École Pour l'Informatique et les Techniques Avancées

Approximation of Functions

Ch2: Series of Functions

Convergence

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KAMEL ATTAR

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$$\forall x \in I, \quad \lim_{n \to +\infty} \sum_{k=0}^{n} f_k(x) = S(x) \quad \Longleftrightarrow \quad \forall x \in I, \quad \lim_{n \to +\infty} S_n(x) = S(x)$$

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$$\lim_{n \to +\infty} \left\| \sum_{k=0}^{n} f_k - S \right\|_{\infty} = 0 \qquad \lim_{n \to +\infty} \sum_{k=0}^{n} \| f_k \|_{\infty} = C(Constant)$$

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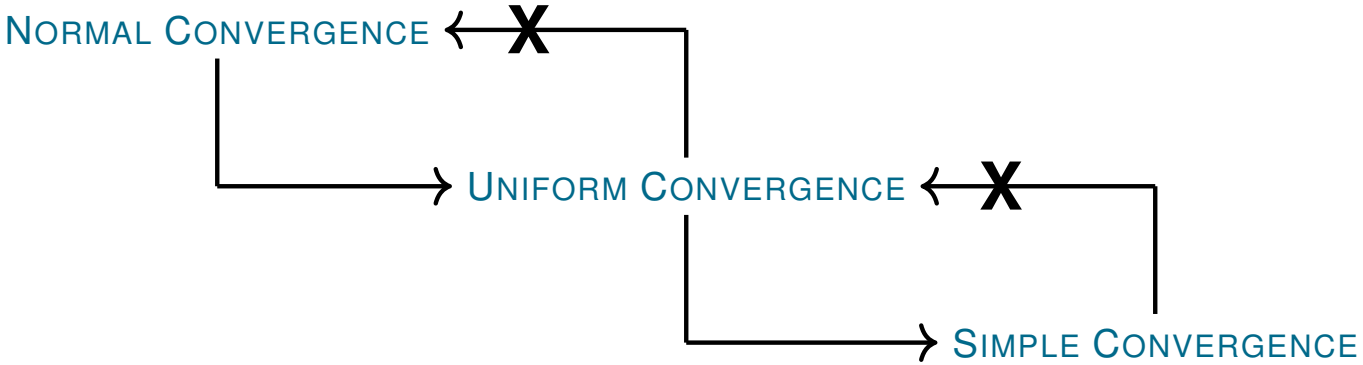

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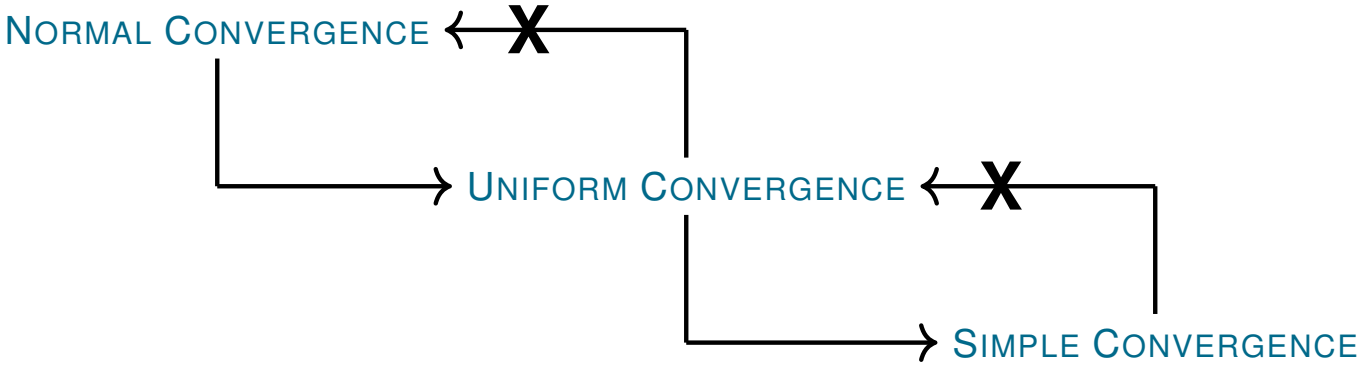

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At the end of this course, students will be able to:

- **Distinguish** the issues associated with the different notions of functional convergence.
- **Validate and study** the nature of series of functions.

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Why is the Region of Convergence a Disk?

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✎ **Wooclap 1.** Among the following series, which are series of functions?

1. $\sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$
2. $\sum_{n\geq 1}\left(\frac{x}{n}-\frac{1}{n+1}\right)$
3. $\sum_{n\geq 1}\frac{(-1)^n}{an}$ where $a\in\mathbb{R}$ is an arbitrary constant
4. $\sum_{n\geq 1}\int_0^{\pi/2}\left(\frac{\cos(nx)}{n^2}\right)dx$

▸ Solution

At this stage, we have only considered the convergence of sequences of functions.

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In the process of approximating a function f, it is common to construct sequences approaching f through a *cumulative* approach.
In other words, we aim to construct a series of functions:

$$\sum_{n=0}^{\infty} f_n(x)$$

where each $f_n(x)$ is a function depending on x and n, so that the partial sum:

$$S_N(x) = \sum_{n=0}^{N} f_n(x)$$

converges to $f(x)$ as $N \to \infty$.

Working with series of functions does not fundamentally alter the theory compared to sequences. However, it offers a new perspective that helps to better understand the modes of convergence.

Example

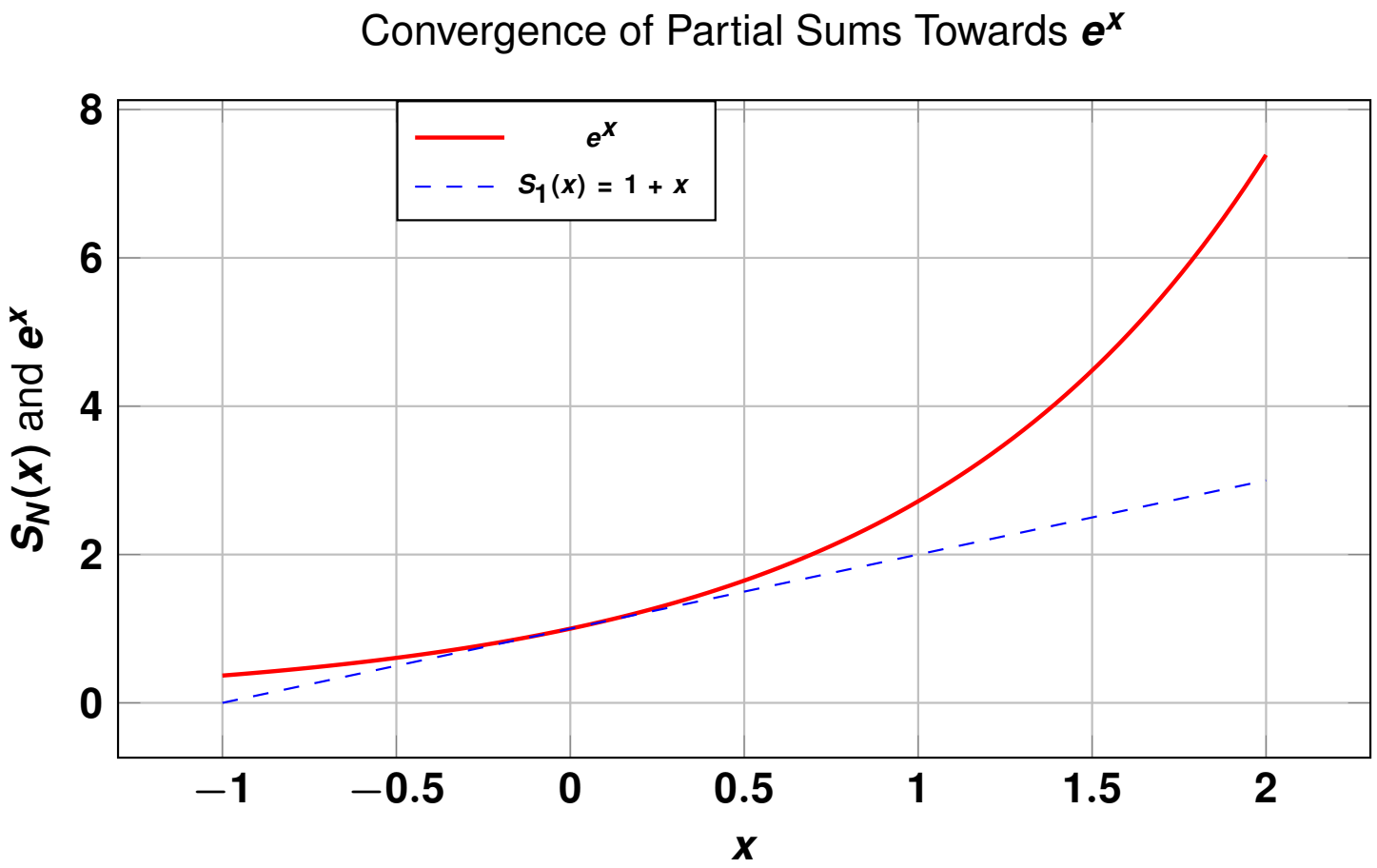
A classic example of approximation is the exponential series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Here, each term of the series $f_n(x) = \frac{x^n}{n!}$ is a function of x, and the series converges uniformly to the function e^x.

