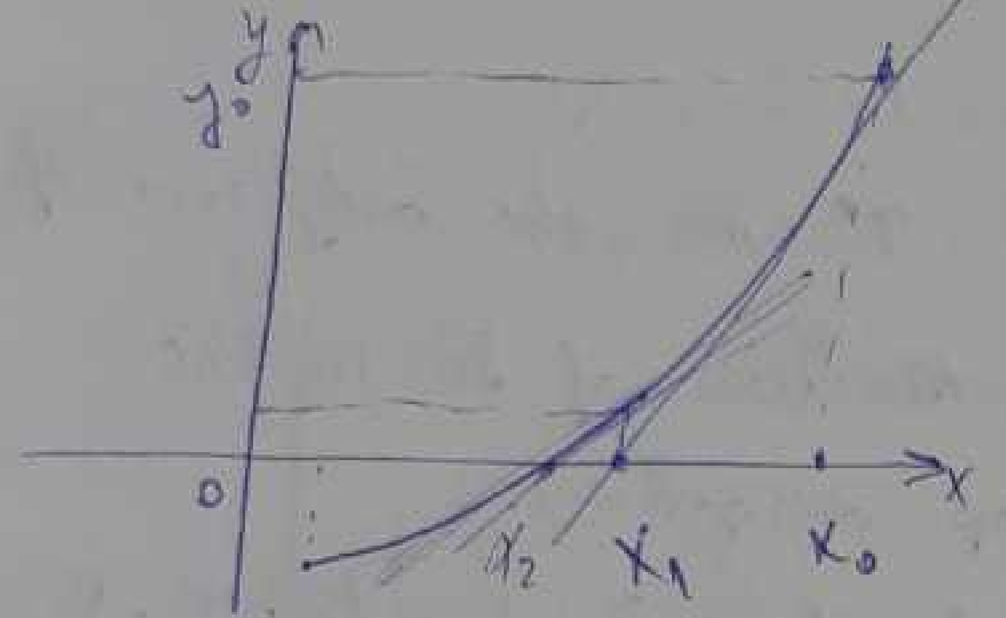
$f'(x) = -4x$

$f'(-1) = (-4)(-1) = 4 > 1 \Rightarrow -1$ is an unstable fixed point.

$f'(\frac{1}{2}) = -2 \Rightarrow |f'(\frac{1}{2})| > 1 \Rightarrow \frac{1}{2}$ is an unstable fixed point.

The Newton-Raphson's method to approximate the zeros of scalar maps.

Let $g: I \to \mathbb{R}$ be a C^1 map, where $I \subset \mathbb{R}$ is nonempty, open interval.



$\exists \eta^* \in I$ s.t. $g(\eta^*) = 0$ and $g'(\eta^*) \neq 0$

the tangent to the graph of g in $(x_0, g(x_0))$ has the eq.:

$y - g(x_0) = g'(x_0) \cdot (x - x_0)$

$\cap Ox \Rightarrow y = 0$

$-g(x_0) = g'(x_0)(x_1 - x_0) \Rightarrow x_1 = x_0 - \dfrac{g(x_0)}{g'(x_0)}$

we assume that $x_0 \in V$, where $V \subset I$ is s.t. $g'(x) \neq 0$, $\forall x \in V$.
(for example, V is a sufficiently small neighbourhood of η^*).

$\Rightarrow x_{k+1} = x_k - \dfrac{g(x_k)}{g'(x_k)}$, $\forall k \geq 0$, $x_0 \in V$ fixed.

we intend to prove that $\lim\limits_{k \to \infty} x_k = \eta^*$, for any x_0 suff. close to η^*.

we define $f: I \to \mathbb{R}$, $f(x) = x - \dfrac{g(x)}{g'(x)}$, $\forall x \in V$

we find the fixed points of f, i.e. $f(x) = x$, $x = V$.

$x - \dfrac{g(x)}{g'(x)} = x \Leftrightarrow g(x) = 0 \Rightarrow \eta^*$ is the only f.p. of f in V.

we want to prove that η^* is an attractor for f.

$f'(x) = 1 - \dfrac{g'(x) \cdot g'(x) - g(x) \cdot g''(x)}{[g'(x)]^2} \Rightarrow f'(\eta^*) = 1 - 1 = 0 \Rightarrow$

$\Rightarrow |f'(\eta^*)| < 1.$

linearization method.

$\overset{L.M}{\Longrightarrow}$ η^* is an attractor of f $\overset{def.}{\to}$ $\exists r>0$ s.t. whenever

$|\eta-\eta^*| < r$.

