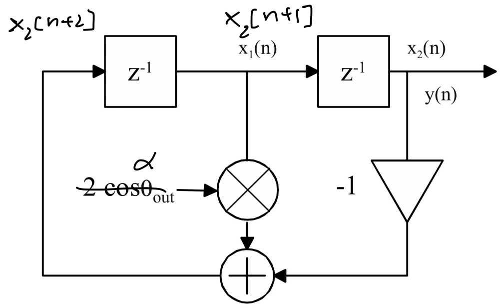
(5.1)

Initial condition: $x_2[n+2] = \alpha x_2[n+1] - x_2[n]$, $\quad x_1[n] = x_2[n+1] \rightarrow X_1[z] = zX_2[z]$

$$X_2[z] = z^{-1}X_1[z]$$



$$X_2(z) = \frac{(z^2 - \alpha z)x_2(0) + z x_1(0)}{z^2 - \alpha z + 1},$$

$$x_2[n+2] = \alpha x_2[n+1] - x_2[n] \quad — (1)$$

$$\Big\downarrow z$$

$$z^2\left(X_2[z] - \sum_{k=0}^{1} x_2[k]z^{-k}\right) = \alpha z\left(X_2[z] - x_2[0]\right) - X_2[z]$$

$$z^2\left(X_2[z] - x_2[0] - z^{-1}x_2[1]\right) = \alpha z\left(X_2[z] - x_2[0]\right) - X_2[z]$$

$$z^2X_2[z] - z^2x_2[0] - zx_2[1] = \alpha zX_2[z] - \alpha zX_2[0] - X_2[z]$$

$$z^2X_2[z] - \alpha zX_2[z] + X_2[z] = z^2x_2[0] - \alpha zx_2[0] + zx_2[1]$$

$$(z^2 - \alpha z + 1)X_2[z] = (z^2 - \alpha z)x_2[0] + zx_2[1]$$

$\boxed{\text{Proof. (5.3)}} \longrightarrow$ $$\therefore Y[z] = X_2[z] = \frac{(z^2 - \alpha z)x_2[0] + zx_1[0]}{z^2 - \alpha z + 1} \quad — (2)$$

if, $\alpha = 2\cos\theta_{out}$, $\quad \theta_{out} = \omega_{out}T = 2\pi f_{out}/f_s$

Post-process of inverse z (z^{-1}) transform.: $y[n] = x_2[n] = A\cos(n\theta_{out})$ for $n \geq 0$

$$\therefore y[n+k] = x_2[n+k] = A\cos(\theta_{out}(n+k))$$

substitute $x_2[n]$ and α in (1); $\quad A\cos(\theta_{out}(n+2)) = 2A\cos(\theta_{out}(n+1)) - A\cos(n\theta_{out})$

if $x_1[0] = A\cos\theta_{out}$, $\quad x_2(0) = A$ $\quad$ substitute into (2)

thus,
$$Y[z] = \frac{(z^2 - 2z\cos\theta_{out})A + zA\cos\theta_{out}}{z^2 - 2z\cos\theta_{out} + 1}$$

$$\therefore Y[z] = \frac{A(z^2 - z\cos\theta_{out})}{z^2 - 2z\cos\theta_{out} + 1}$$

$$= A\left[\frac{z^2 - z\cos\theta_{out}}{z^2 - 2z\cos\theta_{out} + 1}\right]$$

$$= A\left[\frac{1 - z^{-1}\cos\theta_{out}}{1 - 2z^{-1}\cos\theta_{out} + z^{-2}}\right]$$

$$= A\left[\frac{1 - z^{-1}\left[\frac{1}{2}\left(e^{j\theta_{out}} + e^{-j\theta_{out}}\right)\right]}{1 - 2z^{-1}\left[\frac{1}{2}\left(e^{j\theta_{out}} + e^{-j\theta_{out}}\right)\right] + z^{-2}}\right]$$

$$= \frac{A}{2}\left[\frac{2 - z^{-1}e^{j\theta_{out}} - z^{-1}e^{-j\theta_{out}}}{1 - z^{-1}e^{j\theta_{out}} - z^{-1}e^{-j\theta_{out}} + z^{-2}}\right]$$

$$Y[z] = \frac{A}{2}\left[\frac{2 - z^{-1}e^{j\theta_{out}} - z^{-1}e^{-j\theta_{out}}}{(1 - z^{-1}e^{j\theta_{out}})(1 - z^{-1}e^{-j\theta_{out}})}\right]$$

$$= \frac{A}{2}\left[\frac{1}{1 - z^{-1}e^{j\theta_{out}}} + \frac{1}{1 - z^{-1}e^{-j\theta_{out}}}\right]$$

if $\alpha = e^{j\theta_{out}}$ then $\dfrac{1}{1 - \alpha z^{-1}} \xrightarrow{z^{-1}} \alpha^n u(n)$

$$Y[z] \xrightarrow{z^{-1}} y(n) = \frac{A}{2}\left[(\alpha^n + \alpha^{-n})u(n)\right]$$

$$= A\left[\frac{1}{2}\left(e^{jn\theta_{out}} + e^{-jn\theta_{out}}\right)\right]u(n)$$

$(5.8) \rightarrow y(n) = A\cos(n\theta_{out})$, $\quad n \geq 0$

$\therefore y(0) = A$, $\quad y(1) = A\cos(\theta_{out})$

if $y(n)$ has phase offset : $y(n) = A\cos(n\theta_{out} + \varphi_0) = x_2(n)$

and $x_1(n) = x_2(n+1)$; $x_1(0) = A\cos(\theta_{out} + \varphi_0)$

$$x_2(0) = A\cos(\varphi_0)$$

from $x_2(n+2) = \alpha x_2(n+1) + x_2(n) = 2\cos(\theta_{out})\, x_2(n+1) + x_2(n)$

thus, $A\cos(\theta_{out}(n+2)) = 2A\cos(\theta_{out})\cos(\theta_{out}(n+1)) + A\cos(n\theta_{out}) \longleftarrow \boxed{\text{Proof. (5.12)}}$

$$\theta_{out} = 2\pi f_0 / f_s$$

$$\alpha = 2\cos\theta_{out} = 2 - 2^{-b}$$

$$\theta_{min} = \cos^{-1}\left[\underbrace{\tfrac{1}{2}\underbrace{(2 - 2^{-b})}_{=2,\ b\gg}}_{=1}\right] \longrightarrow f_{min} = \frac{\theta_{min}}{2\pi} f_s$$

if $n = n+2$; $y(n+4) = \alpha y(n+3) - y(n+2) + e_2(n+2)$, $\alpha = 2\cos\theta_{out}$

z - transform ; $z^4\left(Y(z) - z^{-3}y(3) - z^{-2}y(2) - z^{-1}y(1) - y(0)\right) = \alpha z^3\left(Y(z) - z^{-2}y(2) - z^{-1}y(1) - y(0)\right)$

$$- z^2\left(Y(z) - z^{-1}y(1) - y(0)\right) + z^2\left(E_2(z) - z^{-1}e_2(1) - e_1(0)\right)$$

$$(z^4 - \alpha z^3 + z^2)\,Y(z) = (z^4 - \alpha z^3)y(0) + (z^3 - \alpha z^2)y(1) + (z^2 - \alpha z)y(2) + z y(3) + z^2 E_2(z) - z e_2(1) - z^2 e_2(0)$$

$$Y(z) = \underbrace{\frac{(z^4 - \alpha z^3)y(0) + (z^3 - \alpha z^2)y(1) + (z^2 - \alpha z)y(2) + z y(3)}{z^4 - \alpha z^3 + z^2}}_{Y_{ideal}(z)} + \underbrace{\frac{z^2 E_2(z) - z e_2(1) - z^2 e_2(0)}{z^4 - \alpha z^3 + z^2}}_{Y_{err}(z)}$$