Graphical Illustration

The graph below shows the convergence of the partial sums $S_N(x)$ towards e^x for different values of N.



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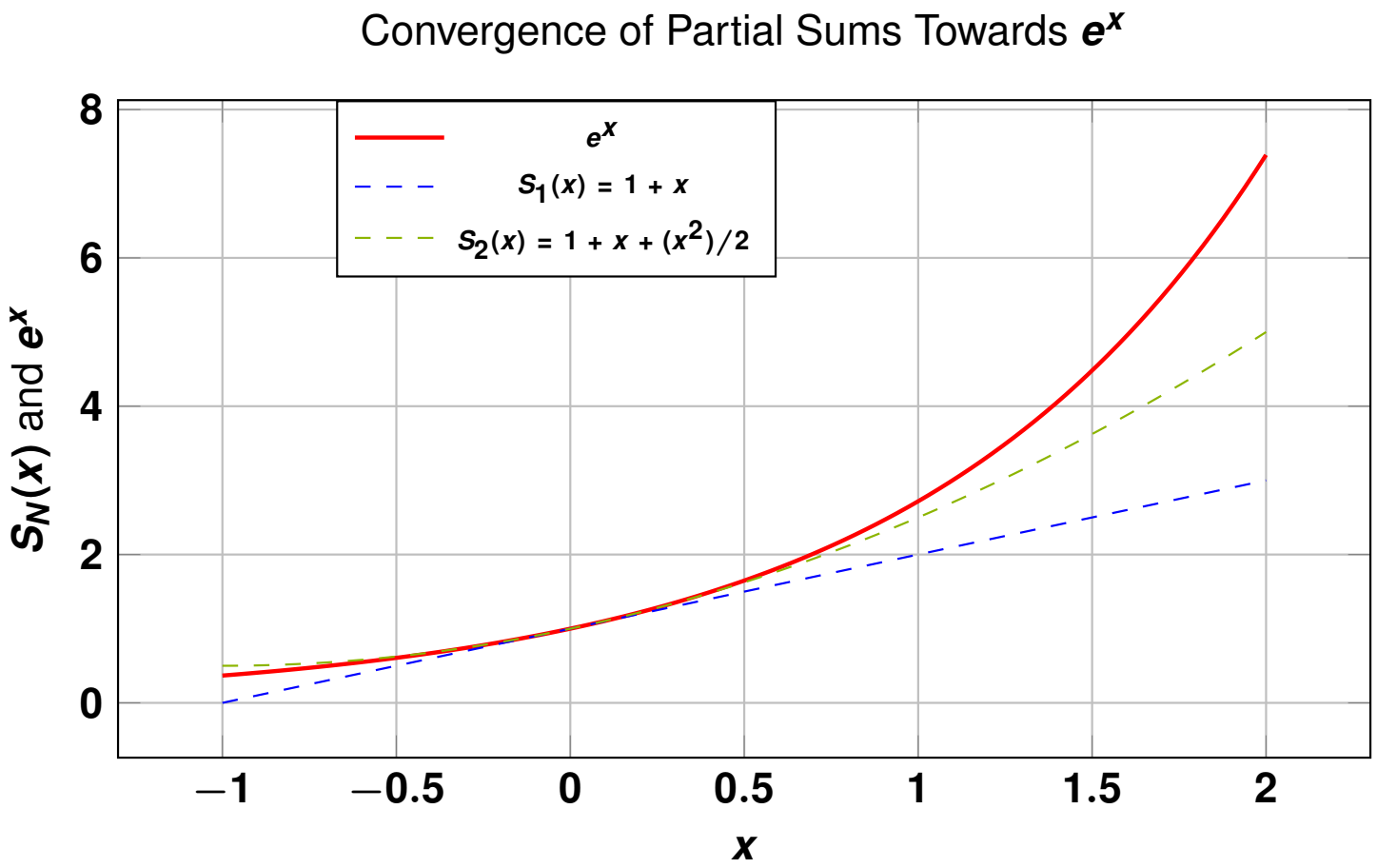

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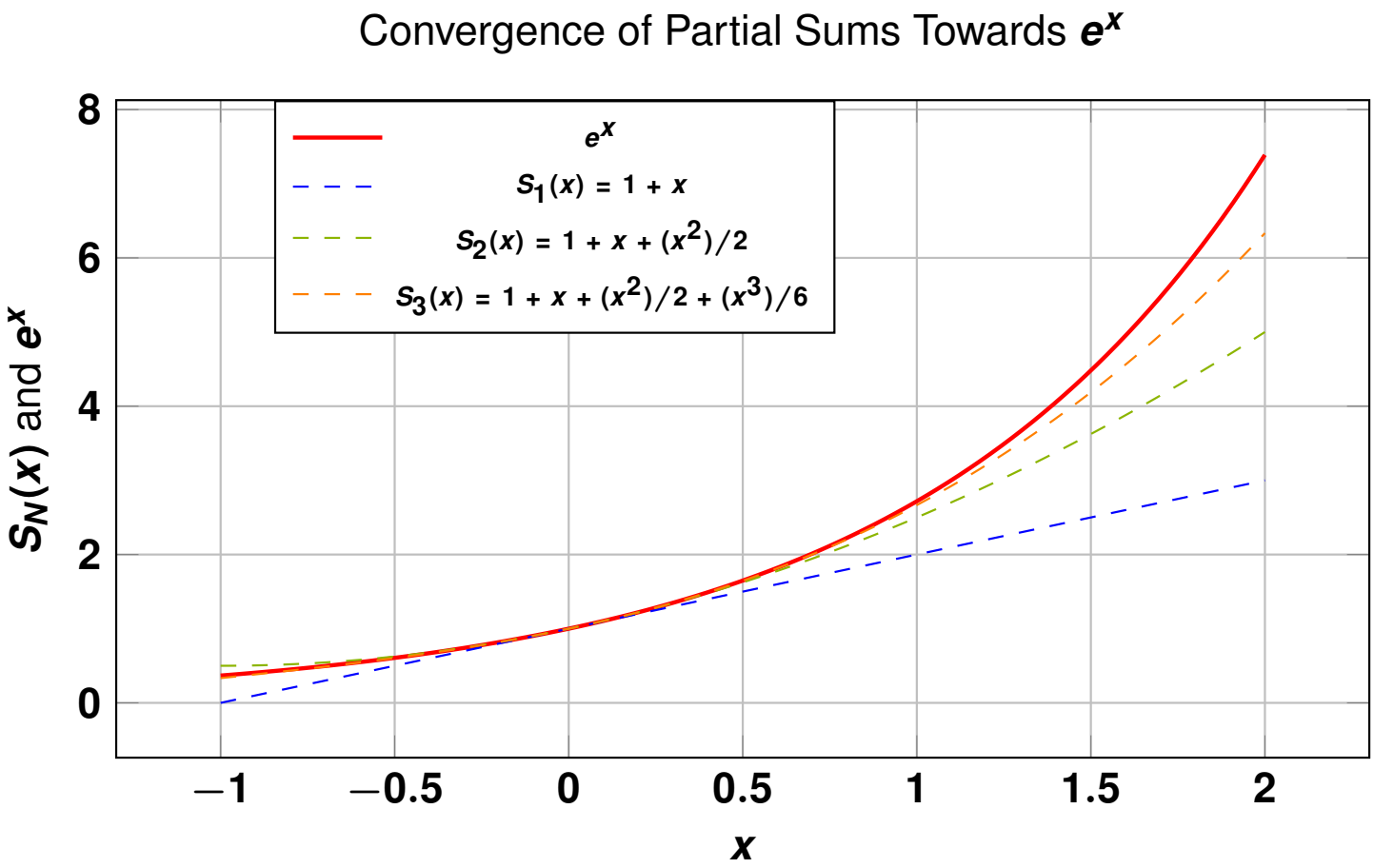

