École Pour l'Informatique et les Techniques Avancées

Approximation of functions

Ch1: Metric Spaces

Distances and Norms

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KAMEL ATTAR

1 Issues
Where Does All of This Start
Proximity as Similarity
Approximation

2 The Notion of Distance
Definition
Notion of neighborhood and convergence

3 Norms p
Concept of Norm on a Vector Space
p-Norm on $\mathbb{R}^n$
Norm p on $\mathcal{C}^0(I, \mathbb{R})$

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- **Understand** the connection between normed spaces and metric spaces.

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You may have had the *opportunity* to study $\underbrace{\text{real-valued functions}}_{f:I\subset\mathbb{R}\longmapsto\mathbb{R}}$ in your previous years. For a typical, sufficiently regular function f, you used to:

- Studying the limits of f at the boundary points of its domain of definition

$$\lim_{x\to-\infty} e^x = 0\,, \quad \lim_{x\to+\infty} \frac{\sin(x)}{x} = 1\,, \quad \lim_{x\to 0} \frac{1}{x} = \infty\,.$$

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- Studying the sign of f' and the table of variation of f specifying notable points (zeros, extrema, asymptotes, etc.).

x	0^+		1		$+\infty$
$\left(\ln x\right)' = \frac{1}{x}$	‖	$+$		$+$	
$\ln x$	$-\infty$	↗	0	↗	$+\infty$

☆ *Where Does All of This Start ?* ☆

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This question hides a real issue:

Replace implies a similarity between f and its *proxy*.

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- **Growth or Decrease:** Observe the intervals where the functions are increasing or decreasing [1].

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- **Local Extrema:** Examine the positions and values of local extrema (maxima or minima) of the functions and compare them.
- **Behavior at Limits:** Observe the behavior of the functions as x tends to infinity or approaches a particular point, and compare these limits.

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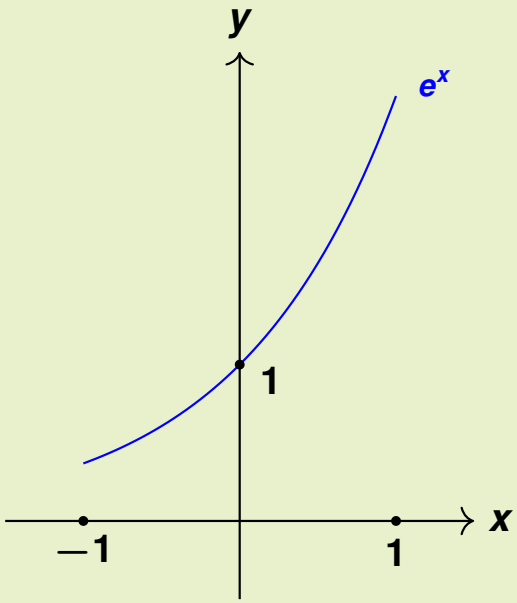
Example

Consider the function $f(x) = e^x$ and suppose we want to study it over the interval $[-1, 1]$, but for some reason, we cannot perform this study directly.



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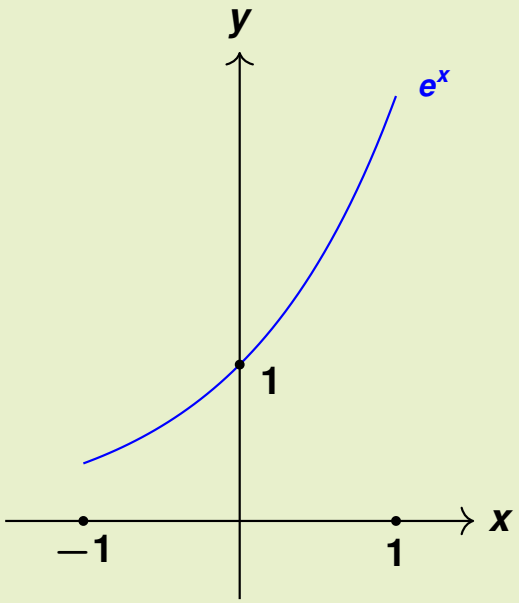

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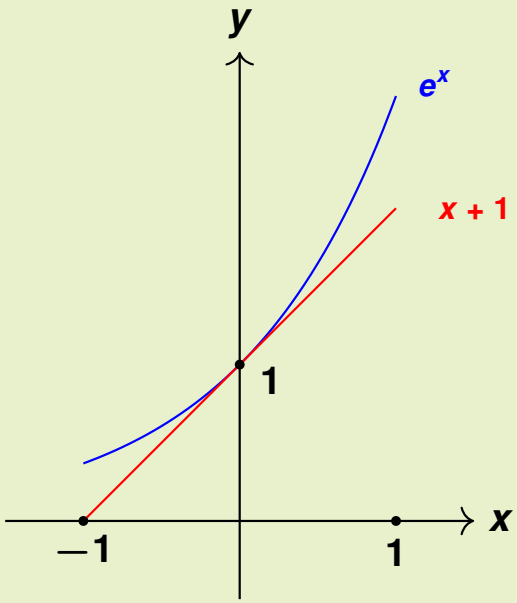
For example, consider the polynomial function $g(x) = 1 + x$. Over the interval $[-1, 1]$, this function shows some similarity to f in terms of general behavior (growth) and values at the ends of the interval.



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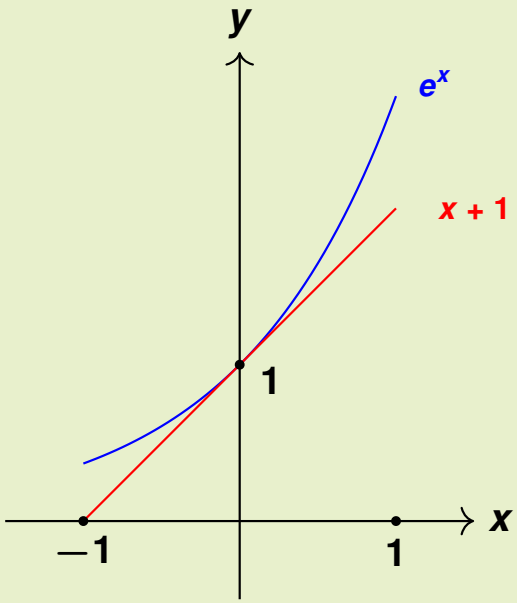

Here are the steps to show this substitution:

1. **Determine the values at the ends of the interval:**
   - $f(-1) = e^{-1} \approx 0.3679$
   - $f(1) = e^1 = 2.7183$
   - $g(-1) = 1 - 1 = 0$
   - $g(1) = 1 + 1 = 2$

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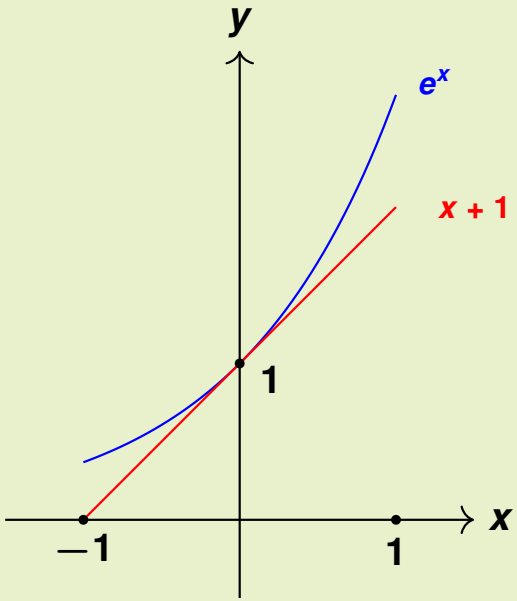

Here are the steps to show this substitution:

2. **Compare the derivatives:**
   - $f'(x) = e^x$
   - $g'(x) = 1$

Over the interval $[-1, 1]$, $f'(x)$ varies from $e^{-1} \approx 0.3679$ to $e \approx 2.7183$, while $g'(x) = 1$ remains constant. This shows that f and g have different growth behaviors, but f can be approximated by g to simplify the analysis.