from. $Y_{err}(z) = Y(z) - Y_{ideal}(z)$

then. $Y_{err}(z) = \dfrac{z^2E_2(z) - z^2 e_2(0) - z e_2(1)}{z^2 - \alpha z + 1}\left(\dfrac{1}{z^2}\right) = \left[\dfrac{z^2E_2(z) - z^2 e_2(0) - z e_2(1)}{z^2 - \alpha z + 1}\right] z^{-2}$

$$Y'_{err}(z) \longrightarrow \boxed{z^{-1}} \longrightarrow \boxed{z^{-1}} \longrightarrow Y_{err}(z) = Y'_{err}(z)\, z^{-2}$$

$$\therefore\ Y'_{err}(z) = \frac{z^2E_2(z) - z^2 e_2(0) - z e_2(1)}{z^2 - \alpha z + 1} \longleftarrow \boxed{\text{Proof. (5.18)}}$$

from $\sin(\Omega_1 n)\,u[n] \underset{z^{-1}}{\overset{z}{\rightleftharpoons}} \dfrac{z^{-1}\sin\Omega_1}{1 - z^{-1}2\cos\Omega_1 + z^{-2}} = \dfrac{z\sin\Omega_1}{z^2 - 2z\cos\Omega_1 + 1}$, $|z| > 1$

where $e_2(0)$ and $e_2(1)$ assume to be zero

then $Y'_{err}(z) = \dfrac{z^2 E_2(z)}{z^2 - 2z\cos\theta_{out} + 1}$

$z^{-k}X(z) \xrightarrow{z^{-1}} x[n-k]$ (time - shifting)

$\boxed{\text{Proof. (5.19)}}$

$$Y'_{err}(z) = \frac{z}{\sin\theta_{out}} \cdot \underbrace{\frac{z\sin\theta_{out}}{z^2 - 2z\cos\theta_{out} + 1}}_{\text{multiplication} \xrightarrow{z^{-1}} \text{convolution}} \xrightarrow{z^{-1}} y'_{err}(n) = \frac{1}{\sin\theta_{out}}\sum_{\underbrace{k=2}}^{n} e_2(n)\sin(\theta_{out}(n-k+1)) ,\ n \geq 2$$

Exept $n = 0$ and $n = 1$
because $e_2(0)$ and $e_2(1)$ assume to be zero

Sirapop Saengthongkam 6301011611224

Chapter 7 : Source of Noise and Spurs in DDS

6 sources
- ตัด j bits เหลือ k bits (truncation) (e_p)
- การบีบอัด sine ใน LUT (e_{COM})
- finite precision of sine samples in LUT (e_A)
- D/A (e_{DA})
- post-filter (e_F)
- phase noise of clock (n_{clk})
- frequency error (Δf)
- frequency error (Δf) cause a frequency offset

7.1) Phase truncation

if Output (j bits) of Phase accumulator : $s(n) = \sin(2\pi \frac{\Delta P}{2^j} n)$

then truncate to k bits : $s(n) = \sin\left(\frac{2\pi}{2^k}\left[\frac{n\Delta P}{2^{j-k}}\right]\right)$, [ ] denotes truncation to integer: $n\Delta P \bmod 2^{j-k}$

$$= \sin\left(\frac{2\pi}{2^j}(n\Delta P - e_p(n))\right), \quad e_p(n) < 2^{j-k}$$

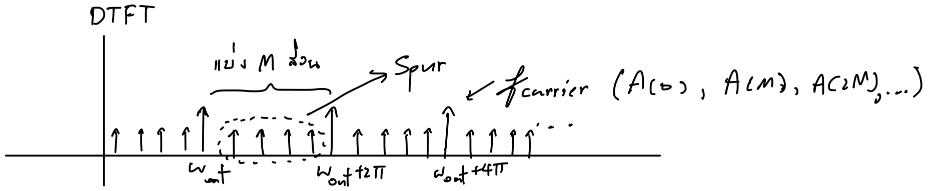
กำหนดให้ Phase increment คือ $\Delta P = W + L/M$ โดย L และ M ไม่มีตัวประกอบร่วม ดังนั้น $\frac{L}{M}$ ไม่ใช่จำนวนเต็ม

M คือจำนวนที่ทำให้ $M\Delta P = M(W + L/M)$ เป็นจำนวนเต็ม

เมื่อ digital spectrum ของ sample waveform คือ

$$G(w) = \frac{1}{T_s}\sum_{r=-\infty}^{\infty} \underset{\downarrow\ \text{DFT}}{A(r)}\, 2\pi\delta\left[w - w_{out} - r(2\pi/MT_s)\right]$$

พิจารณา $A(r) = \sum_{m=0}^{M-1}\left[\frac{1}{M}e^{-j2\pi t_m f_{out}/f_s}\right]e^{-jrm(2\pi/M)}$, $t = t_m T_s$ คือช่วงเวลาของ error



ส่วนเหลือจากการหาร mL ด้วย M

และ $2\pi t_m f_{out}/f_s = \frac{2\pi}{N}\frac{\langle mL\rangle_M}{M}$, $m = 0, 1, 2, \dots, M-1$

ถ้า $\langle mL\rangle_M = m$; $A(r, L, M, N) = \sum_{m=0}^{M-1}\left[\frac{1}{M}e^{-j2\pi m/MN}\right]e^{-jrm2\pi/M} \leftarrow \boxed{(7.15)}$

Sirapop Saengthongkam 6301011611224

from (7.18); $|A(0,L,M,N)|^2 = \left[\frac{\sin^2(\pi/N)}{(\pi/N)^2}\frac{(\pi/MN)^2}{\sin^2(\pi/MN)}\right]$

$$S/N = 10\log\left[\frac{|A(0,L,M,N)|^2}{1-|A(0,L,M,N)|^2}\right] \quad \text{—— } (7.18)$$

For S/N (max) given $M=2$ and fixed N ; $|A(0,L,2,N)|^2 = \frac{1}{4}\frac{\sin^2(\pi/N)}{\sin^2(\pi/2N)}$

from $\sin^2(2x) = 4\sin^2(x)\cos^2(x)$; $\sin^2(y) = 4\sin^2(\frac{y}{2})\cos^2(\frac{y}{2})$; $\sin^2(\frac{y}{2}) = \frac{\sin^2(y)}{4\cos^2(\frac{y}{2})}$, $y=2x$

let $x=\pi/N$; $\frac{\sin^2(x)}{\sin^2(\frac{x}{2})} = \frac{\sin^2(x)}{\frac{\sin^2(x)}{4\cos^2(\frac{x}{2})}} = 4\cos^2(\frac{x}{2})$

thus, $|A(0,L,2,N)|^2 = \frac{1}{4}\frac{\sin^2(x)}{\sin^2(x/2)} = \cos^2(\frac{x}{2})$

$$S/N(max) = 10\log\left[\frac{\cos^2(\frac{x}{2})}{1-\cos^2(\frac{x}{2})}\right]$$

from $\sin^2\theta + \cos^2\theta = 1$; $\sin^2\theta = 1-\cos^2\theta$

then, $S/N(max) = 10\log\left[\frac{\cos^2(\frac{x}{2})}{\sin^2(\frac{x}{2})}\right] = 20\log\left[\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})}\right] = 20\log\left[\cot(\frac{x}{2})\right]$

