HW Problems for Assignment 1 - Lecture 2
Due 6:30 PM Wednesday, September 23, 2020

1. Loss Distributions for a Hedged Call Option. As in the Black-Scholes model, assume the stock price has dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

where $W = \{W_t\}_{t\leq T}$ is a Brownian motion under the physical measure $\mathbb{P}$. The interest rate is $r > 0$. Let T be the maturity and K the strike of a call option, and set $C^{BS}(t,x)$ as the price of the call given $S_t = x$. I.e.

$$C^{BS}(t,x) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\left(S_T - K\right)^+ \big| S_t = x\right]. \tag{0.1}$$

where $\mathbb{Q}$ is the risk neutral measure under which S has drift μ. The famous Black-Scholes formula states (you DO NOT have to prove this)

$$C^{BS}(t,x) = xN(d_1(T-t,x)) - Ke^{-r(T-t)}N(d_2(T-t,x)),$$

where N is the standard normal cdf and

$$\begin{aligned} d_1(\tau,x) &= \frac{1}{\sigma\sqrt{\tau}}\left(\log\left(\frac{x}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(\tau)\right), \\ d_2(\tau,x) &= d_1 - \sigma\sqrt{\tau}. \end{aligned}$$

Furthermore, with ϕ denoting the standard normal pdf we have

$$\begin{aligned} \delta(t,x) &= \partial_x C^{BS}(t,x) = N(d_1), \\ \gamma(t,x) &= \partial_{xx} C^{BS}(t,x) = \frac{\phi(d_1(T-t,x))}{x\sigma\sqrt{T-t}}, \\ \theta(t,x) &= \partial_t C^{BS}(t,x) = -\frac{\sigma}{2\sqrt{T-t}}x\phi(d_1(T-t,x)) \\ &\qquad - Kre^{-r(T-t)}N(d_2(T-t,x)). \end{aligned}$$

These are the "delta", "gamma" and "theta" respectively for the option.

As in class, at time t we are short M call options and long $M\delta(t,S_t)$ shares of S. Over the period $[t,t+\Delta]$ we hold the share position constant, writing $\lambda = \delta(t,S_t)$ to reinforce this fact. With this notation, the values of our portfolio at t and $t+\Delta$ are

$$\begin{aligned} V_t &= M\left(\lambda S_t - C^{BS}(t,S_t)\right), \\ V_{t+\Delta} &= M\left(\lambda S_{t+\Delta} - C^{BS}(t+\Delta,S_{t+\Delta})\right). \end{aligned}$$

(a) **(15 Points)** With $z_t = \ln(S_t)$ identify the full, linearized, and second order loss operators over $[t,t+\Delta]$ as a function of the log return $x = X_{t+\Delta}$. **Notes:**

(i) Make sure to fully evaluate the linearized loss operator - there is a cool answer!.

(ii) For the second order loss operator, only include the second derivative with respect to x: i.e. ignore the second order time derivative and second order time-space derivative.

(b) **(15 Points)** Write a simulation which identifies the loss distribution for the portfolio using the full, linearized and second order (with the adjustments in note (ii)) loss operators. As in class, produce a histogram approximation of the probability density functions. How well do the approximations work?

For parameters use $\mu = 0.15475$, $\sigma = 0.2214$, $r = 0.0132$, $t = 0$, $T = .25$, $\Delta = 10/252$ (ten day horizon), $S_0 = 158.12$, $K = 170$ and $M = 100$ options. Run $N = 100,000$ trials.

2. Practice with VaR. Explicitly compute $\mathrm{VaR}_\alpha(L)$ assuming L has the following distributions/probability distribution functions (pdfs).

(a) **(7 Points)** L is a "double-sided" exponential with threshold l_0 in that L has pdf

$$f(l) = \frac{ab}{ae^{-bl_0} + be^{al_0}} \left(e^{al} 1_{l \leq l_0} + e^{-bl} 1_{l > l_0} \right); \qquad l \in \mathbb{R},$$

where $a, b > 0$. Here, you may assume $\alpha \geq b/(b + ae^{-(a+b)l_0})$.

(b) **(6 Points)** L is a binomial random variable with n number of trials and p probability of success on any given trial. Give an explicit answer when $n = 6$, $p = 1/2$ and $\alpha = .9$.

