ECE310 Final Review - Cramming Carnival Solutions

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1 Questions

Note: Unless stated otherwise, assume all *sequences* of form $f[n]$ are *real valued.*

1.1 Sampling

1.1.1 Frugal Sampling

You are working on a system with Dr. Dee Efty that takes an analog signal as input, converts it to digital, filters the digital signal, and converts it back to analog. He tells you that the input analog signal is bandlimited at 60kHz, but nothing else. A bad sampling rate is an issue for everybody - you'll need more memory to store the larger number of samples, and the hardware to sample faster will undoubtedly cost more. So, you ask Dr. Efty via email what the filter will look like, and he has *yet* to respond.

1. Knowing nothing about the filter, what is the smallest sampling rate for the A/D converter that will result in LTI operation?

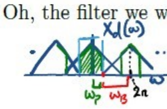
Must use Nyquist. $T < \frac{1}{2B} = \frac{1}{120\text{kHz}}$ $\quad f_s = \frac{1}{T}$ $\quad \boxed{\frac{120}{1}\text{kHz} < f_s}$

2. FINALLY, Dr. Efty gets back to you. Below are a set of possible responses from Dr. Efty. For each, report whether the minimum sampling rate for LTI operation can be made smaller, or whether the sampling rate remains the same.

*The highest frequency must never alias into the passband.

$* \ \omega = \frac{\Omega}{F_s} = T\Omega$

(a) "Oh, the filter we want to implement is a lowpass cutting off at 10kHz."



ω_B: digital freq of bandwidth freq.

ω_p: digital freq of max frequency value after filtering.

$\omega_B + \omega_p < 2\pi$

$2\pi T(60\text{kHz}) + T(10\text{kHz}) < 2\pi$

$2\pi T < \frac{2\pi}{70\text{kHz}}$. Can reduce rate.

$f_s > 70$ kHz

(b) "The filter is a bandpass, permitting frequencies from 20kHz to 30kHz to pass through."

$\Omega_B = 60\text{kHz}$

$\Omega_p = 30\text{kHz}$

$2\pi T(60\text{kHz}) + T(30\text{kHz}) < 2\pi$; $2\pi T < \frac{2\pi}{90\text{kHz}}$ Can reduce rate.

$f_s > 90$ kHz

(c) "The desired filter is a highpass with cutoff at 40kHz."

$\Omega_B = 60\text{kHz}$

$\Omega_p = 60\text{kHz}$.

(we know this is a highpass and that the input's max freq. is 60kHz; so, the max freq that can pass through is 60kHz.)

$2\pi T(60\text{kHz}) + T(60\text{kHz}) < 2\pi$;

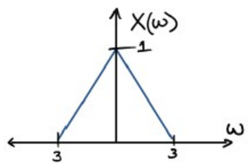
$2\pi T < \frac{2\pi}{2 \cdot 60\text{kHz}}$

$2\pi T < \frac{\pi}{60\text{kHz}}$. CANNOT reduce rate.

$f_s > 120$ kHz

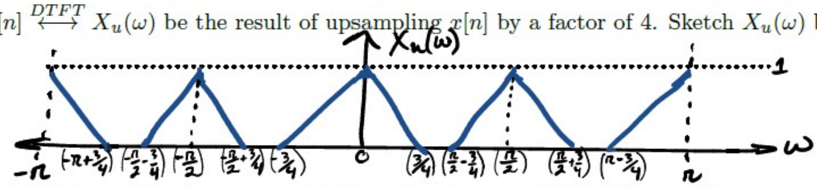
1.1.2 Up Up, Down Down, ...

Suppose we are given the signal $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$ below:



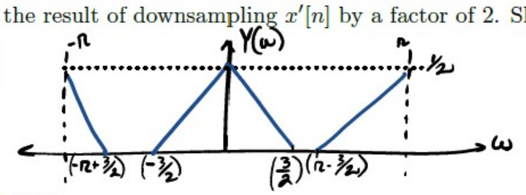
1. Let $x_u[n] \overset{DTFT}{\longleftrightarrow} X_u(\omega)$ be the result of upsampling $x[n]$ by a factor of 4. Sketch $X_u(\omega)$ below.

$X_u(\omega) = X(4\omega)$



2. Let $x'[n]$ be the result of passing $x_u[n]$ through a low pass filter with cutoff frequency $\omega_c = \frac{\pi}{2}$. Let $y[n] \overset{DTFT}{\longleftrightarrow} Y(\omega)$ be the result of downsampling $x'[n]$ by a factor of 2. Sketch $Y(\omega)$ below.

