Functions of several variables

2. Multiple Integrals

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ABSTRACT: The concepts and methods of multiple integrals are discussed in details. We begin with double integrals that we interpret as successive integrals and then introduce the notion of coordinates change to extent the application of double integrals. In a second part, we focus on triple integrals and volumes calculation. We use examples borrowed from geometry and physics throughout. Exercises for each section are provided and they are instrumental for a good understanding of the material.

Contents

1 Double Integrals

In these notes we shows how to integrate functions of two or more variables. First, a double integral is defined as the limit of sums. Second, we find a fast way to compute it. *The key idea is to replace a double integral by two ordinary « single » integrals.*

The double integral $\iint f(x,y)\,dy\,dx$ starts with $\int f(x,y)\,dy$. For each fixed x we integrate with respect to y. The answer depends on x. Now integrate again, this time with respect to x. The limits of integration need care and attention! Frequently those limits on y and x are the hardest part.

There is two main reasons why we use sums and limits to talk about multiple integrals. There has to be a definition and a computation to fall back on, when the single integrals are difficult or impossible. And also, this is important, multiple integrals represent more than area and volume. Those words and the pictures that go with them are the easiest to understand. You can almost see the volume as a « sum of slices » or a « double sum of thin sticks » . The true applications are mostly to other things, but the central idea is always the same: *Add up small pieces and take limits.*

We begin with the area of R and the volume of V, by double integrals.

1.1 A limit of sums

The graph of $z = f(x,y)$ is a curved surface above the xy plane. At the point (x,y) in the plane, the height of the surface is z. (The surface is above the xy plane only when z is positive. Volumes below the plane come with minus signs, like areas below the x axis.) We begin by choosing a positive function, for example $z = 1 + x^2 + y^2$. The base of our solid is a region $\mathbb{R}$ in the xy plane. That region will be chopped into small rectangles (sides Δx and Δy). When $\mathbb{R}$ itself is the rectangle $0 < x < 1$, $0 < y < 2$, the small pieces fit perfectly. For a triangle or a circle, the rectangles miss part of R. But they do fit in the limit, and any region with a piecewise smooth boundary will be acceptable.

Question 1. *Question What is the volume above R and below the graph of $z = f(x,y)$?*

Answer 1. It is a double integral, the integral of $f(x,y)$ over R. To reach it we begin with a sum, as suggested by **Figure 1**.

For single integrals, the interval $[a,b]$ is divided into short pieces of length Δx. For double integrals, R is divided into small rectangles of area $\Delta A = (\Delta x)(\Delta y)$. Above the i-th rectangle is a « thin stick » with small volume. That volume is the base area ΔA times the height above it, except that this height $z = f(x,y)$ varies from point to point. Therefore we select a point (x_i, y_i) in the i-th rectangle, and compute the volume from the height above that point :

$$\text{« volume of one stick »} = f(x_i,y_i)\,\Delta A \qquad \text{« volume of all sticks »} = \sum f(x_i,y_i)\,\Delta A$$

This is the crucial step for any integral, to see it as a sum of small pieces. Now take the limits $\Delta x \to 0$ and $\Delta y \to 0$. The height $z = f(x,y)$ is nearly constant over each rectangle (we assume that f is a continuous function). The sum approaches a limit, which depends only on the base R and the surface above it. The limit is the volume of the solid, and it is the *double integral* of $f(x,y)$ over R :

$$\iint_R f(x,y)\,dA = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \sum_i f(x_i,y_i)\,\Delta A \tag{1.1}$$

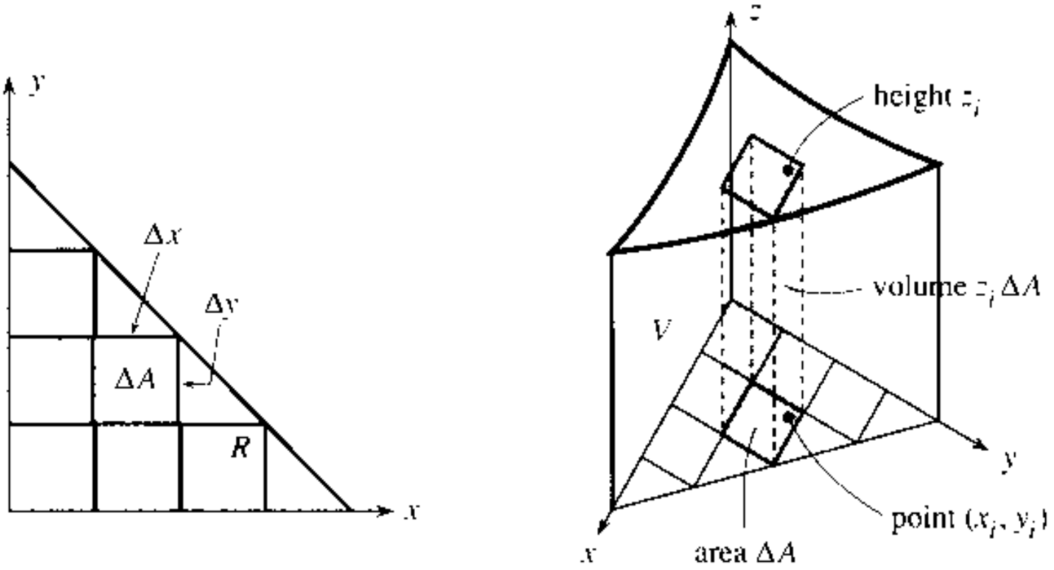

Figure 1: Base R cut into small pieces ΔA. Solid V cut into thin sticks $\Delta V = z\Delta A$.

To repeat : The limit is the same for all choices of the rectangles and the points (x_i, y_i). The rectangles will not fit exactly into R, if that base area is curved. The heights are not exact, if the surface $z = f(x, y)$ is also curved. But the errors on the sides and top, where the pieces don't fit and the heights are wrong, approach zero. Those errors are the volume of the « icing » around the solid, which gets thinner as $\Delta x \to 0$ and $\Delta y \to 0$. A careful proof takes more space than we are willing to give. But the properties of the integral need and deserve attention:

(1) ***Linearity*** :

$$\iint (f+g)\,dA = \iint f\,dA + \iint g\,dA$$

(2) ***Constant comes outside*** :

$$\iint c\,f(x,y)\,dA = c\iint f\,dA$$

(3) R ***splits into*** S ***and*** T ***(not overlapping)*** :

$$\iint f\,dA = \iint_S f\,dA + \iint_T f\,dA$$

In 1 the volume under $f+g$ has two parts. The « thin sticks » of height $f+g$ split into thin sticks under f and under g. In 2 the whole volume is stretched upward by c. In 3 the volumes are side by side. As with single integrals, these properties help in computations.

By writing dA, we allow shapes other than rectangles. Polar coordinates have an extra factor r in $dA = r\,dr\,d\theta$. By writing $dx\,dy$, we choose rectangular coordinates and prepare for the splitting that comes now.

1.2 Splitting a double integral into two single integrals

The double integral $\iint f(x,y)\,dydx$ will now be reduced to single integrals in y and then x (or vice versa, our first integral could equally well be $\int f(x,y)dx$). First came the area of a slice, which is a single integral. Then came a second integral to add up the slices. For solids formed by revolving a curve, all slices are circular disks, now we expect other shapes.

Figure 2 shows a slice of area $A(x)$. It cuts through the solid at a fixed value of x. The cut starts at $y = c$ on one side of R, and ends at $y = d$ on the other side. This particular example goes from $y = 0$ to $y = 2$ (R is a rectangle). The area of a slice is the y integral of $f(x,y)$. Remember that x is fixed and y goes from c to d :

$$A(x) = \textbf{area of slice} = \int_c^d f(x,y)dy \quad \text{(the answer is a function of } x). \tag{1.2}$$

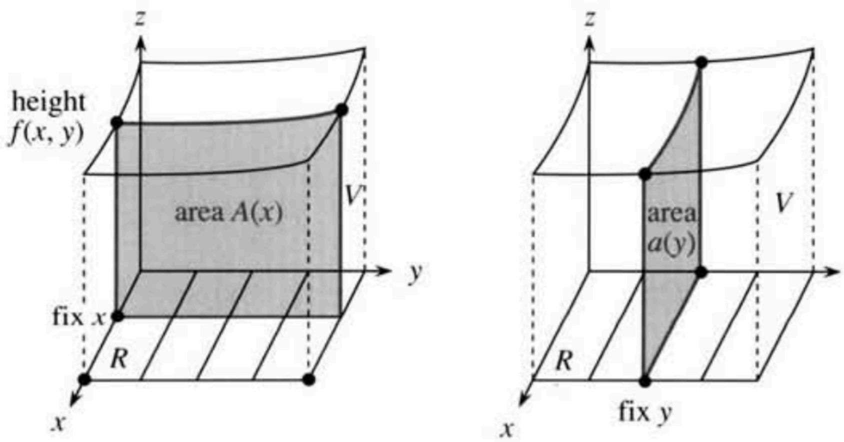

Figure 2: A slice of V at a fixed x has area $A(x) = \int f(x,y)\,dy$.

Example 1. *Consider*

$$A = \int_{y=0}^{2} (1 + x^2 + y^2)\,dy = \left[y + x^2y + \frac{y^3}{3}\right]_{y=0}^{y=2} = 2 + 2x^2 + \frac{8}{3} \tag{1.3}$$

*This is the reverse of a partial derivative! The integral of $x^2\,dy$, with x constant, is x^2y. This «partial integral» is actually called an **inner integral**. After substituting the limits $y = 2$ and $y = 0$ and subtracting, we have the area $A(x) = 2 + 2x^2 + 8/3$. Now the **outer integral** adds slices to find the volume $\int A(x)\,dx$. The answer is a* <u>*number*</u> *:*

$$\text{volume} = \int_{x=0}^{1} \left(y + x^2y + \frac{y^3}{3}\right) dx = \left[2x + \frac{2x^3}{3} + \frac{8}{3}x\right]_0^1 = 2 + \frac{2}{3} + \frac{8}{3} = \frac{16}{3} \tag{1.4}$$

To complete this example, check the volume when the x integral comes first :

$$\textit{inner integral} = \int_{x=0}^{1} (1 + x^2 + y^2)\,dx = \left[x + \frac{x^3}{3} + y^2x\right]_{x=0}^{x=1} = \frac{4}{3} + y^2 \tag{1.5}$$

$$\textit{outer integral} = \int_{y=0}^{2} \left(\frac{4}{3} + y^2\right) dy = \left[\frac{4}{3}y + \frac{y^3}{y}\right]_{y=0}^{y=2} = \frac{8}{3} + \frac{8}{3} = \frac{16}{3} \tag{1.6}$$

The fact that double integrals can be split into single integrals is *Fubini's Theorem.*

Fubini's Theorem

If $f(x,y)$ is continuous on the rectangle R, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \tag{1.7}$$

The inner integrals are the cross-sectional areas $A(x)$ and $a(y)$ of the slices. The outer integrals add up the volumes $A(x)\,dx$ and $a(y)\,dy$. Notice the reversing of limits.

Normally the brackets in (1.7) are omitted. When the y integral is first, dy is written inside dx. ***The limits on y are inside too***. It is recommended that you compute the inner integral on one line and the outer integral on a *separate line.*

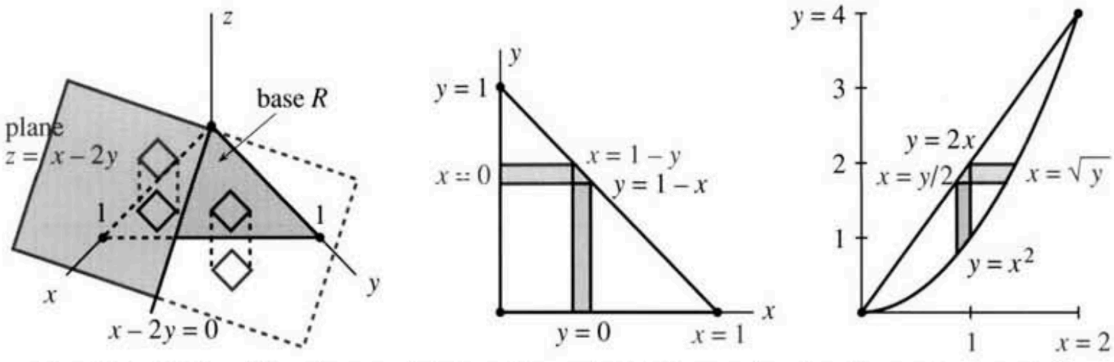

Figure 3: Thin sticks above and below (Example 2). Reversed order (Examples 3 and 4).

