# Generalized Integrals

Chapter 1: Definition and properties

Nasko Karamanov

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#### In this lecture course ...

What is a generalized integral?

How to decide for a convergence : comparaisor

Partial integration

We will introduce and study generalized (improper) integrals

$$\int_{a}^{+\infty} f(x) \, \mathrm{d}x$$

For now, keep in mind the infinity bound +∞

We will introduce and study integrals with a parameter

$$\int_{a}^{+\infty} f(x,t) dx$$

We will study the convergence of sequences of such integrals

$$\lim_{n\to+\infty}\int_{a}^{+\infty}f_{n}(x)\,\mathrm{d}x$$

## **Learning Outcomes**

#### As a direct application of this course :

- · determine if a given generalized integral is well defined
- determine the convergence of a sequence of integrals and find the limit (if it exists)
- identify the properties of a integral depending on a parameter in most usual cases (as Fourier and Laplace transform)
- simplify expressions involving limits of sequences of integrals and parameter integrals
- validate a reasoning implicating questions of convergences of integrals or parameter integrals.

In situations of modelization in mathematics for signal processing, probability and automatics :

- calculate moments and probabilistic quantities related to a random variable with density
- identify hypothesis and arguments used in studying the convergence in probability
- calculate Fourier and Laplace transform of a function

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What is a generalized integral?

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Type 1: what happens at infinity

#### Definition

Let f be a continuous function over  $[a, +\infty[$ .

The generalized integral  $\int_{a}^{+\infty} f(t) dt$  converges if the limit

$$\lim_{x \to +\infty} \int_{a}^{x} f(t) dt$$
 exists and is finite.

In this case we let:

$$\int_{a}^{+\infty} f(t) dt = \lim_{x \to +\infty} \int_{a}^{x} f(t) dt$$

Type 1: what happens at infinity

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#### Example

$$\int_0^{+\infty} e^{-t} dt = \lim_{x \to +\infty} \int_0^x e^{-t} dt = \lim_{x \to +\infty} (1 - e^{-x}) = 1$$

#### Type 2: what happens on finite borders

#### **Definition**

Let f be a continuous function on [a, b[ where f is discontinued/not defined in b.

The generalized integral  $\int_a^b f(t) dt$  converges if the limit

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## Example

$$\int_{0}^{4} \frac{1}{\sqrt{4-t}} \, \mathrm{d}t = \lim_{x \to 4} \int_{0}^{x} \frac{1}{\sqrt{4-t}} \, \mathrm{d}t = \lim_{x \to 4} \left[ -2\sqrt{4-t} \right]_{0}^{x} = \lim_{x \to 4} \left( 4 - 2\sqrt{4-x} \right) = 4$$

In brief

If 
$$-\infty < a < b < = +\infty$$

$$\int_{a}^{b} f(t) dt = \lim_{x \to b} \int_{a}^{x} f(t) dt$$

Memo

Generalized integral = limit (Riemann integral)

## In brief

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#### Memo

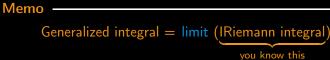
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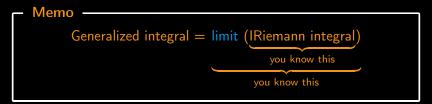


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Remark: Generalized integrals with a problem in the first border *a* are treated in the same manner.

## To follow in this RMD

- Generalized integrals in boths bounderies
- Chasles
- Partial integration
- Change of variables
- Comparaison

Warmup ...

Wooclap[1-2]

# Some properties: Chasles relation

#### Proposition -

Let  $f: [a, b[ \to \mathbb{R}] \to \mathbb{R}$  be coninuous on  $-\infty < a < c < b \le +\infty$ .

The integrals  $\int_a^b f(t) dt$  and  $\int_c^b f(t) dt$  are of same nature.

In the case of convergence :

$$\int_{a}^{b} f(t) dt = \int_{a}^{c} f(t) dt + \int_{c}^{b} f(t) dt$$

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#### Démonstration.

For all x such that  $a < c < x < +\infty$  Chasles relation for Riemann integrals gives

$$\int_{a}^{x} f(t) dt = \int_{a}^{c} f(t) dt + \int_{c}^{x} f(t) dt$$

As f is continuous on [a,c] the integral in the middle is finate. Passing to the limit gives the result.

