Fourier series

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Abstract

The theory of Fourier series looks for approximations of periodic functions under the form of sequences (series)of trigonometric polynomials (that is to say, sums of sinusoidal functions, whose frequence is a multiple of the chosenbasic frequence). Until now, we mostly worked with iterative or recursive approximation methods, whose results couldbe enhanced step by step till the wanted accuracy was reached (for instance, Taylor expansions, power series etc.). Usingthe advantages of the structure of the vector space spanned by the previously evoked sinusoidal functions, the theoryof Fourier series offers a very interesting result : any single term of the decomposition into a Fourier series of a givenfunction can be calculated separately from the others, just by calculating an integral.Originally introduced to write harmonic decompositions of periodic signals, the study of Fourier series has sincebeen used in many fields of research in mathematics, even opening new domains like harmonic analysis or signal theory.Fourier series will then represent an important part of your MASI lesson about Signal Mathematics.

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1 Fourier Series Calculation

The decomposition into Fourier series leverages the orthogonality of the basis functions $[t \mapsto \cos(nt)$ and $t \mapsto \sin(nt)$ in $\mathbb{R}$, or $t \mapsto \mathrm{e}^{int}$ in $\mathbb{C}]$ to determine each coefficient independently through orthogonal projection, using the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{f}(t)g(t)\,\mathrm{d}t$ in the pre-Hilbert complex space generated by the functions $t \mapsto \mathrm{e}^{int}$. The factor $\frac{1}{2\pi}$ is a normalization factor, ensuring that this family of functions is orthonormal.

Although the theory is (as always) simpler with complex numbers, we will primarily use real coefficients here. Real coefficients allow us to exploit the symmetry properties of the signals to be decomposed, reducing the amount of computation required.

Exercice 1.1 — **Absolute Value**

Consider the function f, which is 2π-periodic and continuous on $\mathbb{R}$, such that:

$$\forall x \in [-\pi, \pi], f(x) = |x|$$

1. What can we say about the parity of f?
2. Calculate the real Fourier coefficients of f, and write its real Fourier series.

Absolute Value

☆ ❸ Calculate the complex Fourier coefficients of f.

Exercice 1.2 **Square Wave Function**

Consider the function g, which is 2π-periodic on $\mathbb{R}$, such that:

$$g(x) = \begin{cases} -1 \text{ if } x \in [-\pi, 0[\\ 1 \text{ if } x \in [0, \pi[\end{cases}$$

❶ Is g even or odd? What can we say about its regularization $\tilde{g}$?

❷ Calculate the real Fourier coefficients of g, and write its real Fourier series.

☆ ❸ Calculate the complex Fourier coefficients of g.

2 Application: Dirichlet's Theorem

Dirichlet's theorem provides simple convergence results for Fourier series, assuming certain regularity for the studied functions. It is less precise than Fejér's theorem, which constructs a uniform approximation for a continuous function (Weierstrass theorem), but its result is easier to apply.

In the case of functions with discontinuity points, the simple convergence is not towards the function itself, but towards its regularization. Care must be taken when evaluating points where the original function experiences a jump.

This result can be used to determine sums of numerical series.

Exercice 2.1

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 < x < \pi \end{cases}$$

❶ Graphically represent the function f in an orthogonal coordinate system on the interval $]-3\pi, 3\pi[$.

❷ Calculate the trigonometric Fourier coefficients a_0, a_n, and b_n of f, such that

$$[F(f)](x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

❸ (a) Does f satisfy the hypotheses of Dirichlet's theorem? What can we deduce about the convergence of its Fourier series $F(f)$?

(b) Taking $x = \frac{\pi}{2}$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

(c) Taking $x = 0$, show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Exercice 2.2

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} 1, & -\pi < x \leq 0 \\ 0, & 0 < x < \pi \end{cases}$$

❶ Graphically represent the function f in an orthogonal coordinate system on the interval $]-2\pi, 2\pi[$.

❷ Calculate the trigonometric Fourier coefficients a_0, a_n, and b_n of f.

❸ (a) Does f satisfy the hypotheses of Dirichlet's theorem? What can we deduce about the convergence of its Fourier series $F(f)$?

(b) Taking $x = \dfrac{\pi}{2}$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Exercice 2.3

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be the 2π-periodic function defined by

$$f(x) = \begin{cases} x, & 0 < x \leq \pi \\ \pi, & \pi < x < 2\pi \end{cases}$$

1. Graphically represent the function f in an orthogonal coordinate system on the interval $]-2\pi, 2\pi[$.
2. (a) Calculate the trigonometric Fourier coefficients a_0, a_n, and b_n of f.

   (b) Taking $x = \dfrac{\pi}{2}$, show that

   $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

   (c) Taking $x = 0$, show that

   $$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

3 Convergence

Exercice 3.1 **Absolute Value**

We revisit in this exercise the function f defined in Exercise 1.1.

1. Does f satisfy the hypotheses of Dirichlet's theorem? What can we deduce about the convergence of its Fourier series $F(f)$?
2. Evaluate the Fourier series of f at $t = 0$. Deduce the value of $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$.
3. Deduce from the previous result the value of $\sum_{n=1}^{+\infty} \frac{1}{n^2}$.
4. What do we obtain by evaluating the Fourier series $F(f)$ at $\frac{\pi}{2}$?
5. Evaluate the Fourier series $F(f)$ at $\frac{\pi}{4}$. Write the result as a series by expressing the values taken by the cosines.

Exercice 3.2 **Square Wave Function**

We revisit in this exercise the function g defined in Exercise 1.2.

1. Does g satisfy the hypotheses of Dirichlet's theorem? What can we deduce about the convergence of its Fourier series $S(g)$?
2. What do we obtain by evaluating $S(g)$ at 0?
3. Evaluate the Fourier series $S(g)$ at $\frac{\pi}{2}$. Write the result as a series.
4. Evaluate the Fourier series $S(g)$ at $\frac{\pi}{4}$. Write the result as a series.