$\therefore$ $S/N(max) = 20\log[\cot(\pi/2N)] \longleftarrow \boxed{\text{Proof (7.19)}}$

from $|A(0,L,M,N)|^2 = \left[\frac{\sin^2(\pi/N)}{(\pi/N)^2}\frac{(\pi/MN)^2}{\sin^2(\pi/MN)}\right]$

$$|A(0,L,\infty,N)|^2 = \lim_{M\to\infty}\left[\frac{\sin^2(\pi/N)}{(\pi/N)^2}\frac{(\pi/MN)^2}{\sin^2(\pi/MN)}\right] = \frac{\sin^2(\pi/N)}{(\pi/N)^2}\left[\lim_{M\to\infty}\frac{(\pi/MN)^2}{\sin^2(\pi/MN)}\right]$$

$$\lim_{M\to\infty}\frac{(\pi/MN)^2}{\sin^2(\pi/MN)} = \lim_{M\to\infty}\left[-\frac{2}{M^3}\left(\frac{\pi}{N}\right)^2\cdot\frac{1}{-2(\sin(\pi/MN))(\cos(\pi/MN))(\pi/M^2N)}\right] \; : \text{ L'Hospital}$$

$$= \lim_{M\to\infty}\frac{\pi/MN}{\sin(2\pi/MN)}$$

$$= \lim_{M\to\infty}\frac{-\pi/M^2N}{\cos(2\pi/MN)(-2\pi/M^2N)}$$

$$= \lim_{M\to\infty}\frac{1}{\cos(2\pi/MN)}$$

$$= \frac{1}{\cos(0)} = 1 \qquad \boxed{\text{Proof. (7.20)}}$$

$\therefore$ $|A(0,L,\infty,N)|^2 = \frac{\sin^2(\pi/N)}{(\pi/N)^2}$ และจาก (7.18) จะได้ $S/N(min) = 10\log\left[\frac{[\sin(\pi/N)/(\pi/N)]^2}{1-[\sin(\pi/N)/(\pi/N)]^2}\right]$

Taylor's series of $\cot(x)$ is $\cot(x) \approx \underbrace{\frac{1}{x}}_{\text{first significant}} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \dots - \frac{2^{2n} B_n x^{2n-1}}{(2n)!} - \dots, \; 0 < |x| < \pi$

$$S/N_{(max)} = 20 \log[\cot(\pi/2N)] \approx 20 \log\left[\frac{N}{\pi/2}\right] \text{, first significant}$$

$$\approx 20 \log N - 20 \log(\pi/2)$$

$$\therefore S/N_{(max)} \approx 6.02k - 3.9224 \text{ dB.} \longleftarrow \boxed{\text{Proof. (7.21)}}$$

Taylor's series of $\sin(x)$ is $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \; \forall x$

$$\frac{[\sin(\pi/N)/(\pi/N)]^2}{1 - [\sin(\pi/N)/(\pi/N)]^2} \approx \frac{[1 - (\pi/N)^3/6(\pi/N)]^2}{1 - [1 - (\pi/N)^3/6(\pi/N)]^2}$$

$$\approx \frac{1 - \frac{1}{3}(\pi/N)^2 + (\pi/N)^4}{\frac{1}{3}(\pi/N)^2 - (\pi/N)^4}, \quad \frac{1}{3}(\pi/N)^2 + (\pi/N)^4 << 1, \; N > 1000$$

$$1 - \frac{1}{3}(\pi/N)^2 + (\pi/N)^4 \approx 1$$

$$\approx \frac{1}{\frac{1}{3}(\pi/N)^2} \approx \frac{N^2}{\frac{\pi^2}{3}}$$

$$S/N_{(min)} = 10 \log\left[\frac{[\sin(\pi/N)/(\pi/N)]^2}{1 - [\sin(\pi/N)/(\pi/N)]^2}\right] \approx 10 \log[N^2/(\pi^2/3)]$$

$$\therefore S/N_{(min)} \approx 20 \log N - 10 \log(\pi^2/3) \approx 6.02k - 5.1718 \text{ dB.} \longleftarrow \boxed{\text{Proof. (7.22)}}$$

Signal to Spurs ratio is $SP_{(r)} = 10 \log\left[\frac{|A(0,L,M,N)|}{|A(r,L,M,N)|}\right]$

from. $A(r,L,M,N) = \sum_{m=0}^{M-1} \left[\frac{1}{M} e^{-j2\pi m/MN}\right] e^{-jrm2\pi/M} = \sum_{m=0}^{M-1} \frac{1}{M} e^{-j\frac{2\pi m}{MN}(1+rN)} = \frac{1}{M} \underbrace{\left[\frac{1 - e^{-j\frac{2\pi}{MN}(1+rN)(M)}}{1 - e^{-j\frac{2\pi}{MN}(1+rN)}}\right]}_{\text{Geometric series}}$

$$= \frac{1}{M}\left[\frac{e^{j\frac{\pi}{N}(1+rN)} - e^{-j\frac{\pi}{N}(1+rN)}}{e^{j\frac{\pi}{MN}(1+rN)(M)} - e^{-j\frac{\pi}{MN}(1+rN)}}\right] \cdot \frac{e^{-j\frac{\pi}{N}(1+rN)}}{e^{-j\frac{\pi}{MN}(1+rN)}}$$

$$= \frac{1}{M}\left[\frac{\sin(\frac{\pi}{N}(1+rN))\, e^{-j\frac{\pi}{N}(1+rN)}}{\sin(\frac{\pi}{MN}(1+rN))\, e^{-j\frac{\pi}{MN}(1+rN)}}\right]$$

$$|A(r,L,M,N)|^2 = \frac{\sin^2[\frac{\pi}{N}(1+rN)]}{[\frac{\pi}{N}(1+rN)]^2} \frac{[\frac{\pi}{MN}(1+rN)]^2}{\sin^2[\frac{\pi}{MN}(1+rN)]} = \frac{\text{Sinc}(\frac{1}{N}(1+rN))^2}{\text{Sinc}(\frac{1}{MN}(1+rN))^2} = \frac{\text{Sinc}(r+1/N)^2}{\text{Sinc}(\frac{Nr}{NM} + \frac{1}{NM})^2}$$

$$|A(0,L,M,N)|^2 = \left[\frac{\sin^2(\pi/N)}{(\pi/N)^2}\,\frac{(\pi/MN)^2}{\sin^2(\pi/MN)}\right] = \frac{\mathrm{Sinc}(1/N)^2}{\mathrm{Sinc}(1/MN)^2}$$

$$\therefore\ SP_{(r)} = 10\log\left[\frac{\mathrm{Sinc}(1/N)^2\,\mathrm{Sinc}(Nr/NM + 1/NM)^2}{\mathrm{Sinc}(1/NM)^2\,\mathrm{Sinc}(r+1/N)}\right],\quad r = 1,2,\dots,M-1,\quad \mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$\boxed{\text{Proof. (7.24)}}$

$$SP_{(1)\,min} = 10\log\left[\frac{A(0,L,2,N)^2}{A(1,L,2,N)^2}\right],\quad A(r,L,M,N) = \frac{1}{M}\left[\frac{\sin(\frac{\pi}{N}(1+rN))\,e^{-j\frac{\pi}{N}(1+rN)}}{\sin(\frac{\pi}{MN}(1+rN))\,e^{-j\frac{\pi}{MN}(1+rN)}}\right]$$

$$= 10\log\left[\frac{\frac{1}{M}\,\frac{\sin(\frac{\pi}{N})\,e^{-j\frac{\pi}{N}}}{\sin(\frac{\pi}{2N})\,e^{-j\frac{\pi}{2N}}}}{\frac{1}{M}\,\frac{\sin(\frac{\pi}{N}(1+N))\,e^{-j\frac{\pi}{N}(1+N)}}{\sin(\frac{\pi}{2N}(1+N))\,e^{-j\frac{\pi}{2N}(1+N)}}}\right]^2$$

