HW Problems for Assignment 4 - Part 2
Due 6:30 PM Wednesday, November 11, 2020

1. (30 Points) Approximating a Compound Poisson Random Variable. Let $S_N = \sum_{k=1}^{N} X_k$ where $N \sim \text{Poi}(\lambda)$ and $\{X_k\}$ are i.i.d. (and independent of N) log-normal $LN(\mu, \sigma^2)$ random variables. You will estimate the tail of the c.d.f. three ways: using the normal, translated gamma, and generalized Pareto approximations.

$N \sim \text{Poi}(\lambda)$

Mean λ

Variance λ

(a) For the normal approximation assume

$$S_N \sim E[S_N] + \sqrt{\text{Var}[S_N]}Z; \qquad Z \sim N(0,1).$$

To compute the mean and variance, use the formulas in class.

(b) For the translated gamma approximation assume

$$S_N \sim k + Y; \qquad Y \sim \text{Gamma}(\alpha, \beta).$$

Here, k, α, β are chosen to match the mean, variance, skewness of S_N.

(c) For the generalized Pareto approximation, used the methodology described on slide 19 of the lecture notes. However, take $u = \text{VaR}_{\alpha_0}$ associated to the sampled data empirical cdf so $F(u) = \alpha_0$. As in the slides, set M as the number of simulation runs.

For parameter values use

$$\lambda = 100; \qquad \mu = .10; \qquad \sigma = .4; \qquad M = 1,000,000; \qquad \alpha_0 = .99.$$

For your plot, produce a log-log plot of $1 - F_{S_N}(x)$ versus x with the three above approximations. For comparison purposes, sample M NEW (i.e. not from part (c) which was used to fit the GP distribution) copies of S_N and plot the tail of the empirical c.d.f. as well. For the range of x, choose the low value to be VaR_{α_0} associated to the empirical distribution. For the high value, take $\text{VaR}_{.99999}$ of the empirical distribution. Which approximation works best?

2. (20 Points) LVaR for Random Spreads. The Value at Risk adjustment LVaR, in both the constant and random spread setting, takes the regular VaR estimate and adds on the liquidity cost.

Unfortunately, when spreads are random, this does not accurately reflect the liquidity risk associated to holding the position, because it does not specify what might happen if we liquidate our position at the end of the period, at which point the bid-ask spread might have changed. In this exercise we will estimate LVaR via simulation in the random spread case using a theoretically more justified approach, and compare it to the simple approximation.

Assume at $t = 0$ (today) we own $\lambda > 0$ shares of S. The theoretical value of our portfolio is $V_0 = \lambda S_0$. We hold λ shares until 1, but then liquidate our holding. The value of our portfolio at 1 is thus $V_1 = \lambda S_1^b$, leading to losses

$$L_1 = -\lambda \left(S_1^b - S_0 \right).$$

To estimate the risk of our losses we make the following assumptions:

(i) The theoretical price is $S_1 = S_0 e^{X_1}$, $X_1 \sim N\left(\mu_1, \sigma_1^2\right)$.
(ii) At time 1 the proportional bid-ask spread $s_1 \sim N\left(\nu_1, \zeta_1\right)$ and is independent of X_1.

In this setting complete the following tasks:

(a) Explicitly write the losses L_1 in terms of λ, S_0, X_1, and s_1.
(b) Identify the industry approximation

$$\mathrm{LVaR}_\alpha^{ind} = \mathrm{VaR}_\alpha + \mathrm{LC},$$

where VaR_α is the Value at Risk associated to holding λ shares in a theoretical asset with price process S, and LC is the industry suggested liquidity cost $\mathrm{LC} = \frac{1}{2}\lambda S_0 \left(\nu_1 + k\zeta_1\right)$. Here, compute the exact value of VaR_α as we did in class.
(c) Estimate the risk-adjusted $\mathrm{LVaR}_\alpha^{sim}$ via simulation. I.e. for $m = 1, ..., M$, sample X_1^m, s_1^m, compute the losses $\ell_m = L_1^m$ and then output $\mathrm{LVaR}_\alpha^{sim} = \ell_{(\lceil M\alpha \rceil)}$.

For parameter values use $\lambda = 100$, $S_0 = 59$, $\mu_1 = 0$, $\sigma_1 = .4/\sqrt{252}$, $\nu_1 = .2\%$, $\zeta_1 = .08\%$, $k = 3$, $M = 1,000,000$, and $\alpha = .99$. For the simulation in part (c) output:

(1) The confidence α.
(2) An estimate of $\mathrm{LVaR}_\alpha^{sim}$ found via simulation.
(3) The theoretical VaR_α from part (b) - here you will have to compute this in the simulation.
(4) The estimated liquidity cost $\mathrm{LC}^{sim} = \mathrm{LVaR}_\alpha^{sim} - \mathrm{VaR}_\alpha$.
(5) The estimated percentage increase in the risk measure: $100\left(\frac{\mathrm{LVaR}_\alpha^{sim}}{\mathrm{VaR}_\alpha} - 1\right)$.
(6) The industry approximate $\mathrm{LVaR}_\alpha^{ind}$.
(7) The industry liquidity cost LC.
(8) The industry percentage increase in the risk measure: $100\left(\frac{\mathrm{LVaR}_\alpha^{ind}}{\mathrm{VaR}_\alpha} - 1\right)$.