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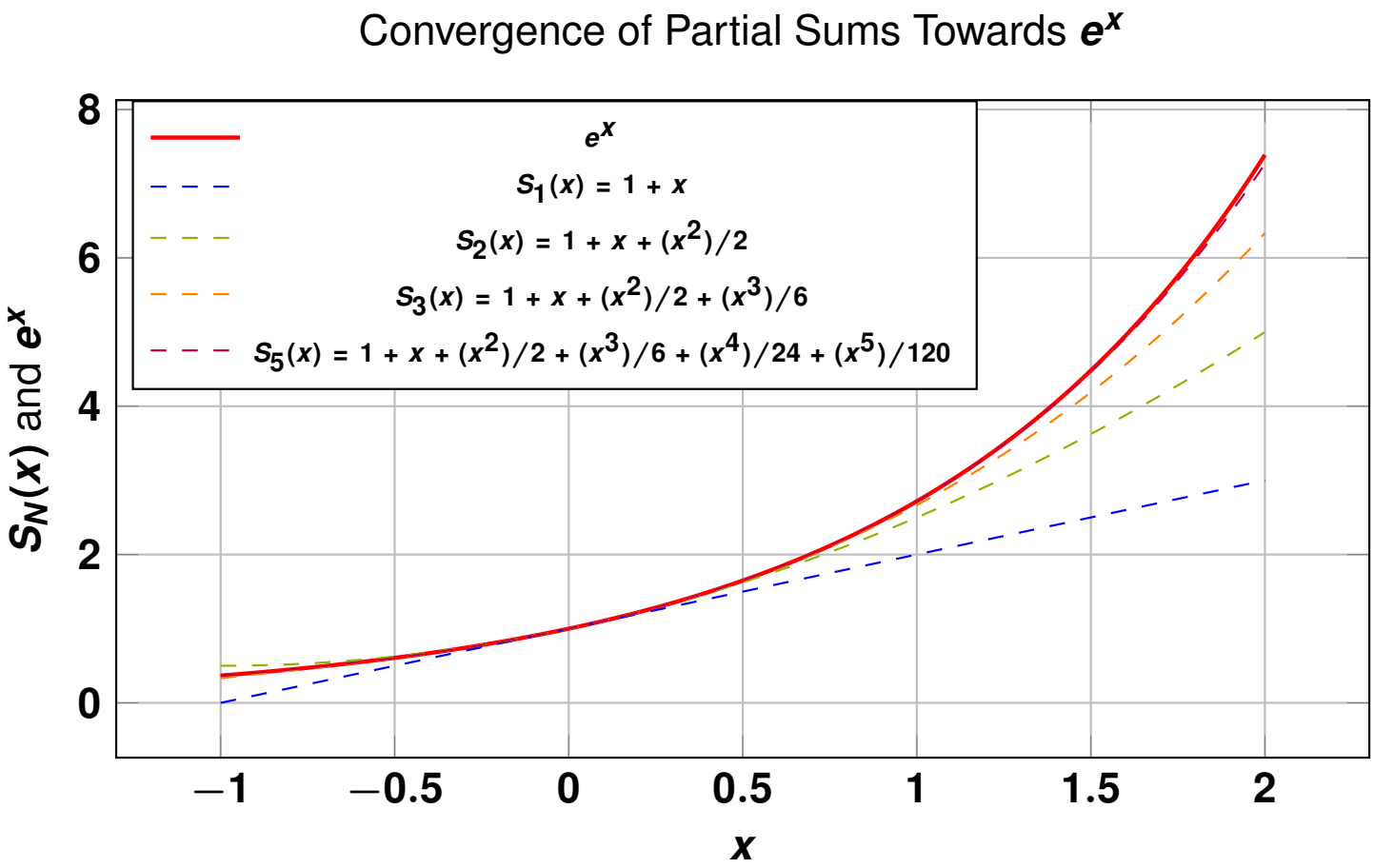

✎ **Wooclap 2.** How can we formally write that the series of functions $\sum_{n\geq 0} f_n$ converges to the function f on I? ▸ Solution

1. $\forall x \in I, \sum_{n\geq 0} f_n(x) = f(x)$
2. $\sum_{n\geq 0} f_n = f$
3. $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in I$
4. $\sum_{n=0}^{\infty} |f_n(x)| < \infty, \forall x \in I$

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Definition

Let $\sum_{n=0}^{+\infty} f_n(x)$ be a series of functions on I. We say that $\sum_{n=0}^{+\infty} f_n(x)$ converges simply to $S(x)$ on I if the sequence of partial sums $S_n = \sum_{k=0}^{n} f_k(x)$ converges simply to $S(x)$ on I.

$$\forall x \in I, \quad \lim_{n\to+\infty} \sum_{k=0}^{n} f_k(x) = S(x) \iff \forall x \in I, \quad \lim_{n\to+\infty} S_n(x) = S(x)$$

✎ **Wooclap 3. Let's study the convergence of the geometric series $\sum_{n\geq 0} x^n$, defined on $\mathbb{R}$.** On which interval of $\mathbb{R}$ does the geometric series $\sum_{n\geq 0} x^n$ converge, and what is its sum in this domain?

1. The series converges for $|x| < 1$ and the sum is given by $\frac{1}{1-x}$.
2. The series converges for $|x| \leq 1$, and the sum is given by $\frac{1}{1-x}$.
3. The series converges for $|x| < 1$, and the sum is $\sum_{n\geq 0} x^n = \infty$.
4. The series converges for all $x \in \mathbb{R}$, and the sum is $\frac{1}{1-x}$.

▸ Solution

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Let $\sum_{n=0}^{+\infty} f_n(x)$ be a series of functions on I. We say that $\sum_{n=0}^{+\infty} f_n(x)$ converges uniformly to $S(x)$ on I if the sequence of partial sums $S_n = \sum_{k=0}^{n} f_k(x)$ converges uniformly to $S(x)$ on I.

$$\boxed{\lim_{n\to+\infty} \left\| \sum_{k=0}^{n} f_k - S \right\|_\infty = 0 \iff \lim_{n\to+\infty} \|S_n - S\|_\infty = 0 \iff \lim_{n\to+\infty} \|R_n\|_\infty = 0}$$

Here $\underbrace{R_n = \sum_{k=n+1}^{+\infty} f_k(x)}_{\text{Remainder}}$.

✎ **Wooclap 4.** Consider the geometric series $\sum_{n=0}^{\infty} x^n$. Which statement is correct regarding its uniform convergence on an interval $I = [-a, a]$, with $a \in [0, 1[$?

1. The series converges uniformly for all $x \in \mathbb{R}$.
2. The series converges uniformly on any interval $I = [-a, a]$, with $a < 1$.
3. The series converges uniformly only for $a = 0$.
4. The series diverges on any interval I.
5. The series converges uniformly for all $x \in [-1, 1]$.

▸ Solution

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  4. CONCLUSION ON SIMPLE CONVERGENCE: If the series $\sum_{n\geq 0} f_n(x)$ converges for all $x \in I$, the series converges **simply** to the function $f(x)$. Otherwise, simple convergence is not assured.

- **Study of Uniform Convergence** Uniform convergence means that the convergence of the series to $\boldsymbol{f(x)}$ is uniform for all $\boldsymbol{x} \in \boldsymbol{I}$, i.e., the rate of convergence does not depend on the point $\boldsymbol{x}$. In other words, the truncation error must become arbitrarily small regardless of the choice of $\boldsymbol{x}$.

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  1. CALCULATE THE TRUNCATION ERROR: The truncation error after N terms is given by:

$$R_N(x) = S(x) - S_N(x) = \sum_{n=N+1}^{\infty} f_n(x)$$

  You need to check if this error tends uniformly to 0 as $N \to \infty$.

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  3. CONCLUSION ON UNIFORM CONVERGENCE: If $||R_N(x)||_\infty \to 0$ as $N \to \infty$, then the series converges **uniformly**.

✎ **Wooclap 5.** Let's study the convergence of the alternating series $\sum_{n\geq 1} \frac{(-1)^n x^n}{\sqrt{n}}$, defined on $]0, 1[$.

▸ Solution

Theorem :

If the series $\sum_{n=0}^{+\infty} f_n(x)$ converges simply (resp. uniformly) on an interval I, then the sequence $(f_n)_n$ converges simply (resp. uniformly) to the zero function on this interval.

This property is often used to demonstrate that a series of functions is not uniformly convergent. If the sequence (f_n) does not converge uniformly to 0, the series does not converge uniformly.

