- **性能指标** 积分型性能指标$\mathcal{J} = \int_{t_0}^{t_f} F[\boldsymbol{x}(t), u(t), t]dt$

■ 最短时间问题
- ➢ $\mathcal{J} = t_f - t_0 = \int_{t_0}^{t_f} dt$
- ➢ $F[\boldsymbol{x}(t), u(t), t] = 1$

■ 最小燃料消耗问题
- ➢ 控制量$u(t)$与燃料消耗能量成正比
- ➢ $\mathcal{J} = \int_{t_0}^{t_f} |u(t)|dt$
- ➢ $F[\boldsymbol{x}(t), u(t), t] = |u(t)|$

■ 最小能量控制问题
- ➢ 考虑与消耗功率成正比
- ➢ $\mathcal{J} = \int_{t_0}^{t_f} u^2(t)dt$
- ➢ $F[\boldsymbol{x}(t), \boldsymbol{u}(t), t] = u^2(t)$

■ 状态镇定问题
- ➢ 稳定：状态$x(t)$/输出$y(t)$渐近收敛到0
- ➢ $\mathcal{J} = \int_{t_0}^{t_f} \frac{1}{2} x^T(t)Qx(t)dt$
- ➢ $F[x(t), u(t), t] = \frac{1}{2} x^T(t)Qx(t)$

■ 状态跟踪问题
- ➢ 跟踪：状态$x(t)$/输出$y(t)$跟踪目标轨线$x_d(t)/y_d(t)$
- ➢ $\mathcal{J} = \int_{t_0}^{t_f} \frac{1}{2} \left(x(t) - x_d(t)\right)^T Q\left(x(t) - x_d(t)\right)dt$
- ➢ $F[\boldsymbol{x}(t), u(t), t] = \frac{1}{2} \left(x(t) - x_d(t)\right)^T Q\left(x(t) - x_d(t)\right)$

■ 调节问题
- ➢ 调节：正则化(避免过拟合)
- ➢ 镇定调节：$\mathcal{J} = \int_{t_0}^{t_f} \frac{1}{2} [x^T(t)Qx(t) + u^T(t)Ru(t)]dt$
- ➢ $F[\boldsymbol{x}(t), \boldsymbol{u}(t), t] = \frac{1}{2} [x^T(t)Qx(t) + u^T(t)Ru(t)]$
- ➢ 跟踪调节：$\mathcal{J} = \int_{t_0}^{t_f} \frac{1}{2} \left[\left(x(t) - x_d(t)\right)^T Q\left(x(t) - x_d(t)\right) + u^T(t)Ru(t)\right] dt$
- ➢ $F[\boldsymbol{x}(t), \boldsymbol{u}(t), t] = \frac{1}{2} \left[\left(x(t) - x_d(t)\right)^T Q\left(x(t) - x_d(t)\right) + u^T(t)Ru(t)\right]$

- **问题简化**

- 变分法求解最优控制问题的特点
  - 没有明确考虑控制能量消耗的问题
  - 针对一般研究对象，不易获取具有解析表达的控制器
  - 所获得的最优控制器难以表达为状态反馈的形式
- 线性系统的简化——二次型性能指标
  - 物理意义明确，可以兼顾系统性能与能量消耗
  - 可以得到统一的解析表达
  - 可以得到状态线性反馈的最优控制律，从而形成闭环

# 06 镇定、跟踪与调节



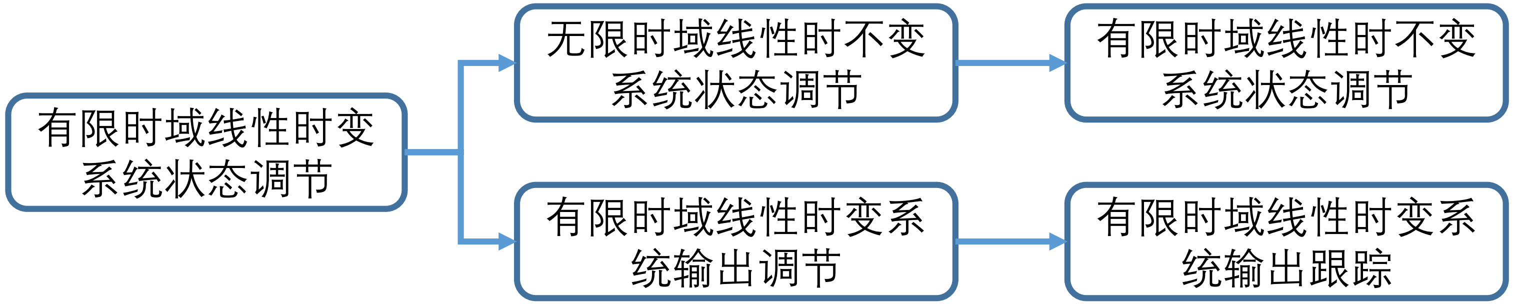
## ● 问题简化

■ 二次型性能指标最优控制的分类

- 状态调节问题：$C(t)=I$，$y_d(t)=0$ 用较小的控制能量使状态维持在零附近
- 输出调节问题：$y_d(t)=0$ 用较小的控制能量使输出维持在零附近
- 输出跟踪问题：$y_d(t)\neq 0$ 用较小的控制能量使输出跟踪理想输出

