ECE310 Final Review - Cramming Carnival Spring 2022 FINAL UPDATE (colored red)

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1 Questions

Note: Unless stated otherwise, assume all *sequences* of form $f[n]$ are *real valued.*

1.1 Sampling

1.1.1 Frugal Sampling

You are working on a system with Dr. Dee Efty that takes an analog signal as input, converts it to digital, filters the digital signal, and converts it back to analog. He tells you that the input analog signal is bandlimited at 60kHz, but nothing else. A bad sampling rate is an issue for everybody - you'll need more memory to store the larger number of samples, and the hardware to sample faster will undoubtedly cost more. So, you ask Dr. Efty via email what the filter will look like, and he has *yet* to respond.

1. Knowing nothing about the filter, what is the smallest sampling rate for the A/D converter that will result in LTI operation?

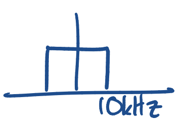
Must use Nyquist. $T < \frac{\pi}{B} = \frac{\pi}{60kHz}$ $\quad f_s = \frac{1}{T}$ $\quad \boxed{\frac{60}{\pi}kHz < f_s}$

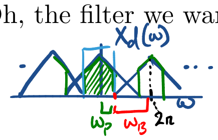
2. FINALLY, Dr. Efty gets back to you. Below are a set of possible responses from Dr. Efty. For each, report whether the minimum sampling rate for LTI operation can be made smaller, or whether the sampling rate remains the same.

*The highest frequency must never alias into the passband.

$\ast\ \omega = \frac{\Omega}{F_s} = T\Omega$

(a) "Oh, the filter we want to implement is a lowpass cutting off at 10kHz."





ω_B: digital freq of bandwidth freq.

ω_p: digital freq of max frequency value after filtering.

$\omega_B + \omega_p < 2\pi$

$T(60kHz) + T(10kHz) < 2\pi$

$T < \frac{2\pi}{70kHz}$. Can reduce rate. $f_s > \frac{35}{\pi}kHz$

(b) "The filter is a bandpass, permitting frequencies from 20kHz to 30kHz to pass through."

$\Omega_B = 60kHz$

$\Omega_p = 30kHz$

$T(60kHz) + T(30kHz) < 2\pi$; $\quad T < \frac{2\pi}{90kHz}$. Can reduce rate.

$f_s > \frac{45}{\pi}kHz$.

(c) "The desired filter is a highpass with cutoff at 40kHz."

$\Omega_B = 60kHz$

$\Omega_p = 60kHz$.

(we know this is a highpass and that the input's max freq. is 60kHz; so, the max freq that can pass through is 60kHz.)

$T(60kHz) + T(60kHz) < 2\pi$;

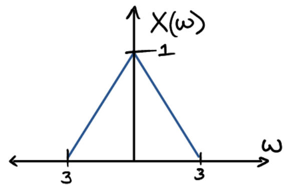
$T < \frac{2\pi}{2 \cdot 60kHz}$

$T < \frac{\pi}{60kHz}$. CANNOT reduce rate.

$f_s > \frac{60}{\pi}kHz$

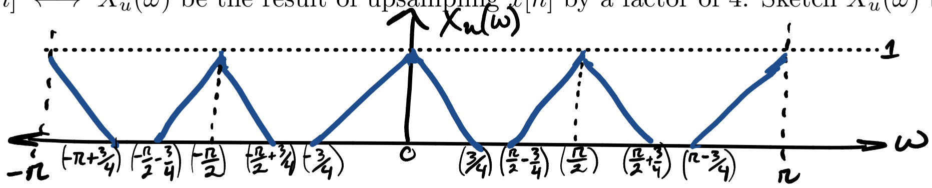
1.1.2 Up Up, Down Down, ...

Suppose we are given the signal $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$ below:

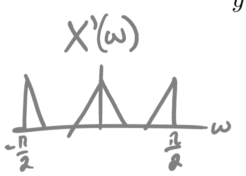


1. Let $x_u[n] \overset{DTFT}{\longleftrightarrow} X_u(\omega)$ be the result of upsampling $x[n]$ by a factor of 4. Sketch $X_u(\omega)$ below.