(c) **(7 Points)** L is an "exponential with exponential mean" random variable. Here, we first sample Y which is exponentially distributed with mean $1/\theta$. Then, given $Y = y$ we sample L off an exponential distribution with mean $1/y$.

$$\mathrm{VaR}_\alpha := \inf [l \in \mathbb{R} \mid \mathbb{P}[L > l] \leq 1-\alpha]$$

1. (a) $V_t = M(\lambda S_t - C^{BS}(t, S_t)) = M(\lambda e^{Z_t} - C^{BS}(t, e^{Z_t}))$

$V_{t+\Delta} = M(\lambda S_{t+\Delta} - C^{BS}(t+\Delta, S_{t+\Delta})) = M(\lambda e^{Z_{t+\Delta}} - C^{BS}(t+\Delta, e^{Z_{t+\Delta}}))$

$X_{t+\Delta} = \frac{\ln S_{t+\Delta}}{\ln S_t} = Z_{t+\Delta} - Z_t$

Full loss : $L_{t+\Delta} = -(V_{t+\Delta} - V_t)$

$= -[M(\lambda(S_{t+\Delta} - S_t) - (C^{BS}(t+\Delta, S_{t+\Delta}) - C^{BS}(t, S_t))]$

$= -[M(\lambda S_t(e^x - 1) - (C^{BS}(t+\Delta, S_{t+\Delta}) - C^{BS}(t, S_t))]$

$= -[M(\lambda S_t(e^x - 1) - (C^{BS}(t+\Delta, S_t e^x) - C^{BS}(t, S_t)))]$

Linearized loss :

$\partial_t V(t, Z_t)\Delta = M(0 - \theta)\Delta = -M\theta\Delta$

$\frac{\partial V(t, Z_t)}{\partial Z_t} = e^{Z_t}\lambda$ $\quad \partial_{Z_t} V(t, Z_t) X_{t+\Delta} = M(\lambda e^{Z_t} - \lambda e^{Z_t}) X_{t+\Delta} = 0$

$\Rightarrow L^{lin}_{t+\Delta} = -(-M\theta\Delta + 0) = M\theta\Delta$

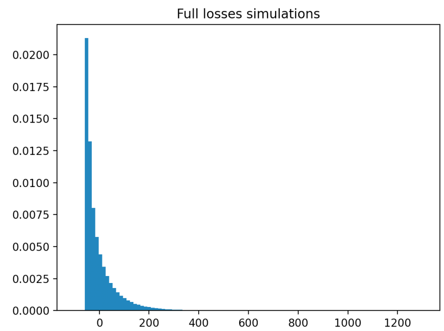
Quadratic loss: $\frac{\partial V}{\partial t}\Delta + \frac{\partial V}{\partial Z} \cdot X + \frac{1}{2}\frac{\partial^2 V}{\partial Z^2} \cdot X^2$

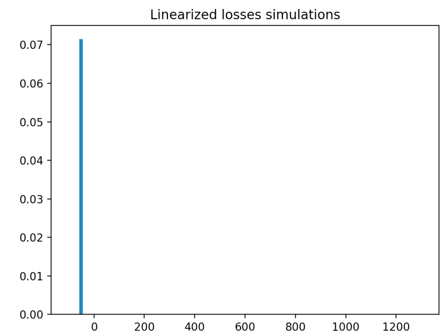
$\frac{\partial_{Z_t} V}{\partial Z_t} = 0 \quad \frac{\partial^2_{Z_t} V}{\partial Z_t} = M(\lambda e^{Z_t} - e^{Z_t}\frac{\partial C}{\partial e^{Z_t}} - e^{2Z_t}\frac{\partial^2 C}{\partial e^{Z_t^2}}) = -M\gamma_t e^{2Z_t}$

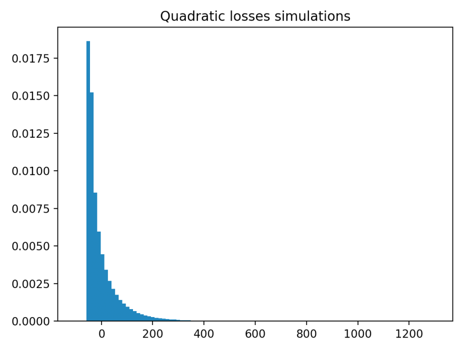
$\Rightarrow L^{Qua}_{t+\Delta} = -(-M\theta\Delta - \frac{1}{2}\gamma_t e^{2Z_t} X^2 M)$

$= M\theta\Delta + \frac{1}{2}\gamma_t e^{2Z_t} X^2 M$

(b) The simulation will depend on Python code.







