Sequences of functions

K.ATTAR

Résumé

This worksheet approaches technical aspects related to notions of convergence for the sequences of functions. Its aim is to let you work on the pointwise and uniform convergence modes and to exacerbate their differences. The questions of convergence in 1-norm will be studied during the `[INTG]` lesson while those of convergence in 2-norm will be partially evoked during this lesson and the following `[MASI]` one.

Table des matières

1 Pointwise convergence

The pointwise convergence is the most naïve approach towards the convergence of sequences of functions. It reminds you of childhood memories [1] about the study of convergence of numerical sequences. However, this type of convergence does not derive from a distance defined over spaces of functions. It furthermore presents the big downside not to keep a good number of interesting properties we would want to preserve while going for the limit, including crucial notions like smoothness. Its main interest is to remain a relatively easy method to determine a limit towards which we will be able to test for more interesting convergence types, among which the uniform convergence.

> **Question 1-1** Study the pointwise convergence of the following sequences of functions ; when they converge, determine their limit.
>
> 1. $f_n : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x^n \end{array} \right.$
>
>   What remark can we make concerning the smoothness of the f_n and their limit function ?

1. Now you can guess which song I was listening to while writing this sheet :-)

2. $g_n : \left\{ \begin{array}{ccc} [0,2] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{1+x^n} \end{array} \right.$
3. $h_n : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x\mathrm{e}^{\frac{x}{n}} \end{array} \right.$
4. $j_n : \left\{ \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & nx^2\mathrm{e}^{-nx} \end{array} \right.$
5. $j_n : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x}{n}\cos\left(\dfrac{x}{n}\right) \end{array} \right.$

Question 1-2 We define the sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ over $\mathbb{R}$ as follows :

$$\forall n \in \mathbb{N}^*, \quad \forall x \in \mathbb{R}, \quad f_n(x) = n\sin\left(\frac{x}{n}\right)$$

1. Does the sequence $(f_n)_{n\in\mathbb{N}^*}$ converge pointwise on $\mathbb{R}$, and if yes, to what function ?
2. Is the convergence of the sequence $(f_n)_{n\in\mathbb{N}^*}$ uniform on $\mathbb{R}$?
3. Is the convergence of the sequence $(f_n)_{n\in\mathbb{N}^*}$ uniform on $[-1,1]$?

2 Estimating infinite norms

We study the uniform convergence of sequences of functions by estimating an infinite norm of functions. This section will be focused on this aspect. Some additional elements are given in annex A.

Question 2-3 Calculate – when possible – the infinite norm of the following functions :

1. $f_n : \left\{ \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x}{n+x} \end{array} \right.$
2. $g_n : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left\{ \begin{array}{l} nx^n\ln(x) \text{ if } x \neq 0; \\ 0 \text{ if } x = 0 \end{array} \right. \end{array} \right.$
3. $h_n : \left\{ \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & nx^k\mathrm{e}^{-nx} \end{array} \right.$ depending on k.
4. $j_n : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \mathrm{e}^{\frac{x}{n}} - 1 \end{array} \right.$

3 Practising convergence

In practice, one does not know *a priori* which type of convergence a sequence of functions will folllow. Only tinkering and intuition fed by experience can help determine it. One can however steer the convergence tests in such a way as to gradually narrow the results, and intermediary steps often give pieces of information that can be used in further calculations. The main process consists in looking for a potential limit by pointwise convergence, before checking whether the convergence towards this limit is or is not uniform.

Question 3-4 Study the convergence of the following sequences of functions :

1. $i_n : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \sqrt{x^2 + \frac{1}{n^2}} \end{array} \right.$

2. $f_n : \left\{ \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & n^\alpha x^2 \mathrm{e}^{-nx} \end{array} \right.$ depending on the parameter α

3. $g_n : \left\{ \begin{array}{ccc} \mathbb{R}_+ & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{n}{n+x} \end{array} \right.$

4. $h_n : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{1}{n}\cos(nx) \end{array} \right.$

5. $j_n : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{x\sqrt{n}}{1+nx^2} \end{array} \right.$

Question 3-5 Let k be a natural number ($k \in \mathbb{N}$) and $\left(f_n\right)_{n\in\mathbb{N}^*}$ defined by $f_n(x) = \frac{x^k}{x^2+n}$. For what values of k does this sequence converge uniformly over $\mathbb{R}$?

Question 3-6

1. Study the pointwise then uniform convergence over $[0,1]$ of the sequence of functions (g_n) defined by :

$$g_n : \left\{ \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \left\{ \begin{array}{l} nx^n \ln(x) \text{ if } x \neq 0; \\ 0 \text{ if } x = 0 \end{array} \right. \end{array} \right.$$

2. Prove that, for every $a \in]0,1[$, the sequence (g_n) converges uniformly over $[0,a]$.

Question 3-7 Study the pointwise then uniform convergence over $\mathbb{R}$ of the sequence of functions defined by $f_n(x) = \frac{x}{\sqrt{x^2+\frac{1}{n^2}}}$.

Question 3-8 Let us consider the sequence of functions $f_n : x \longmapsto n^2 x \mathrm{e}^{-nx}$ over $[0,1]$.

1. Study the pointwise convergence of the sequence (f_n).
2. Using an integration by parts, calculate $\displaystyle\int_0^1 f_n(x)\mathrm{d}x$.
3. From the previous result, draw a conclusion concerning the uniform convergence of (f_n).

A Remark : how to evaluate an infinite norm

It is ***important*** to grab the fact that an upper (resp. lower) bound is often enough to draw a conclusion conccerning the convergence (resp. non-convergence) of a sequence of functions. This matters even more because the explicit calculation of an infinite norm is not always easy. Given a function $f : I \to \mathbb{R}$, as soon as we get an inequality of this kind :

$$\forall x \in I, \quad |f(x)| \leq M$$

we deduce systematically the inequality $\|f\|_\infty \leq M$. It translates that M is an upper bound of $|f|$. Getting such inequalities is influenced by our technical efficiency to determine upper bounds of usual analytic expressions and by our knowledge concerning the behaviour of usual functions.

Here are some reminders of strateges that can be implemented in order to calculate a supremum given a upper bound M (likely candidate to be the supremum) :

- find an element $x \in I$ such that $|f(x)| = M$
- find a sequence (x_n) in $I^{\mathbb{N}}$ such that $\left(|f(x_n)|\right)$ converges towards M

We can else dwell on the following approaches :

- study the function f, if it is differentiable establish its table of variations ; it will help to identify upper bounds and possibly maxima (or more generally extrema)
- determine a sequence (x_n) of $I^{\mathbb{N}}$ such that $\big(|f(x_n)|\big)$ tends towards $+\infty$
- seperate the study into a finite union of segments on which the study of $|f|$ is easier, then rebuild an upper bound of $\|f\|_\infty$ by gathering the different studies.

All these approaches are potentially usable when dealing with approximations or exact calculations of infinite norms of functions, particularly when studying the uniform convergence of sequences of functions. You should have them in mind when you try and solve your exercises, and also during the lessons, since the teachers might use them without reexplaining them.