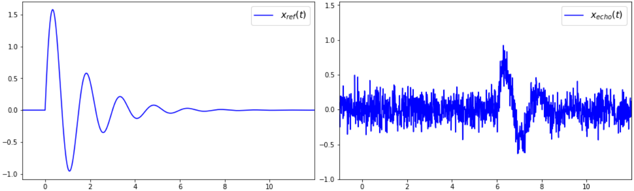
Correlation and convolution

Guillaume Tochon

LRE

Radar detection

Let's revisit the radar example...



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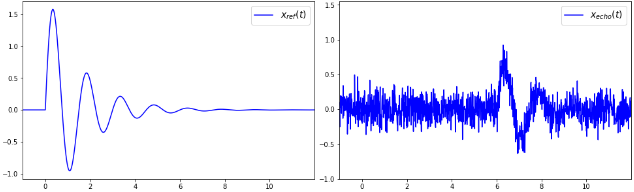

To find the delay between x_{ref} and x_{echo}, it is "sufficient" to measure the similarity (i.e., the **dot product**) between x_{ref} and all versions $x_{echo,\tau} : t \mapsto x_{echo}(t - \tau)$ shifted by a factor τ to find the delay that maximizes this resemblance.

Therefore, we can mathematically expect to manipulate a function $\tau \mapsto \langle x_{ref}, x_{echo,\tau} \rangle$
$\Rightarrow$ This function is called the **cross-correlation** between x_{ref} and x_{echo}

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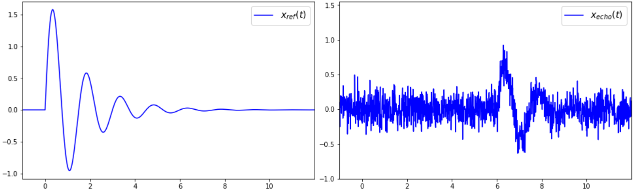

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$\Rightarrow$ This function is called the **cross-correlation** between x_{ref} and x_{echo}

Note: The notation $\langle x_{ref}, x_{echo,\tau} \rangle$ is not easily readable; we will actually prefer $\langle x_{ref}(t), x_{echo}(t-\tau) \rangle$. This is an abuse of notation since the dot product acts on the signals (which are functions) x_{ref} and $x_{echo,\tau}$ and not on their values at time t $x_{ref}(t)$ and $x_{echo}(t-\tau)$.

Autocorrelation

Before tackling cross-correlation, let's simplify things a bit:

Autocorrelation function of signal x

For a signal $x \in \mathcal{L}^2(\mathbb{R})$, we call **autocorrelation** function of x the function

$$\boxed{\Gamma_{xx} : \tau \mapsto \langle x(t), x(t-\tau) \rangle = \int_{\mathbb{R}} x(t)\overline{x(t-\tau)}dt}$$

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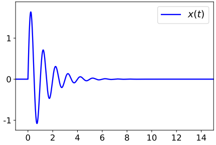

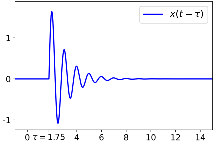

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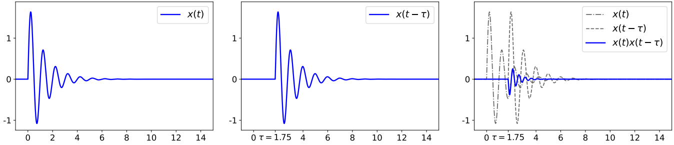

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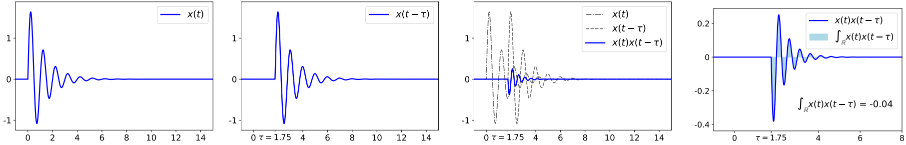

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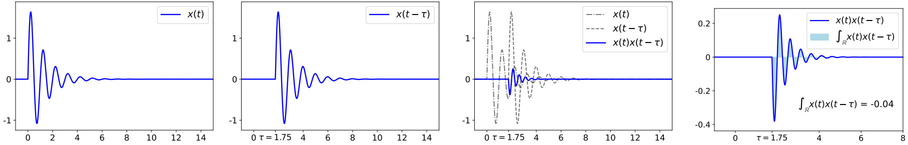

