Function of several variables
Session 4

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EPITA

September 2024

Themes :

- Triple integrals
- Cylindrical and spherical coordinates

Triple Integrals

General

- From a formal point of view, it is **relatively simple** to pass from a double integrable to a triple integral

$$\iiint_R f(x,y,z)\,dV$$

- The integration measure dV corresponds to an **infinitesimal volume** given by (in Cartesian coordinates)

$$dV = dx\,dy\,dz$$

- The definition as **limit of a sum** becomes

$$\iiint_R f(x,y,z)\,dV \;=\; \lim_{\substack{n\to\infty \\ \Delta x,\,\Delta y,\,\Delta z\,\to\,0}} \sum_{i=0}^{n} f(x_i,y_i,z_i)\,\Delta V_i$$

- We are **slicing** the integration region into small volumes ΔV_i (instead of ΔA_i for double integrals).

- The triple integral is then obtained by **summing** over an infinity ($n \to \infty$) of these arbitrarily small volumes.

- We will limit ourselves to **classic volumes** such as boxes, cylinders, prisms, tetrahedra, etc...

- If $f = 1$, the number $\iiint_R dV$ corresponds to a **real volume**.

- For example, the volume of a **box** delimited by

$$0 \leq x \leq 2 \quad , \quad 0 \leq y \leq 3 \quad , \quad 0 \leq z \leq 1$$

is given by the triple integral

$$\iiint_R dV = \int_0^1 \int_0^3 \int_0^2 dx\, dy\, dz = 1 \times 3 \times 2 = 6$$

Example : Mass Calculation

- A **cube** of side 4cm is made of a material of **variable density**. We place the origin at one of its vertices and its sides align with the axes Ox, Oy, Oz.

- Its **density** (in g/cm^3) is given by the function $\delta(x, y, z) = 1 + xyz$. The **mass** of the cube is therefore

$$
\begin{aligned}
M &= \iiint_R \delta \, dV = \int_0^4 \int_0^4 \int_0^4 (1 + xyz) \, dx \, dy \, dz \\
&= \int_0^4 \int_0^4 \left[x + \frac{x^2 yz}{2} \right]_0^4 = \int_0^4 \int_0^4 (4 + 8yz) \, dy \, dz \\
&= \int_0^4 \left[4y + 4y^2 z \right]_0^4 dz = \int_0^4 (16 + 64z) \, dz = 576 \, \mathrm{g}
\end{aligned}
$$

Example : Calculating the volume of a prism

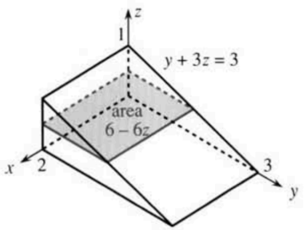
- We consider a **prism** delimited by the planes

$$x = 0 \quad , \quad y = 0 \quad , \quad z = 0 \quad , \quad x = 2 \quad , \quad y + 3z = 3$$

- To calculate $\iiint_R dx\,dy\,dz$, **we choose** to first consider the internal integral $\iint_{R'} dx\,dy$, corresponding to the area d 'a **slice** parallel to the xy plane at a height z.

- Integration according to x will clearly go from plane $x = 0$ to $x = 2$. The one according to y goes from $y = 0$ to plane $y + 3z = 3$ (see figure). So we have

$$\begin{aligned}
\iint_{R'} dx\,dy &= \int_0^{3(1-z)} \int_0^2 dx\,dy \\
&= \int_0^{3(1-z)} 2\,dy \\
&= 6(1-z) \\
\iiint_R dx\,dy\,dz &= \int_0^1 6(1-z)\,dz \\
&= \left[6z - 3z^2\right]_0^1 \\
&= 3
\end{aligned}$$



Example : Calculating the volume of a tetrahedron

- We consider a **tetrahedron** delimited by the planes

$$x = 0 \quad , \quad y = 0 \quad , \quad z = 0 \quad , \quad x + y + z = 1$$

- We choose (once again) to first consider the internal integral $\iint_{R'} dx\, dy$, corresponding to the **triangular area** of a slice parallel to the plane xy at height z.

