École Pour l'Informatique et les Techniques Avancées

Approximation of Functions

Ch2: Sequences of Functions

Convergence

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KAMEL ATTAR

In this lecture · · ·

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$$\forall x \in I,\quad \forall \varepsilon > 0,\quad \exists N_{x,\varepsilon} \in \mathbb{N},\quad \forall n \geq N \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.$$

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At the end of this lecture, the student will be able to:

- **Distinguish** the issues related to defining a notion of functional convergence.
- **Validate and study** the nature of function sequences.

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2 Looking for Limits
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Uniform Convergence

3 Two important theorems on uniform approximation
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4 *p*-Norms on Functions

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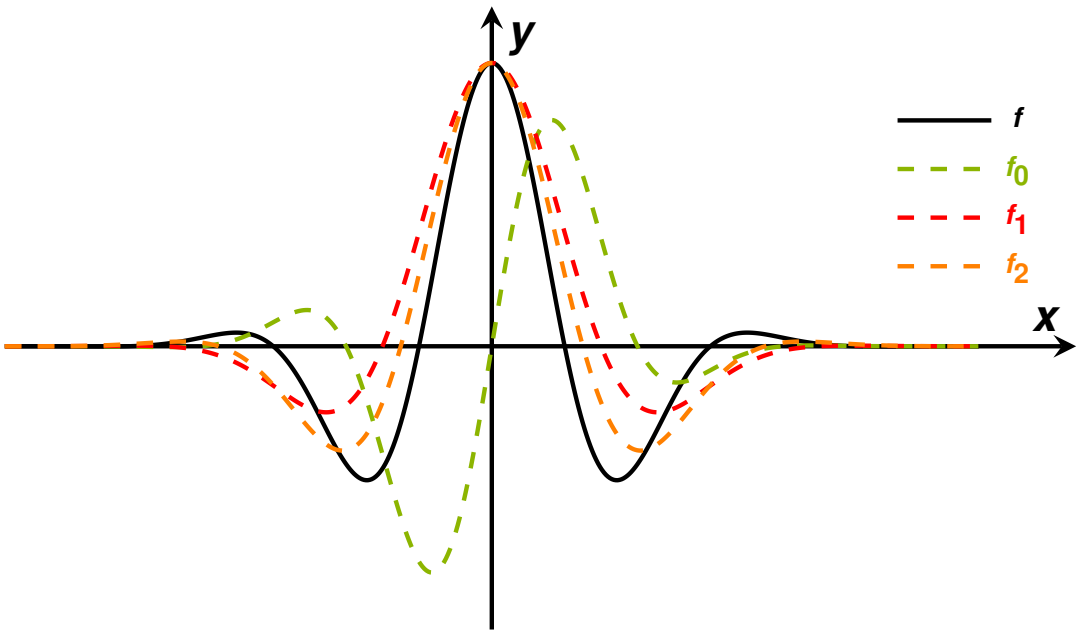
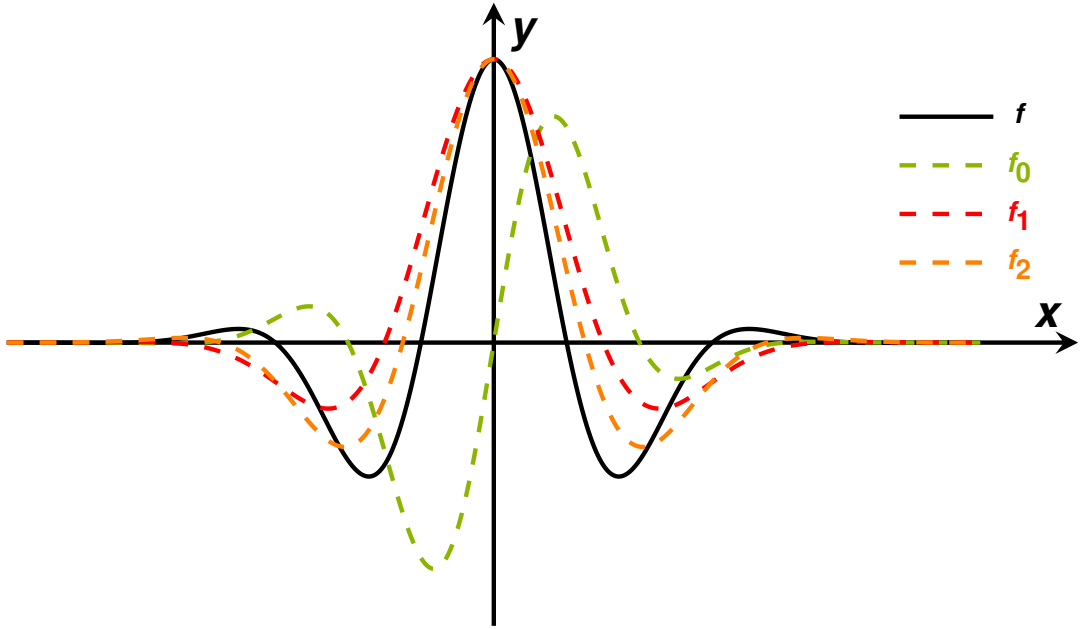
☞ Given a function f, how can we rigorously formulate the idea that a sequence of functions $f_0, f_1, f_2, \ldots$ increasingly approximates f, as precisely as desired?



Given a function f, how can we rigorously formulate the idea that a sequence of functions $f_0, f_1, f_2, \ldots$ increasingly approximates f, as precisely as desired?

We want to be able to say that the sequence of functions (f_n) converges to f when the representative curve of the function f_n approaches, as n tends to infinity, the curve of f.

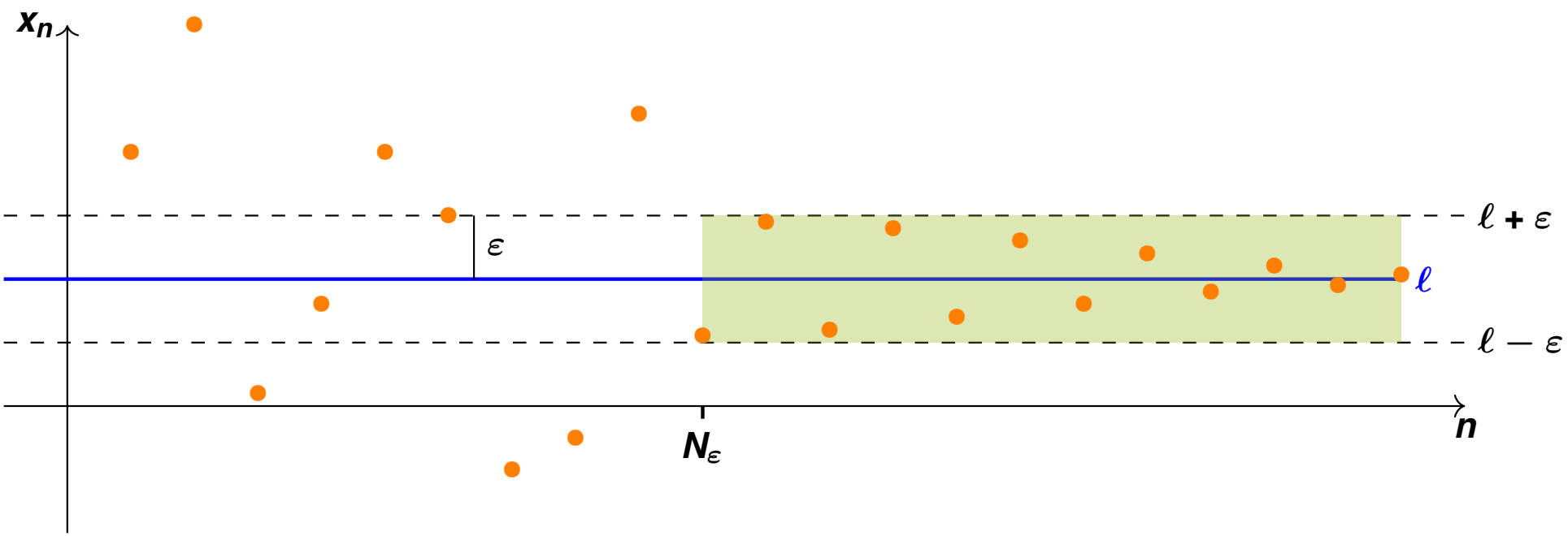


In the context of a metric space (E, d), a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E converges to $\ell \in E$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\quad n \geq N \implies d(x_n, \ell) < \varepsilon$$

x_n is at a *distance* of at most ε from ℓ

For any $\varepsilon > 0$, no matter how small, the distance between x_n and ℓ will eventually stay $< \varepsilon$ from a certain rank onwards.



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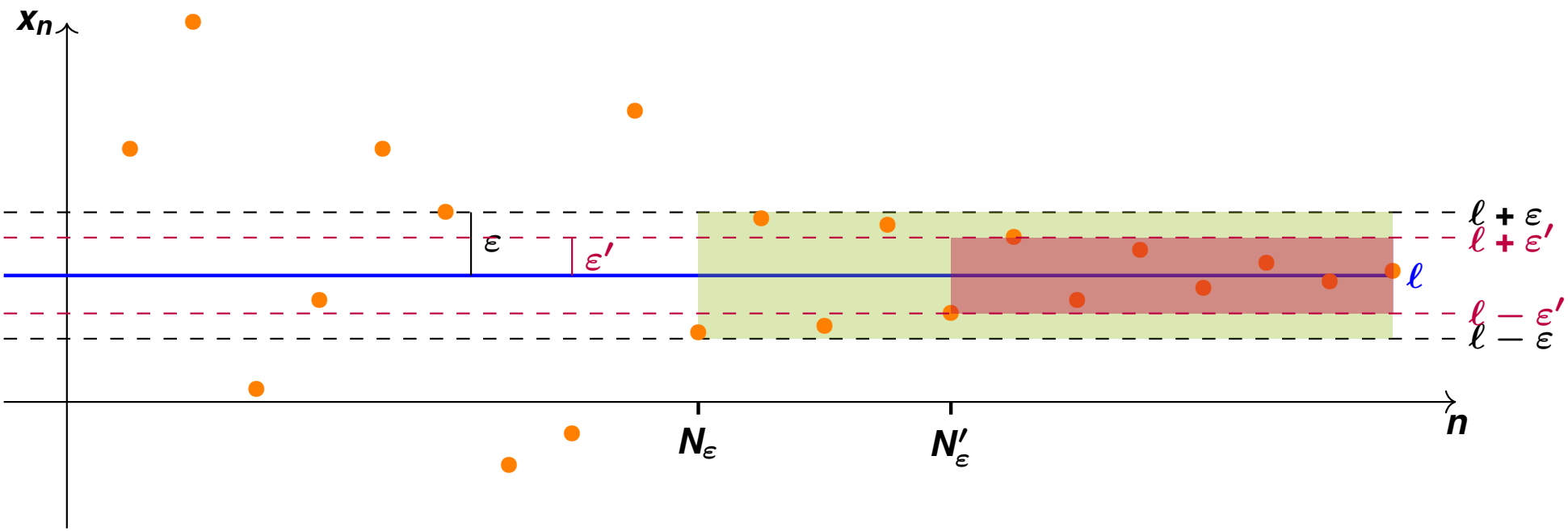

It can be observed that the specific rank generally depends on ε: the smaller ε is, the larger the value of N needs to be before we consistently see the distance between x_n and ℓ remain $< \varepsilon$.

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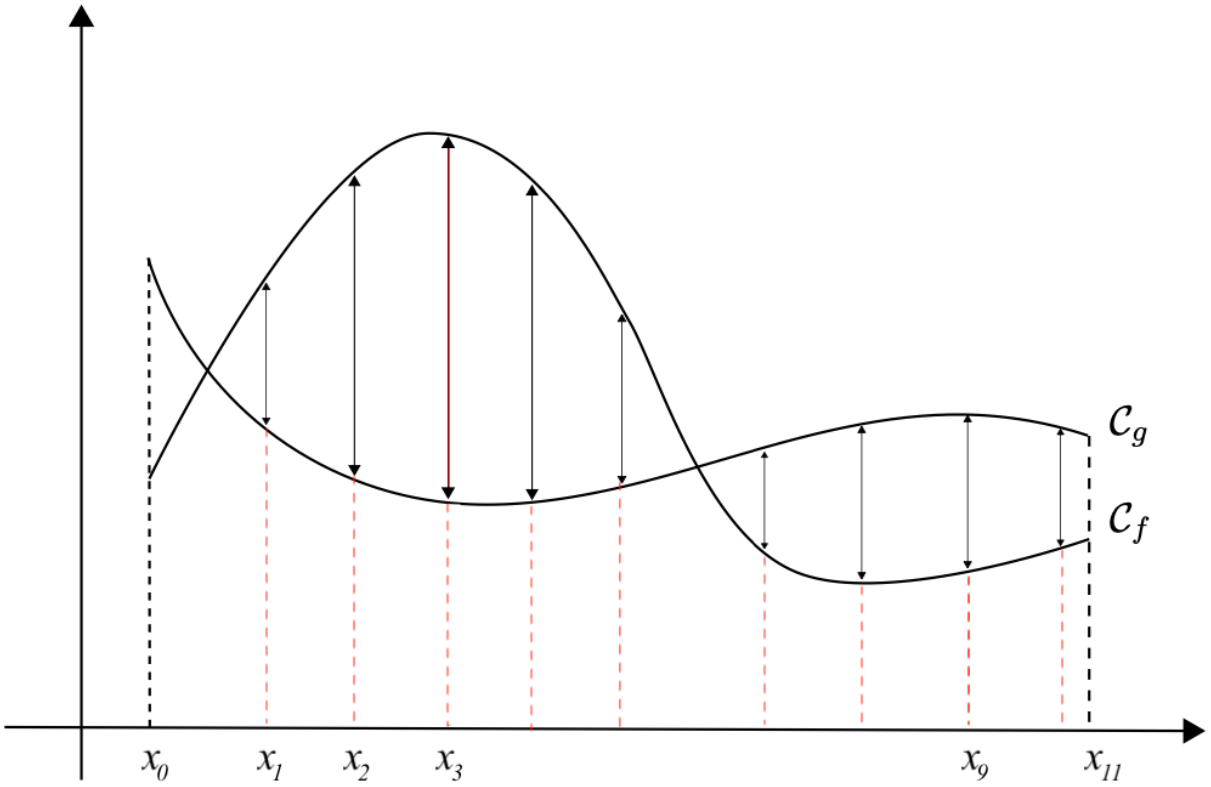
☞ What useful distances can one build on spaces of real-valued functions?