Example 2. *Find the volume below the plane $z = x - 2y$ and above the base triangle R.*

The triangle R has sides on the x and y axes and the line $x + y = 1$. The strips in the y direction have varying lengths (the strips in the x direction as well). This is the main point of the example, the base is not a rectangle. The upper limit on the inner integral changes as x changes. **The top of the triangle is at $y = 1 - x$.**

Figure 3 shows the strips. The region should always be drawn (except for rectangles). Without a figure the limits are hard to find. A sketch of R makes it easy :

$$y \text{ goes from } c = 0 \text{ to } d = 1 - x. \text{ Then } x \text{ goes from } a = 0 \text{ to } b = 1 \tag{1.8}$$

The inner integral has *variable limits* and the outer integral has *constant limits* :

$$\text{inner} \;:\; \int_{y=0}^{y=1-x} (x - 2y)\,dy = \left[xy - y^2\right]_{y=0}^{y=1-x} = x(1-x) - (1-x)^2 = -1 + 3x - 2x^2 \tag{1.9}$$

$$\text{outer} \;:\; \int_{x=0}^{1} (-1 + 3x - 2x^2)\,dx = \left[-x + \frac{3}{2}x^2 - \frac{2}{3}x^3\right]_0^1 = -1 + \frac{3}{2} - \frac{2}{3} = -\frac{1}{6} \tag{1.10}$$

The volume is negative. Most of the solid is below the xy plane. To check the answer, $-1/6$ do the x integral first : x goes from 0 to $1-y$. Then y goes from 0 to 1.

$$\text{inner} \ : \quad \int_{x=0}^{x=1-y} (x-2y)\ dx = \left[\frac{x^2}{2} - 2xy\right]_{x=0}^{x=1-y} = \frac{1}{2}(1-y)^2 - 2(1-y)y = \frac{1}{2} - 3y + \frac{5}{2}y^2 \tag{1.11}$$

$$\text{outer} \ : \quad \int_{y=0}^{1} \left(\frac{1}{2} - 3y - \frac{5}{2}y^2\right) dy = \left[\frac{1}{2}y + \frac{3}{2}y^2 - \frac{5}{6}y^3\right]_0^1 = \frac{1}{2} - \frac{3}{2} + \frac{5}{6} = -\frac{1}{6} \tag{1.12}$$

.

Same answer and therefore probably right. The next example computes $\iint 1\, dxdy =$ area of R.

Example 3. *The area of R is*

$$\int_{x=0}^{1}\int_{y=0}^{1-x} dy\, dx \qquad \textit{and also} \qquad \int_{y=0}^{1}\int_{x=0}^{1-y} dy\, dx \tag{1.13}$$

The first has vertical strips. The inner integral equals $1-x$. Then the outer integral (of $1-x$) has limits 0 and 1, and the area is $1/2$. It is like an indefinite integral inside a definite integral.

Example 4. *Reverse the order of integration in*

$$\int_{x=0}^{2}\int_{y=x^2}^{2x} x^3\, dy\, dx \tag{1.14}$$

Solution 4. Draw a figure! The inner integral goes from the parabola $y = x^2$ up to the straight line $y = 2^x$. This gives vertical strips. The strips sit side by side between $x = 0$ and $x = 2$. They stop where $2x$ equals x^2, and the line meets the parabola.

The problem is to put the x integral first. It goes along horizontal strips. On each line y - constant, we need the *entry value* of x and the *exit value* of x. From the figure, x goes from $y/2$ to $\sqrt{y}$. Those are the inner limits. Pay attention also to the outer limits, because they now apply to y. The region starts at $y = 0$ and ends at $y = 4$. No change in the integrand x^3 that is the height of the solid :

$$\int_{x=0}^{2}\int_{y=x^2}^{2x} x^3\, dy\, dx \qquad \text{is reversed to} \qquad \int_{y=0}^{4}\int_{x=y/2}^{\sqrt{y}} x^3\, dy\, dx \tag{1.15}$$

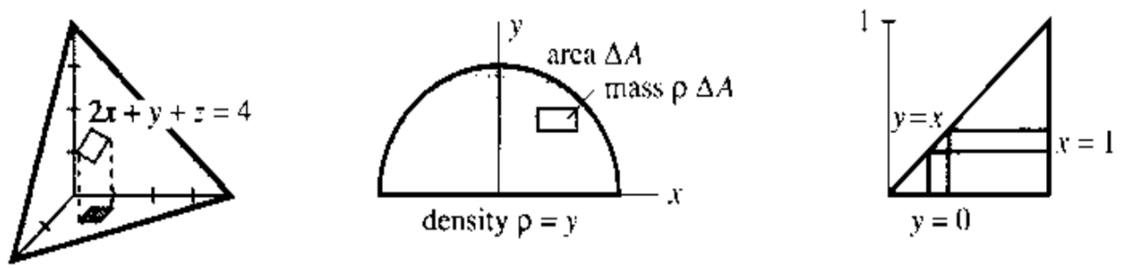

Figure 4: Tetrahedron in Example 5, semicircle in Example 6, triangle in Example 7.

Example 5. *Find the volume bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x + y + z = 4$.*

Solution 5. The solid is a *tetrahedron* (four sides). It goes from $z = 0$ (the xy plane) up to the plane $2x + y + z = 4$. On that plane $z = 4 - 2x - y$. This is the height function $f(x, y)$ to be integrated.

Figure 4 shows the base R. To find its sides, set $z = 0$. The sides of R are the lines $x = 0$ and $y = 0$ and $2x + y = 4$. Taking vertical strips, dy is inner :

$$\text{inner} \;:\; \int_{y=0}^{y=4-2x} (4 - 2x - y)\, dy = \left[(4 - 2x)y - \frac{1}{2}y^2\right]_0^{4-2x} = \frac{1}{2}(4 - 2x)^2 \tag{1.16}$$

$$\text{outer} \;:\; \int_{x=0}^{2} \frac{1}{2}(4 - 2x)^2\, dx = \left[-\frac{(4 - 2x)^3}{2 \cdot 3 \cdot 2}\right]_0^2 = \frac{4^3}{2 \cdot 3 \cdot 2} = \frac{16}{3} \tag{1.17}$$

.

Question 2. *What is the meaning of the inner integral $\frac{1}{2}(4 - 2x)^2$ (and also $\frac{16}{3}$) ?*

Answer 2. The first is $A(x)$, the area of the slice. $\frac{16}{3}$ is the solid volume.

Question 3. *What if the inner integral $\int f(x, y)\, dy$ has limits that depend on y ?*

Answer 3. It can't. Those limits must be wrong, we have to find them again.

Example 6. *Find the mass in a semicircle $0 \le y \le \sqrt{1 - x^2}$ if the density is $\rho = y$.*

This is a new application of double integrals. The total mass is a sum of small masses (ρ times ΔA) in rectangles of area ΔA. The rectangles do not fit perfectly inside the semicircle R, and the density is not constant in each rectangle, but those problems disappear in the limit. We are left with a double integral :

$$\textit{total mass } \boldsymbol{M} = \iint_R \rho\, dA = \iint_R \rho(x, y)\, dx\, dy \tag{1.18}$$

Set $\rho = y$. **Figure 4** shows the limits on x and y (try both $dy\, dx$ and $dx\, dy$) :

$$\boldsymbol{M} = \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} y\, dy\, dx \qquad \text{and also} \qquad \boldsymbol{M} = \int_{y=0}^{1} \int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dx\, dy \tag{1.19}$$

The first inner integral is $\frac{1}{2}y^2$. Substituting the limits gives $\frac{1}{2}(1-x^2)$. The outer integral of $\frac{1}{2}(1-x^2)$ yields the total mass $\boldsymbol{M} = 2/3$.

The second inner integral is xy. Substituting the limits on x gives ______. Then the outer integral is $-\frac{2}{3}(1 - y^2)^{3/2}$. Substituting $y = 1$ and $y = 0$ yields $\boldsymbol{M} =$ ______.

Remark 1. *This same calculation also produces the* ***moment***[1] *around the x axis, when the density is $\rho = 1$. The factor y is the distance to the x axis.* ***The moment is $\boldsymbol{M}_x$*** $= \iint y\, dA = 2/3$*. Dividing by the area of the semicircle (which is $\pi/2$) locates the centroid : $\overline{x} = 0$ by symmetry and*

$$\overline{y} = \textit{height of centroid} = \frac{\textit{moment}}{\textit{area}} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \tag{1.20}$$

This is the « average height » of points inside the semicircle.

[1] Moment is mass times a distance. This quantity is instrumental in determining the position of ***the center of mass*** of a object of density ρ.

Example 7. *Integrate*

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} \cos x^2 \, dx \, dy \tag{1.21}$$

avoiding the impossible $\int \cos x^2 \, dx$.

This is a famous example where reversing the order makes the calculation possible. The base R is the triangle in **Figure 4** (note that x goes from y to 1). ***In the opposite order*** y ***goes from*** 0 ***to*** x. Then $\int \cos x^2 \, dy = x \cos x^2$ contains the factor x that we need :

$$\text{outer integral} \; : \quad \int_0^1 x \cos x^2 \, dx = \left[\frac{1}{2} \sin x^2 \right]_0^1 = \frac{1}{2} \sin 1 \tag{1.22}$$

2 Change to better coordinates

When evaluating double integrals, one quickly encounter the necessity to change variables. Many regions simply do not fit with the x and y axes. Two examples are in **Figure 5**, a tilted square and a ring. Those are classic and useful shapes, in the right coordinates.

We have to be able to answer basic questions like these :

$$\text{Find the area} \iint dA \text{ and moment} \iint x \, dA \text{ and moment of inertia} \iint x^2 \, dA.$$

The problem is *what is* dA ? We are leaving the xy variables where $dA = dx\,dy$. The reason for changing is that the limits of integration in the y direction are needlessly complicated. They are notoriously difficult to determine, even from a figure. For every x we would need the entry point P of the line $x =$ constant, and the exit point Q. The heights of P and Q are the limits on $\int dy$, the inner integral. The geometry of the square and ring are totally missed, if we stick rigidly to x and y.

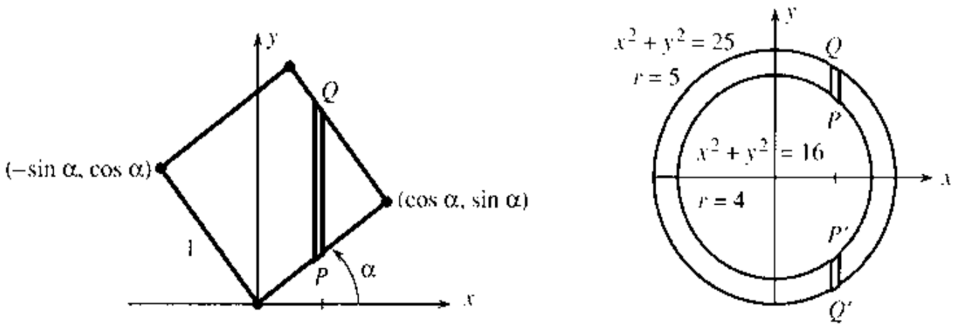

Figure 5: Unit square turned through angle α. Ring with radii 4 and 5.

Which coordinates are better ? One would easily agreed that the area of the tilted square is 1. In that case we just have to « turn »it and the area is obvious. But we have to realise that we may not know the moment or the center of gravity or the moment of inertia. So we actually have to do the turning.

