

# Poisson Regression

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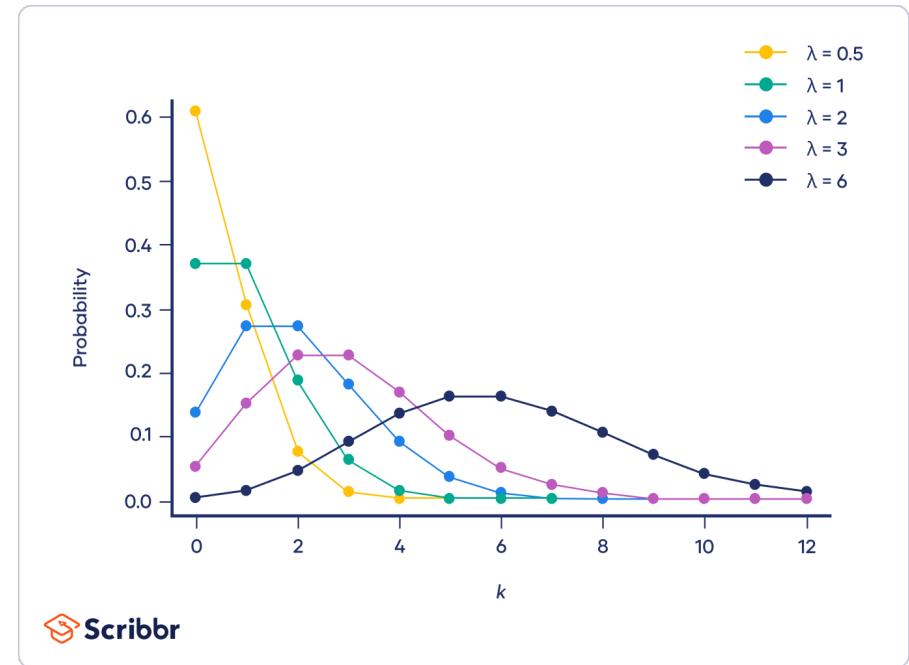
# Poisson Model

- Count response variable: occurrence count recorded for a particular measurement window (length of time, but it can also be a distance, area, etc). Ex: number of calls to a customer service in an hour or number of flaws in a tabletop of a certain area.

- **Poisson distribution:** its minimum value is zero and, in theory, the maximum is unbounded. It models the probability of 0, 1, 2, . . . Events. **The mean of the distribution is equal to the variance.**

$$P(Y = k) = \frac{e^{-\lambda} \lambda^k}{k!} \quad K=0,1,2,\dots$$

$\lambda$  = mean and variance of  $Y$



We'd like to model our main parameter  $\lambda$ , the average number of occurrences per unit of time or space, as a function of one or more covariates.

For the random component, we assume that the response  $Y$  has a Poisson distribution:  $Y_i \sim \text{Poisson}(\lambda_i)$  where the expected count of  $Y_i$  is  $E(Y_i) = \lambda_i$ . The link function is usually the (natural) log, but sometimes the identity function may be used. The systematic component consists of a linear combination of explanatory variables, identical to that for logistic regression. Thus, in the case of a single explanatory, the model is:

