Logical Formalism
Exercise Sheet 1

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August 14, 2024

Main Exercises

The Dumbest Proofs

Find the errors in the following proofs. Note that the result may be true, even if the method isn't.

Exercise 1 - *Proving that* $sin(x + \pi) = -sin(x)$.

$$
\begin{aligned}
sin(x + \pi) &= sin(x) + sin(\pi) && (1)\\
&= sin(x) - 1 && (2)\\
&= (-1)sin(x) && (3)\\
&= -sin(x) && (4)
\end{aligned}
$$

Exercise 2 - *Proving that* $2 = 4$.

$$
\begin{aligned}
& 2 = 4 && (5)\\
\implies\ & 2 - 3 = 4 - 3 && (6)\\
\implies\ & -1 = 1 && (7)\\
\implies\ & (-1)^2 = 1^2 && (8)\\
\implies\ & 1 = 1 && (9)
\end{aligned}
$$

Indeed, $1 = 1$. Hence, $2 = 4$.

Exercise 3 - *The exponential function has an upper bound.* Given a real number x, since the exponential function is strictly increasing, $e^x < e^{x+1}$. Consider $M = e^{x+1}$. Hence for all real numbers x, $e^x < M$. The exponential function therefore has an upper bound.

Sets and Formulas

Exercise 4 - *A strange equation.* Let E be a set and $A, B \in \mathcal{P}(E)$ be two subsets of E. Prove that $\forall X \in \mathcal{P}(E), (A \cap X = B) \iff (B \subseteq X) \wedge (X \subseteq B \cup A^{\complement}) \wedge (B \subseteq A)$.

Exercise 5 - $\forall$ *and* $\vee$. Consider $A = \forall x \in E, P(x) \vee Q(x)$ and $B = (\forall x \in E, P(x)) \vee (\forall x \in E, Q(x))$. Does A imply B, or B imply A? Find a counter-example if one of these implications does not hold.

Exercise 6 - $\exists$ *and* $\wedge$. Consider $A = \exists x \in E, P(x) \wedge Q(x)$ and $B = (\exists x \in E, P(x)) \wedge (\exists x \in E, Q(x))$. Does A imply B, or B imply A? Find a counter-example if one of these implications does not hold.

Extra Exercises

Classical Logics

Exercise 7 - *Negating* $\vee$. Let ABC be a triangle. What is the negation of the proposition *ABC is isosceles*?

Exercise 8 - *An equivalence.* Prove that $\forall n \in \mathbb{N}$, n is a multiple of 3 if and only if the sum of its digits is a multiple of 3 as well.

Set Theory

Exercise 9 - *Properties of* $\emptyset$. Prove that for any set A, $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$.

Exercise 10 - *A triple equivalence.* Prove that the three following propositions are equivalent, given two sets A and B:

1. $A \subseteq B$.
2. $B^{\complement} \subseteq A^{\complement}$.
3. $B^{\complement} \cap A = \emptyset$.

Exercise 11 - *Properties of the complement.* Prove that for any set $A \in \mathcal{P}(E)$, $A \cap A^{\complement} = \emptyset$ and $A \cup A^{\complement} = E$.

Quantifiers

Exercise 12 - *Negating logical formulas.* Write the negation of the following formulas (using as simple an equivalent formula as possible):

1. $A \implies (B \wedge \neg(C \vee D))$.

2. $A \vee B \iff C$.

3. $\forall x \in X, \forall y \in Y, (P(x) \wedge (\exists z \in Z, Q(y, z)))$.