HW Problems for Assignment 2 - Part 2
Due 6:30 PM Tuesday, October 7th, 2020

1. (20 Points, 5 Points Each) Practice with ES. Explicitly compute $\mathrm{ES}_\alpha(L)$ assuming L has the following distributions/probability distribution functions (pdf).

(a) $L \sim \mathrm{Exp}(1/\theta)$ is exponentially distributed with mean θ.

(b) $L = \max\{L_1, L_2\}$ where L_1, L_2 are iid $\mathrm{Exp}(1/\theta)$ random variables. How does your answer compare to (a) as $\alpha \to 1$?

(c) If Y is a positive random variable with strictly positive pdf, and g is a strictly increasing function on $(0, \infty)$, show that $\mathrm{VaR}_\alpha(g(Y)) = g(\mathrm{VaR}_\alpha(X))$.

(d) Let $Y \sim \mathrm{Exp}(1/\theta)$ and $g(y) = y^2$. What is $\mathrm{ES}_\alpha(g(Y))$? Is $\mathrm{ES}_\alpha(g(Y)) = g(\mathrm{ES}_\alpha(Y))$? What about as $\alpha \to 1$?

2. (20 Points) Time aggregated risk measures for a constant weight portfolio of equities. In this exercise you will estimate various risk measures at a $K = 10$ day horizon for a portfolio of equities, assuming the portfolio weights are held constant over the K day horizon.

The portfolio consists of Boeing, McDonald's, Nike and Walmart stock. At the start (time $t = 9/1/2020$) the portfolio value is $V_t = \$1M$. The weights are determined using the time t market capitalizations from (in billions of dollars) of

Boeing : 97.39; McDonald's : 158.20; Nike : 179.01; Walmart : 417.97.

Historical prices from $8/31/2015 - 8/31/2020$ are in the file "Prices.csv". The first column is the date (in Excel numeric format) while columns 2-5 give the stock price data. Data is sorted oldest to newest.

Our goal is to write a simulation to estimate the distribution of the K day losses $L_{t+K\Delta}$, and then use the distribution to estimate the risk measures. We work in the normal log-returns framework, using full losses and EWMA to estimate the conditional mean and covariance. However, as our time horizon is longer than 1 day, there are some subtleties when writing the simulation. Thus, complete the following steps:

(a) (**5 Points**) (pen and paper problem) As an abuse of notation, for a vector $x = (x^{(1)}, ..., x^{(d)}$ write e^x for the vector $(e^{x^{(1)}}, ..., e^{x^{(d)}})$. Show that for constant weights, the full loss over the K day horizon is

$$L_{t+K\Delta} = -V_t \left(\prod_{k=1}^{K} w^{\mathrm{T}} e^{X_{t+k\Delta}} - 1 \right).$$

(b) (**15 Points**) Write the simulation to obtain the time-aggregated risk measures, building into your simulation that (1) weights are held constant and (2) we use EWMA *each day* to obtain a new conditional mean and covariance estimate.

As for (2), use the historical data to obtain an estimate $\mu_{t+\Delta}, \Sigma_{t+\Delta}$ as of the initial time t. Since we have 5 years of data (about 1200 points) it is OK to start the EWMA procedure off with $\mu_{t_0} = 0 = \Sigma_{t_0}$.

Write $T_k = t+k\Delta$. From the historical data we have obtained μ_{T_1}, Σ_{T_1} and can sample $X_{T_1} \sim N(\mu_{T_1}, \Sigma_{T_1})$. Next, for $k = 2, ..., K$ write your simulation to include updating the mean and covariance estimates! Indeed, once we have sampled X_{T_1} we obtain our next estimate by setting

$$\mu_{T_2} = \lambda\mu_{T_1} + (1-\lambda)X_{T_1};$$
$$\Sigma_{T_2} = \lambda\Sigma_{T_1} + (1-\theta)(X_{T_1} - \mu_{T_1})\left(X_{T_1} - \mu_{T_1}\right)^{\mathrm{T}}.$$

We then sample $X_{T_2} \sim N(\mu_{T_2}, \Sigma_{T_2})$ and obtain μ_{T_3}, Σ_{T_3} by EWMA accordingly. We repeat this over the K day horizon.