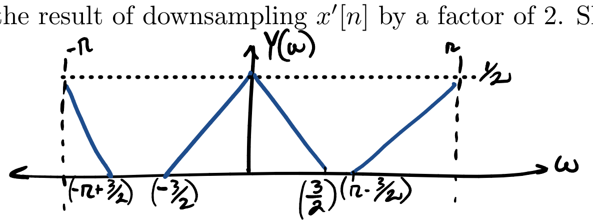
$X_u(\omega) = X(4\omega)$



2. Let $x'[n]$ be the result of passing $x_u[n]$ through a low pass filter with cutoff frequency $\omega_c = \frac{\pi}{2}$. Let $y[n] \overset{DTFT}{\longleftrightarrow} Y(\omega)$ be the result of downsampling $x'[n]$ by a factor of 2. Sketch $Y(\omega)$ below.



$Y(\omega) = \frac{1}{2} X'\left(\frac{\omega}{2}\right)$



1.2 GLP and FIR Filters

1.2.1 FIR, Step-by-Step

Dr. Dee Efty is back with more tasks for you. You are asked to make an FIR filter $h[n]$ that implements $D(\omega)$, a desired frequency response, where on the interval $\omega \in [-\pi, \pi)$:

310 changed its notation/terminology for filters AGAIN. But, when in Rome, do as the Romans do.

$$D(\omega) = \begin{cases} 1 & |\omega| < 0.5 \\ 0 & \text{else} \end{cases}$$

1. Dr. Efty has employed the help of an ECE 408[1] student to write FIR convolution code on a GPU. The student successfully wrote their code, but due to off-by-one errors, it only works on filters with *even* length. Furthermore, due to the limitations of the hardware, the filter can have at most 64 values.

   (a) Is it possible to make this filter, and if so, what is the maximum length of the FIR filter for which the filter is possible? It is possible to make a LPF with even length. The maximum length is N=64.

   (b) Circle the correct choices: The filter as described above must have **odd**/**even** symmetry, and the type must be type I/II/III/IV for the filter with maximal length.

2. You report your results to Dr. Efty, and he decides last minute to change the filter specification to a highpass with cutoff $\omega_c = \frac{1}{2}$, and to implement the filter with **odd** symmetry and **even** length $L = 8$. That is to say, the desired frequency response on the interval $\omega \in [-\pi, \pi)$ is:

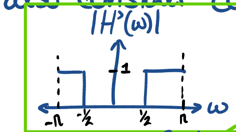
$$D(\omega) = \begin{cases} 0 & |\omega| < 0.5 \\ 1 & \text{else} \end{cases}$$

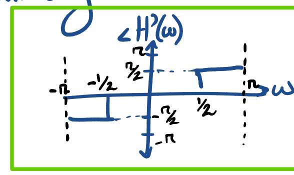
   (a) Circle the correct choice: this new filter is now a type I/II/III/IV filter.

[1] The entire latter half of the class is about implementing block convolution on NVIDIA GPUs. Not even joking.

(b) Let $H'(\omega)$ represent the DTFT of the above filter with the given specifications *without* any delay or windowing applied. Sketch $|H'(\omega)|$ and $\angle H'(\omega)$.

Because this is GLP type II, $\alpha = \frac{\pi}{2}$. However, we must also consider conjugate symmetry: $H'(-\omega) = H'(\omega)^*$. Therefore, $H'(\omega) = \begin{cases} e^{j\frac{\pi}{2}}, & \omega > \frac{1}{2} \\ e^{-j\frac{\pi}{2}}, & \omega < -\frac{1}{2} \\ 0 & \text{else} \end{cases}$





(c) What is the necessary time delay d to apply to $H'(\omega)$? The delay is $\frac{8-1}{2} = 3.5$

(d) After adding this delay, what is the DTFT of the *unwindowed* frequency response of the FIR filter, $H_u(\omega)$? Express it in terms of $H'(\omega)$, and then express it in terms of $D(\omega)$.

$$H_u(\omega) = \boxed{H'(\omega)e^{-j\omega 3.5}} = \boxed{\begin{cases} D(\omega)e^{j(\frac{\pi}{2} - \omega 3.5)} & \text{if } \omega > 0 \\ D(\omega)e^{j(-\frac{\pi}{2} - \omega 3.5)} & \text{if } \omega < 0 \end{cases}}$$

Multiple answers possible here.

(e) What is the expression for $h_u[n]$, the unwindowed filter?

