MF 796: Computational Methods of Mathematical Finance

Professors Christopher Kelliher and Eugene Sorets — Spring 2020

Problem set # 2

Due: Wednesday, February 5, 2020 by 8 am.

Problem 1: Evaluation of a known integral using various quadratures: In this problem we are going to compute the price of a European call option with 3 month expiry, strike 12, and implied vol 20, Assume the underlying is 10 now and the interest rate is 4%.

1. Use Black-Scholes formula to compute the price of the call analytically.
2. Calculate the price of the call numerically using the following 3 quadrature methods:
   (a) Left Riemann rule
   (b) Midpoint rule
   (c) Gauss nodes of your choice (say explicitly why you made that choice)

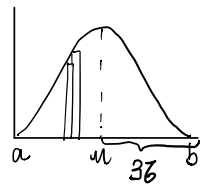
   with the number of nodes $N = 5, 10, 50, 100$ and compute the calculation error as a function of N for each of the methods.
3. Estimate the experimental rate of convergence of each method and compare it with the known theoretical estimate.
4. Which method is your favorite and why?

1. According to the question,

$$S_0 = 10, \quad K = 12, \quad t = 0.25 \left(\frac{3}{12}\right), \quad r = 0.04, \quad \sigma = 0.2$$

The call price is $C \approx 0.0188$ by the Black-Scholes formula, the code at the attachment.

2.



To find a:

$a = \ln K$

$b = \mu + 4 \cdot sd$

$\begin{cases} \mu = \ln(S_0) + (r - \frac{1}{2}\sigma^2) \cdot t \\ sd = \sqrt{\sigma^2 t} \end{cases}$

$$C = \hat{E}[e^{-rt}(S_T - K)_+]$$

$$C \approx e^{-rt} \int_{\ln K}^{\mu + 4 \cdot sd} \frac{e^{\ln x} - K}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{(\ln x - \mu)^2}{2 sd^2}\right) d\ln x$$

We need to computer an approximation to the integral, this means that we must find weights $w_i \in \mathbb{R}$ and nodes $x_i \in [a, b]$ so that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} w_i f(x)$$

Firstly, we have known that $[a, b] \subset [\mu - 3sd, \mu + 3sd]$, so we can get $b > \mu + 3\sigma$, in here I wanna chose b at $\mu + 4sd$, and chose a at $\ln K$.

(a) Left Riemann rule:

we wanna get x by left Riemann rule: $x_i = a + \frac{b-a}{N}(i-1)$ $i = 1, \cdots N$.

In here N is the number of nodes.

Therefore, we can estimate the integration by Riemann sum, the function is

$$\sum_{i=1}^{N} \left(\frac{b-a}{n}\right) \cdot C(x_i)$$

The code at the attachment.

(b) Midpoint rule: $x_i = a + \frac{b-a}{N}(i - \frac{1}{2})$ $i = 1, \cdots N$

Similary, estimate the integration by $\sum_{i=1}^{N} \left(\frac{b-a}{n}\right) \cdot C(x_i)$

(c) Gauss nodes of my choice:

Gauss lengendre: $\int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right) dx$

$$\Rightarrow \quad \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f\left(\frac{b-a}{2}x_i + \frac{a+b}{2}\right)$$

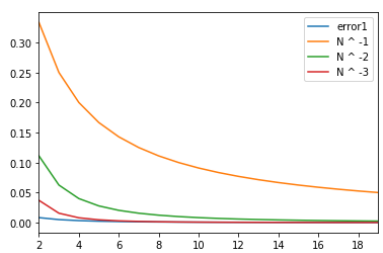
$a = \ln K$ $b = \mu + 4 \cdot sd$ to get the integration.

calculation error: compare the results from Black-Scholes with these three methods. The code at the attachment.

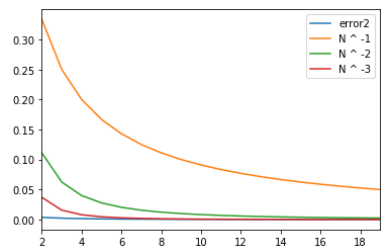
Compute the calculation error and the results show at the table 1:

N	Left-Riemann	Midpoint	Gauss-legendre
5	0.0018	0.0007	0.0001
10	0.0006	0.0001	0.0001
50	0.0001	0.0001	0.0001
100	0.0001	0.0001	0.0001

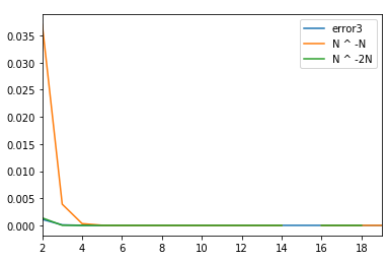
3. By the data frame from programming, we can get three figure, and the code at attachment.



