MF 796: Computational Methods of Mathematical Finance

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Problem set # 5

Due: Wednesday, March 4th, by 8am

1. **Implementation of Breeden-Litzenberger:**

You are given the following volatility data:

Expiry / Strike	1M	3M
10DP	32.25%	28.36%
25DP	24.73%	21.78%
40DP	20.21%	18.18%
50D	18.24%	16.45%
40DC	15.74%	14.62%
25DC	13.70%	12.56%
10DC	11.48%	10.94%

$$C = \hat{E}[e^{-rT}(S_T - K)_+]$$
$$= \Phi(d_1) S_0 - \Phi(d_2) K e^{-rT}$$

$$P = \Phi(-d_2) K e^{-r(T-t)} - N(-d_1) S_t$$

You also know that the current stock price is 100, the risk-free rate is 0, and the asset pays no dividends.

NOTE: The table of strikes is quoted in terms of Deltas, where the DP rows indicate "X Delta Puts" and the DC rows indicate "X Delta Calls".

(a) Using the table of quoted (Black-Scholes) Deltas and volatilities, extract a table of strikes corresponding to each option.

(b) Choose an interpolation scheme that defines the volatility function for all strikes, $\sigma(K)$.

(c) Extract the risk neutral density for 1 & 3 month options. Comment on the differences between the two distributions. Is it what you would expect?

(d) Extract the risk neutral density for 1 & 3 month options using a constant volatility equal to the 50D volatility. Contrast these densities to the densities obtained above.

(e) Price the following European Options using the densities you constructed in (1c).

i. 1M European Digital Put Option with Strike 110.

ii. 3M European Digital Call Option with Strike 105.

iii. 2M European Call Option with Strike 100.

2. **Calibration of Heston Model:**

Recall that the Heston Model is defined by the following system of SDE's:

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{\nu_t}S_t\,dW_t^1 \\ d\nu_t &= \kappa(\theta - \nu_t)\,dt + \sigma\sqrt{\nu_t}\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned} \tag{1}$$

Recall also that the characteristic function for the Heston Model is known to be:

$$\omega(u) = \frac{\exp\left(iu\ln S_0 + iu(r-q)t + \sigma^{-2}\left\{\kappa\theta t\left(\kappa - i\rho\sigma u\right)\right\}\right)}{\left(\cosh\frac{\lambda t}{2} + \frac{\kappa - i\rho\sigma u}{\lambda}\sinh\frac{\lambda t}{2}\right)^{\frac{2\kappa\theta}{\sigma^2}}} \tag{2}$$

$$\Phi(u) = \omega(u)\exp\left(\frac{-(u^2+iu)\nu_0}{\lambda\coth\frac{\lambda t}{2} + \kappa - i\rho\sigma u}\right) \tag{3}$$

$$\lambda = \sqrt{\sigma^2(u^2+iu) + (\kappa - i\rho\sigma u)^2} \tag{4}$$

See the attached spreadsheet for options data. $r = 1.5\%$, $q = 1.77\%$, $S_0 = 267.15$.

Consider the given market prices and the following equal weight least squares minimization function:

$$\vec{p}_{\min} = \min_{\vec{p}}\left\{\sum_{\tau,K}\left(\tilde{c}(\tau,K,\vec{p}) - c_{\tau,K}\right)^2\right\} \tag{5}$$

where $\tilde{c}(\tau,K,\vec{p})$ is the FFT based model price of a call option with expiry τ and strike K.

(a) Check the option prices for arbitrage. Are there arbitrage opportunities at the mid? How about after accounting for the bid-ask spread? Remove any arbitrage violations from the data.

(b) Using the FFT Pricing code from your last homework, find the values of κ, θ, σ, ρ and ν_0 that minimize the equal weight least squared pricing error. You may choose the starting point and upper and lower bounds of the optimization. You may also choose whether to calibrate to calls, puts, or some combination of the two.
NOTE: In R, the optimx function can be used to perform this optimization, which has a parameter named method which can be set to Nelder-Mead.
Note also that you are given data for multiple expiries, each of which should use the same parameter set, but will require a separate call to the FFT algorithm.

(c) Try several starting points and several values for the upper and lower bounds of your parameters. Does the optimal set of parameters change? If so, what does this tell you about the stability of your calibration algorithm?

(d) Instead of applying an equal weight to each option consider the following function

which makes the weights inversely proportional to the quoted bid-ask spread:

$$\omega_{\tau,K} = \frac{1}{c_{\tau,K,\text{ask}} - c_{\tau,K,,\text{bid}}} \tag{6}$$

$$\vec{p}_{\min} = \min_{\vec{p}} \left\{ \sum_{\tau,K} \left(\omega_{\tau,K} \tilde{c}(\tau, K, \vec{p}) - c_{\tau,K} \right)^2 \right\} \tag{7}$$

where as before $\tilde{c}(\tau, K, \vec{p})$ is the FFT based model price of a call option with expiry τ, strike K. Repeat the calibration with this objective function and comment on how this weighting affects the optimal parameters.