✎ Wooclap 1.

1. Give examples of norms on the set $\mathcal{C}^0([0,1],\mathbb{R})$ of continuous real-valued functions on the segment $[0,1]$.
2. Calculate the infinity distance d_∞ between C_f and C_g.



Solution

✎ **Wooclap 2.** Among the following suggestions which ones are true ?

1. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ over $[0, 1]$ by $f_n(x) = x^n$ converges with respect to the norm 1 to the null function.

2. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ over $[0, 1]$ by $f_n(x) = x^n$ converges for the ∞ norm toward the null function.

3. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ over $[0, 1]$ by $f_n(x) = 1 + \frac{x}{n}$ converges for the ∞ norm twoard the constant function equal to 1.

4. The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined for all $n \in \mathbb{N}$ over $[0, 1]$ by $f_n(x) = 1 + \frac{x}{n}$ converges for the 2-norm towards the constant function equal to 1.

▸ Solution

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✎ **Wooclap 3.** Let's analyze each of the numerical sequences to determine their convergence and, if they converge, calculate their limits.

1. $S_n = 1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{n^3}$ Solution

2. $A_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1}$ Solution

The definition of convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in a metric space (E, d) towards a point $\ell \in E$ suggests that we need to know ℓ to show that (x_n) converges to ℓ.

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}, \quad n \geq N \quad \Longrightarrow d(x_n, \ell) < \varepsilon$$

Attention! It is a reality that even in well-known metric spaces, we are rarely able to define generic methods to identify the limit $\ell \in E$ of an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$.

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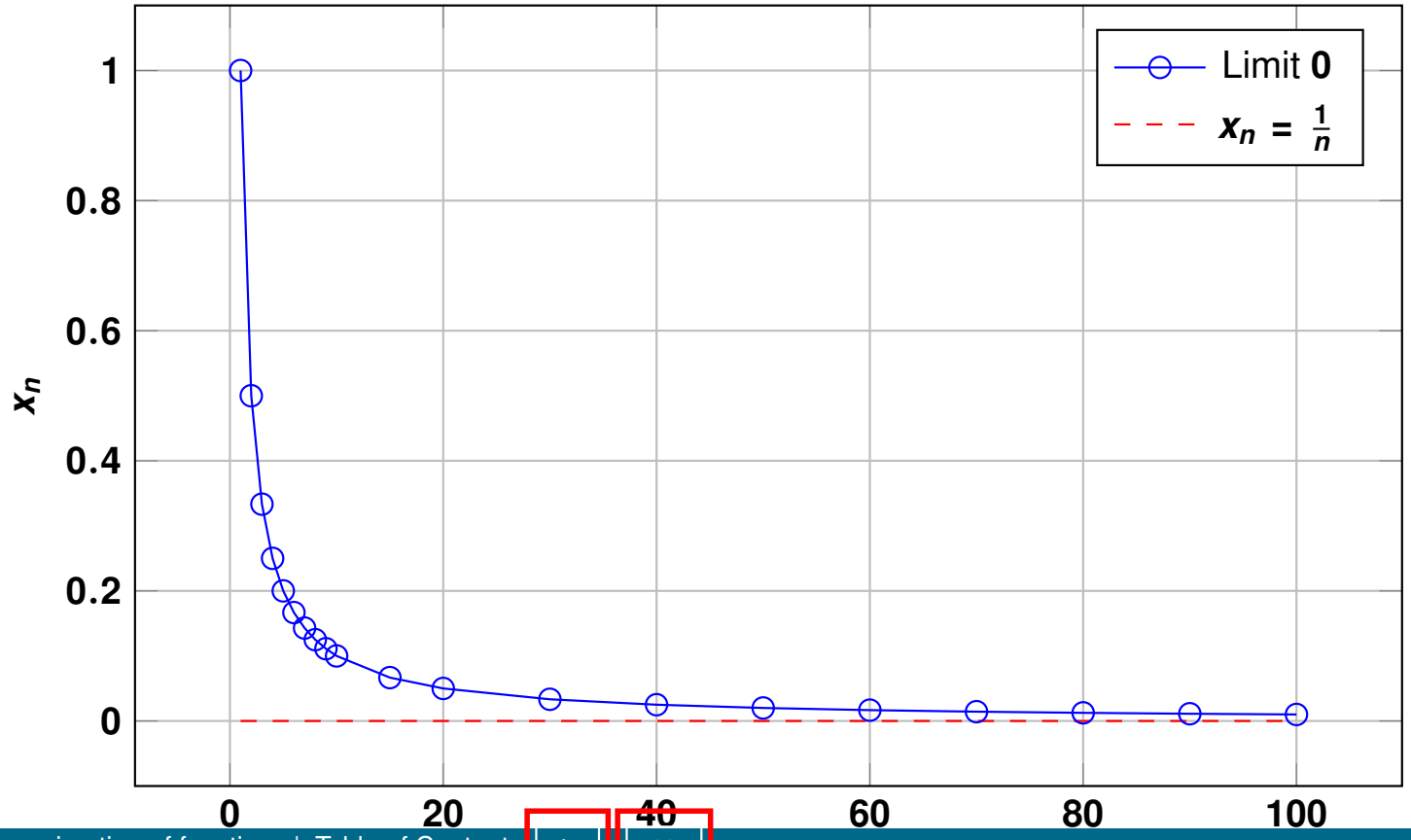
To Better Understand

If a sequence $(x_n)_{n=1}^{\infty}$ seems to get closer to a certain value ℓ as n approaches infinity, ℓ can be considered as a candidate for the limit of the sequence. For example, consider the sequence $x_n = \frac{1}{n}$. It is clear that the terms of this sequence become smaller and smaller as n increases. Intuitively, it seems that these terms approach zero as n tends to infinity.



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To Better Understand

If a sequence $(x_n)_{n=1}^{\infty}$ converges to a limit ℓ but we cannot calculate it exactly, a numerical approximation can be used to obtain a value close to ℓ.

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Assumption

We shall be limiting our study to the case of uniform convergence, given by $\|\cdot\|_\infty$.

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Definition: Pointwise Convergence

A sequence of functions $(f_n)_{n\in\mathbb{N}}$, defined on an interval I, converges pointwise to a function f defined over I if

$$\forall x \in I, \quad (f_n(x))_{n\in\mathbb{N}} \text{ converges to } f(x) \iff \forall x \in I, \quad \lim_{n\to+\infty} f_n(x) = f(x).$$

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For every point x in the interval I, if we consider the values of the sequence of functions $f_n(x)$ for increasingly large indices n, then these values approach the value of the limit function $f(x)$.

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Attention!

- The notion of pointwise convergence *is not* associated with a notion of distance, as in the case for convergence in norm p.
- Showing the pointwise convergence of a sequence (f_n) is equivalent to showing the convergence of the numerical sequence $f_n(x)$ for each x in I.

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- First, disregard the parameter of the function n.
- Fix $x \in I$ and prove that the numerical sequence $(f_n(x))$ converges to $f(x)$.
- Therefore, this is a problem of convergence of real number sequences, not really a problem of convergence of sequences of functions.

Example

The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by

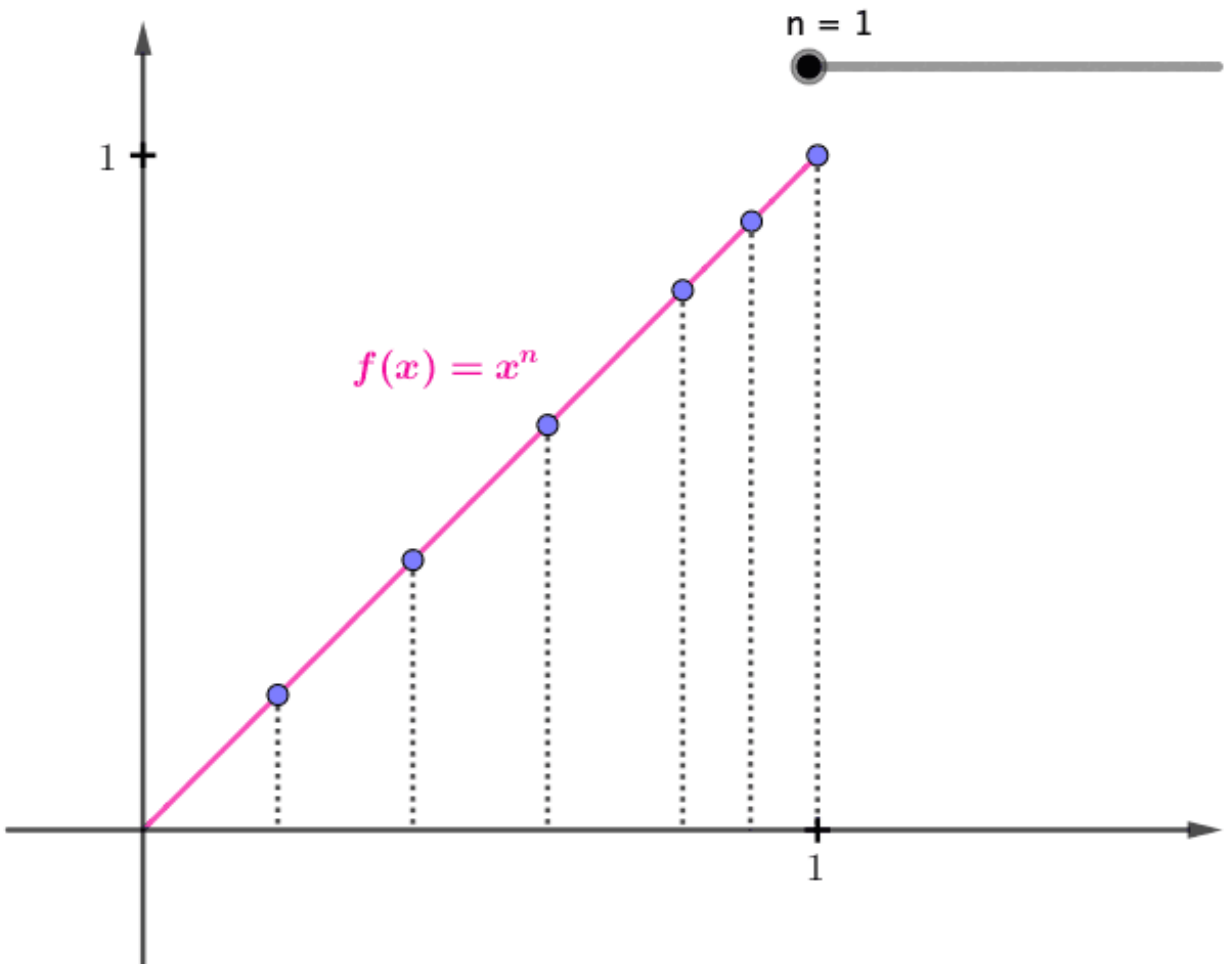
$$f_n(x) = x^n$$

converges pointwise to the function

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

on the interval $[0, 1]$.

