INTG - Chapter 2

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September 14, 2023

In this RMD session ...

The question that will guide this lecture :

Question

If f_n is a family of functions, when do we have

$$\lim \int_I f_n(x)dx = \int_I \lim f_n(x)dx = \int_I f(x)dx?$$

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Towards the answer :

- We recall ***piecewise continuity***
- We define simple convergence of sequences of functions
- Dominated convergence theorem

Piecewise continuity

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- So far we have studied continuous functions defined on an open, semi-open, closed, or infinite interval.

Piecewise continuity

- So far we have studied continuous functions defined on an open, semi-open, closed, or infinite interval.

- Now we will study a larger family of functions by allowing some points of ***discontinuity***. This leads to the notion of ***piecewise continuous function***

Flashback to Video 2

Wooclap 1

Question

How can one integrate a piecewise continuous function over I?

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Réponse

Définition

Let f be piecewise continuous over I. We say that the integral $\int_I f(t)\,\mathrm{d}t$ ***converges*** if $\int_{x_i}^{x_{i+1}} f(t)\,\mathrm{d}t$ converges as a generalized integral on each interval $]x_i, x_{i+1}[\subset I$ on which f is continuous.

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Remarque

If f is piecewise continuous on a closed interval then it has a finite number of points of discontinuity. On any interval it has a countable number of such points.

Wooclap 2

Now we study simple(pointwise) convergence of functions. Recall :

Definition

The sequence of functions is said to converge simply (pointwisely) towards f on I if for each $x \in I$

$$\lim_{n \to +\infty} f_n(x) = f(x)$$

Wooclap 3

Piecewise continuity

Dominated convergence theorem

Question

If f_n is a family of piecewise continuous functions. When do we have

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First attempt :

It f_n is a family of piecewise continuous functions pointwisely converging to f then

$$\lim \int_I f_n(x)dx = \int_I \lim f_n(x)d = \int_I f(x)dx$$

Wooclap 4

Towards $\int \lim f_n = \lim \int f_n$

Deuxième tentative

It f_n is a family of piecewise continuous functions bounded pointwisely converging to f then

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Deuxième tentative

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Wooclap 5

Third attempt

... f_n uniformly bounded ....

Integrable function

Définition

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Remark

When f is contunuous on I sans with no improper borders then this is a classical Riemann integral, thus finite.

Dominated convergence theorem

Théorème

Let f_n be a sequence of piecewise continuous functions over I. If

- $\lim f_n = f$ (pointwisely), f piecewise continuous
- there existe integrablle function φ piecewise continuous over$I$ such that

$$\text{for all } x \in I, \quad \left|f_n(x)\right| \leq \varphi(x)^a$$

Then $\displaystyle\int_I f_n(x)dx$ and $\displaystyle\int_I f(x)dx$ converge absolutely and

$$\lim_{n\to+\infty} \int_I f_n(x)dx = \int_I \lim_{n\to+\infty} f_n(x)d = \int_I f(x)dx$$

[a] domination hypothesis

Wooclap 6-8

To keep in mind

- Dominated convergence theorem allows us to study limites od sequences of integrals of a family of functions
- When using it, one needs to pay attention on all hypothesis.