MF 796: Computational Methods of Mathematical Finance

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Problem set # 6

Due: Wednesday, March 18, by 8am

Problem 1: Numerical PDEs: Suppose that the underlying security SPY evolves according to standard geometric brownian motion. Then its derivatives obey the Black-Scholes equation:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 c}{\partial s^2} + rs\frac{\partial c}{\partial s} - rc = 0 \qquad (1)$$

$\sigma_2 = 28.99\%$

$\sigma_1 = 27.98\%$

Use SPY's closing price on March 4, 2020. 312.86 $K_1 = 315$ Implied volatility $= 28.49\%$

We are going to find the price of a call spread with the right of early exercise. The two strikes of the call spread are $K_1 = 315$ and $K_2 = 320$ and the expiry is September 30, 2020.

$c = S \cdot N(d_1) - Xe^{rt} \cdot N(d_2)$

1. Explain why this instrument is not the same as being long an American call with strike 315 and short an American call with strike 320, both with expiry September 30, 2020.
2. For riskless rate r, use the 3-month US Treasury bill at the close of March 4, 2020. Say where you got the rate and why you consider it a reliable source.
3. Let's assume that we are not able to find σ by calibrating to the European call spread price and must find it by other means. Find a way to pick the σ, explain why you chose this method, and then find the σ. σ
4. Set up an explicit Euler discretization of (1). You will need to make decisions about the choice of $s_{\max}$, h_s, h_t, etc. Please explain how you arrived at each of these choices.
5. Let A be the update matrix that you created in the previous step. Find out its eigenvalues and check their absolute values.
6. Apply your discretization scheme to find today's price of the call spread *without* the right of early exercise. The scheme will produce a whole vector of prices at time 0. Explain how you chose the one for today's price.
7. Modify your code in the previous step to calculate the price of the call spread *with* the right of early exercise. What is the price?
8. Calculate the early exercise premium as the difference between the American and European call spreads. Is it reasonable?

Problem 1

1. According to American calls, when you long an American calls with strike 315, you can decide if exercised an American calls if you want. However, if you short an American calls, you cannot decide the time of exercise, and it depends on buyers.

2. According to FRED, the 3-month US Treasury bill at the close of March 4, 2020 is 0.72% I consider it is a reliable source because the data is collected by government.

3. By the question, we have known

$K_1 = 315 \quad K_2 = 320 \quad S = 312.86 \quad r = 0.72\%$

March 4, 2020 — September 30, 2020 $\Rightarrow T = \frac{142}{252}$

According to Yahoo Finance, the implied volatility is 0.2798 for $K_1 = 315$.
The second implied volatility is 0.2899. for $K_2 = 320$
In this question, I will use the average of these two volatilities, because we can not find σ by calibrating to the European call spread price.
So, $\sigma = \frac{\sigma_1 + \sigma_2}{2} \doteq 0.28485$

4. According to this question, set up an explicit Euler discretization of

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0$$

$$C(S_i, t_{j-1}) = \overbrace{\left(1 - \sigma^2 S_i^2 \frac{h_t}{h_s^2} - rh_t\right)}^{a_i} C(S_i, t_j) + \overbrace{\left(\frac{\sigma^2 S_i^2}{2}\frac{h_t}{h_s^2} - \frac{rS_i h_t}{2h_s}\right)}^{l_i} C(S_{i-1}, t_j) + \overbrace{\left(\frac{\sigma^2 S_i^2}{2}\frac{h_t}{h_s^2} + \frac{rS_i h_t}{2h_s}\right)}^{u_i}.$$

$$C(S_{i+1}, t_j) = a_i C(S_i, t_j) + l_i C(S_{i-1}, t_j) + u_i C(S_{i+1}, t_j)$$

when $i=1$ we need to use the top condition. $i=M-1$ we need to use bottom.

and $c(S_M, t_j) = C(S_{max}, t_j) = S_M - Ke^{-r(t_N - t_j)}$ $\forall j$ (top)

In here, I want to set $S_{max} = 600$, and it is enough.

