Functions of several variables
Session 3

Edouard Marchais

EPITA

September 2024

Themes :

- Double integrals
- Change of coordinates

Double integrals

- From a **geometrical** viewpoint a double integral

$$\iint_R f(x,y)\,dA$$

  corresponds to the volume between the xy plane and the surface $z = f(x,y)$, for a region R of the xy plane.

- We are talking here of a **algebric volume** that can be negative. If there is as much volume above than below the xy plane, for a region R, the intégral is zero.

- The mesure dA represents an **infinitesimal** surface element in the xy plane. If we use the cartesian coordinates x and y then we have $dA = dx\,dy$

- **Limit of a sum**

  - We can also propose a **slicing** of the previous volume in small columns which squared basis are of area $\Delta A = \Delta x\, \Delta y$ and their height is $z = f(x, y)$.

  - We can **fill approximatively** the volume by summing on a finite number of column

    $$V \simeq \sum_{i=0}^{n} z_i\, \Delta A_i$$

    with $z_i = f(x_i, y_i)$ and $\Delta A_i = \Delta x_i \Delta y_i$ (see figure).

  - Then, by taking an **infinite number** of column $(n \to \infty)$ with an infinitesimally small basis $(\Delta A_i \to 0)$, we can **fill completely** the volume corresponding to $\iint_R f\, dA$

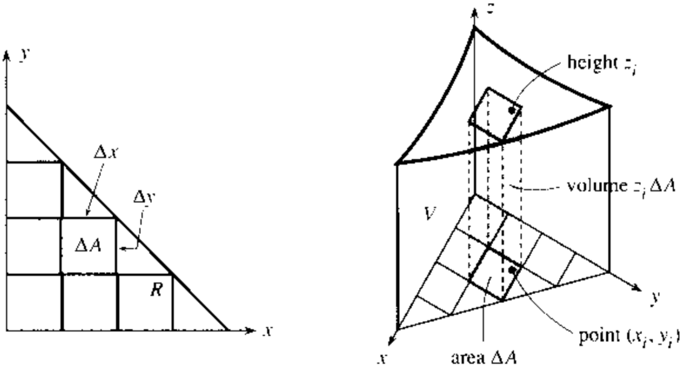
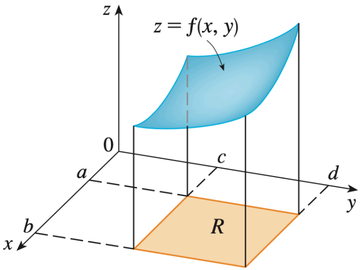
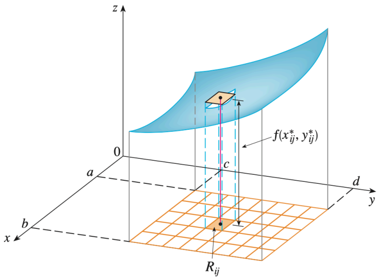


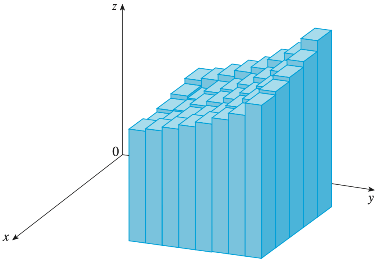
Figure – R basis sliced in small pieces ΔA. The volume V sliced in small columns of infinitesimal volume $\Delta V = z\,\Delta A$.



z
$z = f(x, y)$
0
a
c
d
b
x
y
R



z
0
a
c
d
b
x
y
$f(x_{ij}^*, y_{ij}^*)$
R_{ij}



z
0
x
y

- We can now propose a **definition** of double integrals limits of a sum

$$\iint_R f(x,y)\,dA \;=\; \lim_{\substack{n\to\infty \\ \Delta x\to 0 \\ \Delta y\to 0}} \sum_{i=0}^{n} f(x_i,y_i)\,\Delta A_i$$

- This formula can be used as a starting point for a **numerical evaluation** of a double integral.

- It can also be used to show a number of **basic properties** essential to practical calculation.