$Y(\omega) = \frac{1}{2} X'\left(\frac{\omega}{2}\right)$



1.2 GLP and FIR Filters

1.2.1 FIR, Step-by-Step

Dr. Dee Efty is back with more tasks for you. You are asked to make an FIR filter $h[n]$ that implements $D(\omega)$, a desired frequency response, where on the interval $\omega \in [-\pi, \pi)$:

310 changed its notation/terminology for filters AGAIN. But, when in Rome, do as the Romans do.

$$D(\omega) = \begin{cases} 1 & |\omega| < 0.5 \\ 0 & \text{else} \end{cases}$$

1. Dr. Efty has employed the help of an ECE 408[1] student to write FIR convolution code on a GPU. The student successfully wrote their code, but due to off-by-one errors, it only works on filters with *even* length. Furthermore, due to the limitations of the hardware, the filter can have at most 64 values.

   (a) Is it possible to make this filter, ~~and if so, what is the maximum length of the FIR filter for~~ which the filter is possible? It is possible to make a LPF with even length. The maximum length is $N = 64$.

   (b) Circle the correct choices: The filter as described above must have **odd**/**even** symmetry, and the type must be type I/II/III/IV for the filter with maximal length.

2. You report your results to Dr. Efty, and he decides last minute to change the filter specification to a highpass with cutoff $\omega_c = \frac{1}{2}$, and to implement the filter with **odd** symmetry and **even** length $L = 8$. That is to say, the desired frequency response on the interval $\omega \in [-\pi, \pi)$ is:

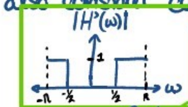
$$D(\omega) = \begin{cases} 0 & |\omega| < 0.5 \\ 1 & \text{else} \end{cases}$$

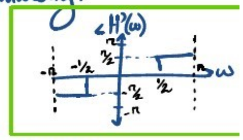
   (a) Circle the correct choice: this new filter is now a type I/II/III/IV filter.

[1]The entire latter half of the class is about implementing block convolution on NVIDIA GPUs. Not even joking.

(b) Let $H'(\omega)$ represent the DTFT of the above filter with the given specifications *without* any delay or windowing applied. Sketch $|H'(\omega)|$ and $\angle H'(\omega)$.

Because this is GLP type II, $\alpha = \frac{\pi}{2}$. However, we must also consider conjugate symmetry: $H'(-\omega) = H'(\omega)^*$. Therefore, $H'(\omega) = \begin{cases} e^{j\frac{\pi}{2}}, & \omega > \frac{1}{2} \\ e^{-j\frac{\pi}{2}}, & \omega < -\frac{1}{2} \\ 0 & \text{else} \end{cases}$





(c) What is the necessary time delay d to apply to $H'(\omega)$? The delay is $\frac{8-1}{2} = 3.5$

(d) After adding this delay, what is the DTFT of the *unwindowed* frequency response of the FIR filter, $H_u(\omega)$? Express it in terms of $H'(\omega)$, and then express it in terms of $D(\omega)$.

$$H_u(\omega) = \boxed{H'(\omega)e^{-j\omega 3.5}} = \boxed{\begin{cases} D(\omega)e^{j(\frac{\pi}{2}-\omega 3.5)} & \text{if } \omega > 0 \\ D(\omega)e^{j(-\frac{\pi}{2}-\omega 3.5)} & \text{if } \omega < 0 \end{cases}}$$

Multiple answers possible here.

(e) What is the expression for $h_u[n]$, the unwindowed filter?

See the next page

3. Let $h'[n] \overset{DTFT}{\longleftrightarrow} H'(\omega)$ be the filter *before* delaying and windowing, as from earlier. Let $h[n]$ be the final, *windowed* and *delayed* FIR filter. Assuming that the window used is **rectangular**, if $h[k] = h'[m]$, then write k under the corresponding box for $h'[m]$. For example, if $h'[3] = h[4]$ but the value of $h'[3.5]$ is never used for $h[n]$, then only write "4":

n for $h'[n]$	3	3.5
n for $h[n]$	4	(leave blank)

Delayed by 3.5 $\equiv$ $h[n] = h'[n-3.5]$

Windowed: only $0 \le n < 8$ used.

n for $h'[n]$	-8.5	-8	-7.5	-7	-6.5	-6	-5.5	-5
n for $h[n]$								
n for $h'[n]$	-4.5	-4	-3.5	-3	-2.5	-2	-1.5	-1
n for $h[n]$			0		1		2	
n for $h'[n]$	-0.5	0	0.5	1	1.5	2	2.5	3
n for $h[n]$	3		4		5		6	
n for $h'[n]$	3.5	4	4.5	5	5.5	6	6.5	7
n for $h[n]$	7							