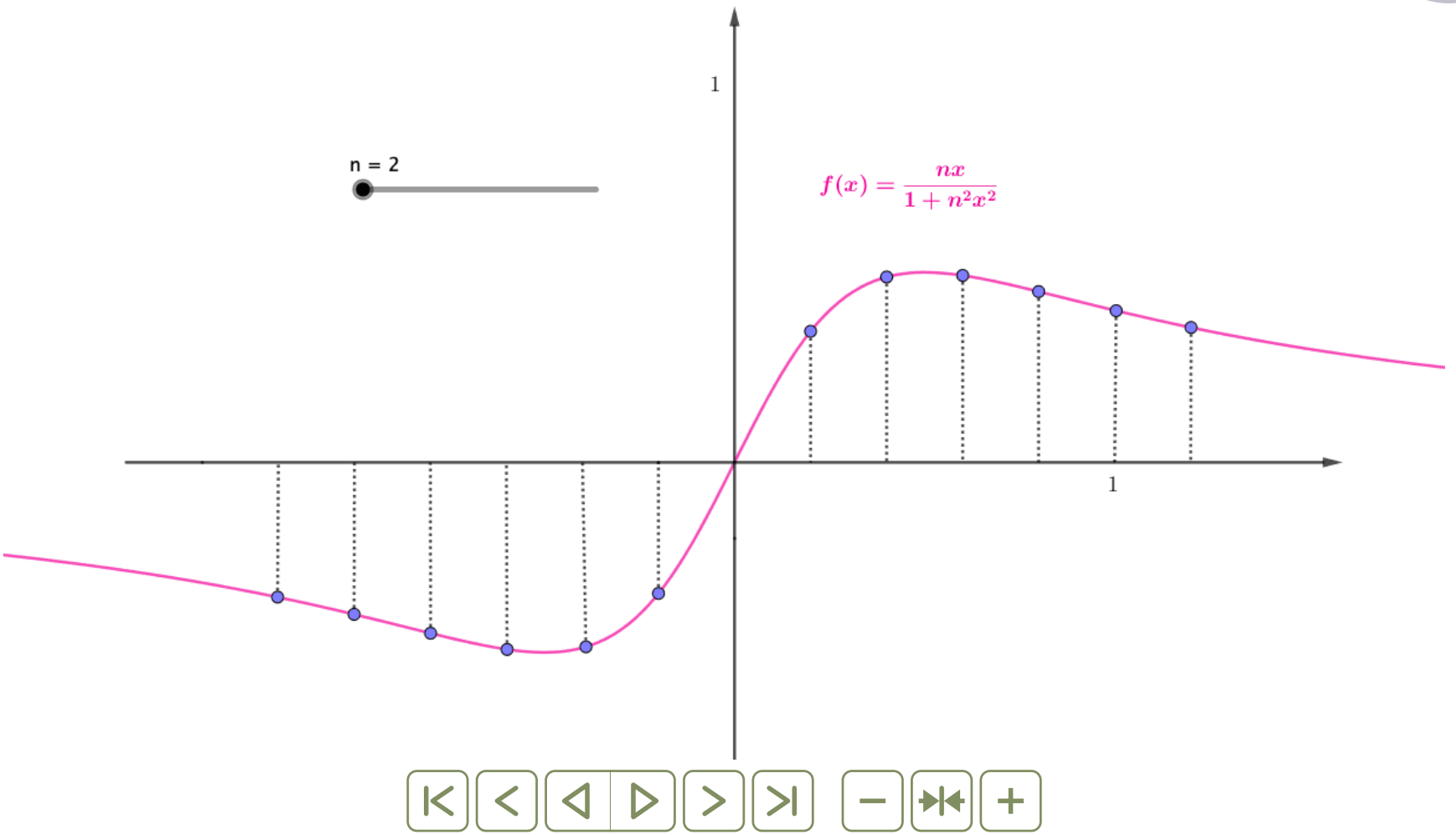
- If $x = 1$, the numerical sequence $\left(f_n(1)\right)$ is constant with a limit of 1.
- If $x \in [0, 1[$, the numerical sequence $\left(f_n(x)\right)$ tends to 0 as $n \to +\infty$.


n = 1
1
$f(x) = x^n$
1

Example

The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined by $f_n(x) = \frac{nx}{1+n^2x^2}$ converges pointwise to the zero function on the real numbers ($\mathbb{R}$).

- If $x = 0$, the numerical sequence $\left(f_n(0)\right)$ is zero.
- If $x \neq 0$, the numerical sequence $f_n(x) \underset{+\infty}{\approx} \frac{nx}{n^2x^2} = \frac{1}{nx}$ approaches zero as $n \to +\infty$.


n = 2
$f(x) = \frac{nx}{1+n^2x^2}$
1
1

The usual properties of compatibility with algebraic operations for numerical sequences hold for pointwise convergence.

Proposition

Given $\lambda \in \mathbb{R}$, $(f_n) \in (\mathbb{R}^I)^{\mathbb{N}}$, and $(g_n) \in (\mathbb{R}^I)^{\mathbb{N}}$ as sequences of functions converging pointwise to $f \in \mathbb{R}^I$ and $g \in \mathbb{R}^I$, respectively, then

- $(f_n + \lambda g_n)$ converges pointwise to $f + \lambda g$ on I
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Important to Know

It's important to know that while pointwise convergence has its advantages due to its ease of use, it also has significant disadvantages. The properties of the terms in the sequence of functions are largely independent of similar properties for the eventual limit.

✎ **Wooclap 4.** We give ourselves a sequence of functions $(f_n)_{n\in\mathbb{N}}$ which pointwise converges to f. We're wondering which properties satisfy the following assertion : if each f_n satisfies (P) then f satisfies (P).

- (P) : "to be continuous"
- (P) : "to be differentiable"
- (P) : "to be periodic"
- (P) : "to be bounded"

Solution

Limitations of Pointwise Convergence

Simple convergence has several drawbacks that may limit its use in practical applications:

- **Lack of Uniform Control:** The functions f_n do not necessarily approach the limit f uniformly.
- **Loss of Continuity:** Even if each function f_n is continuous, the limit f may not be.
- **Properties Not Preserved:** Pointwise convergence does not always preserve properties such as integrability or differentiability.
  - If (f_n) is a sequence of differentiable functions that converges Pointwise, the sequence (f_n') does not necessarily converge. $f_n(x) = \frac{nx}{1+n^2x^2}$, $f_n(x) = x^n$
  - If (f_n) is a sequence of continuous functions on $[a, b]$, the sequence $\int_a^b f_n(x)\,dx$ does not necessarily converge. $f(x) = \begin{cases} 0 & \text{if } x > \frac{1}{n} \\ n & \text{if } 0 \leq x \leq \frac{1}{n} \end{cases}$
- **Limited Use in Calculations:** Insufficient for many numerical or analytical calculations requiring stricter control.

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$$\lim_{n \to +\infty} f_n(x) = f(x)$$

$$\boxed{\forall x \in I, \quad \forall \varepsilon > 0, \quad \exists N_{x,\varepsilon} \in \mathbb{N}, \quad \forall n \geq N_{x,\varepsilon} \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.}$$

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$$\boxed{\forall x \in I, \quad \forall \varepsilon > 0, \quad \exists N_{x,\varepsilon} \in \mathbb{N}, \quad \forall n \geq N_{x,\varepsilon} \quad \Longrightarrow \quad |f_n(x) - f(x)| < \varepsilon.}$$

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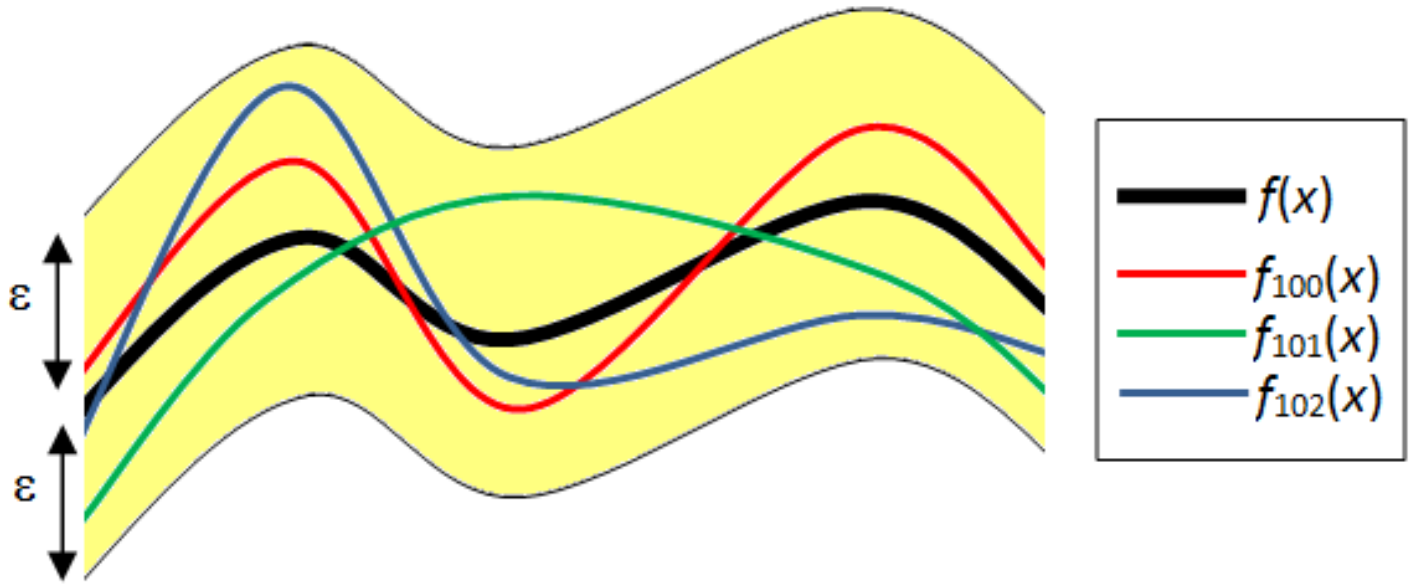

☞ How to prove that (f_n) converges uniformly to f on I?

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- Method 2 : Bound $|f_n(x) - f(x)|$ by a quantity independent of $x \in I$ that tends to 0. In other words:
  For $x \in I$, if we find

$$|f_n(x) - f(x)| \leq a_n \Longrightarrow \|f_n - f\|_\infty \leq a_n,$$

  where a_n does not depend on x. Since the sequence (a_n) tends to 0, it follows that (f_n) converges uniformly to f on I.

Example

Let (f_n) be the sequence of functions on $[0,1]$ defined for all $n \in \mathbb{N}^*$ and all $x \in [0,1]$ by

$$f_n(x) = \frac{ne^{-x}}{n^2 + x}$$

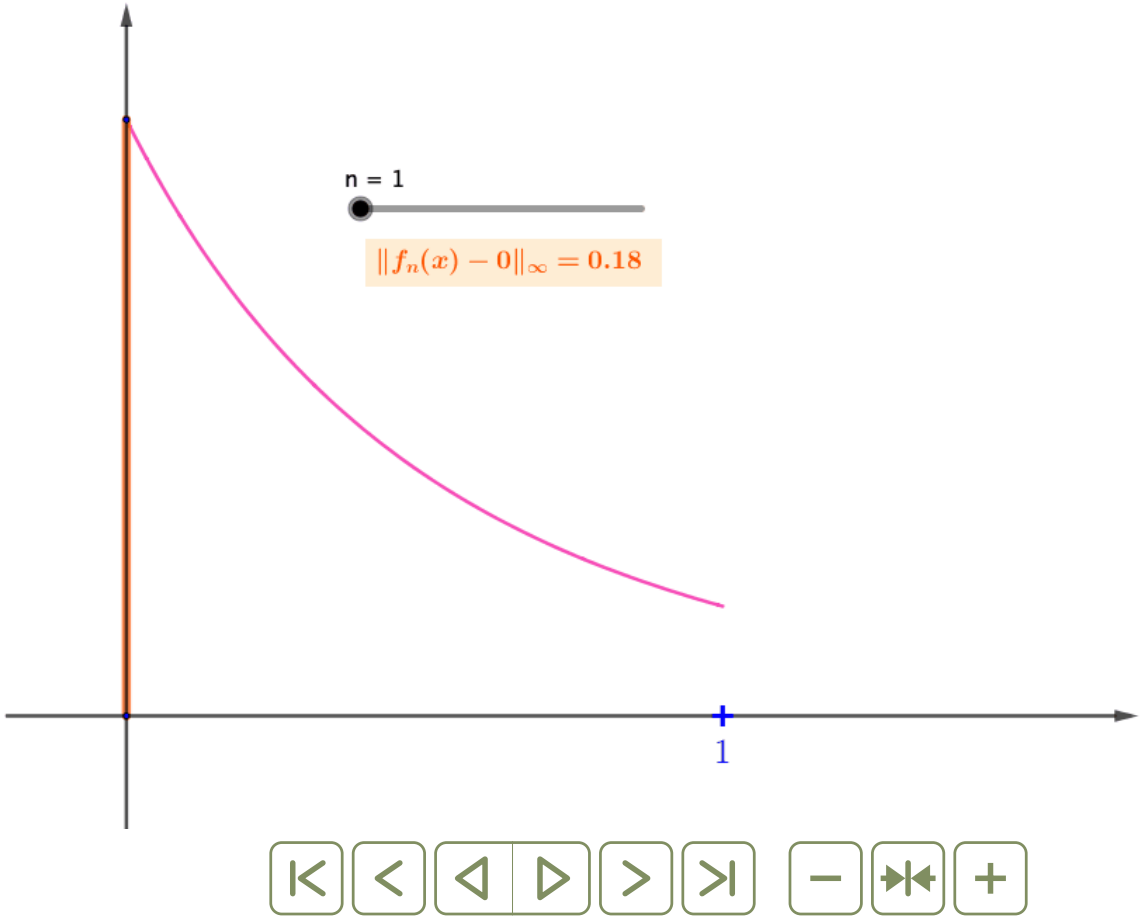
We will show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to the zero function $f(x) = 0$.