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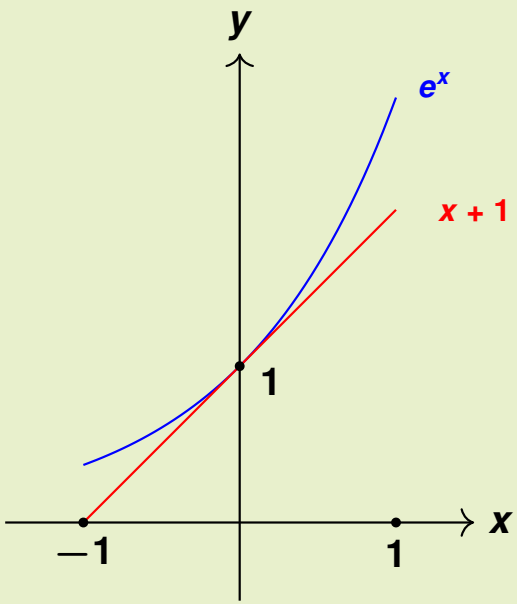

Here are the steps to show this substitution:

3. **Study the variations:**
   - For $f(x) = e^x$, f is increasing over $[-1, 1]$.
   - For $g(x) = 1 + x$, g is also increasing over $[-1, 1]$.

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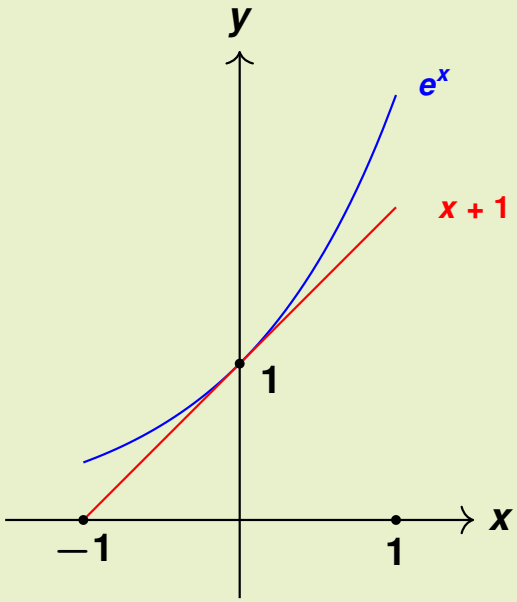

Here are the steps to show this substitution:

4. **Conclusion:**
Although f and g are not identical, the linear function $g(x) = 1 + x$ can serve as a proxy for $f(x) = e^x$ over the interval $[-1, 1]$ due to their similar growth behavior and comparable values at the ends of the interval. This substitution allows for a simplified analysis of f without significant loss of accuracy for qualitative studies.

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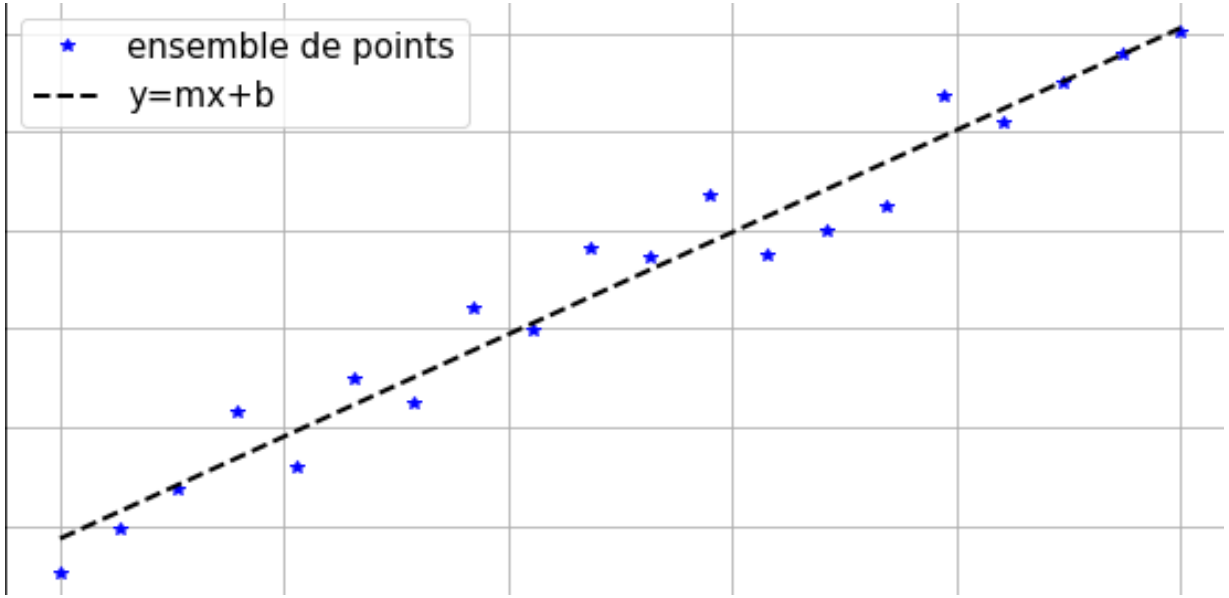
In the generic problem we are interested in, we aim to imitate the behavior of f with another function, that is, to find values *close* to the values taken by f at each of its points.

Here are some common approaches to achieve this goal:

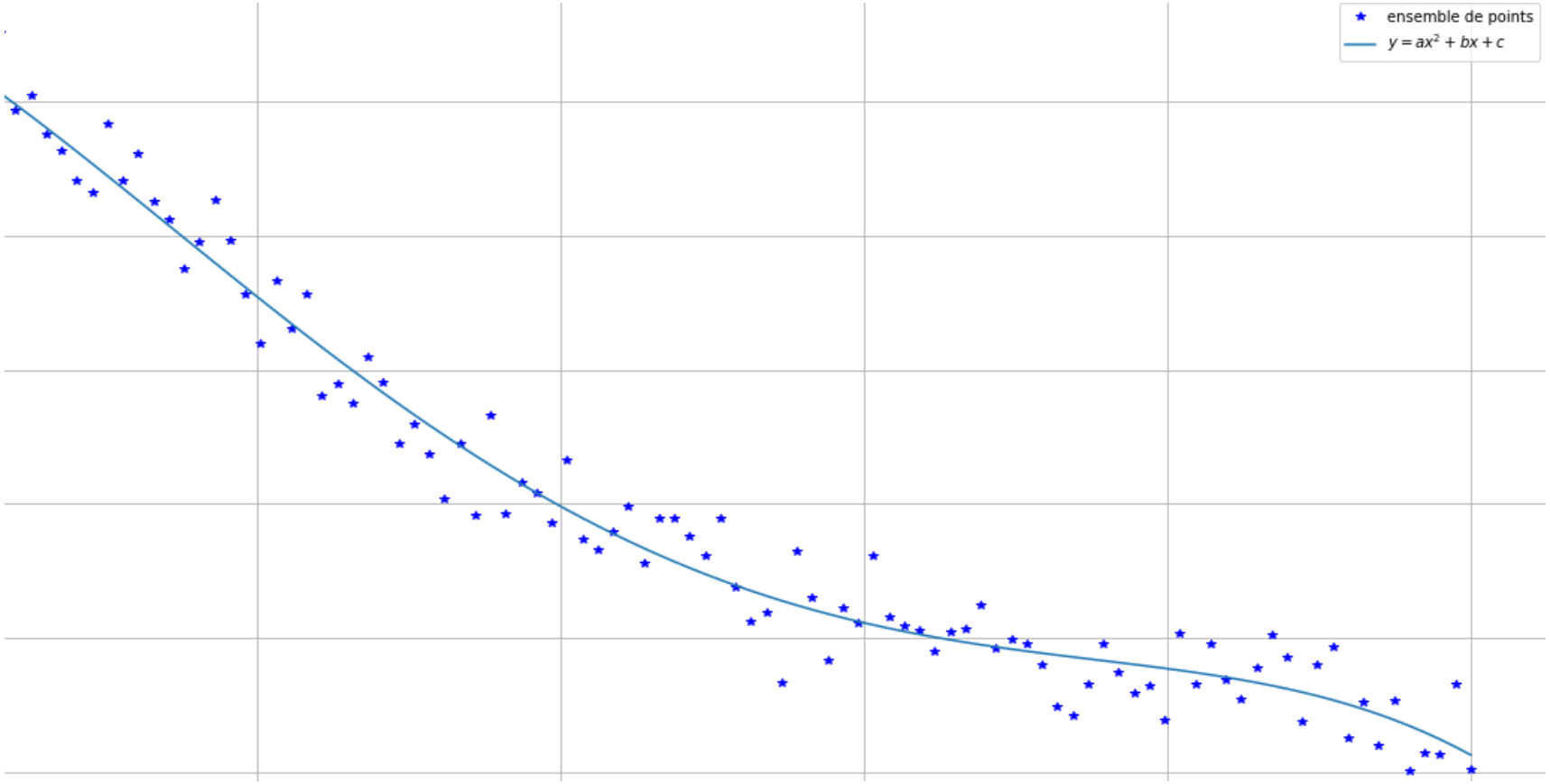
Linear approximation: Linear approximation is the simplest approach. It involves approximating a function (or a cloud of points) with a straight line.

