06 卡尔曼滤波变种——扩展卡尔曼滤波

- **概述**

□ (线性)卡尔曼滤波的缺点

➢ $\boldsymbol{f}$与$\boldsymbol{h}$存在非线性

□ 拓展卡尔曼滤波(EKF)假设

➢ $\boldsymbol{f}$与$\boldsymbol{h}$均可以线性化，线性化后：状态方程：$\boldsymbol{X}_k = \boldsymbol{F}\boldsymbol{X}_{\boldsymbol{k-1}} + \boldsymbol{Q}_k$，观测方程：$\boldsymbol{Y}_k = \boldsymbol{H}\boldsymbol{X}_k + \boldsymbol{R}_k$

➢ $p_{\boldsymbol{Q}}(\boldsymbol{x})$，$p_{\boldsymbol{R}}(\boldsymbol{x})$为均值为0的正态分布，$p_{\boldsymbol{0}}(\boldsymbol{x})$为正态分布

➢ $\boldsymbol{X}_{\boldsymbol{k-1}} \sim \mathcal{N}(\boldsymbol{\mu}_{k-1}^{+}, \boldsymbol{\Sigma}_{k-1}^{+})$

- **拓展卡尔曼滤波1**

  - 一维EKF

  - $Q_k \sim \mathcal{N}(0,\ Q^2)$，$R_k \sim \mathcal{N}(0,\ R^2)$，$x_{k-1}^{+} \sim \mathcal{N}\left(\mu_{k-1}^{+}, \sigma_{k-1}^{2+}\right)$

预测线性化

$$
\begin{aligned}
f(X_{k-1}) &= f(\mu_{k-1}^{+}) + f'(\mu_{k-1}^{+})(X_{k-1} - \mu_{k-1}^{+}) + o(\mu_{k-1}^{+}) \\
&\approx f(\mu_{k-1}^{+}) + f'(\mu_{k-1}^{+})(X_{k-1} - \mu_{k-1}^{+}) \\
&= f'(\mu_{k-1}^{+})X_{k-1} + f(\mu_{k-1}^{+}) - f'(\mu_{k-1}^{+})\mu_{k-1}^{+}
\end{aligned}
$$

函数$f(X_{k-1})$在$X_{k-1} = \mu_{k-1}^{+}$处的一阶Taylor展开

设$A = f'(\mu_{k-1}^{+}), B = f(\mu_{k-1}^{+}) - f'(\mu_{k-1}^{+})\mu_{k-1}^{+}$

预测方程：$X_k = AX_{k-1} + B + Q_k$，其中$A = f'(\mu_{k-1}^{+}), B = f(\mu_{k-1}^{+}) - f'(\mu_{k-1}^{+})\mu_{k-1}^{+}$

- **拓展卡尔曼滤波2**

□ 一维EKF

➢ $Q_k \sim \mathcal{N}(0,\ Q^2)$，$R_k \sim \mathcal{N}(0,\ R^2)$，$x_{k-1}^{+} \sim \mathcal{N}\left(\mu_{k-1}^{+}, \sigma_{k-1}^{2+}\right)$

➢ 线性化预测方程：$X_k = AX_{k-1} + B + Q_k$，其中$A = f'(\mu_{k-1}^{+}), B = f(\mu_{k-1}^{+}) - f'(\mu_{k-1}^{+})\mu_{k-1}^{+}$

预测步

$$f(v) = Av + B$$

$$p_k^{-}(x) = \int_{-\infty}^{\infty} p_{Q_k}\left(x - f(v)\right) p_{f_{k-1}}(v) dv$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}Q} \exp\left(-\frac{(x - Av - B)^2}{2Q^2}\right) \frac{1}{\sqrt{2\pi}\sigma_{k-1}^{+}} \exp\left(-\frac{(v - \mu_{k-1}^{+})^2}{2\sigma_{k-1}^{2+}}\right) dv$$

$$= \mathcal{N}\left(A\mu_{k-1}^{+} + B, A^2\sigma_{k-1}^{2+} + Q^2\right) = \mathcal{N}\left(f(\mu_{k-1}^{+}), A^2\sigma_{k-1}^{2+} + Q^2\right)$$

