HW Problems for Assignment 3 - Part 2
Due 6:30 PM Wednesday, October 21, 2020

1. (25 Points) Scenario Analysis for a hedged portfolio of options in the Black-Scholes Model. Assume the stock price has dynamics

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{0.1}$$

where $\mu \in \mathbb{R}, \sigma > 0$, and W is a standard Brownian motion under the physical probability measure $\mathbb{P}$. We assume a constant interest $r \in \mathbb{R}$. In our portfolio we are long both a call and put option with the same strike K, and we are delta hedging both positions. The arbitrage free price of the call and put are

$$C^{BS}(t, S_t, \sigma_t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(S_T - K)^+ \mid S_t\right];$$
$$P^{BS}(t, S_t, \sigma_t) = E^{\mathbb{Q}}\left[e^{-r(T-t)}(K - S_T)^+ \mid S_t\right].$$

$P^{BS}(t, S_t, \sigma_t)$
$= C^{BS}(t, S_t, \sigma_t) + K \cdot e^{-r(T-t)}$
$- S_t$

Here $\mathbb{Q}$ is the risk neutral measure under which S has drift rate r. Above, by "σ_t" we mean that we have estimated the volatility at time t to be σ_t, and then to price the option, we assume the volatility is kept at σ_t over $[t, T]$. Note, this is not entirely consistent with (0.1), but it is common industry practice.

$h_t = \partial_S C^{BS}(t, S_t, \sigma_t) + \partial_S C^{BS}(t, S_t, \sigma_t) - 1$

We are using the standard delta-hedging strategy. As such, our position at time t is 1 unit notional of each option, and $-h_t$ shares of the stock, where

$$h_t = \partial_S C^{BS}(t, S_t, \sigma_t) + \partial_S P^{BS}(t, S_t, \sigma_t).$$

$\boxed{h_t}$

The portfolio value at time t is thus

$$V_t = C^{BS}(t, S_t, \sigma_t) + P^{BS}(t, S_t, \sigma_t) - h_t S_t.$$

The horizon is 5 days, where one day is $\Delta = 1/252$. We hold a constant share position over $[t, t+5\Delta]$ so that at $t + 5\Delta$ the value of our portfolio is

$$V_{t+5\Delta} = C^{BS}(t+5\Delta, S_{t+5\Delta}, \sigma_{t+5\Delta}) + P^{BS}(t+5\Delta, S_{t+5\Delta}, \sigma_{t+5\Delta}) - h_t S_{t+5\Delta}.$$

Again, by $\sigma_{t+5\Delta}$ we mean the constant volatility, now estimated at $t + 5\Delta$ which we use to price the option over $[t + 5\Delta, T]$. We are interested in running a scenario analysis to see what our losses look like if at the end of 5 days the log return $X_{t+5\Delta} = \log(S_{t+5\Delta}/S_t)$ and volatility $\sigma_{t+5\Delta}$ takes certain values.

$e^{X_{t+5\Delta}} \cdot S_t = S_{t+5\Delta}$

The scenarios under consideration are any combination of

$$X_{t+5\Delta} = \pm 0.20, \pm 0.10, \pm 0.05;$$
$$\frac{\sigma_{t+5\Delta}}{\sigma_t} = .5, .75, 1.25, 1.5, 1.75, 2.$$

Thus, there are 36 possible scenarios $x_1, ..., x_{36}$. For each of the above scenarios x_n, compute the full portfolio loss $\ell_n := L_{t+5\Delta}(x_n) = -(V_{t+5\Delta}(x_n) - V_t)$.

(a) Output the worst case scenario risk measure

$$\varrho(L_{t+5\Delta}) = \max\left\{\ell_n \;\middle|\; n = 1, ..., 36\right\},$$

along with the log return/volatility combination which achieves this measure.