we have $\lim_{k\to\infty} f^k(\eta) = \eta^*$

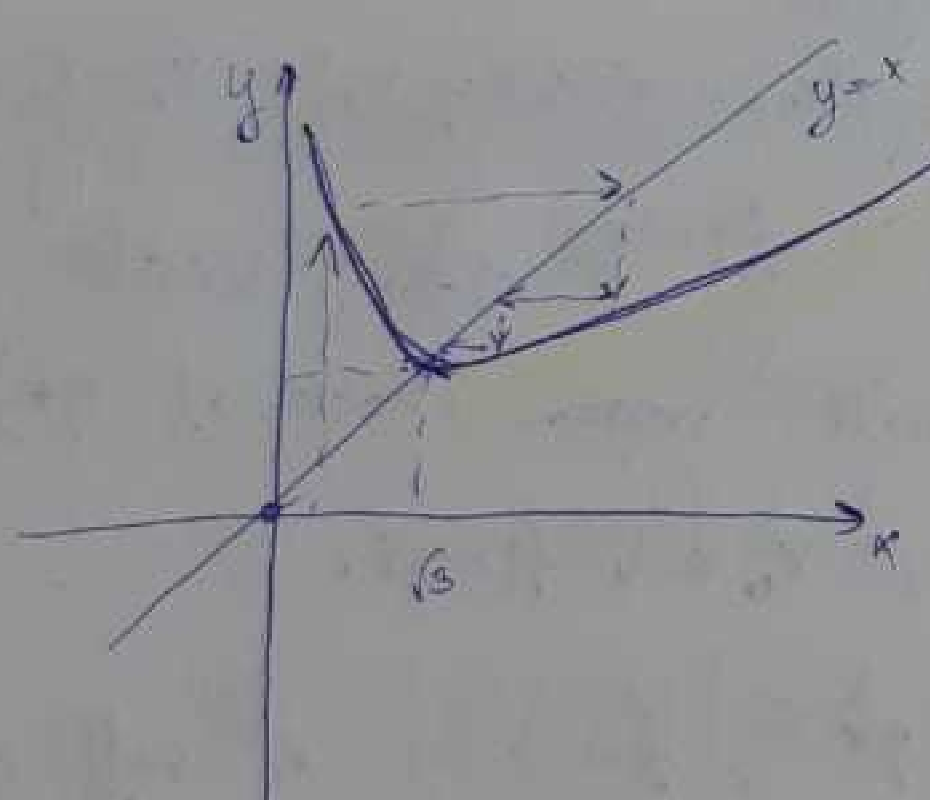
$\eta = x_0$, $f^k(\eta) = x_k$

A particular case

$g:(0,\infty) \to \mathbb{R}$, $g(x) = x^3-3$. with $\eta^* = \sqrt{3}$, the only zero of g.

we intend to estimate the basin of attraction of $\sqrt{3}$ in the Newton's method. (using the stair-step diagram).

$f:(0,\infty)\to\mathbb{R}$, $f(x) = x - \frac{x^2-3}{2x} = x - \frac{1}{2}x + \frac{3}{2}\cdot\frac{1}{x} = \frac{1}{2}x + \frac{3}{2}\cdot\frac{1}{x}$



$f(x) > 0$, $\forall x>0$.

$\lim_{x\searrow 0} f(x) = +\infty$ $\quad$ $\lim_{x\to\infty} f(x) = \infty$

$f'(x) = \frac{1}{2} - \frac{3}{2x^2} = \frac{1}{2}\cdot\frac{x^2-3}{x^2}$

x	0		$\sqrt{3}$		$+\infty$
f'		$- - - -$	0	$+ + + +$	

it seems that $A_{\sqrt{3}} = (0,\infty)$. $y = \frac{1}{2}x$ is an oblic asymptote.

Elaydi Discrete Chaos.

CAD PhD.

We consider the map $T:[0,1]\to\mathbb{R}$, $T(x) = 1-|2x-1|$

a). Represent the graph of T. Find the fixed points of T.

b). Compute the orbits corresponding to the initial values $\eta = \frac{3}{2^n}$, $n\geq 2$

c). Solve the equations $T(x)=0$, $T(x)=1$; $T(x) = \frac{1}{2}$, $T^2(x)=0$, $T(x)=1$; $T^2(x)=\frac{1}{2}$.

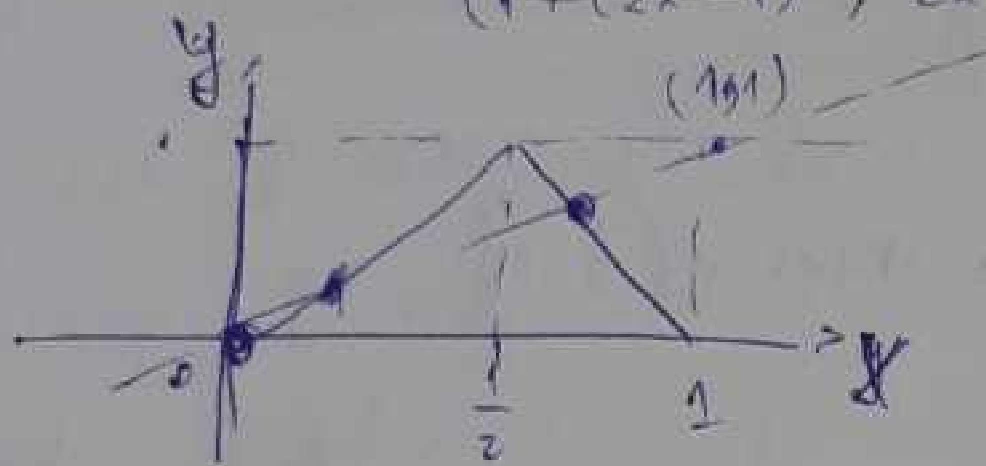
d). Compute T^2

e). Represent the graph of T^2 and T^3. How many fixed points do they have?

f). T has a 2-periodic orbit? T has a 3-p. orbit?