■ 线性系统求解思路

  - 求解有限时域线性时变系统的状态调节问题
  - 推广到无限时域时不变系统的状态问题，然后推广到有限时域时不变系统状态调节问题
  - 推广到有限时域时变系统输出调节问题，然后推广到有限时域时变系统的输出跟踪问题



## ● 二次型性能指标最优控制提出

### ■ 线性时变系统

- 被控对象：$\dot{x}(t) = A(t)x(t) + B(t)u(t)$
  $y(t) = C(t)x(t)$
- 误差向量：$e(t) = y_d(t) - y(t)$
- 二次型性能指标：$\mathcal{J} = \frac{1}{2}e^T(t_f)Fe(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[e^T(t)Q(t)e(t) + u^T(t)R(t)u(t)]dt$
- 第一项$\frac{1}{2}e^T(t_f)Fe(t_f)$终端时刻的控制误差，$F$半正定
- 第二项$\frac{1}{2}e^T(t)Q(t)e(t)$控制过程的控制误差，$Q$半正定
- 第三项$\frac{1}{2}u^T(t)R(t)u(t)$控制过程的能量消耗，$R$正定
- 权值矩阵：衡量各个误差成分和控制分量的重要程度

被控对象：$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

二次型性能指标：$\mathcal{J} = \frac{1}{2}x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt$

- **有限时域线性时变系统状态调节**

■ 极大值原理

➢ 哈密顿函数$H(x,u,\lambda,t) = \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) + \lambda^T(t)A(t)x(t) + \lambda^T(t)B(t)u(t)$

➢ 正则方程 $\dot{x}^*(t) = \frac{\partial H}{\partial \lambda} = A(t)x^*(t) + B(t)u^*(t)$ (1) $\dot{\lambda}^*(t) = -\frac{\partial H}{\partial x} = -Q(t)x^*(t) - A^T(t)\lambda^*(t)$ (2)

➢ 控制方程 $\frac{\partial H}{\partial u} = R(t)u^*(t) + B^T(t)\lambda^*(t)$ => $u^*(t) = -R^{-1}(t)B^T(t)\lambda^*(t)$ (3) ($R(t)$正定的原因)

➢ 正则方程1 $\dot{x}^*(t) = A(t)x^*(t) - B(t)R^{-1}(t)B^T(t)\lambda^*(t)$ (4) $\dot{\lambda}^*(t) = -Q(t)x^*(t) - A^T(t)\lambda^*(t)$(5)

➢ 横截条件 $\lambda^*(t_f) = \frac{\partial}{\partial x(t_f)}\left[\frac{1}{2}x^T(t_f)Fx(t_f)\right] = Fx^*(t_f)$ (6)

➢ 终端时刻参数$\lambda^*(t_f)$与终端状态$x^*(t_f)$成正比，为了简化问题，任意时刻，均满足线性关系

$\lambda^*(t) = P(t)x^*(t)$ (7)，其导数为$\dot{\lambda}^*(t) = \dot{P}(t)x^*(t) + P(t)\dot{x}^*(t)$ (8)

被控对象：$\dot{x}(t) = A(t)x(t) + B(t)u(t)$

二次型性能指标：$\mathcal{J} = \frac{1}{2}x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt$

- **有限时域线性时变系统状态调节**

■ 极大值原理

- (4)和(8)，$\dot{\lambda}^*(t) = [\dot{P}(t) + P(t)A(t) - P(t)B(t)R^{-1}B^T(t)P(t)]x^*(t)$
- (5)和(7)，$\dot{\lambda}^*(t) = [-Q(t) - A^T(t)P(t)]x^*(t)$
- 联立，可得：$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}B^T(t)P(t) - Q(t)$
- 一阶非线性矩阵Riccati微分方程，边界条件：$P(t_f) = F$
- 最优控制解$u^*(t) = -R^{-1}(t)B^T(t)\lambda^*(t) = -R^{-1}(t)B^T(t)P(t)x^*(t)$
- $K(t) = -R^{-1}(t)B^T(t)P(t)$，$u^*(t) = K(t)x^*(t)$ 线性状态反馈

- 最优控制律是线性状态反馈，可以实现闭环最优控制
- 可以证明$P(t)$是对称矩阵，即$P(t) = P^T(t)$
- 最优性能指标$\mathcal{J} = \frac{1}{2}x^T(t_0)P(t_0)x(t_0)$

被控对象：$\dot{x}(t) = Ax(t) + Bu(t)$

二次型性能指标：$\mathcal{J} = \int_{t_0}^{\infty} \frac{1}{2}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt$