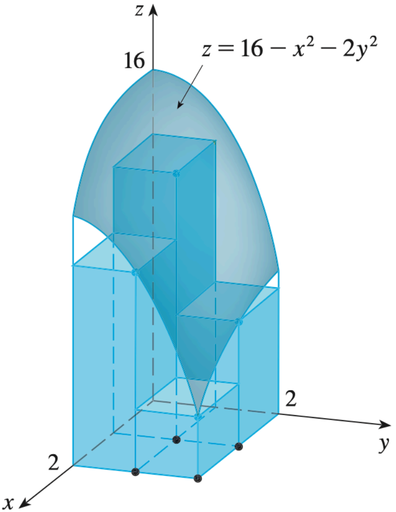
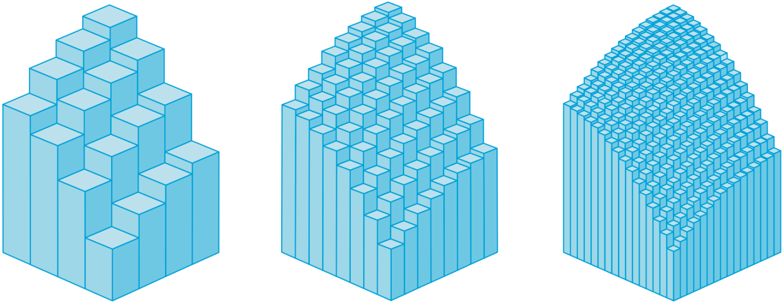


Figure – The slicing by 4 columns with 2×2 basis leads to $V \approx 34$ for the volume between the surface $z = f(x, y)$ and the xy plane.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

- **Some important properties**
  - **Linearity** : f, g are two functions and α, β two numbers

$$\iint_R (\alpha\, f + \beta\, g)\, dA = \alpha \iint_R f\, dA + \beta \iint g\, dA$$

  - **Separation** : $R = S \cup T$ and $S \cap T = \varnothing$

$$\iint_R f\, dA = \iint_S f\, dA + \iint_T f\, dA$$

  - **Order** : $f(x, y) \geq g(x, y)$ on R

$$\iint_R f(x, y)\, dA \ \geq \ \iint_R g(x, y)\, dA$$

  - etc ... (see course document)

- **Practical calculation**

    - It is interesting to consider the integral $\iint_R f\,dx\,dy$ (respect. $\iint_R f\,dy\,dx$) as a **successive integral** on x then on y (respect. on y then on x), we must have

$$\int\left(\int f\,dx\right)dy = \int\left(\int f\,dy\right)dx$$

    - Geometrically, the internal integral (on x or on y) may be viewed as a **slice** of the corresponding volume of $\iint_R f\,dA$ (see figure).

    - Therefore the order of integration possesses an important **technical signification**!

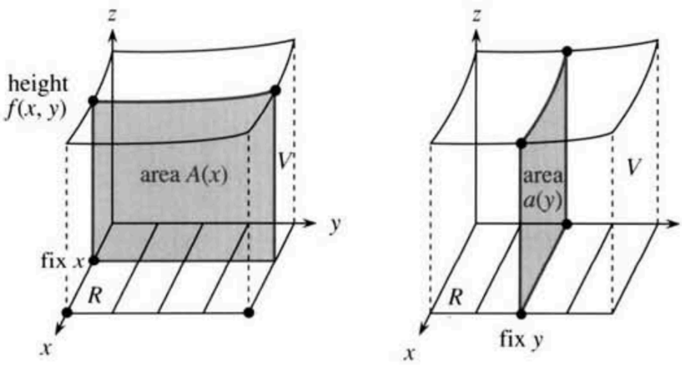


Figure – A slice of V for a given x possesses an area of $A(x) = \int f(x,y)\,dy$.