For example, for a given function f, the linear approximation could be in the form $y = mx + b$.



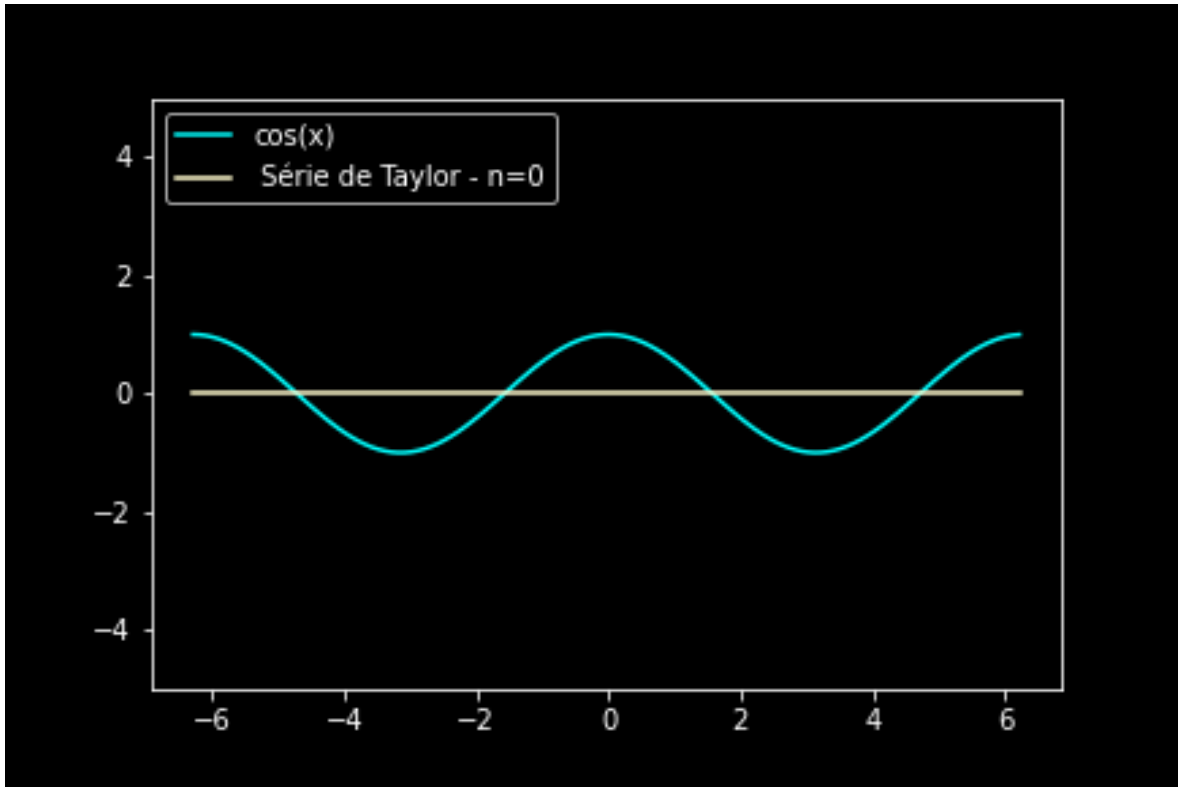
Polynomial Approximation: Polynomial approximation involves using a polynomial to approximate a function (or a cloud of points). A polynomial of degree n can be adjusted to obtain a better approximation.

For example, a polynomial approximation of degree two would be in the form $y = ax^2 + bx + c$.

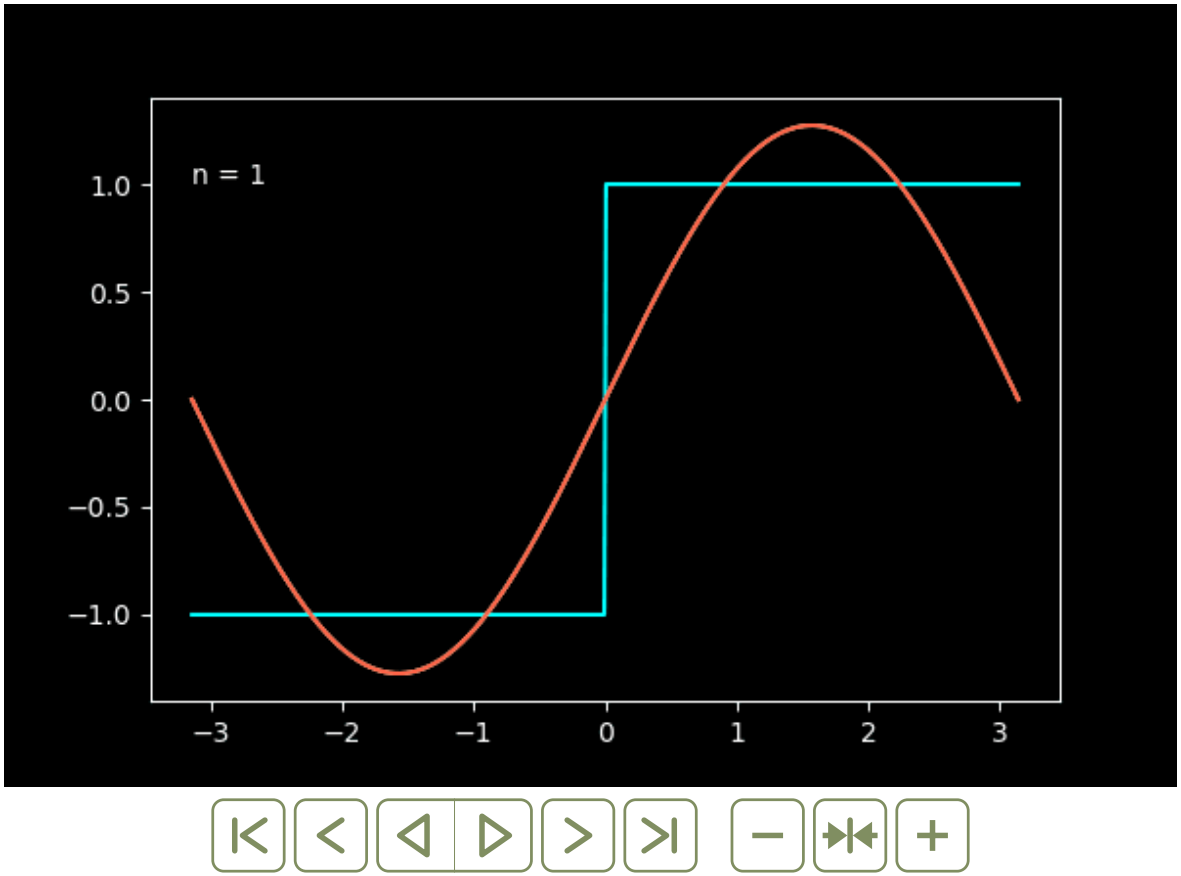


Taylor Series Approximation: Taylor series are a mathematical method that approximates a function by an infinite sum of polynomial terms.

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$



Approximation by Trigonometric Functions: For some periodic functions, Fourier series can be used to approximate them using trigonometric functions (sine and cosine).



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☆ *Being close* depends on a *distance*[3] ☆

[3] This notion of distance is well formalized by mathematicians, and it is the one you worked on in the videos for this session.

