# TD1

## Q1-1

a) $\int_0^{+\infty} e^{\alpha x} dx = \lim_{t \to +\infty} \left[\frac{1}{\alpha} e^{\alpha x}\right]_0^t$

$= \left(\lim_{t \to +\infty} \frac{1}{\alpha} e^{\alpha t}\right) - \left(\frac{1}{\alpha} e^{0\alpha}\right)$

$= \lim_{t \to +\infty} \frac{1}{\alpha} e^{\alpha t} - \frac{1}{\alpha}$

here we can spot two distinct cases:

1. $\alpha < 0 \Rightarrow \lim_{t \to +\infty} \alpha t \to -\infty$

   $\Rightarrow \lim_{t \to +\infty} e^{\alpha t} \to 0$ CVG

2. $\alpha > 0 \Rightarrow \lim_{t \to +\infty} \alpha t \to +\infty$

   $\Rightarrow \lim_{t \to +\infty} e^{\alpha t} \to +\infty$ DVG

In the last case where $\alpha = 0$ we can see quite obviously that $\int_0^{+\infty} e^{\alpha x} = \int_0^{+\infty} 1$ is an integral of a constant function and will be divergent.

b) $\int_{-\infty}^{0} e^{\alpha x} dx = \lim_{t \to -\infty} \left[\frac{1}{\alpha} e^{\alpha x}\right]_t^0$

$= \left(\frac{1}{\alpha} e^{0\alpha}\right) - \left(\lim_{t \to -\infty} \frac{1}{\alpha} e^{\alpha t}\right)$

$= \frac{1}{\alpha} - \lim_{t \to -\infty} \frac{1}{\alpha} e^{\alpha t}$

here we can spot two distinct cases:

1. $\alpha < 0 \Rightarrow \lim_{t \to -\infty} \alpha t \to +\infty$

   $\Rightarrow \lim_{t \to -\infty} e^{\alpha t} \to +\infty$ DVG

$$2.\ \alpha > 0 \quad \Rightarrow \lim_{t \to -\infty} \alpha t \to -\infty$$

$$\Rightarrow \lim_{t \to -\infty} e^{\alpha t} \to 0 \qquad \text{CVG}$$

In the last case where $\alpha = 0$ we can see quite obviously that $\int_{-\infty}^{0} e^{\alpha x} = \int_{-\infty}^{0} 1$ is an integral of a constant function and will be divergent.

## Q1-2

Let $f : [1, +\infty[ \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{1}{x} = 0$$

$$\int_{a}^{+\infty} f(x)\, dx = \int_{1}^{+\infty} \frac{1}{x}\, dx$$

using the Riemann reference integrals we can say that $\int_{1}^{+\infty} \frac{1}{x}\, dx$ diverges

From this counter example we can say that this statement is false.

## Q1-3

a) $\int_{0}^{+\infty} \cos(t)\, dt$

$$\lim_{a \to +\infty} \int_{0}^{a} \cos(t)\, dt$$

$$= \lim_{a \to +\infty} \left[\sin(t)\right]_{0}^{a}$$

$$= \left(\lim_{a \to +\infty} \sin(a)\right) - \left(\sin(0)\right)$$

$$= \lim_{a \to +\infty} \sin(a)$$

Since the sin function is periodic and has no limit in $+\infty$, the integral cannot converge and is therefore divergent.

b) $\int_0^{+\infty} \frac{1}{x^{4/5}}\,dx$ : $\left( \underbrace{\int_0^1 \frac{1}{x^{4/5}}}_{\text{CVG}} + \underbrace{\int_1^{+\infty} \frac{1}{x^{4/5}}}_{\text{DVG}} \right)$ using the Riemann reference integrals we can deduce the nature of both of these integrals since $\frac{4}{5} < 1$

We can split this integral into two parts, one convergent and one divergent therefore it must be divergent.

c) $\int_1^{+\infty} \frac{1}{x^{5/4}}\,dx$ Here we can once again use the Riemann reference integrals to say that this integral converges because $\frac{5}{4} > 1$

Q1-4

a) $\int_1^{+\infty} \frac{\cos^2(t)}{t^2}\,dt$ : $\forall t \in [1, +\infty[ \quad -1 \leq \cos(t) \leq 1$

$$\Rightarrow 0 \leq \cos^2(t) \leq 1$$

$$\Rightarrow 0 \leq \frac{\cos^2(t)}{t^2} \leq \frac{1}{t^2}$$

$\int_1^{+\infty} \frac{1}{t^2}\,dt$ : Using the Riemann reference integrals we can say that this integral converges because $2 > 1$

Therefore, using the comparison theorem we can also say that $\int_1^{+\infty} \frac{\cos^2(t)}{t^2}\,dt$ converges.

b) $\int_0^{+\infty} \frac{e^{-t}}{1+t^2}\,dt$ : $\forall t \in [0,+\infty[ \quad 0 \leq e^{-t} \leq 1$

$$\Rightarrow 0 \leq \frac{e^{-t}}{1+t^2} \leq \frac{1}{1+t^2}$$

because $\begin{cases} 1+t^2 > 1 \\ 1+t^2 > e^{-t} \end{cases} \quad \forall t \in [0,+\infty[$

$\int_0^{+\infty} \frac{1}{1+t^2}\,dt$ : $\lim\limits_{a\to+\infty} \int_0^{a} \frac{1}{1+t^2}\,dt = \lim\limits_{a\to+\infty} \left[\arctan(t)\right]_0^a$

$$= \left(\lim_{a\to+\infty} \arctan(a)\right) - 0$$

$$= \frac{\pi}{2}$$

So this integral converges

Therefore by the comparison theorem we can say that $\int_0^{+\infty} \frac{e^{-t}}{1+t^2}\,dt$ also converges