Example

$$\begin{aligned}
\iint_R f\,dA &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=2}(1+x^2+y^2)\,dy\right)dx \\
&= \int_{x=0}^{x=1}\left[y+x^2y+\frac{y^3}{3}\right]_{y=0}^{y=2}dx \\
&= \int_{x=0}^{x=1}\left(2+2x^2+\frac{8}{3}\right)dx \\
&= \left[2x+\frac{2x^3}{3}+\frac{8x}{3}\right]_{x=0}^{x=1}dx \\
&= \frac{16}{3} \\
&= \int_{y=0}^{y=2}\left(\int_{x=0}^{x=1}(1+x^2+y^2)\,dx\right)dy \\
&= \ldots \quad \text{(to finish)}
\end{aligned}$$

- If $f = 1$ in $\iint_R f\, dA$ then the integral $\iint_R dA$ corresponds to the **geometrical area** of the region R.

- In the case where **integration bounds** are real numbers in a given coordinates system, and f a continuous function, we have the following general result

$$\int_a^b \left[\int_c^d f(x, y)\, dy \right] dx = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

  known as the **Fubini theorem**.

- **Warning** : when the bounds are functions of x or y, the integrations on x and on y are not interchangeables ! We have to **reformulate the bounds** (using a figure).

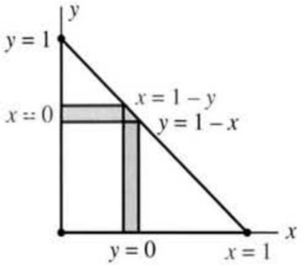
Example

We consider the following double integral

$$\iint_R f\,dA = \int_0^1 \int_0^{1-x} dy\,dx$$

It corresponds to the area of triangle with a right angle at the origin and which side's equations are

$$x = 0 \quad , \quad y = 0 \quad , \quad x = 1 - y \;\text{ or }\; y = 1 - x$$



- If we integrate first **with respect to y**, we have

$$\iint_R f\,dA = \int_0^1 \left[\,y\,\right]_0^{1-x} dx = \int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

- Graphically, the **internal integral** on y, corresponds to an vertical segment starting from the frontier $y = 0$ to the frontier $y = x - 1$.

- Then, the **external integral** on x sums (adds) all the possible segments (of variable length $1 - y$) in order to cover the whole surface of the triangle (which we know).

- If we invert the integration ordre we must then write

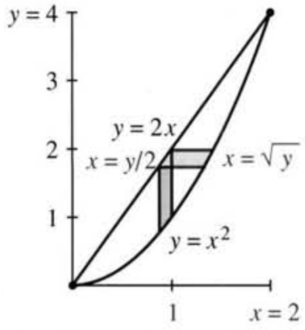
$$\iint_R f\,dA = \int_0^1 \int_0^{1-y} dx\,dy \qquad \text{(to do)}$$

Example

The double integral

$$\iint_R f\,dA = \int_0^2 \int_{x^2}^{2x} x^3\,dy\,dx \qquad \text{(to do)}$$

corresponds to the area between the curves $y = x^2$ and $y = 2x$. We have the following figure



- If we change the order of integration we also have

$$\iint_R f\,dA = \int_0^4 \int_{y/2}^{\sqrt{y}} x^3\,dx\,dy$$

where the horizontal internal integral goes from the « left frontier » $x = y/2$ (coming from $y = 2x$) to the « right frontier » $x = \sqrt{y}$ (coming from $y = x^2$). Then we sum from $y = 0$ to $y = 4$.

- We compute then

$$\iint_R f\,dA = \int_0^4 \left[\frac{x^4}{4}\right]_{y/2}^{\sqrt{y}} dy = \int_0^4 \left(\frac{y^2}{2^2} - \frac{y^4}{2^6}\right) dy$$

$$= \left[\frac{y^3}{3\cdot 2^2} - \frac{y^5}{5\cdot 2^6}\right]_0^{2^2} = \frac{2^4}{3} - \frac{2^4}{5} = \frac{2^5}{15} = \frac{32}{15} \simeq 2.13$$

Wooclap - Question 1

Which of the following assertions are correct ?

(1) The double integral $\iint_A f\,dA$ is always positive.

(2) If $f(x, y) = k$ for all points (x, y) in a region R then $\iint_A f\,dA = k \cdot \mathrm{Area}(A)$

(3) If R is the rectangle $0 \leq x \leq 1, 0 \leq y \leq 1$ then $\iint e^{xy}\,dA > 3$.

(4) Let $\rho(x, y)$ be the population density of a city, in people per km^2. If R is a region in the city, then $\iint_R \rho\,dA$ gives the total number of people in the region R.