Properties:

- $\Gamma_{xx}(0) = E_x$
- The autocorrelation is maximum at 0: $|\Gamma_{xx}(\tau)| \leq \Gamma_{xx}(0) \quad \forall \tau \in \mathbb{R}$
- Γ_{xx} has Hermitian symmetry: $\Gamma_{xx}(-\tau) = \overline{\Gamma_{xx}(\tau)}$
  $\rightarrow$ if x takes real values, then $\Gamma_{xx}(-\tau) = \Gamma_{xx}(\tau)$: the autocorrelation is even

Autocorrelation

For signals of finite mean power

If x has finite mean power ($P_x < +\infty$) but not finite energy ($E_x = +\infty$), the previous definition is no longer valid since the integral $\int_{\mathbb{R}}$ is not convergent.
$\Rightarrow$ use of the dot product of $\mathcal{L}^{pm}(\mathbb{R})$:

$$\boxed{\Gamma_{xx} : \tau \mapsto \langle x(t), x(t-\tau)\rangle = \lim_{T\to+\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}dt}$$

The properties remain the same:

- $\Gamma_{xx}(0) = \langle x(t), x(t)\rangle = P_x$
- $\forall \tau \in \mathbb{R}, |\Gamma_{xx}(\tau)| \leq \Gamma_{xx}(0)$
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Furthermore, if x is T-periodic ($x(t+T) = x(t)\ \forall t$):

- Γ_{xx} is also T-periodic
- The definition of autocorrelation simplifies into

$$\boxed{\Gamma_{xx} : \tau \mapsto \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}dt = \frac{1}{T}\int_{0}^{T} x(t)\overline{x(t-\tau)}dt}$$

The mathematical expression of the autocorrelation function depends on the definition of the used dot product, and thus on the underlying space the signal x belongs to ($\mathcal{L}^2(\mathbb{R})$ or $\mathcal{L}^{pm}(\mathbb{R})$).

Good news 🥳

We can simply retain the formula $\Gamma_{xx}(\tau) = \langle x(t), x(t-\tau) \rangle$ (and be careful to identify the space the signal x belongs to)

$$\boxed{\begin{array}{lll} x \in \mathcal{L}^2(\mathbb{R}) \quad \rightarrow & \Gamma_{xx}(\tau) = \displaystyle\int_{\mathbb{R}} x(t)\overline{x(t-\tau)}dt & \Gamma_{xx}(0) = E_x \\ x \in \mathcal{L}^{pm}(\mathbb{R}) \quad \rightarrow & \Gamma_{xx}(\tau) = \displaystyle\lim_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}dt & \Gamma_{xx}(0) = P_x \\ x \;\; T\text{-periodic} \rightarrow & \Gamma_{xx}(\tau) = \displaystyle\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}dt & \Gamma_{xx}(0) = \Gamma_{xx}(nT) = P_x \end{array}}$$

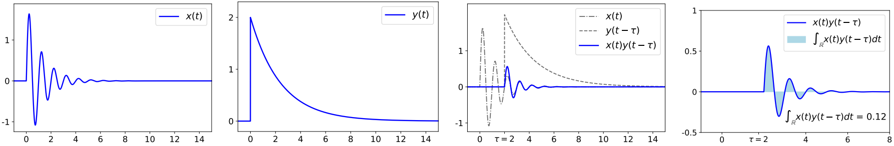
Cross-correlation

We talk about cross-correlation when the shifted signal is different from the fixed signal.

Cross-correlation function of two signals x and y

Let two signals $x, y \in \mathcal{L}^2(\mathbb{R})$, we call **cross-correlation** function of x and y the function

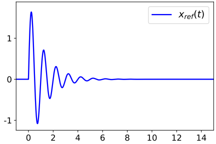
$$\boxed{\Gamma_{xy} : \tau \mapsto \langle x(t), y(t-\tau) \rangle = \int_{\mathbb{R}} x(t)\overline{y(t-\tau)}dt}$$

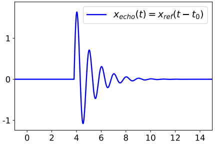


Obviously, $\Gamma_{xy}(\tau) = \lim\limits_{T \to +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{y(t-\tau)}dt$ if $x, y \in \mathcal{L}^{pm}(\mathbb{R})$

And $\Gamma_{xy}(\tau) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{y(t-\tau)}dt$ if x, y are both T-periodic (**with the same period**).