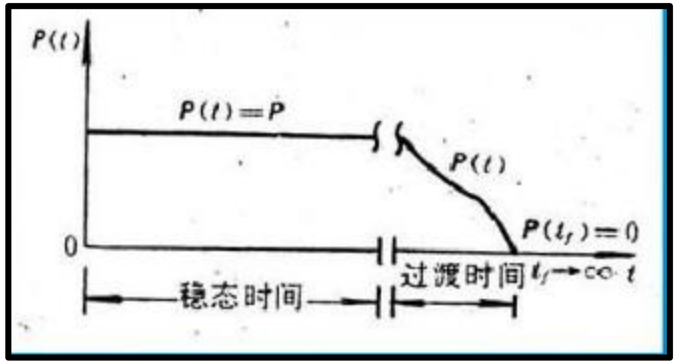
- **无限时域线性时不变系统状态调节**

■ 有限时域线性时变系统状态调节的推广

- 考虑稳态性能，且$P(\infty) = F = 0$
- 系统完全能控
- 有限时域线性时变系统状态调节问题的特殊形式
- 最优控制解$u^*(t) = -R^{-1}(t)B^T(t)P(t)x^*(t)$
- 最优性能指标$\mathcal{J} = \frac{1}{2}x^T(t_0)P(t_0)x(t_0)$，其中$P(\infty) = 0$

■ 最优控制解的讨论

- 当终端时刻$t_f = \infty$时，满足$P(\infty) = 0$
- $P(t)$在接近终端时变化剧烈
- $P(t)$在远离终端时趋近于某个常矩阵$\bar{P}$



被控对象：$\dot{x}(t) = Ax(t) + Bu(t)$

二次型性能指标：$\mathcal{J} = \int_{t_0}^{\infty} \frac{1}{2}[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)]dt$

- **无限时域线性时不变系统状态调节**

■ 最优控制解的讨论

- 有限时域内，系统是远离终端的，从而$P(t)$可以用其稳态值$\bar{P}$代替
- Riccati微分方程可以简化为Riccati代数方程 $\bar{P}A + A^T\bar{P} - \bar{P}BR^{-1}B^T\bar{P} + Q = 0$
- 求解非线性方程，可以得到稳态解
- $\bar{P}$是对称正定阵，容易实现
- 最优控制律是一个定常线性状态反馈，$u^*(t) = -R^{-1}B^T\bar{P}x^*(t)$
- 最优性能指标$\mathcal{J} = \frac{1}{2}x^T(t_0)\bar{P}x(t_0)$

- **有限时域线性时变系统输出调节**

被控对象：$\dot{x}(t) = A(t)x(t) + B(t)u(t)$
$y(t) = C(t)x(t)$

二次型性能指标：$\mathcal{J} = \frac{1}{2}y^T(t_f)Fy(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[y^T(t)Q(t)y(t) + u^T(t)R(t)u(t)]dt$

- 求解过程
  - 将输出调节问题通过系统的输出方程转换为状态调节问题
  - $\mathcal{J} = \frac{1}{2}x^T(t_f)C^T(t)FC(t)x(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[x^T(t)C^T(t)Q(t)C(t)x(t) + u^T(t)R(t)u(t)]dt$
  - 与状态调节问题相比，唯一区别在于权值系数矩阵发生了改变
  - 若这种变换不影响权值系数矩阵的正定性，则可以用状态调节相关结论
  - 定理：当$F$和$Q(t)$半正定时，当且仅当系统完全可观测时，$C^T(t)FC(t)$和$C^T(t)Q(t)C(t)$是半正定
  - 当系统完全可观测时，存在唯一的输出调节最优控制器为$u^*(t) = -R^{-1}(t)B^T(t)P(t)x^*(t)$
  - 其微分Riccati方程为$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}B^T(t)P(t) - C^T(t)Q(t)C(t)$
  - 终端积分条件$P(t_f) = C^T(t_f)FC(t_f)$
  - 最优性能指标$\mathcal{J} = \frac{1}{2}x^T(t_0)P(t_0)x(t_0)$

- **有限时域线性时变系统输出调节**

■ 解的讨论

被控对象：$\dot{x}(t) = A(t)x(t) + B(t)u(t)$
$y(t) = C(t)x(t)$

二次型性能指标：$\mathcal{J} = \frac{1}{2}y^T(t_f)Fy(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[y^T(t)Q(t)y(t) + u^T(t)R(t)u(t)]dt$

➢ 输出调节问题的最优控制并不是输出量的线性反馈，仍然是状态量的线性反馈（最优控制需要完全状态信息，输出仅反映了状态的线性组合，无法提供系统全部的内部特性）

➢ 无限时域线性时不变系统输出调节：$\mathcal{J} = \int_{t_0}^{\infty}\frac{1}{2}[y^T(t)Q(t)y(t) + u^T(t)R(t)u(t)]dt$，只要系统是完全能控能观测的，其最优控制律为$u^*(t) = -R^{-1}B^T\bar{P}x^*(t)$，代数Riccati方程为$\bar{P}A + A^T\bar{P} - \bar{P}BR^{-1}B^T\bar{P} + C^TQC = 0$，最优性能指标为$\mathcal{J} = \frac{1}{2}x^T(t_0)\bar{P}x(t_0)$