$$= 20\log\left[\frac{\sin(\frac{\pi}{N})\,\cancel{e^{-j\frac{\pi}{N}}}}{\sin(\frac{\pi}{2N})\,\cancel{e^{-j\frac{\pi}{2N}}}}\cdot\frac{\sin(\frac{\pi}{2N}(1+N))\,\cancel{e^{-j\frac{\pi}{2N}}}\,e^{-j\frac{\pi}{2}}}{\sin(\frac{\pi}{N}(1+N))\,\cancel{e^{-j\frac{\pi}{N}}}\,e^{-j\pi}}\right]$$

$$= 20\log\left[\frac{\sin(\frac{\pi}{N})\left[\sin(\frac{\pi}{2N})\cancel{\cos(\frac{\pi}{2})}^{0} + \cos(\frac{\pi}{2N})\cancel{\sin(\frac{\pi}{2})}^{1}\right]}{\sin(\frac{\pi}{2N})\left[\sin(\frac{\pi}{N})\cancel{\cos(\pi)}^{1e^{j\pi}} + \cos(\frac{\pi}{N})\cancel{\sin(\pi)}^{0}\right]}\cdot e^{j\frac{\pi}{2}}\right]$$

$$= 20\log\left[\frac{\cancel{\sin(\frac{\pi}{N})}\cos(\frac{\pi}{2N})}{j\sin(\frac{\pi}{2N})\cancel{\sin(\frac{\pi}{N})}}\right] + 20\log\left[e^{-j\frac{\pi}{2}}\right]$$

$$= 20\log\left[\frac{\cos(\frac{\pi}{2N})}{\sin(\frac{\pi}{2N})}\right]$$

$$\therefore\ SP_{(1)\,min} = 20\log\left[\cot\left(\frac{\pi}{2N}\right)\right] \leftarrow \boxed{\text{Proof. (7.27)}}$$

$$SP_{(1)_{max}} = 20\log\left[\frac{A(0,L,\infty,N)}{A(1,L,\infty,N)}\right],\quad A(r,L,M,N) = \frac{1}{M}\left[\frac{\sin(\frac{\pi}{N}(1+rN))\,e^{-j\frac{\pi}{N}(1+rN)}}{\sin(\frac{\pi}{MN}(1+rN))\,e^{-j\frac{\pi}{MN}(1+rN)}}\right]$$

$$= 20\log\left[\lim_{M\to\infty}\frac{\sin(\frac{\pi}{N})\,\cancel{e^{-j\frac{\pi}{N}}}}{\sin(\frac{\pi}{MN})\,\cancel{e^{-j\frac{\pi}{MN}}}}\cdot\frac{\sin(\frac{\pi}{MN}(1+N))\,\cancel{e^{-j\frac{\pi}{MN}}}\,e^{-j\frac{\pi}{M}}}{\sin(\frac{\pi}{N}(1+N))\,\cancel{e^{-j\frac{\pi}{N}}}\,e^{-j\pi}}\right]$$

$$= 20\log\left[\lim_{M\to\infty}\frac{\cancel{\sin(\frac{\pi}{N})}}{\sin(\frac{\pi}{MN})}\cdot\frac{\sin(\frac{\pi}{MN}(1+N))}{\cancel{\sin(\frac{\pi}{N})}\cancel{\cos(\pi)}^{1e^{j\pi}} + \cos(\frac{\pi}{N})\cancel{\sin(\pi)}^{0}}\,e^{j\frac{(M-1)}{M}\pi}\right]$$

$$= 20\log\left[\lim_{M\to\infty}\frac{\sin\left(\frac{\pi}{MN}(1+N)\right)}{\sin\left(\frac{\pi}{MN}\right)}\right] + 20\log\left[e^{-j\frac{\pi}{M}}\right]$$

$$= 20\log\left[\lim_{M\to\infty}\frac{\cos\left(\frac{\pi}{MN}(1+N)\right)\left(\cancel{-\frac{\pi}{M^2N}}(1+N)\right)}{\cos\left(\frac{\pi}{MN}\right)\left(\cancel{-\frac{\pi}{M^2N}}\right)}\right]$$

$$\therefore SP_{(1)_{max}} = 20\log\left[\frac{\cos(0)}{\cos(0)}(1+N)\right] = 20\log(1+N) \longleftarrow \boxed{\text{Proof. } (7.28)}$$

$$F(r) = \left[(1+rN)\cdot\frac{\Delta P}{GCD(\Delta P, 2^j)}\right] \bmod P_e$$

$$F(r) = P_e - F(r) \quad \text{when } F(r) > \frac{P_e}{2}$$

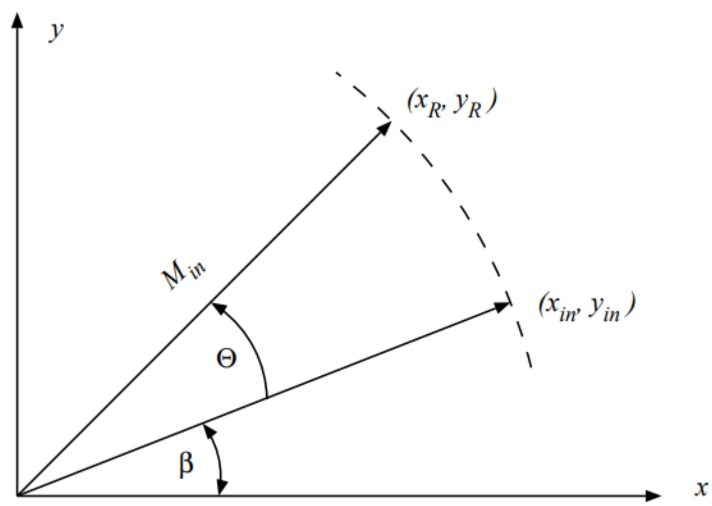
Coordinate Rotation Digit Computer (CORDIC) Algorithm

CORDIC Algorithm เป็นอัลกอลิทึมที่สามารถใช้คำนวณค่าของฟังก์ชันตรีโกณมิติ ไฮเพอร์โบลิก รากการคูณ การหาร เอกส์โพเนนเชียล และ ล็อกกาลิทึมด้วยฐานที่สามารถกำหนดเองได้ โดยการใช้เพียงแค่ การบวกลบ และการเลื่อนบิตเท่านั้น

CORDIC Microrotation Equations:

$$x[j+1] = x[j] - \sigma_j 2^{-j} y[j]$$
$$y[j+1] = y[j] + \sigma_j 2^{-j} x[j]$$
$$z[j+1] = z[j] - \sigma_j tan^{-1}(2^{-j})$$

Equations Proof



รูปที่ 1 Vector Rotation

กำหนดให้เวกเตอร์สองมิติเริ่มต้นมีค่า (X, Y) = (x_{in}, y_{in}) โดยทำมุมกับแกน X เริ่มต้นคือ β โมดูลัสของเวกเตอร์คือ M_{in} เมื่อหมุนเวกเตอร์ดังกล่าวไปด้วยมุมคือ θ ผลลัพธ์ที่ได้จากการหมุนคือ (X, Y) = (x_R, y_R) ซึ่งทำมุมกับแกน X คือ θ + β

