Generalized Integrals

Chapter 1 : Definition and properties

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In this lecture course ...

What is a generalized integral ?

How to decide for a convergence : comparaison

Partial integration

In this lecture course ...

- We will introduce and study generalized (improper) integrals

$$\int_a^{+\infty} f(x)\,\mathrm{d}x$$

For now, keep in mind the infinity bound $+\infty$

- We will introduce and study integrals with a parameter

$$\int_a^{+\infty} f(x, t)\,\mathrm{d}x$$

- We will study the convergence of sequences of such integrals

$$\lim_{n\to+\infty}\int_a^{+\infty} f_n(x)\,\mathrm{d}x$$

Learning Outcomes

As a direct application of this course :

- determine if a given generalized integral is well defined
- determine the convergence of a sequence of integrals and find the limit (if it exists)
- identify the properties of a integral depending on a parameter in most usual cases (as Fourier and Laplace transform)
- simplify expressions involving limits of sequences of integrals and parameter integrals
- validate a reasoning implicating questions of convergences of integrals or parameter integrals.

In situations of modelization in mathematics for signal processing, probability and automatics :

- calculate moments and probabilistic quantities related to a random variable with density
- identify hypothesis and arguments used in studying the convergence in probability
- calculate Fourier and Laplace transform of a function

In this lecture course ...

What is a generalized integral ?

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What is a generalized integral ?

Type 1 : what happens at infinity

Definition

Let f be a continuous function over $[a, +\infty[$.

The generalized integral $\displaystyle\int_a^{+\infty} f(t)\,\mathrm{d}t$ ***converges*** if the limit

$\displaystyle\lim_{x\to+\infty}\int_a^x f(t)\,\mathrm{d}t$ exists and is finite.

In this case we let :

$$\int_a^{+\infty} f(t)\,\mathrm{d}t = \lim_{x\to+\infty}\int_a^x f(t)\,\mathrm{d}t$$

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Example

$$\int_0^{+\infty} e^{-t}\,\mathrm{d}t = \lim_{x\to+\infty}\int_0^x e^{-t}\,\mathrm{d}t = \lim_{x\to+\infty}(1-e^{-x}) = 1$$

What is a generalized integral ?

Type 2 : what happens on finite borders

Definition

Let f be a continuous function on $[a, b[$ where f is discontinued/not defined in b.

The generalized integral $\int_a^b f(t)\,\mathrm{d}t$ ***converges*** if the limit

$\lim_{x \to b} \int_a^x f(t)\,\mathrm{d}t$ exists and is finite.

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Example

$$\int_0^4 \frac{1}{\sqrt{4-t}}\,\mathrm{d}t = \lim_{x\to 4}\int_0^x \frac{1}{\sqrt{4-t}}\,\mathrm{d}t = \lim_{x\to 4}\left[-2\sqrt{4-t}\right]_0^x = \lim_{x\to 4}\left(4-2\sqrt{4-x}\right) = 4$$

In brief

If $-\infty < a < b <= +\infty$

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{x \to b} \int_a^x f(t)\,\mathrm{d}t$$

Memo

Generalized integral = limit (Riemann integral)

In brief

If $-\infty < a < b <= +\infty$

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Remark : Generalized integrals with a problem in the first border a are treated in the same manner.

To follow in this RMD

- Generalized integrals in boths bounderies
- Chasles
- Partial integration
- Change of variables
- Comparaison

Warmup ...

Wooclap[1-2]

Some properties : Chasles relation

Proposition

Let $f : [a,b[\to \mathbb{R}$ be coninuous on $-\infty < a < c < b \leq +\infty$.

The integrals $\int_a^b f(t)\mathrm{d}t$ and $\int_c^b f(t)\mathrm{d}t$ are of same nature.

In the case of convergence :

$$\int_a^b f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t + \int_c^b f(t)\mathrm{d}t$$

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Démonstration.