With this methodology, write a simulation to estimate the K day VaR, ES as well as the spectral risk measure with exponential weighting function $\phi_\gamma(u) = \frac{\gamma}{e^\gamma - 1}e^{\gamma u}, 0 \le u \le 1$. To compare with square root of time, first do this for $K = 1$ to estimate the one day risk measures. Then do it for $K = 10$ for the ten day measures. Output the ten day measures as well as the square root of time approximations. For parameter values take $\alpha = .99$, $\gamma = 30$, $\lambda = .94$, $\theta = .97$. Have your simulations perform $N = 50,000$ trials.

For error checking purposes, I am obtaining around \$70,000, \$90,000 and \$78,000 for the VaR, ES and spectral risk measure respectively.

1. (a) $L \sim Exp(1/\theta)$

$$f(l) = \begin{cases} \frac{1}{\theta} e^{-l/\theta} & l \geq 0 \\ 0 & l < 0 \end{cases}$$

$$\Rightarrow F(l) = \begin{cases} 1 - e^{-l/\theta} & l \geq 0 \\ 0 & l < 0 \end{cases}$$

$$P[L \leq \tau] = F(\tau) = \int_0^\tau f(l)\, dl = 1 - e^{-\tau/\theta}$$

$$\alpha = 1 - e^{-\tau/\theta} \quad 1-\alpha = e^{-\tau/\theta} \quad -\tau/\theta = \ln(1-\alpha) \Rightarrow \tau = -\theta \ln(1-\alpha)$$

$$VaR_\alpha(L) = -\theta \ln(1-\alpha)$$

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 -\theta \ln(1-u)\, du$$

$$= \frac{-\theta}{1-\alpha} \int_\alpha^1 \ln(1-u)\, du$$

$$= \frac{-\theta}{(1-\alpha)} \int_0^{1-\alpha} \ln x\, dx$$

$$= \frac{-\theta}{(1-\alpha)} \left[x\ln x - \int x\, d\ln x \right] \Big|_0^{1-\alpha}$$

$$= \frac{-\theta}{(1-\alpha)} \left[x\ln x - x \right] \Big|_0^{1-\alpha}$$

$$= \frac{-\theta}{(1-\alpha)} \left[(1-\alpha)\ln(1-\alpha) - (1-\alpha) \right]$$

$$= -\theta \ln(1-\alpha) + \theta$$

(2) $L = \max\{L_1, L_2\}$ L_1, L_2 are $Exp(1/\theta)$ (iid)

$$F_L = P\{L < l\} = P\{L_1 < l, L_2 < l\} = P\{L_1 < l\} \cdot P\{L_2 < l\} = P^2\{L < l\} = F^2(l)$$

$$\Rightarrow F_L(l) = \begin{cases} (1 - e^{-l/\theta})^2 & l \geq 0 \\ 0 & l < 0 \end{cases}$$

$$\alpha = (1 - e^{-l/\theta})^2 \Rightarrow VaR_\alpha(L) = -\theta \ln(1 - \sqrt{\alpha})$$

$$ES_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L)\, du = \frac{-\theta}{1-\alpha} \int_\alpha^1 \ln(1 - \sqrt{u})\, du$$

$$= \frac{-\theta}{1-\alpha} \int_{1-\sqrt{\alpha}}^{0} \ln x \, d(1-x)^2 = \frac{-\theta}{1-\alpha} \int_0^{1-\sqrt{\alpha}} \ln x \cdot 2 \cdot (1-x)\, dx$$

$$= \frac{-2\theta}{1-\alpha} \int_0^{1-\sqrt{\alpha}} \ln x \, dx + \frac{2\theta}{1-\alpha} \int_0^{1-\sqrt{\alpha}} x \ln x \, dx$$

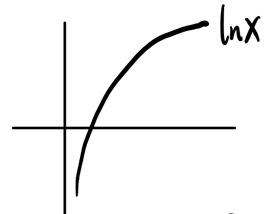
① $\int_0^{1-\sqrt{\alpha}} \ln x \, dx = (1-\sqrt{\alpha})\ln(1-\sqrt{\alpha}) - (1-\sqrt{\alpha})$

② $\int_0^{1-\sqrt{\alpha}} x \ln x \, dx = \left[\frac{1}{2}x^2 \ln(x) - \frac{x^2}{4}\right]\Big|_0^{1-\sqrt{\alpha}} = \frac{1}{2}\ln(-\sqrt{\alpha}+1) + \frac{1}{2}\alpha \ln(1-\sqrt{\alpha}) - \sqrt{\alpha}\ln(-\sqrt{\alpha}+1) - \frac{(\sqrt{\alpha}+1)^2}{4}$

By ① and ② $\Rightarrow$ $ES_\alpha(L) = -\frac{\theta[(1-2\ln(1-\sqrt{\alpha}))\alpha + 2\sqrt{\alpha} + 2\ln(1-\sqrt{\alpha}) - 3]}{2(1-\alpha)}$

According to (a), when $\alpha \to 1$ $ES_\alpha(L) \to \infty$

Similarly, $\alpha \to 1$ in the (b), $ES_\alpha(L)$ also go to ∞.