1.3 I Got 99 Block Diagrams, but...

Suppose we are given the difference equation system H, described by:

$$y[n] + 8y[n-1] + 15y[n-2] = -x[n] + x[n-2]$$

1. What is the z-transform, $H(z)$? Assume the system is causal.

$$y[n] + 8y[n-1] + 15y[n-2] = -x[n] + x[n-2]$$

$$Y(z)(1 + 8z^{-1} + 15z^{-2}) = X(z)(-1 + z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2}-1}{15z^{-2}+8z^{-1}+1} = \boxed{\frac{(z^{-1}-1)(z^{-1}+1)}{(3z^{-1}+1)(5z^{-1}+1)}}, \quad \boxed{|z| > \frac{1}{3}}$$

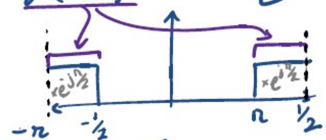
(Poles: $z = -\frac{1}{3}, -\frac{1}{5}$).

1.2.1 P2.e)

Idea: instead of evaluating integrals, just convert your filter into scaled & shifted LPFs & use properties.

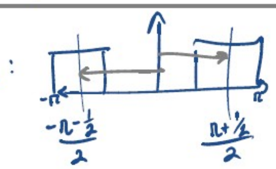
$$H_u(\omega) = H'(\omega)\, e^{-j3.5\omega}$$

Recall $H'(\omega) = \begin{cases} D(\omega) e^{-j\frac{\pi}{2}}, & \omega < 0 \\ D(\omega) e^{j\frac{\pi}{2}}, & \omega > 0 \end{cases}$. $H'(\omega)$ is an HPF with total "width" of $2(\pi - \frac{1}{2})$ with scaling



OR they can be two shifted LPFs scaled by $e^{\pm j\frac{\pi}{2}}$, the LPF having width $\pi - \frac{1}{2}$ (equiv, $\omega_c = \frac{1}{2}(\pi - \frac{1}{2})$). Let $L(\omega)$ represent the freq. response of an LPF with $\omega_c = \frac{1}{2}\pi - \frac{1}{4}$. Then,

Picture:



$$H'(\omega) = e^{-j\pi/2} L\left(\omega + \frac{\pi + \frac{1}{2}}{2}\right) + e^{j\pi/2} L\left(\omega - \frac{\pi + \frac{1}{2}}{2}\right)$$

$$\Updownarrow$$

$$h'[n] = e^{-j\pi/2} e^{-j\left(\frac{\pi+\frac{1}{2}}{2}\right)n} L[n] + e^{j\frac{\pi}{2}} e^{j\left(\frac{\pi+\frac{1}{2}}{2}\right)n} L[n]$$

$$= L[n]\left(e^{-j\left[\left(\frac{\pi+\frac{1}{2}}{2}\right)n + \frac{\pi}{2}\right]} + e^{j\left[\left(\frac{\pi+\frac{1}{2}}{2}\right)n + \frac{\pi}{2}\right]}\right)$$

$$= 2L[n]\cos\left(\left[\frac{\pi+\frac{1}{2}}{2}\right]n + \frac{\pi}{2}\right) = \frac{2\sin\left(\left[\frac{\pi-\frac{1}{2}}{2}\right]n\right)\cos\left(\left[\frac{\pi+\frac{1}{2}}{2}\right]n + \frac{\pi}{2}\right)}{\pi n}$$

(Note: Inv. DTFT for LPF with ω_c is $\frac{\sin(\omega_c n)}{\pi n}$).

Adding back delay ($h_u[n] = h[n-3.5]$):

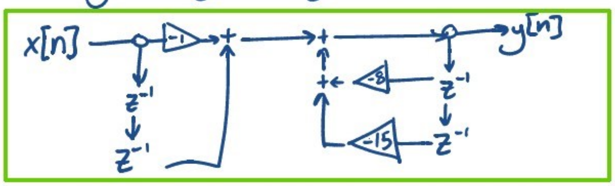
$$\boxed{h_u[n] = \frac{2\sin\left(\left[\frac{\pi-\frac{1}{2}}{2}\right](n-3.5)\right)\cos\left(\left[\frac{\pi+\frac{1}{2}}{2}\right](n-3.5) + \frac{\pi}{2}\right)}{\pi(n-3.5)}}$$

Note: Many possible equivalent expressions possible!

$y[n] = x[n] + 3y[n-1]$

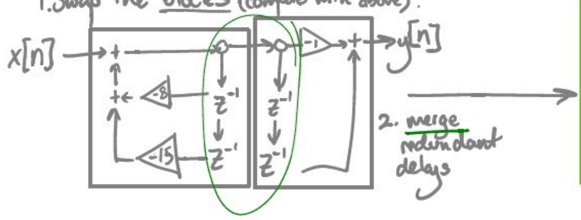
2. Draw below the block diagram in **direct form I**.