Fact: There is a result: "If a map has a 3-per. orbit, then it has a p-per orbit for any $p \in \mathbb{N}^*$"

a). $T(x) = \begin{cases} 1-(2x-1), & 2x-1>0 \\ 1+(2x-1), & 2x-1 \le 0 \end{cases} = \begin{cases} 2(1-x), & x \in (\frac{1}{2}, 1) \\ 2x, & x \in [0, \frac{1}{2}] \end{cases}$



$T(0) = 0$
$T(\frac{1}{2}) = 1$
$T(1) = 0$

the fixed points of T are: 0 and $\frac{2}{3}$.

b). $T^k(\frac{3}{2^n}) = ?$ $\gamma_{\frac{3}{2^n}} = ?$

$n=2$ $q = \frac{3}{4}$, $T(q) = T(\frac{3}{4}) = 2(1-\frac{3}{4}) = \frac{1}{2}$

$T^2(q) = T(\frac{1}{2}) = 1.$

$T^3(q) = T(1) = 0, 0, \dots$

$n=3$ $q = \frac{3}{8}$ $T(q) = T(\frac{3}{8}) = 2 \cdot \frac{3}{8} = \frac{3}{4}.$

$T^2(q) = T(\frac{3}{4}) = \frac{1}{2}, 1, 0, \dots$

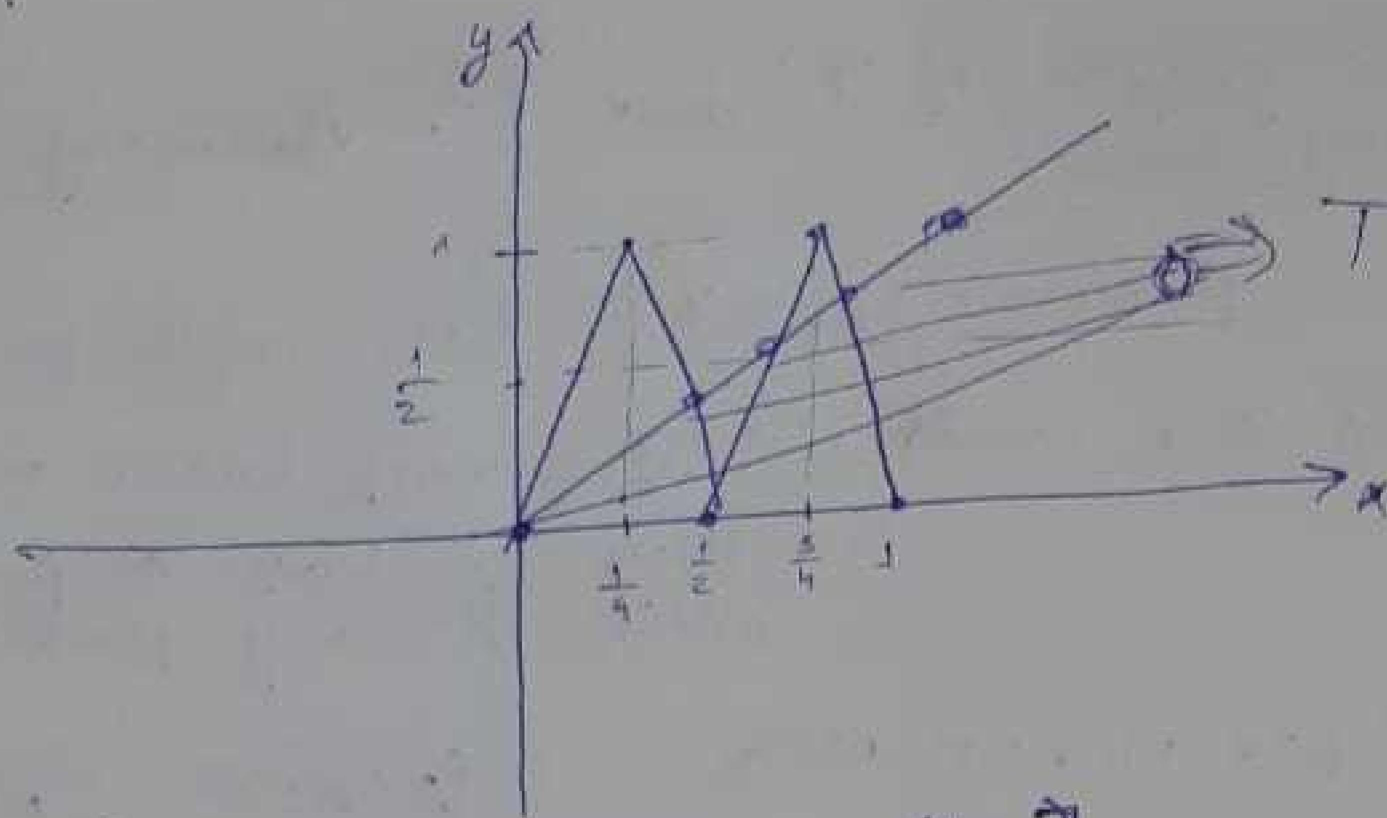
$n \ge 3$ $T(\frac{3}{2^n}) = \frac{3}{2^{n-1}}, \dots, \frac{3}{8}, \frac{3}{4}, \frac{1}{2}, 1, 0, \dots, 0.$