(b) Now, assume the following weights:

$$X_{t+5\Delta} = \begin{cases} \pm 0.20 & \text{weight} = .5 \\ \pm 0.10 & \text{weight} = .75 \\ \pm 0.05 & \text{weight} = 1 \end{cases};$$

$$\frac{\sigma_{t+5\Delta}}{\sigma_t} = \begin{cases} .5, 2 & \text{weight} = .5 \\ .75, 1.75 & \text{weight} = 1.25 \\ 1.25, 1.5 & \text{weight} = .75 \end{cases};$$

so that, for example, the weight of $X_{t+\Delta} = -.1$ and $\sigma_{t+5\Delta} = 2\sigma_t$ is $w = .75 \times .5 = .375$. Compute the scenario risk measure

$$\varrho(L_{t+5\Delta}) = \max\left\{w_n \ell_n \;\middle|\; n = 1, \ldots, 36\right\}.$$

along with the log return/volatility combination which achieves this measure. For this combination, what is the portfolio loss?

For parameter values use

$$r = 1.32\%; \quad \mu = 15.475\%; \quad \sigma_t = 22.14\%; \quad t = 0;$$
$$T = 0.25; \quad S_0 = 158.12; \quad K = 160; \quad \Delta = 1/252.$$

2. (25 Points) Stress Test for a Market-Cap Weighted Portfolio of Microsoft and Apple Stocks. In this exercise you will perform the stress test shown in class, except that we will assume a negative shock to both stocks, not just one, with one negative shock dependent upon the other.

We have a market cap weighted portfolio of Microsoft and Apple stocks, where the caps are computed as of 9/1/16. We have daily log return data from 9/1/11 – 9/1/16. We shock the portfolio by assuming a large negative daily log return for Apple, which leads to a large negative log return for Microsoft. We want to see how this shock affects our losses over a K day horizon. In particular, we wish to estimate the shocked K day Value at Risk, and the frequency at which losses exceed the regulatory capital requirement estimated prior to the shock.

To perform the stress test:

(1) With the historical log return data, estimate the mean vector $\mu_{t+\Delta}$ and covariance matrix $\Sigma_{t+\Delta}$ as of time $t = 9/1/16$ using EWMA. You can

start off with initial values of 0 for both and then update through time via the recursive equations

$$\mu_{s+\Delta} = \lambda\mu_s + (1-\lambda)X_s;$$
$$\Sigma_{s+\Delta} = \theta\Sigma_s + (1-\theta)(X_s-\mu_s)(X_s-\mu_s)^{\mathrm{T}}.$$

(2) At t assume $X_{t+\Delta} \overset{\mathcal{F}_t}{\sim} N\left(\mu_{t+\Delta}, \Sigma_{t+\Delta}\right)$. Estimate $\mathrm{VaR}_\alpha\left(L^{lin}_{t+\Delta}\right)$, and using the square root of time rule, estimate $\mathrm{VaR}_\alpha\left(L^{lin}_{t+K\Delta}\right)$ as well as the regulatory capital change $3\times\mathrm{VaR}_\alpha\left(L^{lin}_{t+K\Delta}\right)$.

(3) Shock the system by assuming a large negative return for Apple, where below (1) corresponds to Microsoft, and (2) corresponds to Apple. Specifically:

(i) Set $X_{t+\Delta,(2)} = x_{(2)} = \mu_{t+\Delta,(2)} - 5\times\sigma_{t+\Delta,(2)}$. Shock

(ii) To obtain $X_{t+\Delta,(1)}$, first use the conditioning result which showed that given $X_{t+\Delta,(2)} = x_{(2)}$, $X_{t+\Delta,(1)}$ is normally distributed with

$$\left\{\begin{aligned} \widetilde{\mu}_{t+\Delta,(1)} &= \mu_{t+\Delta,(1)} + \frac{\rho_{t+\Delta,(1)}\sigma_{t+\Delta,(1)}}{\sigma_{t+\Delta,(2)}}\left(x_{(2)} - \mu_{t+\Delta,(2)}\right);\\ \widetilde{\sigma}^2_{t+\Delta,(1)} &= \sigma^2_{t+\Delta,(1)}\left(1-\rho^2_{t+\Delta}\right).\end{aligned}\right.$$

Then, assume a -5 sigma (conditional) shock for Microsoft by setting $X_{t+\Delta,(1)} = x_{(1)} = \widetilde{\mu}_{t+\Delta,(1)} - 5\widetilde{\sigma}_{t+\Delta,(1)}$. This part varies from the lecture slides, and makes the stress test (with high probability) more stressful.

