Fourier Series

2 - Fourier series

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EPITA

November 4, 2021

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More precisely, we try, starting from a periodic function f, to build an approximation of f under the form of a sequence of trigonometric polynomials $\left(x \longmapsto \sum_{k=0}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)\right)$ or, when working with complex numbers, $\left(x \longmapsto \sum_{k=0}^{n} c_k e^{ikx}\right)$.

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One of the upsides of this method is that it is not really an iterative method: we can actually calculate all the projections separately. This gives a way simpler approach and a lesser complexity of calculations.

Remark: we chose here to use functions with a basic period of 2π. The $\cos(kx)$ and $\sin(kx)$ functions have a frequency of $\frac{k}{2\pi}$ which is a multiple of the basic frequency $\frac{1}{2\pi}$. In order to transpose this study to functions with a different period, a simple substitution will do.

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Indeed, if f is of period T, then $x \longmapsto f\left(\frac{2\pi}{T}x\right)$ has a period of 2π.

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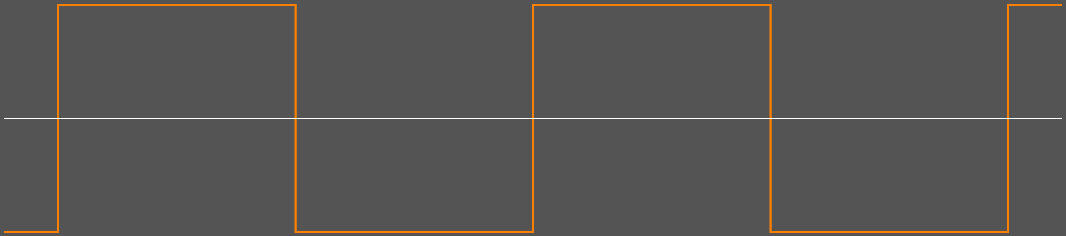

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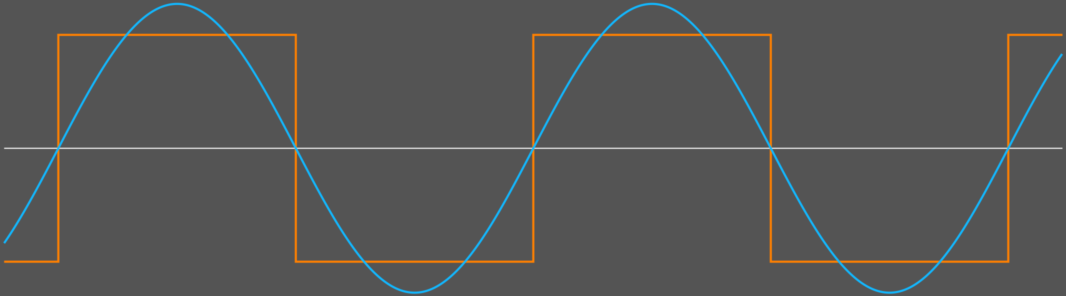

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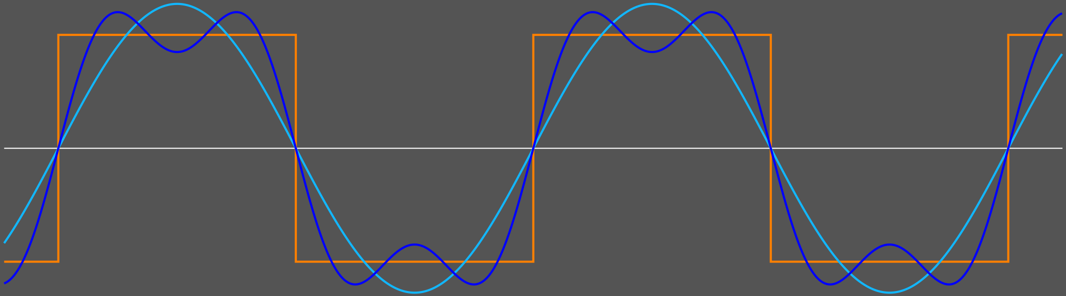