See the next page

3. Let $h'[n] \overset{DTFT}{\longleftrightarrow} H'(\omega)$ be the filter *before* delaying and windowing, as from earlier. Let $h[n]$ be the final, *windowed* and *delayed* FIR filter. Assuming that the window used is **rectangular**, if $h[k] = h'[m]$, then write k under the corresponding box for $h'[m]$. For example, if $h'[3] = h[4]$ but the value of $h'[3.5]$ is never used for $h[n]$, then only write "4":

n for $h'[n]$	3	3.5
n for $h[n]$	**4**	(leave blank)

Delayed by 3.5 $\equiv$ $h[n] = h'[n-3.5]$

Windowed: only $0 \leq n < 8$ used.

n for $h'[n]$	-8.5	-8	-7.5	-7	-6.5	-6	-5.5	-5
n for $h[n]$								
n for $h'[n]$	-4.5	-4	-3.5	-3	-2.5	-2	-1.5	-1
n for $h[n]$			0		1		2	
n for $h'[n]$	-0.5	0	0.5	1	1.5	2	2.5	3
n for $h[n]$			3		4		5	
n for $h'[n]$	3.5	4	4.5	5	5.5	6	6.5	7
n for $h[n]$			6		7			

1.3 I Got 99 Block Diagrams, but...

Suppose we are given the difference equation system H, described by:

$$y[n] + 8y[n-1] + 15y[n-2] = -x[n] + x[n-2]$$

1. What is the z-transform, $H(z)$? Assume the system is causal.

$$y[n] + 8y[n-1] + 15y[n-2] = -x[n] + x[n-2]$$

$$Y(z)(1 + 8z^{-1} + 15z^{-2}) = X(z)(-1 + z^{-2})$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2} - 1}{15z^{-2} + 8z^{-1} + 1} = \boxed{\frac{(z^{-1}-1)(z^{-1}+1)}{(3z^{-1}+1)(5z^{-1}+1)}}, \quad \boxed{|z| > \frac{1}{3}}$$

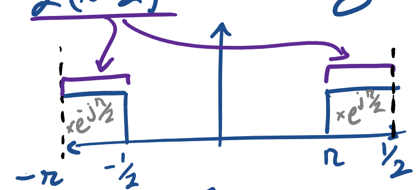
(Poles: $z = -\frac{1}{3}, -\frac{1}{5}$)

1.2.1 P2.e)

Idea: instead of evaluating integrals, just convert your filter into scaled & shifted LPFs & use properties.

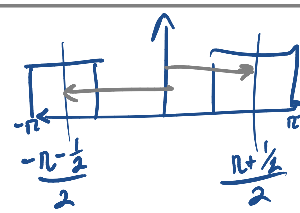
$$H_u(\omega) = H'(\omega)\, e^{-j3.5\omega}$$

Recall $H'(\omega) = \begin{cases} D(\omega)e^{-j\frac{\pi}{2}}, & \omega<0 \\ D(\omega)e^{j\frac{\pi}{2}}, & \omega>0 \end{cases}$. $H'(\omega)$ is an HPF with total "width" of $2(\pi-\frac{1}{2})$ with scaling



OR they can be two shifted LPFs scaled by $e^{\mp j\frac{\pi}{2}}$, the LPF having width $\pi-\frac{1}{2}$ (equiv, $\omega_c = \frac{1}{2}(\pi-\frac{1}{2})$). Let $L(\omega)$ represent the freq. response of an LPF with $\omega_c = \frac{1}{2}\pi - \frac{1}{4}$. Then,

Picture:



$$H'(\omega) = e^{-j\pi/2} L\left(\omega + \frac{\pi+\frac{1}{2}}{2}\right) + e^{j\pi/2} L\left(\omega - \frac{\pi+\frac{1}{2}}{2}\right)$$

$$\Updownarrow$$

$$h'[n] = e^{-j\pi/2} e^{-j\left(\frac{\pi+\frac{1}{2}}{2}\right)n} L[n] + e^{j\frac{\pi}{2}} e^{j\left(\frac{\pi+\frac{1}{2}}{2}\right)n} L[n]$$

$$= L[n]\left(e^{-j\left[\left(\frac{\pi+\frac{1}{2}}{2}\right)n + \frac{\pi}{2}\right]} + e^{j\left[\left(\frac{\pi+\frac{1}{2}}{2}\right)n + \frac{\pi}{2}\right]}\right)$$

$$= 2L[n]\cos\left(\left[\frac{\pi+\frac{1}{2}}{2}\right]n + \frac{\pi}{2}\right) = \frac{2\sin\left(\left[\frac{\pi-\frac{1}{2}}{2}\right]n\right)\cos\left(\left[\frac{\pi+\frac{1}{2}}{2}\right]n + \frac{\pi}{2}\right)}{\pi n}$$

(Note: Inv. DTFT for LPF with ω_c is $\frac{\sin(\omega_c n)}{\pi n}$).

