Approximation of functions

The notion of metric space

October 8, 2021

EPITA

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The aim of this lesson is to guide you into the analysis of such situations, which requires a lot more focus than it seems.

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The expression $|u_n - \ell| < \varepsilon$ expresses that u_n is at a distance of at most[2] ε of ℓ. One can thus say :

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}, \quad n \geq N \Rightarrow u_n \text{ is at a } \textbf{\textit{distance}} \text{ of at most } \varepsilon \text{ of } \ell.$$

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In this first set of slides, we will dwell on that notion of distance.

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What to do from now on?

Starting point

Let E be a given set.

Question

How can we talk about distance between two elements of E ?

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- Given 2 elements x, y of E and an intermediary element z, the sum of distances between x and z then z and y cannot be less than that between x and y.

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These 4 remarks will be the fundamental axioms enabling us to define a notion of distance.

Definition of a distance

Définition

A ***distance*** over a set E is an application $d : E \times E \to \mathbb{R}_+$ satisfying the following:

Separation $\forall x, y \in E,\ d(x,y) = 0 \iff x = y$

Symmetry $\forall x,\ y \in E,\ d(x,y) = d(y,x)$

$\triangle$ **inequality** $\forall x,\ y,\ z \in E,\ d(x,y) \leq d(x,z) + d(z,y)$.

Remarque : Why is the last point called «triangle inequality» ?

This will be the only definition for today!

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You have to think about the following questions:

- What can be said about the equation

$$d(x,y) = \min_{z \in E} d(x,z) + d(z,y)?$$

  How to interpret it?
- Does the path given by the *Waze* application define a distance between two GPS points?

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- Does the path given by the *Waze* application define a distance between two GPS points?
- How to define a distance over the vertices of a directed graph?

Where to find the lesson material?

As for all of your ECUE, all the information about this lesson is to be found on the `21_IS5TC_HM_APXF` ***Moodle*** platform:
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That's All Folks!