- **有限时域线性时变系统输出跟踪**

被控对象 $\dot{x}(t)=A(t)x(t)+B(t)u(t)$, $y(t)=C(t)x(t)$ 性能指标 $\mathcal{J}=\frac{1}{2}\left(z(t_f)-y(t_f)\right)^T F\left(z(t_f)-y(t_f)\right)$

$+\int_{t_0}^{t_f}\frac{1}{2}\left[\left(z(t)-y(t)\right)^T Q(t)\left(z(t)-y(t)\right)+u^T(t)R(t)u(t)\right]dt$

■ 求解过程极大值原理

- 哈密顿函数函数$H(x,u,\lambda,t)=\frac{1}{2}\left(z(t)-C(t)x(t)\right)^T Q(t)\left(z(t)-C(t)x(t)\right)+\frac{1}{2}u^T(t)R(t)u(t)+\lambda^T(t)(A(t)x(t)+B(t)u(t))$
- 正则方程1 $\dot{x}^*(t)=\frac{\partial H}{\partial \lambda}=A(t)x^*(t)+B(t)u^*(t)$ (1)
- 正则方程2 $\dot{\lambda}^*(t)=-\frac{\partial H}{\partial x}=C^T(t)Q(t)\left(z(t)-C(t)x^*(t)\right)-A^T(t)\lambda^*(t)$ (2)
- 边界条件 $x^*(t_0)=x_0$ (3)
- 横截条件 $\lambda^*(t_f)=\frac{\partial}{\partial x(t_f)}\left[\frac{1}{2}\left(z(t_f)-y(t_f)\right)^T F\left(z(t_f)-y(t_f)\right)\right]=-C^T(t_f)F\left(z(t_f)-C(t_f)x^*(t_f)\right)$ (4)
- 控制方程 $\frac{\partial H}{\partial u}=R(t)u^*(t)+B^T(t)\lambda^*(t)$ => $u^*(t)=-R^{-1}(t)B^T(t)\lambda^*(t)$ (5)

- **有限时域线性时变系统输出跟踪**

被控对象 $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $y(t) = C(t)x(t)$ 性能指标 $\mathcal{J} = \frac{1}{2}\left(z(t_f) - y(t_f)\right)^T F\left(z(t_f) - y(t_f)\right)$

$+\int_{t_0}^{t_f} \frac{1}{2}\left[\left(z(t) - y(t)\right)^T Q(t)\left(z(t) - y(t)\right) + u^T(t)R(t)u(t)\right] dt$

  - 求解过程极大值原理
    - 假设存在线性关系$\lambda^*(t) = P(t)x^*(t) - g(t)$，其中$P(t_f) = C^T(t_f)FC(t_f)$，$g(t_f) = C^T(t_f)Fz(t_f)$
    - 其导数为：$\dot{\lambda}^*(t) = \dot{P}(t)x^*(t) + P(t)\dot{x}^*(t) - \dot{g}(t)$，将一次关系代入正则方程
    - $\dot{\lambda}^*(t) = C^T(t)Q(t)\left(z(t) - C(t)x^*(t)\right) - A^T(t)\left(P(t)x^*(t) - g(t)\right)$
    - 两个式子相等：$\dot{P}(t)x^*(t) + P(t)\dot{x}^*(t) - \dot{g}(t) = C^T(t)Q(t)\left(z(t) - C(t)x^*(t)\right) - A^T(t)\left(P(t)x^*(t) - g(t)\right)$
    - 将控制器 $u^*(t) = -R^{-1}(t)B^T(t)\lambda^*(t) = -R^{-1}(t)B^T(t)\left(P(t)x^*(t) - g(t)\right)$代入上式系统方程，再代入上式
    - $\left[\dot{P}(t) + P(t)A(t) - P(t)B(t)R^{-1}(t)B^T(t)P(t)\right] + P(t)B(t)R^{-1}(t)B^T(t)g(t) - \dot{g}(t) =$
      $-[C^T(t)Q(t)C(t) + A^T(t)P(t)]x^*(t) + C^T(t)Q(t)z(t) + A^T(t)g(t)$
    - Riccati微分方程：$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) - C^T(t)Q(t)C(t)$
    - 补充微分方程：$\dot{g}(t) = -[A^T(t) - P(t)B(t)R^{-1}(t)B^T(t)]g(t) - C^T(t)Q(t)z(t)$
    - 边界条件：$P(t_f) = C^T(t_f)FC(t_f)$，$g(t_f) = C^T(t_f)Fz(t_f)$
    - 解的形式： => $u^*(t) = -R^{-1}(t)B^T(t)[P(t)x^*(t) - g(t)]$

- **有限时域线性时不变系统输出跟踪**

被控对象 $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ 性能指标 $\mathcal{J} = \frac{1}{2}\left(z(t_f) - y(t_f)\right)^T F\left(z(t_f) - y(t_f)\right)$