Adding back delay ($h_u[n] = h[n-3.5]$):

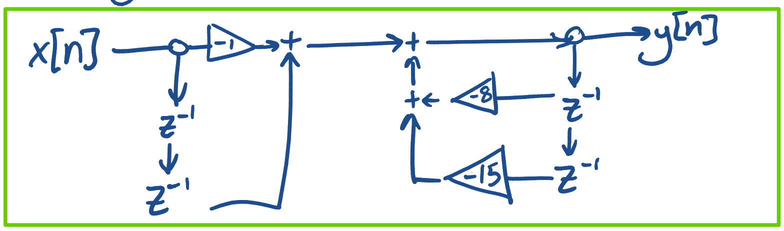
$$\boxed{h_u[n] = \frac{\left(2\sin\left(\left[\frac{\pi-\frac{1}{2}}{2}\right](n-3.5)\right)\cos\left(\left[\frac{\pi+\frac{1}{2}}{2}\right](n-3.5) + \frac{\pi}{2}\right)\right)}{\pi(n-3.5)}}$$

Note: Many possible equivalent expressions possible!

$$y[n] = x[n] + 3y[n-1]$$

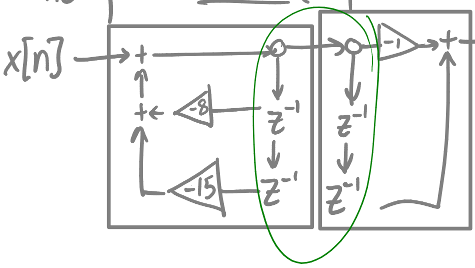
2. Draw below the block diagram in **direct form I**.

$$y[n] = -8y[n-1] - 15y[n-2] - x[n] + x[n-2]$$

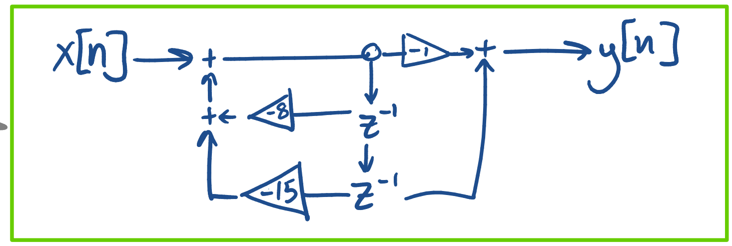


3. Draw below the block diagram in **direct form II**.

1. Swap the blocks (compare with above).



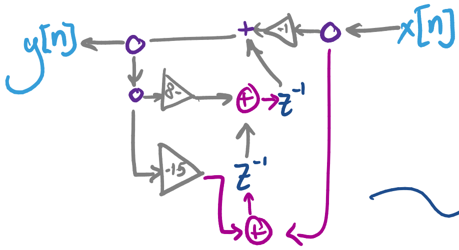
2. merge redundant delays

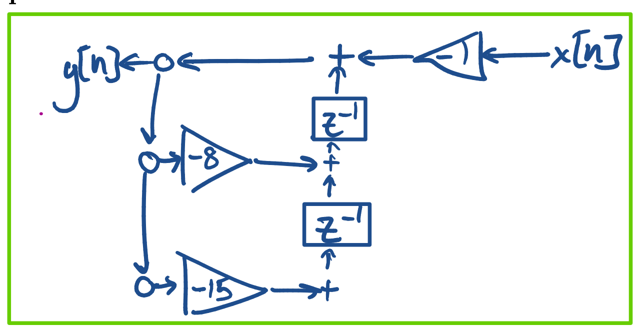


4. Draw below the block diagram in **direct form II, transposed**.

1. Reverse branches 2. Swap branch & + 3. exchange input & output.

4. Make accomodations (multiple inputs to sums)





5. Draw below the block diagram in **cascade for direct form I**.

★ Multiple diagrams possible based on pairing.