✎ **Wooclap 6.** Among the following series of functions, which converge uniformly on the given interval?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$, for $x \in [0, 1]$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nx)$, for $x \in [0, \pi]$.
3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$, for $x \in [0, 1]$.
4. $\displaystyle\sum_{n=1}^{\infty} \frac{nx}{n^2 + x^2}$, for $x \in [1, 2]$.
5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$, for $x \in [0, 2\pi]$.
6. All series do not converge uniformly.

▸ Solution

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Simple Convergence of a Series of Functions
Uniform Convergence of a Series of Functions
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A series of functions $\sum_{n\geq 0} f_n$ converges *normally* on I if the numeric series $\sum_{n\geq 0} \|f_n\|_\infty$ converges on I.

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This convergence is stronger than **uniform convergence**, and implies it, though the reverse is not true.

Theorem :

If a series of functions converges normally, then it converges uniformly.

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Proof NC $\Rightarrow$ UC

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$\Rightarrow$ There exists a convergent numeric series $\sum_{n\geq 0} a_n$ with $a_n \in \mathbb{R}^+$ and $\|f_n\|_\infty \leq a_n$ for all $n \in \mathbb{N}$.

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$\Rightarrow$ There exists a convergent numeric series $\sum_{n\geq 0} a_n$ with $a_n \in \mathbb{R}^+$ and $\|f_n\|_\infty \leq a_n$ for all $n \in \mathbb{N}$.

$\Rightarrow$ For all $x \in I$, we have $\|R_n\|_\infty = \|\sum_{k\geq n+1} f_k(x)\|_\infty \leq \sum_{k\geq n+1} \|f_k(x)\|_\infty \leq \sum_{k\geq n+1} a_n$

Assume that $(f_n)_n$ converges normally on I.

$\Rightarrow$ There exists a convergent numeric series $\sum_{n\geq 0} a_n$ with $a_n \in \mathbb{R}^+$ and $\|f_n\|_\infty \leq a_n$ for all $n \in \mathbb{N}$.

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$\Rightarrow$ Therefore, $\|R_n\|_\infty \to 0$ (Cauchy criterion for uniform convergence).

$\Rightarrow$ Hence, $\sum_{n\geq 0} f_n$ converges uniformly.

✎ **Wooclap 7.** Among the following series of functions, which converge normally?

1. The series of functions $\sum_{n\geq 1} f_n$ where the f_n are functions defined on $[0,1]$ by $f_n(x) = \frac{x^n}{n^\alpha}$, for $\alpha > 1$.
2. The series of functions $\sum_{n\geq 1} f_n$ where the f_n are functions defined on $[0,1]$ by $f_n(x) = \frac{(-1)^n x^n}{n}$.
3. The series of functions $\sum_{n\geq 1} f_n$ where the f_n are functions defined on $[0,a]$ by $f_n(x) = x^n$, for $a \in \mathbb{R}_+^*$
4. The series of functions $\sum_{n\geq 1} f_n$ where the f_n are functions defined on $[0,1/2]$ by $f_n(x) = \frac{x^n}{n^\alpha}$, for $\alpha \in \mathbb{R}$.

▸ Solution

1 Where To Start?

2 Simple and Uniform Convergence
Simple Convergence of a Series of Functions
Uniform Convergence of a Series of Functions
Method to Study Simple and Uniform Convergence of a Series of Functions

3 Normal Convergence
Definition of NC
Proof NC $\Rightarrow$ UC
Brief Review of Numeric Series
UC vs NC

4 Side-talk About Power Series
What is a Power Series?
Why is the Region of Convergence a Disk?

- Studying normal convergence of a series of functions reduces the study of a sequence of functions to that of a numeric series.

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✎ **Wooclap 10.** Provide criteria for the convergence of numeric series.

1. THE *nth*-Term Test for a Divergent Series: If $\lim_{n\to\infty} x_n \neq 0$ or $\lim_{n\to\infty} x_n$ fails to exist, then $\sum_{n=1} x_n$ diverges.

Example

(*a*) $\sum_{n=1}^{\infty}(-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

(*b*) $\sum_{n=1}^{\infty}\frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty}\frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

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2. DEFINITION OF CONVERGENCE: Try to find an explicit formula for the partial sums S_n, and study the sequence (S_n).

$\sum_{n=1}^{\infty} x_n$ converges if and only if the partial sum series $S_k = \sum_{n=1}^{k} x_n$ converges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{k\to\infty}\sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1 - \lim_{k\to\infty}\frac{1}{k+1} = 1$$

❸ p- Series (Riemann series):

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

- If $p > 1$, the series converges.
- If $p \leq 1$, the series diverges.

Example

a) The series $\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$ diverges because it's a p-series with $p = 0.5 < 1$.

b) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it's a p-series with $p = 1$.

c) The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it's a p-series with $p = 2 > 1$.

4. If $u_n \geq 0$ for every n, try to use the following criteria :

① COMPARISON TEST:

- If $x_n \leq y_n$ and $\sum y_n$ converges, then $\sum x_n$ also converges.
- If $z_n \leq x_n$ and $\sum z_n$ diverges, then $\sum x_n$ also diverges.

Hint: Often a series can be compared with one of the particular series above: a GEOMETRIC, HARMONIC, or p SERIES

Example

a) The series $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges. Indeed let $x_n = \frac{5}{5n-1}$ we have

$$5n - 1 < 5n \iff \frac{1}{5n-1} > \frac{1}{5n} \iff \frac{5}{5n-1} > \frac{1}{n}$$

So, let $y_n = \frac{1}{n}$, the series $\sum y_n$ is a p-series with $p = 1$ then it's diverges.
Since $x_n > y_n$ and $\sum y_n$ diverges, we obtain using the comparison test that $\sum x_n$ diverges.

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① COMPARISON TEST:

- If $x_n \leq y_n$ and $\sum y_n$ converges, then $\sum x_n$ also converges.
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Hint: Often a series can be compared with one of the particular series above: a GEOMETRIC, HARMONIC, or p SERIES

Example

b) The series $\sum_0^\infty \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{3!} + \cdots$ converges because its terms are all positive and

$$2^n \leq n! \iff \frac{1}{n!} \leq \frac{1}{2^n}.$$

$\sum \frac{1}{2^n}$ is a geometric series with $q = \frac{1}{2} < 1$ then its converges and we have

$$\sum_0^\infty \frac{1}{2^n} = \frac{1}{1 - 1/2} = 2.$$

② LIMIT COMPARISON TEST:

- If $\lim_{n\to\infty} \frac{x_n}{y_n} = c > 0$ then $\sum x_n$ and $\sum y_n$, both converge or both diverge.
- If $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$ and $\sum y_n$ converges then $\sum x_n$ converges.
- If $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$ and $\sum y_n$ diverges then $\sum x_n$ diverges.

Example

a) $\sum_{n=0}^{\infty} \frac{2n+1}{n^2+2n+1}$ is diverges. Let $x_n = \frac{2n+1}{n^2+2n+1}$. $\lim_{n\to\infty} x_n \approx \frac{2}{n}$, so we let $y_n = \frac{1}{n}$.
Since

$$\sum y_n = \sum \frac{1}{n} \quad \text{diverges}, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{x_n}{y_n} = 2,$$

then $\sum x_n$ diverges by Part **1** of the Limit Comparison Test.

② LIMIT COMPARISON TEST:

- If $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = c > 0$ then $\sum x_n$ and $\sum y_n$, both converge or both diverge.
- If $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = 0$ and $\sum y_n$ converges then $\sum x_n$ converges.
- If $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \infty$ and $\sum y_n$ diverges then $\sum x_n$ diverges.

Example

b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n - 1}$ is converges. Let $x_n = \dfrac{1}{2^n - 1}$. We have $\lim\limits_{n\to\infty} x_n \approx \lim\limits_{n\to\infty} 1/2^n$, so we let $y_n = \dfrac{1}{2^n}$ Since

$$\sum y_n \quad \text{converges}, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1,$$

then $\sum x_n$ converges by Part **1** of the Limit Comparison Test.

② LIMIT COMPARISON TEST:

- If $\lim_{n\to\infty} \frac{x_n}{y_n} = c > 0$ then $\sum x_n$ and $\sum y_n$, both converge or both diverge.
- If $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$ and $\sum y_n$ converges then $\sum x_n$ converges.
- If $\lim_{n\to\infty} \frac{x_n}{y_n} = \infty$ and $\sum y_n$ diverges then $\sum x_n$ diverges.

Example

c) $\sum_{n=0}^{\infty} \frac{1 + n\ln n}{n^2 + 5}$ is diverges. Let $x_n = \frac{1 + n\ln n}{n^2 + 5}$. $\lim_{n\to\infty} x_n \approx \lim_{n\to\infty} \frac{\ln n}{n}$, which is greate than $1/n$, so we let y_n. Since

$$\sum y_n \quad \text{diverges}, \qquad \text{and} \qquad \lim_{n\to\infty} \frac{x_n}{y_n} = \infty,$$

then $\sum x_n$ diverges.