(5) If $\iint f\,dA = 0$ then $f(x, y) = 0$ at all points of R.

Wooclap - Solution 1

(1) **False**. For example if $f(x,y) < 0$ for all (x,y) in R then $\iint_R f\, dA$ is negative.

(2) **True**. The double integral is the limit of the sum $\sum f(x,y)\,\Delta A = \sum k\,\Delta A = k \sum \Delta A$ over rectangles that lie inside the region R. As the area $\Delta A \to 0$, this sum approaches $k \cdot \text{Area}(R)$.

(2) **False**. The function $f(x,y) = e^{xy}$ is largest at the $(1,1)$ corner of R, so for any (x,y) in R we have $e^{xy} \leq e^{1\cdot 1} = e$. Then

$$\iint_R e^{xy} dA = \lim_{\Delta A \to 0} \sum e^{xy} \Delta A \leq \lim_{\Delta A \to 0} \sum e\,\Delta A$$

$$\text{and} \quad \lim_{\Delta A \to 0} \sum e\,\Delta A = e \lim_{\Delta A \to 0} \sum \Delta A = e \cdot \text{Area}(R) = e$$

So $\iint_R e^{xy} dA \leq e \approx 2.7$.

(4) **True**. The double integral is the limit of the sum $\sum_{\Delta A \to 0} \rho(x, y)\, \Delta A$. Each of the terms $\rho(x, y)\, \Delta A$ is an approximation of the total population inside a small rectangle of area ΔA. Thus the limit of the sum of all of these numbers as $\Delta A \to \infty$ gives the total population of the region R.

(5) **False**. If the graph of f has equal volumes above and below the xy-plane over the region R, the double integral is zero without having $f(x, y) = 0$ everywhere.

Change of coordinates

Generalities

- The cartesian coordinates xy are **not always adapted** to describe an integration region R in

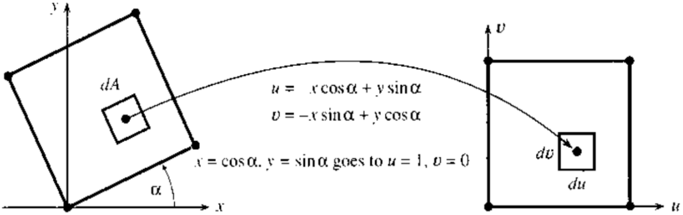
$$\iint_R f(x,y)\,dx\,dy$$

- In such situation it is standard to use alternative coordinates **to facilitate** the description of the integral boundaries.

- Incidentally, this new coordinates can also **simplify** the form of the fonction f to integrate.

A simple example

- We consider here the example of the surface of a unit square making an **angle α** with the axis Ox.



- The **area** of the square is given by the simple (double) integral

$$\iint_R dA = \iint_R dx\, dy = \iint_R dy\, dx$$

whose integration bounds are **difficult to determine**, even with the help of a figure.

- However, the figure tells us that a simple **rotation** of the Ox and Oy axes would simplify the situation.

- The **new coordinates** uv, result of the angle rotation α, are then given by the **linear** relations

$$\begin{cases} u &= x\cos\alpha + y\sin\alpha \\ v &= y\cos\alpha - x\sin\alpha \end{cases}$$

- The area of the square being **invariant**, we must have

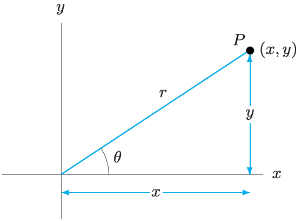
$$\int_?^? \int_?^? dx\,dy = 1 = \int_0^1 \int_0^1 du\,dv$$

where the bounds in the system uv are easy to guess.

- **Careful**, in general we have $\boldsymbol{dx\,dy \neq du\,dv}$. Here, the equality is true because the transformation is an **isometry**.

