HW Problems for Assignment 2 - Part 1
Due 6:30 PM Wednesday, October 7, 2020

1. VaR for a Portfolio of Apple and Amazon Stocks. The file "StockData.csv" contains prices for Apple and Amazon over the period 9/1/18 – 8/31/20. Column 1 is the date; Column 2 Amazon closing prices; and Columns 3 Apple closing prices. Additionally, as of 8/31/20, the market capitalizations of Apple and Amazon were 2,225.98 and 1,725.48 billion dollars respectively. Our goal is to compute the Value at Risk for a market capitalization weighted portfolio of these two stocks. The portfolio size is \$1,000,000 and the risk factor changes are the log returns X.

(a) **(5) Points** Estimate the mean vector and covariance matrix for the daily log returns using EWMA. Start with some initial value μ_0, Σ_0 (e.g. the sample mean and variance or 0) and then update via the EWMA formulas

N=502

$$\begin{cases} \mu_{t+\Delta} = \lambda\mu_t + (1-\lambda)X_t \\ \Sigma_{t+\Delta} = \theta\Sigma_t + (1-\theta)\underline{\overset{2X1}{(X_t-\mu_t)}\overset{1X2}{(X_t-\mu_t)^{\mathrm{T}}}}. \end{cases}$$

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This will give you an estimate for μ, Σ as of 8/31/20, which is then used over the next period 8/31/20 – 9/1/20. Use $\lambda = \theta = 0.97$.

(b) **(15 Points)** Estimate the VaR, for a 95% confidence, of the market cap weighted portfolio in the following ways.

(i) Using the empirical distribution of the log returns and the loss operators $l_{[t]}$, $l_{[t]}^{lin}$, $l_{[t]}^{quad}$.

(ii) Assuming the log returns are normally distributed with the estimated mean vector and covariance matrix, and using loss operators $l_{[t]}$, $l_{[t]}^{lin}$, $l_{[t]}^{quad}$. For the full and quadratic loss operators you will have to run a simulation sampling the normal random variables to obtain the VaR. For the linear loss operator you can obtain an exact VaR estimate using the methodology discussed in class. For the simulation, sample $N = 100,000$ normals.

$l^{lin}_{[t]}$

$l^{quad}_{[t]}$

How do the VaR estimates compare? Is any one (or more than one) estimate different from the rest? If so, please explain why you think this is the case.

2. (20 Points) VaR and Time Aggregation. In this exercise you will replicate the results in lecture, where we estimated the $K = 10$ day VaR for a hedged call option in the Black-Scholes model.

Recall, we have sold a call option with strike κ and maturity T, and are delta hedging in the underlying stock. The asset has dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where μ and $\sigma > 0$ are constant, and W is a Brownian motion under the physical measure $\mathbb{P}$. The risk free rate is the constant $r > 0$.

At the current time t our position in the stock is $h_t = C_x^{BS}(t, S_t)$, where C^{BS} is the Black-Scholes call price. We have no initial position in the money market. As such, the portfolio value is

$$V_t = h_t S_t + 0 - C^{BS}(t, S_t).$$

As h_t is constant over $[t, t+\Delta]$, at the end of the day the portfolio value is

$$V_{t+\Delta} = h_t S_{t+\Delta} + 0 \times e^{rt} - C^{BS}(t+\Delta, S_{t+\Delta}).$$

The one period loss is $L_{t+\Delta} = V_t - V_{t+\Delta}$, and we can think of is as a function of the log return $X_{t+\Delta} = \log(S_{t+\Delta}) - \log(S_t)$.

We then follow the rebalancing rule as described in class. Namely, over $[T_{k-1}, T_k]$, $T_k = t + k\Delta, k = 2, ..., K$ we

- We hold $h_{T_k} = C_S^{BS}(T_{k-1}, S_{T_{k-1}})$ shares of the stock.
- If $Y_{T_{k-1}}$ was the amount in the money market at the end of the previous period, the new amount at the beginning of the current period is $Y_{T_{k-1}}^{new} = Y_{T_{k-1}} + (h_{T_{k-1}} - h_{T_k})S_{T_{k-1}}$ which has end of period value $Y_{T_k} = Y_{T_{k-1}}^{new} e^{r\Delta}$.
- At T_k the portfolio value is

$$V_{T_k} = h_{T_k} S_{T_k} + Y_{T_k} - C^{BS}(T_k, S_{T_k}),$$

$$V_{T_0} = h_{T_0} S_{T_0} + Y_{T_0} - C^{BS}(T, S_0)$$

  and the loss is $L_k = V_{T_{k-1}} - V_{T_k}$.