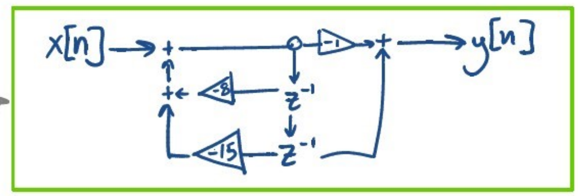
$y[n] = -8y[n-1] - 15y[n-2] - x[n] + x[n-2]$



3. Draw below the block diagram in **direct form II**.

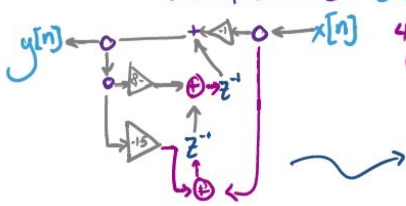
1. Swap the blocks (compare with above).

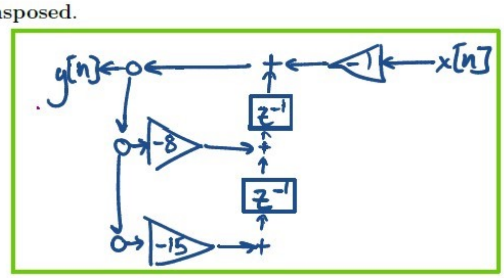




4. Draw below the block diagram in **direct form II, transposed**.

1. Reverse branches 2. Swap branch & + 3. exchange input & output. 4. Make accomodations (multiple inputs to sums)





5. Draw below the block diagram in **cascade for direct form I**.

ignore

$H(z) = \frac{(z^{-1}-1)(z^{-1}+1)}{(3z^{-1}+1)(5z^{-1}+1)} = \underbrace{\left(\frac{z^{-1}-1}{3z^{-1}+1}\right)}_{H_1}\underbrace{\left(\frac{z^{-1}+1}{5z^{-1}+1}\right)}_{H_2}$

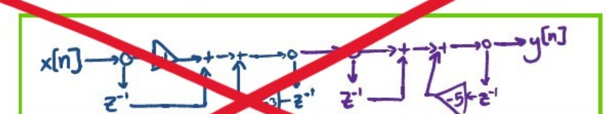
express each as direct form 1, then cascade

$H_1: \frac{z^{-1}-1}{3z^{-1}+1} \longleftrightarrow y[n] + 3y[n-1] = -x[n] + x[n-1]$

$y[n] = -3y[n-1] - x[n] + x[n-1]$

$H_2: \frac{z^{-1}+1}{5z^{-1}+1} \longleftrightarrow y[n] + 5y[n-1] = x[n] + x[n-1]$

$y[n] = -5y[n-1]$ ...



6. Draw below the block diagram in **parallel form direct form I**.

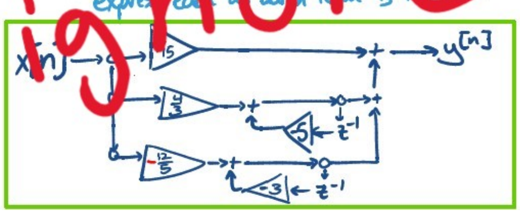
$H(z) = \frac{(z^{-1}-1)(z^{-1}+1)}{(3z^{-1}+1)(5z^{-1}+1)}$ Partial Fraction Expansion $= \underbrace{\frac{1}{15}}_{H_1} + \underbrace{\frac{4/3}{3z^{-1}+1}}_{H_2} + \underbrace{\frac{-12/5}{5z^{-1}+1}}_{H_3}$

express each as direct form 1, then sum.

$H_1: y[n] = \frac{1}{15}x[n]$

$H_2: y[n] + 3y[n-1] = \frac{4}{3}x[n]$

$H_3: y[n] + 5y[n-1] = \frac{12}{5}x[n]$



1.4 DFTs

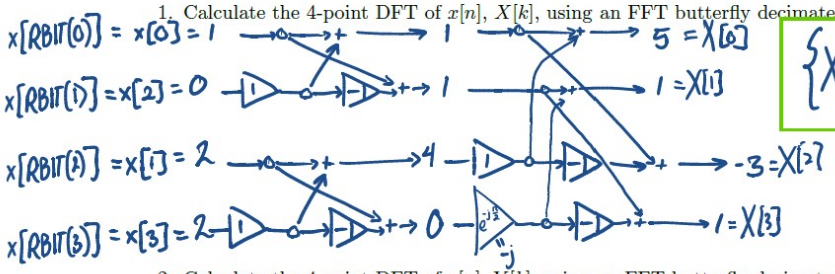
1.4.1 Math Without Calculators

Consider the two below sequences:

Note: 4 values → 2 bits per index.