$A\mu_{k-1}^{+} + B = f'(\mu_{k-1}^{+})\mu_{k-1}^{+} + f(\mu_{k-1}^{+}) - f'(\mu_{k-1}^{+})\mu_{k-1}^{+} = f(\mu_{k-1}^{+})$

只需要计算A，而不需要计算B

预测步结论

➢ $x_k^{-} \sim \mathcal{N}\left(f(\mu_{k-1}^{+}), A^2\sigma_{k-1}^{2+} + Q^2\right)$

➢ (1) $\mu_k^{-} = f(\mu_{k-1}^{+})$

➢ (2) $\sigma_k^{2-} = A^2\sigma_{k-1}^{2+} + Q^2$

➢ $A = f'(\mu_{k-1}^{+})$

- **拓展卡尔曼滤波3**

  - 一维EKF

  - $Q_k \sim \mathcal{N}(0,\ Q^2)$，$R_k \sim \mathcal{N}(0,\ R^2)$，$x_k^+ \sim \mathcal{N}\left(\mu_k^-, \sigma_k^{2-}\right)$

更新线性化

$$h(X_k) = h(\mu_k^-) + h'(\mu_k^-)(X_k - \mu_k^-) + o(\mu_k^-)$$

$$\approx h(\mu_k^-) + h'(\mu_k^-)(X_k - \mu_k^-)$$

$$= h'(\mu_k^-)X_k + h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$$

函数$h(X_k)$在$X_k = \mu_k^-$处的一阶Taylor展开

设$C = h'(\mu_k^-), D = h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$

更新方程：$Y_k = CX_k + D + R_k$，其中$C = h'(\mu_k^-), D = h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$

- **拓展卡尔曼滤波4**

□ 一维EKF

➢ $Q_k \sim \mathcal{N}(0,\ Q^2)$，$R_k \sim \mathcal{N}(0,\ R^2)$，$x_k^+ \sim \mathcal{N}\left(\mu_k^-, \sigma_k^{2-}\right)$

➢ 线性化更新方程：$Y_k = CX_k + D + R_k$，其中$C = h'(\mu_k^-), D = h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$

更新步

$h(x) = Cx + D$

$$p_k^+(x) = \eta * p_{R_k}\left(y_k - h(x)\right) * p_k^-(x)$$

$$= \eta * \frac{1}{\sqrt{2\pi}R}\exp\left(-\frac{(y_k - Cx - D)^2}{2R^2}\right)\frac{1}{\sqrt{2\pi}\sigma_k^-}\exp\left(-\frac{(x-\mu_k^-)^2}{2\sigma_k^{2-}}\right)$$

$$= \mathcal{N}\left(\mu_k^- + \frac{C\sigma_k^{2-}}{C^2\sigma_k^{2-} + R^2}\ (y_k - C\mu_k^-), \left(1 - \frac{C^2\sigma_k^{2-}}{C^2\sigma_k^{2-} + R^2}\right)\sigma_k^{2-}\right)$$

更新步结论

➢ $x_k^+ \sim \mathcal{N}\left(\mu_k^- + K_k(y_k - C\mu_k^- - D), (1 - K_k H)\sigma_k^{2-}\right) = \mathcal{N}\left(\mu_k^- + K_k\left(y_k - h(\mu_k^-)\right), (1 - K_k H)\sigma_k^{2-}\right)$

➢ (4) $\mu_k^+ = \mu_k^- + K_k\left(y_k - h(\mu_k^-)\right)$

➢ (5) $\sigma_k^{2+} = (1 - K_k C)\sigma_k^{2-}$

➢ $C = h'(\mu_k^-)$　　　　(3) 卡尔曼增益$K_k = \frac{C\sigma_k^{2-}}{C^2\sigma_k^{2-}+R^2}$

$C\mu_k^- + D = h'(\mu_k^-)\mu_k^- + h(\mu_k^-) - h'(\mu_k^-)\mu_k^- = h(\mu_k^-)$

只需要计算C，而不需要计算D

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- **拓展卡尔曼滤波5**

□ 一维EKF

➢ $Q_k \sim \mathcal{N}(0,\ \ Q^2)$，$R_k \sim \mathcal{N}(0,\ \ R^2)$，$x_k^+ \sim \mathcal{N}\left(\mu_k^-, \sigma_k^{2-}\right)$

