MF 796: Computational Methods of Mathematical Finance

Professors Christopher Kelliher and Eugene Sorets Spring 2020

Problem set # 7

Due: Wednesday, April 15, by 8am

Problem 1: Simulation in the Heston Model: Suppose that the underlying security SPY evolves according to the Heston model. That is, we know its dynamics are defined by the following system of SDEs:

$$\begin{aligned} dS_t &= (r-q)S_t\,dt + \sqrt{\nu_t}S_t\,dW_t^1 \\ d\nu_t &= \kappa(\theta-\nu_t)\,dt + \sigma\sqrt{\nu_t}\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned} \tag{1}$$

$S_0=282$ $q=0.0177$

You know that the last closing price for SPY was 282. You also know that the dividend yield for SPY is 1.77% and the corresponding risk-free rate is 1.5%. $r=0.015$

Using this information, you want to build a simulation algorithm to price a knock-out option on SPY, where the payoff is a European call option contingent on the option not being knocked out, and the knock-out is an upside barrier that is continuously monitored. We will refer to this as an **up-and-out call**.

This payoff can be written as:

$$c_0 = \mathbb{E}\left[(S_T - K_1)^+ 1_{\{M_T < K_2\}}\right] \tag{2}$$

where M_T is the maximum value of S over the observation period, and $K_1 < K_2$ are the strikes of the European call and the knock-out trigger respectively.

1. Find a set of Heston parameters that you believe govern the dynamics of SPY. You may use results from a previous Homework, do this via a new calibration, or some other empirical process. Explain how you got these and why you think they are reasonable.
2. Choose a discretization for the Heston SDE. In particular, choose the time spacing, ΔT as well as the number of simulated paths, N. Explain why you think these choices will lead to an accurate result.
3. Write a simulation algorithm to price a European call with strike $K = 285$ and time to expiry $T = 1$. Calculate the price of this European call using FFT and comment on the difference in price.
4. Update your simulation algorithm to price an up-and-out call with $T = 1$, $K_1 = 285$ and $K_2 = 315$. Try this for several values of N. How many do you need to get an accurate price?

5. Re-price the up-and-out call using the European call as a control variate. Try this for several values of N. Does this converge faster than before?

Problem 1:

1. By the homework 5, I get $k = 3.51$, $\theta = 0.052$

$\sigma = 1.17$, $\rho = -0.77$, $v_0 = 0.034$ by Calibration.

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$par
[1]  3.51556747  0.05198731  1.17559799 -0.77477588  0.03399850
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I think that they are reasonable, because they are constructed depend on economics meaning and the calibration will make the least squared erros.

2. I wanna use Euler Discretization for the Heston model:

$$\hat{S}_{t_{j+1}} = \hat{S}_{t_j} + (r-q)\hat{S}_{t_j}\Delta t + \sqrt{v_{t_j}}\,\hat{S}_{t_j}\sqrt{\Delta t}\, Z_j^1$$

$$\hat{v}_{t_{j+1}} = \hat{v}_{t_j} + k(\theta - v_{t_j})\Delta t + \sigma\sqrt{v_{t_j}}\sqrt{\Delta t}\, Z_j^2$$

$$Z_j^2 = \rho Z_1 + \sqrt{1-\rho^2}\, Z_2$$

$$Z_j^1 = Z_1$$

The dT and N are choiced should depend on the difference between FFT price and simulation price. Therefore, I will choice $dt = \frac{1}{252}$ $N = 10000$.

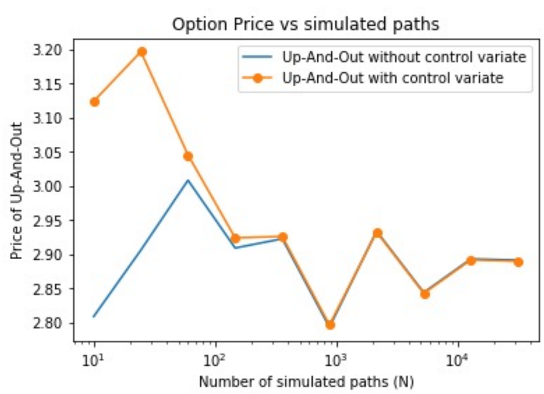
3. By these parameter, I get $p = 17.7850$ by simulation and $p = 17.5288$ by FFT. The difference is 0.2561, and I think it can be accepted and the difference can be further reduced by the adjustion in N and dt.

4. In this question, I will use a list of N = [100, 200, 500, 1000, 5000, 10000, 15000, 20000] and use N = 20000 as benchmark.

N	ABS
100	0.06228
200	0.06513
500	0.02862
1000	0.02645
5000	0.00758
10000	0.008059
15000	0.007232
20000	0.009193

I think when N > 10000 the call price will be accuratly get.

5.



By the figure, the control variate will offer a faster convergence.