We have $\forall x \in [0,1]\,,\ \frac{ne^{-x}}{n^2+x} \leq \frac{n}{n^2} = \frac{1}{n}$

Then

$$\forall x \in [0,1]\,,\ \frac{ne^{-x}}{n^2 + x} \leq \frac{n}{n^2} = \frac{1}{n} \Longrightarrow \|f_n - f\|_\infty = \sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \frac{ne^{-x}}{n^2 + x} \leq \frac{1}{n} \longrightarrow$$

This means that the sequence of functions $(f_n)n \in \mathbb{N}$ converges uniformly to the limit function $f(x) = 0$ on the interval $[0,1]$.


n = 1
$\|f_n(x) - 0\|_\infty = 0.18$
1

Proposition

If a sequence of functions $(f_n)_{n\in\mathbb{N}}$, that are bounded and defined over I converge with respect to $\|\cdots\|_\infty$ to a function f then $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f.

Proof :

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$$n \geq N_\varepsilon \Longrightarrow \sup_{t\in I} \left| f_n(t) - f(t) \right| < \varepsilon$$

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As $x \in I$, by definition of the supremum $\left|f_n(x) - f(x)\right| \leq \sup_{t\in I} \left|f_n(t) - f(t)\right|$ dso, for all $\forall n \in \mathbb{N}$

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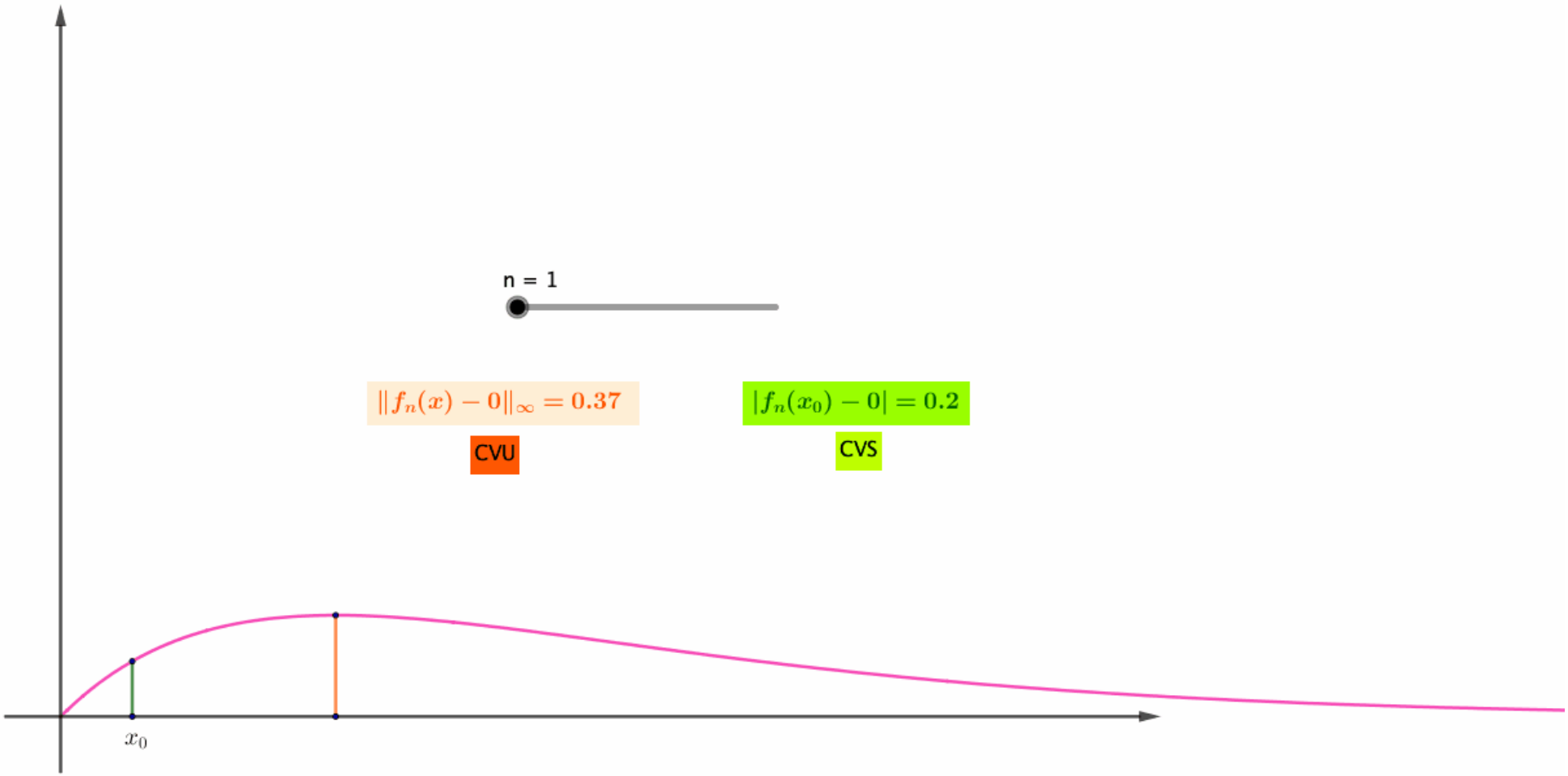
Attention ! The converse of the previous proposition is false; a sequence of functions that converges pointwise to a given limit does not necessarily converge uniformly.


n = 1
$\|f_n(x) - 0\|_\infty = 0.37$
CVU
$|f_n(x_0) - 0| = 0.2$
CVS
x_0

✎ **Wooclap 5.** We give ourselves a sequence of functions $(f_n)_{n\in\mathbb{N}}$ which converges uniformly to f. We're wondering which properties satisfy the following assertion : if each f_n satisfies (P) then f satisfies (P).

- (P) : "to be continuous"
- (P) : "to be differentiable"
- (P) : "to be periodic"
- (P) : "to be bounded"

Solution

- **Uniform convergence preserves the continuity of the limit**: If a sequence of functions (f_n) converges uniformly to f on I, and each (f_n) is continuous on I, then f is continuous on I.

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  - the functions f_n are of class C^1 on I;
  - (f_n) converges (pointwise) on I to f;
  - (f_n') converges uniformly on I to a function g.

  Then, f is differentiable on I, and $f' = g$, which means:

$$\boxed{\frac{d}{dx}\left(\lim_{n\to+\infty} f_n(x)\right) = \lim_{n\to+\infty} \frac{d}{dx}\left(f_n(x)\right)}$$

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- **Uniform Convergence and Integration**: If (f_n) converges uniformly to f on $[a, b]$ (closed and bounded), then: $\boxed{\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b \lim_{n\to+\infty} f_n(t)dt = \int_a^b f_n(t)dt}$

☞ How to prove that (f_n) does not converge uniformly to f on I?

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- Method 1 : Study the behavior of the function $f_n(x) - f(x)$, and deduce that $\|f_n - f\|_\infty = \sup_{x \in I} |f_n(x) - f(x)| = a_n$, and prove that this quantity does not tend to 0.

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- Method 2 : Find a sequence (x_n) within I such that $(f_n(x_n) - f(x_n))$ does not tend to 0.

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- Method 2 : Find a sequence (x_n) within I such that $(f_n(x_n) - f(x_n))$ does not tend to 0.
- Method 3 : Prove that the sequence of functions f_n are continuous, but the limit function f is not continuous.

Example

- **Method 1: Studying the ∞ norm**:
  Let $f_n(x) = nx$ and $f(x) = 0$ for $x \in [0, 1]$. Here, $||f_n - f||_\infty = \sup_{x\in[0,1]} |nx - 0| = n$.
  Since $||f_n - f||_\infty \nrightarrow 0$, the function does not converge uniformly to f.

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  Since $\|f_n - f\|_\infty \not\to 0$, the function does not converge uniformly to f.
- **Method 2: Finding a sequence (x_n) in I**:
  Let $f_n(x) = \chi_{[0,\frac{1}{n}]}(x)$ (the indicator function of $[0, \frac{1}{n}]$) and $f(x) = 0$ for $x \in [0, 1]$. Then,

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

For $x_n = \frac{1}{2n}$, we have $f_n(x_n) = 1$ and $f(x_n) = 0$, so $|f_n(x_n) - f(x_n)| = 1$, which does not tend to 0. This shows that (f_n) does not converge uniformly to f.

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- **Method 3: Discontinuous limit function**:
  Let $f_n(x) = x^n$ for $x \in [0, 1]$. Each f_n is continuous, but the limit

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

  is not continuous. This shows that (f_n) does not converge uniformly to f.

Attention! IF THE SEQUENCE OF FUNCTIONS DOES NOT CONVERGE POINTWISE, THEN IT CANNOT CONVERGE UNIFORMLY.

$$\begin{tikzcd} \text{Uniform Convergence} \arrow[r] & \text{Pointwise Convergence} \arrow[l, bend right, "\times"'] \end{tikzcd}$$

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2 Looking for Limits
Defining and Understanding Convergence
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Compatibility with Algebraic Operations
Uniform Convergence

3 Two important theorems on uniform approximation
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Trigonometric Version

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2 Looking for Limits
Defining and Understanding Convergence
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How to Proceed in Practice
Compatibility with Algebraic Operations
Uniform Convergence

3 Two important theorems on uniform approximation
Polynomial Version
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Theorem : Weierstrass theorem (Polynomial Version)

Let a and b be two real numbers with $a < b$. Any function f that is continuous on the interval $[a, b]$ with real or complex values is the uniform limit on $[a, b]$ of a sequence of polynomial functions.

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Let a and b be two real numbers with $a < b$. Any function f that is continuous on the interval $[a, b]$ with real or complex values is the uniform limit on $[a, b]$ of a sequence of polynomial functions.

Proof: The proof is based on the explicit construction of Bernstein polynomials:

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0, 1],$$

which have the property of exactly reproducing low-degree polynomials and uniformly approximating any continuous function.
This approach clearly demonstrates how polynomials can be used to uniformly approximate continuous functions, thereby proving Weierstrass's theorem.

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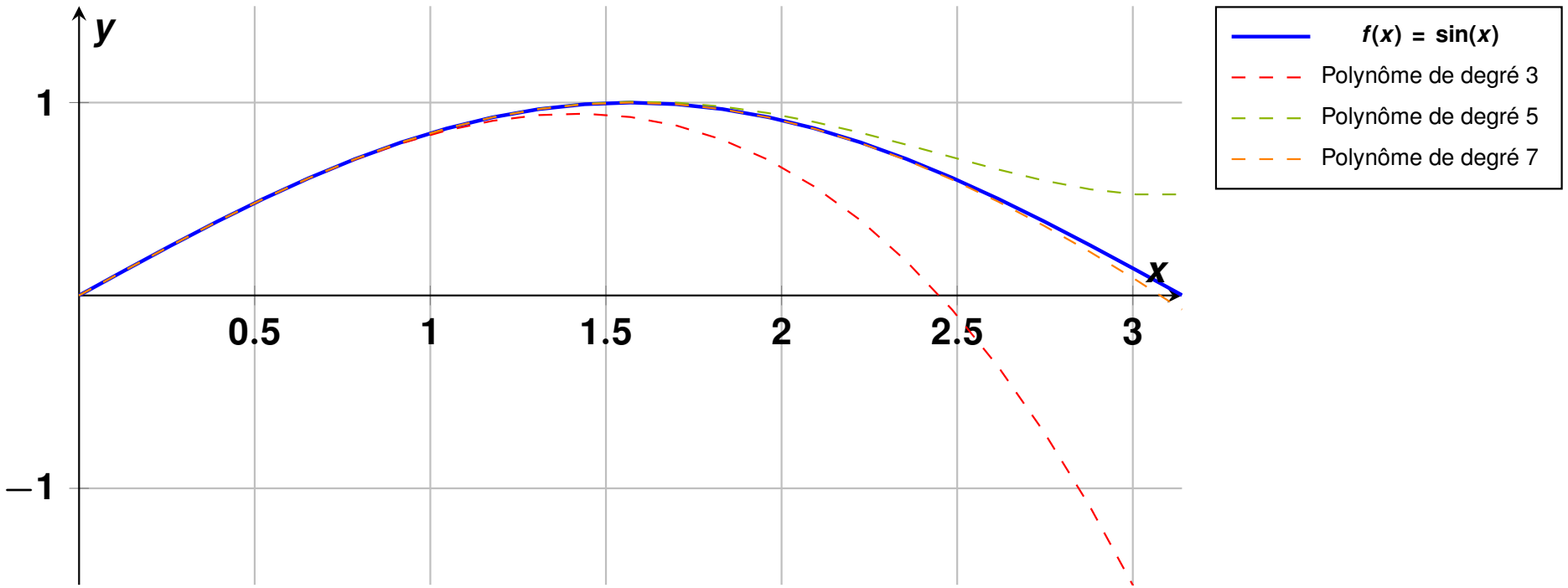

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2 Looking for Limits
Defining and Understanding Convergence
Pointwise Convergence
How to Proceed in Practice
Compatibility with Algebraic Operations
Uniform Convergence

3 Two important theorems on uniform approximation
Polynomial Version
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4 *p*-Norms on Functions

Theorem : Weierstrass theorem (Trigonometric Version)

Any continuous and 2π-periodic function from $\mathbb{R}$ to $\mathbb{C}$ is the uniform limit on $\mathbb{R}$ of a sequence of trigonometric polynomial functions.