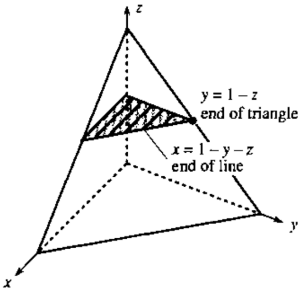
- We follow the **same procedure** as when we calculated the area of a triangle, but it happens in 3D now...

- Following x, we start from the plane $x = 0$ up to the plane of equation $x = 1 - y - z$ (see figure).

- Then we integrate from $y = 0$ up to the maximum value of y for all x (in the plane of the slice), ie $y = 1 - z$. So we have

$$\begin{aligned} \iint_{R'} dx\,dy &= \int_0^{1-z} \int_0^{1-y-z} dx\,dy \\ &= \int_0^{1-z} (1 - y - z)\,dy \\ &= \tfrac{1}{2}(1-z)^2 \end{aligned}$$

$$\begin{aligned} \iiint_R dx\,dy\,dz &= \int_0^1 \tfrac{1}{2}(1-z)^2\,dz \\ &= \tfrac{1}{3} \end{aligned}$$



Question 1 (Wooclap)

Which of the following assertions are correct ?

(1) If $\rho(x, y, z)$ is a mass density of a material in 3-space, then $\iiint_W \rho(x, y, z)\, dV$ gives the volume of the solid region W.

(2) The region of integration of the triple iterated integral $\int_0^1 \int_0^1 \int_0^x f\, dz\, dy\, dx$ lies above a square in the xy-plane and below a plane.

(3) If W is the entire unit ball $x^2 + y^2 + z^2 \leq 1$ then an iterated integral over W has limits $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} f\, dz\, dy\, dx$.

(4) The iterated integrals $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f\, dz\, dy\, dx$ and $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} f\, dz\, dy\, dx$ are equals.

Solution 1 (Wooclap)

(1) **False**. The integral gives the total mass of the material contained in W.

(2) **True**. The region lies above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ and below the plane $z = x$.

(3) **False**. The given limits only cover the part of the unit ball in the first octant where $x \leq 0$, $y \leq 0$ and $z \leq 0$.To cover the entire unit ball the limits are

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f \, dz \, dy \, dx$$

(4) **True**. Both sets of limits describe the solid region lying above the triangle $x + y \leq 1$, $x \geq 0$, $y \geq 0$, $z = 0$ and below the plane $x + y + z = 1$.

Cylindrical and spherical coordinates

General

- Unsurprisingly, the Cartesian coordinates xyz, are also not always suitable for the evaluation of

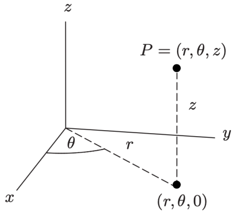
$$\iiint_R f(x,y,z)\,dx\,dy\,dz$$

- The advantage sought through switching to uvw coordinates is always to simplify the description of the **integration domain** but also of the **function f**.

- The two most commonly used alternative coordinate systems are **cylindrical** and **spherical**.

- This is a direct **generalization** of the $r\theta$ polar coordinate system with the addition of the z height.

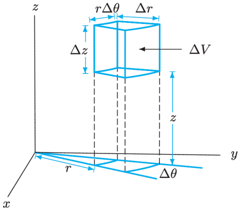
$$\begin{cases} x = r\cos\theta & (0 \le r \le +\infty) \\ y = r\sin\theta & (0 \le \theta \le 2\pi) \\ z = z & (-\infty \le z \le +\infty) \end{cases}$$



$$\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

- The volume dV is deduced from the calculation of dA in **polar**.

$$\begin{aligned} \Delta V &\approx \Delta r \times r\Delta\theta \times \Delta z \\ &\Downarrow \\ dV &= r\,dr\,d\theta\,dz \end{aligned}$$

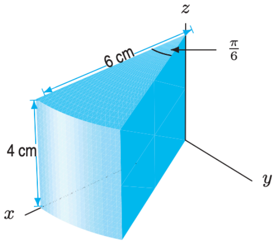


Example : Mass Calculation

- Calculate the mass of a piece of pie, for 12 people, 4cm in height, 6cm in radius and density $\delta = 1.2\,\mathrm{g/cm}^3$.

- This mass is given by $M = \iiint_R \delta\, dV$, i.e.

