HW Problems for Assignment 4 - Part I
Due 6:30 PM Wednesday, November 11, 2020

$n \to \infty$

$\lim_{n\to\infty} F(c_n x + d_n)$

1. (10 Points, 5 points each) GEV for Discrete Distributions. Here, we will see that the limiting statement $\lim_{n\to\infty} F(c_n x + d_n)^n = H(x)$ for some non-trivial cdf H is a bit more restrictive than it might seem.

(a) Let X be a discrete random variable taking values in the finite set $\{1, 2, ..., K\}$ for some fixed integer K. X has pmf

$$p(k) = \mathbb{P}\left[X = k\right] > 0, k = 1, ..., K$$

and hence cdf

$$F(x) = \sum_{k=1}^{\lfloor x \rfloor} p(k), \qquad \lfloor x \rfloor = \max\left\{j \text{ an integer } \;\middle|\; j \leq x\right\}.$$

Show that there are NO $c_n > 0, d_n$ such that $F(c_n x + d_n)^n \to H(x)$ for a non-degenerate cdf H. Here, by "non-degenerate" we mean that H does NOT take the form $H(x) = 1_{x \geq x_0}$ for some $x_0 \in \mathbb{R}$, as this cdf corresponds the non-random constant x_0.

(b) Now, assume X has pmf

$$p(k) = \int_k^{k+1} \frac{1}{y^2} dy = \frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}, \qquad k = 1, 2, \ldots.$$

Show that there is $c_n > 0, d_n$ such that $F(c_n x + d_n)^n \to H(x)$ for a GEV H. Identify the parameters ξ, μ, σ for the GEV distribution.

(c) **Extra Credit: 10 points** Assume $X \sim \text{Geom}(p)$ for $p < 1$ so that

$$p(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots.$$

Are there $c_n > 0, d_n$ and non-degenerate H such that $F(c_n x + d_n)^n \to H(x)$?

2. (10 Points) GP Distributions for $U(0,1)$ random variables. Let $X \sim U(0,1)$. Find a function $\beta(u)$ so that

$$\lim_{u \uparrow 1} \sup_{0 \leq x \leq 1-u} \left|F_u(x) - G_{-1,\beta(u)}(x)\right| = 0.$$

3. (30 Points) Could we have predicted the 1987 crash using extreme value theory? The data file

"SP500_Log_Returns.csv"

contains daily log return data for the S%P 500 index from 6/10/1960 until 10/16/1987. Specifically, column 1 stores the date in numeric format (oldest to newest) and column 2 the log return.

As you may recall, 10/16/1987 was the day before the famous crash in the markets. In this exercise we will see if our extreme value theory-based risk measures could have predicted the crash.

Consider a hypothetical portfolio of $1,000,000 in the S%P 500 index. The losses are estimated via the linearized returns so that $L' = -VX$ where V is the portfolio value, and X is the log return. For each of the methods described below, determine the VaR_α, as a function of α, for the portfolio losses.

(1) Using the empirical distribution for the log returns.

(2) Assuming that as of $t = 10/16/1987$ the log returns $X_{t+\Delta}$ over the next business day are normally distributed with mean $\mu_{t+\Delta}$ and variance $\sigma^2_{t+\Delta}$. To estimate $\mu_{t+\Delta}, \sigma^2_{t+\Delta}$ use an EWMA procedure with parameter $\lambda = \theta = .97$ and a 500 day initialization time (as we have done before).

(3) Assuming a GEV distribution for the maximum and the block maximum method. For the N days of data ($N = 6875$), break the data into blocks of $n = 125$ days for a total of $m = 55$ blocks. Estimate the resultant GEV parameters: in matlab this is done using the 'gevfit' command.

(4) Assume a GP excess loss distribution. Here, take $u = \text{VaR}_{.95}$ as estimated in (1) above. To estimate the GP parameters, you will have to select only those losses above the u threshold, and compute the loss minus u for these selected days. With this data you can then fit the GP distribution. For example, in matlab this is the 'gpfit' command. With the fitted parameters, estimate the VaR_α in terms of α and $F(u)$, which in this case is .95 by construction.