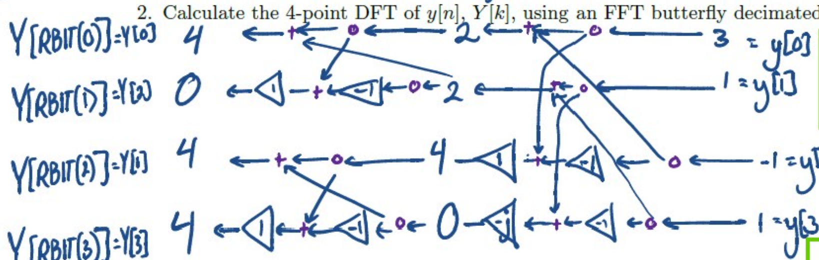
$$\{x[n]\}_{n=0}^{3} = \{1, 2, 0, 2\}\,, \{y[n]\}_{n=0}^{3} = \{3, 1, -1, 1\}$$

1. Calculate the 4-point DFT of $x[n]$, $X[k]$, using an FFT butterfly decimated in time.



$\boxed{\{X[k]\}_{k=0}^{3} = \{5, 1, -3, 1\}}$

2. Calculate the 4-point DFT of $y[n]$, $Y[k]$, using an FFT butterfly decimated in frequency.[2]



$\boxed{\{Y[k]\}_{k=0}^{3} = \{4, 4, 0, 4\}}$

Use the footnote.

3. Calculate $z[n] = x[n] \circledast y[n]$, the circular convolution of $x[n]$ and $y[n]$.

$\boxed{\{z[n]\}_{n=0}^{3} = \{7, 5, 3, 5\}}$

$y[n] = \{3, 1, -1, 1\}$

$y[-n] = \{3, 1, -1, 1\}$ ($y[0]$, $y[3]$, $y[2]$, $y[1]$)

$x[n]: 1, 2, 0, 2$

shift of $y[-n]$:

0 : 3, 1, -1, 1 : 1(3)+2(1)+0(-1)+2(1) : 7

1 : 1, 3, 1, -1 : 1(1)+2(3)+0(1)+2(-1) : 5

2 : -1, 1, 3, 1 : 1(-1)+2(1)+0(3)+2(1) : 3

3 : 1, -1, 1, 3 : 1(1)+2(-1)+0(1)+2(3) : 5

4. Calculate the 4-point DFT of $z[n]$, $Z[k]$, using whatever method you would like. Verify that $Z[k_i] = Y[k_i]X[k_i]$ for $i \in \{0, 1, 2, 3\}$.

Let $W \equiv e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$. W is "twiddle factor." $W^0 = 1, W^1 = -j, W^2 = -1, W^3 = j$. $W^n = W^{n+4}$

$Z[0] = 7W^{0\cdot 0} + 5W^{0\cdot 1} + 3W^{0\cdot 2} + 5W^{0\cdot 3} = 20$

$Z[1] = 7W^{1\cdot 0} + 5W^{1\cdot 1} + 3W^{1\cdot 2} + 5W^{1\cdot 3} = 7 - 5j - 3 + 5j = 4$

$Z[2] = 7W^{2\cdot 0} + 5W^{2\cdot 1} + 3W^{2\cdot 2} + 5W^{2\cdot 3} = 7 - 5 + 3 - 5 = 0$

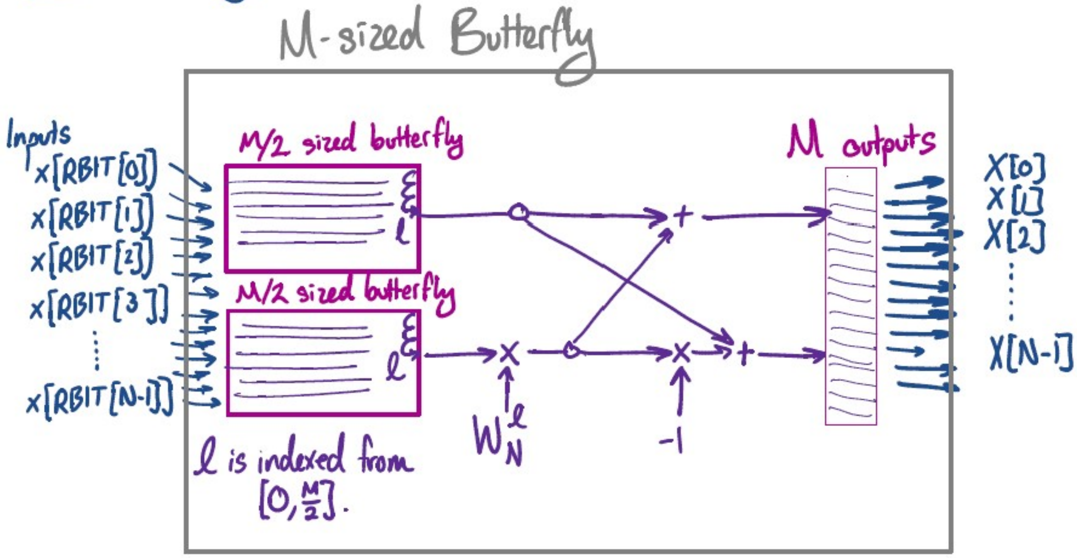
$Z[3] = 7W^{3\cdot 0} + 5W^{3\cdot 1} + 3W^{3\cdot 2} + 5W^{3\cdot 3} = 7 + 5j - 3 - 5j = 4$