In the case without noise:
Let $x_{echo}(t) = x_{ref}(t - t_0)$ with t_0 being the unknown delay to be estimated.
(for simplicity, we write $x \equiv x_{ref}$ and $y \equiv x_{echo}$)





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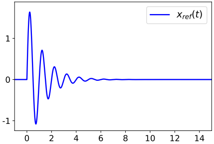

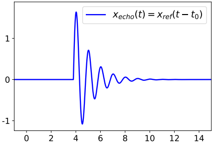

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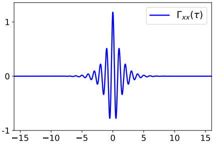
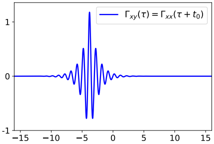
$\rightarrow$ Γ_{xy} is therefore a left-shifted version of Γ_{xx} by t_0

$\rightarrow$ Since Γ_{xx} is maximum at 0, Γ_{xy} is maximum at $-t_0$





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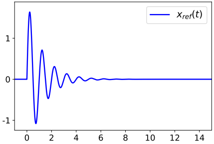

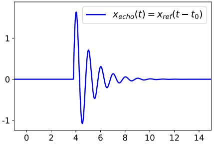

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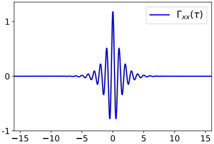
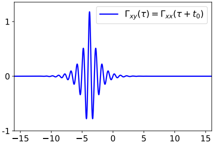




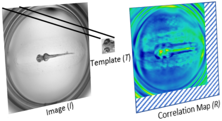
To retrieve the unknown delay t_0 between x and y (with $y(t) = x(t - t_0)$), we then must:

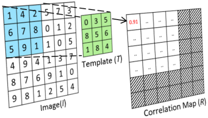
1. Compute the cross-correlation Γ_{xy} between x and y
2. Identify the instant τ_{max} where Γ_{xy} is maximum: $\tau_{max} = \text{argmax}_\tau \, \Gamma_{xy}(\tau)$
3. $\boxed{t_0 = -\tau_{max}}$ $t_0 > 0$ (since it iss a delay), so $\tau_{max} < 0$

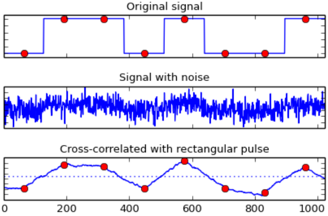
Cross-correlation

In practice

Cross-correlation is useful in practical signal processing applications for **pattern recognition** tasks.







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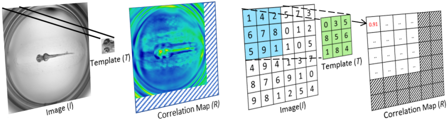
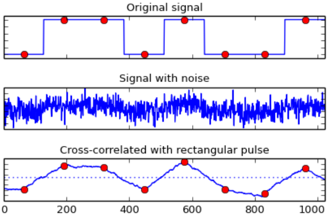
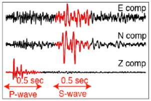
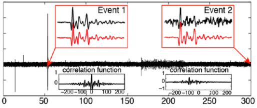




Practical examples of use

- Seismic wave processing
  $\Rightarrow$ Lab Work on convolution/correlation





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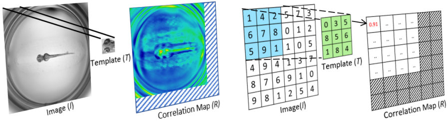

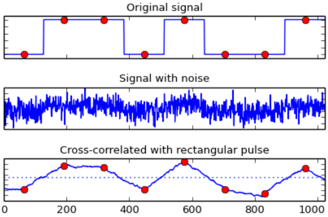

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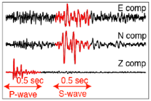

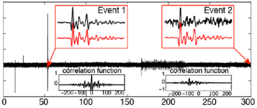

- Where's Waldo?


image


template


detection by cross-correlation

Almost everyone has heard of the convolution operation...