For all x such that $a < c < x < +\infty$ Chasles relation for Riemann integrals gives

$$\int_a^x f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^x f(t)\,\mathrm{d}t$$

As f is continuous on $[a, c]$ the integral in the middle is finate. Passing to the limit gives the result. $\square$

Some properties : linearity

Proposition

Let $f, g : [a, b[\to \mathbb{R}$ be continuous on $-\infty < a < b \leq +\infty$ and $\lambda, \mu \in \mathbb{R}$. If the integrals $\int_a^b f(t)\,\mathrm{d}t$ and $\int_a^b g(t)\,\mathrm{d}t$ converge then $\int_a^b (\lambda f(t) + \mu g(t))\,\mathrm{d}t$ converge too and

$$\int_a^b (\lambda f(t) + \mu g(t))\,\mathrm{d}t = \lambda \int_a^b f(t)\,\mathrm{d}t + \mu \int_a^b g(t)\,\mathrm{d}t$$

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Démonstration.

This is a consequence of the relation for classical Riemann integrals and then taking the limit. $\square$

Integrals over $]-\infty, +\infty[$

Wooclap[3]

Integrals with two improper bounderies $]a, b[$

If there exists $c \in]a, b[$ such that $\int_a^c f(t)\,\mathrm{d}t$ and $\int_c^b f(t)\,\mathrm{d}t$ ***converge*** then we say that the integral $\int_a^b f(t)\,\mathrm{d}t$ ***converge***. In the case of convergence

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^c f(t)\,\mathrm{d}t + \int_c^b f(t)\,\mathrm{d}t$$

The defintion does not depend on c (consequence of Chasles relation).

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Example

$$\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x = \int_{-1}^0 \frac{1}{x^2}\,\mathrm{d}x + \int_0^1 \frac{1}{x^2}\,\mathrm{d}x = \ldots$$

Reference integrals : Riemann

Wooclap[4]

Reference integrals : Riemann

Theorem

Let $\alpha \in \mathbb{R}$

- $$\int_1^{+\infty} \frac{1}{t^\alpha}\,\mathrm{d}t \text{ converges iff } \alpha > 1$$
- $$\int_0^{1} \frac{1}{t^\alpha}\,\mathrm{d}t \text{ converges iff } \alpha < 1$$

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Memo : Riemann series

The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges iff $\alpha > 1$

Reference integrals : Riemann

Wooclap[5]

Reference integrals : exponential (to be done on TD)

Theorem

Let $\alpha \in \mathbb{R}$

- $$\int_0^{+\infty} e^{\alpha t} \,\mathrm{d}t \text{ converges iff } \alpha < 0$$
- $$\int_{-\infty}^{0} e^{\alpha t} \,\mathrm{d}t \text{ converges iff } \alpha > 0$$

In this lecture course ...

What is a generalized integral ?

How to decide for a convergence : comparaison

Partial integration

How to decide for a convergence ?

Question

What if Riemann itnegrals are hard to evaluate (for example a primirive is not known ...) ?

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What if Riemann itnegrals are hard to evaluate (for example a primirive is not known ...) ?

Solution

We compare f with a function g (to be determined depending on f) for which the convergence of its integral is easier to decide.

Comparaison

Proposition

Let f be continuous and **positive** [a] over $[a, b[$ then

$$\int_a^x f(t)\,\mathrm{d}t \text{ bounded } \Leftrightarrow \int_a^b f(t)\,\mathrm{d}t \text{ converges}$$

a. or positive f in a ***neighbourhood*** of b, i.e. on some $]A, b[$

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Wooclap[6]

Comparaison

Proposition

If f and g are positives on $[a,b[$ and $0 \leq f \leq g$ on $[a,b[$

$$\int_a^b g(t)\,\mathrm{d}t \text{ converges } \Rightarrow \int_a^b f(t)\,\mathrm{d}t \text{ converges}^{a}$$

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Démonstration.

