HW Problems for Assignment 2 - Part 2
Due 6:30 PM Tuesday, October 7th, 2020

1. (20 Points, 5 Points Each) Practice with ES. Explicitly compute $\mathrm{ES}_\alpha(L)$ assuming L has the following distributions/probability distribution functions (pdf).

(a) $L \sim \mathrm{Exp}(1/\theta)$ is exponentially distributed with mean θ.

(b) $L = \max\{L_1, L_2\}$ where L_1, L_2 are iid $\mathrm{Exp}(1/\theta)$ random variables. How does your answer compare to (a) as $\alpha \to 1$?

(c) If Y is a positive random variable with strictly positive pdf, and g is a strictly increasing function on $(0, \infty)$, show that $\mathrm{VaR}_\alpha(g(Y)) = g(\mathrm{VaR}_\alpha(X))$.

(d) Let $Y \sim \mathrm{Exp}(1/\theta)$ and $g(y) = y^2$. What is $\mathrm{ES}_\alpha(g(Y))$? Is $\mathrm{ES}_\alpha(g(Y)) = g(\mathrm{ES}_\alpha(Y))$? What about as $\alpha \to 1$?

2. (20 Points) Time aggregated risk measures for a constant weight portfolio of equities. In this exercise you will estimate various risk measures at a $K = 10$ day horizon for a portfolio of equities, assuming the portfolio weights are held constant over the K day horizon.

The portfolio consists of Boeing, McDonald's, Nike and Walmart stock. At the start (time $t = 9/1/2020$) the portfolio value is $V_t = \$1M$. The weights are determined using the time t market capitalizations from (in billions of dollars) of

$$\text{Boeing} : 97.39; \quad \text{McDonald's} : 158.20; \quad \text{Nike} : 179.01; \quad \text{Walmart} : 417.97.$$

Historical prices from $8/31/2015 - 8/31/2020$ are in the file "Prices.csv". The first column is the date (in Excel numeric format) while columns 2-5 give the stock price data. Data is sorted oldest to newest.

Our goal is to write a simulation to estimate the distribution of the K day losses $L_{t+K\Delta}$, and then use the distribution to estimate the risk measures. We work in the normal log-returns framework, using full losses and EWMA to estimate the conditional mean and covariance. However, as our time horizon is longer than 1 day, there are some subtleties when writing the simulation. Thus, complete the following steps:

(a) (**5 Points**) (pen and paper problem) As an abuse of notation, for a vector $x = (x^{(1)}, ..., x^{(d)}$ write e^x for the vector $(e^{x^{(1)}}, ..., e^{x^{(d)}})$. Show that for constant weights, the full loss over the K day horizon is

$$L_{t+K\Delta} = -V_t \left(\prod_{k=1}^{K} w^{\mathrm{T}} e^{X_{t+k\Delta}} - 1 \right).$$

(b) (**15 Points**) Write the simulation to obtain the time-aggregated risk measures, building into your simulation that (1) weights are held constant and (2) we use EWMA *each day* to obtain a new conditional mean and covariance estimate.

As for (2), use the historical data to obtain an estimate $\mu_{t+\Delta}, \Sigma_{t+\Delta}$ as of the initial time t. Since we have 5 years of data (about 1200 points) it is OK to start the EWMA procedure off with $\mu_{t_0} = 0 = \Sigma_{t_0}$.

Write $T_k = t+k\Delta$. From the historical data we have obtained μ_{T_1}, Σ_{T_1} and can sample $X_{T_1} \sim N(\mu_{T_1}, \Sigma_{T_1})$. Next, for $k = 2, ..., K$ write your simulation to include updating the mean and covariance estimates! Indeed, once we have sampled X_{T_1} we obtain our next estimate by setting

$$\mu_{T_2} = \lambda\mu_{T_1} + (1-\lambda)X_{T_1};$$

$$\Sigma_{T_2} = \lambda\Sigma_{T_1} + (1-\theta)(X_{T_1} - \mu_{T_1})\left(X_{T_1} - \mu_{T_1}\right)^{\mathrm{T}}.$$

We then sample $X_{T_2} \sim N(\mu_{T_2}, \Sigma_{T_2})$ and obtain μ_{T_3}, Σ_{T_3} by EWMA accordingly. We repeat this over the K day horizon.

With this methodology, write a simulation to estimate the K day VaR, ES as well as the spectral risk measure with exponential weighting function $\phi_\gamma(u) = \frac{\gamma}{e^\gamma - 1}e^{\gamma u}, 0 \le u \le 1$. To compare with square root of time, first do this for $K = 1$ to estimate the one day risk measures. Then do it for $K = 10$ for the ten day measures. Output the ten day measures as well as the square root of time approximations. For parameter values take $\alpha = .99$, $\gamma = 30$, $\lambda = .94$, $\theta = .97$. Have your simulations perform $N = 50,000$ trials.

For error checking purposes, I am obtaining around \$70,000, \$90,000 and \$78,000 for the VaR, ES and spectral risk measure respectively.