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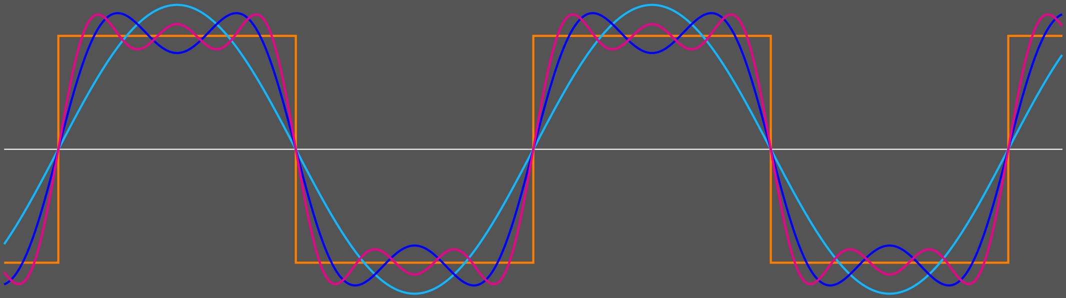

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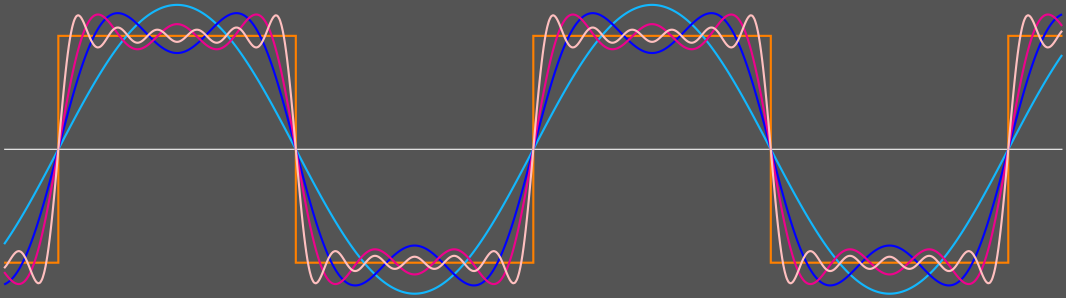

Summary

Orthogonality of trigonometric functions

Fourier coefficients and Fourier series

Dirichlet's convergence theorem

Parseval's identity

Orthogonality of trigonometric functions

We will show here that the trigonometric functions ($\sin(kx)$ and $\cos(nx)$ for real numbers, e^{ikx} for complex numbers) are orthogonal with respect to the standard inner product.

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Since we chose to work with 2π-periodic functions, we can set the study on any interval of length 2π (most of the time, $[0; 2\pi]$ or $[-\pi; \pi]$); this will be sufficient to describe the entire behaviour.

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The inner product between f and g is thus given by the formula:

$$\varphi(f,g) = \int_0^{2\pi} \overline{f(t)}g(t)\mathrm{d}t = \int_{-\pi}^{\pi} \overline{f(t)}g(t)\mathrm{d}t.$$

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We will use this property to define the Fourier coefficients of a function; these are in fact the coordinates of the orthogonal projection of this function along each trigonometric function.

Orthogonality of the e^{ikx}

For $(k, n) \in \mathbb{Z}^2$ we get:

$$\varphi(x \mapsto e^{ikx}, x \mapsto e^{inx}) = \int_0^{2\pi} e^{-ikx} e^{inx} dx = \int_0^{2\pi} e^{i(n-k)x} dx$$

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- if $k = n$, $e^{i(n-k)x} = e^0 = 1$ thus the integral is worth 2π;

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- if $k = n$, $e^{i(n-k)x} = e^0 = 1$ thus the integral is worth 2π;
- if $k \neq n$, a primitive of$e^{i(n-k)x}$ is$\dfrac{e^{i(n-k)x}}{i(n-k)}$; then

$$\int_0^{2\pi} e^{i(n-k)x} dx = \left[\frac{e^{i(n-k)x}}{i(n-k)}\right]_0^{2\pi} = 1 - 1 = 0.$$

Orthogonality of the e^{ikx}

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$$\varphi(x \mapsto e^{ikx}, x \mapsto e^{inx}) = \begin{cases} 2\pi & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n \end{cases}$$

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To further simplify the manipulations, we can turn it into an orthonormal set by normalising the inner product this way:

$$< f, g >= \frac{1}{2\pi} \int_0^{2\pi} \overline{f(t)} g(t) \mathrm{d}t.$$

We will denote by $\| \cdot \|$ the norm induced by $< \cdot, \cdot >$.