For each of the four methods, produce a plot of $\alpha \mapsto \text{VaR}_\alpha$ for α between .99 and .9999 in increments of .000099 (100 values). For $\alpha = .9999$ what are the four VaR_α values? Which is highest?

The actual log return over Monday October 19th, 1987 was $X = -0.099452258$, so that our linearized losses would have been $L' = \$99,452$. Could any of the above methods predicted such a loss via the VaR_α? If so, for what α?

1. (a) Let x be discrete r.v. taking value in finite set $\{1,2,\dots,k\}$ for some fixed integer K.

Consider the sequence $\{C_n x + d_n\}_{n=1}^{\infty}$ $\quad F(C_n x + d_n) = \begin{cases} \text{some constant} <1 \ (\varepsilon) & \text{if } (C_n x + d_n) < K \\ 1 & \text{if } (C_n x + d_n) \geq K \end{cases}$

① when $N>0$ and $\forall n > N \Rightarrow C_n x + d_n > K \Rightarrow \lim\limits_{n\to\infty} F(C_n x + d_n)^n = \lim 1^n = 1 = H(x)$ ② when $N>0$ and $\forall n > N \Rightarrow C_n x + d_n < K \Rightarrow \lim\limits_{n\to\infty} F(C_n x + d_n)^n = \lim\limits_{n\to\infty} (1-\varepsilon)^n = 0 \quad (0<\varepsilon<1)$ ③ when $N>0$ and $p,q > N \Rightarrow C_p x + d_p > k$ and $C_p x + d_p < K \Rightarrow \begin{cases} \lim\limits_{n\to\infty} F(C_{n_i} x + d_{n_i}) = 1 & \{C_{n_i} x + d_{n_i}\}_{i=1}^{\infty} \text{ such } C_{n_i} x + d_{n_i} \geq K \\ \lim\limits_{n\to\infty} F(C_{n_j} x + d_{n_j}) = 0 & \{C_{n_j} x + d_{n_j}\}_{j=1}^{\infty} \text{ such } C_{n_j} x + d_{n_j} < K \end{cases}$

$\Rightarrow H(x)$ doesn't

If $H(x)$ exists for some x, C_n, $d_n \Rightarrow H(x) = 0$ or 1. $\Rightarrow H(x) = 1_{x\geq 0}$ if H is well defined. (right continuous)

Therefore, there are no $C_n > 0$, d_n such that $F(C_n x + d_n)^n \to H(x)$ for a non-degenerate cdf H.

(2) $\quad p(k) = \int_k^{k+1} \frac{1}{y^2}\,dy = \frac{1}{k(k+1)} \Rightarrow F(x) = \begin{cases} 1 - \frac{1}{\lfloor x \rfloor + 1} & \text{for } x>0 \\ 0 & \text{for } x \leq 0 \end{cases}$

① $H(x) = \lim\limits_{n\to\infty} F(C_n x + d_n)^n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{(C_n x + d_n)+1}\right)^n$ if we conside $C_n = n$, $d=-1 \Rightarrow C_n x + d_n > 0 \ \forall n > N$

Then $H(x) = \lim\limits_{n\to\infty} \left(\left(1 - \frac{1}{\lfloor nx \rfloor}\right)^{\lfloor nx \rfloor}\right)^{n/\lfloor nx \rfloor} = \lim\limits_{n\to\infty} e^{-n/\lfloor nx \rfloor} = e^{-\frac{1}{x}}$

② $x \leq 0 \quad \forall n > 0 \quad C_n x + d_n < 0$
$H(x) = \lim\limits_{n\to\infty} (0)^n = 0$ $\quad \Rightarrow H(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x>0 \\ 0 & \text{if } x \leq 0 \end{cases}$

Let $\xi = \mu = \sigma = 1 \quad H(x) = H_{\xi,\mu,\sigma}(x) = H_\xi\left(\frac{x-1}{1}\right) = e^{-\frac{1}{x}}$