Conclusion: $\frac{3}{2^n}$, $(n \ge 2)$ is eventually the fixed point 0.

c). $T(x) = 0 \Leftrightarrow x \in \{0, 1\}$
$T(x) = 1 \Leftrightarrow x = \frac{1}{2}$
$T(x) = \frac{1}{2} \Leftrightarrow x \in \{\frac{1}{4}, \frac{3}{4}\}$
$T^2(x) = 0 \Leftrightarrow T(T(x)) = 0 \Leftrightarrow T(x) = \{0, 1\} \Leftrightarrow x \in \{0, \frac{1}{2}, 1\}$
$T^2(x) = 1 \Leftrightarrow T(T(x)) = 1 \Leftrightarrow T(x) = \frac{1}{2} \Leftrightarrow x \in \{\frac{1}{4}, \frac{3}{4}\}$
$T^2(x) = \frac{1}{2} \Leftrightarrow T(T(x)) = \frac{1}{2} \Leftrightarrow T(x) \in \{\frac{1}{4}, \frac{3}{4}\} \Leftrightarrow$
$\Leftrightarrow x \in \{\frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{5}{8}\}$

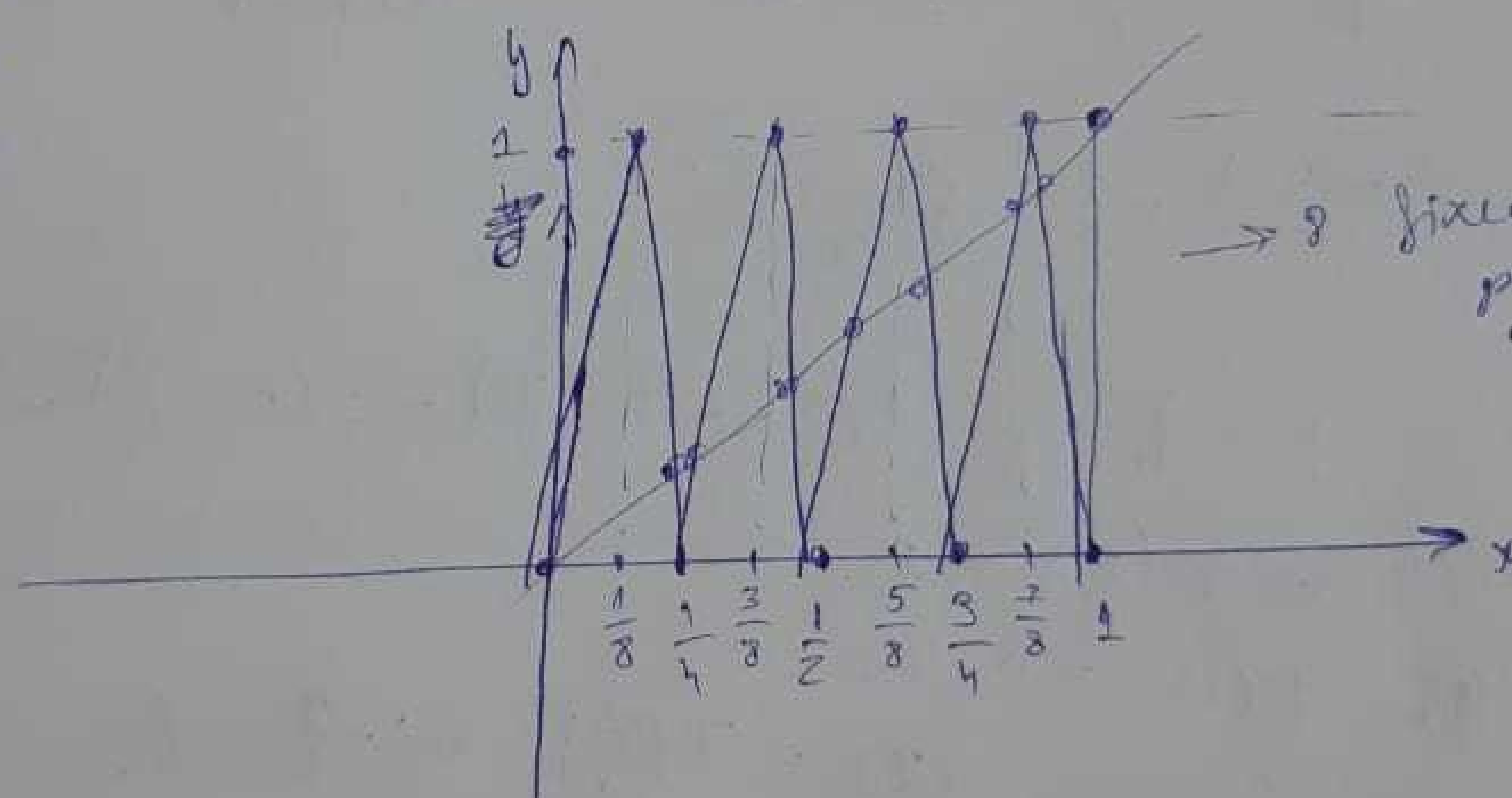
d). HW.