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- **Trigonometric Polynomials** are finite linear combinations of the functions $\sin(kx)$ and $\cos(kx)$ where $k \in \mathbb{Z}$. A trigonometric polynomial of degree n is a function of the form:

$$T_n(x) = \sum_{k=-n}^{n} c_k e^{ikx},$$

where $c_k \in \mathbb{C}$ are complex coefficients and $e^{ikx} = \cos(kx) + i\sin(kx)$ is the exponential representation of trigonometric functions.

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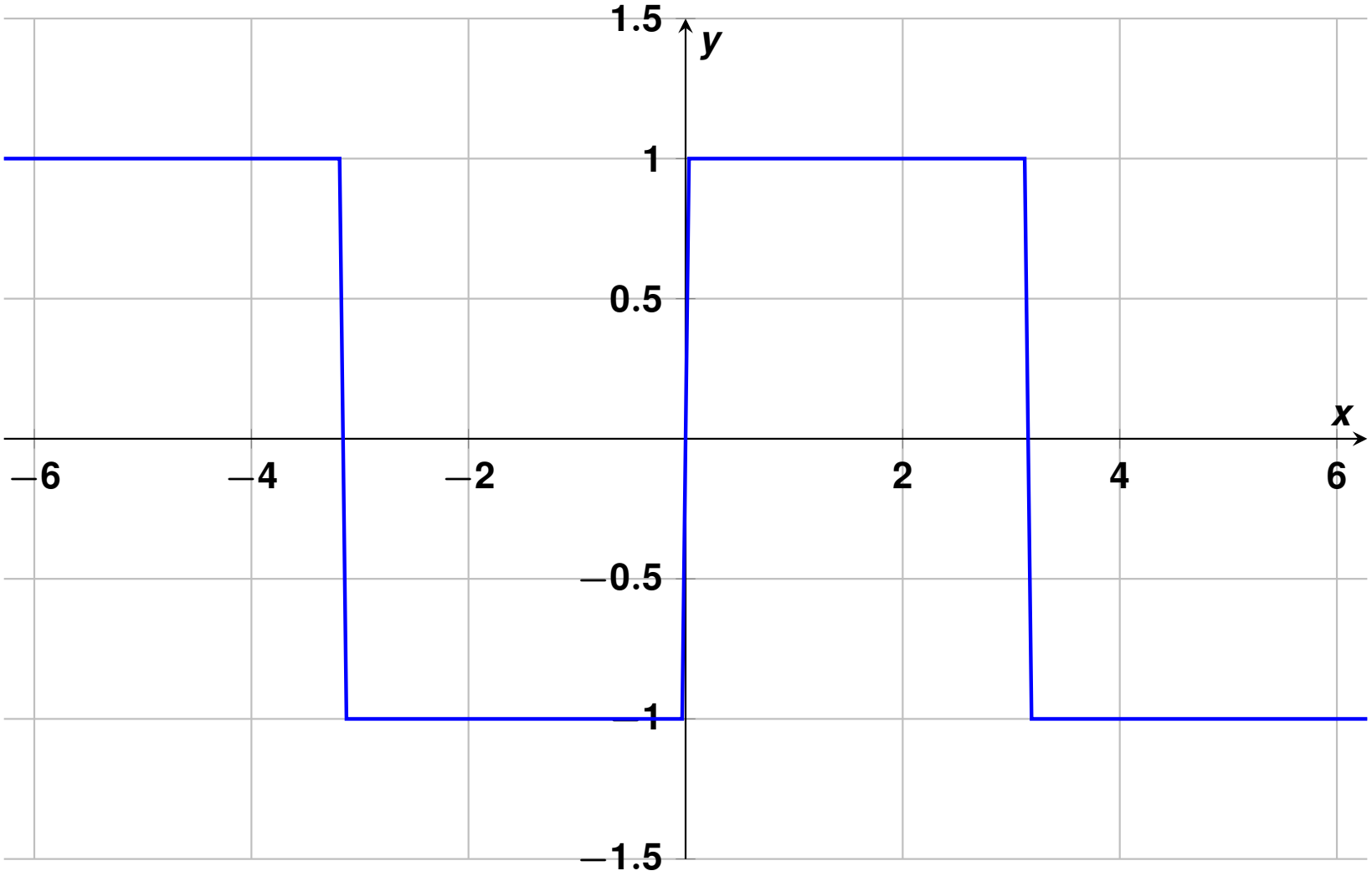
- **Using Fourier Series**
For a continuous and 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$, we can associate the Fourier series of f:

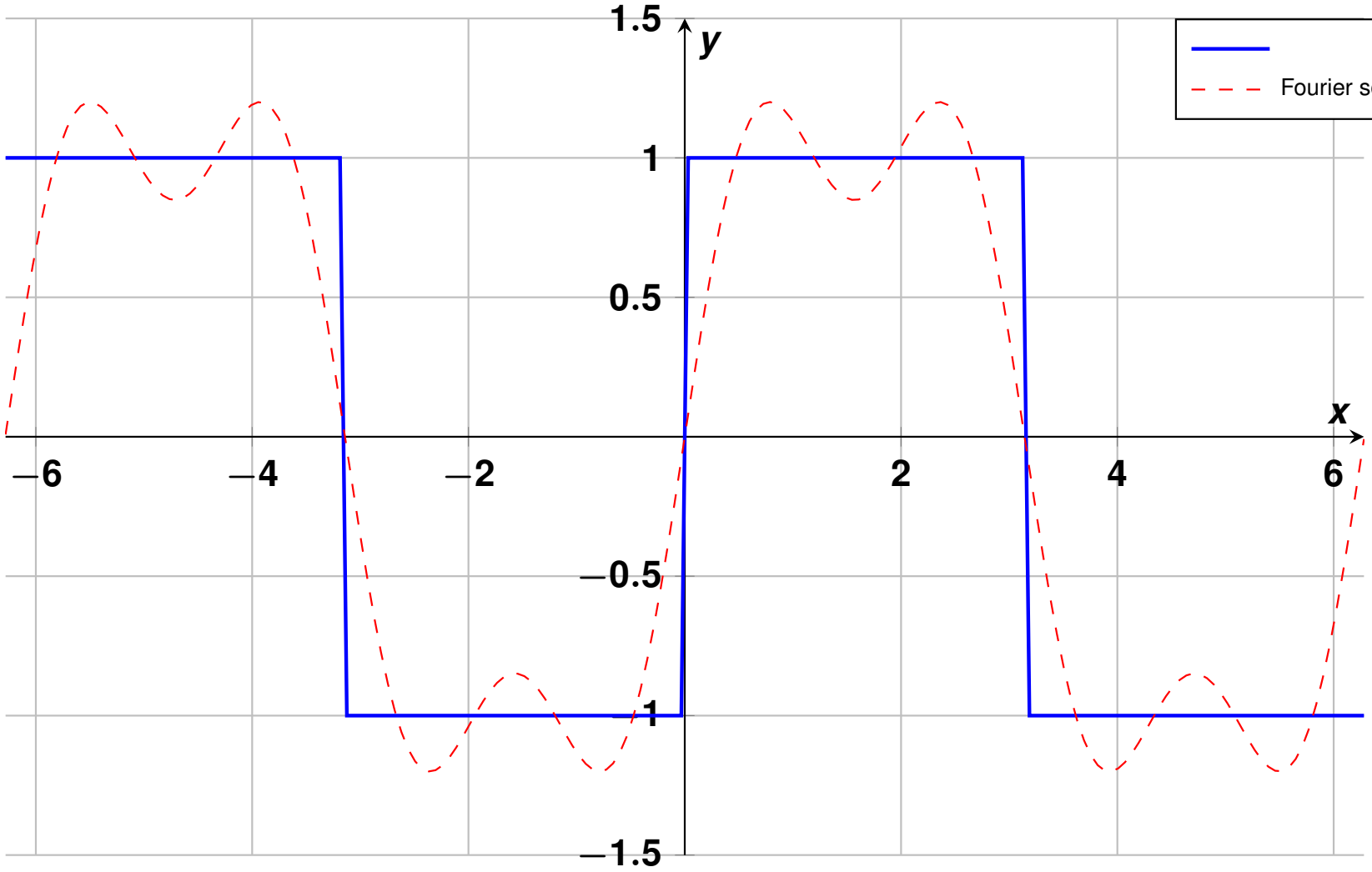
$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where the Fourier coefficients c_k are given by:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt.$$