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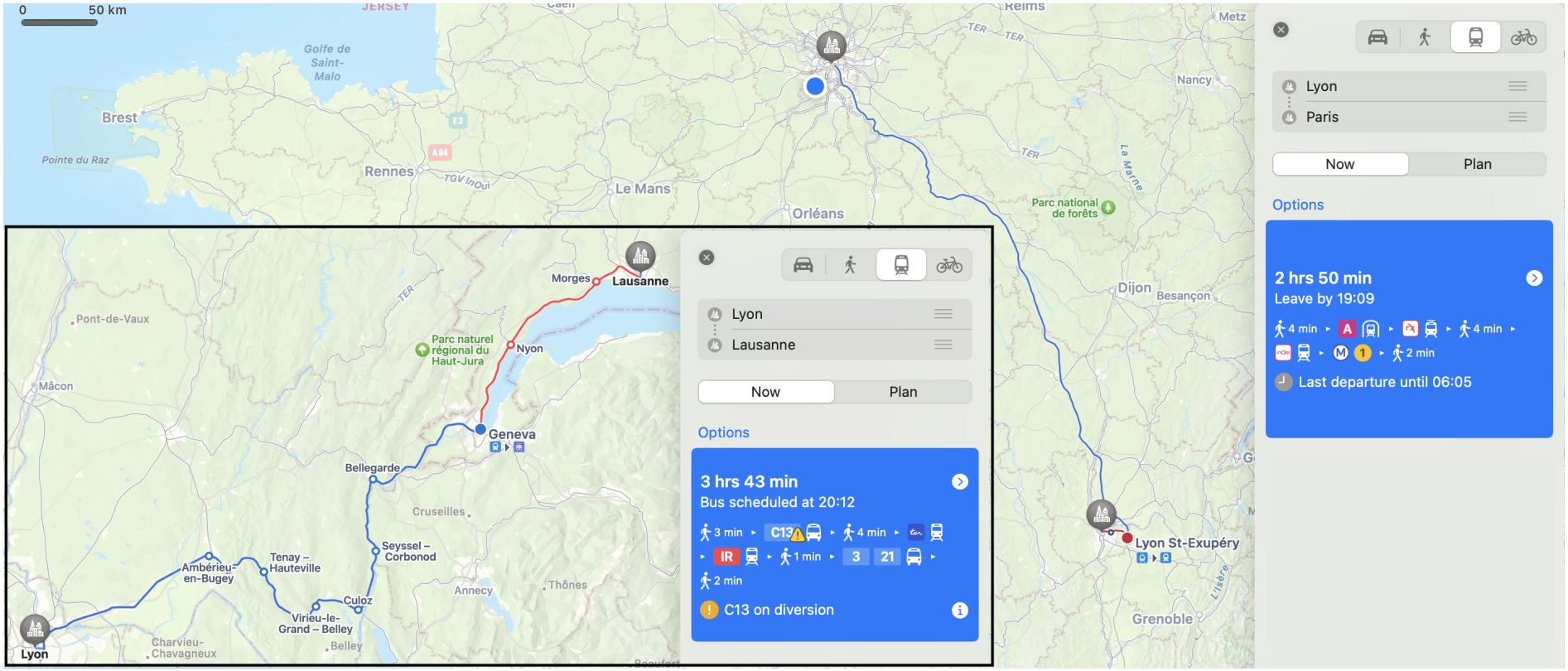
- Lyon is closer to Paris than to Lausanne when considering the train travel time.



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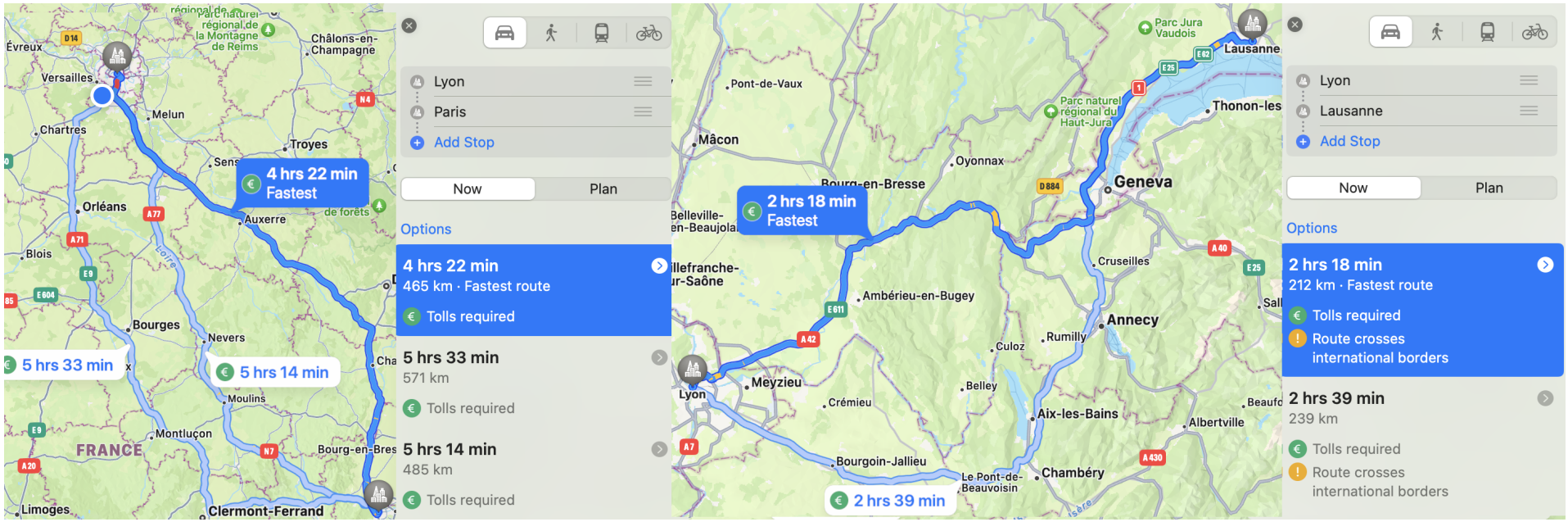
- This is not the case when considering the minimal driving time.



[3] This notion of distance is well formalized by mathematicians, and it is the one you worked on in the videos for this session.

✎ **Wooclap 1.** Among the following properties, which do not define a metric distance:

1. Symmetry: $d(x, y) = d(y, x)$.
2. Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$.
3. Transitivity : if $d(x, y) = 0$ then $x = y$.
4. Distributivity: $d(x, z) \leq d(x, y) \cdot d(y, z)$.

▸ Solution

✎ **Wooclap 2.** Among the following functions, which define distances on the described sets:

1. The absolute value function on $\mathbb{R}$.
2. The function $\boldsymbol{d} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}$ by $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{0}$ if $\boldsymbol{x} = \boldsymbol{y}$ and $\boldsymbol{d}(\boldsymbol{x}, \boldsymbol{y}) = \mathbf{1}$ otherwise.
3. The function that assigns to two vertices of an undirected and connected graph the smallest number of edges between these two vertices.
4. The function that assigns to two vertices of a directed and strongly connected graph the smallest number of edges between these two vertices.
5. The function that assigns to two vertices of an undirected, connected, and weighted graph the sum of the weights of the shortest path (in terms of the number of edges) between these two vertices.

▸ Solution

✎ Wooclap 3.

1. Why does the function $d(x, y) = (x - y)^2$ not define a metric distance on $\mathbb{R}$?

2. For a car, does the path given by Waze between any two road points define a distance ?

3. Given a distance d on a set E, what does this relation

$$d(x, y) = \min_{z \in E} \Big(d(x, z) + d(z, y) \Big) \ ?$$

say to you?

Solution

1 Issues
Where Does All of This Start
Proximity as Similarity
Approximation

2 The Notion of Distance
Definition
Notion of neighborhood and convergence

3 Norms p
Concept of Norm on a Vector Space
p-Norm on $\mathbb{R}^n$
Norm p on $\mathcal{C}^0(I, \mathbb{R})$

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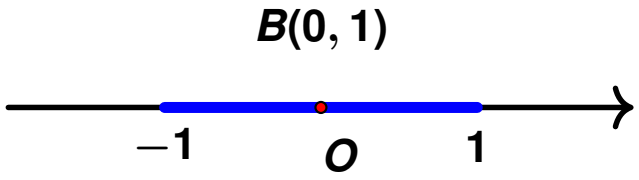
- **Notion of Neighborhood** of a point $x \in E$ with respect to d.