# Some properties: linearity

### Proposition

Let  $f,g:[a,b[ \to \mathbb{R}]$  be continuous on  $-\infty < a < b \le +\infty$  and

 $\lambda, \mu \in \mathbb{R}$ . If the integrals  $\int_a^b f(t) dt$  and  $\int_a^b g(t) dt$  converge

then  $\int_{a}^{b} (\lambda f(t) + \mu g(t)) dt$  converge too and

$$\int_{a}^{b} (\lambda f(t) + \mu g(t)) dt = \lambda \int_{a}^{b} f(t) dt + \mu \int_{a}^{b} g(t) dt$$

# Some properties: linearity

## Proposition

Let  $f,g:[a,b[\to\mathbb{R}$  be continuous on  $-\infty < a < b \le +\infty$  and  $\lambda,\mu\in\mathbb{R}$ . If the integrals  $\int_a^b f(t)\,\mathrm{d}t$  and  $\int_a^b g(t)\,\mathrm{d}t$  converge then  $\int_a^b (\lambda f(t) + \mu g(t))\,\mathrm{d}t$  converge too and

$$\int_{a}^{b} (\lambda f(t) + \mu g(t)) dt = \lambda \int_{a}^{b} f(t) dt + \mu \int_{a}^{b} g(t) dt$$

#### Démonstration.

This is a consequence of the relation for classical Riemann integrals and then taking the limit.  $\Box$ 

Integrals over  $]-\infty,+\infty[$ 

Wooclap[3]

# Integrals with two improper bounderies ]a,b[

If there exists  $c \in ]a,b[$  such that  $\int_a^c f(t)dt$  and  $\int_c^b f(t)dt$  converged then we say that the integral  $\int_a^b f(t)dt$  converge. In the case of convergence

$$\int_{a}^{b} f(t) dt = \int_{a}^{c} f(t) dt + \int_{c}^{b} f(t) dt$$

The defintion does not depend on c (consequence of Chasles relation).

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# Example

$$\int_{-1}^{1} \frac{1}{x^2} dx = \int_{-1}^{0} \frac{1}{x^2} dx + \int_{0}^{1} \frac{1}{x^2} dx = \dots$$

Reference integrals : Riemann

Wooclap[4]

# Reference integrals: Riemann

# Theorem Let $\alpha \in \mathbb{R}$ $\int_{1}^{+\infty} \frac{1}{t^{\alpha}} \, \mathrm{d}t \ converges \ iff} \ \alpha > 1$ $\int_{0}^{1} \frac{1}{t^{\alpha}} \, \mathrm{d}t \ converges \ iff} \ \alpha < 1$

# Reference integrals: Riemann

#### Theorem

Let  $\alpha \in \mathbb{R}$ 

•

$$\int_{1}^{+\infty} \frac{1}{t^{\alpha}} dt \text{ converges iff } \alpha > 1$$

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$$\int_0^1 \frac{1}{t^{\alpha}} dt \ converges \ iff \ \alpha < 1$$

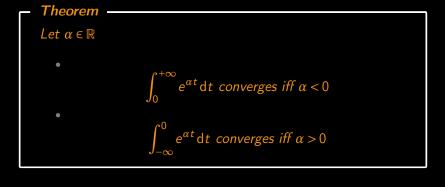
Memo: Riemann series

The series  $\sum_{n\geq 1} \frac{1}{n^{\alpha}}$  converges iff  $\alpha > 1$ 

Reference integrals : Riemann

Wooclap[5]

# Reference integrals : exponential (to be done on TD)



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What is a generalized integral?

How to decide for a convergence : comparaison

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How to decide for a convergence?

#### Question

What if Riemann itnegrals are hard to evaluate (for example a primirive is not known ...)?

# How to decide for a convergence?

#### Question

What if Riemann itnegrals are hard to evaluate (for example a primirive is not known ...)?

#### **Solution**

We compare f with a function g (to be determined depending on f) for which the convergence of its integral is easier to decide.