Onde carrée


1.5
y
1
0.5
x
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−2
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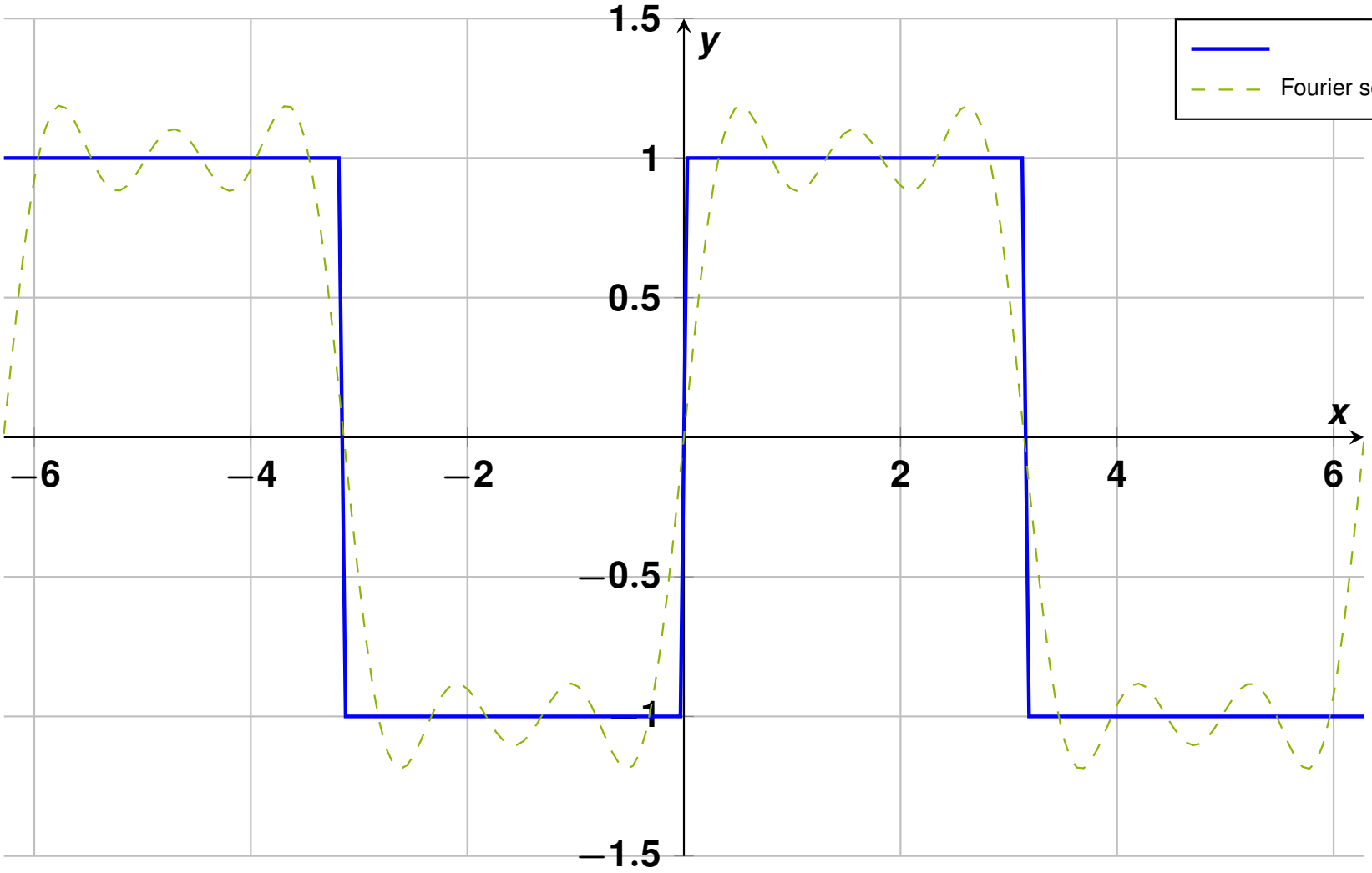
Onde carrée
Fourier somme partielle, degré 3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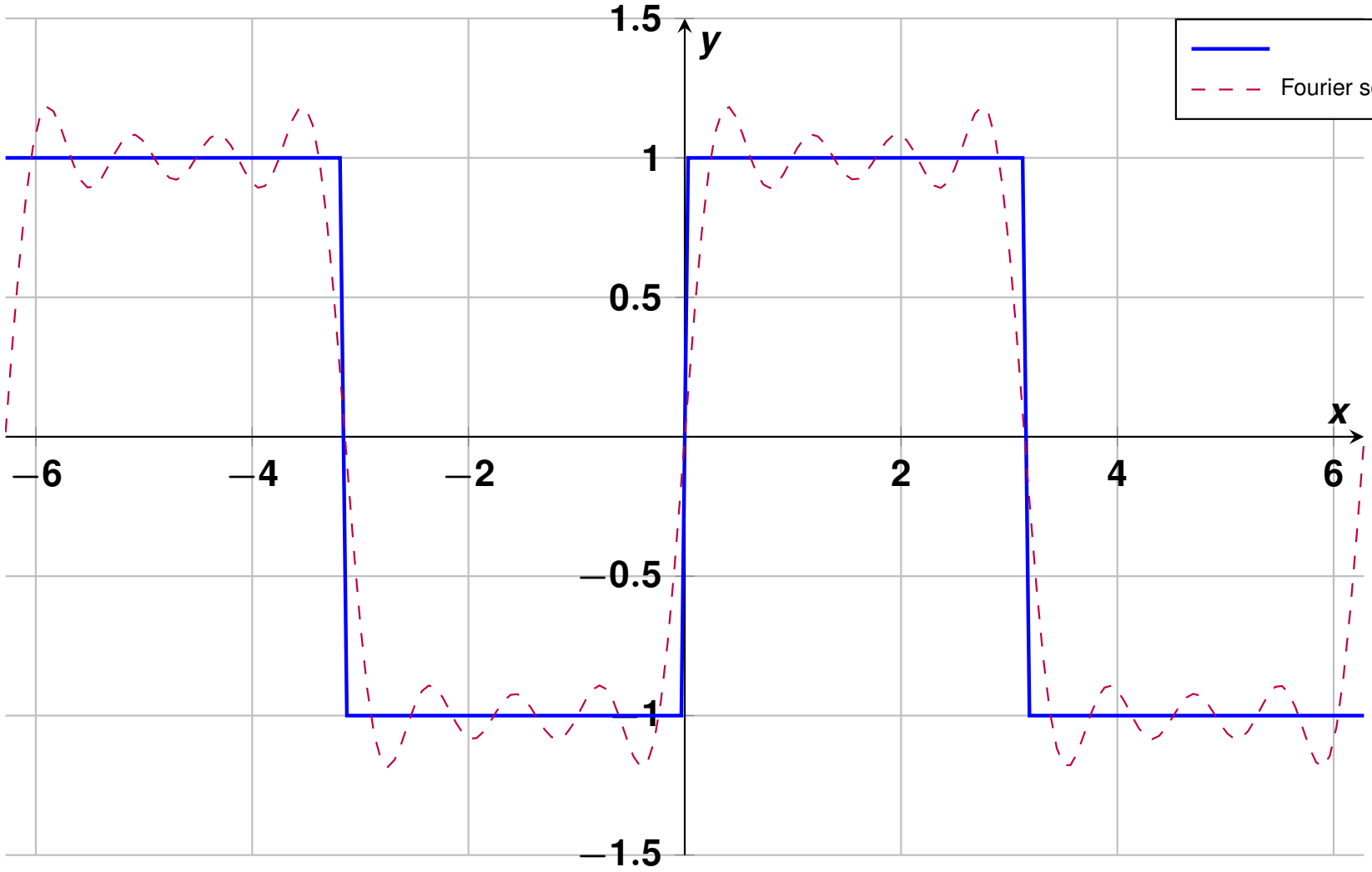
Onde carrée
Fourier somme partielle, degré 5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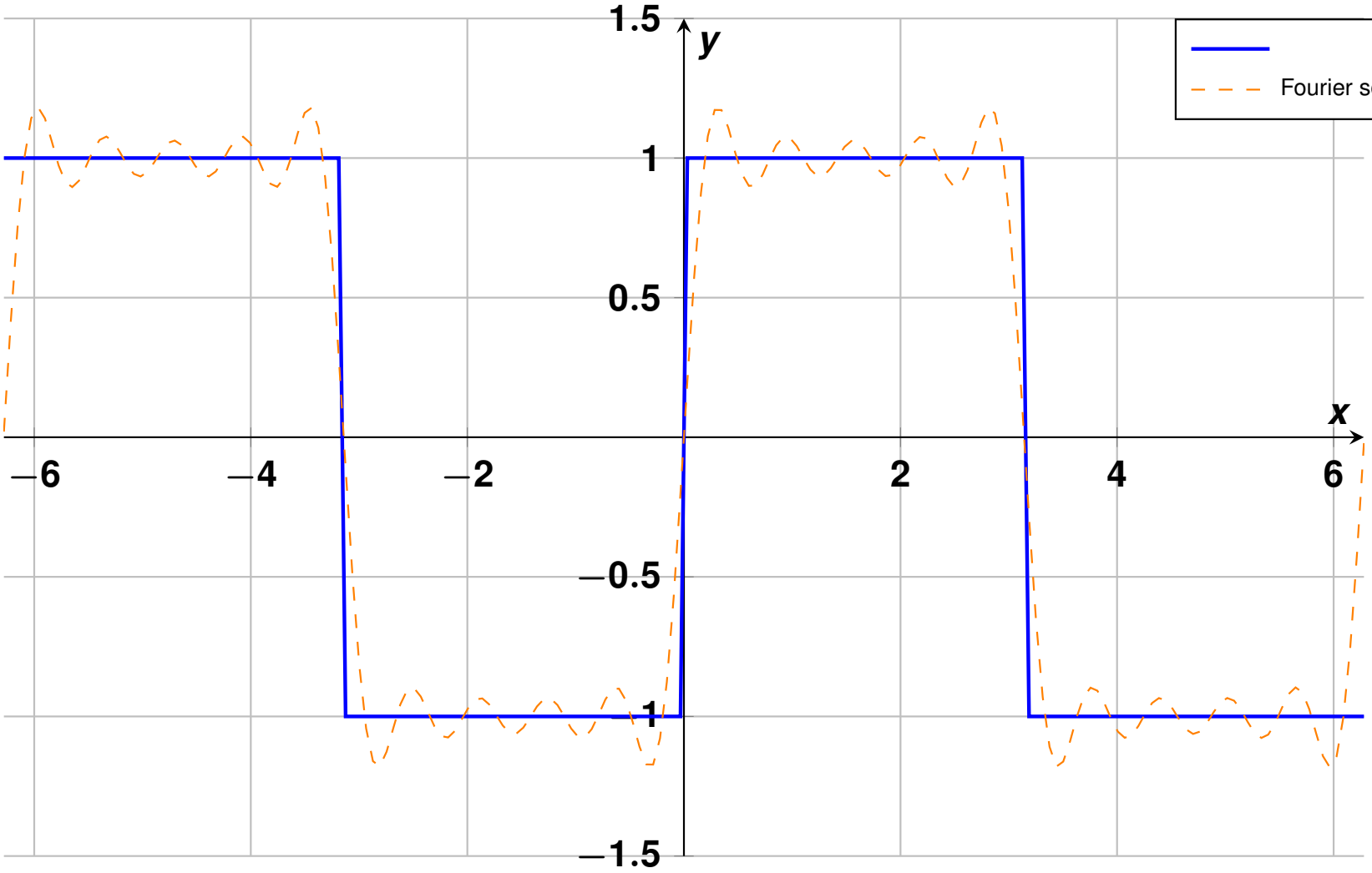
Onde carrée
Fourier somme partielle, degré 7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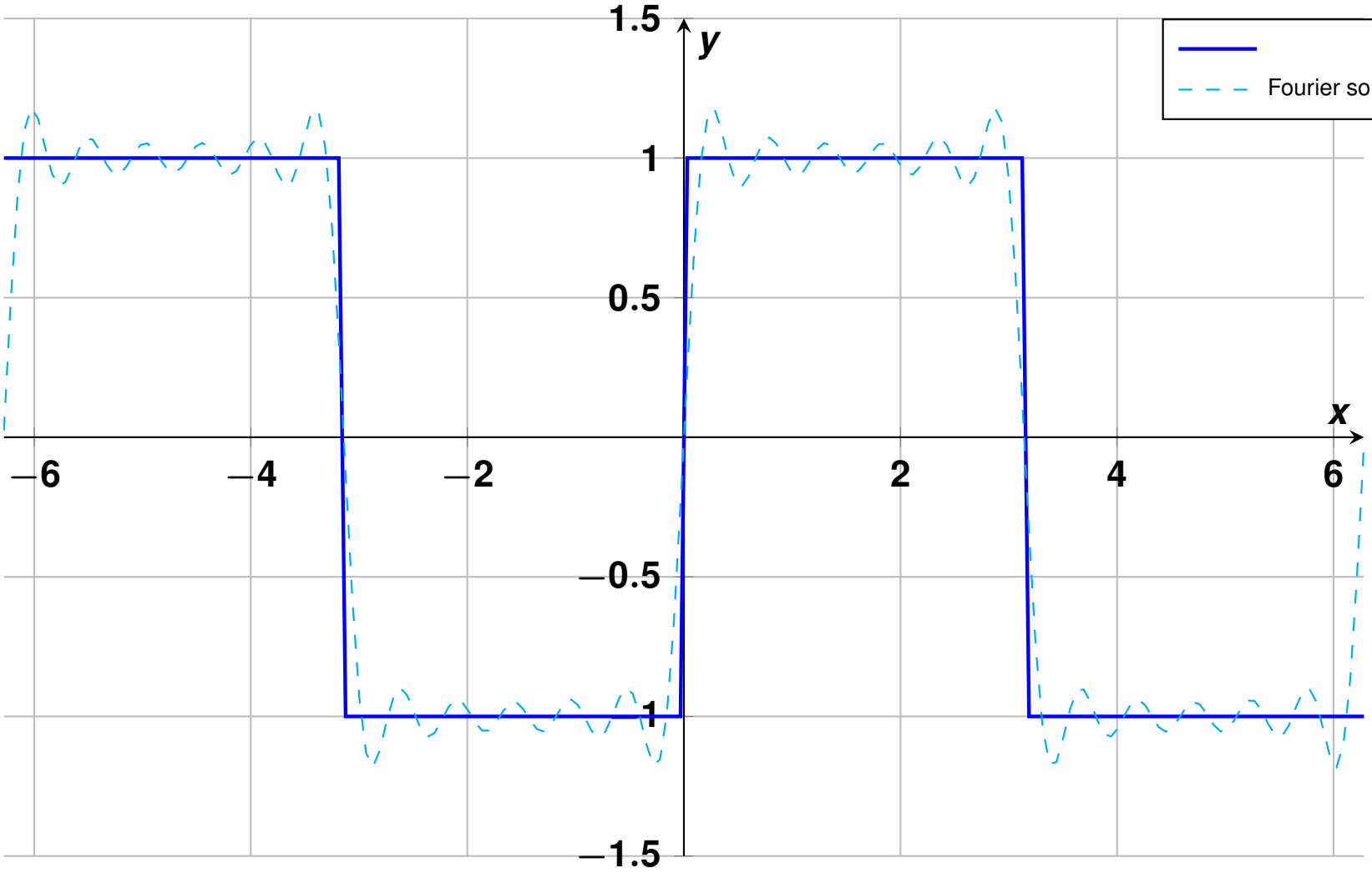
Onde carrée
Fourier somme partielle, degré 9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−1
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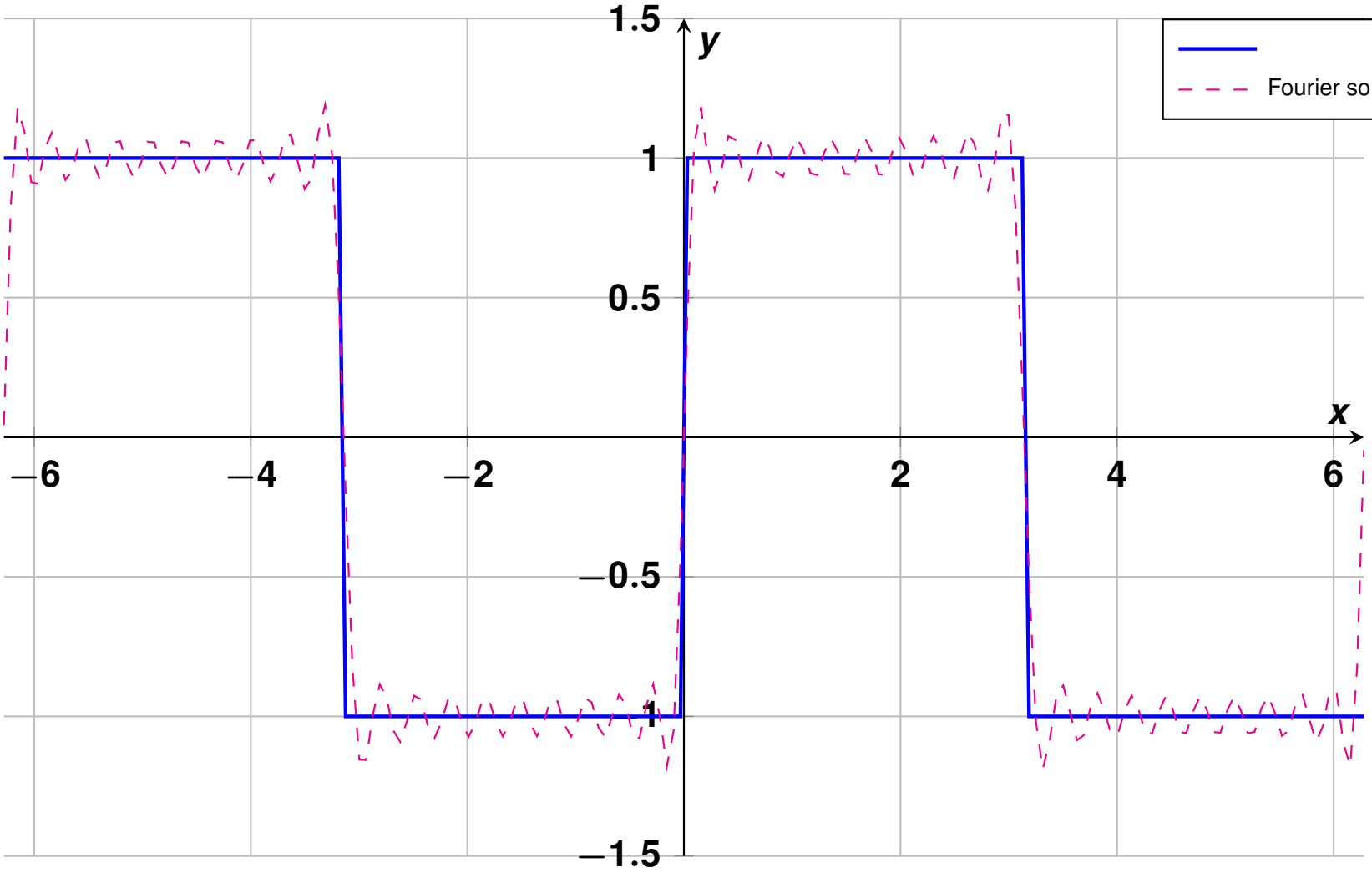
Onde carrée
Fourier somme partielle, degré 11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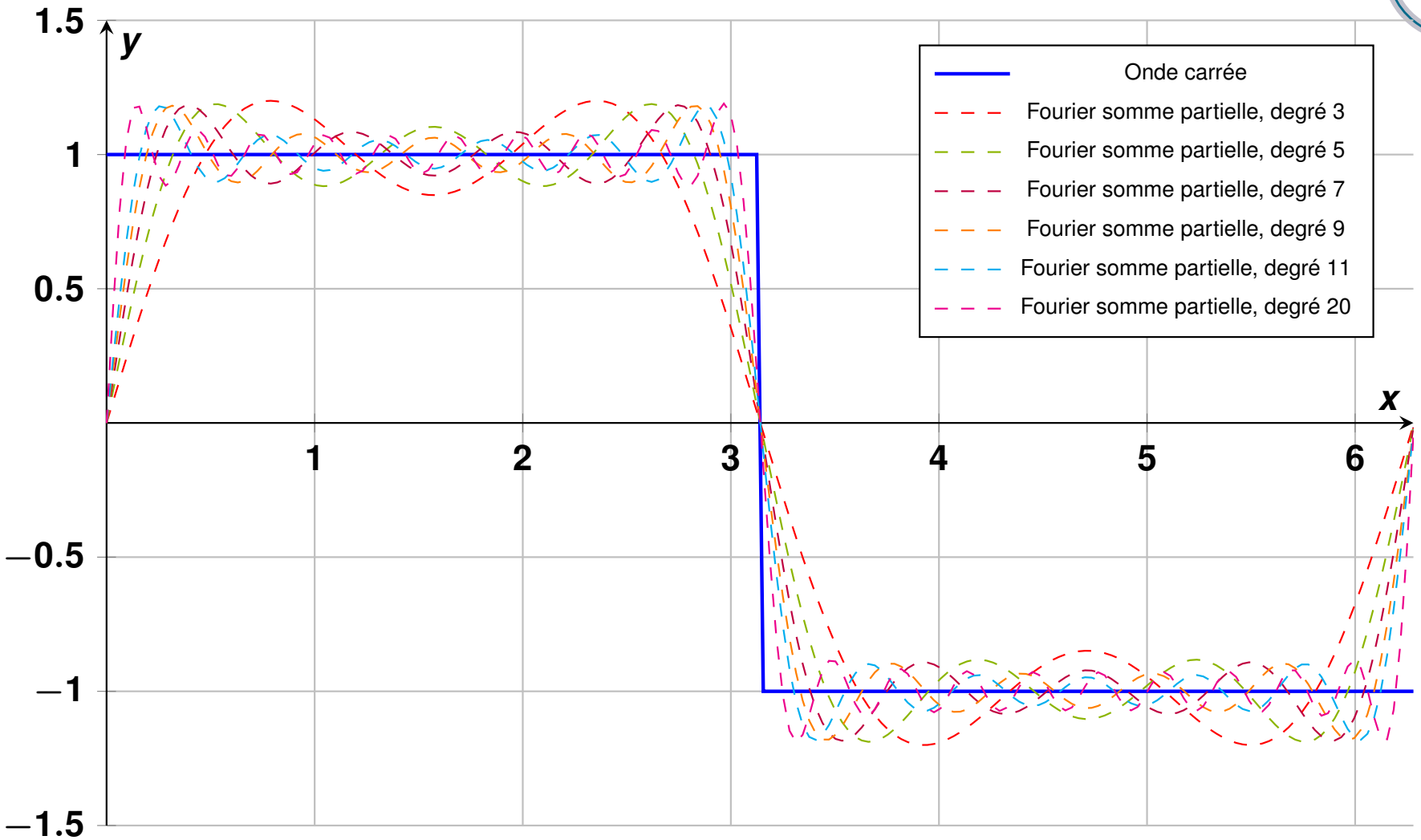
Onde carrée
Fourier somme partielle, degré 20



