HW Problems for Assignment 3 - Part I
Due 6:30 PM Wednesday, October 21, 2020

1. (20 Points) Component risk measures for an equity portfolio. In this exercise, we will compute component risk measures for an equally weighted portfolio of five stocks: Walmart, Target, Costco, Citigroup and JP Morgan. Closing price data is in the file "Five_Stock_Prices.csv". In the file, the first column is the date (in Excel numeric format) while columns 2-6 give the stock price data. Data is sorted oldest to newest.

The hypothetical portfolio is fixed at 15 million dollars, and allocates 20% in each stock. As such the respective dollar positions are kept constant throughout time at $3 million in each stock.

We use linearized losses and normally distributed log returns, with EWMA updating. As before, we use the oldest M returns to obtain an estimate for μ, Σ, and then update there-after according to the EWMA procedure.

In this setting, compute on a daily basis throughout time

(1) The percent component value at risk.
(2) The percent component expected shortfall.
(3) The percent contribution to the loss variance.

Begin your calculations after the first M periods so, with N log returns, you will have $N + 1 - M$ component risk measure estimates.

Note that for each day, the above quantities are vectors with five components (one for each stock). Output a time evolution plot for each of the above values, over the range $t - (N - M)\Delta, \ldots, t$. Each plot will have five graphs.

In addition to the above portfolio value and weights, use $M = 50$ trail days, a confidence of $\alpha = .99$ and EWMA parameters of $\lambda = .94$, $\theta = .96$.

As we discussed in class, the component risk measure plots should be very close to the variance contribution plot. Is this the case?

2. Spherical and elliptical random variables. The next two exercises show that many of the conclusions on component risk measures, and risk-measure based optimal investment, extend beyond the Gaussian setting.

We say a random vector $Z \in \mathbb{R}^d$ is *spherical* if for all $a \in \mathbb{R}^d$ the random variables $a^{\mathrm{T}}Z$ and $|a|Z^{(1)}$ have the same distribution, where $Z^{(1)}$ is the first component of Z. In other words $\mathbb{P}\left[a^{\mathrm{T}}Z \leq \tau\right] = \mathbb{P}\left[|a|Z^{(1)} \leq \tau\right]$ for all $\tau \in \mathbb{R}$. For more information on spherical random variables, see Chapter 6.3 of the class textbook.

We say a random vector $X \in \mathbb{R}^d$ is *elliptical* if $X = \mu + AZ$ where $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$ and Z is spherical. Lastly, denote by $\Sigma = AA^{\mathrm{T}}$.

(a) (**5 Points**) Assume $Z = \sqrt{W} \times \widetilde{Z}$ where $\widetilde{Z} \sim N(0, 1_d)$, and $W \geq 0$ is a random scalar independent of $\widetilde{Z}$. Show that Z is spherical. In particular, taking $W \equiv 1$ shows that $\widetilde{Z}$ is spherical.

(b) Assume a one period model with d stocks, whose log returns X are elliptically distributed. Write θ as the vector of dollar positions.

  (i) (**5 Points**) Show that the linearized losses L^{lin} have the same distribution as

$$-\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times Z^{(1)},$$

  and hence for any cash-additive, positively homogenous risk measure ϱ

$$\varrho(L^{lin}) = -\theta^{\mathrm{T}}\mu + \sqrt{\theta^{\mathrm{T}}\Sigma\theta} \times \varrho(Z^{(1)}).$$

  (ii) (**5 Points**) Argue why part (i) means we can make the same conclusions connecting $\mathcal{R}'_{\mathrm{C},\%}$ to the percentage contribution to variance, as we did when $X \sim N(\mu, \Sigma)$.

3. Markowitz with elliptic returns. Here, we will reproduce the risk-measure based "Markowitz" results when the log returns are elliptically distributed. As we will see, the answers are essentially identical to the case of normal log returns.

Assume ϱ is a cash-additive, positively homogenous, risk measure; the bond has a known log return of $r \geq 0$, and the stocks' log returns X are elliptically distributed, with decomposition $X = r\mathbf{1} + \mu + AZ$ with $\Sigma = AA^{\mathrm{T}}$ and Z is spherical. Here, as in lecture, μ is now the excess return (you DO NOT have to do this, but one can show that Z has 0 mean).

We use portfolio weights in the stocks as the decision variable and write R^{lin} as the linearized portfolio return, which takes the value

$$R^{'^{lin}} = r + w^{\mathrm{T}}(X - r\mathbf{1}).$$

(a) (**5 Points**) Show that

$$\varrho(L^{lin}) = -V(r + w^{\mathrm{T}}\mu) + V\sqrt{w^{\mathrm{T}}\Sigma w}\,\varrho(Z^{(1)}).$$

(b) Given that the mean of R' is $r + w^{\mathrm{T}}\mu$, in the risk-measure constrained optimization problem, the goal is to compute

$$\mathcal{U} := \max_{w \in \mathbb{R}^d} \left\{ r + w^{\mathrm{T}}\mu \;\middle|\; -V(r + w^{\mathrm{T}}\mu) + V\sqrt{w^{\mathrm{T}}\Sigma w}\,\varrho(Z^{(1)}) \leq \overline{\varrho} \right\}.$$

  (i) (**5 Points**)Assume $\varrho(Z^{(1)}) \leq \sqrt{\mu'\Sigma^{-1}\mu}$. By considering strategies of the form $w_k = k \times \Sigma^{-1}\mu$ for $k > 0$, show that $\mathcal{U} = \infty$.

(ii) (**5 Points**) Assume $\varrho(Z^{(1)}) > \sqrt{\mu'\Sigma^{-1}\mu}$. Using Lagrange multipliers show that the optimal strategy is

$$\hat{w}_{\overline{\varrho}} = \hat{k} \times \Sigma^{-1}\mu,$$

for a certain constant $\hat{k}$ which you must explicitly find.

Hint: First, fix a portfolio standard deviation level $\vartheta > 0$, and a portfolio risk measure level $m \leq \overline{\varrho}$. Using Lagrange multipliers find the optimal return for the fixed levels. Next, show the optimal return is increasing with m, and hence it is ok to assume $\varrho(L') = \overline{\varrho}$.