$$H(z) = \frac{(z^{-1}-1)(z^{-1}+1)}{(3z^{-1}+1)(5z^{-1}+1)} = \underbrace{\overset{H_1}{\left(\frac{z^{-1}-1}{3z^{-1}+1}\right)}}\underbrace{\overset{H_2}{\left(\frac{z^{-1}+1}{5z^{-1}+1}\right)}}$$

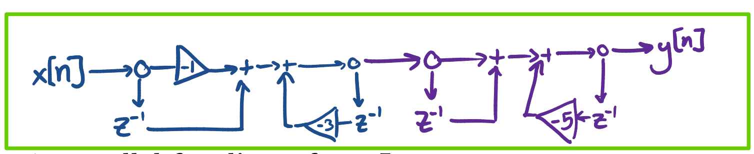
express each as direct form 1, then cascade

$H_1: \frac{z^{-1}-1}{3z^{-1}+1} \longleftrightarrow y[n] + 3y[n-1] = -x[n] + x[n-1]$

$y[n] = -3y[n-1] - x[n] + x[n-1]$

$H_2: \frac{z^{-1}+1}{5z^{-1}+1} \longleftrightarrow y[n] + 5y[n-1] = x[n] + x[n-1]$

$y[n] = -5y[n-1]$



6. Draw below the block diagram in **parallel for direct form I**.

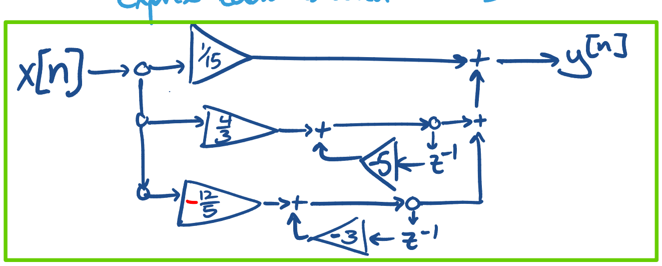
$$H(z) = \frac{(z^{-1}-1)(z^{-1}+1)}{(3z^{-1}+1)(5z^{-1}+1)} \overset{\text{Partial Fraction Expansion}}{=\!=} \underbrace{\frac{1}{15}}_{H_1} + \underbrace{\frac{4/3}{3z^{-1}+1}}_{H_2} + \underbrace{\frac{-12/5}{5z^{-1}+1}}_{H_3}$$

express each as direct form I, then sum.

$H_1: y[n] = \frac{1}{15}x[n]$

$H_2: y[n] + 3y[n-1] = \frac{4}{3}x[n]$

$H_3: y[n] + 5y[n-1] = \frac{12}{5}x[n]$



1.4 DFTs

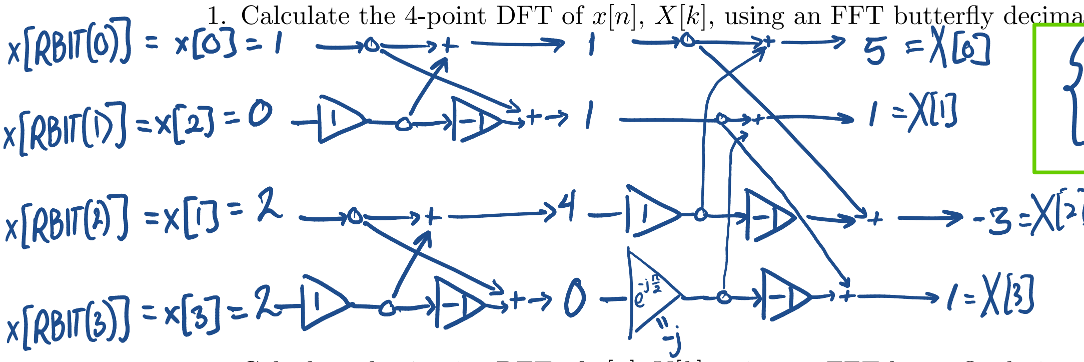
1.4.1 Math Without Calculators

Consider the two below sequences:

Note: 4 values → 2 bits per index.

