Metric spaces

Distances and norms

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EPITA

October 16, 2021

Table of contents

Summary

Generalities about norms

Usual norms in $\mathbb{R}^n$

Usual norms in spaces of functions

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Definition [Norm]

Let E be a vector space over a field $\mathbb{K}$ — here $\mathbb{R}$ or $\mathbb{C}$.

An application $N : E \longrightarrow \mathbb{R}_+$ is called a *norm* over E if it checks the following properties:

$$\left\| \begin{array}{lll} \forall x \in E, & \mathbf{N}(\mathbf{x}) = \mathbf{0} \iff \mathbf{x} = \mathbf{0} & \textit{it is «definite»} \\ \forall x \in E, \forall \lambda \in \mathbb{K}, & \mathbf{N}(\lambda \mathbf{x}) = |\lambda| \mathbf{N}(\mathbf{x}) & \textit{(homogeneity)} \\ \forall (x, y) \in E^2, & \mathbf{N}(\mathbf{x} + \mathbf{y}) \leqslant \mathbf{N}(\mathbf{x}) + \mathbf{N}(\mathbf{y}) & \textit{(triangle inequality)} \end{array} \right.$$

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The homogeneity property goes together well with the structure of vector space.

A norm N over E induces a distance d_N over E through the relation:

$$\forall (x, y) \in E^2, \quad d_N(x, y) = N(x - y).$$

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Hence, the norm of a vector can be viewed as its distance to 0_E :

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Remark: there are also distances over E that do not come from any norm.

A bit of topology:

Definition [Open / Closed ball]

We call *open ball* centred at a of radius r with respect to d the set

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The concept of ball extends the notion of neighbourhood you already used in $\mathbb{R}$ to multidimensional spaces.

Summary

Definition [p-norms over $\mathbb{R}^n$]

Let $p \in [1, +\infty[$. We call *p-norm* over $\mathbb{R}^n$ the norm denoted by $\|\cdot\|_p$ defined for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

This definition is extended to $p = +\infty$ by looking at the limit:

$$\|x\|_\infty = \sup_{1 \leqslant i \leqslant n} |x_i| = \max_{1 \leqslant i \leqslant n} |x_i|$$

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d_1 is called Manhattan distance (or taxicab distance), while d_2 is the Euclidean distance.

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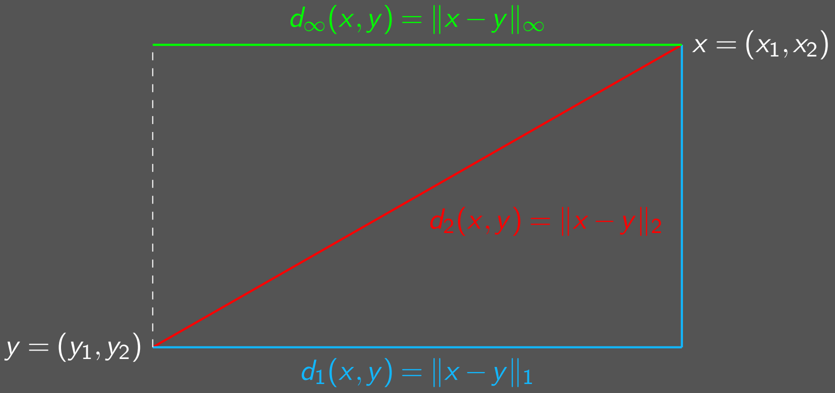

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And more generally:

$$p < q \Longrightarrow \|x\|_p \geqslant \|x\|_q.$$

Definition [Equivalent norms]
Two norms N and N' over E are said to be equivalent if and only if there exist two strictly positive real numbers k_1 and k_2 such that:

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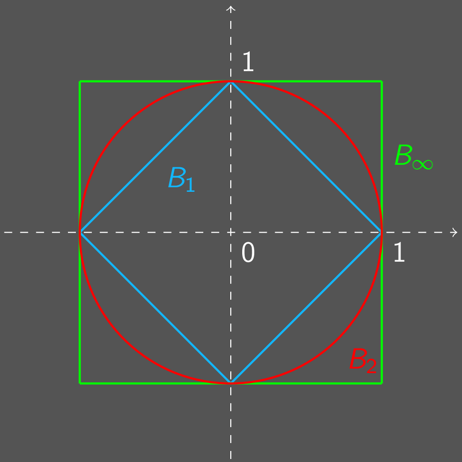
Please remark that k_1 and k_2 do not depend on the chosen x.