## Proposition

Let f be continuous and **positive** a over [a, b[ then

$$\int_{a}^{x} f(t) dt \text{ bounded } \Leftrightarrow \int_{a}^{b} f(t) dt \text{ converges}$$

a. or positive f in a **neighbourhood** of b, i.e. on some ]A, b[

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Wooclap[6]

#### Proposition

If f and g are positives on [a,b[ and  $0 \le f \le g$  on [a,b[

$$\int_{a}^{b} g(t) dt \text{ converges } \Rightarrow \int_{a}^{b} f(t) dt \text{ converges }^{a}$$

a. What thus the opposite result say?

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#### Démonstration.

For each  $x \in [a, b[$  we have  $0 \le f(x) \le g(x)$ . Thus

$$0 \le \int_a^x f(t) dt \le \int_a^x g(t) dt$$

If  $\int_a^b g(t) dt$  converges then  $\int_a^x g(t) dt$  is bounded and we use the previous proposition.

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# Proposition

• If f and g are positives on [a, b[ and f = O(g) or f = o(g) then

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• If  $f \sim g$  then

$$\int_{a}^{b} g(t) dt \text{ converges } \Leftrightarrow \int_{a}^{b} f(t) dt \text{ converges}$$

a. What does the opposite result say?

## Recall

If  $f,g:[a,b[\rightarrow \mathbb{R}]$  are two functions and g is non zero in a neighbourhood of b

$$f = O(g) \Leftrightarrow \frac{f}{g}$$
 is bounded in a neighbourhood of  $b$ 

$$f = o(g) \Leftrightarrow \lim_{x \to b} \frac{f(x)}{g(x)} = 0$$

$$f \approx \lim_{x \to b} \frac{f(x)}{g(x)} = 1$$

# **Examples**

## Question

What is the nature of  $\int_0^1 \ln(t) dt$ ?

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By comparative growth for all  $\alpha > 0$  we have

$$t^{\alpha} \ln(t) \underset{t \to 0}{\longrightarrow} 0$$

In particular, for  $\alpha = \frac{1}{2}$  we have

$$t^{\frac{1}{2}}\ln(t) \underset{t\to 0}{\longrightarrow} 0$$

Thus  $\ln(t)=O(\frac{1}{t^{\frac{1}{2}}})$  (we can also say that in a neighbourhood of 0  $\ln(t)<\frac{1}{t^{\frac{1}{2}}}$ 

But the integral  $\int_0^1 \frac{1}{\frac{1}{2}} dt$  converges.

## Proposition Absolue convergence

$$\int_{a}^{+\infty} |f(t)| dt \text{ converges } \Rightarrow \int_{a}^{+\infty} f(t) dt \text{ converges}$$

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#### Memo

Recall similar properties for series ....

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Wooclap[7-9]

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## **Proposition**

If  $\int_a^b u(t)v'(t)dt$  is a generalized integral (u and v are  $C^1$ ) and if

$$[u(t)v(t)]_a^b = \lim_{x \to b} u(x)v(x) - \lim_{x \to a} u(x)v(x)$$

is finite then the integrals  $\int_a^b u(t)v'(t)\mathrm{d}t$  and  $\int_a^b u'(t)v(t)\mathrm{d}t$  are of same nature.

In the case of convergence:

$$\int_{a}^{b} u(t)v'(t) dt = [u(t)v(t)]_{a}^{b} - \int_{a}^{b} u'(t)v(t) dt$$

Remark: We always start by checking if  $[u(t)v(t)]_a^b$  has a finite limite.

PI : example

 $\mathsf{Wooclap}[10]$ 

# Change of variables

## Proposition

If  $\int_a^b f(t) dt$  is a generalized integral and  $\varphi: I = ]\alpha, \beta[\rightarrow]a, b[$  bijective of class  $C^1$  such that

$$\lim_{t \to \alpha} \varphi(t) = a \text{ et } \lim_{t \to \beta} \varphi(t) = b$$

Then

$$\int_a^b f(t) dt$$
 et  $\int_a^\beta f(\varphi(t)) \varphi'(t) dt$ 

are of same nature nature. In the case of convergence both integrals have same value.

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# Wooclap[11]

## What we learned

- Generalized integral = limit (Riemann Integral)
- Same properties and techniques: linarity, Chasles, Pi, change of variables
- Pay attention when dealing with such integrals, check convergence before doing calculation ...

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#### To follow:

- Sequence of generalized integrals
- Parameter integrals