(iii) Use $x_{(1)}, x_{(2)}$ to update $\mu_{t+2\Delta}, \Sigma_{t+2\Delta}$ via EWMA.

(4) For days $k = 2,\dots,K$ sample $X_{t+k\Delta} \sim N(\mu_{t+k\Delta}, \Sigma_{t+k\Delta})$ and update $\mu_{t+(k+1)\Delta}, \Sigma_{t+(k+1)\Delta}$ via EWMA (the mean and covariance need not be updated for $k=K$).

(5) Estimate the linearized K day portfolio loss by assuming the shares the stocks were held constant: i.e. by setting $L^{lin}_{t+K\Delta} = -\theta_t^{\mathrm{T}}\left(\sum_{k=1}^K X_{t+k\Delta}\right)$ where $\theta_t = S_t\lambda_t$ is the dollar position in the stocks at time t.

You will then repeat steps $(3)-(5)$ above $\hat{M}$ times to get the shocked losses

$$\ell_m = L'^{,m}_{t+K\Delta} = -\theta_t^{\mathrm{T}}\left(\sum_{k=1}^K X^m_{t+k\Delta}\right).$$

Using the losses $\{\ell_m\}_{m=1}^{\hat{M}}$ compute

(a) The average K-day portfolio loss.
(b) An estimate of the K-day VaR_α.
(c) The frequency with which the losses exceeded the initial K day Value at Risk found using the square root of time rule.
(d) The frequency with which the losses exceed the regulatory capital found using the square root of time rule.

To obtain the frequencies, take the number of exceedances and multiply by $100/\hat{M}$ to get the percentage of days the losses exceeded the given values.

The log return data is in the file

"MSFT_AAPL_Log_Returns.csv".

This file has three columns: the date in numeric format sorted oldest to newest; the daily log return for MSFT; the daily log return for APPL. In order to help out the grader, you MUST write your program to use this data file, in this specific format. The market caps as of 9/1/16 are

$$\text{MSFT: } 448.77(b); \qquad \text{AAPL: } 575.11(b).$$

For the confidence use $\alpha = .95$. For the number of days use $K = 10$. For the portfolio value use $\$1,000,000$. For the initial days of data use $M = 100$. For the EWMA procedure use $\lambda = \theta = .97$. For the number of paths to run in your simulation use $\hat{M} = 50,000$.

$$L^{lin} = -\theta^T \mu + \sqrt{\theta^T \Sigma \theta} \cdot Z$$

1. $$\ell_n := L_{t+5\Delta}(X_n) = -(V_{t+5\Delta}(X_n) - V_t)$$

$$-(V_{t+5\Delta}(X_n) - V_t) = -\left(C^{BS}(t+5\Delta, S_{t+5\Delta}, \sigma_{t+5\Delta}) - C^{BS}(t, S_t, \sigma_t)\right.$$
$$\left. + P^{BS}(t+5\Delta, S_{t+5\Delta}, \sigma_{t+5\Delta}) - P^{BS}(t, S_t, \sigma_t) - h_t(S_{t+5\Delta} - S_t)\right)$$

$$h_t = \partial_S C^{BS}(t, S_t, \sigma_t) + \partial_S P^{BS}(t, S_t, \sigma_t)$$

(a) According to calculation is python code, we get

```
Worst Case Scenario Risk Measure 27.6186
Worst Case Log Return 0.2000
Worst Case Beta 2.0000
```

(b) After the impacts of weights, we can get the result like

```
Weighted Worst Case Scenario Risk Measure 15.8051
Weighted Worst Case Log Return 0.2000
Weighted Worst Case Beta 1.7500
Weighted Worst Case Loss 25.2882
```

2. After the calculation, we can get the result as this picture. And the code will show in Python.

```
Confidence: 0.95
Number of days: 10
Initial one day VaR:   14383.85
Initial 10 day VaR:   45485.72
3x Initial 10 day VaR:  136457.15
10 day VaR (no shock, shock): 44358.70 105892.78

Average 10 day loss: (no shock, shock)  -11501.96   39978.22

Pct exceedances for 10 day VaR over inital 10 day VaR (no shock, shock):        4.66      44.19

Pct exceedances for 10 day VaR over 3x inital 10 day VaR (no shock, shock):       0.01       0.94
```