(c) $X \sim Geom(p)$ $p<1$ $p(k) = (1-p)^{k-1}p$

$$F(x) = \begin{cases} 1-(1-p)^x & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$H(x) = \lim_{n\to\infty} F(c_n x + d_n)^n = \lim_{n\to\infty} (1-(1-p)^{(c_n + d_n)})^n$$

$$= \lim_{n\to\infty} (1 - \frac{1}{n}(1-p)^{(c_n x + d_n) - \log_{1-p}\frac{1}{n}})^n$$

$$= \exp(-(1-p)^a) \quad a = \lim_{n\to\infty} (c_n x + d_n) - \log_{1-p}\frac{1}{n}$$

when $a = \infty$ $H(x) = 1$ $\Rightarrow$ if $\exists$ $c_n > 0$, d_n such a exits

$a = -\infty$ $H(x) = 0$ Then we can prove if H is degenerate.

by contradiction $\Rightarrow$ $\nexists$ $c_n > 0$ and d_n such a exists.

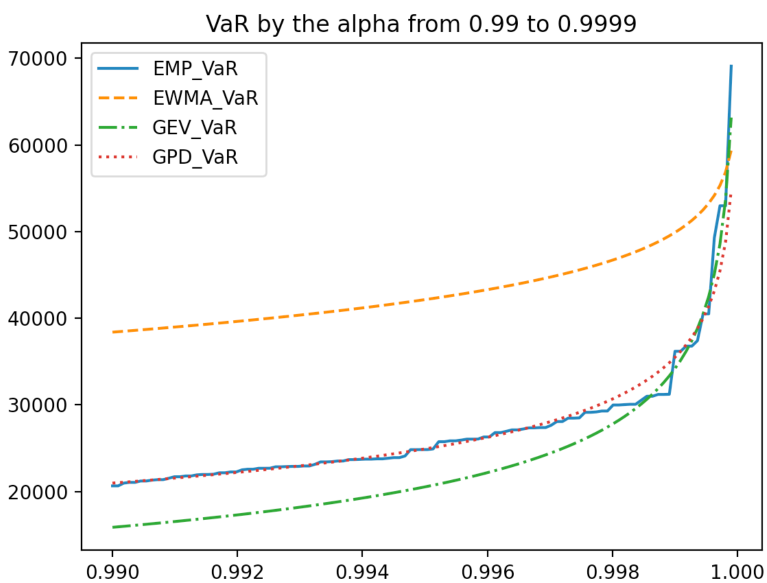
$\Rightarrow$ there is no $c_n > 0$, d_n $\Rightarrow$ $F(c_n x + d_n)^n \to H(x)$ for non-degenerate cdf H.

2. $0<u<1 \qquad 0<x\le 1-u$

$$F_u(x) = P[X\le x+u \mid X>u] = \frac{P(u<X\le x+u)}{P(X>u)} = \frac{x+u-u}{1-u} = \frac{x}{1-u}$$

$$G_{-1,\beta(u)}(X) = \frac{x}{\beta(u)} \Rightarrow \beta(u) = \frac{1}{1-u}$$

3.

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Crash Loss: 99452.26
Empirical VaR see Crash? No
Closest Empirical VaR alpha: 0.999900
Closest Empirical VaR:   69088.98
EWMA VaR see Crash? No
Closest EWMA VaR alpha: 0.999900
Closest EWMA VaR:   59361.88
GEV VaR see Crash? No
Closest GEV VaR alpha: 0.999900
Closest GEV VaR:   63158.41
GPD VaR see Crash? No
Closest GPD VaR alpha: 0.999900
Closest GPD VaR:   54751.99
alpha = 0.9999 Empirical VaR:   69088.98
alpha = 0.9999 EWMA VaR:   59361.88
alpha = 0.9999 GEV VaR:   63158.41
alpha = 0.9999 GPD VaR:   54751.99
The Highest VaR is generated by the empirical method,
and the loss when X = -0.99452258 can not be predicted by these methods
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