➢ 线性化预测方程：$X_k = AX_{k-1} + B + Q_k$，其中$A = f'(\mu_{k-1}^+), B = f(\mu_{k-1}^+) - f'(\mu_{k-1}^+)\mu_{k-1}^+$

➢ 线性化更新方程：$Y_k = CX_k + D + R_k$，其中$C = h'(\mu_k^-), D = h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$

拓展卡尔曼滤波

➢ 计算预测系数$A = f'(\mu_{k-1}^+)$

➢ (1) 一步预测状态：$\mu_k^- = f(\mu_{k-1}^+)$

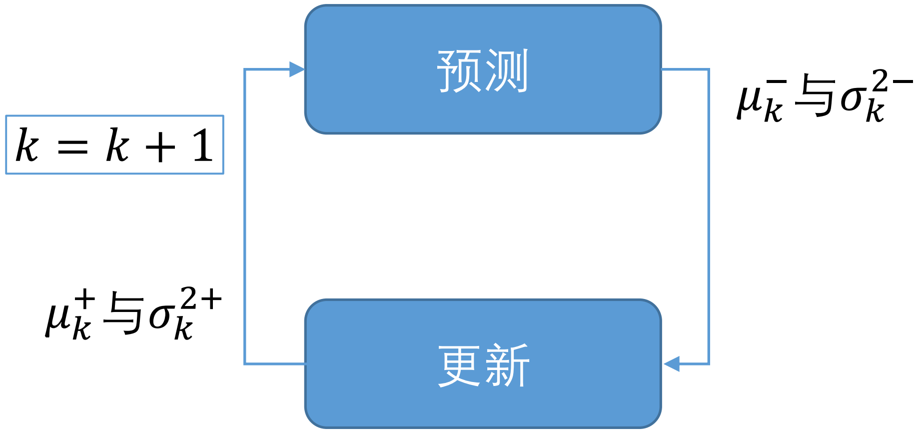
➢ (2) 一步预测方差：$\sigma_k^{2-} = A^2\sigma_{k-1}^{2+} + Q^2$

➢ 计算更新系数$C = h'(\mu_k^-)$

➢ (3) 更新卡尔曼增益：$K_k = \frac{C\sigma_k^{2-}}{C^2\sigma_k^{2-} + R^2}$

➢ (4) 更新状态：$\mu_k^+ = \mu_k^- + K_k\left(y_k - h(\mu_k^-)\right)$

➢ (5) 更新方差：$\sigma_k^{2+} = (1 - K_kC)\sigma_k^{2-}$



- **拓展卡尔曼滤波6**

□ 多维EKF

➢ $\boldsymbol{Q}_k \sim \mathcal{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{Q}_k}\right)$，$\boldsymbol{R}_k \sim \mathcal{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{R}_k}\right)$，$\boldsymbol{x}_{k-1}^{+} \sim \mathcal{N}(\boldsymbol{\mu}_{k-1}^{+}, \boldsymbol{\Sigma}_{k-1}^{+})$

预测线性化

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{X}_{k-1}) &= \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+}) + \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})(\boldsymbol{X}_{k-1} - \boldsymbol{\mu}_{k-1}^{+}) + \boldsymbol{o}(\boldsymbol{\mu}_{k-1}^{+}) \\ &\approx \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+}) + \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})(\boldsymbol{X}_{k-1} - \boldsymbol{\mu}_{k-1}^{+}) \\ &= \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})\boldsymbol{X}_{k-1} + \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+}) - \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})\boldsymbol{\mu}_{k-1}^{+}\end{aligned}$$

函数$\boldsymbol{f}(\boldsymbol{X}_{k-1})$在$\boldsymbol{X}_{k-1} = \boldsymbol{\mu}_{k-1}^{+}$处的一阶Taylor展开

设$\boldsymbol{A} = \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+}), \boldsymbol{B} = \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+}) - \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})\boldsymbol{\mu}_{k-1}^{+}$

$$\boldsymbol{A} = \left[\frac{\partial f_i}{\partial x_{k-1}^{j}}\right]_{ij} \Bigg|_{\boldsymbol{X}_{k-1} = \boldsymbol{\mu}_{k-1}^{+}}$$