จากรูปที่ 1 จะได้

$$\cos(\beta) = \frac{x_{in}}{M_{in}} \rightarrow x_{in} = M_{in}\cos(\beta)$$
$$\sin(\beta) = \frac{y_{in}}{M_{in}} \rightarrow y_{in} = M_{in}\sin(\beta)$$

$$x_R = M_{in}\cos(\theta + \beta) \tag{1}$$

$$y_R = M_{in}\sin(\theta + \beta) \tag{2}$$

จากสมการที่ (1) และ (2) เมื่อใช้เอกลักษณ์ตรีโกณ ดังนั้นจะได้

$$\begin{aligned} x_R &= M_{in}(\cos\theta\cos\beta - \sin\theta\sin\beta) \\ &= M_{in}\cos\theta\cos\beta - M_{in}\sin\theta\sin\beta \\ &= x_{in}\cos\theta - y_{in}\sin\theta \end{aligned}$$

$$\begin{aligned} y_R &= M_{in}(\sin\theta\cos\beta + \cos\theta\sin\beta) \\ &= M_{in}\sin\theta\cos\beta + M_{in}\cos\theta\sin\beta \\ &= x_{in}\sin\theta + y_{in}\cos\theta \end{aligned}$$

สามารถเขียนให้อยู่ในรูปแบบของเมตริกซ์ได้ ดังนี้

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} = ROT(\theta)\begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}$$

การหมุนไปของเวกเตอร์ดังกล่าว สามารถเขียนสมการของผลลัพธ์ได้ ดังนี้

$$x_R = M_{in}\cos(\beta + \theta) = x_{in}\cos\theta - y_{in}\sin\theta \tag{3}$$

$$y_R = M_{in}\sin(\beta + \theta) = x_{in}\sin\theta + y_{in}\cos\theta \tag{4}$$

จากนั้น แบ่ง θ ออกเป็นมุมย่อย ๆ j มุม โดยที่ j มีค่าตั้งแต่ 0 ถึง ∞ คือ α_j

จะได้

$$\theta = \sum_{j=0}^{\infty} \alpha_j$$
$$\boldsymbol{ROT}(\theta) = \prod_{j=0}^{\infty} \boldsymbol{ROT}(\alpha_j)$$

<u>พิสูจน์</u> กำหนดให้ j = 0, 1 จะได้

$$\prod_{j=0}^{1}\boldsymbol{ROT}(\alpha_j)=\boldsymbol{ROT}(\alpha_0)\boldsymbol{ROT}(\alpha_1)=\begin{bmatrix}\cos(\alpha_0) & -\sin(\alpha_0)\\ \sin(\alpha_0) & \cos(\alpha_0)\end{bmatrix}\begin{bmatrix}\cos(\alpha_1) & -\sin(\alpha_1)\\ \sin(\alpha_1) & \cos(\alpha_1)\end{bmatrix}$$

$$=\begin{bmatrix}\cos\alpha_0\cos\alpha_1-\sin\alpha_0\sin\alpha_1 & -(\sin\alpha_0\cos\alpha_1+\cos\alpha_0\sin\alpha_1)\\ \sin\alpha_0\cos\alpha_1+\cos\alpha_0\sin\alpha_1 & \cos\alpha_0\cos\alpha_1-\sin\alpha_0\sin\alpha_1\end{bmatrix}$$

จากเอกลักษณ์ตรีโกณมิติ จะได้

$$\prod_{j=0}^{1}\boldsymbol{ROT}(\alpha_j)=\boldsymbol{ROT}\left(\sum_{j=0}^{1}\alpha_j\right)=\begin{bmatrix}\cos(\alpha_0+\alpha_1) & -\sin(\alpha_0+\alpha_1)\\ \sin(\alpha_0+\alpha_1) & \cos(\alpha_0+\alpha_1)\end{bmatrix}$$

$$\prod_{j=0}^{\infty}\boldsymbol{ROT}(\alpha_j)=\boldsymbol{ROT}\left(\sum_{j=0}^{\infty}\alpha_j\right)=\begin{bmatrix}\cos\left(\sum_{j=0}^{\infty}\alpha_j\right) & -\sin\left(\sum_{j=0}^{\infty}\alpha_j\right)\\ \sin\left(\sum_{j=0}^{\infty}\alpha_j\right) & \cos\left(\sum_{j=0}^{\infty}\alpha_j\right)\end{bmatrix}$$

$$\therefore\ \boldsymbol{ROT}(\theta)=\prod_{j=0}^{\infty}\boldsymbol{ROT}(\alpha_j),\qquad \theta=\sum_{j=0}^{\infty}\alpha_j$$

ดังนั้นแล้ว จากสมการที่ (3) และ (4) เมื่อ θ ถูกแบ่งเป็นมุม α_j แล้ว จะได้

$$x_R[j+1]=x_R[j]\cos(\alpha_j)-y_R[j]\sin(\alpha_j)$$
$$y_R[j+1]=x_R[j]\sin(\alpha_j)+y_R[j]\cos(\alpha_j)$$

เมื่อต้องการหลีกเลี่ยงการคูณกัน

$$x_R[j+1]=\cos(\alpha_j)\left(x_R[j]-y_R[j]\tan(\alpha_j)\right)\tag{5}$$
$$y_R[j+1]=\cos(\alpha_j)\left(y_R[j]+x_R[j]\tan(\alpha_j)\right)\tag{6}$$

กำหนดให้ $\tan\alpha_j=\sigma_j\left(2^{-j}\right)\rightarrow\ \alpha_j=tan^{-1}\left(\sigma_j\left(2^{-j}\right)\right)=\sigma_j tan^{-1}\left(2^{-j}\right)$ เมื่อ $\sigma_j\epsilon\{-1,1\}$

จากสมการที่ (5) และ (6) จะได้

$$x_R[j+1] = \cos(\alpha_j)\,(x_R[j] - \sigma_j(2^{-j})y_R[j]) \tag{7}$$

$$y_R[j+1] = \cos(\alpha_j)\,(y_R[j] + \sigma_j(2^{-j})x_R[j]) \tag{8}$$

ดังนั้น

$$\begin{aligned}
\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \prod_{j=1}^{\infty} \boldsymbol{ROT}(\alpha_j) \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} &= \prod_{j=1}^{\infty} \begin{bmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \\
&= \prod_{j=1}^{\infty} \cos\alpha_j \begin{bmatrix} 1 & -\tan\alpha_j \\ \tan\alpha_j & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix} \\
&= \prod_{j=1}^{\infty} \cos\alpha_j \prod_{j=1}^{\infty} \begin{bmatrix} 1 & -\tan\alpha_j \\ \tan\alpha_j & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}
\end{aligned}$$

* เนื่องจาก $\tan\alpha_j = \frac{\sigma_j(2^{-j})}{1}$ ดังนั้น $\cos(\alpha_j) = \frac{1}{\sqrt{1+\sigma_j{}^2(2^{-2j})}} = \frac{1}{\sqrt{1+2^{-2j}}} = (1+2^{-2j})^{-1/2}$

จะได้

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \prod_{j=1}^{\infty} (1+2^{-2j})^{-1/2} \prod_{j=1}^{\infty} \begin{bmatrix} 1 & -\sigma_j(2^{-j}) \\ \sigma_j(2^{-j}) & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}$$

กำหนดให้ $K = \prod_{j=1}^{\infty}(1+2^{-2j})^{1/2}$ ซึ่งจะได้ $K \approx 1.6468$

ดังนั้น

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \frac{1}{K} \prod_{j=1}^{\infty} \begin{bmatrix} 1 & -\sigma_j(2^{-j}) \\ \sigma_j(2^{-j}) & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}$$

$$\begin{bmatrix} x_R \\ y_R \end{bmatrix} = \frac{1}{K} \prod_{j=1}^{\infty} \begin{bmatrix} 1 & -\sigma_j(2^{-j}) \\ \sigma_j(2^{-j}) & 1 \end{bmatrix} \begin{bmatrix} x_{in} \\ y_{in} \end{bmatrix}$$