e).



$$T^2(x) = \begin{cases} 2(1-T(x)), & T(x) \in (\frac{1}{2}, 1] \\ 2(T(x)), & T(x) \in [0, \frac{1}{2}] \end{cases} =$$

$$= \begin{cases} 2(1-2+2x), & x \in (\frac{1}{2}, 1] \wedge T(x) \in (\frac{1}{2}, 1] \\ \vdots \end{cases}$$



$T^3(x) = 0 \Rightarrow T(T^2(x)) = 0 \Leftrightarrow T^2(x) = \{0, 1\}$

$\Leftrightarrow x \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$

f). T^2 has 4 fixed points, 2 of them are 0 and $\frac{2}{3}$, the fixed points of T. The other 2 fixed points of T^2 form a 2-periodic orbit of T. So, T has one 2-per. orbit.

T^3 has 8 fixed points. — II — ...

the other 6 fixed points of T^3 form two 3-periodic. So, T has two 3-per. orbits.

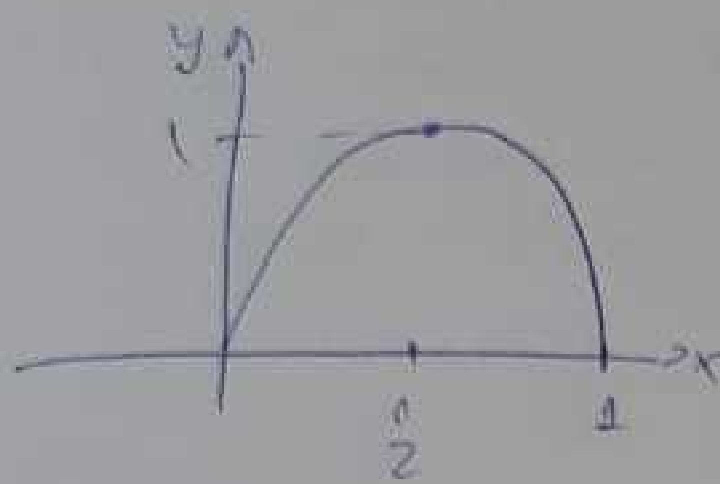
Remark:

Let $f : [0,1] \to \mathbb{R}$

$f(x) = 4x(1-x) = 4x - 4x^2$

$f'(x) = 4 - 8x = 4(1-2x)$

$f(\frac{1}{2}) = 1.$



The Linearization Method

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-map ($n \geq 1$, $n \in \mathbb{N}$) and $\eta^* \in \mathbb{R}^n$ be a fixed point of f.

If $|\lambda| < 1$ for any eigenvalue λ of $Jf(\eta^*)$ then η^* is an attractor.

If there exists an eigenvalue λ of $Jf(\eta^*)$ such that $|\lambda| > 1$ then η^* is unstable.

Newton's method

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^2-map and $V \subset \mathbb{R}^n$ be open, nonempty such that $\exists\ \eta^* \in V \cdot \{\eta^*\}$ with $g(\eta^*) = 0$, $g(\eta) \neq 0$, $\forall \eta \in V$ and $Jf(\eta)$ is invertible $\forall \eta \in V$

we consider the sequence $(X_k)_{k \geq 0}$ such that

$X_{k+1} = X_k - [Jg(X_k)]^{-1} \cdot g(X_k)$, $\forall k \geq 0$ with

$X_0 \in V$ fixed.

If V is sufficiantly small, then $\lim\limits_{k \to \infty} X_k = \eta^*$

Proof:

Let $f : V \to \mathbb{R}^2$ $\quad f(X) = X - [Jg(X)]^{-1} g(X)$

we have to prove that: η^* is the only fixed point of f.

η^* is an attractor for f.

$f(X) = X \to X - [Jg(X)]^{-1} g(X) = X \Leftrightarrow g(X) = 0.$

$Jf(\eta^*) = I_n - [Jg(\eta^*)]^{-1} \cdot Jg(\eta^*)$ – "deriv of Jg" $\cdot\, g(\eta^*) = 0.$

$\Rightarrow \mathcal{J}f(\eta^*) = O_n \Rightarrow \mathcal{J}f(\eta^*)$ has the eigenvalue 0 of multiplicity $n \xRightarrow{CH} \eta^*$ attractor on

1. We consider the ivP $x' = -10^3 x$, $x(0) = 1$.
a). find the solution and its limit ~~to infinity~~ as $t \to \infty$
b). write the Euler's ~~fo~~ numerical formula with constant stepsize h.
c) find the range of values for the stepsize h, s.t. the sol. ~~of the~~ $(x_k)_{k \geq 0}$ of ~~of~~ the difference equation found at b) satisfies $\lim_{k \to \infty} x_k = 0$.