The new coordinates u and v are in **Figure 6**. The limits of integration on v are 0 and 1. So are the limits on u. *But when you change variables, you do not change the limits only.* Two other changes come with new variables :

(1) The small area $dA = dx\,dy$ becomes $dA = du\,dv$.

(2) The integral of x becomes the integral of ______

Substituting $u = \sqrt{x}$ in a single integral. We make the same changes. Limits $x = 0$ and $x = 4$ become $u = 0$ and $u = 2$. Since x is u^2, dx is $2u\,du$. The purpose of the change is to find an antiderivative. For double integrals, the usual purpose is to improve the limits - but we have to accept the whole package.

To turn the square, there are formulas connecting x and y to u and v. The geometry is clear - ***rotate axes by*** α - but it has to be converted into algebra :

$$\begin{cases} u = x\cos\alpha + y\sin\alpha \\ v = -x\sin\alpha + y\cos\alpha \end{cases} \quad \text{and in reverse} \quad \begin{cases} x = u\cos\alpha - v\sin\alpha \\ y = u\sin\alpha + v\cos\alpha \end{cases} \tag{2.1}$$

Figure 6 shows the rotation. As points move, the whole square turns. A good way to remember equation (2.1) is to follow the corners as they become $(1,0)$ and $(0,1)$. The change from $\iint x\,dA$ to $\iint$ ______$du\,dr$ is partly decided by equation (2.1). It gives x as a function of u and r. We also need dA. For a pure rotation the first guess is correct : ***The area*** $dx\,dy$ ***equals the area*** $du\,dv$***. For most changes of variable this is false***. The general formula for dA comes after the examples.

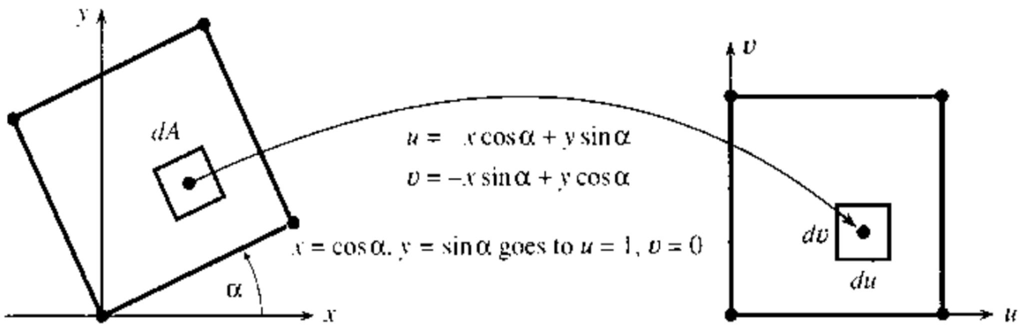

Figure 6: Change of coordinates - axes turned by α. For notation dA is $du\,dv/$

Example 8. *Find* $\iint dA$ *and* $\iint x\,dA$ *and* x *and also* $\iint x^2\,dA$ *for the tilted square.*

Solution 8. The area of the square is $\int_0^1 \int_0^1 du\,dv = 1$. Notice the good limits. Then

$$\iint x\,dA = \int_0^1 \int_0^1 (u\cos\alpha - v\sin\alpha)\,dv\,du = \frac{1}{2}\cos\alpha - \frac{1}{2}\sin\alpha \tag{2.2}$$

This is the *moment around the* y *axis*. The factors $\frac{1}{2}$ come from $\frac{1}{2}u^2$ and $\frac{1}{2}v^2$. The x coordinate of the center of gravity is

$$\overline{x} = \frac{\iint x\,dA}{\iint dA} = \frac{\frac{1}{2}\cos\alpha - \frac{1}{2}\sin\alpha}{1} \tag{2.3}$$

Similarly the integral of y leads to $\overline{y}$. The answer is no mystery - the point $(\overline{x}, \overline{y})$ is at the center of the square! Substituting $x = u\cos\alpha - v\sin\alpha$ made $x\,dA$ look worse, but the limits 0 and 1 are much better.

The moment of inertia I_y around the y axis is also simplified :

$$\iint x^2 \, dA = \int_0^1 \int_0^1 (u \cos\alpha - v \sin\alpha)^2 \, dv \, du = \frac{\cos^2\alpha}{3} - \frac{\cos\alpha \sin\alpha}{2} + \frac{\sin^2\alpha}{3} \tag{2.4}$$

You know this next fact but we write it anyway : *The answers do not contain u or v.* Those are dummy variables like x and y. The answers do contain α because the square has turned (the area is fixed at 1). The moment of inertia $I_x = \iint y^2 \, dA$ is the same as equation (2.4) but with all plus signs.

Question 4. *The sum $I_x + I_y$, simplifies to $2/3$ (a constant). Why no dependence on α?*

Answer 4. $I_x + I_y$ equals I_0. This moment of inertia around $(0,0)$ is unchanged by rotation. We are turning the square around one of its corners.

2.1 Change to polar coordinates

The next change is to r and θ. A small area becomes $dA = r \, dr \, d\theta$ (definitely not $dr \, d\theta$). Area always comes from multiplying two lengths, and $d\theta$ is not a length. **Figure 7** shows the crucial region - a «polar rectangle» cut out by rays and circles. Its area ΔA is found in two ways, both leading to $r \, dr \, d\theta$:

(***Approximate***) The straight sides have length Δr. The circular arcs are close $r \, \Delta\theta$. The angles are $90°$. So ΔA is close to $(\Delta r)(r \, \Delta\theta)$.

(***Exact***) A wedge has area $\frac{1}{2} r^2 \, \Delta\theta$. The difference between wedges is ΔA :

$$\Delta A = \frac{1}{2}\left(r + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r - \frac{\Delta r}{2}\right)^2 \Delta\theta = r \, \Delta r \, \Delta\theta \tag{2.5}$$

The exact method places r dead center (see figure). The approximation says : Forget the change in $r \, \Delta\theta$ as you move outward. Keep only the first-order terms.

A third method exists which requires no picture and no geometry. We can always use a third method! The change of variables $x = r\cos\theta, y = r\sin\theta$ will go into a general formula for dA, and out will come the area $r \, dr \, d\theta$.

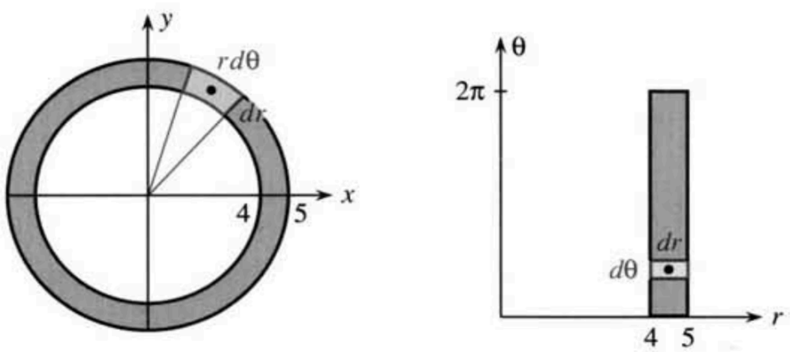

Figure 7: Ring and polar rectangle in xy and $r\theta$, with stretching factor $r = 4.5$

Example 9. *Find the area and center of gravity of the ring. Also find $\iint x^2 \, dA$.*

Solution 9. The limits on r are 4 and 5. The limits on θ are 0 and 2π. *Polar coordinates are perfect for a ring.* Compared with limits like $x = \sqrt{25 - y^2}$, the change to $r\,dr\,d\theta$ is a small price to pay :

$$\text{area} = \int_0^{2\pi} \int_4^5 r\,dr\,d\theta = 2\pi \left[\frac{1}{2}r^2\right]_4^5 = \pi 5^2 - \pi 4^2 = 9\pi \tag{2.6}$$

The θ integral is 2π (full circle). Actually the ring is a giant polar rectangle. We could have used the exact formula $r\,\Delta r\,\Delta\theta$, with $\Delta\theta = 2\pi$ and $\Delta r = 5 - 4$. When the radius r is centered at 4.5, the product $r\,\Delta r\,\Delta\theta$ is $(4.5)(1)(2\pi) = 9\pi$ as above.

Since the ring is symmetric around $(0, 0)$, the integral of $x\,dA$ must be zero :

$$\iint_R x\,dA = \int_0^{2\pi} \int_4^5 (r\cos\theta)\,r\,dr\,d\theta = \left[\frac{1}{3}r^3\right]_4^5 [\sin\theta]_0^{2\pi} = 0 \tag{2.7}$$

Notice $r\cos\theta$ from x, the other r is from dA. The moment of inertia is

$$\iint_R x^2\,dA = \int_0^{2\pi} \int_4^5 r^2\cos^2\theta\,r\,dr\,d\theta = \left[\frac{1}{4}r^4\right]_4^5 \int_0^{2\pi} \cos^2\theta\,d\theta = \frac{1}{4}(5^4 - 4^4)\pi \tag{2.8}$$

This θ integral is π not 2π, because the average of $\cos^2\theta$ is 4 not 1.

For reference here are the moments of inertia when the density is $\rho(x, y)$:

$$I_y = \iint x^2\rho\,dA \qquad I_x = \iint y^2\rho\,dA \qquad I_0 = \iint r^2\rho\,dA = \textbf{\textit{polar moment}} = I_x + I_y \tag{2.9}$$

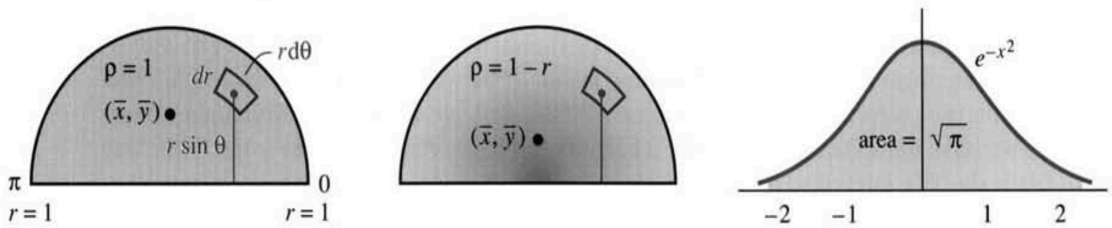

Figure 8: (Left and middle) Semicircles with density piled above them. (Right) Bell-shaped curve.

Example 10. *Find masses and moments for semicircular plates : $\rho = 1$ and $\rho = 1 - r$.*

Solution 10. The semicircles in **Figure 8** have $\rho = 1$. The angle goes from 0 to π (the upper half-circle). Polar coordinates are best. ***The mass is the integral of the density*** ρ :

$$\boldsymbol{M} = \int_0^{\pi} \int_0^1 r\,dr\,d\theta = \frac{1}{2}\pi \qquad \text{and} \qquad \boldsymbol{M} = \int_0^{\pi} \int_0^1 (1 - r)r\,dr\,d\theta = \frac{1}{6}\pi \tag{2.10}$$

The first mass $\pi/2$ equals the area (because $\rho = 1$). The second mass $\pi/6$ is smaller (because $\rho < 1$). Integrating $\rho = 1$ is the same as finding a volume when the height is $z = 1$ (part of a cylinder). Integrating $\rho = 1 - r$ is the same as finding a volume when the height is $z = 1 - r$ (part of a cone). Volumes of cones have the extra factor 1/3.