กำหนดให้ z[j] คือมุมที่ยังเหลืออยู่ของการหมุน จะได้ว่า

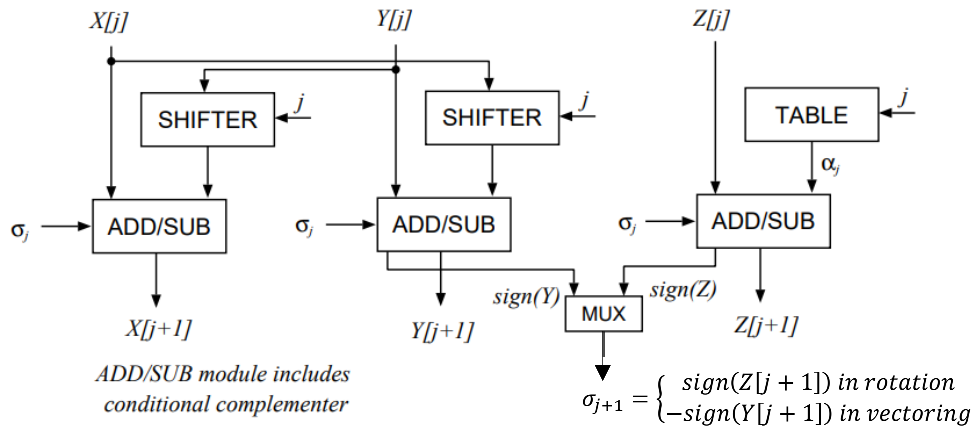
$$z[j+1] = z[j] - \sigma_j tan^{-1}(2^{-j})$$

สามารถเขียน linear combination ได้ดังนี้

$$\begin{aligned}
x[j+1] &= x[j] - \sigma_j 2^{-j} y[j] \\
y[j+1] &= y[j] + \sigma_j 2^{-j} x[j] \\
z[j+1] &= z[j] - \sigma_j tan^{-1}(2^{-j})
\end{aligned}$$

โดยที่เมื่อเป็น Rotation Mode; $\sigma_j = \begin{cases} 1 & ,z[j] \geq 0 \\ -1 & ,z[j] < 0 \end{cases}$, Vectoring Mode; $\sigma_j = \begin{cases} 1 & ,y[j] < 0 \\ -1 & ,y[j] \geq 0 \end{cases}$

CORDIC Block Diagram



รูปที่ 2 CORDiC Block Diagram

CORDIC Rotation Mode

Rotation Mode คือ การหมุนเวกเตอร์เริ่มต้น (x_{in}, y_{in}) ด้วยมุม θ

โดยที่กำหนดเงื่อนไขเริ่มต้น ดังนี้

$$z[0] = \theta \qquad x[0] = x_{in} \qquad y[0] = y_{in}$$

$$\sigma_j = \begin{cases} 1 & , z[j] \geq 0 \\ -1 & , z[j] < 0 \end{cases}$$

เมื่อทำ iteration ของ Microrotation แล้วจะได้ค่าสุดท้าย ดังนี้

$$\begin{aligned} x_f &= K(x_{in} \cos\theta - y_{in} \sin\theta) \\ y_f &= K(x_{in} \sin\theta + y_{in} \cos\theta) \\ z_f &= 0 \end{aligned}$$

หากต้องการหาค่าของ $\cos\theta$ และ $sin\,\theta$ สามารถคำนวณหาได้โดยกำหนดเงื่อนไขเริ่มต้น คือ

$$z[0] = \theta \qquad x[0] = \frac{1}{K} \qquad y[0] = 0$$

แล้วจะได้ค่าสุดท้าย ดังนี้

$$\begin{aligned} x_f &= \cos\theta \\ y_f &= \sin\theta \\ z_f &= 0 \end{aligned}$$

Homework 2: Rotation Mode

Rotate (x_{in}, y_{in}) by 67° using n = 12 Micro-Rotations

Initial Coordinates: x_{in} = 1, y_{in} = 0.125

MATLAB implementation

```
% ======================================================================== %
% This program was built by Sirapop Saengthongkam to study Cordic
% algorithm.
%
% This program can compute CORDIC algorithm with 2 Mode
% Mode = 0: Vectoring Mode --> Input: Xin, Yin, Zin = Angle
%                              / Output: Xf/K = M, Zf = arctan(Yin/Xin)
% Mode = 1: Rotation  Mode --> Input: Xin = 1/K, Yin = 0, Zin = Angle
%                              / Output: Xf = cos(Angle), Yf = sin(Angle)
% n is Iteration index if n is increase then the accuracy is increase.
% ======================================================================== %
clear; clc; close all;

% Constant
K     = 1.6468;             % K = sqrt(1+(2^-2n))

% Initial Conditions
Mode  = 1;                  % 0 is Vectoring Mode, 1 is Rotation Mode.
Xin   = 1;                  % Initial Coordinate-x
Yin   = 0.125;              % Initial Coordinate-y
Zin   = 67;                 % Initial Angle
n     = 12;                 % Iteration index

% Pre-Calculation
Theta = Zin * pi/180;
X     = zeros(n,1);
X(1)  = Xin;
Y     = zeros(n,1);
Y(1)  = Yin;
Z     = zeros(n,1);
Z(1)  = Theta;
sigma = zeros(n,1);
if (Mode)
    if (Z(1) < 0)
        sigma(1) = -1;
    else
        sigma(1) = 1;
    end
else
    if (Y(1) < 0)
        sigma(1) = 1;
    else
        sigma(1) = -1;
    end
```

```matlab
% CORDIC - Iteration
for j = 1:n
    [signX, X(j+1)] = ADD_SUB(X(j), SHIFTER(Y(j), j-1), sigma(j), 0);
    [signY, Y(j+1)] = ADD_SUB(Y(j), SHIFTER(X(j), j-1), sigma(j), 1);
    [signZ, Z(j+1)] = ADD_SUB(Z(j), arctanLUT(j-1),     sigma(j), 0);
    sigma(j+1) = MUX2to1(signY, signZ, Mode);
end

% Display Values
j = (0:1:n)';
if (Mode)
    T = table(j, Z, sigma, X, Y)
    if ((Xin == 1/K)&&(Yin == 0))
        fprintf("======================================================\n")
        fprintf("\t\t\tXf = cos(%.1f°) = %.4f\n\t\t\t" + ...
            "Yf = sin(%.1f°) = %.4f\n", Zin, X(n+1), Zin, Y(n+1))
        fprintf("======================================================\n")
    end
else
    T = table(j, Y, sigma, X, Z)
    if (Zin == 0)
        fprintf("======================================================\n")
        fprintf("\tXf = Modulus = %.4f\n\t" + ...
            "Zf = arctan(%.4f/%.4f) = %.4f = %.1f°\n", X(n+1)/K, Yin, ...
            Xin, Z(n+1), Z(n+1)*180/pi)
        fprintf("======================================================\n")
    end
end
```

Results:

j	z[j]	σ_j	x[j]	y[j]
0	1.1694	1	1.0000	0.1250
1	0.3840	1	0.8750	1.1250
2	-0.0797	-1	0.3125	1.5625
3	0.1653	1	0.7031	1.4844
4	0.0409	1	0.5176	1.5723
5	-0.0215	-1	0.4193	1.6046
6	0.0098	1	0.4695	1.5915
7	-0.0059	-1	0.4446	1.5988
8	0.0020	1	0.4571	1.5954
9	-0.0019	-1	0.4508	1.5972
10	0.0000	1	0.4540	1.5963
11	-0.0010	-1	0.4524	1.5967
12	-0.0005	-1	0.4532	1.5965

