Logical Formalism
Exercise Sheet 2

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Main exercises

Exercise 1 - *Higher order functions.* Are the following binary relations functions? If they are, determine their domain, whether they are injective or surjective, and their image.

1. Let $n, p, q \in \mathbb{N}$. $n \sim_f (p, q)$ if and only $n = p + q$.
2. Let $E \in \mathcal{P}(\mathbb{N})$ and $m \in \mathbb{N}$. $E \sim_f m$ if and only if m is E's minimum.
3. Let $u \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. $u \sim_f g$ if and only if $\forall x \in \mathbb{R}$, $g(x) = x + u$.
4. Let $u \in \mathbb{N}$ and $E \in \mathcal{P}(\mathbb{N})$. $n \sim_f E$ if and only if $\forall n \in E$, $u \mid n$.
5. Let $g, h : \mathbb{R} \to \mathbb{R}$. $g \sim_f h$ if and only if $\forall x \in \mathbb{R}$, $h(x) = g(x)^2$.

Exercise 2 - *A simple inequality.* Prove that for any $x \in \mathbb{R}$ such that $x > 1$ and any $n \in \mathbb{N}$, $(1 + x)^n \geq 1 + nx$.

Exercise 3 - *A road trip.* A country has n cities $C_1, \ldots, C_n$, where n is an integer greater than or equal to 1. Any two distinct cities, C_i and C_j, are connected by a one-way road either from C_i to C_j or from C_j to C_i. Prove that there is a route which passes through every city, first using a strong induction, then a simple induction.

Extra Exercises

Various Functions

Let E be a subset of a set F. The characteristic function of E is a function $\chi_E : F \to \{0, 1\}$ such that $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ otherwise.

Exercise 4 - ***Set-theoretic operations on characteristic functions.*** Consider two subsets U and V of F. Prove that $\chi_{U\cap V} = \chi_U \cdot \chi_V$ and $\chi_{U\cup V} = \chi_U + \chi_V - \chi_U \cdot \chi_V$.

Exercise 5 - ***Sets as characteristic functions.*** Let $\{0,1\}^F$ denote the set of all functions $F \to \{0,1\}$. Consider the function $\chi : E \in \mathcal{P}(F) \to \chi_E \in \{0,1\}^F$. Prove that χ is a bijection.

Inductive Proofs

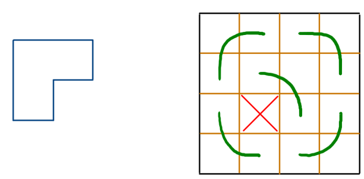

Figure 1: A grid of size 2^2 covered by triominoes.

Exercise 6 - ***A game of triominoes.*** Given an integer $n \in \mathbb{N}^*$, we consider a square grid of size $2^n \times 2^n$ from which a single cell has been removed and crossed out. Prove by induction that such a grid can be entirely covered by non-overlapping triominoes (L-shaped dominoes), as shown in Figure 1.

Exercise 7 - ***Triominoes, again.*** Prove that $\forall n \in \mathbb{N}$, $4^n - 1$ is a multiple of 3. You can use induction directly, or use indirectly a previous property that was already proven by induction.