The center of gravity involves the moment $\boldsymbol{M_x} = \iint y\,\rho\,dA$. ***The distance from the*** x ***axis is*** y***, the mass of a small piece is*** $\rho\,dA$***, integrate to add mass times distance***. Polar coordinates are still best, with $y = r\sin\theta$. Again $\rho = 1$ and $\rho = 1 - r$:

$$\iint y\,dA = \int_0^{\pi}\int_0^1 r\,\sin\theta\,r\,dr\,d\theta = \frac{2}{3} \qquad \iint y(1-r)\,dA = \int_0^{\pi}\int_0^1 r\,\sin\theta\,(1-r)r\,dr\,d\theta = \frac{1}{6}$$

The height of the center of gravity is $\overline{y} = \boldsymbol{M_x/M}$ = ***moment divided by mass*** :

$$\overline{y} = \frac{2/3}{\pi/2} = \frac{4}{3\pi} \quad \text{when } \rho = 1 \qquad \overline{y} = \frac{1/6}{\pi/6} = \frac{1}{\pi} \quad \text{when } \rho = 1 - r \tag{2.11}$$

Question 5. *Compare* $\overline{y}$ *for* $\rho = 1$ *and* $\rho =$ *other positive constants and* $\rho = 1 - r$*.*

Answer 5. Any constant ρ gives $\overline{y} = \frac{4}{3\pi}$. Since $1 - r$ is dense at $r = 0$, $\overline{y}$ drops to $1/\pi$.

Question 6. *How is* $\overline{y} = \frac{4}{3\pi}$ *related to the « average » of* y *in the semicircle?*

Answer 6. They are identical. This is the point of $\overline{y}$. Divide the integral by the area :

$$\textbf{\textit{The average value of a function is}} \quad \frac{\iint f(x,y)\,dA}{\iint dA} \tag{2.12}$$

The integral of f is divided by the integral of 1 (the area). In one dimension $\int_a^b v(x)\,dx$ was divided by $\int_a^b 1\,dx$ (the length $b - a$). That gave the average value of $v(x)$. Equation (2.12) is the same idea for $f(x,y)$.

Example 11. *Compute*

$$A = \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad \textit{from} \qquad A^2 = \int_{-\infty}^{+\infty} e^{-x^2}\,dx \int_{-\infty}^{+\infty} e^{-y^2}\,dy \tag{2.13}$$

A is the area under a « *Bell-shaped curve* » - see **Figure 8**. This is the most important definite integral in the study of probability. It is difficult because a factor $2x$ is not present. Integrating $2x\,e^{-x^2}$ gives $-e^{-x^2}$, but integrating e^{-x^2} is impossible - except approximately by a computer. How can we hope to show that A is exactly $\sqrt{\pi}$? The trick is to go from an area integral A to a volume integral A^2. This is unusual (and hard to like), but the end justifies the means :

$$A^2 = \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} e^{-x^2}\,e^{-y^2}\,dy\,dx = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta \tag{2.14}$$

The double integrals cover the whole plane. The r^2 comes from x^2+y^2, and the key factor r appears in polar coordinates. It is now possible to substitute $u = r^2$. The r integral is $\frac{1}{2}\int_0^{\infty} e^{-u}\,du = 1/2$. The θ integral is 2π. The double integral is $(1/2)(2\pi)$. Therefore $A^2 = \pi$ and the single integral is $A = \sqrt{\pi}$.

Example 12. *Apply Example 11 to the « normal distribution »* $p(x) = e^{-x^2/2}/\sqrt{2\pi}$*.*

In a probability, the importance of this particular $p(x)$ is emphasized. But we are not always able to verify that $\int p(x)\,dx = 1$. Now we can :

$$x = \sqrt{2}y \quad \text{yields} \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = 1 \tag{2.15}$$

Question 7. *Why include the 2's in* $p(x)$ *? The integral of* $e^{-x^2}/\sqrt{\pi}$ *also equals 1.*

Answer 7. With the 2's the « **variance** » is $\int x^2\,p(x)\,dx = 1$. This is a convenient number.

2.2 Change to other coordinates

A third method was promised, to find $r\,dr\,d\theta$ without a picture and without geometry. The method works directly from $x = r\cos\theta$ and $y = r\sin\theta$. It also finds the 1 in $du\,dv$, after a rotation of axes. Most important, this new method finds the factor J in the area $dA = J\,du\,dv$, for any change of variables. The change is from xy to uv.

For single integrals, the « stretching factor » J between the original dx and the new du is (not surprisingly) the ratio dx/du. Where we have dx, we write $(dx/du)du$. Where we have $(du/dx)dx$, we write du. That was the idea of substitutions - the main way to simplify integrals.

For double integrals the stretching factor appears in the area : $dx\,dy$ becomes $|J|\,du\,dv$. ***The old and new variables are related by*** $x = x(u, v)$ ***and*** $y = y(u, v)$. The point with coordinates u and v comes from the point with coordinates x and y. A whole region S, full of points in the uv plane, comes from the region R full of corresponding points in the xy plane. A small piece with area $|J|\,du\,dv$ comes from a small piece with area $dx\,dy$. *The formula for J is a two-dimensional version of dx/du.*

Jacobian determinant

The stretching factor for area is the 2 by 2 *Jacobian determinant* $J(u, v)$:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \tag{2.16}$$

An integral over R in the xy plane becomes an integral over S in the uv plane :

$$\iint_R f(x, y)\,dx\,dy = \iint f(x(u, v), y(u, v))\,|J|\,du\,dv \tag{2.17}$$

The determinant J is often written $\partial(x, y)/\partial(u, v)$, as a reminder that this stretching factor is like dx/du. *We require $J \neq 0$*. That keeps the stretching and shrinking under control.

You naturally ask : Why take the absolute value $|J|$ in equation (2.17) ? Good question - it wasn't done for single integrals. The reason is in the limits of integration. The single integral $\int_0^1 dx$ is $\int_0^{-1}(-du)$ after changing x to $-u$. We keep the minus sign and allow single integrals to run backward. Double integrals could too, but normally they go left to right and down to up. We use the absolute value $|J|$ and run forward.

Example 13. *Polar coordinates have $x = u\cos v = r\cos\theta$ and $y = u\sin v = r\sin\theta$. With no geometry :*

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{2.18}$$

Example 14. *Find J for the linear change to $x = au + by$ and $y = cu + dv$. We have the ordinary determinant :*

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{2.19}$$

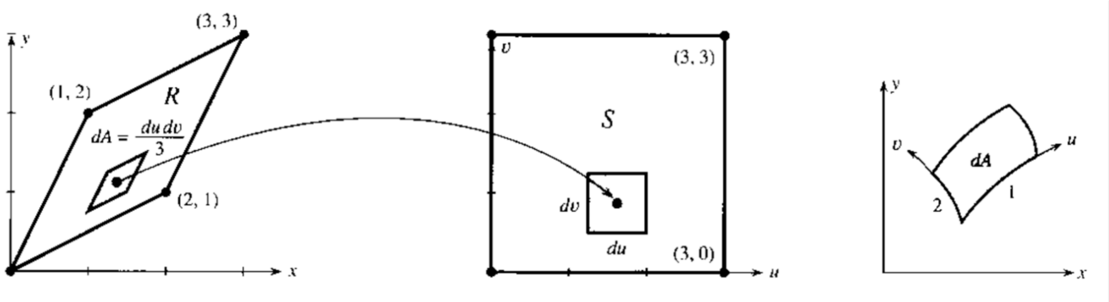

Figure 9: Change from xy to uv has $J = 1/3$ (left and middle). Curved areas are also $dA = |J|\,du\,dv$ (on the right).

Why make this simple change, in which a, b, c, d are all constant ? It straightens parallelograms into squares (and rotates those squares). **Figure 9** (left and middle) is typical.

Common sense indicated $J = 1$ for pure rotation - no change in area. Now $J = 1$ comes from equations (2.1) and (2.19), because $ad - bc$ is $\cos^2\alpha + \sin^2\alpha$.

In practice, xy rectangles generally go into uv rectangles. The sides can be curved (as in polar rectangles) but the angles are often 90°. The change is « orthogonal ». The next example has angles that are not 90° and J still gives the answer.

Example 15. *Find the area of R in Figure* **Figure 9** *(left). Also compute $\iint_R e^x\,dx\,dy$.*

Solution 15. The figure shows $u = 2x + y$ and $v = 2y + x$ hence $x = \frac{2}{3}u - \frac{1}{3}v$ and $y = \frac{2}{3}v - \frac{1}{3}u$. The determinant is then

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3} \tag{2.20}$$

The area of the xy parallelogram becomes an integral over the uv square :

$$\iint_R dx\,dy = \iint_S |J|\,du\,dv = \int_0^3 \int_0^3 \frac{1}{3}\,du\,dv = \frac{1}{3}\cdot 3\cdot 3 = 3 \tag{2.21}$$

The square has area 9, the parallelogram has area 3. We do not know if $= 1/3$ is a stretching factor or a shrinking factor. The other integral $\iint e^x\,dx\,dy$ is

$$\int_0^3 \int_0^3 e^{2u/3 - v/3}\frac{1}{3}\,du\,dv = \frac{1}{3}\left[\frac{3}{2}e^{2u/3}\right]_0^3 \left[-3e^{-v/3}\right]_0^3 = \frac{3}{2}\left(e^2 - 1\right)\left(1 - e^{-1}\right) \tag{2.22}$$

The main point is that the change to u and v makes the limits easy (just 0 and 3).

Why is the stretching factor J a determinant ? With straight sides, this goes back to vectors. *The area of a parallelogram is a determinant.* Here the sides are curved, but that only produces $(du)^2$ and $(dv)^2$, which we ignore.

A change du gives one side of **Figure 9** (right) - it is $(\frac{\partial x}{\partial u}\boldsymbol{i} + \frac{\partial y}{\partial u}\boldsymbol{j})du$. Side 2 $(\frac{\partial x}{\partial v}\boldsymbol{i} + \frac{\partial y}{\partial v}\boldsymbol{j})du$. The curving comes from second derivatives. The area (the cross product of the sides) is $|J|\,du\,dv$.

Final remark We can easily look at the change in the reverse direction. Now the rectangle is in xy and the parallelogram is in uv. In all formulas, exchange x for u and y for v :

$$\textbf{\textit{new}}\ J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\textbf{\textit{old}}\ J} \tag{2.23}$$

This is exactly like $du/dx = 1/(dx/du)$. It is the derivative of the inverse function. The product of slopes is 1 - stretch out, shrink back. From xy to uv we have 2 by 2 matrices, and the identity matrix I takes the place of 1 :

$$\frac{dx}{du}\frac{du}{dx} = 1 \quad \text{becomes} \quad \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.24}$$

The first row times the first column is $\frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial x}{\partial x} = 1$. The first row times the second column is $\frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial x}{\partial y} = 0$. ***The matrices are inverses of each other***. The determinants of a matrix and its inverse obey our rule : old J times new $J = 1$. Those J's cannot be zero, just as $\frac{dx}{du}$ and $\frac{du}{dx}$ were not zero (inverse functions increase steadily or decrease steadily). In two dimensions, an area $dx\,dy$ goes to $J\,du\,dv$ and comes back to $dx\,dy$.

3 Triple integrals

At this point you can guess what triple integrals are like. Instead of a small interval or a small rectangle, there is a small box. Instead of length dx or area $dx\,dy$, the box has volume $dV = dx\,dy\,dz$. That is length times width times height. The goal is to put small boxes together (by integration). The main problem will be to discover the correct limits on x, y, z.

We could imagine more and more complicated regions in three-dimensional space. But you cannot, most of the time, see the method clearly without seeing the region clearly. In practice six shapes are the most important :

box *prism* *cylinder* *cone* *tetrahedron* *sphere*

The box is easiest and the sphere may be the hardest (but no problem in spherical coordinates). Circular cylinders and cones fall in the middle, where xyz coordinates are possible but $r\theta z$ are the best. We start with the box and prism and xyz.

Example 16. *By triple integrals find the volume of a box and a prism (**Figure 10**).*

$$\iiint_{box} dV = \int_{z=0}^{1}\int_{y=0}^{3}\int_{x=0}^{2} dx\,dy\,dz \qquad \text{and} \qquad \iiint_{prism} dV = \int_{z=0}^{1}\int_{y=0}^{3-3z}\int_{x=0}^{2} dx\,dy\,dz$$

The inner integral for both is $\int dx = 2$. Lines in the x direction have length 2, cutting through the box and the prism. The middle integrals show the limits on y (since dy comes second) :

$$\int_{y=0}^{3} 2\,dy = 6 \qquad \text{and} \qquad \int_{y=0}^{3-3z} 2\,dy = 6 - 6z \tag{3.1}$$

After two integrations these are ***areas***. The first area 6 is for a plane section through the box. The second area $6 - 6z$ is cut through the prism. The shaded rectangle goes from $y = 0$ to $y = 3 - 3z$ - we needed and used the equation $y + 3z = 3$ for the boundary of the prism. *At this point z is still constant*! But the area depends on z, because the prism gets thinner going upwards. The base area is $6 - 6z = 6$, the top area is $6 - 6z = 0$.