$+\int_{t_0}^{t_f} \frac{1}{2}\left[\left(z(t) - y(t)\right)^T Q\left(z(t) - y(t)\right) + u^T(t)Ru(t)\right] dt$

■ 推广到线性定常系统的输出跟踪

- 解的形式： => $u^*(t) = -R^{-1}B^T\bar{P}x^*(t) + R^{-1}B^T\bar{g}$
- Riccati代数方程 $\bar{P}A + A^T\bar{P} - \bar{P}BR^{-1}B^T\bar{P} + C^TQC$ =0
- 补充代数方程 $\bar{g} = [\bar{P}BR^{-1}B^T - A^T]^{-1}C^TQz_0$

- **总结**

■ 有限时域线性时变系统输出跟踪问题的结论

➢ 线性时变系统 $\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t)\\ y(t) &= C(t)x(t)\end{aligned}$

➢ 二次型指标 $\mathcal{J} = \frac{1}{2}e^T(t_f)Fe(t_f) + \int_{t_0}^{t_f}\frac{1}{2}[e^T(t)Q(t)e(t) + u^T(t)R(t)u(t)]dt$

➢ 其中 $e(t) = z(t) - y(t)$， $z(t)$是期望输出，初值为$z_0$； $x(t)$的初值为$x_0$

➢ 最优控制解 $u^*(t) = -R^{-1}(t)B^T(t)[P(t)x^*(t) - g(t)]$

➢ 最优性能解 $\mathcal{J} = \frac{1}{2}e^T(t_0)P(t_0)e(t_0)$

➢ Raccati微分方程 $\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) + P(t)B(t)R^{-1}(t)B^T(t)P(t) - C^T(t)Q(t)C(t)$

➢ 补充微分方程 $\dot{g}(t) = -[A^T(t) - P(t)B(t)R^{-1}(t)B^T(t)]g(t) - C^T(t)Q(t)z(t)$

➢ 边界条件 $P(t_f) = C^T(t_f)FC(t_f)$， $g(t_f) = C^T(t_f)Fz(t_f)$

## ● 总结

■ 结论与推广

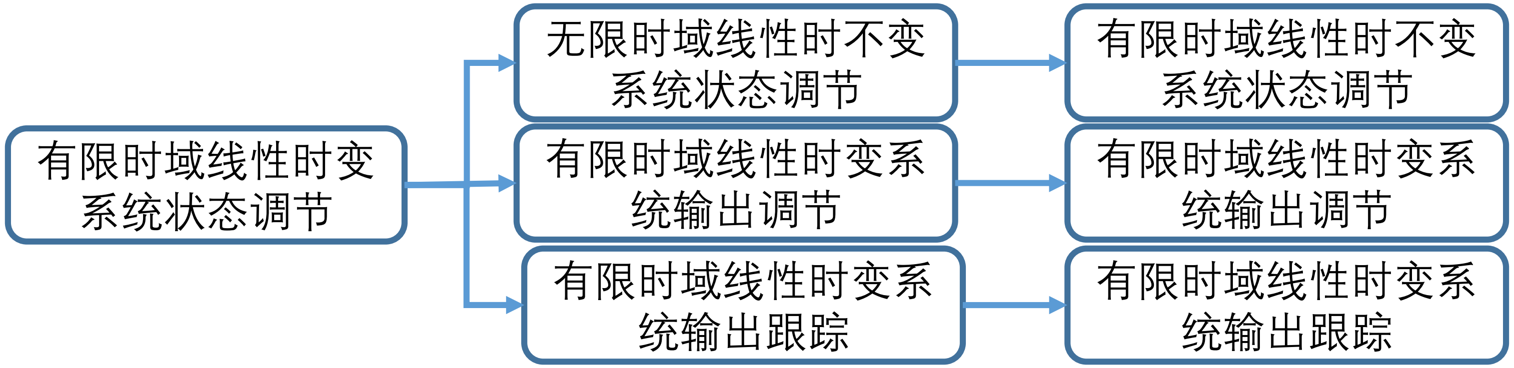
➢ 时变→时不变：$A(t)=A$，$B(t)=B$，$C(t)=C$，$z(t)=z_0$，

$Q(t)=Q$，$R(t)=R$，$P(t)=\bar{P}$，$g(t)=\bar{g}$，$\dot{P}(t)=0$，$\dot{g}(t)=0$

➢ 跟踪→调节：$z(t)=0$

➢ 输出→状态：$C(t)=I$

➢ 有限时域→无限时域：$t_f=\infty$，$F=0$



## 性能指标

- 最优问题
  - 积分型性能指标$\mathcal{J} = \int_{t_0}^{t_f} F[\boldsymbol{x}(t), u(t), t] dt$
- 时间最优问题
  - $\mathcal{J} = t_f - t_0 = \int_{t_0}^{t_f} dt$
  - $F[\boldsymbol{x}(t), u(t), t] = 1$
- 能量最优问题
  - $\mathcal{J} = \int_{t_0}^{t_f} u^2(t) dt$
  - $F[\boldsymbol{x}(t), \boldsymbol{u}(t), t] = u^2(t)$

