Fourier Series

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Résumé

The theory of Fourier series looks for approximations of periodic functions under the form of sequences (series) of trigonometric polynomials (that is to say, sums of sinusoidal functions, whose frequence is a multiple of the chosen basic frequence). Until now, we mostly worked with iterative or recursive approximation methods, whose results could be enhanced step by step till the wanted accuracy was reached (for instance, Taylor expansions, power series etc.). Using the advantages of the structure of the vector space spanned by the previously evoked sinusoidal functions, the theory of Fourier series offers a very interesting result : any single term of the decomposition into a Fourier series of a given function can be calculated separately from the others, just by calculating an integral.

Originally introduced to write harmonic decompositions of periodic signals, the study of Fourier series has since been used in many fields of research in mathematics, even opening new domains like harmonic analysis or signal theory. Fourier series will then represent an important part of your `MASI` lesson about Signal Mathematics.

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1 Calculating Fourier series

The decomposition into a Fourier series uses the orthogonality of the basic functions [$t \mapsto \cos(nt)$ and $t \mapsto \sin(nt)$ in $\mathbb{R}$, $t \mapsto \mathrm{e}^{int}$ in $\mathbb{C}$] to determine every coefficient independently by orthogonal projection, using the inner product $< f, g >= \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \bar{f}(t)g(t)\mathrm{d}t$ in the complex Hilbert space spanned by the functions $t \mapsto \mathrm{e}^{int}$ (the $\dfrac{1}{2\pi}$ is a normalising factor, used so that the set of exponential functions becomes orthonormal).

If, as always, the theory is easier with complex numbers, we will mostly use real coefficients here. These allow to use the properties of parity of the functions to limit the calculations.

Question 1-1 Absolute value

We consider the function f, 2π-periodic and continuous over $\mathbb{R}$, such that :

$$\forall x \in [-\pi, \pi], f(x) = |x|.$$

- What can we say about the parity of f ?
- Calculate the real Fourier coefficients of f, and write down its real Fourier series.
- Calculate the complex Fourier coefficients of f.

Question 1-2 Rectangle wave

We consider the function g, 2π-periodic over $\mathbb{R}$, such that :

$$g(x) = \begin{cases} -1 & \text{if } x \in [-\pi, 0[\\ 1 & \text{if } x \in [0, \pi[\end{cases}.$$

- Is g even or odd ? What can be said about its regularised function $\tilde{g}$?
- Calculate the real Fourier coefficients of g, and write down its real Fourier series.
- Calculate the complex Fourier coefficients of g.

Question 1-3 Linearisation

Determine the Fourier series of the function $x \longmapsto \cos^5(x)$.

2 Application : Dirichlet's theorem

Dirichlet's convergence theorem give results about the pointwise convergence of Fourier series, given that the studied functions are smooth enough. It is less accurate than Fejér's theorem building a uniform approximation of a continuous function (Weierstrass approximation theorem), but its result is easier to use.

When the studied functions offer contain discontinuity points, the pointwise convergence is not towards the function but towards its regularised function. So, take care when evaluating Dirichlet's identities at points where the starting functions "jumps".

This result can be used to determine sums of numerical sequences.

Question 2-4 Absolute value : convergence

In this exercise, we use again the function f defined at question 1-1.

- Does f check the hypotheses of Dirichlet's theorem ? What can we deduce about the convergence of its Fourier series $S(f)$?
- Evaluate the Fourier series of f at $t = 0$. Deduce the value of $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$.
- Deduce from the previous result the value of $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2}$.
- What do we get when evaluating $S(f)$ at $\dfrac{\pi}{2}$?
- Evaluate the Fourier series $S(f)$ at $\dfrac{\pi}{4}$. Write down the result as a series by calculating the different values taken by the cosine functions.

Question 2-5 Rectangle wave : convergence

In this exercise, we use again the function g defined at question 1-2.

- Does g check the hypotheses of Dirichlet's theorem ? What can we deduce about the convergence of its Fourier series $S(g)$?
- What do we get when evaluating $S(g)$ at 0 ?

- Evaluate the Fourier series $S(g)$ at $\frac{\pi}{2}$. Write down the result as a series.
- Evaluate the Fourier series $S(g)$ at $\frac{\pi}{4}$. Write down the result as a series.

3 Application : Parseval's identity

The Parseval identity gives a relation between the Fourier coefficients of the studied function and its norm induced by the inner product that was evoked in the introduction. It extends Bessel's inequality; particularly, it states that the numerical series associated with the squares of moduli of the Fourier coefficients of a function checking the right hypotheses is convergent.

This result is kind of a generalisation of Pythagoras's theorem, it allows to calculate the sum of many numerical series using the computation of an integral.

Question 3-6 Rectangle wave : Parseval's identity

- What do we get when applying the result of Parseval's identity to the function g of question 1-2 ?
- Determine the value of $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}$ (this value has already been calculated by another method in exercise 2-4).
- Deduce the value of $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ (same remark).

Question 3-7 Absolute value : Parseval's identity

- What do we get when applying the result of Parseval's identity to the function f of question 1-1 ?
- Determine the value of $\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4}$.
- Deduce the value of $\sum_{n=1}^{+\infty} \frac{1}{n^4}$.

Question 3-8 Fonction escalier

Let h be the 2π-periodic function defined by $h(x) = \begin{cases} 0 & \text{if} \quad x \in [0, \frac{\pi}{2}[; \\ 1 & \text{if} \quad x \in [\frac{\pi}{2}, \pi[; \\ 2 & \text{if} \quad x \in [\pi, \frac{3\pi}{2}[; \\ 3 & \text{if} \quad x \in [\frac{3\pi}{2}, 2\pi[. \end{cases}$.

- Determine the real Fourier coefficients $a_n(h)$ and $b_n(h)$.
- Explain why $\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = 2\sum_{n=0}^{+\infty} \frac{1}{(4n+1)(4n+3)}$. Using Dirichlet's theorem at $x = \frac{\pi}{2}$, calculate this sum.
- What do we get if we use Dirichlet's theorem at $x = \frac{\pi}{4}$?
- Using Parseval's identity, calculate $\sum_{n=0}^{+\infty} \left(\frac{1}{(4n+1)^2} + \frac{1}{(4n+2)^2} + \frac{1}{(4n+3)^2} \right)$.
- Deduce the value of $\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

A Reminders about complex numbers

For every θ in $\mathbb{R}$, we denote by $e^{i\theta}$ the complex of modulus 1 and of argument θ. This number belongs to the trigonometric circle : its algebraic form is

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Using the parity of the functions sin and cos, we can derive a "converse" formula expressing these two functions with complex exponentials :

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

(These formulae are called Euler formulae).

The Euler formulae enable to decomposer any polynomial of variable $\cos(x)$ or/and $\sin(x)$ into a linear combination of harmonic functions, that is to say $\cos(nx)$ or $\sin(nx)$: this process is called linearisation. It is realised using de Moivre's formula :

$$\forall n \in \mathbb{Z}, \left(e^{i\theta}\right)^n = e^{in\theta} \quad \text{that is to say} \quad \big(\cos(\theta) + i\sin(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta).$$

Beware, this seemingly-obvious formula actually is not obvious, and only holds if the power is an integer : do not forget that $e^{i\theta}$ is a complex number, and non-integer powers of a complex number are not defined.

The linearisation is thus a way to get, for polynomials of the variables cos and sin, their Fourier series, which by the way is finite.