The outer integral multiplies those areas by dz, to give the volume of slices. They are horizontal slices because z came last. Integration adds up the slices to find the total volume :

$$\text{box volume} = \int_{z=0}^{3} 6\,dz = 6 \qquad \text{prism volume} = \int_{z=0}^{3} (6-6z)dz = [6z - 3^2]_0^1 = 3 \tag{3.2}$$

The box volume $2 \cdot 3 \cdot 1$ did not need calculus. The prism is half of the box, so its volume was sure to be 3 - but it is satisfying to see how $6z - 3z^2$ gives the answer. Our purpose is to see how a triple integral works.

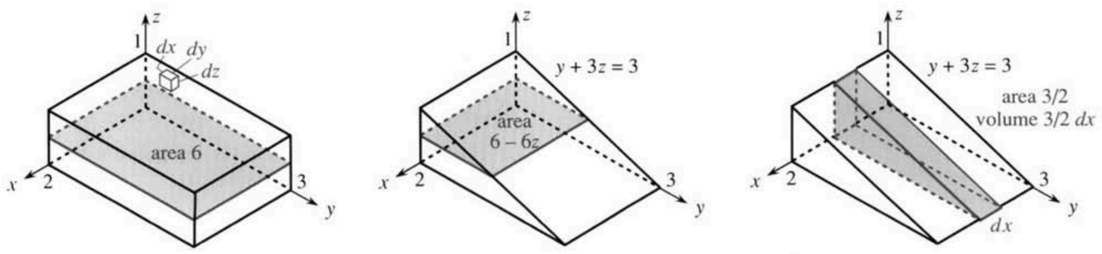

Figure 10: Box with 2,3,1. The prism is half of the box : volume $\int(6-6z)dz$ or $\int(3/2)dx$.

Question 8. *Find the prism volume in the order $dz\,dy\,dx$ (**six orders are possible**).*

Answer 8.

$$\int_0^2 \int_0^3 \int_0^{(3-y)/3} dz\,dy\,dx = \int_0^2 \int_0^3 \left(\frac{3-y}{3}\right) dy\,dx = \int_0^2 \frac{3}{2}\,dx = 3 \tag{3.3}$$

To find those limits on the z integral, follow a line in the z direction. It enters the prism at $z = 0$ and exits at the sloping face $y + 3z = 3$. That gives the upper limit $z = (3 - y)/3$. It is the height of a thin stick as in section . This section writes out $\int dz$ for the height, but a quicker solution starts at the double integral.

What is the number 3/2 in the last integral ? It is the ***area of a vertical slice***, cut by a plane $x =$ constant. The outer integral adds up slices.

$$\iiint f(x,y,z)\,dV \quad \textit{is computed from three single integrals} \quad \int \left[\int \left[\int f\,dx\right] dy\right] dz \tag{3.4}$$

That step cannot be taken in silence - some basic calculus is involved. The triple integral is the limit of $\sum f_i \Delta V$, a sum over small boxes of volume ΔV. Here f_i, is any value $f(x, y, z)$ in the i-th box. (In the limit, the boxes fit a curved region.) Now take those boxes in a certain order. Put them into lines in the x direction and put the lines of boxes into planes. The lines lead to the inner x integral, whose answer depends on y and z. The y integral combines the lines into planes. Finally the outer integral accounts for all planes and all boxes.

Example 17 is important because it displays more possibilities than a box or prism.

Example 17. *Find the volume of a tetrahedron (4-sided pyramid). Locate the center of mass whose position is $(\overline{x}, \overline{y}, \overline{z})$.*

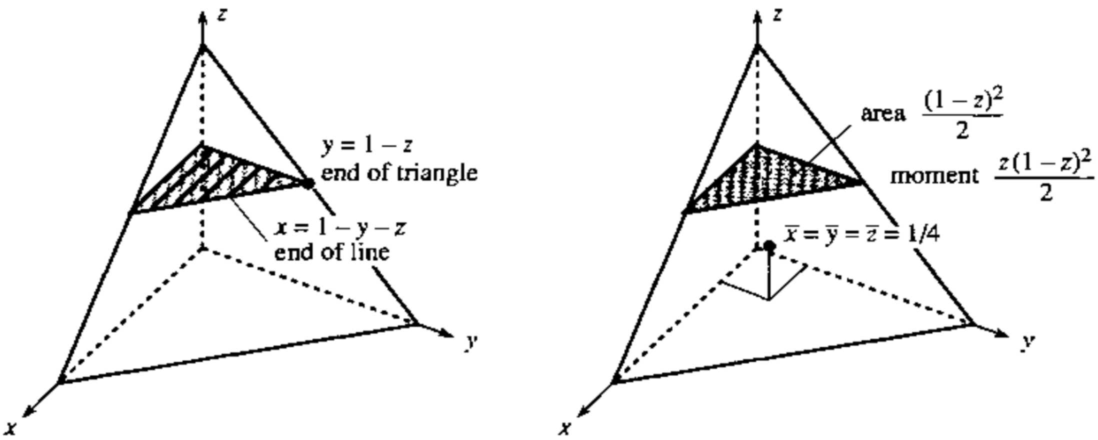

Figure 11: Lines end at plane $x+y+z=1$. Triangles end at edge $y+z=1$. The average height is $\overline{z} = \iiint z\,dV / \iiint dV$.

Solution 17. A tetrahedron has four flat faces, all triangles. The fourth face in **Figure 11** is on the plane $x+y+z=1$. A line in the x direction enters at $x=0$ and exits at $x=1-y-z$ (The length depends on y and z. The equation of the boundary plane gives x). Then those lines are put into plane slices by the y integral :

$$\int_{y=0}^{1-z}\int_{x=0}^{1-y-z} dx\,dy = \int_{y=0}^{1-z}(1-y-z)\,dy = \left[y-\frac{1}{2}y^2-zy\right]_0^{1-z} = \frac{1}{2}(1-z)^2 \tag{3.5}$$

What is this number $\frac{1}{2}(1-z)^2$? ***It is the area at height*** z. The plane at that height slices out a right triangle, whose legs have length $1-z$. The area is correct, but look at the limits of integration. ***If x goes to*** $1-y-z$***, why does*** y ***go to*** $1-z$? Reason : We are assembling lines, not points. The figure shows a line at every y up to $1-z$.

Adding the slices gives the volume : $\int_0^1 \frac{1}{2}(1-z)^2\,dz = [\frac{1}{6}(z-1)^3]_0^1$. This agrees with $1/3$ (base times height), the volume of a pyramid.

The height $\overline{z}$ of the centroid is « z_{average} ». We compute $\iiint z\,dV$ and divide by the volume. Each horizontal slice is multiplied by its height z, and the limits of integration don't change :

$$\iiint z\,dV = \int_0^1\int_0^{1-y}\int_0^{1-y-z} z\,dx\,dy\,dz = \int_0^1 \frac{z(1-z)^2}{2}\,dz = \frac{1}{24} \tag{3.6}$$

This is quick because z is constant in the x and y integrals. Each triangular slice contributes z times its area $\frac{1}{2}(1-z)^2$ times dz. Then the z integral gives the moment $1/24$. To find the average height, divide $1/24$ by the volume :

$$\overline{z} = \text{height of centroid} = \frac{\iiint z\,dV}{\iiint dV} = \frac{1/24}{1/6} = \frac{1}{4} \tag{3.7}$$

By symmetry $\overline{x} = 1/4$ and $\overline{y} = 1/4$. The centroid is the point $(1/4, 1/4, 1/4)$. Compare that with the centroid of the standard right triangle. Compare also with $1/2$, the center of the unit interval. There must be a five-sided region in four dimensions centered at $(1/5, 1/5, 1/5, 1/5)$.

For area and volume we meet another pattern. Length of standard interval is 1, area of standard triangle is 1/2, volume of standard tetrahedron is a hypervolume in four dimensions must be ______. The interval reaches the point $x = 1$, the triangle reaches the line $x+y = 1$, the tetrahedron reaches the plane $x + y + z = 1$. The four-dimensional region stops at the hyperplane ______ $= 1$.

Example 18. *Find the volume* $\iiint dx\,dy\,dz$ *inside the unit sphere* $x^2 + y^2 + z^2 = 1$.

Solution 18. First question : What are the limits on x? If a needle goes through the sphere in the x direction, where does it enter and leave ? Moving in the x direction, the numbers y and z stay constant. The inner integral deals only with x. The smallest and largest x are at the boundary where $x^2 + y^2 + z^2 = 1$. This equation does the work - we solve it for x. Look at the limits on the x integral :

$$\text{volume of sphere} = \int_?^? \int_?^? \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} dx\,dy\,dz = \int_?^? \int_?^? 2\sqrt{1-y^2-z^2}\,dy\,dz \tag{3.8}$$

The limits on y are $-\sqrt{1-z^2}$ and $\sqrt{1-z^2}$. You can use algebra on the boundary equation $x^2 + y^2 + z^2 = 1$. But notice that x is gone! We want the smallest and largest y, for each z. It helps very much to draw the plane at height z, slicing through the sphere in **Figure 12**. The slice is a circle of radius $r = \sqrt{1-z^2}$. So the area is πr^2, which must come from the y integral :

$$\int 2\sqrt{1-y^2-z^2}\,dy = \text{area of slice} = \pi(1-z^2) \tag{3.9}$$

We do not need to integrate to get this result. We simply use the formula πr^2. Mathematics is hard enough, and we don't have to work blindfolded. The goal is understanding and if you know the area then use it. Of course the integral of $\sqrt{1-y^2-z^2}$ can be done if necessary.

The triple integral is down to a single integral. We went from one needle to a circle of needles and now to a sphere of needles. The volume is a sum of slices of area $\pi(1-z^2)$. The South Pole is at $z = -1$, the North Pole is at $z = +1$, and the integral is the volume $4\pi/3$ inside the unit sphere :

$$\int_{-1}^{1} \pi(1-z^2)\,dz = \left[\pi\left(z - \frac{1}{3}z^3\right)\right]_{-1}^{1} = \frac{2}{3}\pi - \left(-\frac{2}{3}\pi\right) = \frac{4}{3}\pi \tag{3.10}$$

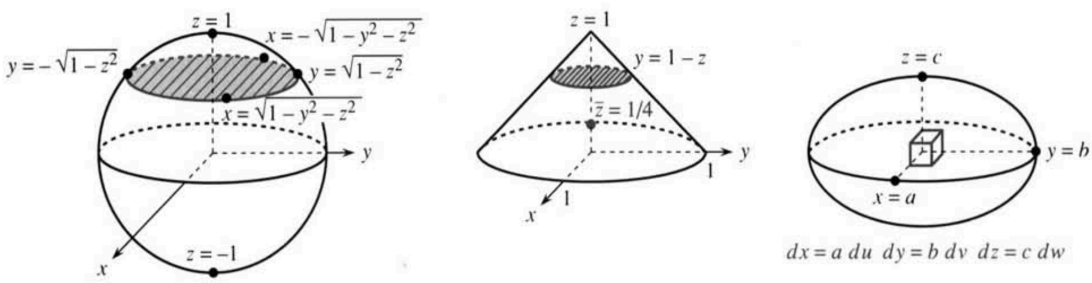

Figure 12: $\int dx$ = length of needle, $\iint dx\,dy$ = area of slice. Ellipsoid is a stretched sphere.

Question 9. *A cone also has circular slices. How is the last integral changed?*

Answer 9. The slices of a cone have radius $1-z$. Integrate $(1-z)^2$ not $\sqrt{(1-z)^2}$.