◆ 求解思路
- 解析法：变分法与Hamilton法
- 数值法：直接配置法
- New Method？

□ 时间尺度的概念
- 路径：路径规划通常会在构型空间生成一条二阶连续可微的曲线，即$Q, s: [0,1] \to q(s)$
- 时间尺度：标量函数：$f, t: [0, t_f] \to s(t)$: $[0,1]$
- 轨迹：为路径赋予时间尺度，即：$Qf, t: [0, t_f] \to q(s(t))$

- 特点：
- (1) 不失一般性认为，$s(t)$是单调递增函数，$\dot{s}(t) \geq 0$，进一步认为$\dot{s}(t) > 0$(不考虑某些拐点)
- (2) $s(0) = 0$，$s(T) = 1$，$\dot{s}(0) = \dot{s}_0$，$\dot{s}(T) = \dot{s}_T$

- **时间尺度约束(动力学)**
  - 问题描述
    - 给定机器人一条无碰撞的路径$q(s)$，沿着这条路径最快的可行轨迹是什么？(时间最优)
    - 给定机器人一条无碰撞的路径$q(s)$，沿着这条路径最省能量的可行轨迹是什么？(能量最优)
    - 受驱动器约束的时间最优的时间尺度$s(t)$是什么？(时间最优的时间尺度问题)
    - 受驱动器约束的能量最优的时间尺度$s(t)$是什么？(能量最优的时间尺度问题)

一些推导：$q = q(s)$

- $\dot{q} = \frac{dq}{ds}\dot{s}$
- $\ddot{q} = \frac{d^2q}{ds^2}\dot{s}^2 + \frac{dq}{ds}\ddot{s}$

动力学：

- $M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u$

约束：

- $u_{\min}(q,\dot{q}) \le u \le u_{\max}(q,\dot{q})$

将时间尺度代入动力学：

- $M\big(q(s)\big)\frac{dq}{ds}\ddot{s} + \left(M(q)\frac{d^2q}{ds^2} + C\big(q(s),\dot{q}(s)\big)\frac{dq}{ds}\right)\dot{s}^2 + G\big(q(s)\big) = u$

基于时间尺度的动力学：

- $m(s)\ddot{s} + c(s)\,\dot{s}^2 + g(s) = u$

约束：

- $u_{\min}(s,\dot{s}) \le m(s)\ddot{s} + c(s)\,\dot{s}^2 + g(s) \le u_{\max}(s,\dot{s})$

注：在参考文献Chapter11中介绍的，动力学项是$\dot{s}^2$，可以仔细看看是为什么？

Choset H M, Lynch K M, Hutchinson S, et al. Principles of robot motion: theory, algorithms, and implementation[M]. MIT press, 2005.

## ● 时间最优

一些推导：$s = s(t)$

- $\dot{s} = \frac{ds}{dt}$
- $T = \int_0^T dt = \int_{s(0)}^{s(T)} \frac{1}{\dot{s}} ds = \int_0^1 \frac{1}{\dot{s}} ds$

◆ 求解思路

- 不凸为什么？微分项
- 不凸怎么办？变凸

一些推导：$a(s) = \ddot{s}$，$b(s) = \dot{s}^2$

- $\dot{b}(s) = \frac{d(\dot{s}^2)}{dt} = 2\dot{s}\ddot{s}$
- $\dot{b}(s) = b'(s)\dot{s}$
- 所以$b'(s) = 2a(s)$

■ 时间最优

- 目标函数

$$\min_{T,s,u} \int_0^1 \frac{1}{\dot{s}} ds$$

- 约束条件
- (1) 边界条件：$s(0) = 0$，$s(T) = 1$，$\dot{s}(0) = \dot{s}_0$，$\dot{s}(T) = \dot{s}_T$
- (2) 动力学：$m(s)\ddot{s} + c(s)\,\dot{s}^2 + g(s) = u$
- (3) 动力学约束：$u_{\min}(s,\dot{s}) \le m(s)\ddot{s} + c(s)\,\dot{s}^2 + g(s) \le u_{\max}(s,\dot{s})$

■ 代数重构时间最优

- 目标函数

$$\min_{T,s,u} \int_0^1 \frac{1}{\sqrt{b(s)}} ds$$

- 约束条件
- (1) 边界条件：$s(0) = 0$，$s(T) = 1$，$\dot{s}(0) = \dot{s}_0$，$\dot{s}(T) = \dot{s}_T$
- (2) 动力学：$m(s)a(s) + c(s)\,b(s) + g(s) = u$
- (3) 动力学约束：$u_{\min}(s) \le m(s)a(s) + c(s)\,b(s) + g(s) \le u_{\max}(s)$
- (4) 附加约束：$b(s) \ge 0$，$0 \le s \le 1$

重构的意义：将非凸问题转换为凸问题，局部解为全局解

胡友忠. 基于动力学的机械臂最优轨迹规划. 浙江大学, 2016.

