Logical Formalism

A Proof Pattern Compendium

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$\Longrightarrow$

Goal. Prove that $P \implies Q$ is true.

If P is true ...
... then Q is true.

$\wedge$ as a Conclusion

Goal. Prove that $P \implies (Q \wedge R)$ is true.

Suppose that P is true ...

Subgoal 1. Prove that Q is true.

Subgoal 2. Prove that R is true.

$\wedge$ as a Hypothesis

Goal. Prove that $(P \wedge Q) \implies R$ is true.

If P is true ...
... and Q is true ...
... then R is true.

$\vee$ as a Conclusion

Goal. Prove that $P \implies (Q \vee R)$ is true.

If P is true ...

Assume that Q is true.

Then obviously $Q \vee R$ is true. No further proof is needed.

Assume that Q is false.

If P is true and Q is false ...

... then R must be true.

$\vee$ as a Hypothesis

Goal. Prove that $(P \vee Q) \implies R$ is true.

Subgoal 1. Prove that $P \implies R$ is true.

If P is true ...
... then R is true.

Subgoal 2. Prove that $Q \implies R$ is true.

If Q is true ...
... then R is true.

$\iff$

Goal. Prove that $P \iff Q$ is true.

Subgoal 1. Prove that $P \implies Q$ is true.

If P is true ...
... then Q is true.

Subgoal 2. Prove that $Q \implies P$ is true.

If Q is true ...
... then P is true.

Proof by Contradiction

Goal. Prove that $P \implies Q$ is true.

If P is true ...

Assume that $\neg Q$ is true ...

... then prove something that contradicts P or a true property.

Therefore Q cannot be false, thus must be true.

Proof by Contraposition

Goal. Prove that $P \implies Q$ is true.

Equivalent goal. Prove that $\neg Q \implies \neg P$ is true.

If $\neg Q$ is true ...
... then $\neg P$ is true.

Case Disjunction

Goal. Prove that $P \implies Q$ is true.

Consider a proposition R.

Subgoal 1. Prove that $(P \wedge R) \implies Q$.

If P and R are true ...
... then Q is true.

Subgoal 2. Prove that $(P \wedge \neg R) \implies Q$.

If P is true and R is false ...
... then Q is true.

Simple Inclusion

Goal. Prove that $A \subseteq B$.

Let $x \in A$ ...
... then $x \in B$.

Double Inclusion

Goal. Prove that $A = B$.

Subgoal 1. Prove that $A \subseteq B$.

Let $x \in A$ ...
... then $x \in B$.

Subgoal 2. Prove that $B \subseteq A$.

Let $x \in B$ ...
... then $x \in A$.

Goal. Prove that $\forall x \in E, P(x)$.

Let $x \in E$ ...
...then $P(x)$ is true.

$\forall$ in a Conclusion

Goal. Prove that $P \implies (\forall x \in E, Q(x))$.

Assume that P is true.
Let $x \in E$ ...
... then $Q(x)$ is true.

$\forall$ in a Hypothesis

Goal. Prove that $(\forall x \in E, P(x)) \implies Q$.

Assume that $\forall x \in E, P(x)$ is true.
Consider one or more particular $x_0, x_1 \ldots \in E$.
Obviously, $P(x_0), P(x_1) \ldots$ are true ...
... then Q is true.

$\exists$

Goal. Prove that $\exists x \in E, P(x)$.

Exhibit a particular value $x_0 \in E$ ...
...then $P(x_0)$ is true.

$\exists$ in a Hypothesis

Goal. Prove that $(\exists x \in E, P(x)) \implies Q$.

Assume that $\exists x \in E, P(x)$ is true.
Consider a particular $x_0 \in E$ such that $P(x_0)$ is true ...
... then Q is true.

$\exists$ in a Conclusion

Goal. Prove that $P \implies (\exists x \in E, Q(x))$.

Assume that P is true.
Exhibit a particular value $x_0 \in E$ ...
... $Q(x_0)$ is proven true.

$\exists!$

Goal. Prove that $\exists! x \in E, P(x)$.

Subgoal 1. Prove that $\exists x \in E, P(x)$.

As described previously . . .

Subgoal 2. Prove that $\forall x_1, x_2 \in E, \boxed{P(x_1) \wedge P(x_2)} \implies \boxed{x_1 = x_2}$.

We can rely on the usual **implication pattern**, a **proof by contradiction**, or a **proof by contraposition** . . .

$\emptyset$

Goal. Prove that $E = \emptyset$.

Let us suppose that $E \neq \emptyset$.
Consider $\exists x \in E$ ...
...then prove something that is blatantly false.
Thus by **contradiction** $E = \emptyset$.