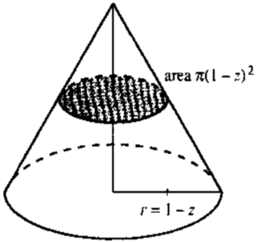
$$
\begin{aligned}
M &= \int_0^4 \int_0^{\pi/6} \int_0^6 (1.2)\, r\, dr\, d\theta\, dz \\
&= \int_0^4 \int_0^{\pi/6} \left[(0.6)\, r^2\right]_0^6 d\theta\, dz \\
&= (21.6) \int_0^4 \int_0^{\pi/6} d\theta\, dz \\
&= (21.6) \times 4 \times \frac{\pi}{6} \\
&= 45.239\,\mathrm{g}
\end{aligned}
$$



- The surface of a **cone** of height 1 and invariant under rotation around the axis Oz is described by $r = 1 - z$ in cylindrical form.

- **We choose** to express the volume of the cone as

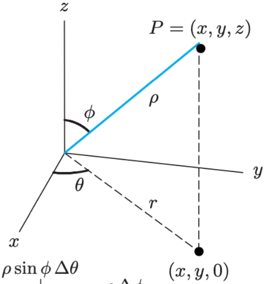
$$
\begin{aligned}
\iiint_R dV &= \int_0^1 \int_0^{2\pi} \int_0^{1-z} r\,dr\,d\theta\,dz \\
&= \int_0^1 \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{1-z} d\theta\,dz \\
&= \int_0^1 \pi(1-z)^2 dz \\
&= \frac{\pi}{3}
\end{aligned}
$$



- Note that in this **specific breakdown**, the intermediate result $\pi(1-z)^2$ corresponds to the area of a **slice** parallel to the plane xy and of height z.

- Naturally this system allows to describe objects invariant by rotation in 3D.

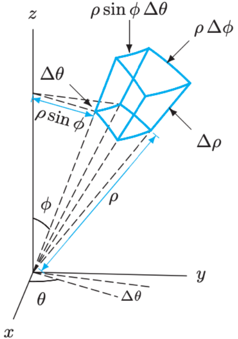
$$\begin{cases} x = \rho \sin\phi \cos\theta & (0 \leq \rho \leq +\infty) \\ y = \rho \sin\phi \sin\theta & (0 \leq \theta \leq 2\pi) \\ z = \rho \cos\phi & (0 \leq \phi \leq \pi) \end{cases}$$



$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\phi = r/z \\ \tan\theta = y/x \end{cases}$$

- The volume dV gives

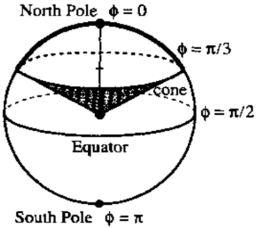
$$\begin{aligned} \Delta V &\approx \Delta\rho \times (\rho\Delta\phi) \times (\rho \sin\phi\Delta\theta) \\ &= \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta \\ &\Downarrow \\ dV &= \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \end{aligned}$$



Example : Volume above a cone

- We want to evaluate the volume above a **cone** using the figure below. From this we write

$$
\begin{aligned}
\iiint_R dV &= \int_0^{2\pi} \int_0^{\pi/3} \int_0^R \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{R^3}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi \, d\phi \, d\theta \\
&= \frac{R^3}{3} \int_0^{2\pi} \big[-\cos\phi \big]_0^{\pi/3} \, d\theta \\
&= \frac{R^3}{3} \left(-\frac{1}{2} + 1 \right) \int_0^{2\pi} d\phi \\
&= \frac{\pi R^3}{3}
\end{aligned}
$$



General case

- For a **change of variable** from the Cartesian system xyz to an arbitrary coordinate system uvw, we have

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

- The **Jacobian** determinant $\boldsymbol{J}$, characterizing the deformation of the area $dA = dx\,dy\,dz$, is defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

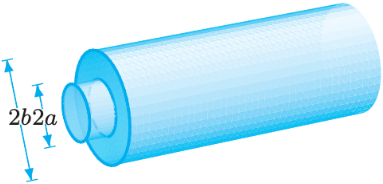
- Pour que cette transformation ait un sens il faut que $\boldsymbol{J} \neq \mathbf{0}$.

- Le détail des démonstrations des formules ci-dessus est donné en **appendice** du document de cours.

- A **coaxial cable** consisting of two conductors, of **constant permittivity ε**, cylindrical centered on the same axis, with radii $a < b$.