③ ROOT TEST:

Suppose $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \rho$.

- If $\rho < 1$ then $\sum\limits_{n=1}^{\infty} x_n$ converges.
- If $\rho > 1$ or $\rho = \infty$ then $\sum\limits_{n=1}^{\infty} x_n$ diverges.
- If $\rho = 1$ then the test is inconclusive.

Example

a) $\sum\limits_{n=1}^{\infty} \frac{n^2}{2^n}$ converges because $\lim\limits_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \lim\limits_{n\to\infty} \frac{n^{2/n}}{2} = \frac{1}{2} < 1$.

b) $\sum\limits_{n=1}^{\infty} \frac{2^n}{n^3}$ diverges because $\lim\limits_{n\to\infty} \sqrt[n]{\frac{2^n}{n^3}} = \lim\limits_{n\to\infty} \frac{2}{n^{3/n}} = 2 > 1$

④ RATIO TEST:

Suppose $\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \rho$.

- If $\rho < 1$ then $\sum\limits_{n=1}^{\infty} x_n$ converges.
- If $\rho > 1$ or $\rho = \infty$ then $\sum\limits_{n=1}^{\infty} x_n$ diverges.
- If $\rho = 1$ then the test is inconclusive.

Example

$\sum\limits_{n=1}^{\infty} \dfrac{n^2(n+2)!}{n!3^{2n}}$ is converges. Let $x_n = \dfrac{n^2(n+2)!}{n!3^{2n}}$ and $x_{n+1} = \dfrac{(n+1)^2(n+3)!}{(n+1)!3^{2n+2}}$ then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)^2(n+3)!}{(n+1)!3^{2n+2}} \times \frac{n!3^{2n}}{n^2(n+2)!} = \lim_{n\to\infty} \frac{(n+1)(n+3)}{(3n)^2} = \frac{1}{3} < 1$$

Thus $\sum x_n$ converges.

5. If u_n does not have a constant sign, try

1. ABSOLUTE CONVERGENCE:

- If $\sum_{n=1}^{\infty} |x_n|$ converges then $\sum_{n=1}^{\infty} x_n$ converges.

Example

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

We have $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ is a p-series with $p = 2 > 1$ then it's converges, this implies that $\sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right|$ converges and so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ converges.

② ALTERNATING SERIES TEST:

- If $x_n > 0$ is decreasing and if $\lim_{n\to\infty} x_n \to 0$ then $\sum_{n=1}^{\infty}(-1)^n x_n$ converges and $R_n \leq |x_{n+1}|$.

  Hint: this is usually the simplest test if you see a $(-1)^n$ in the expression.

Example

The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

clearly satisfies the three requirements of the theorem; it therefore converges.
Indeed, let $x_n = \frac{1}{n}$. We have

$$x_{n+1} - x_n < 0 \qquad \text{and} \qquad \lim_{n\to\infty} x_n = 0\,.$$

Commonly Occurring Limits

① $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

② $\lim_{n\to\infty} \ln(n) = \infty$

③ $\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0)$

④ $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

⑤ $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a \quad (\text{any } a)$

⑥ $\lim_{n\to\infty} n! = \infty$

⑦ $\lim_{n\to\infty} a^n = \infty \quad (|a| > 1)$

⑧ $\lim_{n\to\infty} a^n = 0 \quad (|a| < 1)$

⑨ $\lim_{n\to\infty} \frac{n!}{a^n} = \infty \quad (\text{any } a)$

⑩ $\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (\text{any } a)$

①① $\lim_{n\to\infty} \frac{a^n}{n^c} = \infty \quad (\text{any } a > 1,\ c > 0)$

①② $\lim_{n\to\infty} \frac{n^c}{\ln(n)} = \infty \quad (\text{any } c > 0)$

☞ How do we know which test to use?

1. If you can see easily that $\lim_{n\to\infty} x_n \neq 0$, apply the **Test for Divergence**.
2. Is $\sum x_n$ a p-**series** or **geometric series**? If yes, apply those tests.
3. If the series $\sum x_n$ appears to be a **telescoping sum**, then find a closed formula for the partial sum s_n and use $\sum x_n = \lim_{n\to\infty} s_n$.
4. Is $\sum x_n$ similar to a p-**series** or **geometric series**? If yes, try one of the **Comparison Tests** . You will compare $\sum x_n$ with the series $\sum y_n$ that $\sum x_n$ is similar to.
   a) If the inequality works out the way you need it to, you will you the **Comparison Test**.
   b) If the inequality does not work out the way you need it to, try the **Limit Comparison**.
5. If the series is of the form $\sum (-1)^{n+1} x_n$ or $\sum (-1)^n x_n$ try the **Alternating Series Test**.
6. When earlier tests can not be used simply because some of the terms may be negative, try using the **Absolute Convergence Test**.
7. Series involving factorials (e.g. $n!$) or n-th powers of a constant (e.g. 4^n) can often be studied with the **Ratio Test** (if an easier test does not work.)
8. When it looks like $(..)^n$, and the term inside the parentheses also involves n, try the **Root Test**.

1 Where To Start?

2 Simple and Uniform Convergence
Simple Convergence of a Series of Functions
Uniform Convergence of a Series of Functions
Method to Study Simple and Uniform Convergence of a Series of Functions

3 Normal Convergence
Definition of NC
Proof NC $\Rightarrow$ UC
Brief Review of Numeric Series
UC vs NC

4 Side-talk About Power Series
What is a Power Series?
Why is the Region of Convergence a Disk?

☞ What about cases of uniform convergence that are not normal?

- Uniform convergence of a series of functions does **not guarantee** normal convergence.

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- More formally, a series of functions $\sum f_n(x)$ converges uniformly but not normally on a set I if the following conditions are satisfied:
  - There exists a subset I' of I on which the series $\sum f_n(x)$ converges uniformly.
  - The series $\sum f_n(x)$ does not converge uniformly on I.

Example

The series $\sum_{n\geq 1} \frac{x^n}{n}$ converges uniformly on every closed subinterval $[0, a]$ of $[0, 1]$ $(0 < a < 1)$, but it does not converge uniformly on the entire interval $[0, 1]$ according to the Cauchy criterion for uniform convergence.

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- Uniform convergence that is not normal can be more complex to establish than normal uniform convergence, as it often requires a more detailed analysis of the properties of the functions and the set of convergence.

$$\begin{tikzcd}
\textsc{Normal Convergence} \arrow[dr] & & \\
& \textsc{Uniform Convergence} \arrow[ul, "\times"'] \arrow[dr] & \\
& & \textsc{Simple Convergence} \arrow[ul, "\times"']
\end{tikzcd}$$

1 Where To Start?

2 Simple and Uniform Convergence
Simple Convergence of a Series of Functions
Uniform Convergence of a Series of Functions
Method to Study Simple and Uniform Convergence of a Series of Functions

3 Normal Convergence
Definition of NC
Proof NC $\Rightarrow$ UC
Brief Review of Numeric Series
UC vs NC

4 Side-talk About Power Series
What is a Power Series?
Why is the Region of Convergence a Disk?

1 Where To Start?

2 Simple and Uniform Convergence
Simple Convergence of a Series of Functions
Uniform Convergence of a Series of Functions
Method to Study Simple and Uniform Convergence of a Series of Functions

3 Normal Convergence
Definition of NC
Proof NC $\Rightarrow$ UC
Brief Review of Numeric Series
UC vs NC

4 Side-talk About Power Series
What is a Power Series?
Why is the Region of Convergence a Disk?

Definition

A power series S is a series of functions of the form $\sum_{n \geq 0} a_n x^n$, where the a_n are real numbers called the coefficients of S.

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Power series extend polynomial functions by allowing an infinite sum of terms. They are relatively simple to manipulate and have frequent applications in discrete probability, combinatorial analysis, and solving differential equations.

1 Where To Start?

2 Simple and Uniform Convergence
Simple Convergence of a Series of Functions
Uniform Convergence of a Series of Functions
Method to Study Simple and Uniform Convergence of a Series of Functions

3 Normal Convergence
Definition of NC
Proof NC $\Rightarrow$ UC
Brief Review of Numeric Series
UC vs NC

4 Side-talk About Power Series
What is a Power Series?
Why is the Region of Convergence a Disk?

☞ Why is the region of convergence of a power series a disk?