Orthogonality of the $\sin(kx)$ and $\cos(nx)$

There are many more cases to split the study into here. We can determine the following results using properties of trigonometrics, the Euler formulae to use the results on the e^{ikx}, or the parity of functions and even some substitutions. Finally we get:

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if $k = n$:

$$\varphi(\cos(nx), \cos(nx)) = \begin{cases} 2\pi & \text{si} \quad n = 0 \\ \pi & \text{if} \quad n \neq 0 \end{cases}$$

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and in any case:

$$\varphi(\sin(nx), \cos(kx)) = 0$$

Summary

Orthogonality of trigonometric functions

Fourier coefficients and Fourier series

- Definition
- Real Fourier coefficients and parity
- Link between complex and real coefficients
- Fourier coefficients of a derivative

Dirichlet's convergence theorem

Parseval's identity

All the previous calculations will help us define properly the Fourier coefficients of a function, and write its associated Fourier series.
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Let us denote by e_k the function $x \mapsto \mathrm{e}^{ikx}$.
We use the formula of orthogonal projection along a vector to calculate the component of the decomposition of a function f along the e_k function. Thus, we get:

$$\frac{\varphi(e_k, f)}{\varphi(e_k, e_k)} e_k$$

$\dfrac{\varphi(e_k, f)}{\varphi(e_k, e_k)}$ will be called complex Fourier coefficient of f associated with the integer k, and will be denoted by $c_k(f)$.
The Fourier series of f will be the function $S(f) = \sum_{k \in \mathbb{Z}} c_k e_k$.

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Remark: using the new convention for the inner product $< \cdot, \cdot >$, we can even get $c_k(f) =< e_k, f >$.

Fourier series (in $\mathbb{C}$)

We suppose that the function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is 2π-periodic and moreover piecewise continuous over $[0; 2\pi]$.

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We call (complex) Fourier series associated with the function f, the series of functions

$$S(f) = x \mapsto \sum_{k \in \mathbb{Z}} c_k(f) \mathrm{e}^{ikx}$$

with:

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The Fourier series of f is thus the projection of f onto the subspace spanned by the set of functions $(e_k)_{k \in \mathbb{Z}}$.

Fourier series (in $\mathbb{R}$)

Definition [Real Fourier series]
We call real Fourier series associated with the function f, the series of functions

$$S(f) = x \mapsto \sum_{k \in \mathbb{N}} \left(a_k(f)\cos(kx) + b_k(f)\sin(kx)\right)$$

with:

- $a_0(f) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(t)\,\mathrm{d}t;$
- $b_0(f) = 0;$
- $\forall k \in \mathbb{N}^*, \quad a_k(f) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\cos(kt)\,\mathrm{d}t;$
- $\forall k \in \mathbb{N}^*, \quad b_k(f) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\sin(kt)\,\mathrm{d}t.$

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- if f is odd, its decomposition into a Fourier series will only contain odd functions $(\sin(kx))$, so for every $n \in \mathbb{N}, a_n(f) = 0$.

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- if f is odd, its decomposition into a Fourier series will only contain odd functions $(\sin(kx))$, so for every $n \in \mathbb{N}, a_n(f) = 0$.

Remark: if f is not continuous, the calculation of Fourier coefficients (via integrals) does not take into account the values taken by f at its discontinuity points. Thus, these results still stand if f is "almost everywhere" odd or even.

Link between complex and real coefficients

Knowing that $\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$, using the Euler formulae

$$\cos(\theta) = \frac{\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i},$$

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we can establish the link between the real and complex Fourier coefficients:

- $c_0(f) = a_0(f)$;
- for $n \geqslant 1$: $\begin{cases} a_n(f) &= c_n(f) + c_{-n}(f); \\ b_n(f) &= i(c_n(f) - c_{-n}(f)) \end{cases}$

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Thus, for every $n \geqslant 1$:

- $c_n(f) = \dfrac{a_n(f) - ib_n(f)}{2}$;
- $c_{-n}(f) = \dfrac{a_n(f) + ib_n(f)}{2}$.