and $c(S_0, t_j) = C(0, t_j) = 0$ $\forall j$ (bottom)

$$\Rightarrow C(S_1, t_{j-1}) = a_1 C(S_1, t_j) + l_1 \underbrace{C(S_0, t_j)}_{=0} + u_1 C(S_1, t_j)$$

$$= a_1 C(S_1, t_j) + u_1 C(S_1, t_j)$$

$$C(S_{M-1}, t_{j-1}) = a_{M-1} C(S_{M-1}, t_j) + l_{M-1} C(S_{M-2}, t_j) + u_{M-1} C(S_M, t_j)$$

$$= a_{M-1} C(S_{M-1}, t_j) + l_{M-1} C(S_{M-2}, t_j) + \underbrace{u_{M-1}(S_M - Ke^{-r(t_N - t_j)})}_{b_j},$$

$$\Rightarrow (C(S_1, t_{j-1}), C(S_2, t_{j-1}), \cdots, C(S_{M-2}, t_{j-1}), C(S_{M-1}, t_{j-1}))^T$$

$$= \begin{bmatrix} a_1 & u_1 & & & \\ l_2 & a_2 & u_2 & & \\ & \ddots & \ddots & \ddots & \\ & & l_{M-2} & a_{M-2} & u_{M-2} \\ & & & l_{M-1} & a_{M-1} \end{bmatrix} \begin{bmatrix} C(S_1, t_j) \\ C(S_2, t_j) \\ \vdots \\ C(S_{M-2}, t_j) \\ C(S_{M-1}, t_j) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ b_j \end{bmatrix}$$

$$b_j = u_{M-1}(S_M - Ke^{-r(t_N - t_j)})$$

$$\Rightarrow \vec{C}(j-1) = A\vec{C}(j) + b(\vec{j})$$

Firstly, choice $h_s = 2$. since the matrix A will be raised to high power, we want to make sure its eigenvalues stay below 1. $h_s = \frac{600}{300} = 2$

$$|a_i| < 1$$

$$a_i = \left| \sigma^2 S_{max}^2 \frac{h_t}{h_s^2} + r h_t \right| < 2$$

$\sigma = 0.2849$ $S_{max} = 600$ $r = 0.0072$ to decide h_s

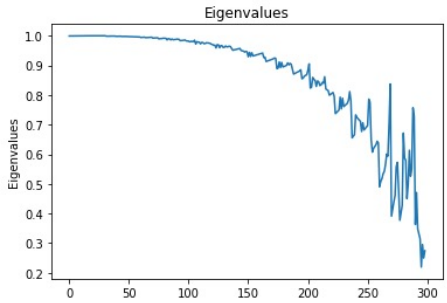
$$\left| \sigma^2 S_{max}^2 \frac{h_t}{h_s^2} \right| < 1$$

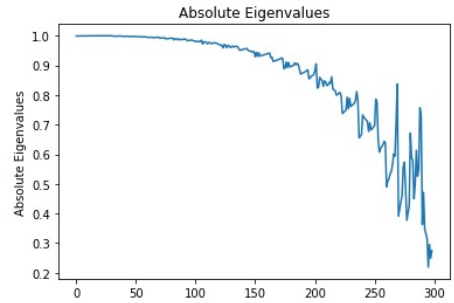
$$\Rightarrow h_t < \left(\frac{1}{\sigma^2 S_{max}^2} \right) h_s^2 = O(h_s^2)$$

$$h_t < 0.13689 \qquad \Rightarrow h_t = \frac{1}{10000}$$

$h_t = \frac{T}{10000}$ $h_s = 2$ to make sure that A matrix has all eigenvalues between -1 and 1.

5.





By the parameters, the eigenvalues between -1 and 1.

$h_s = 2$ $h_t = \frac{T}{10000}$

6. According to linear interpolation, I can get the call spread price when SPY equals to 311 or 313, so I can estimate any call spread price corresponding to SPY price between 311 and 313 use linear interpolation. Therefore, the price is 2.17.

7. With the right of early exercise, I will change $C_{spread}(S_i, t_j)$ to $\max(C_{spread}(S_i, t_j), ((S_i - 315)^+ - (S_i - 320)^+))$. My price is 4.53.

8. By calculation, I get premium is 2.36. And it will be 50% of the American call spread. I think that it is reasonable.