$\boxed{\{Z[k]\}_{n=0}^{4} = \{20, 4, 0, 4\}}$

$= \underbrace{[5, 1, -3, 1]}_{X[k]} \cdot \underbrace{[4, 4, 0, 4]}_{Y[k]}$ ✓

[2]Fun fact: applying the transposition operation that you know from block diagrams to the decimation-in-time butterfly, yields the decimation-in-frequency butterfly! Same with the other way around.

BONUS: A generalized Decimation-in-Time Butterfly Template



1 Famous FFTs

After a wonderful dinner with Dr. Efty, where the two of you discussed important applications of the FFT, you retire back to your room with an important napkin. On the napkin is a beautiful diagram that Dr. Efty had given to you as a proof for a most illuminating theorem. Unfortunately, during the scramble to figure out who was going to pay, some Sprite got on the napkin. Below is remainder of the diagram.

You recall that the diagram was once part of a 64-bit decimation in time radix-2 FFT. You must find the values of the red letters and greek letters so that you can publish another paper.

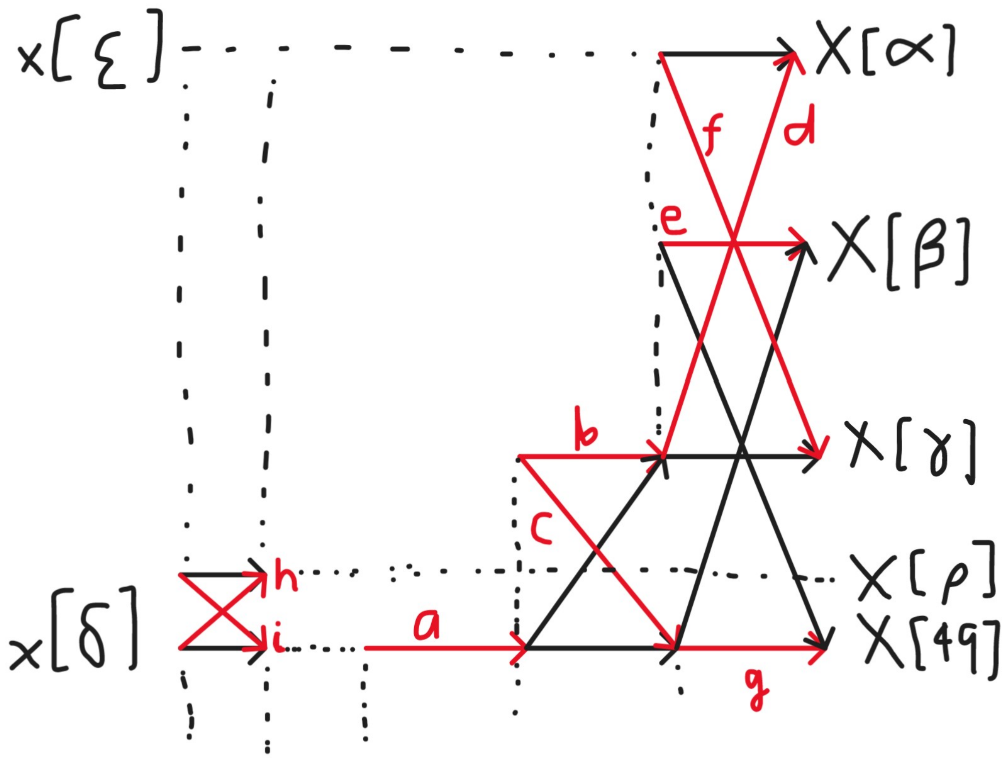


Figure 1: The Remains of the Napkin

1.1 Solution:

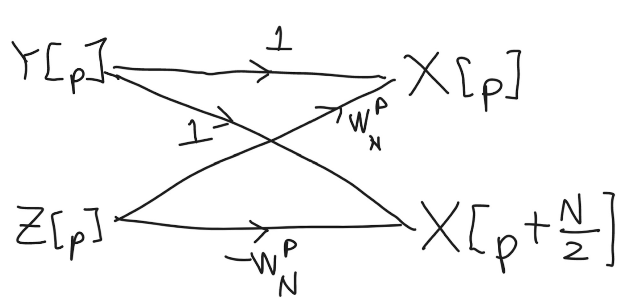


Figure 2: FFT Decimation-in-Time Butterfly

Figure 2 shows the FFT butterfly. Let us now solve the problem.