预测方程：$\boldsymbol{X}_k = \boldsymbol{A}\boldsymbol{X}_{k-1} + \boldsymbol{B} + \boldsymbol{Q}_k$，其中$\boldsymbol{A} = \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+}), \boldsymbol{B} = \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+}) - \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})\boldsymbol{\mu}_{k-1}^{+}$

- **拓展卡尔曼滤波7**

  - 多维EKF

  - $Q_k \sim \mathcal{N}\left(0, \Sigma_{Q_k}\right)$，$R_k \sim \mathcal{N}\left(0, \Sigma_{R_k}\right)$，$x_{k-1}^{+} \sim \mathcal{N}(\mu_{k-1}^{+}, \Sigma_{k-1}^{+})$

更新线性化

函数$h(X_k)$在$X_k = \mu_k^-$处的一阶Taylor展开

$$h(X_k) = h(\mu_k^-) + h'(\mu_k^-)(X_k - \mu_k^-) + o(\mu_k^-)$$

$$\approx h(\mu_k^-) + h'(\mu_k^-)(X_k - \mu_k^-)$$

$$= h'(\mu_k^-)X_k + h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$$

设$C = h'(\mu_k^-), D = h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$

$$C = \left[\frac{\partial h_i}{\partial x_k^j}\right]_{ij} \Bigg|_{X_k=\mu_k^-}$$

更新方程：$Y_k = CX_k + D + R_k$，其中$C = h'(\mu_k^-), D = h(\mu_k^-) - h'(\mu_k^-)\mu_k^-$

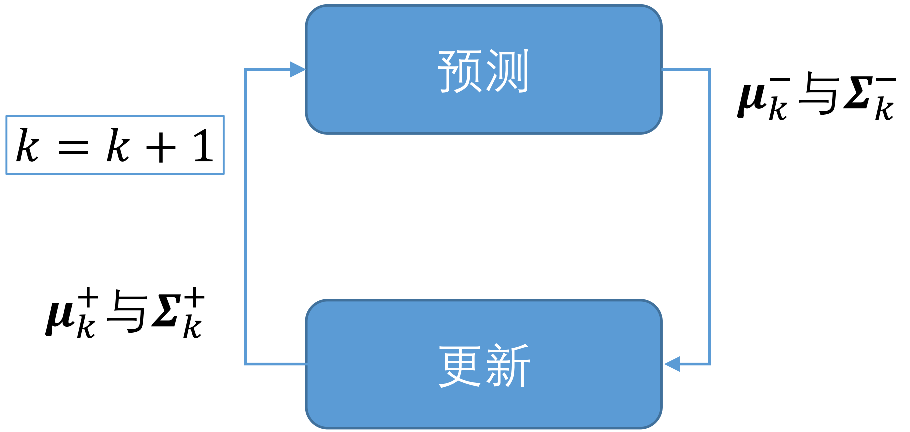
● 拓展卡尔曼滤波8

□ 多维EKF

- ➢ $\boldsymbol{Q}_k \sim \mathcal{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{Q}_k}\right)$，$\boldsymbol{R}_k \sim \mathcal{N}\left(0, \boldsymbol{\Sigma}_{\boldsymbol{R}_k}\right)$，$\boldsymbol{x}_{k-1}^{+} \sim \mathcal{N}(\boldsymbol{\mu}_{k-1}^{+}, \boldsymbol{\Sigma}_{k-1}^{+})$
- ➢ 线性化预测方程：$\boldsymbol{X}_k = \boldsymbol{A}\boldsymbol{X}_{k-1} + \boldsymbol{B} + \boldsymbol{Q}_k$，其中$\boldsymbol{A} = \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+}), \boldsymbol{B} = \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+}) - \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})\boldsymbol{\mu}_{k-1}^{+}$
- ➢ 线性化更新方程：$\boldsymbol{Y}_k = \boldsymbol{C}\boldsymbol{X}_k + \boldsymbol{D} + \boldsymbol{R}_k$，其中$\boldsymbol{C} = \boldsymbol{h}'(\boldsymbol{\mu}_k^{-}), \boldsymbol{D} = \boldsymbol{h}(\boldsymbol{\mu}_k^{-}) - \boldsymbol{h}'(\boldsymbol{\mu}_k^{-})\boldsymbol{\mu}_k^{-}$