The convolution product

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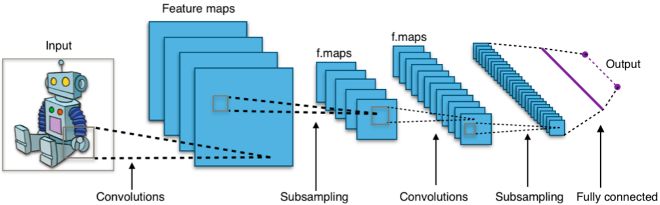

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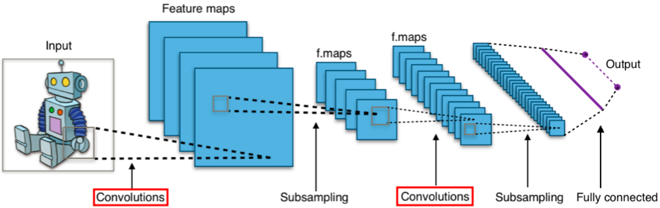

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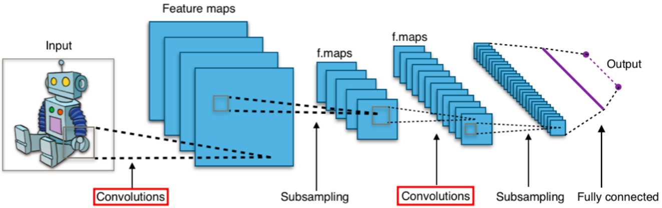

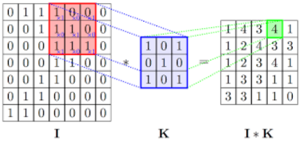

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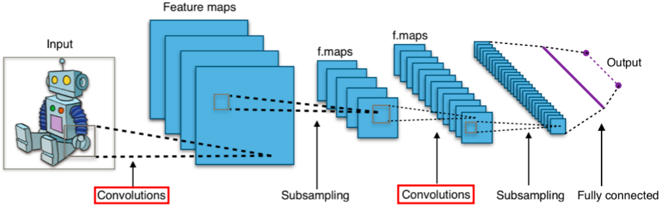

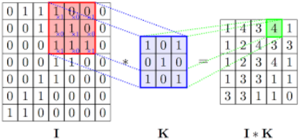

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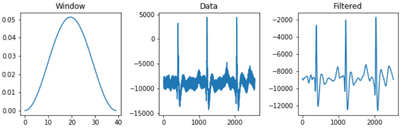
Convolution is the mathematical operation used to model **filtering**.

It is an essential concept in signal processing.



Convolution product between two signals x and y

The convolution product $x * y$ between two signals x and y is the function

$$\boxed{(x * y) : \tau \mapsto \int_{\mathbb{R}} x(t) y(\tau - t) dt}$$

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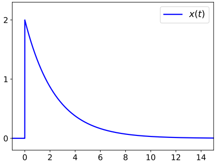
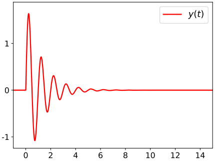
$\rightarrow$ Take two signals x and y.





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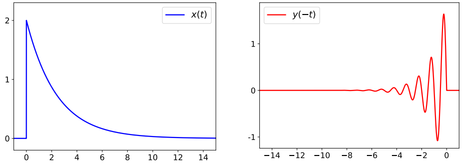
→ Transform y into $y^- : t \mapsto y(-t)$.



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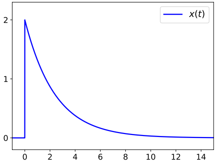
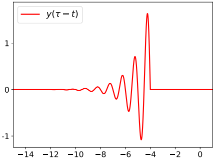
- $\rightarrow$ Transform y into $y^{-} : t \mapsto y(-t)$.
- $\rightarrow$ Shift y^{-} by a factor τ: $y(-t) \rightarrow y(\tau - t)$.





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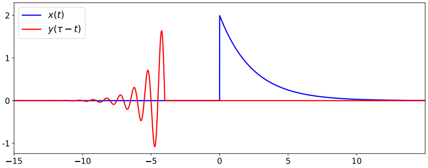

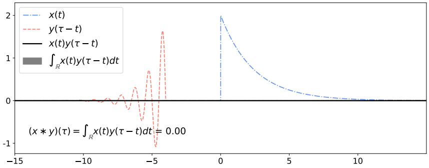

The convolution product
Definition

Convolution product between two signals x and y

The convolution product $x * y$ between two signals x and y is the function

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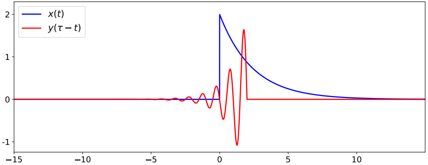

