EPITA
ÉCOLE D'INGENIEURS EN INFORMATIQUE

Generalized integrals

1 Integrals with a parameter

Learning outcomes :

— determine if a given generalized integral is well defined
— identify the properties of a integral depending on a parameter in most usual cases (as Fourier and Laplace transform)
— simplify expressions involving limits of sequences of integrals and parameter integrals
— validate a reasoning implicating questions of convergences of integrals or parameter integrals.

Question 1-1

a) Let

$$F(x) = \int_0^{+\infty} \sin(xt) e^{-t^2}\, \mathrm{d}t$$

1. Show that F is well defined and continuous on $\mathbb{R}$.
2. Show that F is C^1 on $\mathbb{R}$.
3. Give a differential equation that satisfies F (you can use partial integration) !

b) Let

$$G(x) = \int_0^1 \frac{t^2}{\sqrt{1+x^4t^2}}\, \mathrm{d}t$$

Show that G is well defined and continuous on $\mathbb{R}$ and calculate $\lim_{x \to 0} G(x)$.

Question 1-2 Let f be continuous and integrable on $\mathbb{R}$. The Fourier transform of f is defined as follows :

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-i\omega t}\, \mathrm{d}t$$

a) Show that $\hat{f}$ is continuous over $\mathbb{R}$.

b) Let $g(t) = tf(t)$. Suppose that g is integrable on $\mathbb{R}$. Show that $\hat{f}$ is C^1 on $\mathbb{R}$ and

$$\hat{f}'(\omega) = -i\hat{g}(\omega)$$

c) Suppose that the functions $t^k f(t)$ are integrable on $\mathbb{R}$ for each $k \in \mathbb{N}$. Show that $\hat{f}$ is C^∞.

Question 1-3 Let

$$F(x) = \int_0^\pi \sin(x\sin(t))\,\mathrm{d}t$$

a) Show that F is C^1 on $\mathbb{R}$. What is the value of $F(0)$?

b) Deduce the value of :

$$\lim_{x\to 0} \frac{1}{x} \int_0^\pi \sin(x\sin(t))\,\mathrm{d}t$$

Question 1-4 Concider :

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\,\mathrm{d}t$$

a) Dermine the domain of Γ by studing the nature of the integral depending on x.

b) Using partial integration show that $\Gamma(x+1) = x\Gamma(x)$.

c) Deduce the value of $\Gamma(n)$ for all $n \in N^*$.

Question 1-5 We define Gauss integral by :

$$I = \int_0^{+\infty} e^{-t^2}\,\mathrm{d}t$$

and functuons f and g defined over $\mathbb{R}$ by :

$$f(x) = \int_0^x e^{-t^2}\,\mathrm{d}t \text{ and } g(x) = \int_0^1 \frac{e^{-(t^2+1)x^2}}{1+t^2}\,\mathrm{d}t$$

a) Show that g is C^1 sur $\mathbb{R}$.

b) Let $h = g + f^2$ defined over $\mathbb{R}$. Using change of variable calculate $h'(x)$.

c) What is the value of $h(0)$?

d) Using the coninuity of g show that $\lim_{x\to+\infty} g(x) = 0$.

e) Deduce the value of $\lim_{x\to+\infty} f^2(x)$

f) Deduce the value of I.