MF 796: Computational Methods in Mathematical Finance

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Problem set # 3

Due: Wednesday, February 12, by 8am

1. **Option Pricing via FFT Techniques** The Heston Model is defined by the following system of stochastic differential equations:

$$\begin{aligned} dS_t &= rS_t\,dt + \sqrt{\nu_t}S_t\,dW_t^1 \\ d\nu_t &= \kappa(\theta - \nu_t)\,dt + \sigma\sqrt{\nu_t}\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned}$$

The characteristic function for the Heston Model is known to be:

$$\omega(u) = \frac{\exp\left(iu\ln S_0 + iu(r-q)t + \frac{\kappa\theta t(\kappa - i\rho\sigma u)}{\sigma^2}\right)}{\left(\cosh\frac{\lambda t}{2} + \frac{\kappa - i\rho\sigma u}{\lambda}\sinh\frac{\lambda t}{2}\right)^{\frac{2\kappa\theta}{\sigma^2}}} \tag{1}$$

$$\Phi(u) = \omega(u)\exp\left(\frac{-(u^2+iu)\nu_0}{\lambda\coth\frac{\lambda t}{2} + \kappa - i\rho\sigma u}\right) \tag{2}$$

$$\lambda = \sqrt{\sigma^2(u^2+iu) + (\kappa - i\rho\sigma u)^2} \tag{3}$$

Assume the risk-free rate is 2%, the initial asset price is 250 and that the asset pays no dividends.

(a) **Exploring FFT Technique Parameters** Consider a European Call Option with strike 250 expiring in six months.
Additionally, assume you know that the parameters of the Heston Model are:

$$\begin{aligned} \sigma &= 0.2 \\ \nu_0 &= 0.08 \\ \kappa &= 0.7 \\ \rho &= -0.4 \\ \theta &= 0.1 \end{aligned}$$

i. Calculate the price of the European Call option with many values for the damping factor, α. What values of α seem to lead to the most stable price?

ii. Using the results above, choose a reasonable value of α and calculate the price of the same European Call with various values of N and Δk (or equivalently N and B). Comment on what values seem to lead to the most accurate prices, and the efficiency of each parameterization.

iii. Calculate the price of a European Call with strike 260 using various values of N and Δk (or N and B). Do the same sets of values for N, B and Δk produce the best results? Comment on any differences that arise.

(b) **Exploring Heston Parameters** Assume the risk-free rate is 2.5%, the initial asset price is 150 and that the asset pays no dividends.

$$\begin{aligned} \sigma &= 0.4 \\ \nu_0 &= 0.09 \\ \kappa &= 0.5 \\ \rho &= 0.25 \\ \theta &= 0.12 \end{aligned}$$

i. Using these parameters, calculate Heston Model prices for three-month options at a range of strikes and extract the implied volatilities for each strike. Plot the implied volatility $\sigma(K)$ as a function of strike.

ii. Use the FFT pricing technique to obtain prices of 150 strike calls at many expiries. Extract the implied volatility for each and plot the term structure of volatility by plotting time to expiry on the x-axis and implied volatility on the y-axis.

iii. Holding all other parameters constant, vary each of the model parameters and plot the updated volatility skews and term structures. Comment on the impact that each parameter has on the skew and term structure.

Problem 3

1. (a) i. By Heston Model:

$$x_j = \frac{[(2-\delta_{j-1})\Delta v]\, e^{-\int_0^T r_u du}}{2(\alpha+iv_j)(\alpha+iv_j+1)}\, e^{-i(\ln S_0 - \frac{\Delta k N}{2})v_j}\, \Phi(v_j-(\alpha+1)i)$$

$v_j = (j-1)\Delta v \qquad \Delta v = B/N$

$w_j = \frac{\Delta v}{2}(2-\delta_{j-1})$ {δ_{j-1} is an indictor function equal to 1 if $j=1$ and 0 otherwise}

$$\begin{cases} k_m = \beta + (m-1)\Delta k \\ \beta = \ln S_0 - \frac{\Delta k N}{2} \\ v_j = (j-1)\Delta v \\ \Delta v = B/N \\ \frac{2\pi}{N} = \Delta v \Delta k \end{cases}$$

and it is important to know

j can be a complex number

i can be a complex number

for Fast Fourier Transform $N = 2^n$

upper bound $B \Rightarrow$ (in here B is enough larger)

$\Rightarrow$ obtain an output vector y to get call price $B = 250 \times 5\sqrt{\Delta v}$

$$C_T(k_j) = \frac{e^{-\alpha[\ln S_0 - \Delta k(\frac{N}{2}-(j-1))]}}{\pi} Re(y_j)$$

In order to find α such that a stable price generates we can use python code to get the changing. The code at the attachment.

Figure 1: the price changing by α. $\alpha(0\sim5)$

Figure 2: the price changing by α $\alpha(30\sim40)$

It's is obvious that a stable condition exits, and after the condition here is a sharply increasing. We can use the python to get a table to find the changing point.

Alpha	0.5	0.8	1	1.5	30	35	40
price	21.27	21.27	21.27	21.27	21.27	21.27	2.37×10^{17}

Therefore, when $\alpha > 0.5$, the price will up to stable condition.

The price will increasing after $\alpha = 35$.

(ii) By the i, I wanna choice $\alpha = 1$

Meanwhile, when we calculate the price of the same European Call, the N and Δk also be important factors.

In order to find appropriate value, the N and B will be choice from two list. then choice a best combo with a highest efficiency.

The estimation of efficiency will apply the formular:

$$\frac{1}{MSE * RunTime}$$

(RunTime will be the run time of python code).

In this question, N will be choice from [7, 8, 9, 10, 11, 12, 13, 14]

B will be choice from [1.3×250, 1.5×250]

Then we can plot two figure to find which point is better for the calculation in price.

Figure 1: x: N Y: B z: Price

Figure 2: x: N y: B z: efficiency

By the Figure 1; I can find the price will be stable when $N \in [10, 14]$, and by the Figure 2; I find that when $N=10$ $B=375$, the efficiency is highest.

(iii) $N=10$ $B=375$ $\Rightarrow$ the price is. 16.74.
Meanwhile, when $N=8$ $B=375$, the efficiency is highest.

(b) (i) To the begin, I also choice the $\alpha = 1.5$
Use the similar method in (a)(i). I find $\alpha \in [1.5, 20]$ the option price is stable. so I use $\alpha = 1.5$ in here. Use $B = 405$ and K [70, 230]

Figure 3: Volatility $\sigma(K)$ as a function of strike.

(ii) Similarly $N=9$, $B=405$ T from 1 month to 2 years.

(iii) Only updated volatility

① change v_0 to 0.2

From two figure, I can find the volatility curves are shifted up, and the range of volatility increase and the relation between expiry T and volatility become negative.

② Change K to 1

The range of volatility appoximatly same and the term structure decrease.

③ change ρ to −0.2.

The volatility skew move right, the term structure curve moves down a little.

④ Change θ to 0.2

The term structure become increasing, the range of volatility increase.