ผลลัพธ์หลังจาก normalize ด้วย K = 1.6468 ดังตารางด้านล่างนี้

j	z[j]	σ_j	x[j]	y[j]
0	1.1694	1	0.6072	0.0759
1	0.3840	1	0.5313	0.6831
2	-0.0797	-1	0.1898	0.9488
3	0.1653	1	0.4270	0.9014
4	0.0409	1	0.3143	0.9547
5	-0.0215	-1	0.2546	0.9744
6	0.0098	1	0.2851	0.9664
7	-0.0059	-1	0.2700	0.9709
8	0.0020	1	0.2776	0.9688
9	-0.0019	-1	0.2738	0.9699
10	0.0000	1	0.2757	0.9693
11	-0.0010	-1	0.2747	0.9696
12	-0.0005	-1	0.2752	0.9695

ดังนั้น จะได้ค่าสุดท้าย คือ

$$x_f = 0.2752$$
$$y_f = 0.9695$$
$$z_f = 0.00$$

CORDIC Vectoring Mode

Vectoing Mode คือ คือ การหมุนเวกเตอร์เริ่มต้น (x_{in}, y_{in}) จนกระทั่ง y = 0

โดยที่กำหนดเงื่อนไขเริ่มต้น ดังนี้

$$z[0] = z_{in} \quad x[0] = x_{in} \quad y[0] = y_{in}$$

$$\sigma_j = \begin{cases} 1 & , y[j] < 0 \\ -1 & , y[j] \geq 0 \end{cases}$$

ทำการบวกทบค่าของมุม z ไปจนกระทั่ง y = 0

เมื่อทำ iteration ของ Microrotation แล้วจะได้ค่าสุดท้าย ดังนี้

$$\begin{aligned} x_f &= K(x_{in}^2 + y_{in}^2)^{1/2} \\ y_f &= 0 \\ z_f &= z_{in} + tan^{-1}(\frac{y_{in}}{x_{in}}) \end{aligned}$$

Extra-Homework: Vectoring Mode

Initial Vector ($x_{in} = 0.75$, $y_{in} = 0.43$)

y forced to zero in $n = 12$ Micro-Rotations

MATLAB implementation

```
% ======================================================================= %
% This program was built by Sirapop Saengthongkam to study Cordic
% algorithm.
%
% This program can compute CORDIC algorithm with 2 Mode
% Mode = 0: Vectoring Mode --> Input: Xin, Yin, Zin = Angle
%                              / Output: Xf/K = M, Zf = arctan(Yin/Xin)
% Mode = 1: Rotation  Mode --> Input: Xin = 1/K, Yin = 0, Zin = Angle
%                              / Output: Xf = cos(Angle), Yf = sin(Angle)
% n is Iteration index if n is increase then the accuracy is increase.
% ======================================================================= %
clear; clc; close all;

% Constant
K     = 1.6468;             % K = sqrt(1+(2^-2n))

% Initial Conditions
Mode  = 0;                  % 0 is Vectoring Mode, 1 is Rotation Mode.
Xin   = 0.75;               % Initial Coordinate-x
Yin   = 0.43;               % Initial Coordinate-y
Zin   = 0;                  % Initial Angle
n     = 12;                 % Iteration index

% Pre-Calculation
Theta = Zin * pi/180;
X     = zeros(n,1);
X(1)  = Xin;
Y     = zeros(n,1);
Y(1)  = Yin;
Z     = zeros(n,1);
Z(1)  = Theta;
sigma = zeros(n,1);
if (Mode)
    if (Z(1) < 0)
        sigma(1) = -1;
    else
        sigma(1) = 1;
    end
else
    if (Y(1) < 0)
        sigma(1) = 1;
    else
        sigma(1) = -1;
    end
end
```

```
% CORDIC - Iteration
for j = 1:n
    [signX, X(j+1)] = ADD_SUB(X(j), SHIFTER(Y(j), j-1), sigma(j), 0);
    [signY, Y(j+1)] = ADD_SUB(Y(j), SHIFTER(X(j), j-1), sigma(j), 1);
    [signZ, Z(j+1)] = ADD_SUB(Z(j), arctanLUT(j-1),     sigma(j), 0);
    sigma(j+1) = MUX2to1(signY, signZ, Mode);
end

% Display Values
j = (0:1:n)';
if (Mode)
    T = table(j, Z, sigma, X, Y)
    if ((Xin == 1/K)&&(Yin == 0))
        fprintf("=======================================================\n")
        fprintf("\t\t\tXf = cos(%.1f°) = %.4f\n\t\t\t" + ...
            "Yf = sin(%.1f°) = %.4f\n", Zin, X(n+1), Zin, Y(n+1))
        fprintf("=======================================================\n")
    end
else
    T = table(j, Y, sigma, X, Z)
    if (Zin == 0)
        fprintf("=======================================================\n")
        fprintf("\tXf = Modulus = %.4f\n\t" + ...
            "Zf = arctan(%.4f/%.4f) = %.4f = %.1f°\n", X(n+1)/K, Yin, ...
            Xin, Z(n+1), Z(n+1)*180/pi)
        fprintf("=======================================================\n")
    end
end
```

Results:

j	y[j]	σj	x[j]	z[j]
0	0.4300	-1	0.7500	0.0000
1	-0.3200	1	1.1800	0.7854
2	0.2700	-1	1.3400	0.3218
3	-0.0650	1	1.4075	0.5667
4	0.1109	-1	1.4156	0.4424
5	0.0225	-1	1.4226	0.5048
6	-0.0220	1	1.4233	0.5360
7	0.0002	-1	1.4236	0.5204
8	-0.0109	1	1.4236	0.5282
9	-0.0053	1	1.4236	0.5243
10	-0.0025	1	1.4237	0.5224
11	-0.0011	1	1.4237	0.5214
12	-0.0005	1	1.4237	0.5209

ดังนั้น จะได้ค่าสุดท้าย คือ

$$x_f = 1.4237 \rightarrow Modulus = \frac{x_f}{K} = \sqrt{x_{in}^2 + y_{in}^2} = \frac{1.4237}{1.6468} = 0.8645$$