拓展卡尔曼滤波

- ➢ 预测雅克比矩阵：$\boldsymbol{A} = \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^{+})$
- ➢ (1) 一步预测状态：$\boldsymbol{\mu}_k^{-} = \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^{+})$
- ➢ (2) 一步预测方差：$\boldsymbol{\Sigma}_k^{-} = \boldsymbol{A}\boldsymbol{\Sigma}_{k-1}^{+}\boldsymbol{A}^T + \boldsymbol{\Sigma}_{\boldsymbol{Q}_k}$
- ➢ 更新雅克比矩阵：$\boldsymbol{C} = \boldsymbol{h}'(\boldsymbol{\mu}_k^{-})$
- ➢ (3) 更新卡尔曼增益：$\boldsymbol{K}_k = \boldsymbol{\Sigma}_k^{-}\boldsymbol{C}^{\mathrm{T}}\left(\boldsymbol{C}\boldsymbol{\Sigma}_k^{-}\boldsymbol{C}^T + \boldsymbol{\Sigma}_{\boldsymbol{R}_k}\right)^{-1}$
- ➢ (4) 更新状态：$\boldsymbol{\mu}_k^{+} = \boldsymbol{\mu}_k^{-} + \boldsymbol{K}_k\left(\boldsymbol{y}_k - \boldsymbol{h}(\boldsymbol{\mu}_k^{-})\right)$
- ➢ (5) 更新方差：$\boldsymbol{\Sigma}_k^{+} = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{C})\boldsymbol{\Sigma}_k^{-}$



- **KF与EKF**
  - 不同点
    - 线性化的问题：EKF在预测状态与更新状态时均无需线性化

卡尔曼滤波

- (1) 一步预测状态：$\boldsymbol{\mu}_k^- = \boldsymbol{F}\boldsymbol{\mu}_{k-1}^+$
- (2) 一步预测方差：$\boldsymbol{\Sigma}_k^- = \boldsymbol{F}\boldsymbol{\Sigma}_{k-1}^+\boldsymbol{F}^T + \boldsymbol{\Sigma}_{\boldsymbol{Q}_k}$
- (3) 更新卡尔曼增益：$\boldsymbol{K}_k = \boldsymbol{\Sigma}_k^-\boldsymbol{H}^{\mathrm{T}}\left(\boldsymbol{H}\boldsymbol{\Sigma}_k^-\boldsymbol{H}^T+\boldsymbol{\Sigma}_{\boldsymbol{R}_k}\right)^{-1}$
- (4) 更新状态：$\boldsymbol{\mu}_k^+ = \boldsymbol{\mu}_k^- + \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{H}\boldsymbol{\mu}_k^-)$
- (5) 更新方差：$\boldsymbol{\Sigma}_k^+ = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H})\boldsymbol{\Sigma}_k^-$

拓展卡尔曼滤波

- 预测雅克比矩阵：$\boldsymbol{A} = \boldsymbol{f}'(\boldsymbol{\mu}_{k-1}^+)$
- (1) 一步预测状态：$\boldsymbol{\mu}_k^- = \boldsymbol{f}(\boldsymbol{\mu}_{k-1}^+)$
- (2) 一步预测方差：$\boldsymbol{\Sigma}_k^- = \boldsymbol{A}\boldsymbol{\Sigma}_{k-1}^+\boldsymbol{A}^T + \boldsymbol{\Sigma}_{\boldsymbol{Q}_k}$
- 更新雅克比矩阵：$\boldsymbol{C} = \boldsymbol{h}'(\boldsymbol{\mu}_k^-)$
- (3) 更新卡尔曼增益：$\boldsymbol{K}_k = \boldsymbol{\Sigma}_k^-\boldsymbol{C}^{\mathrm{T}}\left(\boldsymbol{C}\boldsymbol{\Sigma}_k^-\boldsymbol{C}^T+\boldsymbol{\Sigma}_{\boldsymbol{R}_k}\right)^{-1}$
- (4) 更新状态：$\boldsymbol{\mu}_k^+ = \boldsymbol{\mu}_k^- + \boldsymbol{K}_k\left(\boldsymbol{y}_k - \boldsymbol{h}(\boldsymbol{\mu}_k^-)\right)$
- (5) 更新方差：$\boldsymbol{\Sigma}_k^+ = (\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{C})\boldsymbol{\Sigma}_k^-$