- **能量最优**
  - 代数重构能量最优
    - 目标函数

$$\min_{T,s,u} \int_0^1 \frac{u^2}{\sqrt{b(s)}} ds$$

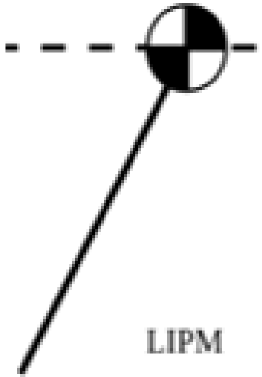
    - 约束条件
    - (1) 边界条件：$s(0) = 0$，$s(T) = 1$，$\dot{s}(0) = \dot{s}_0$，$\dot{s}(T) = \dot{s}_T$
    - (2) 动力学：$m(s)a(s) + c(s)\, b(s) + g(s) = u$
    - (3) 动力学约束：$u_{\min}(s) \leq m(s)a(s) + c(s)\, b(s) + g(s) \leq u_{\max}(s)$
    - (4) 附加约束： $b(s) \geq 0$， $0 \leq s \leq 1$

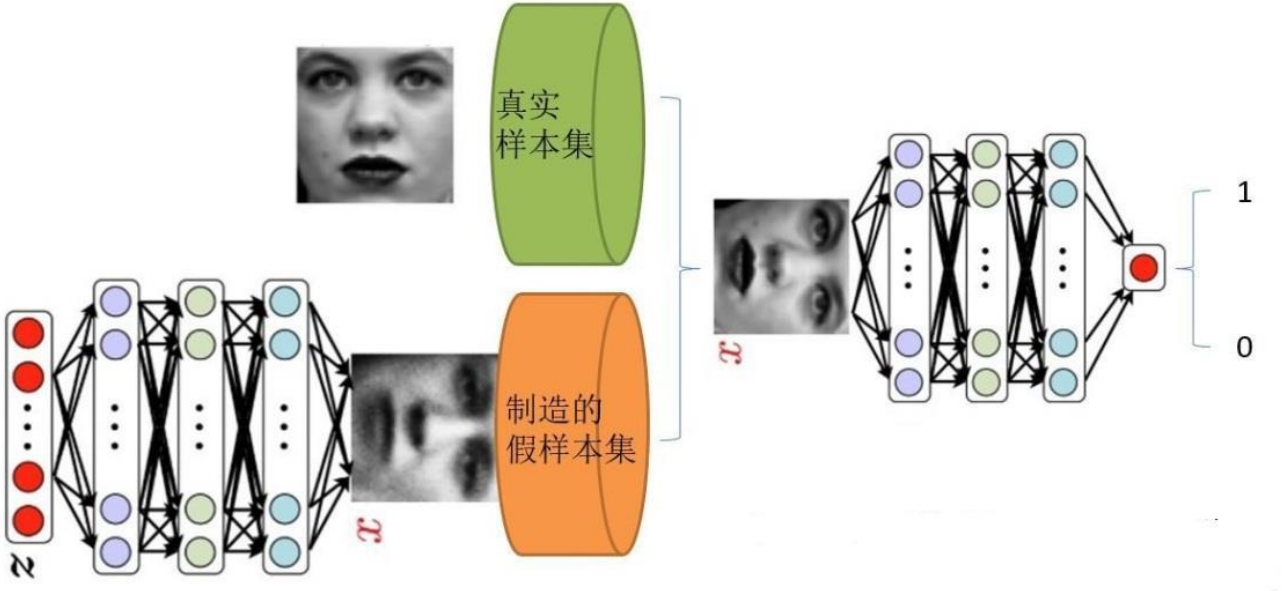
注：重构并不能转化为凸优化问题，一般求解方法是直接配置法

胡友忠. 基于动力学的机械臂最优轨迹规划. 浙江大学, 2016.

- **升维与降维**

  - 简化模型降维(线性倒立摆)与冗余模型升维(七自由度机械臂)
  - 深度学习自编码器的编码降维与解码升维
  - 力学的Newton力学(降维)与Lagrange力学(升维)


https://www.zhihu.com/question/49361447/answer/932775249.

- **再谈优化问题**

**Goal:**

Want to find the maximum or minimum of a function subject to some constraints.

**Formal Statement of Problem:**

Given functions $f$, $g_1, \ldots, g_m$ and $h_1, \ldots, h_l$ defined on some domain $\Omega \subset \mathbf{R}^n$ the optimization problem has the form

$$\boxed{\min_{\mathbf{x} \in \Omega} f(\mathbf{x}) \;\; \text{subject to} \;\; g_i(\mathbf{x}) \leq 0 \;\; \forall i \;\; \text{and} \;\; h_j(\mathbf{x}) = 0 \;\; \forall j}$$

- 约束条件类型
  - 无约束
  - 只有等式约束
  - 只有不等式约束
  - 等式约束 + 不等式约束

忘了哪篇文献.(尴尬)

- **再谈优化问题1. 无约束条件**

  - 充要条件
  - 一阶梯度等于0
  - 二阶梯度大于等于0

**Assume:**

Let $f : \Omega \to \mathbb{R}$ be a continuously differentiable function.