How do the risk measures and liquidity costs compare?

1. $N \sim Poi(\lambda) \Rightarrow \mu = \lambda \quad var = \lambda$

$X \sim LN(\mu, \sigma^2) \Rightarrow$ mean $= \exp(\mu + \frac{\sigma^2}{2})$

$$Var = \exp(2\mu + \frac{\sigma^2}{2}) \cdot [\exp(\sigma^2) - 1]$$

$$S_N = \sum_{k=1}^{N} X_k$$

$$\Rightarrow \mu_{S_N} = E[X]\, E[N] = \lambda \cdot \exp(\mu + \frac{\sigma^2}{2})$$

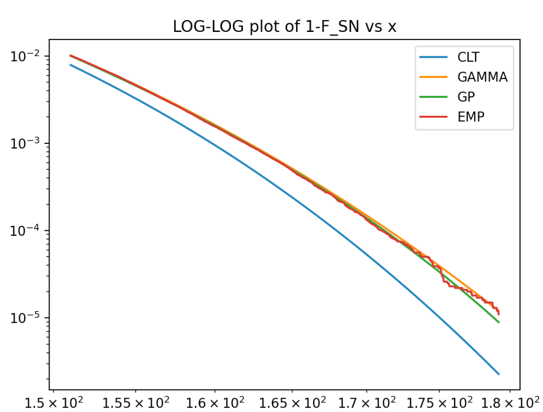
$$\sigma^2_{S_N} = \sigma_X^2 \mu_N + \mu_X^2 \sigma_N^2$$

$$\text{skewness} = \frac{E[X^3]}{\sqrt{\lambda E[X^2]}^3}$$

$$S_N \approx k + \text{Gamma}(\alpha, \beta)$$

$$\alpha = \frac{4}{\text{skewness}^2} \qquad \beta = \frac{\alpha}{\sqrt{\lambda E[X^2]}}$$

$$k = \lambda \cdot \mu_X - \frac{\alpha}{\beta}$$



According to the figure, the fit of GAMMA and GP is better.

2. (a) $$S_1^b = S_1(1-\tfrac{1}{2}s_1)$$

$$L_1 = -\lambda(S_1^b - S_0)$$

$$= -\lambda S_1(1-\tfrac{1}{2}s_1) + \lambda S_0$$

$$= -\lambda S_0\left(e^{X_1}(1-\tfrac{1}{2}s_1) - 1\right)$$

(b) The theoretical losses are

$$\hat{L}_1 = -\lambda S_0(e^{X_1} - 1)$$

$$X_1 \sim N(\mu_1, \sigma_1)$$

$$P[\hat{L}_1 \le \tau] = P\left[Z \ge \frac{1}{\sigma_1}\left(\log\left(1-\frac{\tau}{\lambda S_0}\right) - \mu_1\right)\right]$$

$$= 1 - N\left(\frac{1}{\sigma_1}\left(\log\left(1-\frac{\tau}{\lambda S_0}\right) - \mu_1\right)\right)$$

so $\tau = VaR_\alpha \Rightarrow VaR_\alpha = \lambda S_0\left(1 - e^{\mu_1 + \sigma_1 N^{-1}(1-\alpha)}\right)$

because $1-\alpha = N\left(\frac{1}{\sigma_1}\left(\log\left(1-\frac{\tau}{\lambda S_0}\right) - \mu_1\right)\right)$

$$N^{-1}(1-\alpha) = \frac{1}{\sigma_1}\left(\log\left(1-\frac{\tau}{\lambda S_0}\right) - \mu_1\right)$$

$$\Rightarrow \tau = \lambda S_0\left(1 - \exp(\mu_1 + \sigma_1 N^{-1}(1-\alpha))\right)$$

$$LVaR_\alpha^{ind} = VaR_\alpha + LC$$

$$= \lambda S_0\left(1 - \exp(\mu_1 + \sigma_1 N^{-1}(1-\alpha))\right) + \tfrac{1}{2}\lambda S_0(\mu_1 + k\xi_1)$$

(c)

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Confidence:      0.990
Simulated Liquidty VaR:     341.27
Theoretical VaR:     335.91
Simulated Liquidity Cost:       5.36
Simulated Percentage Liquidity VaR Increase:       1.60
Industry Approximate Liquidity VaR:     348.89
Industry Approximate Liquidity Cost:      12.98
Industry Approximate Percentage Liquidity VaR Increase:       3.86
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The industry approximation is more conservative, because the spreads and log returns are dependent in reality.