# INTG - Improper integrals with a parameter

September 20, 2023

# What we learned last time ...

**Theorem Dominated Convergence Theorem**

Let $f_n$ be piecewise continous functions on an interval $I$. Let

- $\lim f_n = f$ (pointwise) with $f$ is piecewise continous
- There exists $\varphi$ an integrable function on $I$ such that

$$\forall x \in I, \quad \left|f_n(x)\right| \leq \varphi(x)$$

Then $\int_I f_n(x)dx$ and $\int_I f(x)dx$ converge absolutely and

$$\lim \int_I f_n(x)dx = \int_I \lim f_n(x)d = \int_I f(x)dx$$

There are many situations where we are intersted in integrating functions of two variables with respect to one of the varaibles

$$\int_J f(x,t)dt \quad \text{where} \quad f(x,t): I \times J \to \mathbb{R}$$

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**Example**

- Convolution product

$$f * g(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt)$$

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**Example**

- Convolution product

$$f * g(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt$$

- Fourier transformation

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(t)e^{-i\xi t}dt$$

Thus we are insterested in the function

$$F : I \to \mathbb{R} \qquad x \mapsto \int_J f(x,t)dt$$

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**Question**

But is $F$ always a well defined function?

Wooclap 1

# Conclusion

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**Conclusion**

We need for each $x \in I$ the integral $\int_J f(x,t)dt$ to be converging

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**Properties we would like $F$ to have**

- $F$ to be continous
- $F$ to be $C^1$
- $F$ to be $C^k$

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Let's start with the continuity

**Question**

What conditions would be sufficient for $F$ to be continuous?

Wooclap [2]

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Let's start with the continuity

**Conclusion**

- The continouity of $x \mapsto f(x,t)$ is not sufficient

## How it works ... : definition and continuity

**Theorem**

Let

$$f : I \times J \to \mathbb{R} \quad (x,t) \mapsto f(x,t)$$

$$F : I \to \mathbb{R} \quad x \mapsto \int_J f(x,t)dt$$

- $\forall x \in I,\ t \mapsto f(x,t)$ be ***piecewise continous*** on $J$
- $\forall t \in J,\ x \mapsto f(x,t)$ be ***continous*** on $I$
- there exists a piecewise continuous function $\varphi$ ***integrable*** on $J$ such that

$$\forall (x,t) \in I \times J, \quad \left| f(x,t)) \right| \leq \varphi(t).$$

Then the function $F$ is ***well*** defined and ***continuous***.

**Example**

Let

$$F(x) = \int_0^{+\infty} \sin(xt) e^{-t^2} \, \mathrm{d}t$$

Let we show that $F$ is well defined and continuous on $\mathbb{R}$.

# Application : Fourier transformation

Let $f:\mathbb{R}\to\mathbb{C}$ be piecewise continuous and integrable on $\mathbb{R}$. We would like to define the ***Fourier transform*** of $f$ as :

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1. $\forall x \in \mathbb{R}$ the function $t \mapsto e^{-ixt} f(t)$ is piecewise continuous over $\mathbb{R}$.

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Let we show that it is well definied and continuous on $\mathbb{R}$.

Let $g(x,t) = e^{-ixt} f(t)$ defined over $\mathbb{R} \times \mathbb{R}$.

1. $\forall x \in \mathbb{R}$ the function $t \mapsto e^{-ixt} f(t)$ is piecewise continuous over $\mathbb{R}$.
2. $\forall t \in \mathbb{R}$ the function $t \mapsto e^{-ixt} f(t)$ is continuous over $\mathbb{R}$.
3. $\forall (x,t) \in \mathbb{R} \times \mathbb{R}, \quad |g(x,t)| = |f(t)|$ and $f$ is integrable.

The continuity theorem thus implies that $\widehat{f}$ is well defined and continuous.

# Differentiation

Thus we are insterested in the function

$$F : I \to \mathbb{R} \qquad x \mapsto \int_J f(x,t)dt$$

**Question**

What conditions would be sufficient for $F$ to be $\mathrm{C}^1$?

Wooclap [4]

# How it works ...

## Theorem (Leibniz)

$$f : I \times J \to \mathbb{R} \qquad (x,t) \mapsto f(x,t)$$

$$F : I \to \mathbb{R} \qquad x \mapsto \int_J f(x,t)dt$$

- $\forall x \in I,\ t \mapsto f(x,t)$ be piecewise continous on $J$
- $\forall x \in I,\ \int_J f(x,t)$ converges
- $\forall t \in J,\ x \mapsto f(x,t)$ is $C^1$ on $I$
- $\forall x \in I,\ t \mapsto \frac{\partial f}{\partial x}(x,t)$ is piecewise continuous on $J$.
- there exists a piecewise continuous function $\varphi$ integrable on $J$:

$$\forall (x,t) \in I \times J, \quad \left|\frac{\partial f}{\partial x}(x,t)\right| \leq \varphi(t).$$

Then the function $F$ is $C^1$ on $I$ and $F'(x) = \int_J \frac{\partial f}{\partial x}(x,t)dt$.

# Laplace transform of a function

**Example**

Let $f$ be continuous function on $[0,+\infty[$. We the ***the Laplace transform*** of $f$ :

$$L(f)(x)=\int_0^{+\infty} e^{-xt} f(t)\,\mathrm{d}t$$

Let $f(t)=\cos(\omega t)$ defined over $[0,+\infty[$

Let we show that $L(f)$ is $C^1$.

# Generalization

**Theorem ( Here $n \in \mathbb{N} \cup \{+\infty\}$ )**

$$f : I \times J \to \mathbb{R} \quad (x,t) \mapsto f(x,t)$$

$$F : I \to \mathbb{R} \quad x \mapsto \int_J f(x,t)dt$$

- $\forall k \in [\![0,n]\!]$, $\forall x \in I$ , $t \mapsto \frac{\partial^k f}{\partial x^k}(x,t)$ be piecewise continous on $J$
- $\forall k \in [\![0,n]\!]$, $\forall t \in J$ , $x \mapsto \frac{\partial^k f}{\partial x^k}(x,t)$ be continous on $I$
- $\forall k \in [\![0,n]\!]$, there exists a function $\varphi_k$ integrable on $J$ such that

$$\forall (x,t) \in I \times J, \quad \left|\frac{\partial^k f}{\partial x^k}(x,t)\right| \leq \varphi_k(t).$$

Then the function $F$ is $C^k$ on $I$ and $F^{(k)}(x) = \int_J \frac{\partial^k f}{\partial x^k}(x,t)dt$.

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Let we show that $L(f)$ is $C^{\infty}$.

# To be used in S5, S6 ... : MASI, PBS1 , PBS2, ERO2 ...

What we have learnd in this lecture course.

- Generalized integrals
- Dominated convergence theorem
- Integral with a parameter and properties of functions depending on the parameter.