✎ **Wooclap 11.** Find the domain of convergences of the followings:

$$1 + x + x^2 + \cdots + x^n + \cdots \quad \text{and} \quad 1 - \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(-\frac{1}{2}\right)^n x^n + \cdots$$

▸ Solution

Theorem :

Let x_0 be a real number such that the numerical sequence $\left(a_n x_0^n\right)_n$ is bounded. Then, for all numbers x such that $|x| < |x_0|$, the power series $\sum n \geq 0 a_n x^n$ converges normally.

Demonstration :

Theorem :

Let x_0 be a real number such that the numerical sequence $\left(a_n x_0^n\right)_n$ is bounded. Then, for all numbers x such that $|x| < |x_0|$, the power series $\sum n \geq 0 a_n x^n$ converges normally.

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By hypothesis, there exists M such that for all $n \geq 0$, we have: $a_n x_0^n \leq M$

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Demonstration :

By hypothesis, there exists M such that for all $n \geq 0$, we have: $a_n x_0^n \leq M$

Let $x \in]-|x_0|, |x_0|[$. We have :

$$\left|a_n x^n\right| = \left|a_n x_0^n \frac{x^n}{x_0^n}\right| \leq M \left|\frac{x}{x_0}\right|^n \Longrightarrow \left\|a_n x^n\right\|_\infty \leq M \left|\frac{x}{x_0}\right|^n$$

Theorem :

Let x_0 be a real number such that the numerical sequence $\left(a_n x_0^n\right)_n$ is bounded. Then, for all numbers x such that $|x| < |x_0|$, the power series $\sum n \geq 0 a_n x^n$ converges normally.

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$$\left|a_n x^n\right| = \left|a_n x_0^n \frac{x^n}{x_0^n}\right| \leq M \left|\frac{x}{x_0}\right|^n \Longrightarrow \left\|a_n x^n\right\|_\infty \leq M \left|\frac{x}{x_0}\right|^n$$

As $\frac{x}{x_0} < 1$, $M \left|\frac{x}{x_0}\right|^n$ is the general term of a (geometric) series that converges to $\frac{M}{1-|x/x_0|}$.
Then $\sum_{n\geq 0} a_n x^n$ converges normally over $]-|x_0|, |x_0|[$.

✎ **Wooclap 12.** How do you compute the convergence radius of a power series ?

The main methods for calculating the radius of convergence of a power series are the following two characterizations:

- **First characterization of the radius of convergence:** The radius of convergence $\boldsymbol{R}$ of the power series $\sum \boldsymbol{a_n x^n}$ is equal to $\boldsymbol{R}$ if, and only if,
  - for $|\boldsymbol{x}| < \boldsymbol{R}$, the sequence $\boldsymbol{a_n x^n}$ is bounded, and
  - for $|\boldsymbol{x}| > \boldsymbol{R}$, the sequence $\boldsymbol{a_n x^n}$ is unbounded.

The main methods for calculating the radius of convergence of a power series are the following two characterizations:

- **First characterization of the radius of convergence:** The radius of convergence R of the power series $\sum a_n x^n$ is equal to R if, and only if,
  - for $|x| < R$, the sequence $a_n x^n$ is bounded, and
  - for $|x| > R$, the sequence $a_n x^n$ is unbounded.
- **Second characterization of the radius of convergence:** The radius of convergence R of the power series $\sum a_n x^n$ is equal to R if, and only if,
  - for $|x| < R$, the series $\sum a_n x^n$ is absolutely convergent, and
  - for $|x| > R$, the series $\sum a_n x^n$ diverges significantly.

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- **Second characterization of the radius of convergence:** The radius of convergence $\boldsymbol{R}$ of the power series $\sum \boldsymbol{a_n x^n}$ is equal to $\boldsymbol{R}$ if, and only if,
  - for $|\boldsymbol{x}| < \boldsymbol{R}$, the series $\sum \boldsymbol{a_n x^n}$ is absolutely convergent, and
  - for $|\boldsymbol{x}| > \boldsymbol{R}$, the series $\sum \boldsymbol{a_n x^n}$ diverges significantly.
- If the sequence $\sum \boldsymbol{a_n x^n}$ tends to $\mathbf{0}$ for $|\boldsymbol{x}| < \boldsymbol{R}$ and if the series $\boldsymbol{a_n x^n}$ diverges significantly for $|\boldsymbol{x}| > \boldsymbol{R}$, then the radius of convergence is equal to $\boldsymbol{R}$. $\sum \boldsymbol{n^\alpha x^n}$ with $\alpha \in \mathbb{R}$, $\boldsymbol{R = 1}$

The main methods for calculating the radius of convergence of a power series are the following two characterizations:

- **First characterization of the radius of convergence:** The radius of convergence $\boldsymbol{R}$ of the power series $\sum \boldsymbol{a_n x^n}$ is equal to $\boldsymbol{R}$ if, and only if,
  - for $|\boldsymbol{x}| < \boldsymbol{R}$, the sequence $\boldsymbol{a_n x^n}$ is bounded, and
  - for $|\boldsymbol{x}| > \boldsymbol{R}$, the sequence $\boldsymbol{a_n x^n}$ is unbounded.
- **Second characterization of the radius of convergence:** The radius of convergence $\boldsymbol{R}$ of the power series $\sum \boldsymbol{a_n x^n}$ is equal to $\boldsymbol{R}$ if, and only if,
  - for $|\boldsymbol{x}| < \boldsymbol{R}$, the series $\sum \boldsymbol{a_n x^n}$ is absolutely convergent, and
  - for $|\boldsymbol{x}| > \boldsymbol{R}$, the series $\sum \boldsymbol{a_n x^n}$ diverges significantly.
- If the sequence $\sum \boldsymbol{a_n x^n}$ tends to $\mathbf{0}$ for $|\boldsymbol{x}| < \boldsymbol{R}$ and if the series $\boldsymbol{a_n x^n}$ diverges significantly for $|\boldsymbol{x}| > \boldsymbol{R}$, then the radius of convergence is equal to $\boldsymbol{R}$. $\sum \boldsymbol{n^\alpha x^n}$ with $\alpha \in \mathbb{R}$, $\boldsymbol{R = 1}$
- **Using D'Alembert's Ratio Test**

e^x	$=$	$\sum_{n\geq 0} \frac{x^n}{n!}$,	$R = +\infty$
$\cos x$	$=$	$\sum_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!}$,	$R = +\infty$
$\sin x$	$=$	$\sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$,	$R = +\infty$
$\cosh x$	$=$	$\sum_{n\geq 0} \frac{x^{2n}}{(2n)!}$,	$R = +\infty$
$\sinh x$	$=$	$\sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}$,	$R = +\infty$
$\frac{1}{1-x}$	$=$	$\sum_{n\geq 0} x^n$,	$R = 1$
$\ln(1+x)$	$=$	$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n$,	$R = 1$
$\arctan(x)$	$=$	$\sum_{n\geq 0} \frac{(-1)^n}{2n+1} x^{2n+1}$,	$R = 1$
$(1+x)^\alpha$	$=$	$1 + \sum_{n\geq 1} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$,	$R = 1$

Before You Leave

Before You Leave

1. Review the videos and remediation materials.

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2. Try to solve the exercises from TD..