Left Riemann rule error



Midpoint rule error



Gauss Legendre error

Firstly by the left Riemann rule and midpoint rule the error will be order between $O(N^{-2})$ and $O(N^{-3})$. Secondly, the error of Gauss legendre error can be $O(N^{-iN})$ $i \in N$ by Stirling's asymptotic formular.

4. My favourite method is midpoint rule, because the method offers relative low error, and it also be a easier method than Gauss legendre rule.

Problem 2: Calculation of Contingent Options: Let S_1 be a random variable that takes on the value of SPY one year from now and let S_2 take on the values of SPY 6 months from now. Assume that they are jointly normally distributed with $S_1: \tau_1 = 1$

$$\begin{aligned} \sigma_1 &= 20 \\ \sigma_2 &= 15 \\ \rho &= 0.95 \end{aligned}$$

$S_2: \tau_2 = \frac{1}{2} = 0.5$

By ρ here we mean correlation between S_1 and S_2. Also, assume that interest rate is zero. Please specify where you got the current price of the underlying.

1. Evaluate the price of the one year call on SPY with the strike $K_1 = 370$. This is an example of a vanilla option.
2. Evaluate the price of the one year call on SPY with the strike $K_1 = 370$, contingent on SPY at 6 months being below 365. This is a contingent option.
3. Calculate the contingent option again, but with $\rho = 0.8$, $\rho = 0.5$, and $\rho = 0.2$.
4. Does dependence on ρ make sense?
5. Calculate the contingent option again, but with SPY at 6 months below $360, 350$, and 340.
6. Does the dependence on the 6 month value make sense?
7. Under what conditions do you think the price of the contingent option will equal the price of the vanilla one?

1. Get the SPY data from Yahoo Finance (2019-02-02 – 2020-02-02), and the data at the attachment. From the data, we can get the price of SPY at 2020-01-31 is 321.73. Assume the interest rate is 0, $K_1 = 370$, $\sigma_1 = 0.2$, we can get the price of the one year call on SPY is 0.0511 (by the programming at the attachment).

2. Contingent Options: (contingent on SPY at 6 months being below 365)

$\Rightarrow S_{\frac{1}{2}T} \leq 365$

We should caculate the price of the one year call on SPY by

$$C = E^{Q}[e^{-rT}(S_{1T} - K_1)^{+} 1_{\{S_{\frac{1}{2}T} \leq 365\}}]$$

In here $1_{\{S_{\frac{1}{2}T} \leq 365\}} = \begin{cases} 1 & S_{\frac{1}{2}T} \leq 365 \\ 0 & S_{\frac{1}{2}T} > 365 \end{cases}$

By the Integration estimated formular,

$$C = \int_{-\infty}^{365} \int_{370}^{+\infty} e^{-rT}(S_1 - K_1)\, \phi(S_1, S_2)\, dS_1\, dS_2$$

S_1 after 1 year

S_2 after $\frac{1}{2}$ year.

By the joint density formular, two correlation variables:

$$\phi(S_1, S_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(S_1-\mu_1)^2}{\sigma_1^2} + \frac{(S_2-\mu_2)^2}{\sigma_2^2} - \frac{2\rho(S_1-\mu_1)(S_2-\mu_2)}{\sigma_1\sigma_2}\right)\right]$$

Therefore, we need to change the density function to the new one at the programming.

In here S_1, S_2 should be independly choice by mid point rule:

$$S_1 = a_1 + \frac{(b_1 - a_1)}{N} \cdot (i - \tfrac{1}{2}) \qquad i = 1, \cdots, N$$

$$S_2 = a_2 + \frac{(b_2 - a_2)}{N} \cdot (i - \tfrac{1}{2}) \qquad i = 1, \cdots, N$$

By the programming, we can get the price is 0.0283.

3. $\rho = 0.8 \quad c = 0.0394$

$\rho = 0.5 \quad c = 0.0481$

$\rho = 0.2 \quad c = 0.0506$

4. By the caculation in 3, it is clealy to find that the

contingent option price will increase with the decreasing in the correlation between S_1 and S_2. I think that the correlation between S_1 and S_2 decrease will lead the fluctuation of price will have less relation with last price. So the option price increase.

5. con1 = 360 c = 0.0136

con2 = 350 c = 0.0005

con3 = 340 c = 0.0000

6. Similarly, when the increasing in threshold, the contingent condition is less restriction, so it is a increasing at the option.

7. when $K_2 = +\infty$, the contingent option price will equal the price of the vanilla one.