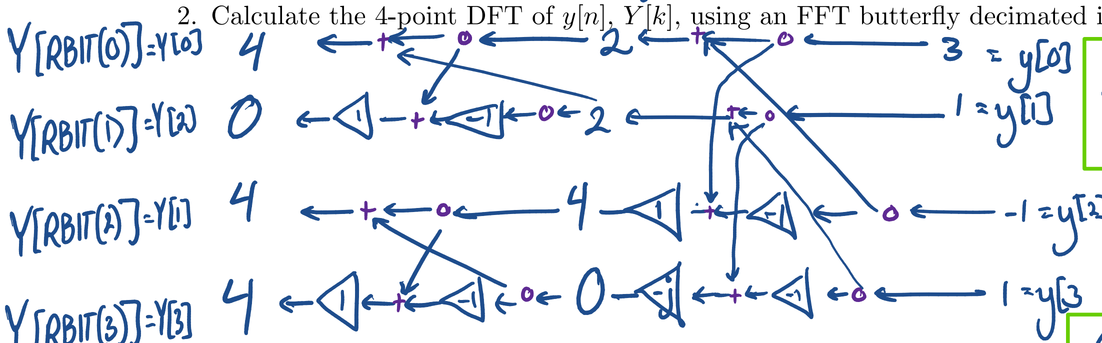
$$\{x[n]\}_{n=0}^{3} = \{1, 2, 0, 2\}, \{y[n]\}_{n=0}^{3} = \{3, 1, -1, 1\}$$

1. Calculate the 4-point DFT of $x[n]$, $X[k]$, using an FFT butterfly decimated in time.



$$\boxed{\{X[k]\}_{k=0}^{3} = \{5, 1, -3, 1\}}$$

2. Calculate the 4-point DFT of $y[n]$, $Y[k]$, using an FFT butterfly decimated in frequency.[2]



$$\boxed{\{Y[k]\}_{k=0}^{3} = \{4, 4, 0, 4\}}$$

Use the footnote.

3. Calculate $z[n] = x[n] \circledast y[n]$, the circular convolution of $x[n]$ and $y[n]$.

$$\boxed{\{z[n]\}_{n=0}^{3} = \{7, 5, 3, 5\}}$$

$y[n] = \{3, 1, -1, 1\}$

$y[-n] = \{3, 1, -1, 1\}$

y[0] y[3] y[2] y[1]

$x[n]$: 1, 2, 0, 2

shift of $y[-n]$

0 : 3, 1, −1, 1 : 1(3)+2(1)+0(−1)+2(1) : 7

1 : 1, 3, 1, −1 : 1(1)+2(3)+0(1)+2(−1) : 5

2 : −1, 1, 3, 1 : 1(−1)+2(1)+0(3)+2(1) : 3

3 : 1, −1, 1, 3 : 1(1)+2(−1)+0(1)+2(3) : 5

4. Calculate the 4-point DFT of $z[n]$, $Z[k]$, using whatever method you would like. Verify that $Z[k_i] = Y[k_i]X[k_i]$ for $i \in \{0, 1, 2, 3\}$.

Let $W \equiv e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$. W is "twiddle factor." $W^0 = 1, W^1 = -j, W^2 = -1, W^3 = j$. $W^n = W^{n+4}$

$Z[0] = 7W^{0\cdot 0} + 5W^{0\cdot 1} + 3W^{0\cdot 2} + 5W^{0\cdot 3} = 20$

$Z[1] = 7W^{1\cdot 0} + 5W^{1\cdot 1} + 3W^{1\cdot 2} + 5W^{1\cdot 3} = 7 - 5j - 3 + 5j = 4$

$Z[2] = 7W^{2\cdot 0} + 5W^{2\cdot 1} + 3W^{2\cdot 2} + 5W^{2\cdot 3} = 7 - 5 + 3 - 5 = 0$

$Z[3] = 7W^{3\cdot 0} + 5W^{3\cdot 1} + 3W^{3\cdot 2} + 5W^{3\cdot 3} = 7 + 5j - 3 - 5j = 4$

$$\boxed{\{Z[k]\}_{n=0}^{4} = \{20, 4, 0, 4\}}$$

$$\overset{\shortparallel}{\underbrace{[5, 1, -3, 1]}_{X[k]} \cdot \underbrace{[4, 4, 0, 4]}_{Y[k]}} \quad ✓$$

[2]Fun fact: applying the transposition operation that you know from block diagrams to the decimation-in-time butterfly, yields the decimation-in-frequency butterfly! Same with the other way around.

BONUS: A generalized Decimation-in-Time Butterfly Template