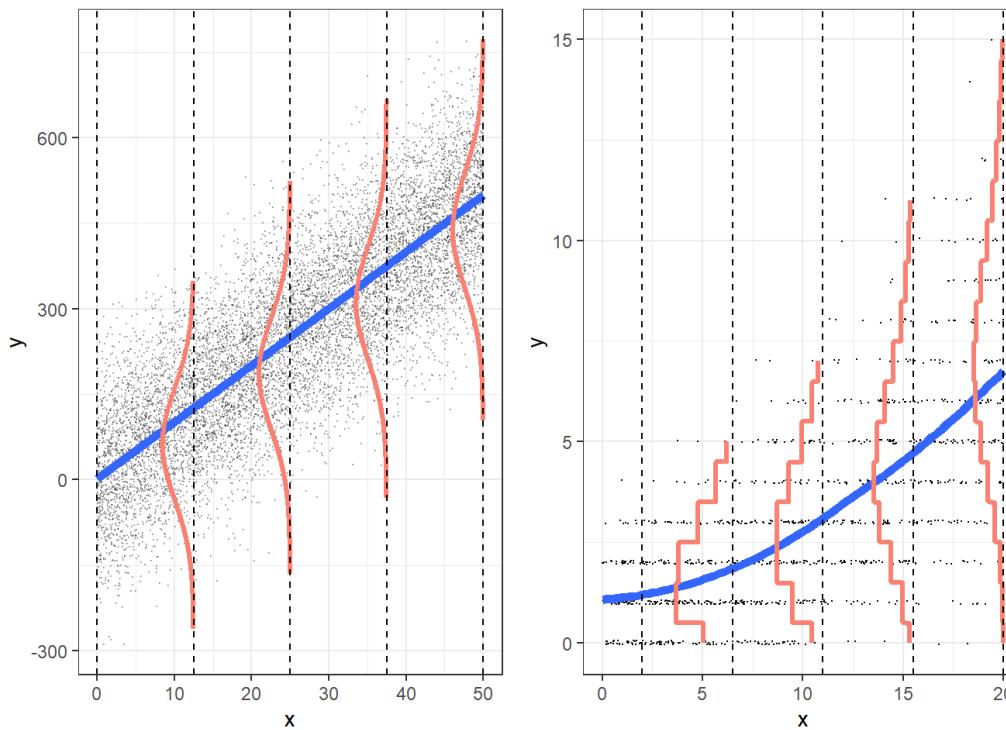
$$\log(\lambda) = \alpha + \beta x$$

This is equivalent to:

$$\lambda = \exp(\alpha + \beta x) = \exp(\alpha) \exp(\beta x)$$

Interpretations of these parameters are similar to those for logistic regression:  $\exp(\alpha)$  is the effect on the mean of  $Y$  when  $x=0$ , and  $\exp(\beta)$  is the multiplicative effect on the mean of  $Y$  for each 1-unit increase in  $x$ .

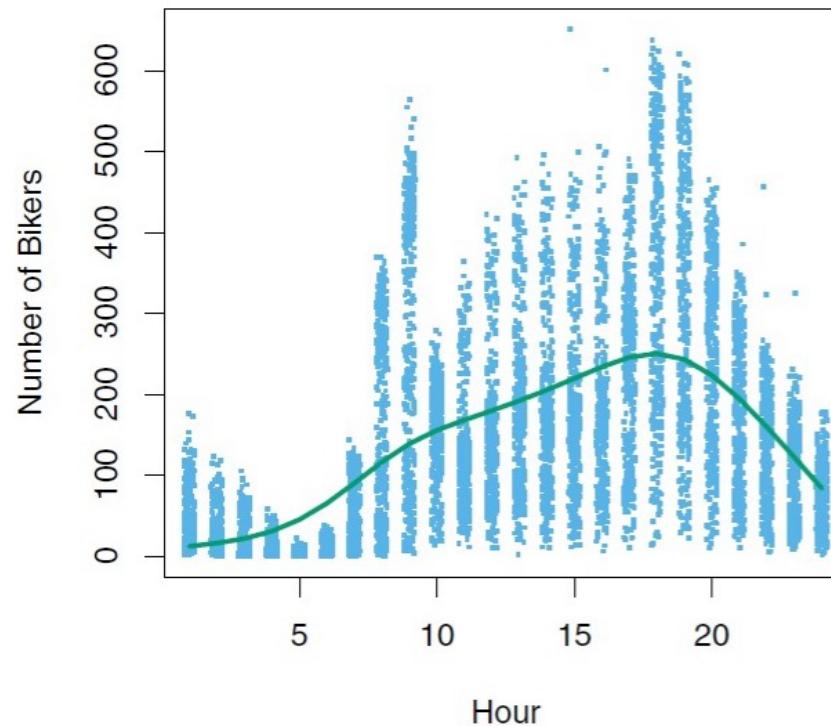
## Regression models: Linear regression (left) and Poisson regression (right)



Linear regression: for each level of  $X$ , the responses are approximately normal. Poisson regression: for each level of  $X$ , the responses follow a Poisson distribution. For Poisson regression, small values of  $\lambda$  are associated with a distribution that is noticeably skewed with lots of small values and only a few larger ones. As  $\lambda$  increases the distribution of the responses begins to look more and more like a normal distribution. For Poisson regression the responses at each level of  $X$  become more variable with increasing means

Number of bikers per hour. The variance of biker numbers changes as the mean number changes:

- during worse conditions, there are few bikers, and little variation in the number of bikers
- during better conditions, there are many bikers on average, but also larger variation in the number of bikers



If the observations recorded correspond to different measurement windows, a scale adjustment has to be made to put them on equal terms, and we model the rate or count per measurement unit.

Example: Gardner, Mulvey, & Shaw (1995), Psychological Bulletin, 118.

$Y$  = Number of violent incidents exhibited over a 6 month period by patients who had been treated in the ER of a psychiatric hospital.

