Generalized integrals

Chapter 1

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In this lecture course ...

Flashback on Riemann Integrals

Why generalized (improper) integrals ?

What is a generalized integral ?

In this lecture course ...

- We will introduce and study generalized (improper) integrals

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- We will study the convergence of sequences of such integrals

$$\lim_{n\to+\infty}\int_a^{+\infty} f_n(x)\,\mathrm{d}x$$

Learning Outcomes

As a direct application of this course :

- determine if a given generalized integral is well defined
- determine the convergence of a sequence of integrals and find the limit (if it exists)
- identify the properties of a integral depending on a parameter in most usual cases (as Fourier and Laplace transform)
- simplify expressions involving limits of sequences of integrals and parameter integrals
- validate a reasoning implicating questions of convergences of integrals or parameter integrals.

In situations of modelization in mathematics for signal processing, probability and automatics :

- calculate moments and probabilistic quantities related to a random variable with density
- identify hypothesis and arguments used in studying the convergence in probability
- calculate Fourier and Laplace transform of a function

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Riemann integrals

If f is a continuous fonction on $[a, b]$ and F a primitive of f then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) = [F(x)]_a^b$$

And

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{1}$$

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- Application : surface calculations, mean value of a function on an interval ...
- Techniques : Chasles, partial integration, change of variables, linarity

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You will encounter phonomena modelised by a ***density distribution*** $f : I \subseteq \mathbb{R} \to \mathbb{R}$:

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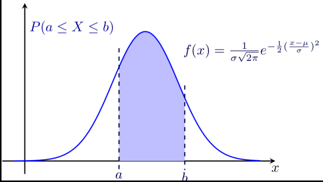
Examples : normal distribution



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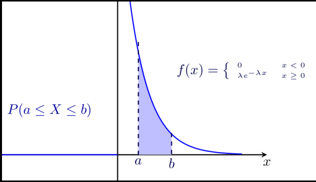
Examples : exponential distribution (modelises interarrival times of agents in a waiting line, to be seen in S9 : ERO2)



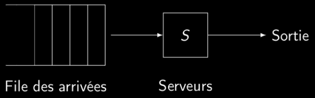


Figure 1 – A glimpse of ERO2

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$$P(X \in [a,b]) = \int_a^b f(x)\,\mathrm{d}x$$

- In both cases one needs to calculate total probability i.e. integrate over $]-\infty, +\infty[$:

$$P(X \in \,]-\infty,+\infty[) = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$$

- The mean value X is given by

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,\mathrm{d}x$$

To be seen in PBS1 and PBS2 (Probability and statistics)

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Mathematics for signal (S5 : MASI)

- a signal is represented by a function

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- energy of a signal :

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

- autocorrelation of signal of period T : (ressemblance of a signal with itself) :

$$\Gamma_{x,x}(T) = \int_{-\infty}^{+\infty} x(t)\overline{x(t-T)} \, dt$$

- Intercorrelation of two signals (ressemblance between x and y)

$$\Gamma_{x,y}(T) = \int_{-\infty}^{+\infty} x(t)\overline{y(t-T)} \, dt$$

- convolution (usefull for filtering)

$$x * y(T) = \int_{-\infty}^{+\infty} x(t)y(t-T) \, dt$$

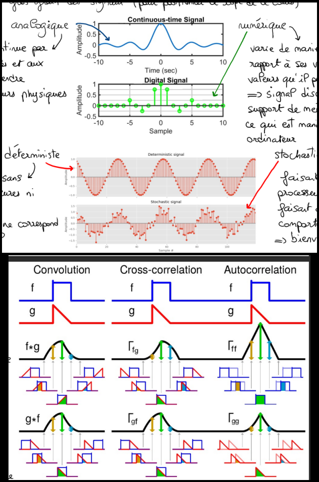

Figure 2 – A glimpse of MASI

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What is a generalized integral ?

Type 1 : what happens at infinity

Definition

Let f be a continuous function over $[a, +\infty[$.

The generalized integral $\displaystyle\int_a^{+\infty} f(t)\,\mathrm{d}t$ ***converges*** if the limit

$\displaystyle\lim_{x\to+\infty}\int_a^x f(t)\,\mathrm{d}t$ exists and is finite.

In this case we let :

$$\int_a^{+\infty} f(t)\,\mathrm{d}t = \lim_{x\to+\infty}\int_a^x f(t)\,\mathrm{d}t$$

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Example

$$\int_0^{+\infty} e^{-t}\,\mathrm{d}t = \lim_{x\to+\infty}\int_0^x e^{-t}\,\mathrm{d}t = \lim_{x\to+\infty}(1-e^{-x}) = 1$$

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Example

$$\int_0^4 \frac{1}{\sqrt{4-t}}\,\mathrm{d}t = \lim_{x \to 4} \int_0^x \frac{1}{\sqrt{4-t}}\,\mathrm{d}t = \lim_{x \to 4} \left[-2\sqrt{4-t}\right]_0^x = \lim_{x \to 4} \left(4 - 2\sqrt{4-x}\right) = 4$$

In brief

If $-\infty < a < b <= +\infty$

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{x \to b} \int_a^x f(t)\,\mathrm{d}t$$

Memo

Generalized integral = limit (Riemann integral)

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Remark : Generalized integrals with a problem in the first border a are treated in the same manner.

Training session

Study the convergence of the following integrals :

$$\int_0^1 \frac{1}{x}\,\mathrm{d}x, \quad \int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x, \quad \int_0^{+\infty} \cos(x)\,\mathrm{d}x$$

Question

What do you think about the following statement :

If f is a continuous function on $[0,+\infty[$ then

$$\lim_{x\to+\infty} f(x) = 0 \Longrightarrow \int_0^{+\infty} f(x)\,\mathrm{d}x \text{ converges}$$

To follow in RMD

- Generalized integrals in boths bounderies
- Chasles
- Partial integration
- Change of variables
- Comparaison