For each $x \in [a, b[$ we have $0 \leq f(x) \leq g(x)$. Thus

$$0 \leq \int_a^x f(t)\,\mathrm{d}t \leq \int_a^x g(t)\,\mathrm{d}t$$

If $\int_a^b g(t)\,\mathrm{d}t$ converges then $\int_a^x g(t)\,\mathrm{d}t$ is bounded and we use the previous proposition. $\square$

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Proposition

- If f and g are positives on $[a, b[$ and $f \underset{b}{=} O(g)$ or $f \underset{b}{=} o(g)$ then

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- If $f \underset{b}{\sim} g$ then

$$\int_a^b g(t)\,\mathrm{d}t \text{ converges } \Leftrightarrow \int_a^b f(t)\,\mathrm{d}t \text{ converges}$$

---

a. What does the opposite result say ?

Recall

If $f, g : [a, b[\to \mathbb{R}$ are two functions and g is non zero in a neighbourhood of b

$$f \underset{b}{=} O(g) \Leftrightarrow \frac{f}{g} \text{ is bounded in a neighbourhood of } b$$

$$f \underset{b}{=} o(g) \Leftrightarrow \lim_{x \to b} \frac{f(x)}{g(x)} = 0$$

$$f \underset{b}{\sim} g \Leftrightarrow \lim_{x \to b} \frac{f(x)}{g(x)} = 1$$

Question

What is the nature of $\int_0^1 \ln(t)\,\mathrm{d}t$?

Examples

Question

What is the nature of $\int_0^1 \ln(t)\,\mathrm{d}t$?

By comparative growth for all $\alpha > 0$ we have

$$t^{\alpha}\ln(t) \underset{t\to 0}{\longrightarrow} 0$$

In particular, for $\alpha = \frac{1}{2}$ we have

$$t^{\frac{1}{2}}\ln(t) \underset{t\to 0}{\longrightarrow} 0$$

Thus $\ln(t) = O(\frac{1}{t^{\frac{1}{2}}})$ (we can also say that in a neighbourhood of 0 $\ln(t) < \frac{1}{t^{\frac{1}{2}}}$

But the integral $\int_0^1 \frac{1}{t^{\frac{1}{2}}}\,\mathrm{d}t$ converges.

Comparaison

Proposition ***Absolue convergence***

$$\int_a^{+\infty} |f(t)|\,\mathrm{d}t \text{ converges } \Rightarrow \int_a^{+\infty} f(t)\,\mathrm{d}t \text{ converges}$$

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Memo

Recall similar properties for series ....

Comparaison

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Wooclap[7-9]

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What is a generalized integral ?

How to decide for a convergence : comparaison

Partial integration

Proposition

If $\int_a^b u(t)v'(t)\,\mathrm{d}t$ is a generalized integral (u and v are C^1) and if

$$[u(t)v(t)]_a^b = \lim_{x\to b} u(x)v(x) - \lim_{x\to a} u(x)v(x)$$

is finite then the integrals $\int_a^b u(t)v'(t)\,\mathrm{d}t$ and $\int_a^b u'(t)v(t)\,\mathrm{d}t$ are of same nature.

In the case of convergence :

$$\int_a^b u(t)v'(t)\,\mathrm{d}t = [u(t)v(t)]_a^b - \int_a^b u'(t)v(t)\,\mathrm{d}t$$

Remark : We always start by checking if $[u(t)v(t)]_a^b$ has a finite limite.

PI : example

Wooclap[10]

Change of variables

Proposition

If $\int_a^b f(t)\,\mathrm{d}t$ is a generalized integral and $\varphi : I =]\alpha, \beta[\to]a, b[$ bijective of class C^1 such that

$$\lim_{t\to\alpha} \varphi(t) = a \text{ et } \lim_{t\to\beta} \varphi(t) = b$$

Then

$$\int_a^b f(t)\,\mathrm{d}t \text{ et } \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,\mathrm{d}t$$

are of same nature nature. In the case of convergence both integrals have same value.

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Wooclap[11]

What we learned

- Generalized integral = limit (Riemann Integral)
- Same properties and techniques : linarity, Chasles, Pi, change of variables
- Pay attention when dealing with such integrals, check convergence before doing calculation ...

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To follow :

- Sequence of generalized integrals
- Parameter integrals