Question 10. *How does this compare with a circular cylinder (height 1, radius 1)?*

Answer 10. Now all slices have radius 1. Above $z=0$, a cylinder has volume it and a half-sphere has volume in and a cone has volume in. For solids with equal surface area, the sphere has largest volume.

Question 11. *What is the average height $\overline{z}$ in the cone and half-sphere and cylinder?*

Answer 11.

$$\overline{z} = \frac{\int z\,(\text{slice area})\,dz}{\int (\text{slice area})\,dz} = \frac{1}{4} \quad \text{and} \quad \frac{3}{8} \quad \text{and} \quad \frac{1}{2} \tag{3.11}$$

Example 19. *Find the volume $\iiint dx\,dy\,dz$ inside the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$.*

Solution 19. The limits on x are now $\pm\sqrt{1-y^2/b^2-z^2/c^2}$. The algebra looks terrible. The geometry is better - all slices are ellipses. A ***change of variable*** is absolutely the best.

Introduce $u=x/a$ and $v=y/b$ and $w=z/c$. Then the outer boundary becomes $u^2+v^2+w^2=1$. In these new variables the shape is a sphere. The triple integral for a sphere is $\iiint du\,dv\,dw = 4\pi/3$. But what volume dV in xyz space corresponds to a small box with sides du and dv and dw ?

Every uvw box comes from an xyz box. The box is stretched with no bending or twisting. Since u is x/a, the length dx is a du. Similarly $dy=b\,dv$ and $dz=c\,dw$. The volume of the xyz box (Figure 12) is $dx\,dy\,dz=(abc)\,du\,dv\,dw$. The stretching factor $J=abc$ is a constant, and the volume of the ellipsoid is

$$\overset{\text{bad limits}}{\underset{\text{ellipsoid}}{\iiint}} dx\,dy\,dz = \overset{\text{bettes limits}}{\underset{\text{sphere}}{\iiint}} (abc)\,du\,dv\,dw = \frac{4\pi}{3}\,abc \tag{3.12}$$

You realize that this is special - other volumes are much more complicated. The sphere and ellipsoid are curved, but the small xyz boxes are straight. The next section introduces spherical coordinates, and we can finally write « good limits ». But then we need a different J.

4 Cylindrical and Spherical Coordinates

Cylindrical coordinates are appropriate, as you may guess, for describing solids that are ***symmetric around an axis***. The solid is three-dimensional, so there are three coordinates r,θ,z :

$$r \;:\; \text{out from the axis} \qquad \theta \;:\; \text{around the axis} \qquad z \;:\; \text{along the axis}$$

This is a mixture of polar coordinates $r\theta$ in a plane, plus z upward. You will not find $r\theta z$ difficult to work with. Start with a cylinder centered on the z axis :

$$\text{solid cylinder} \;:\; 0<r<I \qquad \text{flat bottom and top} \;:\; 0\le z\le 3 \qquad \text{half-cylinder} \;:\; 0\le\theta\le\pi \tag{4.1}$$

Integration over this half-cylinder is $\int_0^3\int_0^\pi\int_0^1 \underline{\quad ? \quad} dr\,d\theta\,dz$. There limits on r,θ,z are especially simple. Two other axially symmetric solids are almost as convenient :

cone : integrate to $r+z=1$ ***sphere*** : integrate to $r^2+z^2=R^2$

We would not use cylindrical coordinates for a box. Or a tetrahedron.

The integral needs one thing more - the volume dV. The movements dr and $d\theta$ and dz give a «curved box» in xyz space, drawn in **Figure 13** (right). The base is a polar rectangle, with area $r\,dr\,d\theta$. The new part is the height dz. The volume of the curved box is $r\,dr\,d\theta\,dz$. Then r goes in the blank space in the triple integral - the stretching factor is $J = r$. There are six orders of integration (we give two) :

$$\text{volume} = \int_z \int_\theta \int_r r\,dr\,d\theta\,dz = \int_\theta \int_z \int_r r\,dr\,dz\,d\theta \tag{4.2}$$

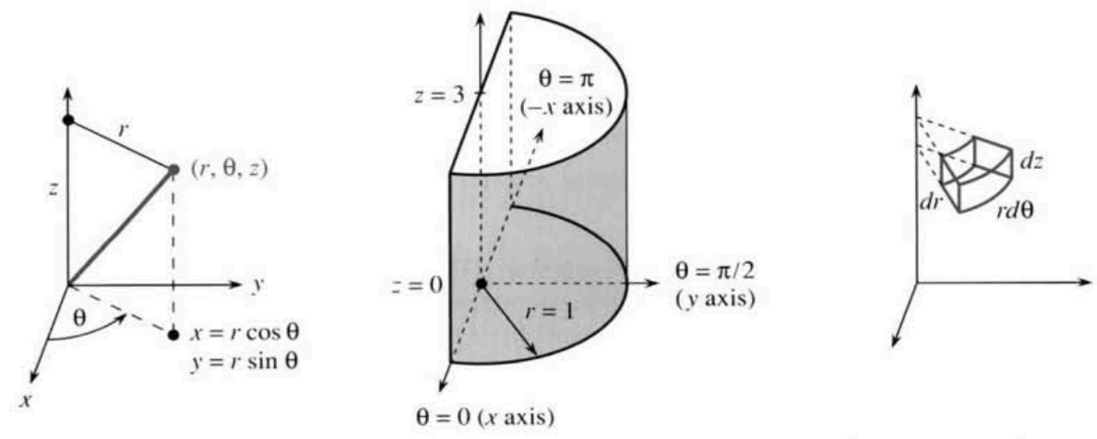

Figure 13: Cylindrical coordinates for a point and a half-cylinder. Small volume $r\,dr\,d\theta\,dz$.

Example 20. *(Volume of the half-cylinder). The integral of $r\,dr$ from 0 to 1 is $1/2$. The θ integral is π and the z integral is 3. The volume is $3\pi/2$.*

Example 21. *The surface $r = 1 - z$ encloses the cone in* **Figure 14**. *Find its volume.*

First solution 21. Since r goes out to $1 - z$, the integral of $r\,dr$ is $\frac{1}{2}(1 - z)^2$. The θ integral is 2π (a full rotation). Stop there for a moment.

We have reached $\iint r\,dr\,d\theta = \frac{1}{2}(1 - z)^2\,2\pi$. This is the ***area of a slice at height*** z. The slice is a circle, its radius is $1 - z$. its area is $\pi(1 - z)^2$. The z integral adds those slices to give $\pi/3$. That is correct, but it is not the only way to compute the volume.

Second solution 21. Do the z and θ integrals first. Since z goes up to $1 - r$, and θ goes around to 2π, those integrals produce $\iint r\,dz d\theta = r(1-r)\,2\pi$. Stop again - this must be the area of something.

After the z and θ integrals we have a sheil at radias r. The height is $1 - r$ (the outer shells are shorter). This height times $2\pi\,r$ gives the area around the shell. Different orders of integration give different ways to cut up the solid. The volume of the shell is area times thickness dr. The volume of the complete cone is the integral of shell volumes: $\int_0^1 r(1 - r)^2\,2\pi\,dr = \pi/3$.

Third solution 21. Do the r and z integrals first: $\iint r\,dr\,dz = 1/6$. Then the θ integral is $\int \frac{1}{6} d\theta$, which gives $\frac{1}{6}$ times 2π. This is the volume $\pi/3$ - but what is $\frac{1}{6}\,d\theta$?

The third cone is cut into wedges. The volume of a wedge is $\frac{1}{6}\,d\theta$. It is quite common to do the θ integral last, especially when it just multiplies by 2π. It is not so common to think of wedges.

Question 12. *Is the volume $\frac{1}{6}\,d\theta$ equal to an area $\frac{1}{6}$ times a thickness $d\theta$?*

Answer 12. No! The triangle in the third cone has area $\frac{1}{2}$ not $\frac{1}{6}$. Thickness is never $d\theta$.

This cone is typical of a ***solid of revolution***. The axis is in the z direction. The θ integral yields 2π, whether it cornes first, second, or third. The r integral goes out to a radius $f(z)$, which is 1 for the cylinder and $1-z$ for the cone. The integral $\iint r\,dr\,d\theta$ is $\pi(f(z))^2$ = area of circular slice. This leaves the z integral $\int \pi(f(z))^2\,dz$.

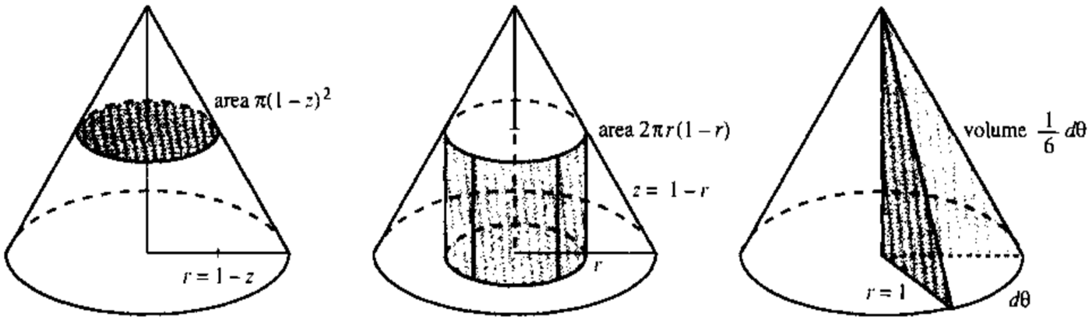

Figure 14: A cone cut three ways : slice at height z, shell at radius r, wedge at angle θ.

Example 22. *The **moment of inertia** around the z axis is $\iiint r^3\,dr\,d\theta\,dz$. The extra r^2 is (distance to axis)^2. For the cone this triple integral is $\pi/10$.*

Example 23. *The **moment** around the z axis is $\iiint r^2\,dr\,d\theta\,dz$. For the cone this is $\pi/6$. The average distancer $\overline{r}$ is (moment)/(volume) $= (\pi/6)/(\pi/3) = 1/2$.*

Example 24. *A sphere of radius R has the boundary $r^2+z^2=R^2$, in cylindrical coordinates. The outer limit on the r integral is $\sqrt{R^2-z^2}$. That is not acceptable in difficult problems. To avoid it we now change to coordinates that are natural for a sphere.*

4.1 Spherical coordinates

The Earth is a solid sphere (or near enough). On its surface we use two coordinates - latitude and longitude. To dig inward or fly outward, there is a third coordinate - *the distance ρ front the center.* This Greek letter ρ replaces r to avoid confusion with cylindrical coordinates. Where r is measured from the z axis, ρ is measured from the origin. Thus $r^2 = x^2+y^2$ and $\rho^2 = x^2+y^2+z^2$.

The angle θ is the same as before. It goes from 0 to 2π. It is the longitude, which increases as you travel east around the Equator.

The angle ϕ is new. It equals 0 at the North Pole and π (not 2π) at the South Pole. It is the ***polar angle***, measured down from the z axis. The Equator has a latitude of 0 but a polar angle of $\pi/2$ (halfway down). Here are some typical shapes :

solid sphere (or bail) : $0 \le \rho \le R$	surface of sphere : $\rho = R$
upper half-sphere : $0 \le \phi \le \pi/2$	eastern half-sphere : $0 \le \theta \le \pi$

The angle θ is constant on a cone from the origin. It cuts the surface in a circle (**Figure 15**), but not a great circle. The angle θ is constant along a half-circle from pole ta pole. The distance ρ

is constant on each inner sphere, starting at the center $\rho = 0$ and moving out to $\rho = R$.

ln spherical coordinates the volume integral is :

$$\iiint \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \tag{4.3}$$

To explain that surprising factor $J = \rho^2 \sin\phi$, start with $x = r\cos\theta$ and $y = r\sin\theta$. In spherical coordinates r is $\rho\sin\theta$ and z is $\rho\cos\theta$ - see the triangle in the figure. So substitute $\rho\sin\phi$ for r :

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi \tag{4.4}$$

Remember those two steps, $\rho\phi\theta$ to $r\theta z$ to xyz. We check that $x^2 + y^2 + z^2 = \rho^2$:

$$\rho^2 (\sin^2\phi \cos^2\phi + \sin^2\phi \sin^2\phi + \cos^2\phi) = \rho^2 (\sin^2\phi + \cos^2\phi) = \rho^2 \tag{4.5}$$

The volume integral is explained by **Figure 15** (right). That shows a « spherical box » with right angles and curved edges. Two edges are $d\rho$ and $\rho\, d\theta$. The third edge is horizontal. The usual $r\, d\theta$ becomes $\rho \sin\theta\, d\theta$. Multiplying those lengths gins dV.