If f is piecewise of class C^1, we can calculate the Fourier series of its derivative; we then get:

$$c_k(f') = \frac{1}{2\pi}\int_0^{2\pi} f'(t)\mathrm{e}^{-ikt}\mathrm{d}t.$$

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Using an integration by parts:

$$c_k(f') = \frac{1}{2\pi}\left[f(t)\mathrm{e}^{-ikt}\right]_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} f(t)\cdot\left(-ik\mathrm{e}^{-ikt}\mathrm{d}t\right).$$

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As the function between brackets is 2π-periodic, the left term is zero. Thus:

$$c_k(f') = 0 + ik \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{-ikt} \mathrm{d}t = ik\ c_k(f)$$

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Regularised function

The point of this part of the lesson is to answer this question about the Fourier series: can we be sure that the Fourier series converges, and if so towards which function? Unfortunately, there is no general result on this topic; yet under certain hypotheses we know how to evaluate the pointwise convergence of the Fourier series.

Regular / regularised functions

Careful: the terms “regular” and “regularised” are applied to different concepts in different mathematical fields.

Definition [Regular functions]
We say that a piecewise continuous function is regular if, at any point a, $f(a)$ is the average of the limits at the left and at the right of f in the neighbourhood of a.

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So, for a function to be regular its values at discontinuity points are determined. Obviously, continuous functions are regular (this is how we define continuity).

Definition [Regularised function]
Let f be a piecewise continuous function; we call regularised function of f the regular function that coincides with f almost everywhere, except eventually at its discontinuity points. It is denoted by $\tilde{f}$.

Then: $\forall a \in \mathbb{R}, \quad \tilde{f}(a) = \dfrac{1}{2}\left(\lim\limits_{x \to a^+} f(x) + \lim\limits_{x \to a^-} f(x) \right).$

Dirichlet's convergence theorem

Theorem [Dirichlet's convergence theorem]
Let f be a 2π-periodic function, piecewise of class C^1.
Then the Fourier series of f converges pointwise towards its regularised function $\tilde{f}$:

$$\forall x \in \mathbb{R}, \sum_{k=-n}^{n} c_k(f)\mathrm{e}^{ikx} \underset{n\to+\infty}{\longrightarrow} \tilde{f}(x).$$

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The smoothness hypotheses we chose in the statement are not the loosest ones, but the easiest to use.
So, in favourable cases, we can get a result of pointwise convergence. However, be careful with the discontinuous functions: their Fourier series does not always converge towards them but towards their regularised function.

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It is in fact a way to rewrite Pythagoras's theorem to match our interests in Fourier series.
As you will see during the tutorial sessions, this result, as well as Dirichlet's convergence theorem, enables us to calculate sums of numerical series.
Careful though: the statement is easy for the complex numbers, but there is an added factor for real numbers that is not to be forgotten.

Parseval's identity

Theorem [Parseval's identity]
Let f be a 2π-periodic function over $\mathbb{R}$, that is also piecewise continuous over $[0;2\pi]$.
Then:

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |c_k(f)|^2.$$

With real coefficients we get:

$$\|f\|^2 = |a_0(f)|^2 + \frac{1}{2}\sum_{k\in\mathbb{N}^*} \left(|a_k(f)|^2 + |b_k(f)|^2\right).$$

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Reminder: $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \mathrm{d}t$. ($\|\cdot\|$ is the norm induced by $< \cdot, \cdot >$).

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Let f be a 2π-periodic function over $\mathbb{R}$, that is also piecewise continuous over $[0;2\pi]$.
Then:

$$\|f\|^2 = \sum_{k\in\mathbb{Z}} |c_k(f)|^2.$$

With real coefficients we get:

$$\|f\|^2 = |a_0(f)|^2 + \frac{1}{2}\sum_{k\in\mathbb{N}^*}\left(|a_k(f)|^2 + |b_k(f)|^2\right).$$

Reminder: $\|f\|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \mathrm{d}t$. ($\|\cdot\|$ is the norm induced by $<\cdot,\cdot>$).

Careful not to forget the $\frac{1}{2}$ scaling factor for real numbers.

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One of the many consequences of this theorem is to ensure that the series associated with the squared moduli of Fourier coefficients is convergent. Particularly, we can deduce that c_n tends towards 0 when n tends towards $\pm\infty$.