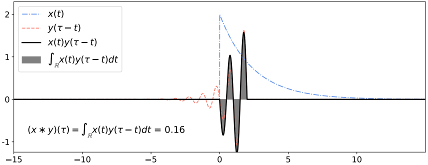

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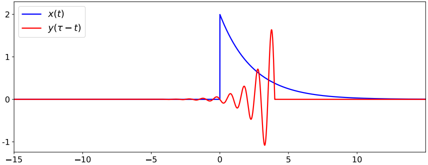

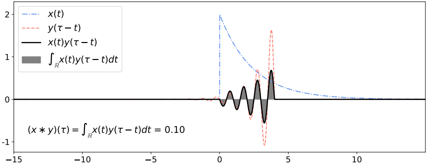

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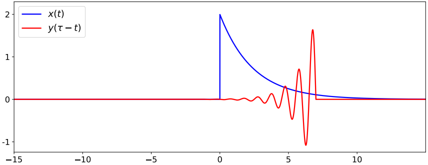

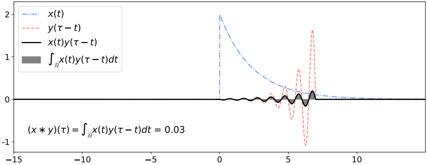

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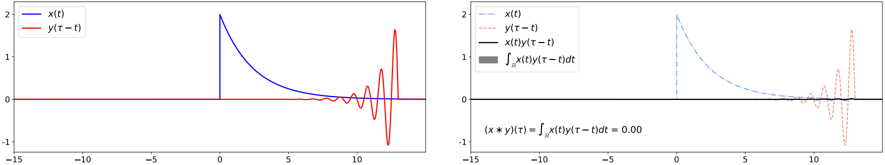

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y plays the role of a sliding window through which we observe x.

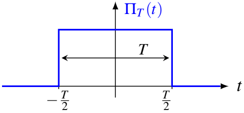
The convolution product can be seen as a **generalized sliding average** for functions.

Here are various useful properties of the convolution product $(x*y)(\tau) = \int_{\mathbb{R}} x(t)y(\tau-t)dt$

- Existence: If $x, y \in \mathcal{L}^1(\mathbb{R})$, then $(x*y)$ exists and $(x*y) \in \mathcal{L}^1(\mathbb{R})$
- Commutativity: The convolution product is commutative: $\boxed{(x*y) = (y*x)}$ jupyter
- Linearity: $x*(y+\lambda z) = x*y + \lambda(x*z)$
- Associativity: $x*(y*z) = (x*y)*z$
- Identity Element: There exists a function e such that $\forall x \in \mathcal{L}^1(\mathbb{R})$, $x*e = e*x = x$
- Symmetries: If x and y are even or odd signals
  - $\rightarrow$ $x*y$ is even if x and y have the same parity
  - $\rightarrow$ $x*y$ is odd if x and y have opposite parity
- Differentiation: If x and y are differentiable and $x, x', y, y' \in \mathcal{L}^1(\mathbb{R})$, then $x*y$ is differentiable and $\boxed{(x*y)' = (x'*y) = (x*y')}$.

Let's compute the convolution of the window function with itself

$$\Pi_T(t) = \begin{cases} 1 & t \in [-\frac{T}{2}, \frac{T}{2}] \\ 0 & \text{otherwise} \end{cases}$$



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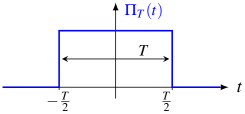

Since one window is "fixed" ($\Pi_T(t)$) and the other slides along the x-axis ($\Pi_T(\tau - t)$), there are thus two cases:

1. either $\Pi_T(t)$ and $\Pi_T(\tau - t)$ do not overlap, in which case $(\Pi_T * \Pi_T)(\tau) = 0$
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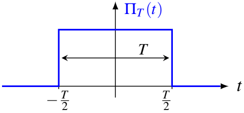

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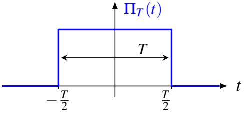

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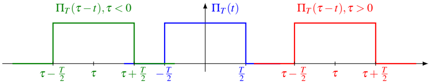

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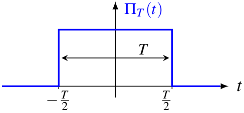

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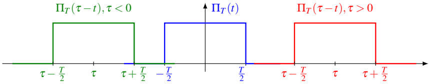

$\rightarrow$ if $\tau > 0$, no overlap between $\Pi_T(t)$ and $\Pi_T(\tau - t)$ if $\tau - \frac{T}{2} > \frac{T}{2} \Rightarrow \tau > T$

