Workshop session 2 (September 2024)

Directional derivatives and gradients

Exercice 22. Use the figure below to estimate the following directional derivatives.

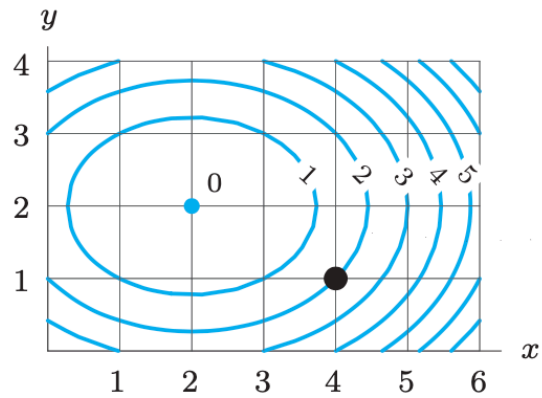
(1) $f_{\boldsymbol{i}}(4,1)$

(2) $f_{\boldsymbol{j}}(4,1)$

(3) $f_{\boldsymbol{u}}(4,1)$ with $\boldsymbol{u} = (\boldsymbol{i} - \boldsymbol{j})/\sqrt{2}$

(4) $f_{\boldsymbol{u}}(4,1)$ with $\boldsymbol{u} = (-\boldsymbol{i} + \boldsymbol{j})/\sqrt{2}$

(5) $f_{\boldsymbol{u}}(4,1)$ with $\boldsymbol{u} = (-2\boldsymbol{i} + \boldsymbol{j})/\sqrt{2}$



Exercice 20. Calculate $\boldsymbol{\nabla} f$, then $D_{\boldsymbol{u}} f = \boldsymbol{\nabla} f \cdot \boldsymbol{u}$, then $D_{\boldsymbol{u}} f$ at P.

(1) $f(x,y) = x^2 - y^2$ avec $\boldsymbol{u} = (\sqrt{3}/2, 1/2)$ et $P(1,0)$

(2) $f(x,y) = 3x + 4y + 7$ avec $\boldsymbol{u} = (3/5, 4/5)$ et $P(0, \pi/2)$

(3) $f(x,y) = e^x \cos y$ avec $\boldsymbol{u} = (0,1)$ et $P(0, \pi/2)$

(4) $f(x,y) = y^{10}$ avec $\boldsymbol{u} = (0,-1)$ et $P(1,-1)$

(5) $f(x,y) =$ distance de (x,y) à $(0,3)$, avec $\boldsymbol{u} = (1,0)$ et $P(1,1)$

Exercice 24. We consider the following function of several variables

$$f(\boldsymbol{x}) = (\boldsymbol{a}^\intercal \boldsymbol{x})\,(\boldsymbol{b}^\intercal \boldsymbol{x})$$

where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{x}$ are vectors in $\mathbb{R}^n$. Determine $\boldsymbol{\nabla} f(\boldsymbol{x})$ and the Hessian $\boldsymbol{H}$ associated to f.

Chain rules

Exercice 29. We consider the functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $\boldsymbol{g} : \mathbb{R} \to \mathbb{R}^2$ such that

$$f(\boldsymbol{x}) = \frac{x_1^2}{6} + \frac{x_2^2}{4} \qquad \text{et} \qquad \boldsymbol{g}(t) = [3t+5, 2t-6]^\intercal$$

Let $F : \mathbb{R} \to \mathbb{R}$ given by $F(t) = f(\boldsymbol{g}(t))$. Compute $\frac{dF}{dt}(t)$ using chain rules.

Exercice 30. We consider the functions

$$\boldsymbol{x}(t) = [e^t + t^3, t^2, t+1]^\intercal \quad (t \in \mathbb{R}) \qquad \text{et} \qquad f(\boldsymbol{x}) = x_1^3 x_2 x_3^2 + x_1 x_2 + x_3$$

with $\boldsymbol{x} = [x_1, x_2, x_3]^\intercal \in \mathbb{R}^3$. Find $\frac{d}{dt} f(\boldsymbol{x}(t))$ in function of t.

Maxima, minima and saddle points

Exercice 40. We consider the functions $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(\boldsymbol{x}) = \boldsymbol{x}^\intercal \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\intercal \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6$$

(1) Find the gradient and the Hessian matrix of f at the point $[1, 1]^\intercal$.

(2) Find the directional derivative of f at $[1, 1]^\intercal$ with respect to a unit vector in the direction of maximal rate of increase.

(3) Find a point satisfying the condition for an extremum. Is it a maximum or a minimum?

Exercice 41. We consider the function

$$f(\boldsymbol{x}) = f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1$$

(a) In what direction does the function f decrease most rapidly at the point $\boldsymbol{x}^{(0)} = [2, 1]^\intercal$?

(b) What is the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction of maximum decrease of f?

(c) Find the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction $\boldsymbol{d} = [3, 4]^\intercal$.

Exercice 42. On considère la fonction $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\intercal \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\intercal \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7$$

(a) Find the directional derivative of f at $[0, 1]^\intercal$ in the direction $[1, 0]^\intercal$.

(b) Find all points that satisfy the condition for an extremum of f . Does f have a minimizer ? If it does, then find all minimizer(s); otherwise, explain why it does not.