  In our case, we limit ourselves to neighborhoods of the form $B(x, r)$, $r > 0$, referred to as the *open ball centered at* x *with radius* r for the distance d

$$\boxed{B(x, r) = \{x \in E \mid d(a, x) < r\}}$$



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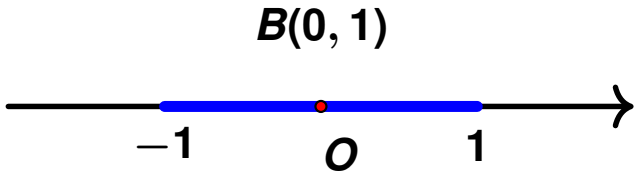

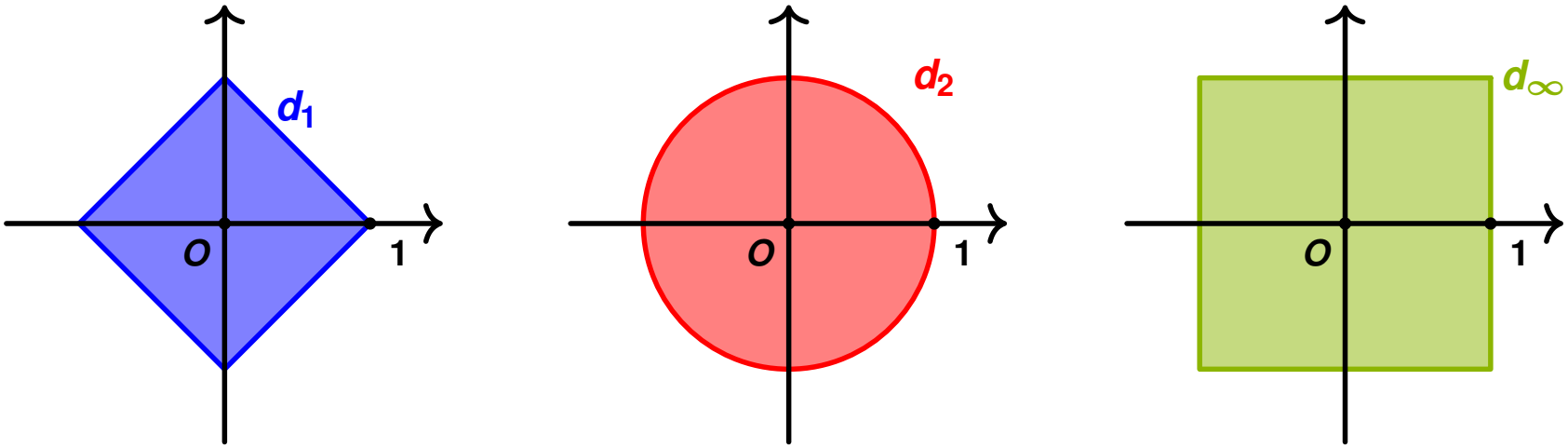

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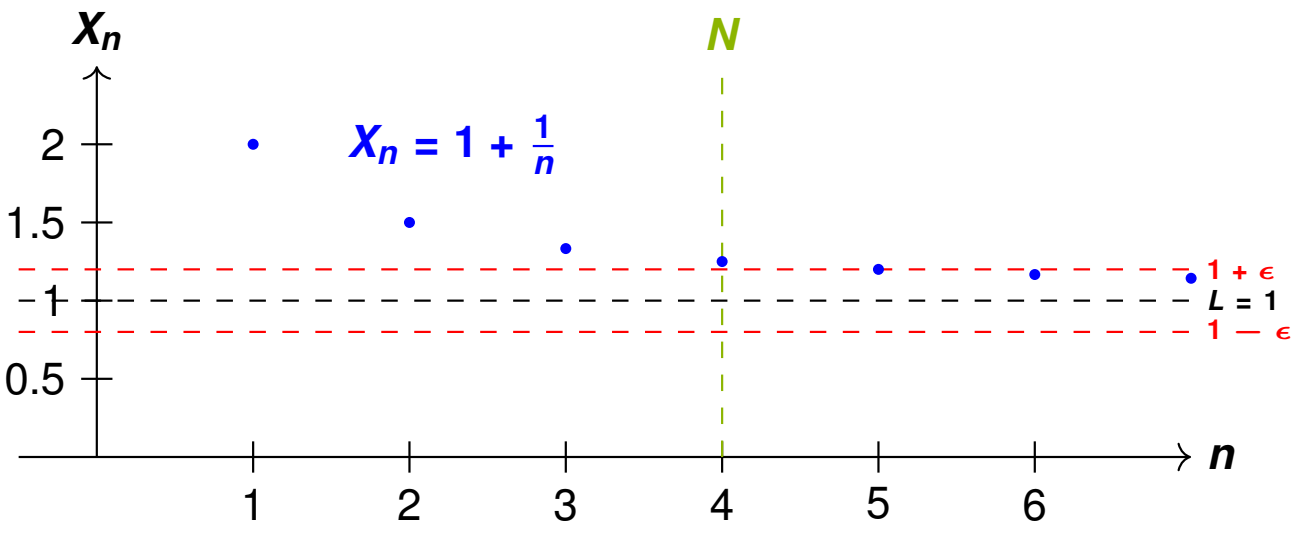
- **Notion of Convergence** of a sequence $(x_n)_{n\in\mathbb{N}}$ to a point x with respect to d

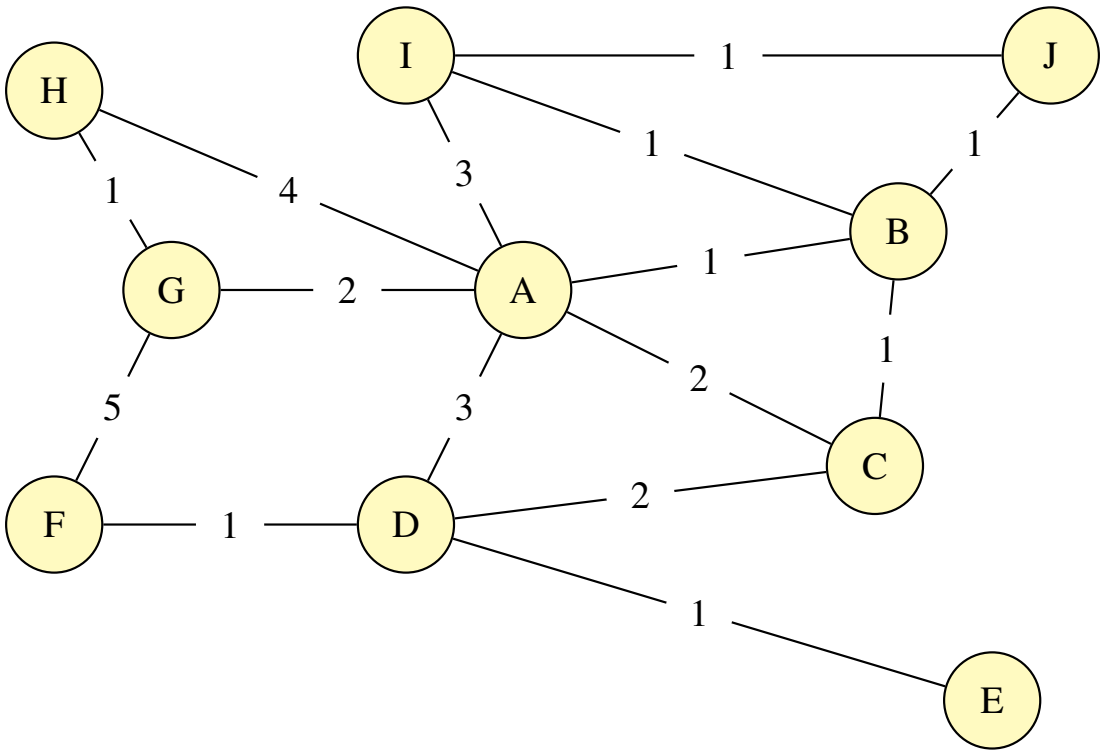
$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\quad n \geq N \implies d(x_n, x) < \varepsilon$$

x_n is at a *distance* at most ε from x



✎ Wooclap 4.

Indicate the list of vertices associated with the open balls centered at ***A***, with the distance given by the shortest path, as shown below.



❶ The ball centered at ***A*** with radius **1/2**

❷ The ball centered at ***A*** with radius **1**

❸ The ball centered at ***A*** with radius **3**

❹ The ball centered at ***A*** with radius **4**

(a) ***A***, ***B***, ***C***, ***G***, ***I***, ***J***.

(b) ***A***

(c) ***A***, ***B***, ***C***, ***D***, ***G***, ***H***, ***I***, ***J***.

(d) ***A***

Solution

✎ **Wooclap 5.** Among the following distances on $\mathbb{R}$, which ones imply that the sequence $(1/n)_{n\geq 1}$ does not converge to 0?