3. **Hedging Under Heston Model:** Consider a 3 month European call with strike 275 on the same underlying asset.

   (a) Calculate this option's Heston delta using finite differences. That is, calculate a first order central difference by shifting the asset price, leaving all other parameters constant and re-calculating the FFT based Heston model price at each value of S_0.

      i. Compare this delta to the delta for this option in the Black-Scholes model. Are they different, and if so why? If they are different, which do you think is better and why? Which would you use for hedging?

      ii. How many shares of the asset do you need to ensure that a portfolio that is long one unit of the call and short x units of the underlying is delta neutral?

   (b) Calculate the vega of this option numerically via the following steps:

      i. Calculate the Heston vega using finite differences. To do this, shift θ and ν_0 by the same amount and calculate a first order central difference leaving all other parameters constant and re-calculating the FFT based Heston model price at each value of θ and ν_0.

      ii. Compare this vega to the vega for this option in the Black-Scholes model. Are they different, and if so why?

1.(1) By Black-Scholes formular equation:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + rS\frac{\partial c}{\partial S} - rc = 0 \quad \Delta_c = \frac{\partial c}{\partial S} = N(d_1)$$

for the call: $d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right]$ $\quad N(d_1) = \frac{\partial c}{\partial S} \quad d_1 = N^{-1}\left(\frac{\partial c}{\partial S}\right)$

$S_t = 100$

$\Rightarrow \quad N^{-1}\left(\frac{\partial c}{\partial S}\right) = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\frac{100}{K} + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right]$ $\quad r=0$ no dividends

$N^{-1}\left(\frac{\partial c}{\partial S}\right) = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{100}{K} + \frac{\sigma^2}{2}T\right]$

$N^{-1}\left(\frac{\partial c}{\partial S}\right)\sigma\sqrt{T} - \frac{\sigma^2}{2}T = \ln\frac{100}{K} \quad \Rightarrow K = 100\, e^{\frac{\sigma^2}{2}T - N^{-1}\left(\frac{\partial c}{\partial S}\right)\sigma\sqrt{T}}$

for the put:

$$\frac{\partial P}{\partial S} = -N(-d_1) \qquad -d_1 = N^{-1}\left(-\frac{\partial P}{\partial S}\right) \qquad \Delta_P = \frac{\partial P}{\partial S} = -N(-d_1)$$

$-d_1 = -\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r+\frac{\sigma^2}{2}\right)(T-t)\right]$

$N^{-1}\left(-\frac{\partial P}{\partial S}\right) = -\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{100}{K}\right) + \frac{\sigma^2}{2}T\right]$

$-N^{-1}\left(-\frac{\partial P}{\partial S}\right)\sigma\sqrt{T} - \frac{\sigma^2}{2}T = \ln\frac{100}{K}$

$\Rightarrow \quad K = 100\, e^{\frac{\sigma^2}{2}T + N^{-1}\left(-\frac{\partial P}{\partial S}\right)\sigma\sqrt{T}}$

Then use python code to handle data. and the code at the attachment.

Expiry/Strike	1M	3M
0.10	89.138758	84.225674
0.25	95.542103	93.470685
0.40	98.700642	98.127960
0.50	100.138720	100.338826
0.40	101.262273	102.141761
0.25	102.783751	104.532713
0.10	104.395838	107.422225

Significantly, when we use $\Delta_c = \Delta_p = 50D$

K_p and K_c will be same.

(b) In this question, I will check the relation between sigma and strike price, and I can find a roughly linear between them by these two figure.

So, I will use sklearn to get a linearregression, In the end I get the relation equation in 1M and 3M

$$\sigma_{1m} = 1.557788 - 0.013788\,K \qquad \text{(The code at attachment)}$$

$$\sigma_{3m} = 0.9322345 - 0.00767355\,K.$$

(c) By implementation of Breeden-Litzenberger in Practive

$$\partial c / \partial K = \frac{c(K+h) - c(K)}{\Delta h} \qquad C = e^{-\int_0^T r_u du} \int_K^{+\infty} (S_T - K)\,\phi(S_T)\,dS_T$$

$$\frac{\partial^2 c}{\partial k^2} = \frac{c(k+h) - c(k) - c(k) + c(k-h)}{\Delta h^2} = \frac{c(k+h) - 2c(k) + c(k-h)}{\Delta h^2}$$

$$\frac{\partial^2 c}{\partial k^2} = e^{-\int_0^T r_u du}\,\phi(k) \Rightarrow \phi(k) = e^{\int_0^T r_u du}\,\frac{c(k-h) - 2c(k) + c(k+h)}{h^2}$$

$r=0 \Rightarrow \phi(k) = \dfrac{c(k-h) - 2c(k) + c(k+h)}{h^2}$ in order to increase correction I chose $h=0.1$

h is smaller to get density.