---

a) $r = -10^3 \Rightarrow r = -1000 \Rightarrow x = c \cdot e^{-1000t}$, $c \in \mathbb{R}$

$x(0) = 1 \Leftrightarrow c = 1$.

$\Rightarrow x = e^{-1000t}$ → the unique sol. of the ivP.

$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} e^{-1000t} = \lim_{t \to \infty} \frac{1}{e^{1000t}} = 0$.

theory

b) $\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$

$x_0 \quad x_{k+1} = x_k + h, \quad k \geq 0$

$y_0 \quad y_{k+1} = y_k + h f(x_k, y_k), \quad k \geq 0$.

$x' = f(t, x) \quad f(t, x) = -10^3 \cdot x$

$t_0 = 0, \quad x_0 = y$

$t_{k+1} = t_k + h$

$x_{k+1} = x_k + h f(t_k, x_k)$

✓ $\boxed{x_{k+1} = x_k - h \cdot 10^3 \cdot x_k}$ → first ord LH dif. eq.

c). ivP $\begin{cases} x_{k+1} = x_k (1 - h \cdot 10^3) \\ x_0 = 1 \end{cases}$

$r = 1 - h \cdot 10^3 \Rightarrow x_k = c \cdot (1 - h \cdot 10^3)^k$

$x_0 = 1 \Rightarrow \boxed{c = 1}$

$x_k = (1 - h \cdot 10^3)^k$

$\lim_{k \to \infty} (1 - h \cdot 10^3)^k = 0 \Leftrightarrow |1 - h \cdot 10^3| < 1$

$\Rightarrow -1 < 1 - h \cdot 10^3 < 1 \quad | -1$

$-2 < -h \cdot 10^3 < 0 \quad | \cdot (-1)$

$2 > h \cdot 10^3 > 0 \quad | : 10^3$

$\frac{2}{10^3} > h > 0$

$h \in \left(0, \frac{2}{10^3}\right)$

② we consider

$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) = f(x,y) = -y+x-x^3-xy^2 \\ \dot{y} = x+y(1-x^2-y^2) = f_2(x,y) = x+y-x^2y-y^3 \end{cases}$$

a) Stability of $(0,0)$

b). Check that $\varphi(t,1,0) = (\cos t, \sin t)$, $\forall t \in \mathbb{R}$.

c). Pass to ~~pla~~ polar coordinates

d). Represent the phase portrait.

! in order to study stabil $\to$ use linearization method

a).

$$J_f(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 1-3x^2-y^2 & -1-2xy \\ 1-2xy & 1-x^2-3y^2 \end{pmatrix}$$

$$J_f(0,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \overset{\text{noted}}{=} B$$

$$\det(B-\lambda I_2) = 0 \iff \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \iff$$

$$\iff (1-\lambda)^2+1 = 0 \iff (1-\lambda)^2 = -1 \iff 1-\lambda = \pm 1$$

$$1-\lambda = 1 \qquad 1-\lambda_2 = -i$$

$$\boxed{\lambda_1 = 1-i} \qquad \boxed{\lambda_2 = 1+i}$$

$Re(\lambda_1) = Re(\lambda_2) = 1 \neq 0 \Rightarrow (0,0)$ is hyperbolic.

focus repeller $(0,0)$. — focus.

b). φ — usual not. for flow.

We have the IVP
$$\begin{cases} (\text{System}) \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

replace the val. of the x & y with

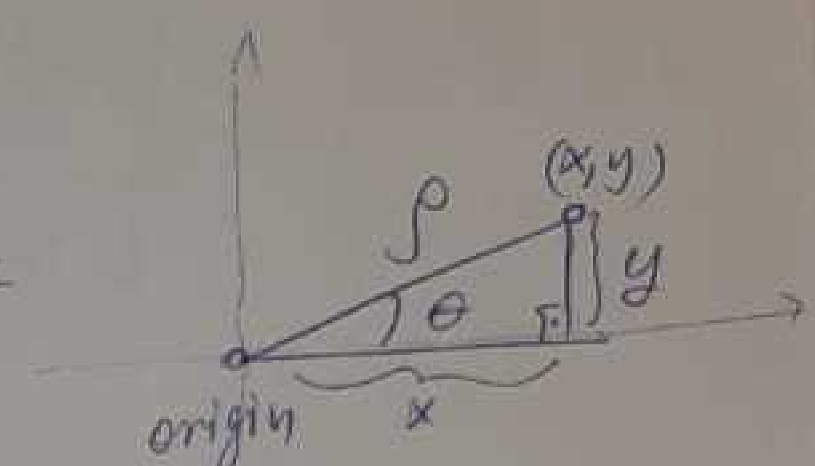
$\varphi(t,1,0) \to$ unique sol. of

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow \iff \begin{cases} -\sin t = -\sin t + \cos(\underbrace{1-\sin^2 t-\cos^2 t}_{0}) \\ \cos t = \cos t + \sin t\,(\underbrace{1-\sin^2 t - \cos^2 t}_{0}) \\ \cos(0) = 1 \quad \text{true} \\ \sin(0) = 0 \quad \text{true} \end{cases}$$

c). $(x,y) \to (\rho, \theta)$

theory.