1. $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
2. $d(x,y) = |x - y|$
3. $d(x,y) = |x^{-1} - y^{-1}|$
4. $d(x,y) = |e^{-x} - e^{-y}|$

Solution

1 Issues
Where Does All of This Start
Proximity as Similarity
Approximation

2 The Notion of Distance
Definition
Notion of neighborhood and convergence

3 Norms p
Concept of Norm on a Vector Space
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1 Issues
Where Does All of This Start
Proximity as Similarity
Approximation

2 The Notion of Distance
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- Once a norm $\mathcal{N}$ is established on a vector space E, a distance d on this space can be defined using the following relation:

$$d(x, y) = \mathcal{N}(x - y)$$

- Real vector spaces are fundamental in engineering, as they provide the framework for mathematical models that simplify complex problems through linear approximations.
- In this context, there is a way to define distances on a vector space while respecting the structure of addition and scalar multiplication: the concept of a norm.
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$$d(x, y) = \mathcal{N}(x - y)$$

Notice that this generalizes what you usually do on $\mathbb{R}$.
$$d(x, y) = |x - y|$$

✎ **Wooclap 6.** Which of the following functions define norms?

1. The function on $\mathbb{R}^n$ that assigns to a vector the number of non-zero components.
2. The function on $\mathbb{C}$ that assigns to a complex number z the value $z\bar{z}$.
3. The function on $\mathbb{R}_n[X]$ that assigns to a polynomial P the value $\sum_{i=0}^{n} |P(i)|$.
4. The function on the set of piecewise continuous functions on $[0, 1]$ that assigns to a function f the value $\int_0^1 |f(x)|\, dx$.

▸ Solution

✎ **Wooclap 7.** Let (x_n) be a sequence in $\mathbb{R}^n$ and $\|\cdot\|$ be any norm on $\mathbb{R}^n$. What is the definition of the convergence of (x_n) to x in this norm?

1. (x_n) converges to x if there exists $\epsilon > 0$ and an integer N such that for all $n \geq N$, $\mathcal{N}(x_n - x) < \epsilon$.
2. (x_n) converges to x if $\lim\limits_{n\to\infty} x_n = x$.
3. (x_n) converges to x if $\mathcal{N}(x_n)$ is bounded.
4. (x_n) converges to x if for all $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, $\mathcal{N}(x_n - x) < \epsilon$.

Solution

1 Issues
Where Does All of This Start
Proximity as Similarity
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2 The Notion of Distance
Definition
Notion of neighborhood and convergence

3 Norms p
Concept of Norm on a Vector Space
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p-norms, also known as ℓ_p norms, are functions defined on vector spaces that generalize the concept of distance. They are commonly used in mathematics, particularly in **functional analysis** and **geometry**.

Definition

Let $p \in [1, +\infty]$. The *p-norm* on $\mathbb{R}^n$, denoted by $\|\cdot\|_p$, is defined for any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

The distance associated with the p-norm on $\mathbb{R}^n$ is denoted by d_p.

p-norms allow us to define different geometries on the vector space, thus influencing various applications in optimization, signal processing, machine learning, and many other fields.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Special Cases:

1. **ℓ_1 Norm (Manhattan Norm or Taxi Distance):**

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \, .$$

This norm represents the sum of the absolute values of the vector components.

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This norm corresponds to the classic Euclidean distance.

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3. **ℓ_∞ Norm (Maximum Norm):**

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| .$$

This norm represents the largest absolute value among the vector components.

✎ **Wooclap 8.** What is the shape of each open ball of radius 1 centered at $(0, 0)$ for the norms given below on $\mathbb{R}^2$?

1. p-Norm **2**
2. Infinity Norm
3. p-Norm **1**

Solution

Proof in the cases $p = 1, 2$.

Demonstration

✎ **Wooclap 9.** The p-norm of a vector $x \in \mathbb{R}^n$ is defined by:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

What is the ℓ_2 norm of the vector $x = (3, 4)$?

1. 5
2. 7
3. 6
4. 25

Solution

1 Issues
Where Does All of This Start
Proximity as Similarity
Approximation

2 The Notion of Distance
Definition
Notion of neighborhood and convergence

3 Norms p
Concept of Norm on a Vector Space
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The norm p can be extended to continuous functions on an interval $I \subseteq \mathbb{R}$. Let $\mathcal{C}^0(I, \mathbb{R})$ denote the space of continuous functions from I to $\mathbb{R}$.

Definition

For $f \in \mathcal{C}^0(I, \mathbb{R})$ and $1 \leq p < \infty$, the norm $\|\cdot\|_p$ is defined by

$$\|f\|_p = \left(\int_I |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Special case $p = +\infty$:

$$\|f\|_\infty = \sup_{x \in I} |f(x)|$$

The distance associated with the p-norm on $\mathcal{C}^0(I, \mathbb{R})$ is denoted d_p.

- Norms p define metrics that measure the "size" or "length" of a continuous function over the interval I.
- The use of p-norms in $\mathcal{C}^0(I, \mathbb{R})$ allows for generalizing and working with concepts of convergence.

✎ **Wooclap 10.** For the function $f(x) = x^2$ on $I = [0, 1]$, what is the norm $\|f\|_1$?

1. $\frac{1}{3}$
2. $\frac{1}{2}$
3. $\frac{1}{4}$
4. 1

Solution

Before You Leave

Before You Leave

1. Review the videos and remediation materials.

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2. Try to solve the exercises from TD..

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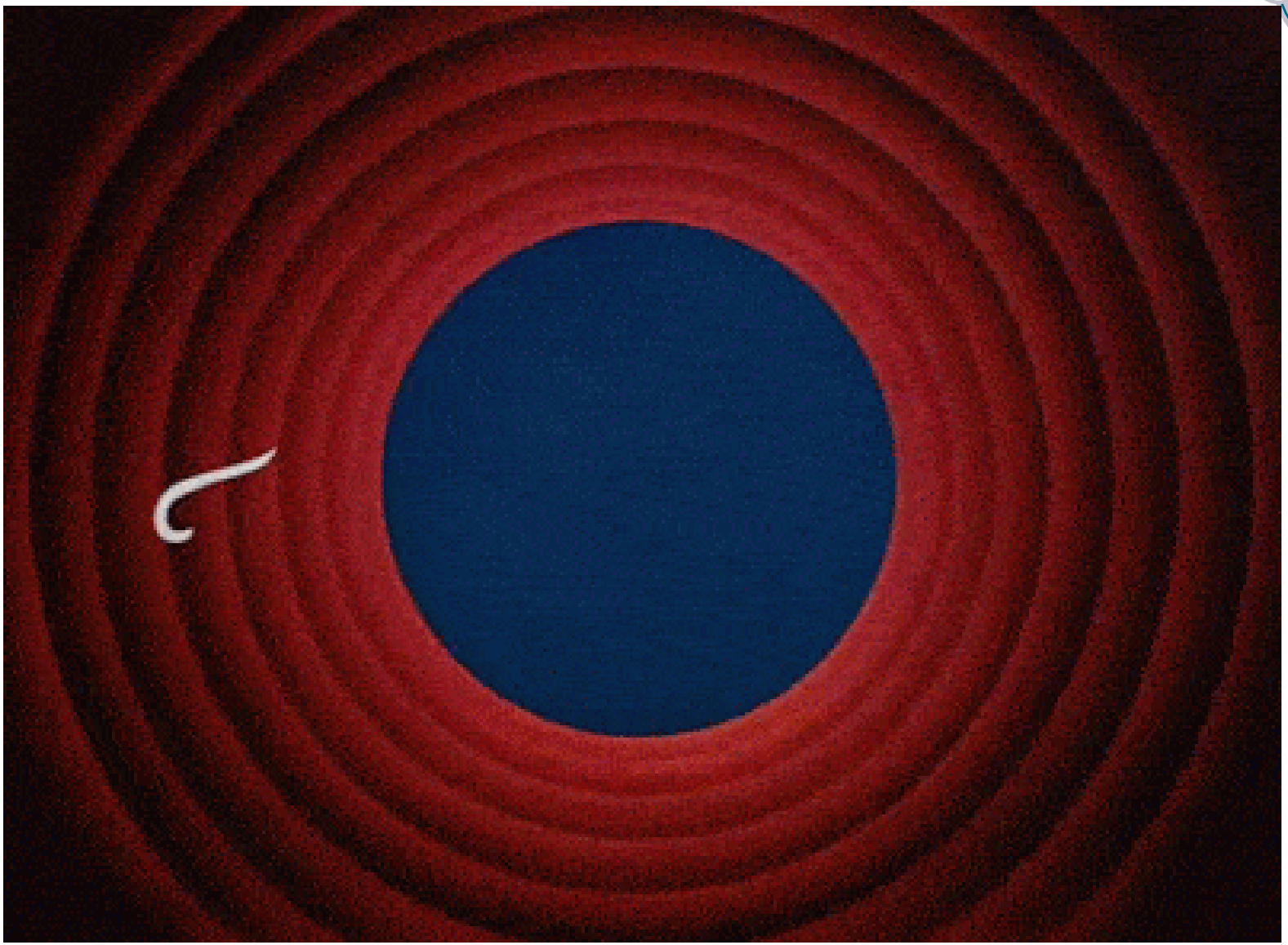
Reminder: You have a formative assessment (to complete at home) by the end of the week.



