ÉCOLE D'INGENIEURS EN INFORMATIQUE

Generalized integrals

1 Definition and properties

Learning outcomes :

— determine if a given generalized integral is well defined
— calculate moments and probabilistic quantities related to a random variable with density

Question 1-1 Reference integrals : exponential
Determine the nature of the integral depending on $\alpha \in \mathbb{R}$:

a) $\displaystyle\int_0^{+\infty} e^{\alpha x}\,\mathrm{d}x$

b) $\displaystyle\int_{-\infty}^{0} e^{\alpha x}\,\mathrm{d}x$

Question 1-2 What can you say about the following statement :

If f is continuous on $[a, +\infty[$ then

$$\lim_{x \to +\infty} f(x) = 0 \implies \int_a^{+\infty} f(x)\,\mathrm{d}x \text{ converges}$$

Question 1-3 Determine the nature of the following integrals. If they converge calculate their value.

a) $\displaystyle\int_0^{+\infty} \cos(t)\,\mathrm{d}t$

b) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^{\frac{4}{5}}}\,\mathrm{d}x$

c) $\displaystyle\int_{1}^{+\infty} \frac{1}{x^{\frac{5}{4}}}\,\mathrm{d}x$

Question 1-4 Using comparison theorems determine the nature of

a) $\displaystyle\int_{1}^{+\infty} \frac{\cos^2(t)}{t^2}\,\mathrm{d}t$

b) $\displaystyle\int_{0}^{+\infty} \frac{e^{-t}}{1+t^2}\,\mathrm{d}t$

c) $\displaystyle\int_{1}^{+\infty} \frac{e^{\sin(t)}}{t}\,\mathrm{d}t$

d) $\displaystyle\int_{\pi}^{+\infty} \frac{1}{t+e^t}\,\mathrm{d}t$

e) $\displaystyle\int_{\pi}^{+\infty} \frac{1}{t-e^{-t}}\,\mathrm{d}t$

f) Gauss integrals : $\displaystyle\int_{-\infty}^{+\infty} e^{-\alpha x^2}\,\mathrm{d}x$ avec $\alpha > 0$

g) $\displaystyle\int_{0}^{+\infty} \ln(t)e^{-t}\,\mathrm{d}t$

h) $\displaystyle\int_{0}^{+\infty} \frac{\ln(1+x^2)}{x^2}\,\mathrm{d}x$

Question 1-5 A signal is represented by

$$x : I \subseteq \mathbb{R} \to \mathbb{R} \text{ where } \mathbb{C}$$

The energy of a signal is given by

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2\,\mathrm{d}t$$

Determine in the following signals are of finite energy.

a) $x : \mathbb{R} \to \mathbb{R}, t \mapsto \cos(\omega t)$

b) $x : \mathbb{R} \to \mathbb{R}, t \mapsto Ae^{-\alpha t}\cos(\omega t)$ on $[0, +\infty[$ and 0 elsewhere. (Ici $A > 0$ est l'amplitude du signal, ω la fréquence angulaire et $\alpha > 0$ la facteur de dégradation (perte, amortissement) du signal. On retrouve ce signal dans des systèmes mechaniques, acoustique, séismologie etc.)

Question 1-6 Calculate the following integrals :

a) $\displaystyle\int_{0}^{+\infty} \frac{1}{1+x^2}\,\mathrm{d}x$

b) $\displaystyle\int_{0}^{+\infty} \frac{1}{2+x^2}\,\mathrm{d}x$

c) En déduire $\displaystyle\int_{0}^{+\infty} \frac{3+2x^2}{(1+x^2)(2+x^2)}\,\mathrm{d}x$

Question 1-7

a) Study the nature of $\int_1^{+\infty} \frac{\sin(x)}{x^{3/2}} \, dx$

b) Deduce the nature of $\int_1^{+\infty} \frac{\cos(x)}{x^{1/2}} \, dx$ (PI)

c) Deduce the nature of $\int_1^{+\infty} \cos(x^2) \, dx$ ($u = x^2$).

Question 1-8 The exponential distribution has density f where

$$f(x) = \begin{cases} 0 & x < 0 \\ \lambda e^{-\lambda x} & x \geq 0 \end{cases}$$

a) Check that f satisfies one of the density properties :

$$P(X \in \mathbb{R}) = \int_{-\infty}^{+\infty} f(x) \, dx = 1$$

b) Calculate the mean value of the random variable X having exponential distribution with parameter λ :

$$E(X) = \int_{-\infty}^{+\infty} x f(x) \, dx$$

c) Calculate the variance of X :

$$V(X) = \int_{-\infty}^{+\infty} (x - E(X))^2 f(x) \, dx$$