$$\begin{cases}\rho = \sqrt{x^2+y^2} \\ \tan\theta = \frac{y}{x}\end{cases} \Leftrightarrow \begin{cases}\rho^2 = x^2+y^2 \\ \tan\theta = \frac{y}{x}\end{cases}$$



$$\begin{cases} x = \rho \cdot \cos\theta \\ y = \rho \cdot \sin\theta \end{cases}$$

derivin → obtain

$$\begin{cases} 2\cdot\rho\cdot\dot\rho = 2\cdot x\cdot\dot x + 2y\cdot\dot y \\ \frac{\dot\theta}{\cos^2\theta} = \frac{\dot y x - \dot x y}{x^2}\end{cases}$$

→ can be used for any system.

$$\Leftrightarrow \begin{cases} \rho\dot\rho = x(-y+x(1-x^2-y^2)) + y\cdot(x+y(1-x^2-y^2)) \\ \frac{\dot\theta}{\cos^2\theta} = \frac{x(x+y(1-x^2-y^2)) - y(-y+x(1-x^2-y^2))}{x^2}\end{cases}$$

$$\Leftrightarrow \begin{cases} \rho\dot\rho = (1-x^2-y^2)(x^2+y^2) \\ \frac{\dot\theta}{\cos^2\theta} = \frac{x^2+y^2}{x^2}\end{cases} \Leftrightarrow \begin{cases} \rho\dot\rho = (1-\rho^2)\cdot\rho^2 \\ \frac{\dot\theta}{\cos^2\theta} = \frac{\rho^2}{\rho^2\cdot\cos^2\theta}\end{cases} \Leftrightarrow$$

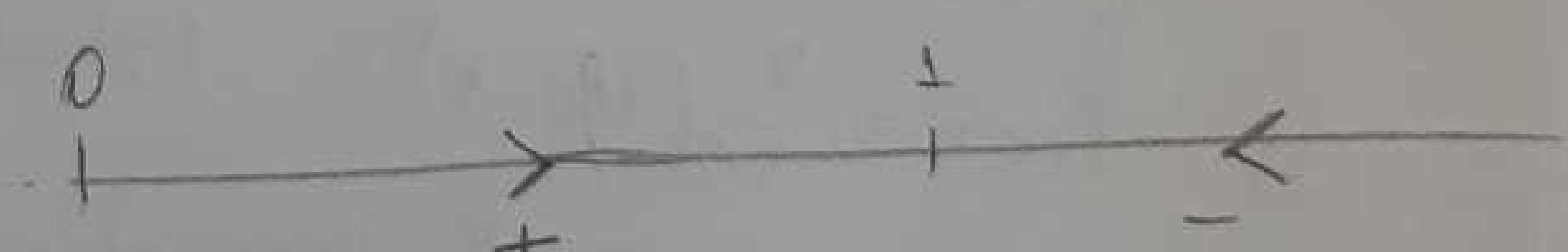
$$\Leftrightarrow \begin{cases} \dot\rho = \rho(1-\rho^2) \\ \dot\theta = 1 \end{cases}$$

→ interpret: $\Rightarrow \dot\theta = 1 > 0 \Rightarrow$ unghiul crește
increases with time
along the orbit