Functions

Goal. Prove that the relation f is a **function** $E \to F$.

Subgoal 1. Prove that an image always **exists**.

Let $x \in E$ ...
... then $\exists y \in F$, $(x, y) \in f$.

Subgoal 2. Prove that the image is **unique**.

Let $(x, y), (x, z) \in f$ ...
... then $y = z$.

Injectivity

Goal. Prove that a function $f : E \to F$ is injective.

Consider any $x, y \in \mathrm{Dom}(f)$.
Assume that $f(x) = f(y) \ldots$
$\ldots$ then $x = y$.

Surjectivity

Goal. Prove that a function $f : E \to F$ is surjective.

Consider any $y \in F$ ...
... then exhibits x such that $y = f(x)$.

Invertibility

Goal. Prove that $f : E \to F$ is a bijection.

Introduce a **candidate** $g \subseteq F \times E$ for the inverse function.
First **prove** that g is indeed a function $F \to E$. Then:

Subgoal 1. Prove that $f \circ g = \mathrm{Id}_F$.

Let $y \in F$ ...
...then $f(g(y)) = y$.

Subgoal 2. Prove that $g \circ f = \mathrm{Id}_E$.

Let $x \in E$ ...
...then $g(f(x)) = x$.

Simple Induction

Goal. Prove that $\forall n \geq n_0, P(n)$.

We are going to use a proof by **simple induction** on n.

Base case. Prove that $P(n_0)$ holds.

Usually straightforward . . .

Inductive case. Prove that $\forall n \geq n_0, \boxed{P(n)} \implies \boxed{P(n+1)}$.

Assume that $P(n)$ holds for some $n \geq n_0$ . . .
. . . then $P(n+1)$ holds.

Depth k Induction

Goal. Prove that $\forall n \geq n_0, P(n)$.

We are going to use a proof by **induction of depth** k.

Base case. Prove that $P(n_0), \ldots, P(n_0 + k - 1)$ hold.

Usually straightforward . . .

Inductive case. Prove that $\forall n \geq n_0, \boxed{P(n) \wedge \ldots \wedge P(n + k - 1)} \implies \boxed{P(n + k)}$.

Assume that $P(n), \ldots, P(n + k - 1)$ hold for some $n \geq n_0$ . . .
. . . then $P(n + k)$ holds.

Strong Induction

Goal. Prove that $\forall n \geq n_0, P(n)$.

We are going to use a proof by **strong induction** on n.

Base case. Prove that $P(n_0)$ holds.

Similar to simple inductive proofs ...

Inductive case. Prove that $\forall n \geq n_0, \boxed{\forall k \in \{n_0, \dots, n\}, P(k)} \implies \boxed{P(n+1)}$.

Assume that for some $n \geq n_0$, $\forall k \in \{n_0, \dots, n\}$, $P(k)$ holds ...
... then $P(n+1)$ holds.

Goal. Prove that E and F are equipotent.

Introduce a **candidate** relation f.

Subgoal 1. Prove that f is a well-defined **function** $E \to F$.

As described previously ...

Subgoal 2. Prove that f is a bijection.

Apply the definition. Prove that f is **injective** and **surjective**.

Find the inverse. Exhibit g such that $g = f^{-1}$.

Partitions

Goal. Prove that $\mathfrak{P}$ is a **partition** of E.

Subgoal 1. Prove that $\forall X \in \mathfrak{P}, X \subseteq E$ and $X \neq \emptyset$.

Let $X \in \mathfrak{P}$. Prove that $X \subseteq E$ and $X \neq \emptyset$.

Subgoal 2. Prove that $\forall X, Y \in \mathfrak{P}, \boxed{X \neq Y} \implies \boxed{X \cap Y = \emptyset}$.

By contradiction, consider $X, Y \in \mathfrak{P}$ such that $\exists x \in X \cap Y$. Then prove that $X = Y$.

Subgoal 3. Prove that $\forall x \in E, \exists X \in \mathfrak{P}, x \in X$.

Let $x \in E$. Exhibit $P \in \mathfrak{P}$ such that $x \in P$.

Analysis and Synthesis

Goal. Given $P(x)$ of variable x, find a set S such that $P(x) \iff x \in S$.

Analysis step. Assume that $P(x)$ is true ...

... from there prove that $x \in R$ for some set R.

Find a proper subset $S \subseteq R$.

Synthesis step. Assume that $x \in R$.

Prove that if $x \in R \setminus S$ then $\neg P(x)$.
Also prove that if $x \in S$ then $P(x)$.