By the observation of Figure 1, 1M will have a thinner tail than 3M. So, it is easily to find that the the price will have positive relation with maturity that means that the probability of price will decrease with the decrease in maturity.

(d) Use the 50 0 volatility. 1M: 0.1824 3M: 0.1645

The figure 2 show that 1M and 3M will have same tail when we constant the volatility.

(e) (i) 0.9913335 (ii) 0.3123886 (iii) 2.7111046

2. (a) In order to check the option for arbitrage, three main test should be involved included that monotonically, rate of change and convex.

if it exits arbitrage
- monotonically
  - call option prices decreasing in strike
  - put option prices increasing in strike
- rate of change
  - $-1 < \frac{\text{call option prices}}{\text{strike}} < 0$
  - $0 < \frac{\text{put option prices}}{\text{strike}} < 1$
- call and put prices are convex.

By python we us mid-price to check these condition by "bool function". and we can find there are no arbitrage. opportunities.

Monotonically:	Rate of change:	Convexity:
[1] TRUE	[1] TRUE	[1] TRUE
[1] TRUE	[1] TRUE	[1] TRUE
[1] TRUE	[1] TRUE	[1] TRUE
[1] TRUE	[1] TRUE	[1] TRUE
[1] TRUE	[1] TRUE	[1] TRUE
[1] TRUE	[1] TRUE	[1] TRUE

(b) In this question, I will chose Nelder-Mead at R. and the code at attachment.

(c)

Starting Point	Lower Bounds	Upper Bounds	Squared Error	Calibrated
(1.5, 0.1, 0.25, -0.5, 0.1)	(0.001, 0, 0, -1, 0)	(5.0, 2.0, 2.0, 1, 1)	27.774	(4.67, 0.05, 1.64, -0.82, 0.04)
(0.2, 0.1, 0.5, 0.1, 0.5)	(0.001, 0, 0, -1, 0)	(5, 2, 2, 1, 1)	28.403	(4.99, 0.036, 1.96, -0.8, 0.042)
(0.2, 0.1, 0.5, 0.1, 0.5)	(0.001, 0, 0, -1, 0)	(2.5, 1, 1, 0.5, 0.5)	28.927	(1.76, 0.070, 1, -8.4, 0.034)

According to this table, the calibrated value will change with the starting point, lower bounds, upper bounds. But it is small. Therefore, we should chose an appropriate range for the parameters to implement the data calibration.

(d) By the table 2, we can find the option will have larger weights with larger liquidity.

Starting Points	Lower Bounds	Upper Bonds	Squared Error	Calibrated
(0.05, 0.2, 0.2, -0.6, 0.03)	(0.001, 0, 0, -1, 0)	(5, 2, 2, 1, 1)	244.037	(0.814, 0.101, 0.855, -0.754, 0.031)
(0.05, 0.2, 0.2, -0.6, 0.03)	(0.001, 0.0, -1, 0)	(2.5, 1, 2, 0.2, 0.5)	234.94	(1.35, 0.07, 0.86, -0.76, 0.03)
(0.25, 0.06, 0.8 -0.25, 0.09)	(0.001, 0, 0, -1, 0)	(5, 2, 2, 1, 1)	221.71	(3.52, 0.05, 1.18, -0.775, 0.034)

3. (a) (i) $\Delta_C = \frac{\partial C}{\partial S}$ $\Delta_H = \frac{C(S+dS) - C(S-dS)}{2dS}$

$\Delta_{BS} = e^{-q\tau} N(d_1)$ $d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\tau}$

by 2(d) when parameters are (3.52, 0.05, 1.18, -0.775, 0.034), the ones with least squared errors.

Let $dS = 0.01$

we can get $\Delta_H = 0.4766$

$\Delta_{BS} = 0.337355$

They are different. I that that this is because volatility will affect the delta, and Heston model is better than the Black-Scholes model. So the impact of volatility on the Heston model will be smaller.

(ii) We can ensure a portfolio that is delta neutral because one unit of the call will give a delta of Δ_C, we need to short Δ_C units of the underlying asset, which will give a delta of $-\Delta_C$.

(b) (i) vega can be defined as $Vega_C = \frac{\partial C}{\partial \sigma}$

$Vega_{BS} = e^{-q\tau} S_0 \sqrt{\tau} N'(d_1)$ where $d_1 = \frac{\ln(\frac{S}{K}) + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$

$(\kappa, \theta, \sigma, \rho, v_0) = (3.52, 0.05, 1.18, -0.775, 0.034)$

$dv = 0.5 v_0$

$$Vega_H = \frac{C(v_0 + dv, \theta + dv) - C(v_0 - dv, \theta - dv)}{2dv}$$

$= 133.13$

$$Vega_{BS} = 45.06$$

(ii) They are different because the negative calibrated value of the parameter ρ will affect the vega. The underlying asset and the volatility will have a negative relationship as the ρ is negative when we use Heston moder. So, the Heston model will have a smaller increasing than that in the Black-Scholes model when they have same increasing in volatility.