Series of functions

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EPITA

October 31, 2021

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We now look at the notion of convergence of a *series* of functions $\sum f_n$, where the f_n are defined over an interval I of $\mathbb{R}$.

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So, we have to devise new tools that are specific to series in order to prove in a swifter way that our requirements to use the wanted calculation properties are met.

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Not knowing how to easily calculate the limit of a numerical series, and of course of a series of functions, makes it really hard to check if its convergence is uniform by evaluating the difference between the partial sum and the limit in infinite norm. We have to get around this problem.

Uniform convergence ($\sum$)

Likewise, the notion of uniform convergence is similar to that of sequences of functions: we say a series of functions $\sum f_n$ to be uniformly convergent over I towards its limit $\mathcal{S}$ iff:

$$\sup_{x \in I} \left| \left(\sum_{i=0}^{n} f_n(x) \right) - \mathcal{S}(x) \right| \xrightarrow[n \to +\infty]{} 0, \quad \text{i.e. :} \quad \left\| \left(\sum_{i=0}^{n} f_n \right) - \mathcal{S} \right\|_{\infty} \xrightarrow[n \to +\infty]{} 0.$$

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Using for the remainder the notation $R_n(x)=\sum_{i=n+1}^{\infty} f_n(x)=\mathcal{S}(x)-\sum_{i=0}^{n} f_n(x)$ we get:

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Thus, the series of functions converges uniformly if and only if its remainder R_n converges uniformly towards the zero function over I.
We can then study the uniform convergence of series without having identified their limit, only using their associated sequence.
However, finding upper bounds for the remainders is often quite difficult ...

Uniform convergence ($\sum$) - alternating series

Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions from I to $\mathbb{R}$ such that for every x in I, the numerical sequence $\left(g_n(x)\right)_{n\in\mathbb{N}}$ is monotonic and tends towards 0.

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Moreover, for every x we can find an upper bound for the remainder $R_n(x)$: we get indeed

$$\forall x \in I, \forall n \in \mathbb{N}, \left|R_n(x)\right| \leqslant \left|g_{n+1}(x)\right|.$$

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$$\forall x \in I, \forall n \in \mathbb{N}, |R_n(x)| \leqslant |g_{n+1}(x)|.$$

Such an upper bound can help determine whether the convergence is uniform: if $\|g_n\|_\infty \underset{n\to+\infty}{\longrightarrow} 0$ over I, then the convergence is uniform.

Uniform convergence ($\sum$) - necessary condition

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For example, if we take f_n being the constant function equal to $\frac{1}{n}$ over $[0,1]$, the sequence (f_n) converges uniformly towards the zero function $\left(\|f_n\|_\infty = \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0\right)$ yet for every x, $\sum f_n(x) = \sum \frac{1}{n}$ is a divergent Riemann series.

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This result thus gives a necessary but not sufficient condition. It can be used by contraposition to prove that a series does not converge uniformly.

Uniform convergence ($\sum$) - properties

Continuity

As for sequences, the interest of uniform convergence is to give sufficient hypotheses to enable the use of results on the limits of series. We will reformulate here the three results we got previously to adapt them to series.

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Theorem [Uniform convergence and continuity]
Let (f_n) be a sequence of continuous functions over I.
We suppose that the series $\sum f_n$ converges uniformly over I.
Then its limit $\mathcal{S} = \sum_{n=0}^{+\infty} f_n$ is continuous over I.

Uniform convergence ($\sum$) - properties

Swapping sums and integrals

Theorem [Swapping sums and integrals]
In this case, I is a segment $[a, b]$.
We suppose that (f_n) is a sequence of continuous functions over $[a, b]$.
If the series $\sum f_n$ converges uniformly over $[a, b]$ then the series $\sum \int_a^b f_n(t)\mathrm{d}t$ converges and:

$$\sum_{n=0}^{+\infty} \int_a^b f_n(t)\mathrm{d}t = \int_a^b \left(\sum_{n=0}^{+\infty} f_n(t) \right) \mathrm{d}t$$

Uniform convergence ($\sum$) - properties

Swapping sums and derivatives

Theorem [Swapping sums and derivatives]

We suppose that:

- the f_n functions are of class C^1 over I ;
- $\sum f_n$ converges (pointwise) over I ;
- $\sum f_n'$ converges uniformly over I.