1. We know that $X[\beta]$ and $X[49]$ are connected by the final FFT. This means that $\beta + \frac{64}{2} = 49$. Solving, we get $\boxed{\beta = 17}$.

2. We know that $X[\gamma]$ and $X[49]$ are connected by the second to last FFT. This means that $\gamma + \frac{64}{4} = 49$. Solving, we get $\boxed{\gamma = 33}$.

3. We know that $X[\gamma]$ and $X[\alpha]$ are connected by the final FFT. This means that $\alpha + \frac{64}{2} = \gamma$. Using the previous part's solution, we get $\boxed{\alpha = 1}$.

4. FFT input indices are the bit reversal of their output index. Thus, ϵ is the bit-reversal of α and δ is the bit reversal of 49. Solving, we get $\boxed{\epsilon = 32, \delta = 35}$.

5. We know that $x[\delta]$ and the element below it are connected by the first FFT. This means that $X[49]$ and $X[\rho]$ are also right next to each other. Since 49 is odd, we then know that $\rho + 1 = 49$, so $\boxed{\rho = 48}$.

6. Let's do the easier letters first, i.e. the letters that are on the top of the butterfly or pointing upwards. Matching the arrows with the corresponding arrows in the FFT butterfly diagram, we get $\boxed{b = 1, e = 1, f = 1, c = 1, i = 1}$.

7. We know that d is the up-diagonal of the 1st butterfly (remember that butterfly counts are 0-indexed) since we know that $\alpha = 1$. Thus, we use $N = 64, p = 1$ for our coefficient. $\boxed{d = W_{64}^{1} = e^{-j\frac{2\pi}{64}1}}$.

8. We know that g is the bottom arrow of the 17th butterfly (remember that butterfly counts are 0-indexed) since we know that $\beta = 17$. Thus, we use $N = 64, p = 17$ for our coefficient. $\boxed{g = W_{64}^{17} = e^{-j\frac{2\pi}{64}17}}$.

9. In the last FFT we're doing 64-length butterflies. In the second-to-last FFT we're doing 32-length butterflies. In the third-to-last FFT we're doing 16-length butterflies.

   Note also that elements 0 through 15 of our output are involved in the first length-16 FFT, while elements $16 - 31$ are involved in the second, etc.

   Following this pattern, elements 48 through 63 are in the fourth butterfly. We then notice that element 49 is in the upper-half of this length-16 butterfly. This implies that a is actually the top arrow of the butterfly, giving us $\boxed{a = 1}$.

10. The very first chunk of the FFT uses length-2 butterflies. From previous parts we know that elements 48 and 49 make up this particular FFT that we're interested. Thus, we know that h is the up-diagonal of the 0th butterfly. We then use $N = 2, p = 0$ for our coefficient. $\boxed{h = W_{2}^{0} = e^{-j\frac{2\pi}{2}0}}$.

2 Lethargic Linear Algebra

Later in the night, you and Dr. Efty engage in a bit of merriment, and Dr. Efty's memory starts to get a little fuzzy. Dr. Efty wants to develop an adaptive filter to fix his noisy radio, as his favorite songs are now distorted, which is ruining his mood. To do this, he knows he will need some linear algebra, but is having some trouble remembering concepts. It is up to you to help him.

1. Firstly, he remembers that he will need to solve an optimization problem, namely that of:

$$\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} ||\boldsymbol{X}^\top \boldsymbol{w} - \boldsymbol{t}||^2$$

He vaguely remembers that the optimal solution to this problem is given as :

$$\boldsymbol{w}^* = (\boldsymbol{X}\boldsymbol{X}^\top)^{-1}\boldsymbol{X}\boldsymbol{t}$$

He would like you to prove that this is the case (Hint: use gradients).

Solution: We can use the rules of gradients here. The function we are minimizing with respect to $\boldsymbol{w}$ is $||\boldsymbol{X}^\top \boldsymbol{w} - \boldsymbol{t}||^2$, which we can also call $J(\boldsymbol{w})$. So, let us take its gradient, and we will set the gradient equal to 0 to find an optimum. (if you've forgotten about gradients, it can be thought of as just a fancy derivative. There are some special rules which you will see in this solution about how to take the gradient a matrix vector multiplication.)