d). → $\dot\rho = \rho(1-\rho^2)$ — ecuație scalară neliniară

we consider $\rho > 0$

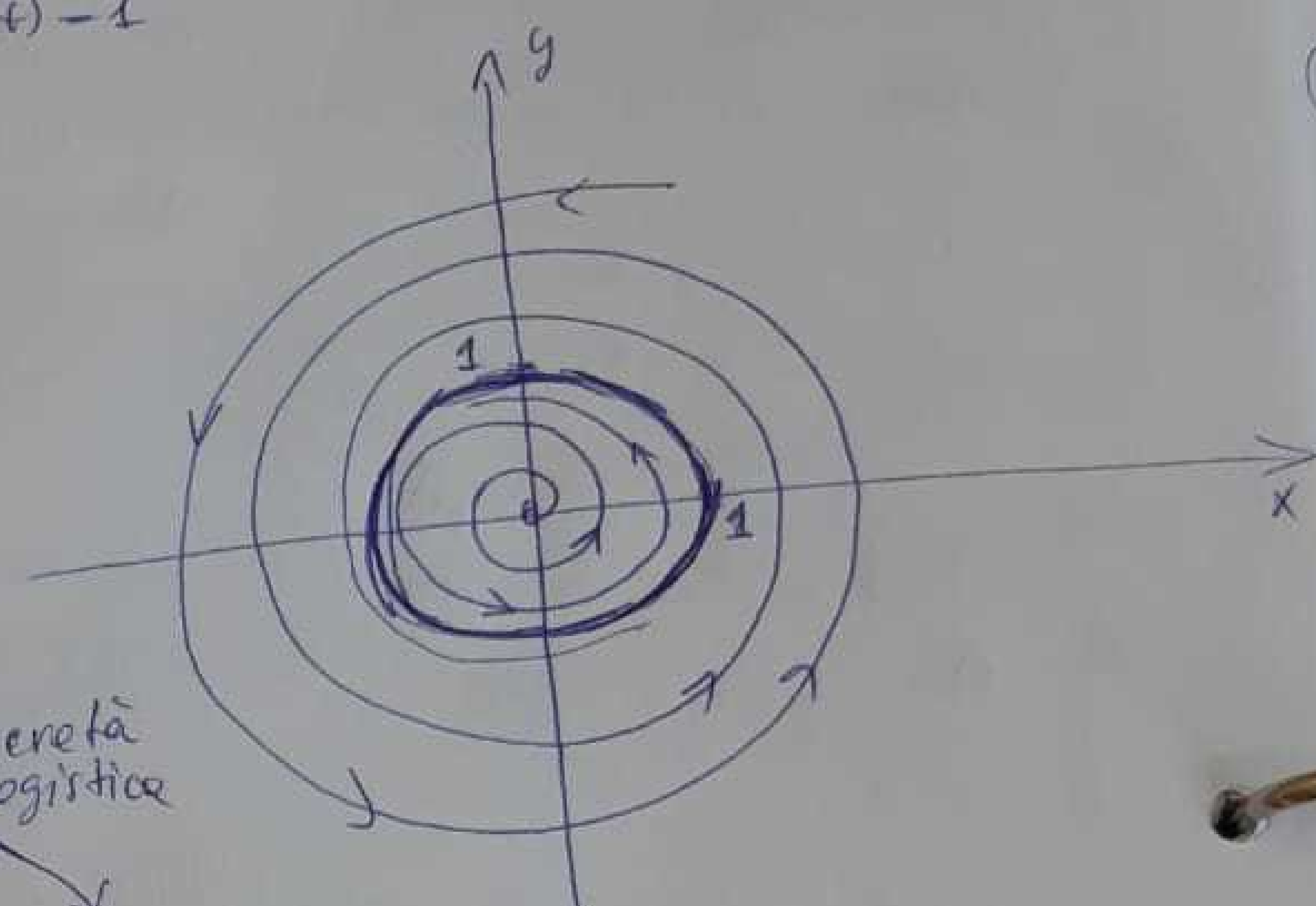
portret fazic al



if $\rho_0 = 1 \Rightarrow \rho_t = 1, \ \forall t \in \mathbb{R}$.

if $\rho_0 \in (0,1) \Rightarrow \begin{cases} \rho(t) \in (0,1) \ \forall t \in \mathbb{R}. \\ \lim_{t\to\infty} \rho(t) = 1 \\ \rho(t) \nearrow \end{cases}$

if $\rho_0 > 1 \to \begin{cases} \rho(t) > 1, \ \forall t \in \mathbb{R} \\ \lim_{t\to\infty} \rho(t) - 1 \\ \rho(t) \searrow \end{cases}$



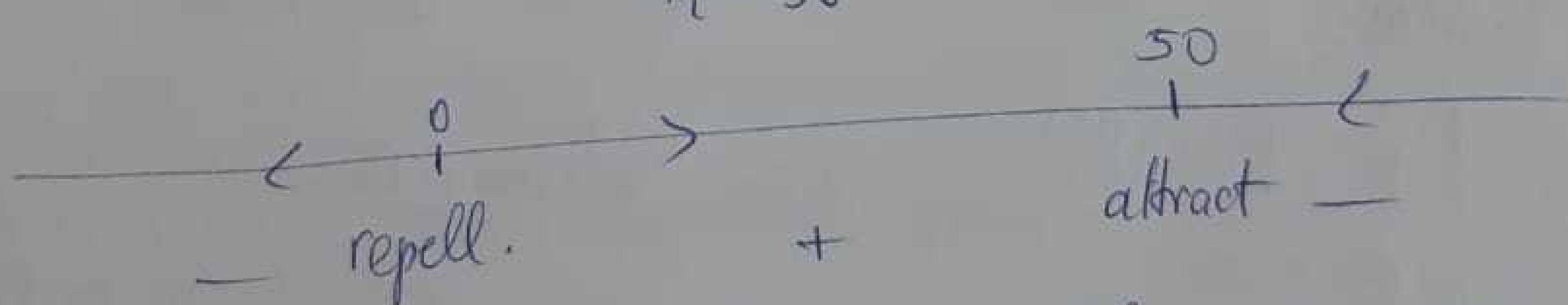
diff eg. logistică

eg. discretă logistica

③ (1) $\begin{cases} \dot{x} = x(x+50) \\ x(0) = \eta \end{cases}$ (2) $\begin{cases} x_{k+1} = \frac{x_k}{50}(100 - x_k) \\ x_0 = \eta \end{cases}$

when $\eta \in \{10, 80\}$

(1) $x(-x+50) = 0 \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 50 \end{array}$



for $\eta = 10 \Rightarrow$ the number increases, until reaches 50

$\eta = 80 \to$ decreases, until 50.

vert $\Rightarrow$ graph
ovit $\Rightarrow$ 1st bis.

(2) $f(x) = \frac{x}{50}(100 - x)$ (study fix points)

$f(x) = x \iff \frac{x}{50}(100 - x) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 50. \end{array}$

for $\eta = 10 \to$ same as (1)

$\eta = 80 \Rightarrow$ decrease in first months, then increases a little of time arround 50.