2. (a) $F(l) = \frac{ab}{ae^{-bl_0} + be^{al_0}} \left(\frac{1}{a} e^{al} 1_{l \le l_0} - \frac{1}{b} e^{-bl} 1_{l > l_0} \right) \quad l \in \mathbb{R}$

$$\Rightarrow \quad F(l_0) = \frac{be^{al_0}}{ae^{-bl_0} + be^{al_0}} = \frac{b}{ae^{-(a+b)l_0} + b}$$

By the question, we have known

$$\alpha \geq \frac{b}{ae^{-(a+b)l_0} + b} \quad \Rightarrow \quad \alpha \geq F(l_0)$$

$$\mathrm{VaR}_\alpha(L) := \inf [l \in \mathbb{R} \mid P[L > l] \leq 1 - \alpha]$$

because $F(l)$ is increasing function

$$\Rightarrow \mathrm{VaR}_\alpha(L) \geq l_0 \quad \Rightarrow \quad l \geq l_0$$

So

$$F(l) = \frac{ab}{ae^{-bl_0} + be^{al_0}} \frac{1}{a} e^{al_0} + \frac{ab}{ae^{-bl_0} + be^{al_0}} \left[-\frac{1}{b} e^{-bl} - \left(-\frac{1}{b} e^{-bl_0}\right) \right]$$

$$= \frac{be^{al_0}}{ae^{-bl_0} + be^{al_0}} + \frac{ae^{-bl_0} - ae^{-bl}}{ae^{-bl_0} + be^{al_0}}$$

$$= \frac{b}{b + ae^{-(a+b)l_0}} + \frac{ae^{-(a+b)l_0} - ae^{-bl - al_0}}{b + ae^{-(a+b)l_0}}$$

$$= \frac{b + ae^{-(a+b)l_0} [1 - e^{-bl - al_0 + al_0 + bl_0}]}{b + ae^{-(a+b)l_0}}$$

$$= \frac{b + ae^{-(a+b)l_0} [1 - e^{-b(l - l_0)}]}{b + ae^{-(a+b)l_0}}$$

$$\Rightarrow \quad \alpha = \frac{b + ae^{-(a+b)l_0}\left[1 - e^{-b(VaR_\alpha(L) - l_0)}\right]}{b + ae^{-(a+b)l_0}}$$

$$\alpha(b + ae^{-(a+b)l_0}) = b + ae^{-(a+b)l_0} - ae^{-(a+b)l_0 - b(VaR_\alpha - l_0)}$$

$$\frac{1}{a}(1-\alpha)(b + ae^{-(a+b)l_0}) = e^{-(a+b)l_0 - b(VaR_\alpha - l_0)}$$

$$\frac{1}{a}(1-\alpha)(b + ae^{-(a+b)l_0}) = e^{-al_0 - bVaR_\alpha}$$

$$e^{al_0}\frac{1}{a}(1-\alpha)(b + ae^{-(a+b)l_0}) = e^{-bVaR_\alpha}$$

$$\Rightarrow -bVaR_\alpha = \ln\left[e^{al_0}\frac{1}{a}(1-\alpha)(b + ae^{-(a+b)l_0})\right]$$

$$\Rightarrow \quad VaR_\alpha(L) = -\frac{1}{b}\ln\left[e^{al_0}\frac{1}{a}(1-\alpha)(b + ae^{-(a+b)l_0})\right]$$

(b) L is a binomial random variable

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$n=6 \qquad p=\frac{1}{2} \qquad \alpha = 0.9$

$F(0) = \frac{1}{2}^6 = \frac{1}{64}$

$F(1) = \frac{1}{2}^6 + 6 \times \frac{1}{2}^6 = \frac{7}{64}$

$F(2) = (1+6+15) \times \frac{1}{2}^6 = \frac{22}{64}$

$F(3) = (1+6+15+20) \times \frac{1}{2}^6 = \frac{42}{64}$

$F(4) = (1+6+15+20+15) \times \frac{1}{2}^6 = \frac{57}{64}$

$F(5) = (1+6+15+20+15+6) \times \frac{1}{2}^6 = \frac{63}{64}$

$F(6) = (1+6+15+20+15+6+1) = 1$

$k=5 \quad F(5) > \alpha = 0.9$

$VaR_\alpha(L) = 5$

(c) L is an "exponential with exponential mean"

Given $Y=y$

$$F(L) = 1 - e^{-yx} \qquad x \geq 0$$

$$P(L \leq VaR_\alpha(L)) = \alpha = E[\mathbb{1}_{L \leq VaR_\alpha(L)}]$$

$$= E[E[\mathbb{1}_{L \leq VaR_\alpha(L)} / Y]] = E[1 - e^{-y\, VaR_\alpha(L)}]$$

$$\alpha = 1 - \frac{\theta}{VaR_\alpha + \theta}$$

$$\Rightarrow 1-\alpha = \frac{\theta}{VaR_\alpha + \theta} \Rightarrow \frac{\theta}{1-\alpha} - \theta = VaR_\alpha(L)$$

$$\Rightarrow VaR_\alpha(L) = \frac{\alpha\theta}{1-\alpha}$$