- The **electric field** between conductors has an amplitude $E = q/(2\pi\varepsilon r)$, where r is the distance to the axis and q is the charge per unit in length on the cable.

- **Question.** Show that the energy stored per unit length is proportional to $\ln(b/a)$.



- The energy U_e stored in a conductor is

$$U_e = \frac{1}{2} \iiint_R \varepsilon\, E^2 \, dV$$

where ε is the permittivity of the material and E is the modulus of the electric field inside the conductor.

- For a portion of cable of unit length, we therefore have

$$\begin{aligned} U_e &= \frac{1}{2} \int_a^b \int_0^1 \int_0^{2\pi} \varepsilon\, E^2\, r\, d\theta\, dz\, dr \\ &= \frac{q^2}{8\pi^2 \varepsilon} \int_a^b \int_0^1 \int_0^{2\pi} \frac{1}{r}\, d\theta\, dz\, dr \\ &= \frac{q^2}{4\pi\varepsilon} \int_a^b \frac{1}{r}\, dr \\ &= \frac{q^2}{4\pi\varepsilon} \Big(\ln b - \ln a \Big) \\ &= \frac{q^2}{4\pi\varepsilon} \ln \frac{b}{a} \end{aligned}$$

- The **electric charge** of a sphere is distributed with a density **inversely proportional** to the distance from the origin.

- **Question.** Show that the total charge inside a sphere of radius R is proportional to R^2.

- We set a density of charge $\delta = k/\rho$, where k is a proportionality constant, the **total charge** $Q = \iiint_R \delta \, dV$ is then given by the triple integral

$$
\begin{aligned}
Q &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{k}{\rho} \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= k \int_0^{2\pi} \int_0^{\pi} \frac{R^2}{2} \, \sin\phi \, d\rho \, d\phi \, d\theta \\
&= 4\pi \, k \, \frac{R^2}{2} \; = \; 2\pi k R^2
\end{aligned}
$$

Example : Calculation of a force of gravity

- Calculate the **force of gravity** exerted by a solid cylinder of radius R, height H and constant density δ on a unit of mass at the center of the base of the cylinder.
- The base of the cylinders is assumed to lie on the xy plane with the center at the origin. Since the cylinder is symmetrical about the z axis, the force in the horizontal direction x or y is 0.
- Thus, we only need to calculate the vertical z component of the force. We will therefore use cylindrical coordinates. Since the force is

  $$G \cdot \mathsf{mass}/(\mathsf{distance})^2$$

  a piece of the cylinder of volume dV located at (r, θ, z) exerts on the unit mass a force of magnitude

  $$F = G(\delta\, dV)/(r^2 + z^2)$$

We have

$$
\begin{aligned}
F &= \int_0^H \int_0^{2\pi} \int_0^R \frac{G\,\delta\,z\,r}{(r^2+z^2)^{3/2}}\,dr\,d\theta\,dz \\
&= \int_0^H \int_0^{2\pi} (G\,\delta\,z) \left[-\frac{1}{\sqrt{r^2+z^2}}\right]_0^R d\theta\,dz \\
&= \int_0^H \int_0^{2\pi} (G\,\delta\,z) \left(-\frac{1}{\sqrt{R^2+z^2}} + \frac{1}{z}\right) d\theta\,dz \\
&= \int_0^H \int_0^{2\pi} (2\pi\,G\,\delta) \left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) d\theta\,dz \\
&= (2\pi\,G\,\delta) \left[z - \sqrt{R^2+z^2}\right]_0^H \\
&= (2\pi\,G\,\delta)\left(H - \sqrt{R^2+H^2} + R\right) \\
&= (2\pi\,G\,\delta)\left(H + R - \sqrt{R^2+H^2}\right)
\end{aligned}
$$

Question 2 (Wooclap)

Which of the following integrals give the volume of the unit sphere ?

(1) $\displaystyle\int_0^{2\pi}\int_0^{2\pi}\int_0^1 d\rho\, d\theta\, d\phi$

(2) $\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^1 d\rho\, d\theta\, d\phi$

(3) $\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$

(4) $\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

(5) $\displaystyle\int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2\, d\rho\, d\phi\, d\theta$

Solution 2 (Wooclap)

(1) **False.**

(2) **False.**

(3) **True.**

(4) **False.**

(5) **False.**

(6) **False.**