During the 6 months period of the study, the individuals were primarily residing in the community. The number of violent acts depends on the opportunity to commit them; that is, the number of days out of the 6 month period in which a patient is in the community (as opposed to being locked up in a jail or hospital).

$Y$  = count (number violent acts).

$t$  = index of the time or space (days in the community).

The sample **rate** of occurrence is  $Y/t$ .

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The sample **rate** of occurrence is  $Y/t$ .

The expected value of the rate is:

$$E(Y/t) = \frac{1}{t}E(Y) = \lambda/t$$

Therefore the Poisson log-linear regression model for the expected rate:

$$\log(\lambda) = \alpha + \beta x + \log(t)$$

The term “**- log(t)**” is an adjustment term and each individual may have a different value of  $t$ . **- log(t)** is referred to as an “**offset**”.

$$\log(\lambda/t) = \alpha + \beta x \quad \text{The expected value of counts depends on both } t \text{ and } x$$

$$\lambda = te^\alpha e^{\beta x}$$

If the variance doesn't behave as the mean structure suggests for that distribution, then we have a problem with **overdispersion** (or occasionally, underdispersion).

Situations of overdispersion:

- ✓ Dispersion up to a constant term: A simple situation when the true variance is proportional to what the model predicts. For Poisson model:  $\text{var}(Y)=\theta\lambda$ ,  $\theta>1$
- ✓ Structural zeros: some subjects are not subject to any risk. E.g., number of cigarettes smoked per day among the non-smokers. The probability of having outcomes greater than 0 for non-smokers is 0, but it could follow a Poisson for smokers.

All the above situations will lead to a problem of over-dispersion.

# Issues related to the Poisson regression

## B) Quasi Maximum Likelihood estimation

"quasi-maximum likelihood" (quasi-ML) is a method used to estimate parameters in situations where the likelihood function is not strictly adhering to the assumptions of standard maximum likelihood estimation.

If you face **overdispersion** then quasi-ML can be used to account for this by using a different likelihood function or incorporating extra parameters into the model.

Consequence of overdispersion:

- Standard errors will be underestimated
- Reject H<sub>0</sub> when we shouldn't

If zero mass and overdispersion (for the non-zero observations): ZINB might help

# Issues related to the Poisson regression

## A) Issues with count data

- A1) Zero Inflated Distribution: distribution of the Yis characterized by a peak on the 0  
→ Zero Inflated Poisson (ZIP): mixture distribution that separately models the zeros.

Probability Mass Function of the Zero-Inflated Poisson regression model

When  $y_i = 0$ :

$$P(y_i = 0) = \phi_i + (1 - \phi_i) * e^{-\lambda_i}$$

When  $y_i = 1, 2, 3, \dots$ :

$$P(y_i = k) = (1 - \phi_i) \frac{e^{-\lambda_i} * \lambda_i^k}{k!}$$

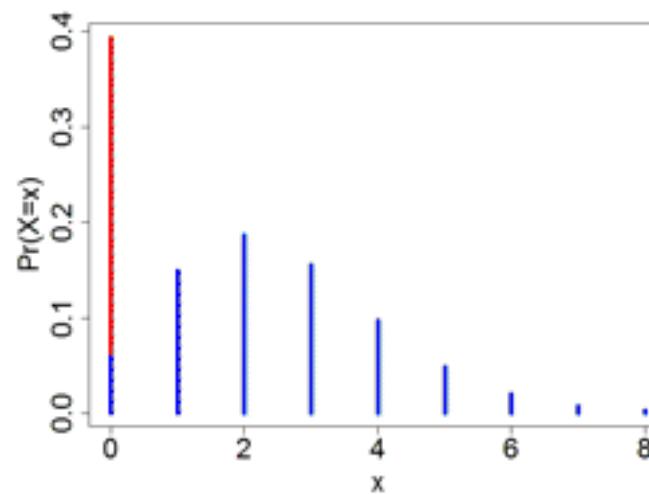
Where  $\lambda_i = e^{x_i \beta}$

$\phi_i$ = proportion of zeros for the  $i^{\text{th}}$  row in the data set

$x_i$ =Regression variables in the  $i^{\text{th}}$  row

$\beta$ =Vector of regression coefficients

Two groups of people: always zero and not always zero. Mixture of probability from the two groups



Alternative models:

Zero-inflated negative binomial (ZINB)

Hurdle models (two-parts)

## Issues related to the Poisson regression

Truncated Poisson: The experimental designs suggest that there cannot be 0s, by construction.

→ Work with a slightly different probability function and model  $P(Y_i=y | Y_i > 0)$