The above methodology produces losses $\{L_k\}_{k=1}^K$ which are then aggregated into the total loss $L_{T_K} = \sum_{k=1}^K L_k$. We can sample this loss by sampling the log returns $\{X_{t+k\Delta}\}_{k=1}^K$.

$$Y_{T_k} = V_{T_k} - h_{T_k} S_{T_k} + C^{BS}(T_k, S_{T_k})$$

Using the above methodology, for a lot of 100 calls (i.e. multiply your losses by 100):

(1) Estimate the one day VaR over $[t, t+\Delta]$.
(2) Estimate the K day VaR over $[t, t+K\Delta]$.
(3) Compare the K day VaR with $\sqrt{K}$ times the one day VaR, to see how the square root of time rule holds up.

For parameters, use $\alpha = .95$, $t = 0$, $K = 10$, $\Delta = 1/252$, $T = .25$, $S_t = 158.12$, $\kappa = 170$, $r = 1.32\%$, $\sigma = 22.14\%$ and $\mu = 15.475\%$. For your simulations, use $25,000$ runs.

For error checking purposes, I obtained $\text{VaR}_{.95}^{10} \approx 37.65$ and a $\sqrt{10}\text{VaR}_{.95} \approx 42.77$.

3. (20 Points) Backtesting VaR. In this exercise, you will replicate the results given in lecture on the backtesting of VaR. Specifically, we will back-test two different methods for estimating VaR, for a portfolio which keeps a constant \$1 in the S&P 500 index throughout time.

The historical prices are given in the file

"SP_Prices.csv".

The first column contains the date, the second the closing price for that date. Data is from 8/31/07 - 8/31/17 sorted oldest to newest.

Using the above data set, produce a series of one-day VaR estimates (using the full loss operator) and exceedances using

(a) The empirical distribution method.

(b) The normal log returns method with EWMA updating.

Follow the lecture notes to implement the above. For both methods above use a four year rolling window, so the first VaR estimate will take place after the 1010^{th} oldest log return.

Notes:

(1) You will need an initial estimate for $\hat{\mu}, \hat{\sigma}$ to obtain the first VaR estimate. To obtain this, it is fine to take the sample mean and sample variance for the first 1010 data points.

Obtain the total number of exceedances for each method and compare them with what was obtained in class. Additionally, produce the $1-\beta$ confidence interval and see if the number of exceedances falls within this range. Use parameter values $\alpha = .95$, $\beta = .05$, and $\lambda = \theta = .97$ EWMA values, to fit the models.

1\. (a) By the calculation in Python code, we got.

$$\mu = [0.00479037,\ 0.00747457]$$

$$\Sigma = \begin{bmatrix} 0.00044944, & 0.00028451 \\ 0.00028451, & 0.00058059 \end{bmatrix}$$

(b)

VaR_emp_full	34098.951308
VaR_emp_lin	34697.393361
VaR_emp_quad	34091.930317
Simulation_VaR_full	26104.918261
Simulation_VaR_lin	26545.627059
Simulation_VaR_quad	26100.158187

According to the calculation, the difference between three operators is small for the empircal or the simulation by EWMA. However, there are large difference between empircal method and the simulation method. In my opinion, I think that the condition mainly depend on the real variance of stock price should larger than the simulation.

2\. Calculation by the Python code.

VaR_one_day	VaR_Kday	aggregation_VaR
13.334193	42.16642	37.765051

3. Calculation by the Python code.

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Mean: 75.40
Low confidence interval: 58.81
High confidence interval: 91.99
Empirical number of exceedences: 44.00
EWMA number of exceedences: 79.00
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