Firstly, we take the gradient. We can use something similar to both the power rule and chain rule from Calc 1. First using the power rule, we see $2(\boldsymbol{X}^\top \boldsymbol{w} - \boldsymbol{t})$, and by the chain rule, we apply the factor of $\boldsymbol{X}^\top$ as a left multiplication of the transpose of this matrix, so we end up with $2\boldsymbol{X}(\boldsymbol{X}^\top \boldsymbol{w} - \boldsymbol{t}) = 0$. We set this equal to 0, as the squared norm is a convex function, and where its gradient = 0, we have optimized the mean squared error. We can therefore ignore the constant 2, and rearrange our equation to look like

$$\boldsymbol{X}\boldsymbol{X}^\top \boldsymbol{w} = \boldsymbol{X}\boldsymbol{t}$$

.

Now, $\boldsymbol{X}\boldsymbol{X}^\top$ is a matrix (and it will be an invertible matrix) so we can solve for the optimal $\boldsymbol{w}^*$ as $\boldsymbol{w}^* = (\boldsymbol{X}\boldsymbol{X}^\top)^{-1}\boldsymbol{X}\boldsymbol{t}$.

2. Dr. Efty would now like you to create an ideal filter satisfying this constraint. He plays you the tune coming out of his radio, and it goes like this:

$$\{x[n]\}_{n=0}^{1} = \{1, 2\}$$

He listens to rather simple music. He knows that this is a noisy signal, and hums you what the song should sound like, and hums a slightly longer section of the song. It goes like so:

$$\{t[n]\}_{n=0}^{2} = \{2, 7, 6\}$$

Given this information, construct a length-2 filter $\boldsymbol{w}[n]$ that performs optimal filtering, using the equation derived in the previous question.

Solution: The first thing is to come up with our matrix X. The dimensions of this matrix will be $L + H - 1$ columns (with L being the length of the signal, and H being the length of the filter), by H rows. For the first row, we put in our signal as a row vector, and pad with zeros at the right. For the next rows, we shift the signal right by one and add a zero at the left. Our matrix will look like this:

$$X = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Now we can find the product $XX^\top$,

$$XX^\top = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

We now have to find the inverse of this matrix. There are several ways to find the inverse of a 2x2 matrix, however Dr. Efty simply knew the answer to this one:

$$(XX^\top)^{-1} = \frac{1}{21} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

Now we have everything for our matrix multiplication:

$$w^* = \frac{1}{21} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$$

If you solve this, you should get

$$w^* = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

3. Dr. Efty knows that using the DFT will help him analyze the filter, or something. To practice, this he looks at a filter given by:

$$\{h[n]\}_{n=0}^{3} = \{1, 2, 0, 2\}$$

(This is the same as the signal from problem 1.4) He would like you to take the DFT by writing out and performing the matrix multiplication, $\boldsymbol{F}^*\boldsymbol{h}$.

Solution: We will write out the matrix multiplication as follows:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0.5 \\ -1.5 \\ 0.5 \end{bmatrix}$$

You will notice that this is exactly half of the output DFT of problem 1.4.1. The additional half is due to the normalizing factor $\frac{1}{\sqrt{N}}$

4. Dr. Efty now wants you to prove that the basis vectors of the Fourier transform matrix $\boldsymbol{F^*}$ form an orthonormal basis (Hint: the row vectors happen to form the basis vectors). For the sake of this problem, it is fine merely to show that the matrix $\boldsymbol{F^*}$ from the last problem is orthonormal.

Solution:

To show the matrix from the previous problem is orthonormal, simply multiply it by its transpose (don't forget the factor of 1/2 !!!), and show that the result is the identity matrix.

More formally (but not very formally), we can show this in the general case by showing that the inner product of any two different rows of $\boldsymbol{F^*}$ are 0, and that the inner product of the same row is always 1. To show this, we can write out the inner product based on the definition of the matrix, for the kth and nth row:

$$\sum_{i=0}^{N-1} \frac{1}{\sqrt{N}} \exp(\frac{-2\pi}{N} jki) \frac{1}{\sqrt{N}} \exp(\frac{-2\pi}{N} jni)$$

We can rewrite this simply as

$$\frac{1}{N} \sum_{i=0}^{N-1} \exp(\frac{-2\pi}{N} j(n-k)i)$$

When we are looking at the inner product when $n = k$, we see that the exponent turns into 0 ,so everything in the summation is simply 1. The summation evaluates to N, and the result is therefore 1. We have shown the first condition. In the case that $n \neq k$, we can replace this value with some integer $C \neq 0$:

$$\frac{1}{N} \sum_{i=0}^{N-1} \exp(\frac{-2\pi}{N} jCi)$$

If we represent $\exp(\frac{-2\pi}{N} jC)$ as ω, we can rewrite the sum and use a summation property

$$\frac{1}{N} \sum_{i=0}^{N-1} \omega^i = \frac{\omega^N - 1}{N(\omega - 1)}$$

We note that $\omega^N = \exp(-2\pi jC)$, and since C is an integer, this value will always be 1, and the numerator of the expression we found will be 0. Therefore, we showed the second condition from earlier, and the matrix is orthonormal.