(3) $\alpha = P\{g(Y) \leq VaR_\alpha(g(Y))\} = P\{Y \leq g^{-1}(VaR_\alpha(g(Y)))\}$

$$\Rightarrow VaR_\alpha(Y) = g^{-1}(VaR_\alpha(g(Y)))$$

$$\Rightarrow g(VaR_\alpha(Y)) = VaR_\alpha(g(Y))$$

(d) According to (c), $VaR_\alpha(g(Y)) = g(VaR_\alpha(X))$

$$\Rightarrow g(VaR_\alpha(X)) = [-\theta\ln(1-\alpha)]^2 = \theta^2[\ln(1-\alpha)]^2$$

$$\Rightarrow ES_\alpha(g(Y)) = \frac{1}{1-\alpha}\int_\alpha^1 \theta^2[\ln(1-u)]^2\,du$$

$$= \frac{\theta^2}{1-\alpha}\cdot\int_0^{1-\alpha}[\ln x]^2\,dx$$

$$= \frac{\theta^2}{1-\alpha}\left[x[\ln x]^2 - 2x\ln x + 2x\right]\Big|_0^{1-\alpha}$$

$$= \frac{\theta^2}{1-\alpha}\left[(1-\alpha)[\ln(1-\alpha)]^2 - 2(1-\alpha)\ln(1-\alpha) + 2(1-\alpha)\right]$$

$$= \theta^2\left[[\ln(1-\alpha)]^2 - 2\ln(1-\alpha) + 2\right] = \theta^2\left[[\ln(1-\alpha)]^2 - 2\ln(1-\alpha) + 1\right] + \theta^2$$

$g(ES_\alpha(Y))$

$$= \theta^2[1-\ln(1-\alpha)]^2$$

$$= \theta^2[1 + [\ln(1-\alpha)]^2 - 2\ln(1-\alpha)]$$

$$\Rightarrow g(ES_\alpha(Y)) = ES_\alpha(g(Y)) - \theta^2$$

So, it is not same.

when $\alpha \to 1$, $\ln(1-\alpha) \to \infty$, the $ES_\alpha(g(Y))$ and $g(ES_\alpha(Y))$ will go to infinity.

2.(a) $$L_{t+k\Delta} = -V_t \sum_{i=1}^{d} w^{(i)} \left(e^{\sum_{k=1}^{K} X_{T_k}^{(i)}} - 1 \right)$$

$$= -V_t \left(\sum_{i=1}^{d} w^{(i)} e^{\sum_{k=1}^{K} X_{T_k}^{(i)}} - 1 \right)$$

$$= -V_t \left(w_1 \cdot e^{\sum_{k=1}^{K} X_{T_k}^{1}} + w_2 e^{\sum_{k=1}^{K} X_{T_k}^{2}} + \cdots + w_d e^{\sum_{k=1}^{K} X_{T_k}^{d}} - 1 \right)$$

$$= -V_t \left(w^T \cdot e^{\sum_{k=1}^{K} X_{T_k}} - 1 \right)$$

$$= -V_t \left(\prod_{k=1}^{K} w^T e^{X_{t+k\Delta}} - 1 \right)$$

$$e^{\sum_{k=1}^{K} X_{T_k}^{1}}$$

$$= e^{X_{t+\Delta}^{1} + X_{t+2\Delta}^{1} + \cdots X_{t+k\Delta}^{1}}$$

$$= \prod_{k=1}^{K} e^{X_{t+k\Delta}^{1}}$$

(b) The Calculation in the Python code.

```
K day VaR: 94704.97
sqrt(K) times one day VaR: 90897.60
K day ES:: 112831.38
sqrt(K) times one day ES: 105238.29
K day Spectral: 78428.17
sqrt(K) times one day Spectral: 77964.83
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