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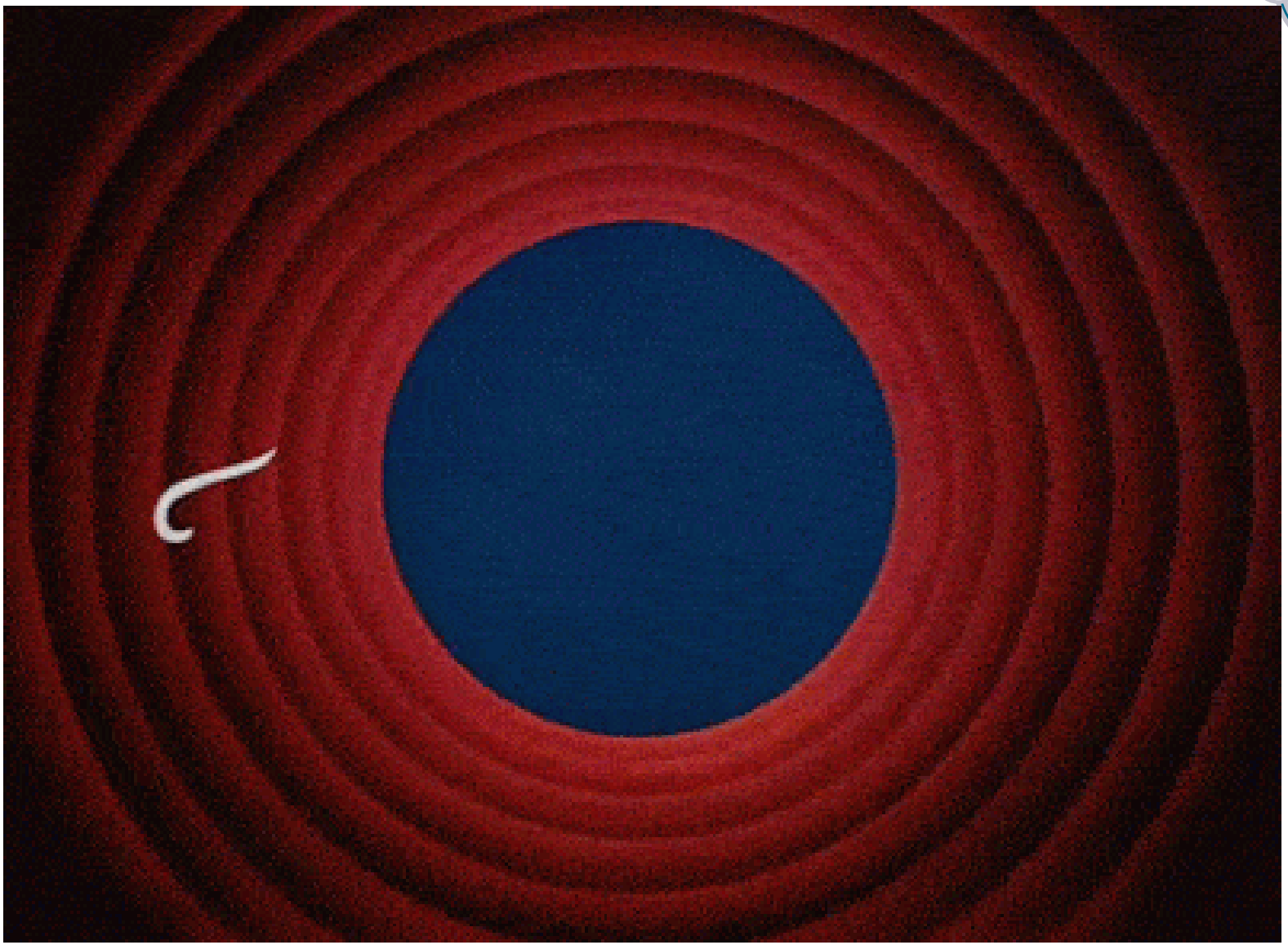
2. Try to solve the exercises from TD..

Reminder: You have a formative assessment (to complete at home) by the end of the week.



Solution 1.

1. $\sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

   n'est pas une série de fonctions, car chaque terme est une constante (indépendant de x).

2. $\sum_{n\geq 1}\left(\frac{x}{n}-\frac{1}{n+1}\right)$

   est une série de fonctions, car les termes dépendent de x .

3. $\sum_{n\geq 1}\frac{(-1)^n}{an}$ avec $a \in \mathbb{R}$ une constante arbitraire

   n'est pas une série de fonctions, car chaque terme dépend d'une constante a et non d'une variable.

4. $\sum_{n\geq 1}\int_0^{\pi/2}\left(\frac{\cos(nx)}{n^2}\right)dx$ n'est pas une série de fonctions, bien que l'intégrale soit évaluée sur un intervalle, chaque terme $\frac{\cos(nx)}{n^2}$ est une fonction de n.

Go Back

Solution 2.

1. $\forall x \in I, \sum_{n\geq 0} f_n(x) = f(x)$
2. $\sum_{n\geq 0} f_n = f$
3. $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in I$
4. $\sum_{n=0}^{\infty} |f_n(x)| < \infty, \forall x \in I$

$$\forall x \in I, \quad \forall \varepsilon > 0, \quad \exists N_\varepsilon \in \mathbb{N}, \quad n \geq N_\varepsilon \Longrightarrow \left| \sum_{k=0}^{n} f_k(x) - f(x) \right| < \varepsilon.$$

Cela signifie que pour tout x dans le domaine de convergence I et pour des valeurs de n suffisamment grandes, la somme partielle de la série des fonctions $\sum_{k=0}^{n} f_k$ se rapproche arbitrairement de la fonction f dans le sens où la différence entre la somme partielle et la fonction f devient arbitrairement petite (c'est-à-dire inférieure à ϵ) lorsque ϵ devient arbitrairement petit.

Solution 3.

1. La série géométrique $\sum_{n\geq 0} x^n$ converge ponctuellement pour $|x| < 1$, et la somme est bien $\frac{1}{1-x}$, la formule classique de la série géométrique.

2. La série diverge pour $x = 1$ et $x = -1$, donc elle ne converge pas pour $|x| \leq 1$. La somme n'est pas définie pour ces valeurs.

3. Cette réponse prétend que la somme est infinie, ce qui est incorrect car la série géométrique converge et a une somme finie dans le domaine $|x| < 1$.

4. La série ne converge pas pour tout $x \in \mathbb{R}$, elle diverge pour $|x| \geq 1$.

Go Back

La série géométrique $\sum_{n\geq 0} x^n$ est une série très classique et importante en analyse. Elle est définie par la somme infinie des puissances de x, c'est-à-dire : $S(x) = \sum_{n=0}^{\infty} x^n$

- **Convergence absolue et rayon de convergence**: Pour étudier la convergence de cette série, on commence par appliquer le critère du **rapport** ou **quotient**, souvent utilisé pour les séries géométriques.

  Application du critère du rapport :

$$\left|\frac{x^{n+1}}{x^n}\right| = |x|$$

  Pour que la série converge, il est nécessaire que le rapport des termes successifs tende vers un nombre strictement inférieur à 1. Ainsi, la série converge si :

$$|x| < 1$$

  Cela signifie que la série géométrique $\sum_{n\geq 0} x^n$ converge absolument pour $|x| < 1$. En revanche, pour $|x| \geq 1$, la série diverge.

- **Somme de la série**

  Lorsque $|x| < 1$, la série géométrique admet une somme finie. En effet, on sait que la somme d'une série géométrique de raison x pour $|x| < 1$ est donnée par la formule classique :

$$S(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

  Cette formule est valide uniquement pour $|x| < 1$.

- **Comportement aux bornes**
  - Cas $x = 1$:

$$S(1) = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \dots$$

    La série diverge pour $x = 1$, car la somme des termes devient infinie.
  - Cas $x = -1$:

$$S(-1) = \sum_{n=0}^{\infty} (-1)^n$$

    La série alterne entre 1 et -1, et donc elle ne tend pas vers une valeur finie. Elle diverge également.

- **Conclusion : rayon de convergence**
  - La série géométrique $\sum_{n\geq 0} x^n$ converge absolument pour $|x| < 1$.
  - La somme de la série dans ce domaine est $\frac{1}{1-x}$.
  - Pour $|x| \geq 1$, la série diverge.

  Le **rayon de convergence** de cette série est donc $R = 1$, et la série converge dans le domaine $]-1, 1[$.

Go Back

Solution 4. Considérons la série géométrique $\sum_{n=0}^{\infty} x^n$. Quelle affirmation est correcte concernant sa convergence uniforme sur un intervalle $I = [-a, a]$, avec $a \in [0, 1[$?

1. La série converge uniformément pour tout $x \in \mathbb{R}$.
   la réponse est fausse, car la série ne converge uniformément que sur les intervalles où $|x| < 1$.

2. La série converge uniformément sur tout intervalle $I = [-a, a]$, avec $a < 1$.

3. La série converge uniformément uniquement pour $a = 0$.
   Elle est fausse, car la série converge uniformément pour des intervalles plus larges que $a = 0$.

4. La série diverge sur tout intervalle I.
   Elle est fausse, car la série ne diverge pas sur tout intervalle ; elle converge pour $|x| < 1$.

5. La série converge uniformément pour tout $x \in [-1, 1]$.
   Elle est fausse, car la série ne converge pas uniformément sur $[-1, 1]$, mais seulement sur des intervalles strictement inclus dans $(-1, 1)$.

Pour étudier la **convergence uniforme** de la série géométrique $\sum_{n\geq 0} x^n$ sur un intervalle I en utilisant la définition de la convergence uniforme et en cherchant la norme infinie, nous suivons les étapes suivantes.

- **Rappel de la convergence uniforme** La série $\sum_{n\geq 0} f_n(x)$ converge uniformément vers une fonction $f(x)$ sur un intervalle I si et seulement si, pour tout $\epsilon > 0$, il existe un $N \in \mathbb{N}$ tel que pour tout $x \in I$ et pour tout $n \geq N$:

$$\left| \sum_{k=N}^{n} f_k(x) \right| < \epsilon$$

Cela signifie que la différence entre la somme partielle et la somme de la série devient arbitrairement petite indépendamment du point x considéré sur l'intervalle I.
Dans notre cas, nous étudions la série géométrique $\sum_{n\geq 0} x^n$, où chaque terme est $f_n(x) = x^n$.