The volume of the box is $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. This is a *distance cubed*, from $\rho^2 \, d\rho$.

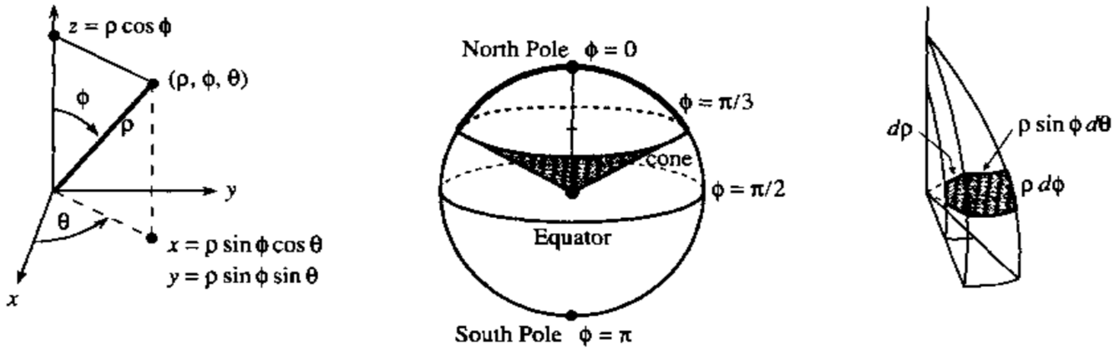

Figure 15: Spherical coordinates $\rho\phi\theta$. The volume $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ of a spherical box.

Example 25. *A solid bail of radius R has known volume $V = \frac{4}{3}\pi R^3$. Notice the limits :*

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \left[\frac{1}{3} \rho^3 \right]_0^R [-\cos\phi]_0^{\pi} \, [\theta]_0^{2\pi} = \frac{1}{3} R^3 \cdot 2 \cdot 2\pi \tag{4.6}$$

Question 13. *What is the volume above the cone in* **Figure 15** *?*

Answer 13. The ϕ integral stops at $[-\cos\phi]_0^{\pi/3} = 1/2$. The volume is $\frac{1}{3} R^3 \cdot \frac{1}{2} \cdot 2\pi$.

Example 26. *The surface area of a sphere is $A = 4\pi R^2$. Forget the ρ integral :*

$$A = \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi \, d\phi \, d\theta = R^2 \, [-\cos\phi]_0^{\pi} \, [\theta]_0^{2\pi} = R^2 \cdot 2 \cdot 2\pi \tag{4.7}$$

After those examples front geometry, here is the real thing from science. We want to compute one of the most important triple integrals in physics - « the gravitation attraction of a solid sphere ». For some reason Isaac Newton had trouble with this integral. He refused to publish his masterpiece on astronomy until he had solved it. He probably did not use spherical coordinates - and the integral is not easy even now.

The answer that Newton finally found is beautiful. The sphere acts as if all its mass were concentrated at the center. At an outside point $(0, 0, D)$, the force of gravity is proportional to $1/D^2$. The force from a uniform solid sphere equals the force from a point mass, at every outside point P. That is exactly what Newton wanted and needed, to explain the solar system and to prove Kepler's laws.

Here is the difficulty. Some parts of the sphere are closer than D, some parts are farther away. The actual distance q, from the outside point P to a typical inside point, is shown in **Figure 16**. The average distance $\overline{q}$ to all points in the sphere is not D. But what Newton needed was a diffèrent average, and by good luck or some divine calculus it works perfectly : ***The average of*** $1/q$ ***is*** $1/D$. This gives the potential energy :

$$\textbf{\textit{potential at point}}\ P \quad = \quad \iiint\limits_{\text{sphere}} \frac{1}{q}\, dV \quad = \quad \frac{\text{volume of sphere}}{D} \tag{4.8}$$

A small volume dV at the distance q contributes dV/q to the potential. The integral adds the contributions from the whole sphere. Equation (4.8) says that the potential at $r = D$ is not changed when the sphere is squeezed to the center. The potential equals the whole volume divided by the single distance D.

Important point : The average of $1/q$ is $1/D$ and not $1/\overline{q}$. The average of $1/2$ and $1/4$ is not $1/3$. Smaller point : we wrote « sphere » where we should have written « ball ». The sphere is solid : $0 \leq \rho \leq R$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

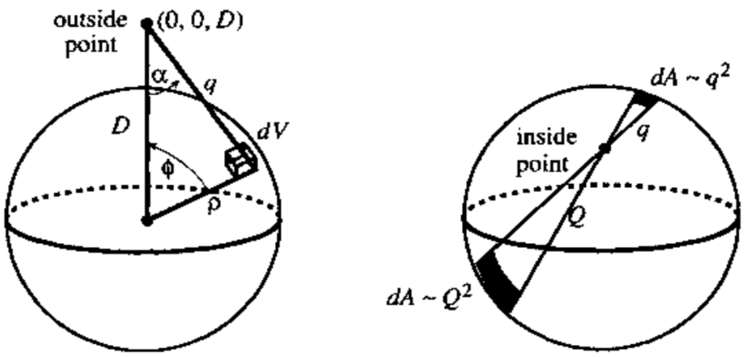

Figure 16: Distance q from outside point to inside point. Distance q and Q to surface.

What about the force ? For the small volume it is proportional to dV/q^2 (this is the inverse square law). But force is a vector, pulling the outside point toward dV - not toward the center of the sphere. The figure shows the geometry and the symmetry. We want the z component of the force (by symmetry the overall x and y components are zero). The angle between the force vector and the z axis is α, so for the z component we multiply by $\cos\alpha$. The total force comes from the integral that Newton discovered :

$$\textbf{\textit{force at point}}\ P \quad = \quad \iiint\limits_{\text{sphere}} \frac{\cos\alpha}{q^2}\, dV \quad = \quad \frac{\text{volume of sphere}}{D^2} \tag{4.9}$$

We will compute the integral (4.8) and leave the privilege of solving (4.9) as an Exercise and that is to be taken seriously. If you have come this far, you deserve the pleasure of doing what at one time only Isaac Newton could do. Problem 1 offers a suggestion (just the law of cosines) but the integral is yours.

The law of cosines also helps with (4.8). For the triangle in the figure it gives $q^2 = D^2 - 2\rho D\cos\phi + \rho^2$. Call this whole quantity u. We do the surface integral first ($d\phi$ and $d\theta$ with ρ fixed). Then $q^2 = u$ and $q = \sqrt{u}$ and $du = 2\rho D\sin\phi\, d\phi$:

$$\int_0^{2pi}\int_0^{\pi}\frac{\rho^2\sin\phi\, d\phi\, d\theta}{q} = \int\frac{2\pi\rho^2}{2\rho D}\frac{du}{\sqrt{u}} = \left[\frac{2\pi\rho}{D}\sqrt{u}\right]_{\phi=0}^{\phi=\pi} \tag{4.10}$$

2π came from the θ integral. The integral of $du/\sqrt{u}$ is $2\sqrt{u}$. Since $\cos\phi = -1$ at the upper limit, u is $D^2+2\rho D+\rho^2$. *The square root of u is $D+\rho$.* At the lower limit $\cos\phi = +1$ and $u = D^2-2\rho D+\rho^2$. This is another perfect square - its square root is $D-\rho$. The surface integral (4.10) with fixed ρ is

$$\iint\frac{dA}{q} = \frac{2\pi\rho}{D}\Big[(D+\rho)-(D-\rho)\Big] = \frac{4\pi\rho^2}{D} \tag{4.11}$$

Last cornes the ρ integral : $\int_0^R 4\pi\rho^2\, d\rho/D = \frac{4}{3}\pi R^3/D$. This proves formula (4.8) : ***potential equals volume of the sphere divided by*** D.

Note 1 Physicists are happy about equation (4.11). The average of $1/q$ is $1/D$ not only over the solid sphere but over each spherical shell of area $4\pi\rho^2$. The shells can have different densities, as they do in the Earth, and still Newton is correct. This also applies to the force integral (4.9) - each separate shell acts as if its mass were concentrated at the center. Then the final ρ integral yields this property for the solid sphere.

Note 2 Physicists also know that force is minus the derivative of potential. The derivative of (4.8) with respect to D produces the force integral (4.9). Problem 2 explains this shortcut to equation (4.9).

Example 27. ***Everywhere inside a hollow sphere the force of gravity is zero.***

When D is smaller than ρ, the lower limit $\sqrt{u}$ in the integral (4.10) changes from $D-\rho$ to $\rho - D$. That way the square root stays positive. This changes the answer in (4.11) to $4\pi\rho^2/\rho$, so the potential no longer depends on D. *The potential is constant inside the hollow shell.* Since the force cornes from its derivative, the force is zero.

A more intuitive proof is in the second figure. The infinitesimal areas on the surface are proportional to q^2 and Q^2. But the distances to those areas are q and Q, so the forces involve $1/q^2$ and $1/Q^2$ (the inverse square law). Therefore the two areas exert equal and opposite forces on the inside point, and they cancel each other. The total force from the shell is zero.

This zero integral is the reason that the inside of a car is safe from lightning. Of course a car is not a sphere. But electric charge distributes itself to keep the surface at constant potential. The potential stays constant inside - therefore no force. The tires help to prevent conduction of current (and electrocution of driver).

P.S. Don't just step out of the car. Let a metal chain conduct the charge to the ground. Otherwise you could be the conductor.

4.2 Change of coordinates - stretching factor J

Once more we look to calculus for a formula. We need the volume of a small curved box in any uvw coordinate system. The $r\theta z$ box and the $\rho\phi\theta$ box have right angles, and their volumes were read off frorn the geometry (stretching factors $J = r$ and $J = \rho^2 \sin\phi$ in **Figures 13** and **15**). Now we change from xyz to other coordinates uvw - which are chosen to fit the problem.

Going from xy to uv, the area $dA = J\,du\,dv$ was a 2 by 2 determinant. In three dimensions the determinant is 3 by 3. The matrix is always the « Jacobian matrix » containing first derivatives. There were four derivatives from xy to uv, now there are nine from xyz to uvw.

Jacobian in three dimensions

Suppose x, y, z are given in ternis of u, v, w. Then a small box in uvw space (sides du, dv, dw) comes from a volume $dV = J\,du\,dv\,dw$ in xyz space :

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \text{stretching factor } \frac{\partial(x,y,z)}{\partial(u,v,w)} \tag{4.12}$$

The volume integral $dx\,dy\,dz$ becomes $\iiint |J|\,du\,dv\,dw$, with limits on uvw.

Remember that a 3 by 3 determinant is the sum of six terms. One terni in J is $(\frac{\partial x}{\partial u})(\frac{\partial y}{\partial u})(\frac{\partial z}{\partial u})$, along the main diagonal. This cornes from pure stretch-ing, and the other five terms allow for rotation. The best way to exhibit the formula is for spherical coordinates - where the fine derivatives are easy but the determinant is not :

Example 28. *Find the factor J for $x = \rho\,\sin\phi\,\cos\theta$, $y = \rho\,\sin\phi\,\sin\theta$, $z = \rho\,\cos\theta$.*

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \sin\phi\,\cos\theta & \rho\,\sin\phi\,\cos\theta & -\rho\,\sin\phi\,\cos\theta \\ \sin\phi\,\sin\theta & \rho\,\sin\phi\,\sin\theta & \rho\,\sin\phi\,\cos\theta \\ \cos\theta & -\rho\,\sin\theta & 0 \end{vmatrix} \tag{4.13}$$

The determinant has six terms, but two arc zero - because of the zero in the corner. The other four terms are $\rho^2\,\sin\phi\,\cos^2\phi\,\sin^2\theta$ and $\rho^2\,\sin\phi\,\cos^2\phi\,\cos^2\theta$ and $\rho^2\,\sin^3\phi\,\sin^2\theta$ and $\rho^2\,\sin^3\phi\,\cos^2\theta$. Add the first two (note $\sin^2\theta + \cos^2\theta$) and separately add the second two. Then add the sums to reach $J = \rho^2\,\sin\phi$.