$\rightarrow$ if $\tau < 0$, no overlap between $\Pi_T(t)$ and $\Pi_T(\tau - t)$ if $\tau + \frac{T}{2} < -\frac{T}{2} \Rightarrow \tau < -T$

$$\Rightarrow \text{No overlap if } \boxed{|\tau| > T}$$

The convolution product

Example of computation 2/2

Second case: overlap

$\rightarrow$ maximum when $\tau = 0$ (both windows are completely superimposed on each other)
$\Rightarrow (\Pi_T * \Pi_T)(\tau = 0) = \int_{-T/2}^{T/2} \Pi_T(t)dt = \int_{-T/2}^{T/2} 1dt = T$

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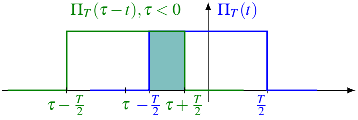

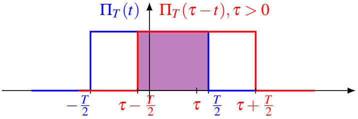

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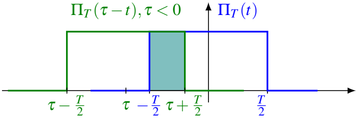

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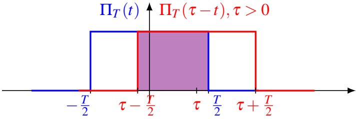

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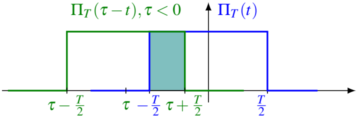

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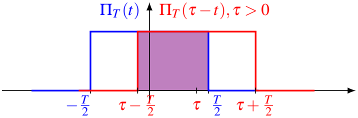

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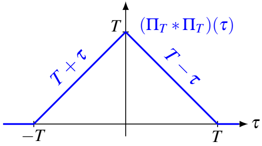
Eventually: $\boxed{(\Pi_T * \Pi_T)(\tau) = \begin{cases} 0 & \forall |\tau| > T \\ T & \text{if} \tau = 0 \\ T + \tau & \text{if } -T < \tau < 0 \\ T - \tau & \text{if } 0 < \tau < T \end{cases}}$



Link between correlation and convolution

What is the relationship between the cross-correlation Γ_{xy} and the convolution $(x * y)$?

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Let x, y be signals of finite energy and integrable: $x, y \in \underbrace{\mathcal{L}^2(\mathbb{R})}_{\text{existence of } \Gamma_{xy}} \cap \underbrace{\mathcal{L}^1(\mathbb{R})}_{\text{existence of } (x*y)}$

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If y is real-valued, $(\bar{y} = y)$ $\quad \Rightarrow \quad \boxed{\Gamma_{xy} = \left(x * y^-\right)}$

Furthermore, if y is even $(y^- = y)$ then $\Gamma_{xy} = (x * y) \rightarrow$ cross-correlation $\equiv$ convolution

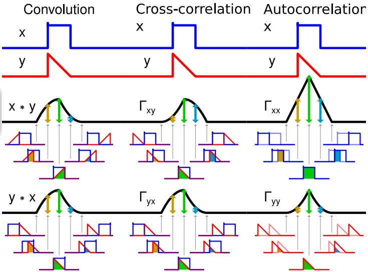
In summary

Autocorrelation $\boxed{\Gamma_{xx}(\tau) = \langle x(t), x(t-\tau)\rangle}$

- Similarity of x with itself
- The exact formula depends on the expression of $\langle\ ,\ \rangle$
- Γ_{xx} is even and maximal at 0
- $\Gamma_{xx}(0) = E_x$ (if $x \in \mathcal{L}^2(\mathbb{R})$) or P_x (if $x \in \mathcal{L}^{pm}(\mathbb{R})$)

Cross-correlation $\boxed{\Gamma_{xy}(\tau) = \langle x(t), y(t-\tau)\rangle}$

- Similarity between x and y
- The exact formula depends on the expression of $\langle\ ,\ \rangle$
- Generally not commutative: $\Gamma_{xy} \neq \Gamma_{yx}$
- Pattern recognition in signal processing



Convolution $\boxed{(x * y)(\tau) = \int_{\mathbb{R}} x(t)y(\tau - t)dt}$

- Sliding weighted average of one function seen through another
- Commutative $x * y = y * x$
- Filtering operation in signal processing
- Link with cross-correlation: $\Gamma_{xy} = (x * y^-)$