- **Somme partielle et erreur de troncature**
  La somme partielle de la série géométrique jusqu'au terme N est donnée par :

$$S_N(x) = \sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}, \quad \text{pour } |x| < 1$$

La somme infinie est :

$$S(x) = \frac{1}{1 - x}, \quad \text{pour } |x| < 1$$

L'erreur de troncature après N termes est donc :

$$R_N(x) = S(x) - S_N(x) = \frac{1}{1 - x} - \frac{1 - x^{N+1}}{1 - x} = \frac{x^{N+1}}{1 - x}$$

- **Chercher la norme infinie**
  Pour vérifier la convergence uniforme, nous devons examiner comment se comporte le terme d'erreur $|R_N(x)|$ sur un intervalle $I \subset (-1, 1)$. Nous calculons la **norme infinie** de l'erreur, c'est-à-dire la borne supérieure de $|R_N(x)|$ sur cet intervalle.
  Prenons $I = [-a, a]$, où $a \in [0, 1[$, et considérons $|R_N(x)|$:

$$|R_N(x)| = \left| \frac{x^{N+1}}{1 - x} \right|$$

Pour $|x| \leq a < 1$, on a :

$$|R_N(x)| \leq \frac{|x|^{N+1}}{1 - |x|} \leq \frac{a^{N+1}}{1 - a}$$

La norme infinie sur $I = [-a, a]$ est donc :

$$\|R_N(x)\|_\infty = \sup_{x \in [-a,a]} |R_N(x)| = \frac{a^{N+1}}{1 - a}$$

- **Vérification de la convergence uniforme** Go Back
  Pour vérifier la convergence uniforme, on veut que la norme infinie de l'erreur tende vers 0 lorsque $N \to \infty$, c'est-à-dire que :

$$\frac{a^{N+1}}{1-a} \to 0 \quad \text{lorsque } N \to \infty$$

  Comme $a \in [0, 1[$, on a $a^{N+1} \to 0$ lorsque $N \to \infty$, et donc :

$$\lim_{N\to\infty} \|R_N(x)\|_\infty = 0$$

  Cela montre que pour tout $\epsilon > 0$, il existe un N suffisamment grand tel que :

$$\|R_N(x)\|_\infty = \sup_{x\in I} |R_N(x)| < \epsilon$$

  Donc la série géométrique $\sum_{n\geq 0} x^n$ converge **uniformément** sur tout intervalle $I = [-a, a]$ avec $a < 1$.
- **Conclusion**
  - La série géométrique $\sum_{n\geq 0} x^n$ converge **uniformément** sur tout intervalle $I = [-a, a]$ avec $a < 1$.
  - Cette convergence est assurée car la norme infinie de l'erreur $|R_N(x)|$ tend vers 0 lorsque N tend vers l'infini.
  - Le rayon de convergence reste $R = 1$, et la série converge uniformément dans tout sous-intervalle fermé de $(-1, 1)$.

Solution 5. Soit $x \in]0,1[$. La série numérique $\sum_{n\geq 1} \frac{(-1)^n x^n}{\sqrt{n}}$ est une série alternée qui vérifie le critère spécial des séries alternées. Il en découle deux choses:

- La série $\sum_{n\geq 1} \frac{(-1)^n x^n}{\sqrt{n}}$ converge
- On a l'inégalité $|R_n(x)| \leq \left| \frac{x^{n+1}}{\sqrt{n+1}} \right|$.

Donc, pour tout $x \in]0,1[$

$$|R_n(x)| \leq \left| \frac{1}{\sqrt{n+1}} \right| \Longrightarrow \|R_n\|_\infty \leq \frac{1}{\sqrt{n+1}} \underset{n\to\infty}{\longrightarrow} 0$$

La série $\sum_{n\geq 1} \frac{(-1)^n x^n}{\sqrt{n}}$ converge simplement et uniformement sur $]0,1[$.

Go Back

Solution 6.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$ est la série de Taylor du logarithme naturel $\ln(1-x)$, qui converge simplement pour $x \in [0,1)$ mais **ne converge pas uniformément** à cause de la singularité en $x = 1$.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos(nx)$ est une série oscillante avec des termes qui ne tendent pas uniformément vers 0. Par conséquent, elle **ne converge pas uniformément**.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ converge simplement, mais elle **ne converge pas uniformément** sur $x \in [0,1]$. Le comportement près de $x = 1$ cause des problèmes de convergence uniforme.

4. $\sum_{n=1}^{\infty} \frac{nx}{n^2+x^2}$ ne converge pas uniformément sur $x \in [1,2]$, car les termes de la série ne tendent pas uniformément vers 0.

5. $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ est la série de Fourier d'une fonction en dents de scie, qui converge simplement mais **ne converge pas uniformément** sur $x \in [0, 2\pi]$ en raison des oscillations de la série (phénomène de Gibbs).

6. Toutes les séries ne convergent pas uniformément

Solution 7.

1. On a, pour tous $1 < k < \alpha$ et $n \in \mathbb{N}^*$, $\|f_n\|_\infty \leq \frac{1}{n^k}$.

   La série $\sum_{n\geq 1} \frac{1}{n^k}$ converge (c'est une série de Riemann convergente lorsque $k > 1$), par le critère de convergence de Weierstrass, la série $\sum_{n\geq 1} f_n$ converge normalement sur $[0, 1]$.

   La série de fonctions $\sum_{n\geq 1} f_n$ où les f_n sont des fonctions définies sur $[0, 1]$ par $f_n(x) = \frac{x^n}{n^\alpha}$, pour $\alpha > 1$.

2. On a, pour tout $n \in \mathbb{N}^*$, $\|f_n\|_\infty = \frac{1}{n}$.

   Maintenant, nous pouvons examiner la série $\sum_{n\geq 1} \|f_n\|_\infty$, qui est la série harmonique $\sum_{n\geq 1} \frac{1}{n}$. Cette série est bien connue pour diverger.

■

3. La série de fonctions $\sum_{n\geq 1} f_n$ où les f_n sont des fonctions définies sur $[0, a]$ par $f_n(x) = x^n$, pour $a \in \mathbb{R}_+^*$ converge simplement pour x dans l'intervalle $[0, 1[$. Alors elle ne converge pas normalement sur l'intervalle $[0, a]$ si $a \geq 1$. Prenons $x = a = 1$ comme contre exemple.
4. On a, $\|f_n\|_\infty = \frac{1}{2^n n^\alpha}$, D'apres la critère d'Alembert elle converge

Go Back

Solution 8.

■

Solution 9.

1.

$$1 + x + x^2 + \cdots + x^n + \cdots$$

For a fixed value of x. It is a power series that matches Equation **(1)** with $a_0 = 1$, $a_1 = 1, \cdots, a_n = 1$. This is just a geometric series

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n$$

with $q = x$. It converges for

$$|x| < 1 \Leftrightarrow -1 < x < 1\,.$$

We know that it sums to $\dfrac{1}{1-x}$ when $x \in]-1, 1[$. For values of x outside the interval $]-1, 1[$, the series diverges.

■

Thus, the power series

$$1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n = \begin{cases} \dfrac{1}{1-x} & \text{for } x \in]-1,1[\\ \text{diverges} & \text{for } x \notin]-1,1[\end{cases}$$

$]-1,1[$ called the interval of converges of the power series $\sum_{n=0}^{\infty} x^n$

Go Back

2. The power series

$$1 - \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(-\frac{1}{2}\right)^n x^n + \cdots$$

Here, $a_0 = 1$, $a_1 = -\frac{1}{2}, \cdots, a_n = \left(-\frac{1}{2}\right)^n$.
This is a geometric series like

$$1 - \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(-\frac{1}{2}\right)^n x^n + \cdots = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$$

with $q = -\frac{x}{2}$. It converges for

$$\left|\frac{x}{2}\right| < 1 \Leftrightarrow -1 < \frac{x}{2} < 1 \Leftrightarrow -2 < x < 2$$

Thus the sum is

$$\frac{1}{1-q} = \frac{1}{1+\frac{x}{2}} = \frac{2}{2+x}$$

and so

$$1 - \frac{1}{2}x + \frac{1}{4}x^2 + \cdots + \left(-\frac{1}{2}\right)^n x^n + \cdots = \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \frac{2}{2+x},$$

where

$$-2 < x < 2.$$

For values of x outside the interval $]-2, 2[$, the series diverges. Go Back