Definition (espace métrique Go Back)

Un espace métrique est un couple (E, d) où E est un ensemble non vide et d est une **distance** sur E, c'est-à-dire une application $d : E \times E \to \mathbb{R}_+$ vérifiant les axiomes suivants de Fréchet :

1. Positivité: $d(x, y) \geq 0$ et
2. Identité de la distance nulle : $d(x, y) = 0$ si et seulement si $x = y$.
3. Symétrie: $d(x, y) = d(y, x)$.
4. Inégalité triangulaire: $d(x, z) \leq d(x, y) + d(y, z)$.

Definition ($\mathbb{R}-$Espace vectoriel Go Back)

Soit E un ensemble muni d'une loi interne notée " $+$ " et d'une loi externe " $\cdot$ " définie sur un couple (α, u) de $\mathbb{R} \times E$, et notée $\alpha \cdot u$. On dit que $(E, +, \cdot)$ est un espace vectoriel sur $\mathbb{R}$ si:

1. $(E, +)$ est un groupe abélien:
   - A1. $+$ est commutative : $\forall u,\ v \in E,\ u + v = v + u$.
   - A2. $+$ est associative: $\forall u,\ v,\ w \in E,\ u + (v + w) = (u + v) + w$.
   - A3. Il existe un (et un seul) élément neutre, noté 0_E vérifiant $\forall u \in E$, $u + 0_E = 0_E + u = u$.
   - A4. Tout élément a un symétrique: $\forall u \in E, \exists v \in E,\ u + v = v + u = 0_E$ on note $v = -u$.
2. l'operation " $\cdot$ " **multiplication par scalaires** vérifiant les propriétés suivantes:
   - M1. $\cdot$ est associative : $\forall \alpha, \beta \in \mathbb{R}$ et $\forall u \in E$, $\alpha(\beta \cdot u) = (\alpha \cdot \beta) \cdot u$.
   - M2. Il existe un (et un seul) élément neutre, noté 1 vérifiant $\forall u \in E$, $1 \cdot u = u$.
   - M3. $\cdot$ est distributive par rapport à l'addition: $\forall \alpha \in \mathbb{R}, \forall u,\ v \in E$, $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$.
   - M4. $+$ est distributive par rapport à la loi $\cdot$: $\forall \alpha, \beta \in \mathbb{R}, \forall u \in E$, $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$.

Definition (**Notion de norme** Go Back)

Soit E un espace vectoriel (sur $\mathbb{R}$ ou $\mathbb{C}$). Une norme sur E est une fonction $\|\cdot\| : E \to [0, +\infty[$ satisfaisant les conditions suivantes :

1. Séparation $\forall x \in E, \|x\| = 0 \iff x = 0$
2. Inégalité triangulaire $\forall x, y \in E, \|x + y\| \leq \|x\| + \|y\|$
3. homogénéité $\forall x \in E, \forall \lambda \in \mathbb{R}$ où $\mathbb{C}, \|\lambda x\| = |\lambda| \cdot \|x\|$

On dit alors que $(E, \|\cdot\|)$ est un espace normé .

Pour prouver que la norme $p = 1$ définit bien une norme sur l'espace vectoriel $\mathbb{R}^n$, nous devons vérifier que cette norme satisfait les trois propriétés fondamentales des normes :

1. **Positivité**: $\|x\|_1 \geq 0$ et $\|x\|_1 = 0 \iff x = 0$.
2. **Homogénéité** : Pour tout scalaire α et tout vecteur x, $\|\alpha x\|_1 = |\alpha| \|x\|_1$.
3. **Inégalité triangulaire** : Pour tous vecteurs x et y, $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$.

Soit $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. La norme $p = 1$, appelée la norme ℓ_1, est définie par :

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|.$$

Vérifions chacune des propriétés :

1. **Positivité**
   - **Non-négativité** : Par définition, chaque $|x_i| \geq 0$. Ainsi, la somme de nombres non négatifs est également non négative :
   $$\|x\|_1 = \sum_{i=1}^{n} |x_i| \geq 0.$$
   - **Nulle uniquement pour le vecteur nul** : Si $\|x\|_1 = 0$, alors cela implique que $\sum_{i=1}^{n} |x_i| = 0$. Étant donné que chaque $|x_i| \geq 0$ et que la somme de ces termes est nulle, cela implique que chaque $|x_i| = 0$. Donc, $x_i = 0$ pour tout i. Par conséquent, $x = 0$.

❷ **Homogénéité**

Soit $\alpha \in \mathbb{R}$ et $x \in \mathbb{R}^n$. Alors, $\|\alpha x\|_1 = \sum_{i=1}^n |\alpha x_i|$.
Utilisons la propriété de la valeur absolue : $|\alpha x_i| = |\alpha||x_i|$. Ainsi,

$$\|\alpha x\|_1 = \sum_{i=1}^n |\alpha||x_i| = |\alpha| \sum_{i=1}^n |x_i| = |\alpha|\|x\|_1.$$

❸ **Inégalité triangulaire**

Pour $x, y \in \mathbb{R}^n$, considérons $x = (x_1, x_2, \ldots, x_n)$ et $y = (y_1, y_2, \ldots, y_n)$. Nous avons :

$$\|x + y\|_1 = \sum_{i=1}^n |x_i + y_i|.$$

En utilisant l'inégalité triangulaire pour les valeurs absolues, nous obtenons :

$$|x_i + y_i| \leq |x_i| + |y_i|.$$

En sommant cette inégalité pour tous les i de 1 à n, nous obtenons :

$$\sum_{i=1}^n |x_i + y_i| \leq \sum_{i=1}^n (|x_i| + |y_i|) = \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| = \|x\|_1 + \|y\|_1.$$

Donc,

$$\|x + y\|_1 \leq \|x\|_1 + \|y\|_1.$$

Ces trois propriétés étant vérifiées, nous pouvons conclure que $\|\cdot\|_1$ est bien une norme sur $\mathbb{R}^n$

Pour démontrer que $||\mathbf{v}||_2$ est effectivement une norme, nous devons vérifier qu'elle satisfait les trois propriétés suivantes de la norme :

1. **Non-négativité** :

$$||\mathbf{v}||_2 \geq 0 \qquad \text{et} \qquad ||\mathbf{v}||_2 = 0 \iff \mathbf{v} = 0\,.$$

En effet, puisque la somme des carrés des composantes d'un vecteur est toujours non-négative et que la racine carrée d'un nombre non-négatif est également non-négative, $||\mathbf{v}||_2 \geq 0$.

De plus, $||\mathbf{v}||_2 = 0$ implique que $\sum_{i=1}^{n} v_i^2 = 0$, ce qui signifie que chaque $v_i = 0$, donc $\mathbf{v} = 0$.

2. **Homogénéité (ou propriété de l'échelle)** : Pour tout scalaire $\alpha \in \mathbb{R}$ et tout vecteur $\mathbf{v}$,

$$||\alpha\mathbf{v}||_2 = |\alpha|\,||\mathbf{v}||_2$$

En effet,

$$||\alpha\mathbf{v}||_2 = \left(\sum_{i=1}^{n}(\alpha v_i)^2\right)^{1/2} = \left(\alpha^2\sum_{i=1}^{n} v_i^2\right)^{1/2} = |\alpha|\left(\sum_{i=1}^{n} v_i^2\right)^{1/2} = |\alpha|\,||\mathbf{v}||_2$$

❸ **Inégalité triangulaire** : Pour tous vecteurs **u** et **v**,

$$\|\mathbf{u}+\mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$$

Cette propriété peut être démontrée en utilisant l'inégalité de Cauchy-Schwarz :

$$(\mathbf{u}\cdot\mathbf{v})^2 \leq (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})$$

En utilisant cette inégalité, nous avons :

$$\|\mathbf{u}+\mathbf{v}\|_2^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \|\mathbf{u}\|_2^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|_2^2$$

En utilisant l'inégalité de Cauchy-Schwarz,

$$2(\mathbf{u}\cdot\mathbf{v}) \leq 2\|\mathbf{u}\|_2\|\mathbf{v}\|_2$$

Donc,

$$\|\mathbf{u}+\mathbf{v}\|_2^2 \leq \|\mathbf{u}\|_2^2 + 2\|\mathbf{u}\|_2\|\mathbf{v}\|_2 + \|\mathbf{v}\|_2^2 = (\|\mathbf{u}\|_2 + \|\mathbf{v}\|_2)^2$$

En prenant la racine carrée des deux côtés, nous obtenons :

$$\|\mathbf{u}+\mathbf{v}\|_2 \leq \|\mathbf{u}\|_2 + \|\mathbf{v}\|_2$$

Ainsi, nous avons démontré que la norme $\|\mathbf{v}\|_2$ satisfait toutes les propriétés d'une norme et est effectivement la norme euclidienne. Go Back

Solution 1.