$$y_f = 0.00$$

$$z_f = tan^{-1}\left(\frac{y_{in}}{x_{in}}\right) = 0.5209 = 29.8°$$

Homework Week 3 (Program): Spurs location and Power Spectrum

Sirapop Saengthongkam 6301011611224

Discrete Fourier transform of the DDS output sequence

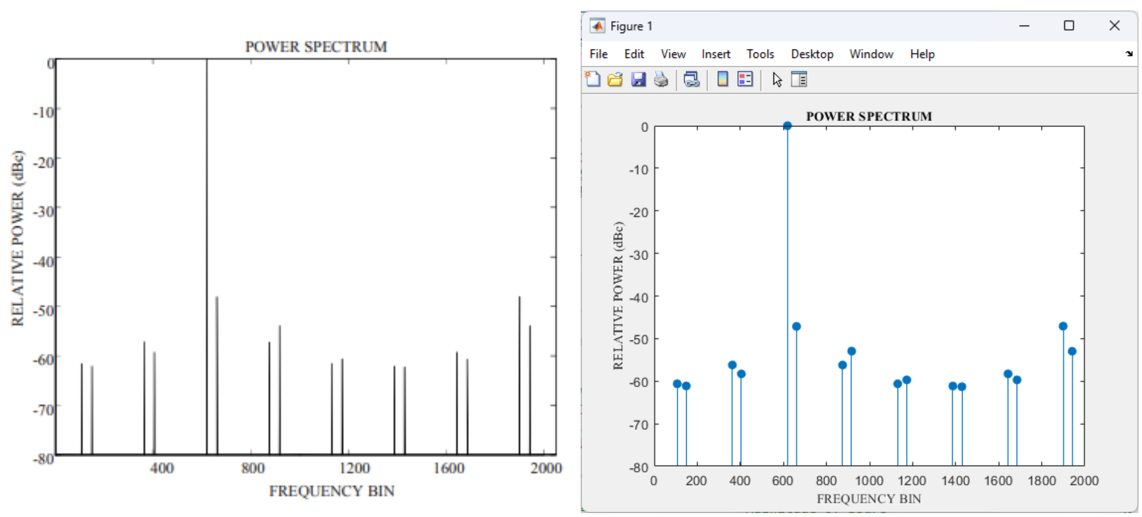
MATLAB implementation

```
% ======================================================================= %
% This program was built by Sirapop Saengthongkam to study
% 1. Behavior of Spur
% 2. Spur locations for quadrature DDS (Complex in-input to Discrete
% Fourier Transform (DFT)) [Bin]
% 3. Power Spectrum of Carrier-to-Spur Relative Power [dBc].
% ======================================================================= %
clear, clc, close all;

% ======================= Initial Conditions ========================== %
j = 12;
k = 8;
dP = 619;
Pe = 4096;
N = 2^k;
M = (2^(j-k))/gcd(dP, 2^(j-k));
Y = M-1;
% ======================================================================= %

% ==================== Position of Spurs in bin ======================== %
fprintf("Number of Spurs are %d\n", Y);
r = [0:1:Y];
Fr = mod((dP/gcd(dP, 2^j)) + r*(N*dP/gcd(dP, 2^j)), Pe);

% Fourier Series De-Aliasing of spurs position
for n = 1:M
    if (Fr(n) > (Pe/2))
        Fr_new = Pe - Fr(n);
        Fr(n) = Fr_new;
    else
        Fr(n) = Fr(n);
    end
end

% Show the Fourier bin position of Carrier and Spurs
fprintf("Carrier bin[Fr(0)] = %d \n", Fr(1));
for i = 2:M
    fprintf("Spur#%d bin[Fr(%d)] = %d \n", i-1, i-1, Fr(i));
end
% ======================================================================= %

% ====================== Magnitude of Spurs =========================== %
SP = [1:M];
SP(1) = (sin(pi/N)^2)*((pi/(M*N))^2)/(((pi/N)^2)*(sin(pi/(M*N))^2));
for r = 1:Y
    SP(r+1) = 10*log10((sinc(1/N)^2 * sinc(N*r/(N*M) + 1/(N*M))^2) / ...
        (sinc(1/(N*M))^2 * sinc(r + 1/N)^2));
end
% ======================================================================= %
```

```
% =================== Relative Power to Carrier ======================= %
for r = 1:M
    Crr2Spr_dB(r) = SP(1) - SP(r);
end
% ==================================================================== %

% =========================== Plotting ============================== %
Plot_data = stem(Fr, Crr2Spr_dB, "filled");
Plot_data.BaseValue = -80;
ylim([-80, 0])
title("POWER SPECTRUM", 'fontsize', 10, 'fontname', 'Times New Roman')
xlabel("FREQUENCY BIN", 'fontsize', 10, 'fontname', 'Times New Roman')
ylabel("RELATIVE POWER (dBc)", 'fontsize', 10, 'fontname', ...
    'Times New Roman')
% ==================================================================== %
```

Results #1:

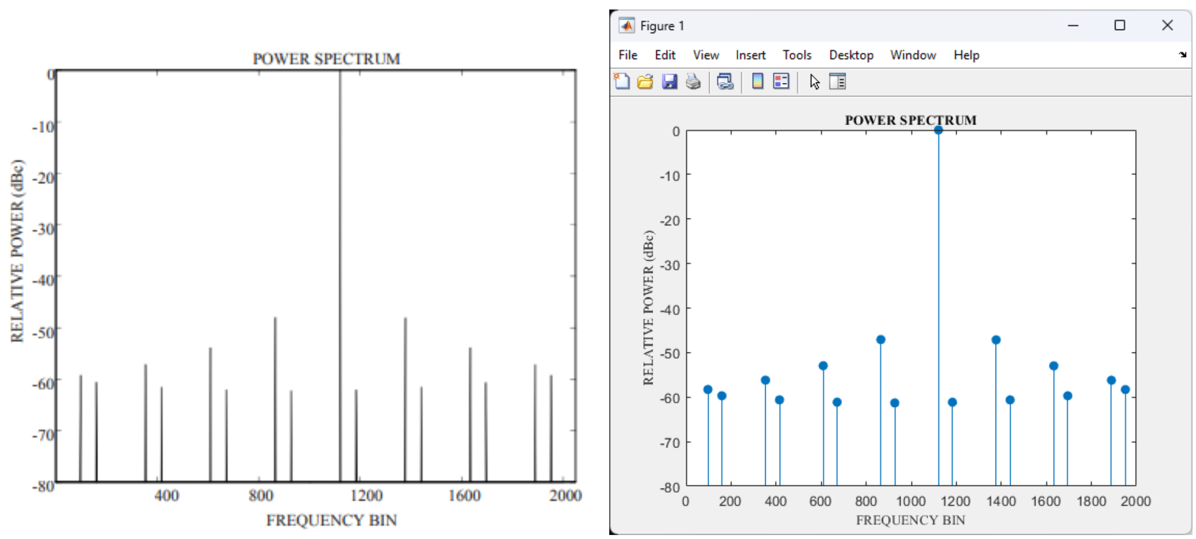
```
Number of Spurs are 15
Carrier bin[Fr(0)] = 619
Spur#1 bin[Fr(1)] = 661
Spur#2 bin[Fr(2)] = 1941
Spur#3 bin[Fr(3)] = 875
Spur#4 bin[Fr(4)] = 405
Spur#5 bin[Fr(5)] = 1685
Spur#6 bin[Fr(6)] = 1131
Spur#7 bin[Fr(7)] = 149
Spur#8 bin[Fr(8)] = 1429
Spur#9 bin[Fr(9)] = 1387
Spur#10 bin[Fr(10)] = 107
Spur#11 bin[Fr(11)] = 1173
Spur#12 bin[Fr(12)] = 1643
Spur#13 bin[Fr(13)] = 363
Spur#14 bin[Fr(14)] = 917
Spur#15 bin[Fr(15)] = 1899
```



รูปที่ 1 Discrete Fourier transform of the DDS output sequence for j = 12, k = 8 and ΔP = 619.

Results #2:

```
Number of Spurs are 15
Carrier bin[Fr(0)] = 1121
Spur#1 bin[Fr(1)] = 1377
Spur#2 bin[Fr(2)] = 1633
Spur#3 bin[Fr(3)] = 1889
Spur#4 bin[Fr(4)] = 1951
Spur#5 bin[Fr(5)] = 1695
Spur#6 bin[Fr(6)] = 1439
Spur#7 bin[Fr(7)] = 1183
Spur#8 bin[Fr(8)] = 927
Spur#9 bin[Fr(9)] = 671
Spur#10 bin[Fr(10)] = 415
Spur#11 bin[Fr(11)] = 159
Spur#12 bin[Fr(12)] = 97
Spur#13 bin[Fr(13)] = 353
Spur#14 bin[Fr(14)] = 609
Spur#15 bin[Fr(15)] = 865
```



รูปที่ 2 Discrete Fourier transform of the DDS output sequence for j = 12, k = 8 and ΔP = 1121.