Sequences of functions

Pointwise and uniform convergence

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Interest of the convergence

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The notion of convergence as we can guess it intuitively unfortunately does not allow to keep a lot of interesting properties of functions. Thus, we will try to refine it a little.
The existence of several ways to define a distance over a space of functions can also offer several ways to look at the problem, it is good to know how to determine which one is the most interesting.

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However, dealing with each point separately lets us lose the global properties of the function: we do not use the topology of I which links the values the function takes at its different arguments.

Pointwise convergence

Definition [Pointwise convergence of a sequence of functions]
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Which can be written as:

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The sequence of function thus converges pointwise towards the function

$$f : x \longmapsto \begin{cases} 0 & \text{if } 0 \leqslant x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$

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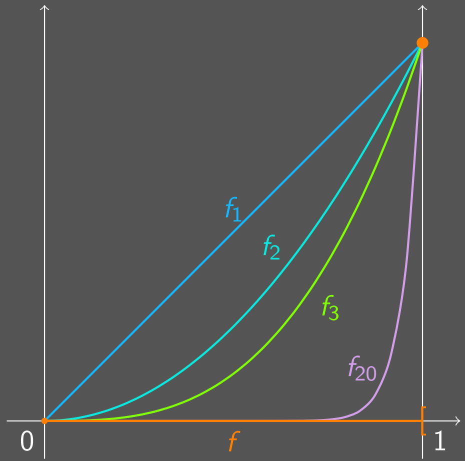

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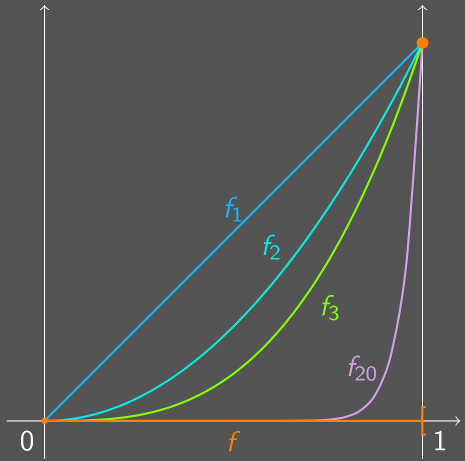

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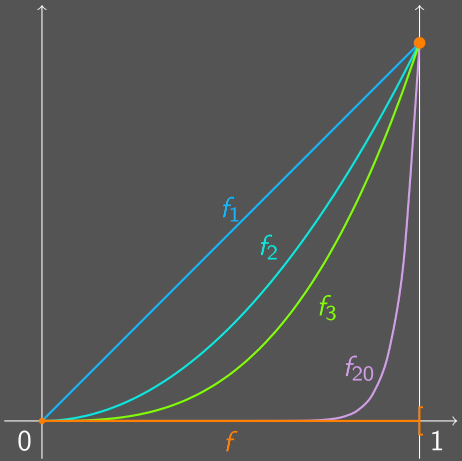

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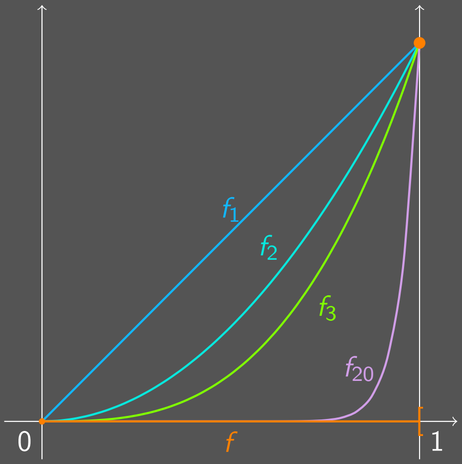

Slight disappointment: all the f_n functions are smooth, yet the limit function f isn't even continuous.
The pointwise convergence does not guarantee the conservation of smoothness properties.
And that is far from being the only problem ...