As n grows larger, we observe that the sequence of functions becomes increasingly closer.

✎ **Wooclap 6.**
What is a typical example of a function that can be approximated by a trigonometric series according to Weierstrass' theorem?

1. $f(x) = e^x$
2. $f(x) = \sqrt{x}$
2. $f(x) = \sin(x)$
4. $f(x) = \ln(x)$

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Trigonometric Version

4 *p*-Norms on Functions

THIS SECTION IS NOT CENTRAL IN THE COURSE ; IT IS HERE BECAUSE IT SHEDS A BROADER LIGHT ON WHAT STUDENTS WILL ENCOUNTER DURING THIS COURSE AND UPCOMING ONES THIS YEAR.

We give ourselves a sequence of continous and bounded functions $(f_n)_{n\in\mathbb{N}}$, over an interval $I \subset \mathbb{R}$. Among the following propositions which ones are true ?

- If (f_n) converges to f for the ∞ norm, then it converges for the **1**-norm
- If (f_n) converges to f for the ∞ norm, then it converges for the **2**-norm
- If (f_n) converges to f for the **1** norm, then it converges for the **2**-norm
- If (f_n) converges to f for the **2** norm, then it converges for the **1**-norm
- If (f_n) converges to f for the **1** norm, then it converges for the ∞-norm
- If (f_n) converges to f for the **2** norm, then it converges for the ∞-norm

Solution

Before You Leave

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1. Review the videos and remediation materials.

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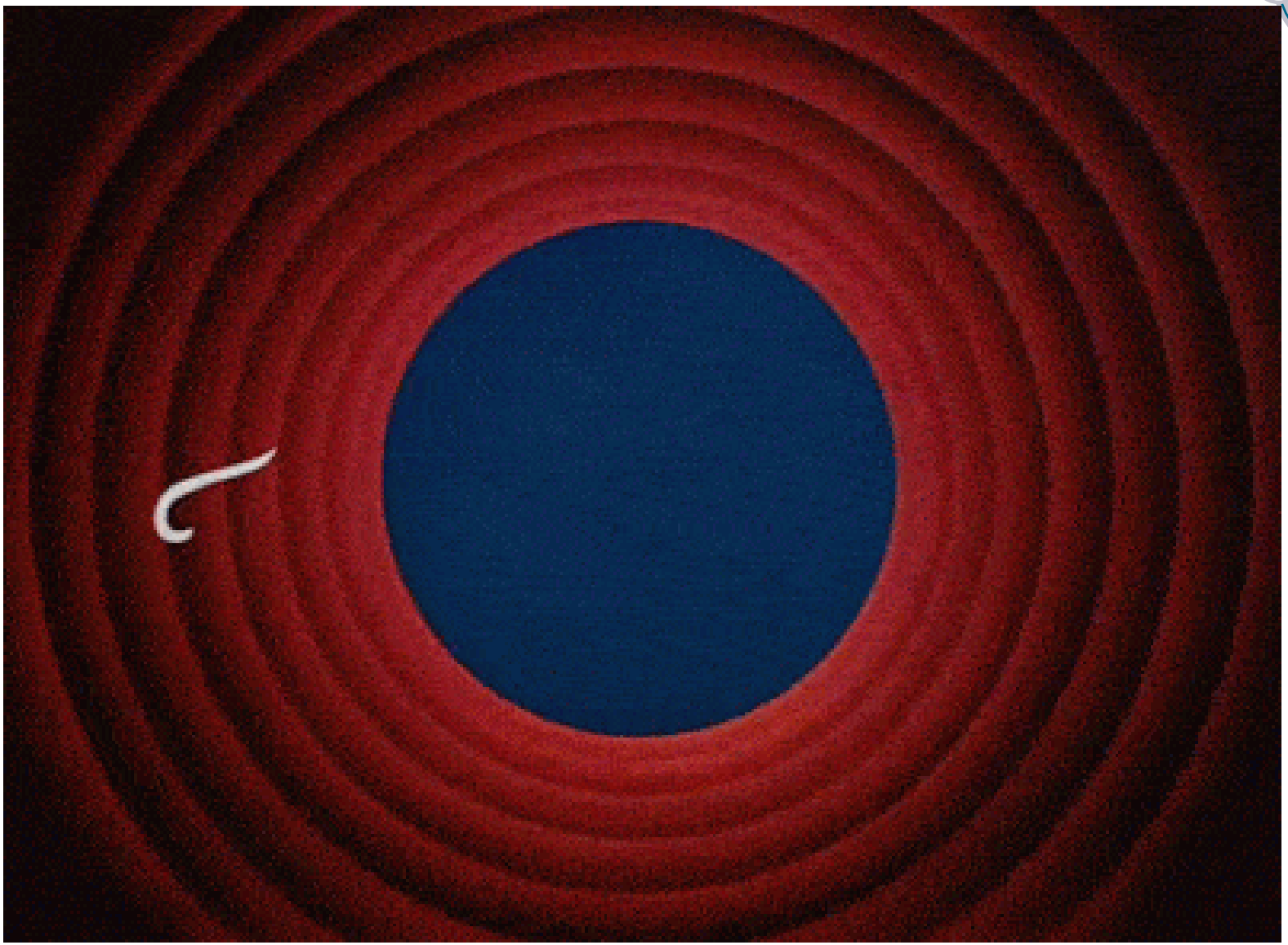
Reminder: You have a formative assessment (to complete at home) by the end of the week.



Solution 1.

- $\|f\|_p = \left(\int_I |f|^p\right)^{\frac{1}{p}}$ et $\|f\|_\infty = \sup_{x\in I} |f(x)|$

Go Back

Solution 2.

1. La norme 1 d'une fonction f sur $[0, 1]$ est définie par $\|f\|_1 = \int_0^1 |f(x)|\, dx$. Pour $f_n(x) = x^n$, calculons la limite de $\|f_n\|_1$:

$$\|f_n\|_1 = \int_0^1 x^n\, dx = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}.$$

Comme $\frac{1}{n+1} \to 0$ quand $n \to \infty$, $\|f_n\|_1 \to 0$. Donc, $f_n \to 0$ en norme 1.

2. La norme ∞ d'une fonction f sur $[0, 1]$ est définie par $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Pour $f_n(x) = x^n$, nous avons :

$$\|f_n\|_\infty = \sup_{x\in[0,1]} x^n.$$

Comme $x^n \leq 1$ pour tout $x \in [0, 1]$ et $x^n \to 0$ quand $n \to \infty$ pour tout $x \in [0, 1)$, mais $(1^n = 1)$ pour $x = 1$, on obtient $\|f_n\|_\infty = 1$. Donc, f_n ne converge pas vers 0 en norme ∞.