- When the region to be integrated follows a « curved geometry », the **polar coordinates** $r\theta$ can be used.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



- We have important relations

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

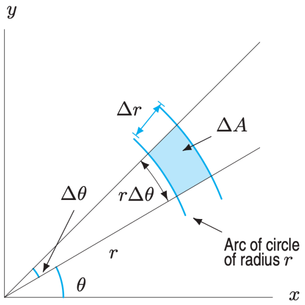
- Now we evaluate the **elementary area** $dA = dx\,dy$ in the system $r\theta$. This takes the form of a *polar rectangle*.

- The (by far) easiest method is to make a **figure**.

$$\Delta A \approx \Delta r \times r\Delta\theta$$

$$\Downarrow$$

$$dA = r\,dr\,d\theta = dx\,dy$$



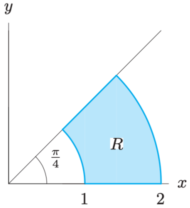
- We can also find the expression of dA by an **exact** method (see course).

- Note that here $dx\,dy \neq\ dr d\theta$

A simple example

- We want to evaluate $\iint_R f\,dA$ for the function below.

$$f(x,y) = \frac{1}{(x^2+y^2)^{3/2}}$$



- The region is given (in polar terms) by $1 \leq r \leq 2$ and $0 \leq \theta \leq \pi/4$. Moreover, since $r = \sqrt{x^2+y^2}$ we have

$$f = \frac{1}{(r^2)^{3/2}} = \frac{1}{r^3} \quad \text{so} \quad \iint_R f\,dA = \int_0^{\pi/4}\int_1^2 \frac{1}{r^3}\,r\,dr\,d\theta$$

- The calculation is now **much easier** to perform

$$\iint_R f\,dA = \int_0^{\pi/4}\left[-\frac{1}{r}\right]_1^2 d\theta = \int_0^{\pi/4}\frac{1}{2}\,d\theta = \frac{\pi}{8}$$

- We want to evaluate the following **Gaussian integral**

$$\mathcal{I} = \int_{-\infty}^{+\infty} e^{-x^2}\, dx$$

where the function e^{-x^2} has no primitive.

- The trick is to calculate $\mathcal{I}^2$ instead...

$$\begin{aligned} \mathcal{I}^2 &= \left(\int_{-\infty}^{+\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\, dy\right) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy \end{aligned}$$

- Using $r^2 = x^2 + y^2$ and the fact that $dx\,dy = r\,dr\,d\theta$, we can perform an **analytical integration**.

- The region of integration corresponds to the **whole plane**, so it is described in terms of polar coordinates by

$$0 \leq r \leq +\infty \qquad \text{and} \qquad 0 \leq \theta \leq 2\pi$$

- The double integral $\mathcal{I}^2$ then becomes

$$\begin{aligned}
\mathcal{I}^2 &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{e^{-r^2}}{2} \right]_0^{+\infty} d\theta \\
&= \int_0^{2\pi} \frac{1}{2}\,d\theta = \left[\frac{\theta}{2} \right]_0^{2\pi} = \frac{1}{2} \times 2\pi = \pi
\end{aligned}$$

- The initial integral thus gives

$$\mathcal{I} = \sqrt{\pi} \simeq 1.772\ldots$$

General case

- For a **change of variable** from the Cartesian system xy to an arbitrary coordinate system uv, we have

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(u,v)\,|J(u,v)|\,du\,dv$$

- We have introduced the **Jacobian** determinant $\boldsymbol{J}$, characterizing the deformation of the infinitesimal area $dA = dx\,dy$, defined by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- For this transformation to make sense, $\boldsymbol{J \neq 0}$ must be.

- The details of the demonstrations of the above formulas are given in **appendix** of the course document.

Wooclap - Question 2

Which of the following integrals give the area of the unit circle ?

(1) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dydx$$

(2) $$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dxdy$$

(3) $$\int_{0}^{2\pi}\int_{0}^{1} r\,drd\theta$$

(4) $$\int_{0}^{2\pi}\int_{0}^{1} drd\theta$$

(5) $$\int_{0}^{1}\int_{0}^{2\pi} r\,d\theta dr$$

(6) $$\int_{0}^{1}\int_{0}^{2\pi} d\theta dr$$

Wooclap - Solution 2

(1) **True.**

(2) **False.**

(3) **True.**

(4) **False.**

(5) **False.**

(6) **False.**