Logical Formalism
Exercise Sheet 3

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Main exercises

Exercise 1 - *A simple bijection.* Let $f : \mathbb{R} \to \mathbb{R}$ be a bijection. Prove that $g : x \in \mathbb{R} \mapsto \big(2 \cdot f(x) + 3\big) \in \mathbb{R}$ is a bijection.

Exercise 2 - *Even subsets.* Given $n \in \mathbb{N}^*$, let E be a finite set such that $\mathrm{Card}(E) = n$ and let $\mathcal{P}_e(E)$ be the set of all subsets of E of even cardinality. Prove that $\mathrm{Card}(\mathcal{P}_e(E)) = 2^{n-1}$.

Exercise 3 - *Cardinality of a generic union.* Given two finites sets E and F such that $\mathrm{Card}(E) = n$ and $\mathrm{Card}(F) = m$, prove that $\mathrm{Card}(E \cup F) \leq n + m$, then that $\mathrm{Card}(E \cup F) = n + m$ if and only $E \cap F = \emptyset$.

Exercise 4 - *Analysis and synthesis.* Find two real numbers a and b, knowing their sum and their product.

Extra Exercises

Analysis and Synthesis

Exercise 5 - *A decomposition.* Given $f : \mathbb{R} \to \mathbb{R}$, find all the functions $g, h : \mathbb{R} \to \mathbb{R}$ such that g is even (that is, $\forall x \in \mathbb{R}$, $g(x) = g(-x)$), h is odd (that is, $\forall x \in \mathbb{R}$, $h(x) = -h(-x)$), and $f = g + h$.

Exercise 6 - *A functional equation.* Find all the functions $f : \mathbb{R} \to \mathbb{R}$ such that $\forall x \in \mathbb{R}$, $f(x) + x \cdot f(1 - x) = 1 + x$.

Exercise 7 - *An application on* $\mathbb{R}^2$. Given $a, b \in \mathbb{R}$, let $f : (x, y) \in \mathbb{R}^2 \to ax + by \in \mathbb{R}$. First prove that f is not injective; then that f is surjective if and only if $(a, b) \neq (0, 0)$.

Combinatorics

Exercise 8 - ***A not so friendly dinner.*** Assume I, a popular man, have 10 friends, and want to invite 5 of them to my birthday party.

1. What is the number of possible guest lists?
2. What if we assume that two of my friends are married and must always be invited together?
3. What if two of my friends cannot stand each other and should under no circumstances be invited together?

Exercise 9 - ***The average subset.*** Let E be a set of cardinality n. What is the average cardinality $m = \frac{1}{\mathrm{Card}(\mathcal{P}(E))} \cdot \sum_{F \in \mathcal{P}(E)} \mathrm{Card}(F)$ of E's subsets?

You may rely on the equality $n \cdot (x+y)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} \cdot k \cdot x^{k-1} \cdot y^{n-k}$.

Exercise 10 - ***Inclusions.*** Let E be a set of cardinality n. Consider the set:

$$F = \{(X, Y) \mid X, Y \in \mathcal{P}(E), X \subseteq Y\}$$

Determine $\mathrm{Card}(F)$.