1. Symétrie: $d(x, y) = d(y, x)$.
2. Inégalité triangulaire: $d(x, z) \leq d(x, y) + d(y, z)$.
3. Transitivité : Si $d(x, y) = 0$ alors $x = y$.
4. Distributivité: $d(x, z) \leq d(x, y) \cdot d(y, z)$.

Go Back

Solution 2.

1. La valeur absolue sur $\mathbb{R}$. Il faut que l'ensemble de départ soit un produit cartésian $E \times E$

2. La fonction $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ définie pour tout $x, y \in \mathbb{R}$ par $d(x, y) = 0$ si $x = y$ et $d(x, y) = 1$ sinon.

3. La fonction qui associe à deux sommets d'un graphe non-orienté et connexe le plus petit nombre d'arêtes entre ces deux sommets.

4. La fonction qui associe à deux sommets d'un graphe orienté et fortement connexe le plus petit nombre d'arêtes entre ces deux sommets. $d(x, y) \neq d(y, x)$

5. La fonction qui associe à deux sommets d'un graphe non-orienté, connexe et valué la somme des valuations du chemin ayant le plus petit nombre d'arêtes entre ces deux sommets. $d(x, y) = 0$ avec $x \neq y$

Go Back

Solution 3.

1. Car, cela ne satisfait pas l'inégalité triangulaire. Choisissez $x = 3$, $y = 1$ et $z = 2$, puis

$$d(3, 1) = (3 - 1)^2 = 2^2 = 4 \quad \text{et} \quad d(3, 2) + d(2, 1) = (3 - 2)^2 + (2 - 1)^2 = 2.$$

2. Analysons si le trajet donné par Waze satisfait ces propriétés :

   Non-négativité : Oui, le trajet calculé par Waze entre deux points a toujours une longueur non négative, car il mesure une distance physique ou temporelle sur un réseau routier.

   Identité des indiscernables : Oui, le trajet entre un point et lui-même est de longueur nulle. Si $x = y$ alors $d(x, y) = 0$.

   Symétrie : Non nécessairement. Le trajet entre deux points A et B peut ne pas être le même dans les deux sens en raison de sens uniques, restrictions de circulation, ou autres facteurs. Donc, $d(A, B)$ peut être différent de $d(B, A)$.

   Inégalité triangulaire : Oui, en général, le chemin direct entre deux points est au moins aussi court que de passer par un troisième point, donc cette propriété est satisfaite.

   En conclusion, bien que le trajet donné par Waze satisfait les propriétés de non-négativité, d'identité des indiscernables, et d'inégalité triangulaire, il ne satisfait pas nécessairement la propriété de symétrie. Par conséquent, le trajet donné par Waze entre deux points routiers ne définit pas toujours une distance au sens mathématique strict.

3. La distance entre deux points x et y est égale à la somme minimale des distances entre x et tout point intermédiaire z, plus la distance entre z et y.

Solution 4.

1. La boule centrée en A de rayon $1/2$ $\longrightarrow$ (a) A
2. La boule centrée en A de rayon 1 $\longrightarrow$ (b) B
3. La boule centrée en A de rayon 3 $\longrightarrow$ (c) A, B, C, G, I, J.
4. La boule centrée en A de rayon 4 $\longrightarrow$ (d) A, B, C, D, G, H, I, J.

Go Back

Solution 5.

- Pour la distance ❶, $d(x, y)$ est 1 si $x \neq y$, donc la suite $(1/n)$ ne peut jamais "s'approcher" de 0 dans cette métrique, car pour tout n, $d(1/n, 0) = 1$.

- Pour la distance ❸ , $d(x, y) = |x^{-1} - y^{-1}|$, la distance entre $1/n$ et 0 n'est pas définie car 0^{-1} n'est pas défini. Pour $x \neq 0$ et $y = 0$, la distance est infinie, donc la suite ne peut pas converger vers 0.

Go Back

Solution 6.

1. La fonction sur $\mathbb{R}^n$ qui à un vecteur associe le nombre de composantes non nulles. $N(\lambda\vec{v}) = N(\vec{v})$

2. La fonction sur $\mathbb{C}$ qui a un nombre complexe z associe $z\bar{z}$. $N(\lambda z) = \lambda^2 N(z)$

3. La fonction sur $\mathbb{R}_n[X]$ qui a un polynôme P associe $\sum_{i=0}^{n} |P(i)|$.

4. La fonction sur l'ensemble des fonctions continues par morceaux sur $[0, 1]$, qui a une fonction f associe $\int_0^1 |f(x)|dx$. Prenons $f(x) = \begin{cases} x, & \text{si } 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \text{si } \frac{1}{2} < x \leq 1 \end{cases}$ et $g(x) = \begin{cases} x, & \text{si } 0 \leq x \leq \frac{1}{2} \\ 0, & \text{si } \frac{1}{2} < x \leq 1 \end{cases}$.
On a $N(f + g) = \frac{5}{8} > N(f) + N(g) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

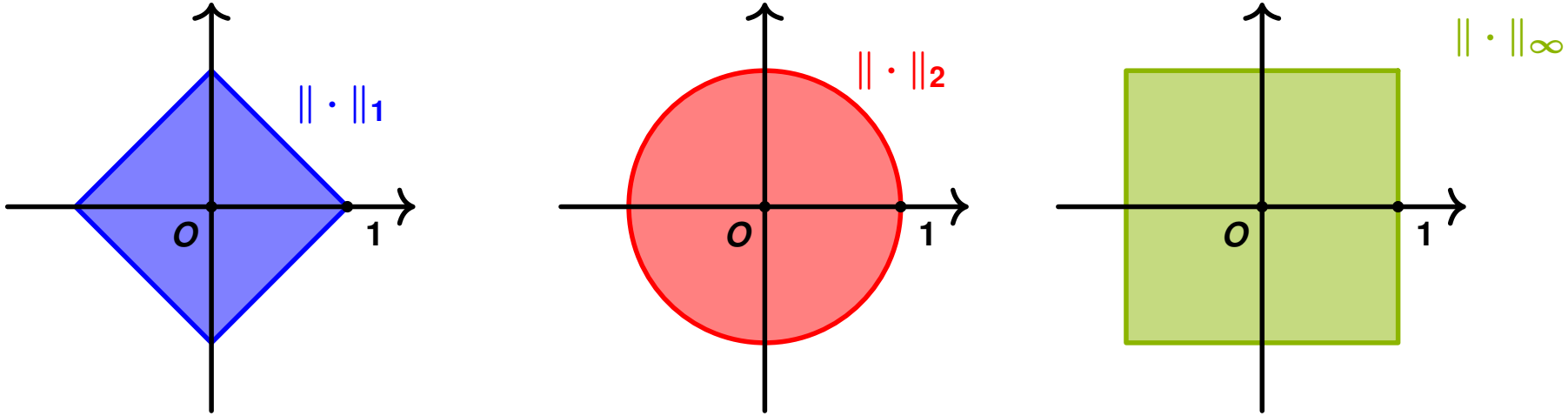
Go Back

Solution 7.

1. (x_n) converge vers x s'il existe $\epsilon > 0$, il existe un entier N tel que pour tout $n \geq N$, $\mathcal{N}(x_n - x) < \epsilon$.
2. (x_n) converge vers x si $\lim\limits_{n \to \infty} x_n = x$.
3. (x_n) converge vers x si $\mathcal{N}(x_n)$ est borné.
4. (x_n) converge vers x si pour tout $\epsilon > 0$, il existe un entier N tel que pour tout $n \geq N$, $\mathcal{N}(x_n - x) < \epsilon$.

Go Back

Solution 8.



Go Back

Solution 9.

1. 5
2. 7
3. 6
4. 25

Go Back

Solution 10.

1. $\frac{1}{3}$
2. $\frac{1}{2}$
3. $\frac{1}{4}$
4. 1

Go Back