Theorem [Equivalence of norms in finite dimension]
If E is a vector space of finite dimension, then all norms over E are equivalent.

Balls of radius 1 for norms 1, 2, ∞ :



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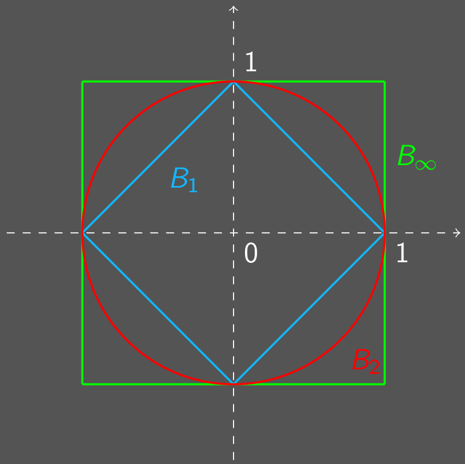

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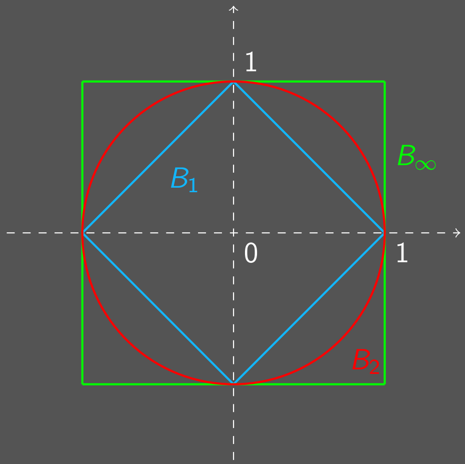

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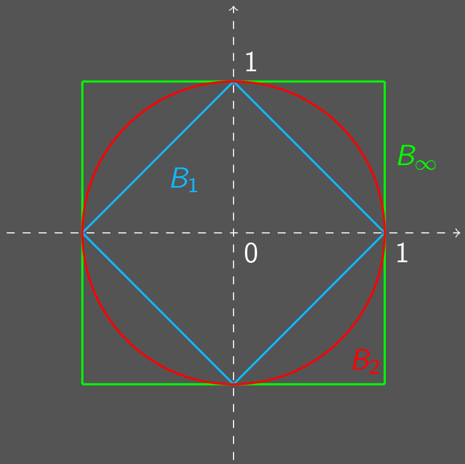

- If we choose another radius, we get the same graph up to the scale (homogeneity).
- The ball inclusions translate the inequalities between norms; these can be determined by comparing the radii in the optimal framing of the ball of a given norm by two balls of another one norm.
- This is also topologically interesting: all this norms depict the same structure.

Summary

Generalities about norms

Usual norms in $\mathbb{R}^n$

Usual norms in spaces of functions

Definition [p-norms over sets of functions]

Let I be a segment of $\mathbb{R}$ not reduced to a single element, let us look at the space $\mathcal{C} = C^0(I, \mathbb{R})$ of continuous functions over I with values in $\mathbb{R}$.

Let $p \in [1, +\infty[$. We call *p-norm* over $\mathcal{C}$ the norm denoted by $\|\cdot\|_p$ defined for every $f \in \mathcal{C}$ by

$$\|f\|_p = \left(\int_I |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

For $p = +\infty$, we set:

$$\|f\|_\infty = \sup_{x \in I} |f(x)|.$$

We denote by d_p the distance induced by $\|\cdot\|_p$.

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Moreover, the theorem of equivalence of norms in finite dimension unfortunately does not hold anymore. We even get the result:

If $p \neq q$ then $\|\cdot\|_p$ and $\|\cdot\|_q$ are not equivalent over $\mathcal{C}$.