3. Calculons la norme ∞ de $f_n(x) = 1 + \frac{x}{n}$:

$$\|f_n - 1\|_\infty = \sup_{x\in[0,1]} \left|1 + \frac{x}{n} - 1\right| = \sup_{x\in[0,1]} \left|\frac{x}{n}\right| = \frac{1}{n}.$$

Comme $\frac{1}{n} \to 0$ quand $n \to \infty$, on a $\|f_n - 1\|_\infty \to 0$. Donc, $f_n \to 1$ en norme ∞.

Go back

4. La norme **2** d'une fonction f sur $[0, 1]$ est définie par $\|f\|_2 = \left(\int_0^1 |f(x)|^2\, dx\right)^{1/2}$. Calculons $\|f_n - 1\|_2$:

$$\|f_n - 1\|_2 = \left(\int_0^1 \left|1 + \frac{x}{n} - 1\right|^2 dx\right)^{1/2} = \left(\int_0^1 \left(\frac{x}{n}\right)^2 dx\right)^{1/2}.$$

Cela donne :

$$\|f_n - 1\|_2 = \left(\frac{1}{n^2}\int_0^1 x^2\, dx\right)^{1/2} = \left(\frac{1}{n^2}\cdot\frac{1}{3}\right)^{1/2} = \frac{1}{n\sqrt{3}}.$$

Comme $\frac{1}{n\sqrt{3}} \to 0$ quand $n \to \infty$, on a $\|f_n - 1\|_2 \to 0$. Donc, $f_n \to 1$ en norme **2**.

Go back

Solution 3.

1. Cette suite est la somme partielle de la série numérique $\sum_{k=1}^{\infty} \frac{1}{k^3}$.

Convergence de la série:
Pour déterminer si la suite S_n converge, on doit examiner la convergence de la série associée $\sum_{k=1}^{\infty} \frac{1}{k^3}$. Il s'agit d'une série p-adique de la forme $\sum \frac{1}{k^p}$, où $p = 3 > 1$.
Il est bien connu que la série p-adique $\sum \frac{1}{k^p}$ converge si et seulement si $p > 1$. Donc, dans notre cas, puisque $p = 3 > 1$, la série converge.

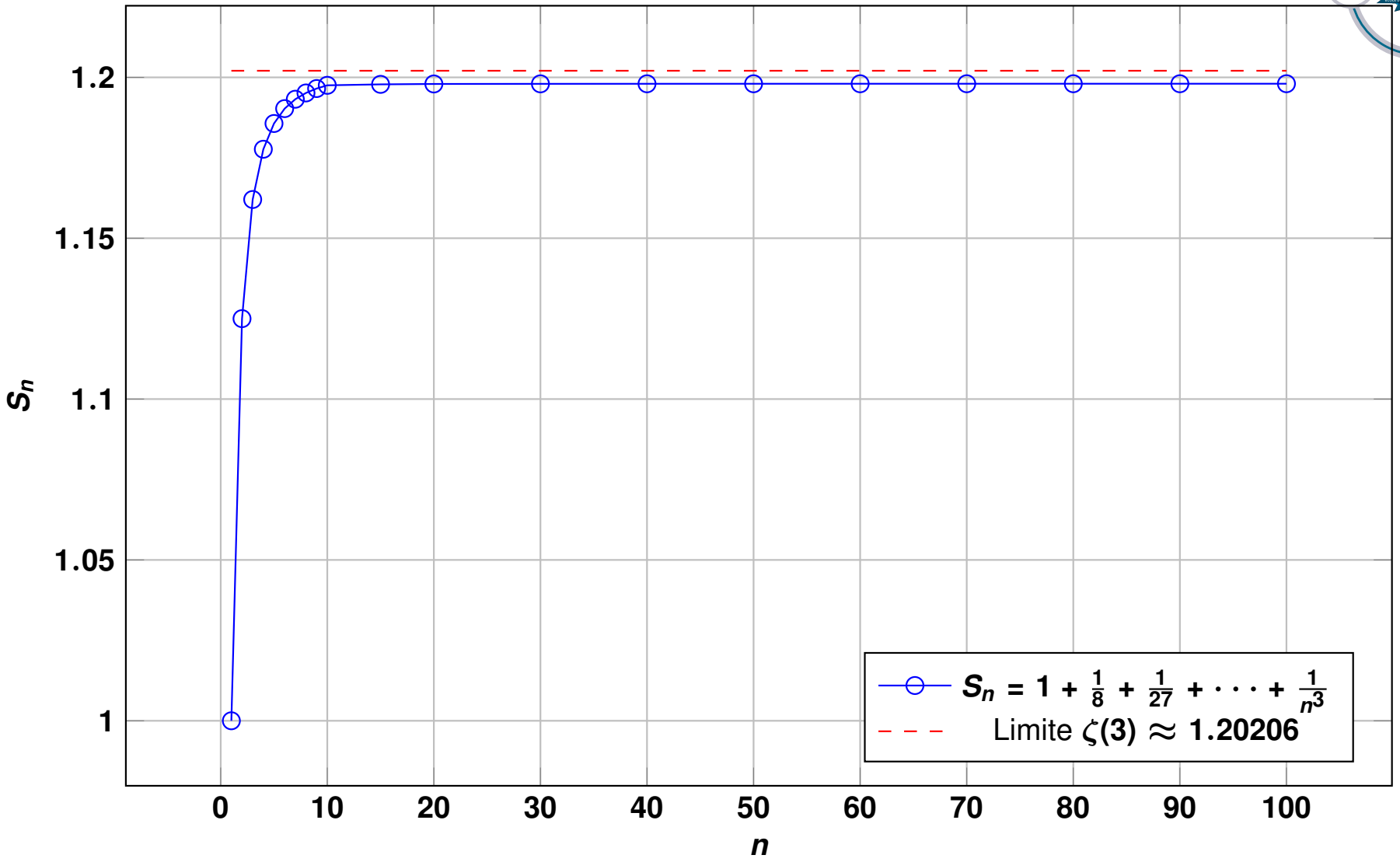
Limite de la suite
La limite de la suite S_n est simplement la somme de la série $\sum_{k=1}^{\infty} \frac{1}{k^3}$. Cette somme est une constante connue sous le nom de constante d'Apéry :

$$\lim_{n \to \infty} S_n = \zeta(3),$$

où $\zeta(3)$ est la valeur de la fonction zêta de Riemann en 3, approximativement égale à 1.20206.

Conclusion : La suite S_n converge, et sa limite est $\zeta(3) \approx 1.20206$.

Go back


1.2
1.15
1.1
1.05
1
S_n
0
10
20
30
40
50
60
70
80
90
100
n
$S_n = 1 + \frac{1}{8} + \frac{1}{27} + \cdots + \frac{1}{n^3}$
Limite $\zeta(3) \approx 1.20206$

Go back

❷ Cette suite est la somme partielle de la série alternée $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$. (suite de Gregory-Leibniz)

Convergence de la série:
La série est de la forme $\sum_{k=0}^{\infty}(-1)^k a_k$ avec $a_k = \frac{1}{2k+1}$. Pour appliquer le **critère de Leibniz** pour la convergence des séries alternées, nous devons vérifier deux conditions :

- $a_k = \frac{1}{2k+1}$ est décroissante. ($\left|\frac{a_{k+1}}{a_k}\right| = \frac{2k+1}{2k+3} \leq 1$)
- $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{2k+1} = 0$.

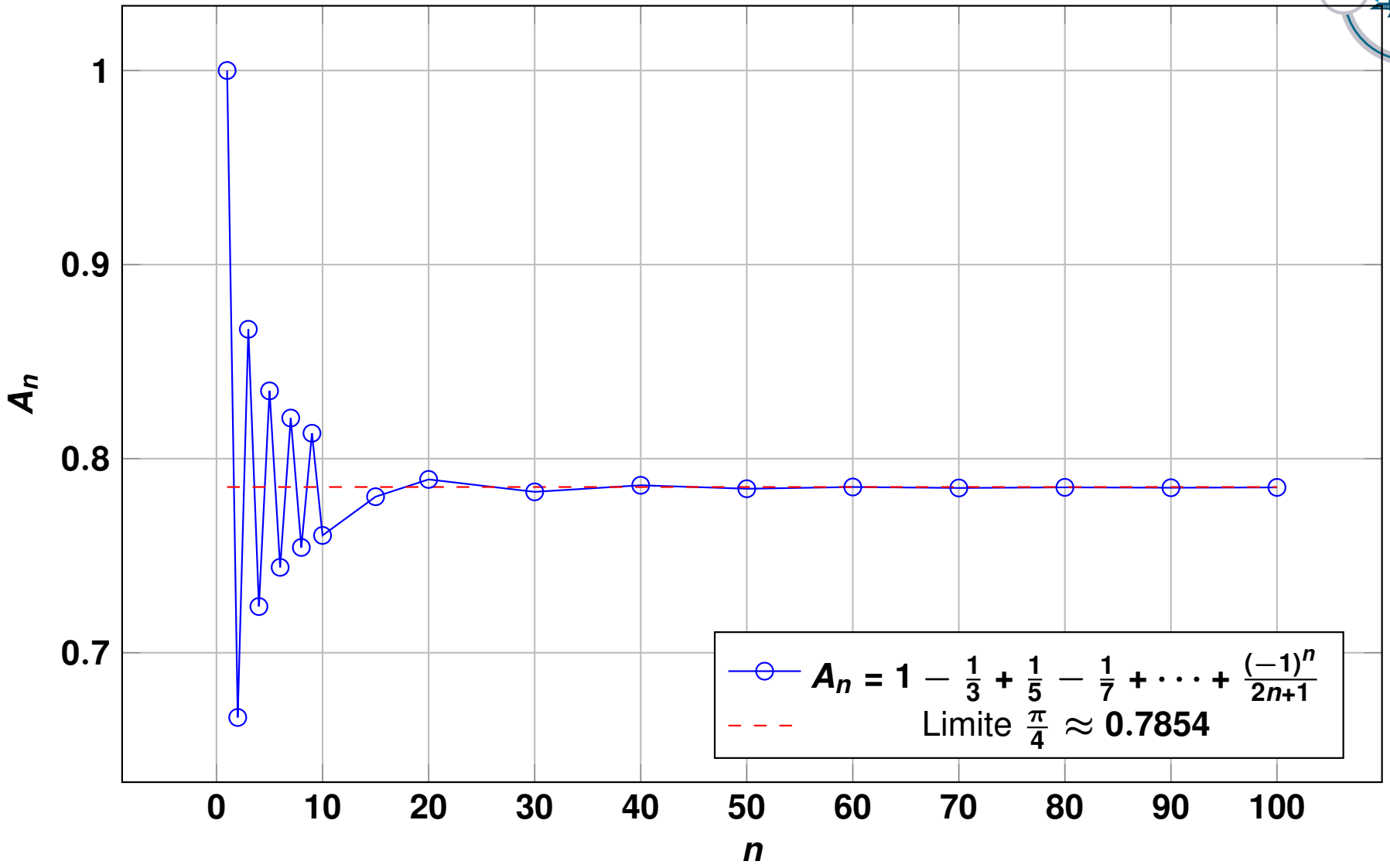
Ces deux conditions sont satisfaites. Donc, par le critère de Leibniz, la série converge.

Limite de la suite
La série $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ est bien connue pour être la série qui donne $\frac{\pi}{4}$:

$$\lim_{n\to\infty} A_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Conclusion : La suite A_n converge, et sa limite est $\frac{\pi}{4} \approx 0.7854$.

Go back


1
0.9
0.8
0.7
A_n
0
10
20
30
40
50
60
70
80
90
100
n
$A_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1}$
Limite $\frac{\pi}{4} \approx 0.7854$

Go back

Solution 4.

- (P) : "être continue" Prenons $f_n(x) = x^n$
- (P) : "être dérivable" Prenons $f_n(x) = \sqrt{x} + \frac{1}{n}$
- (P) : "être périodique"
- (P) : "être borné"

Go back

Solution 5. On se donne une suite de fonctions $(f_n)_{n\in\mathbb{N}}$ qui converge uniformement vers f. On se pose la question de connaître les propriétés (P) qui satisfont l'assertion suivante : si chaque f_n satisfait (P) alors f satisfait (P).

- (P) : "être continue"
- (P) : "être dérivable"
- (P) : "être périodique"
- (P) : "être borné"

Go back

Solution 6. On se donne une suite de fonctions $(f_n)_{n\in\mathbb{N}}$, continues et bornées sur un intervalle $I \subset \mathbb{R}$
Parmi les propositions qui suivent lesquelles sont justes?

- Si (f_n) converge vers f pour la norme ∞, alors elle converge pour la norme 1
- Si (f_n) converge vers f pour la norme ∞, alors elle converge pour la norme 2
- Si (f_n) converge vers f pour la norme 1, alors elle converge pour la norme 2
- Si (f_n) converge vers f pour la norme 2, alors elle converge pour la norme 1
- Si (f_n) converge vers f pour la norme 1, alors elle converge pour la norme ∞
- Si (f_n) converge vers f pour la norme 2, alors elle converge pour la norme ∞

Go back