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$$\lim_{n\to+\infty}\int_I f_n(t)\mathrm{d}t \overset{?}{=} \int_I f(t)\mathrm{d}t \left(= \int_I \lim_{n\to+\infty} f_n(t)\mathrm{d}t\right)$$

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$$\lim_{n\to+\infty} f_n'(t) \stackrel{?}{=} f'(t) = \left(\lim_{n\to+\infty} f_n\right)'$$

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Thus, we will need to devise more powerful ways to evaluate the convergence of sequences of functions, particularly in regards to the properties at the core of our problems.

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Definition [Uniform convergence of a sequence of functions] We say that a sequence (f_n) *converges uniformly* towards a function f over I iff, as long as the following quantities are defined:

$$\operatorname*{Sup}_{x\in I}|f_n(x) - f(x)| \underset{n\to+\infty}{\longrightarrow} 0.$$

that is to say

$$\|f_n - f\|_\infty \underset{n\to+\infty}{\longrightarrow} 0.$$

Which can be written as

$$\forall\varepsilon \in \mathbb{R}_+^*, \exists N_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}, (n \geqslant N_\varepsilon \Longrightarrow \forall x \in I, |f_n(x) - f(x)| < \varepsilon).$$

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Remark: the uniform convergence is the convergence with respect to the infinite norm (or distance).

Uniform convergence: in practice

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- Some properties are useful to work with (smoothness, monotonicity etc.)
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List not exhaustive. One can also imagine using sequential criteria, particularly when one wants to come up with a simple counter-example.

Properties of the uniform convergence

Uniform $\implies$ Pointwise

Theorem

Uniform convergence implies pointwise convergence (towards the same limit).

Properties of the uniform convergence

Uniform $\Longrightarrow$ Pointwise

Theorem
Uniform convergence implies pointwise convergence (towards the same limit).

In other words, if (f_n) converges uniformly towards f over I then it converges pointwise towards f.
Obviously, the reciprocal is false (use the previous example of $f_n : x \longmapsto x^n$ over $[0,1]$, where $\|f_n - f\|_\infty = 1$ for every n).
The pointwise convergence remains insufficient, yet it is still a necessary step to find the limit towards which we want to determine whether the convergence is uniform or not.

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Remark: the contraposition of this property can be used to show that a convergence is not uniform.

Properties of the uniform convergence

Swapping integrals and limits

Theorem

In this case, I is a segment $[a, b]$.

If (f_n) converges uniformly towards f over $[a, b]$, then:

$$\lim_{n\to+\infty}\int_a^b f_n(t)\mathrm{d}t = \int_a^b \lim_{n\to+\infty} f_n(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t.$$

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Thus, if the convergence is uniform, we can «swap» the signs of integration and limit.

Remark: this result becomes false if I is not bounded. Particularly, let us consider the sequence (g_n) with $\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R}, g_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \ [-n; n] \\ 0 & \text{else} \end{cases}$ then (g_n) converges uniformly towards the zero function over $\mathbb{R}$, yet for every n, $\int_{\mathbb{R}} g_n(t)\mathrm{d}t = 2 \underset{n\to+\infty}{\not\longrightarrow} 0$.

Properties of the uniform convergence

Swapping derivatives and limits

Beware: here the hypotheses are way more restrictive.

Theorem

Let us suppose that:

- the f_n functions are of class C^1 over I;
- (f_n) converges (pointwise) over I towards f;
- (f_n') converges uniformly over I towards a function g.

Then f is differentiable over I and $f' = g$, that is to say:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\lim_{n\to+\infty} f_n(x)\right) = \lim_{n\to+\infty}\frac{\mathrm{d}}{\mathrm{d}x}(f_n(x))$$

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So, here again we can «swap» the signs of limit and differentiation.

Remark: for all the properties we talked about here, the uniform convergence is a sufficient yet not necessary condition. There exists indeed sequences of continuous functions that converge pointwise towards continuous functions, and so on.
The uniform convergence is just a relatively simple way to make sure that these important results can be applied.