Geometry already gave this answer. For most uvw variables, use the determinant.

A Exercises

A.1 Double Integrals

Exercice 1. *Compute the double integrals by two integrations.*

(1) $\displaystyle\int_{y=0}^{1}\int_{x=0}^{2} x^2\,dx\,dy$ *and* $\displaystyle\int_{y=0}^{1}\int_{x=0}^{2} y^2\,dx\,dy$ **(3)** $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi/4} \sin(x+y)\,dx\,dy$ *and* $\displaystyle\int_{1}^{2}\int_{0}^{2} \frac{dx\,dy}{(x+y)^2}$

(2) $\displaystyle\int_{y=2}^{2e}\int_{x=1}^{e} 2xy\,dx\,dy$ *and* $\displaystyle\int_{y=2}^{2e}\int_{x=1}^{e} \frac{dx\,dy}{xy}$ **(4)** $\displaystyle\int_{0}^{1}\int_{1}^{2} y\,e^{xy}\,dx\,dy$ *and* $\displaystyle\int_{-1}^{1}\int_{0}^{3} \frac{dy\,dx}{\sqrt{3+2x+y}}$

Exercice 2. *Draw the region and compute the area. Then reverse the order of integral (and find the new limits) and compute, again, the area.*

(1) $\displaystyle\int_{x=1}^{2}\int_{y=1}^{2x} dy\,dx$ **(3)** $\displaystyle\int_{0}^{\infty}\int_{e^{-2x}}^{e^{-x}} dy\,dx$ **(5)** $\displaystyle\int_{-1}^{1}\int_{y^2}^{1} dx\,dy$

(2) $\displaystyle\int_{0}^{1}\int_{x^3}^{x} dy\,dx$ **(4)** $\displaystyle\int_{-1}^{1}\int_{x^2-1}^{1-x^2} dy\,dx$ **(6)** $\displaystyle\int_{-1}^{1}\int_{x=y}^{|y|} dx\,dy$

Exercice 3. *Evaluate the following integrals*

(1) $\displaystyle\int_{0}^{b}\int_{0}^{a} \frac{\partial^2 f}{\partial x\partial y}\,dx\,dy$ **(2)** $\displaystyle\int_{0}^{b}\int_{0}^{a} \frac{\partial f}{\partial x}\,dx\,dy$

Exercice 4. *Divide a unit square R into triangles S and T and verify $\iint_R f\,dA = \iint_S f\,dA + \iint_T f\,dA$.*

(1) $f(x,y) = 2x - 3y + 1$ **(2)** $f(x,y) = x\,e^y - y\,e^x$

Exercice 5. *We consider a rectangle with corners $(1,1),(1,3),(2,1),(2,3)$ and that has density $\rho(x,y) = x^2$. The moments are $\boldsymbol{M}_y = \iint x\,\rho\,dA$ and $\boldsymbol{M}_x = \iint y\,\rho\,dA$. Find*

(1) *the mass* **(2)** *the center of mass*

Exercice 6. *Write dow a program to compute $\int_0^1\int_0^1 f(x,y)\,dxdy$ by the Midpoinrule*[2] *(points in the middle of small square n^2). What are the functions $f(x,y)$ that can be exactly integrated by your program?*

Exercice 7. *Apply the Midpoint rule program to integrate x^2 et xy et y^2. The error is decreasing in which power of $\Delta x = \Delta y = 1/n$?*

Exercice 8. *We consider a region which is a circular wedge of radius 1 between the lines $y = x$ and $y = -x$. Find*

(1) *the area* **(2)** *the centroid $(\overline{x}, \overline{y})$.*

Exercice 9. *A city localized by a circle which equation is $x^2 + y^2 = 100$ possesses a population density of $\rho(x,y) = 10(100 - x^2 - y^2)$. Integrate it to find its population.*

Exercice 10. *We want to find the volume between the planes $x = 0, y = 0, z = 0$ and $ax+by+cz = 1$.*

[2] *The Midpoint Rule* en anglais

A.2 Change to better coordinates

Exercice 11. *R is a pie-shaped wedge : $0 \le r \le 1$ and $\pi/4 \le \theta \le 3\pi/4$.*

(1) *What is the area of R ? Check by integration in polar coordinates.*

(2) *Find limits on $\iint dy\,dx$ to yield the area of R, and integrate. Extra : Find limits on $\iint dx\,dy$.*

(3) *Equation (2.1) with $\alpha = \pi/4$ rotates R into the uv region $S =$ ______. Find limits on $\iint du\,dv$.*

(4) *Compute the centroid height $\overline{y}$ of R by changing $\iint y\,dx\,dy$ to polar coordinates. Divide by the area of R.*

(5) *The region R has $\overline{x} = 0$ because ______. After rotation through $\alpha = \pi/4$, the centroid $(\overline{x}, \overline{y})$ of R becomes the centroid ______ of S.*

(6) *Find the centroid of any wedge $0 \le r \le a$, $0 \le \theta \le b$.*

Exercice 12. *Change four-sided regions to squares.*

(1) *R has straight sides $y = 2x$, $x = 1$, $y = 1 + 2x$, $x = 0$. Locate its four corners and draw R. Find its area by geometry.*

(2) *Choose a, b, c, d so that the change $x = au + bv$, $y = cu + dv$ takes the previous R into S, the unit square $0 \le u \le 1$, $0 \le v \le 1$. From the stretching factor $J = ad - bc$ find the area of R.*

Exercice 13. *Using polar coordinates, find the volume under $z = x^2 + y^2$ above the unit disk $x^2 + y^2 \le 1$.*

Exercice 14. *In **Example 11** we integrate e^{-x^2} from 0 to ∞ (answer: $\sqrt{\pi}$). Also, $B = \int_0^1 e^{-x^2} dx$ leads to $B^2 = \int_0^1 e^{-x^2} dx \int_0^1 e^{-y^2} dy$. Change this double integral on the square unit for the coordinates r and θ, and find the limits on r making the exact integration impossible.*

Exercice 15. *Drw teh region R: $0 \le x \le 1, 0 \le y < \infty$ and describe it using polar coordinates (give the limits on r and θ). Compute $\iint_R (x^2 + y^2)^{-3/2} dxdy$ using polar coordinates.*

Exercice 16. *We want to find the mass of the inclined square in **Example 8** if the density is $\rho = xy$.*

Exercice 17. *Find the mass of the ring in example 9 if the density is $\rho = x^2 + y^2$. This is the same as which moment of inertia with which density ?*

Exercice 18. *In the square $-1 \le x \le 2$, $-2 \le y \le 1$, where could you distribute a unit mass (with $\iint \rho\,dx\,dy = 1$) to maximize the following integrals*

(1) $\iint x^2 \rho\,dA$ **(2)** $\iint y^2 \rho\,dA$ **(3)** $\iint r^2 \rho\,dA$

A.3 Triple Integrals

Exercice 19.

(1) *For the solid region $0 \le x \le y \le z \le 1$, find the limits in $\iiint J\,dx\,dy\,dz$ and compute the volume.*

(2) *Reverse the order of integration in the previous question to $\iiint J\,dz\,dy\,dx$ and find the limits of integration. The four faces of this tetrahedron are the planes $x = 0$ and $y = x$ and ______.*

Exercice 20.

(1) *For the solid region $0 \le x \le y \le z \le 1$, find the limits in $\iiint dx\,dy\,dz$ and compute the volume.*

(2) *Invert the order of integration in the previous question in $\iiint dz\,dy\,dx$ and find the limits on the integrals. The four faces of the tetraedra are the planes $x = 0$ et $y = x$ and ______.*

Exercice 21. *Find the limits in $\iiint dx\,dy\,dz$ or $\iiint dz\,dy\,dx$. Compute the volume.*

(1) *A circular cylinder with height 6 and base $x^2 + y^2 \le 1$.*

(2) *The part of that cylinder below the plane $z = x$. Watch the base. Draw a picture.*

Exercice 22.

(1) *Find the volume and centroid of the region bounded by $x = 0$, $y = 0$, $z = 0$ and $x/a + y/b + z/c = 1$.*

(2) *Based on the text, what is the volume inside $x^2+4y^2+9z^2 = 16$? What is the « hypervolume » of the 4-dimensional pyramid that stops at $x + y + z + w = 1$?*

Exercice 23. *Find the partial derivatives $\frac{\partial I}{\partial x}$, $\frac{\partial I}{\partial y}$, $\frac{\partial^2 I}{\partial y \partial z}$ of*

(1) $I = \int_0^z \int_0^y dx\,dy$ **(2)** $I = \int_0^z \int_0^y \int_0^x f(x', y', z')\,dx'\,dy'\,dz'$

Exercice 24. *Define the mean value of $f(x, y, z)$ over a volume V.*

Exercice 25. *Give the upper limits to produce the volume of a unit cube frm small cubes in the triple sum : $\sum_{i=1} \sum_{j=1} \sum_{k=1} (\Delta x)^3 = 1$.*

Exercice 26. *Find the limit when $\Delta x \to 0$ de $\sum_{i=1}^{3/\Delta x} \sum_{j=1}^{2/\Delta x} \sum_{k=1}^{j} (\Delta x)^3$.*

A.4 Cylindrical and Spherical Coordinates

Exercice 27. *Convert the xyz coordinates to $r\theta z$ and $\rho\phi\theta$ (watch θ in the third case).*

(1) $(D, 0, 0)$ **(2)** $(0, -D, 0)$ **(3)** $(0, 0, D)$ **(4)** $(3, 4, 5)$

Exercice 28. *From the limits of integration describe each region and find its volume. The inner integral has the inner limits.*

(1) $\int_{\theta=0}^{2\pi} \int_{r=0}^{1/\sqrt{2}} \int_{z=r}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta$

(2) $\int_{\theta=0}^{\pi} \int_0^1 \int_0^{1+r^2} r\,dz\,dr\,d\theta$

(3) $\int_{\theta=0}^{2\pi} \int_{z=0}^{1} \int_{r=0}^{2-z} r\,dr\,dz\,d\theta$

(4) $\int_{\theta=0}^{\pi} \int_0^{\pi} \int_0^{\pi} r\,d\theta\,dr\,dz$

(5) $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(6) $\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^{2} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(7) $\int_0^{\pi} \int_0^{\pi} \int_0^{\sin\phi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(8) $\int_0^{2\pi} \int_0^{\pi/4} \int_1^3 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(9) $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

(10) $\int_0^1 \int_0^1 \int_0^1 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

Exercice 29. ***Example*** *23 gave the volume integral for a sphere in $r\theta z$ coordinates. What is the area of the circular slice at height z ? What is the area of the cylindrical shell at radius r ? Integrate over slices (dz) and over shells (dr) to reach $4\pi R^3/3$.*

Exercice 30. *The following problems are on the attraction of a sphere, use* **Figure 16** *and the law of cosines $q^2 = D^2 - 2\rho D \cos\phi + \rho^2 = u$.*

1. *(***Newton's achievement***) Show that $\iiint (\cos\alpha)\, dV/q^2$ equals volume / D^2.* One hint only*: Find $\cos\alpha$ from a second law of cosines $\rho^2 = D^2 - 2qD\cos\alpha + q^2$. The ϕ integral should involve $1/q$ and $1/q^3$, Equation (4.8) integrates $1/q$, leaving $\iiint dV/q^3$ still to do.*

2. *Compute $\partial q/\partial D$ in the first cosine law snd show from* **Figure 16** *that it equals $\cos\alpha$. Then the derivative of equation (4.8) with respect to D is a shortcut to Newton's equation (4.9) .*