Then $\sum_{n=0}^{+\infty} f_n$ is differentiable over I and:

$$\left(\sum_{n=0}^{+\infty} f_n(x)\right)' = \sum_{n=0}^{+\infty} f_n'(x)$$

Summary

Pointwise and uniform convergences ($\sum$)

Absolute convergence

Normal convergence

Reminders about power series

Applications: Weierstrass approximation theorems

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The absolute convergence for series of functions extends the notion of absolute convergence for numerical series. Thus, like the pointwise convergence, it is a convergence that looks at each point separately, and that will not keep the interesting properties of the functions.

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As for numerical series:

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The absolute convergence implies the pointwise convergence.

Absolute and uniform convergences

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Indeed, there exist series for which the convergence is uniform yet not absolute: *example : the series* $\sum g_n$ *where the* g_n *functions are constant, equal to* $\dfrac{(-1)^n}{n}$ *converges uniformly (Leibniz's test for alternating series with* $\|R_n\|_\infty \leqslant \|g_{n+1}\|_\infty = \dfrac{1}{n+1} \underset{n\to+\infty}{\longrightarrow} 0$*), yet not absolutely since* $\sum |g_n| = \sum \dfrac{1}{n}$ *is a divergent Riemann series*;

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and others for which the convergence is absolute yet not uniform (for instance, any series of positive functions that converges pointwise yet not uniformly).

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Definition [Normal convergence of a series of functions]
Let (f_n) be a sequence of bounded[a] functions over I.
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This characterisation is thus simpler, yet way less accurate.

Properties of the normal convergence

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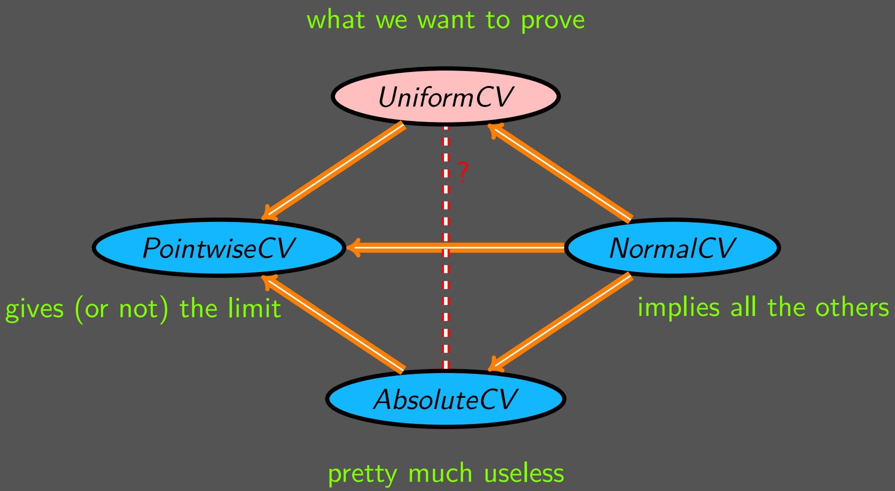
The normal convergence implies the uniform convergence.

The study of the normal convergence is a quick way, when it gives a favourable outcome, to determine that the convergence of a series of functions is uniform. However, this is not a necessary condition: when the convergence is not normal, we have to study the series more in detail.

Links between the different convergences



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Then, the series is absolutely convergent over the open ball of radius R centred at 0 (over $\mathbb{R}$, it is the interval $]-R;R[$) and on any segment included in this interval, the convergence is normal thus uniform ($[r_1, r_2]$ with $-R < r_1 < r_2 < R$).

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Thus, the convergence being uniform, power series are an excellent way to approximate functions while keeping their interesting properties.

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$$P_n : x \longmapsto \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left(\begin{array}{c} n \\ k \end{array}\right) x^k (1-x)^{n-k}.$$

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The right part of the expression is inspired from the binomial law of probabilities. The proof uses many results from the theory of probabilities, among which the weak law of large numbers; it enables to prove the pointwise and uniform convergences.

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We call trigonometric polynomial any function with complex values under the form $P : x \longmapsto \sum_{k=-n}^{n} c_k e^{ikx}$. Such a function, when taking only real values, can also be written $P : x \longmapsto a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)$.

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This result derives from the theory of Fourier series; the Hungarian mathematician Lipót Fejér proved that for every 2π-periodic continuous function f, the sequence of Cesàro means $\frac{1}{p+1} \sum_{n=0}^{p} S_n(f)$ of the sequence of partial sums of its Fourier series $S_n(f) : x \longmapsto \sum_{k=-n}^{n} c_k(f) e^{ikx}$ with $c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$ converges uniformly towards f.