Fourier Series

1 - Inner product spaces

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It is an extension of the notion of vector space. On a vector space E over a field $\mathbb{K}$, is added an inner operation $E \times E \to \mathbb{K}$ that, to two elements of E, maps a scalar from $\mathbb{K}$. This operation is called inner product.

Summary

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In order to be an inner product, an application $\varphi : E \times E \to \mathbb{K}$ has to check several properties.
These will ensure that this operation is an interesting one to add to a structure of vector space, allowing us particularly to induce a norm.
These properties vary slightly depending on the chosen field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Bilinearity (and sesquilinearity)

The first property facilitates the manipulations with respect to the vector space structure: it is the bilinearity / sesquilinearity.

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Reminder: a function f over a $\mathbb{K}$-vector space E is said to be linear if

$$\forall (x,y) \in E^2, \forall \lambda \in \mathbb{K}, \quad f(x+y) = f(x) + f(y) \quad \text{and} \quad f(\lambda x) = \lambda f(x)$$

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If $\mathbb{K} = \mathbb{C}$, we say that g is semi-linear if

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We will see later why it is necessary to use conjugates when working with complex numbers.

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Take care, it does not mean that φ only takes positive values. Indeed, for $x \neq y$, we can have $\varphi(x,y) < 0$ over $\mathbb{R}$, and if $\mathbb{K} = \mathbb{C}$ we can get any complex number. But, if we take the same argument both on the left and the right, the result will be a positive real number.

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This property is to be paralleled with the separation axiom of distances and norms.

Inner product: definition

We say that φ is an inner product over $\mathbb{K}$ if φ is:

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- symmetric
- positive
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If so, we say that $(E, +, \cdot, \varphi)$ is a structure of inner product space.
If E is of finite dimension, we call it an Euclidean space if $\mathbb{K} = \mathbb{R}$ and a Hermitian space if $\mathbb{K} = \mathbb{C}$.

Standard inner products - $\mathbb{R}^n$ and $\mathbb{C}^n$

On $\mathbb{R}^n$, the standard inner product $< \cdot, \cdot >$ maps to the vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ the scalar

$$< x, y >= \sum_{i=1}^{n} x_i y_i .$$

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On $\mathbb{C}^n$, the definition is almost similar: $< \cdot, \cdot >$ maps to the vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ the scalar

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Remark: the formula over $\mathbb{C}$ still stands over $\mathbb{R}$, since the conjugate of a real number is itself.
We can see that these functions check all the needed properties if we remember that for every $z \in \mathbb{C}$, $z \cdot \bar{z} = |z|^2 \in \mathbb{R}_+$.

Standard inner products - spaces of functions

Over spaces of smooth enough functions, we generally define the standard inner product as being, for functions with real values:

$$< f, g >= \int_I f(t)g(t)\mathrm{d}t$$

and for fonctions with complex values:

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Of course, in order to be allowed to do so, the integrals must be defined, which happens particularly if we choose continuous or piecewise continuous functions defined over a segment.

Induced norm

The way we defined inner products help us get the following interesting result:

Proposition [Norm induced by an inner product]
Let E be a vector space, together with an inner product φ.
Then, the map N_φ defined for every $x \in E$ by $N_\varphi(x) = \sqrt{\varphi(x,x)}$ is a norm over E.
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The structure of inner product space thus comes together with built-in notions of distance and norm.

Euclidean norms

- Over $\mathbb{R}^n$ or $\mathbb{C}^n$,the norm induced by the standard inner product is the Euclidean norm $\|\cdot\|_2$:

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- Likewise, over $C^0([a,b],\mathbb{R})$ or another such space, the norm induced by the standard inner product is the Euclidean norm $\|\cdot\|_2$:

$$\|x\|_2 = \left(\int_a^b |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}$$

Summary

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Orthogonality

The two following results enable us to prove that the application N_φ induced by φ is a norm over E.

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They had already been talked about during the tutorial about distances. Now you will know whence they come :-)

Cauchy-Schwarz inequality

Theorem [Cauchy-Schwarz inequality]
Let E be an inner product space (over $\mathbb{R}$ or $\mathbb{C}$); we denote by $< \cdot, \cdot >$its inner product and by $\| \cdot \|$ the induced norm. Then:

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The proof of this result is interesting: if we evaluate the expression $< x + ty, x + ty >$, we get an degree 2 equation of the variable t, that cannot take strictly negative values (positivity), meaning that its discriminant is negative (possibly 0 iff $x + ty = 0$.). This discriminant gives the statement of the Cauchy-Schwarz inequality.

Minkowski inequality

The Minkowski inequality can be derived from the Cauchy-Schwarz inequality:

Theorem [Minkowski inequality]
With the notations used for the previous theorem:

$$\sqrt{< x+y, x+y >} \leqslant \sqrt{< x, x >} + \sqrt{< y, y >}$$

That is to say:

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The Minkowski inequality establishes that the norm induced by an inner product checks the property of triangle inequality.
You already proved it for particular cases during the first tutorial, when proving the triangle inequality for the Euclidean norms.

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We for instance will not dwell on the notions of orthogonality to a space and their problems, but instead will focus on the orthogonality between vectors, only to introduce the orthogonal projection.
Similarly, we will not talk about the Gram-Schmidt process, it is to be studied later during the lessons about matrix decompositions.

Orthogonality

Definition [Orthogonality]
In an inner product space E together with an inner product $< \cdot, \cdot >$, we say that two vectors x and y are orthogonal if $< x, y >= 0$.
We say that a set of vectors $(e_1, \ldots, e_n)$ is (pairwise) orthogonal if, for every (i, j) such that $i \neq j$, $< e_i, e_j >= 0$.
If, moreover, for every i then $< e_i, e_i >= 1$ then the set is said to be orthonormal.

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In $\mathbb{R}^2$, you already know orthogonality as perpendicularity.
Orthogonality is an important property when it comes to get decompositions (in a basis for instance): it ensures that no mixed products will pollute the calculations involving the inner product.

Orthogonal supplementary subspace

Proposition [Orthogonal supplementary subspace]
Let F be a vector subspace of finite dimension of E. Then the set of vectors of E that are orthogonal to every vector of F, called orthogonal of F and denoted by $F^{\perp}$, is a supplementary subspace of F in E.

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Remark: the Gram-Schmidt process is a constructive method to build an orthonormal basis of a vector subspace of finite dimension, ensuring that such bases exist.

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If F is of finite dimension, we can even write:
if $(f_1, \dots, f_p)$ is an orthogonal basis of F, then

$$p_F(x) = \sum_{i=1}^{p} \frac{< x, f_i >}{< f_i, f_i >} f_i.$$

If the basis $(f_1, \dots, f_p)$ is orthonormal, we get

$$p_F(x) = \sum_{i=1}^{p} < x, f_i > f_i.$$

Pythagoras's theorem

Theorem [Pythagoras's theorem]
If x and y are orthogonal, then $< x+y, x+y > = < x, x > + < y, y >$, that is to say:

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This result can be generalised to any orthogonal set: if the set $(e_1, \ldots, e_n)$ is orthogonal then

$$\left\| \sum_{i=1}^{n} e_i \right\|^2 = \sum_{i=1}^{n} \|e_i\|^2.$$

Pythagoras's theorem

Theorem [Pythagoras's theorem]
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It is a rewriting of the Pythagoras's theorem you already know for right-angled triangles, but in a general inner product space and more than two dimensions.