**Necessary and sufficient conditions for a local minimum:**

$\mathbf{x}^*$ is a local minimum of $f(\mathbf{x})$ if and only if

1. $f$ has zero gradient at $\mathbf{x}^*$:

$$\boxed{\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \mathbf{0}}$$

2. and the Hessian of $f$ at $\mathbf{w}^*$ is positive semi-definite:

$$\boxed{\mathbf{v}^t \left(\nabla^2 f(\mathbf{x}^*)\right) \mathbf{v} \geq 0, \ \forall \mathbf{v} \in \mathbb{R}^n}$$

where

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix}$$

# 08 硬约束 & 软约束 (1)



- **再谈优化问题2. 等式约束条件**

Remember our constrained optimization problem is

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \text{ subject to } h(\mathbf{x})=0}$$

Define the **Lagrangian** as note $\mathcal{L}(\mathbf{x}^*,\mu^*)=f(\mathbf{x}^*)$
↓

$$\boxed{\mathcal{L}(\mathbf{x},\mu)=f(\mathbf{x})+\mu\, h(\mathbf{x})}$$

Then $\mathbf{x}^*$ a local minimum $\iff$ there exists a unique $\mu^*$ s.t.

1. $\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}^*,\mu^*)=\mathbf{0}$ ← encodes $\nabla_{\mathbf{x}} f(\mathbf{x}^*)=\mu^*\nabla_{\mathbf{x}} h(\mathbf{x}^*)$
2. $\nabla_{\mu}\mathcal{L}(\mathbf{x}^*,\mu^*)=0$ ← encodes the equality constraint $h(\mathbf{x}^*)=0$
3. $\mathbf{y}^t(\nabla^2_{\mathbf{xx}}\mathcal{L}(\mathbf{x}^*,\mu^*))\mathbf{y}\geq 0 \quad \forall\mathbf{y}$ s.t. $\nabla_{\mathbf{x}}h(\mathbf{x}^*)^t\mathbf{y}=0$

↑
Positive definite Hessian tells us we have a local minimum

- 局部充要
- 一阶梯度条件
- 等式约束
- 二阶梯度条件

The constrained optimization problem is

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \text{ subject to } h_i(\mathbf{x})=0 \text{ for } i=1,\dots,l}$$

Construct the **Lagrangian** (introduce a multiplier for each constraint)

$$\boxed{\mathcal{L}(\mathbf{x},\boldsymbol{\mu})=f(\mathbf{x})+\sum_{i=1}^{l}\mu_i\, h_i(\mathbf{x})=f(\mathbf{x})+\boldsymbol{\mu}^t\,\mathbf{h}(\mathbf{x})}$$

Then $\mathbf{x}^*$ a local minimum $\iff$ there exists a unique $\boldsymbol{\mu}^*$ s.t.

1. $\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}^*,\boldsymbol{\mu}^*)=\mathbf{0}$
2. $\nabla_{\boldsymbol{\mu}}\mathcal{L}(\mathbf{x}^*,\boldsymbol{\mu}^*)=\mathbf{0}$
3. $\mathbf{y}^t(\nabla^2_{\mathbf{xx}}\mathcal{L}(\mathbf{x}^*,\mu^*))\mathbf{y}\geq 0 \quad \forall\mathbf{y}$ s.t. $\nabla_{\mathbf{x}}h(\mathbf{x}^*)^t\mathbf{y}=0$

忘了哪篇文献.(尴尬)

- **再谈优化问题3. 不等式约束条件**

■ 局部充要

➢ KKT条件

Given the optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \text{ subject to } g(\mathbf{x}) \le 0$$

Define the **Lagrangian** as

$$\mathcal{L}(\mathbf{x},\lambda) = f(\mathbf{x}) + \lambda\, g(\mathbf{x})$$

Then $\mathbf{x}^*$ a local minimum $\iff$ there exists a unique $\lambda^*$ s.t.

1. $\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}^*,\lambda^*) = \mathbf{0}$
2. $\lambda^* \ge 0$
3. $\lambda^*\, g(\mathbf{x}^*) = 0$
4. $g(\mathbf{x}^*) \le 0$
5. Plus positive definite constraints on $\nabla_{\mathbf{xx}}\mathcal{L}(\mathbf{x}^*,\lambda^*)$.

These are the **KKT conditions**.

**Case 1 - Inactive constraint:**

- When $\lambda^* = 0$ then have $\mathcal{L}(\mathbf{x}^*,\lambda^*) = f(\mathbf{x}^*)$.
- Condition KKT 1 $\Longrightarrow$ $\nabla_{\mathbf{x}} f(\mathbf{x}^*) = \mathbf{0}$.
- Condition KKT 4 $\Longrightarrow$ $\mathbf{x}^*$ is a feasible point.

**Case 2 - Active constraint:**

- When $\lambda^* > 0$ then have $\mathcal{L}(\mathbf{x}^*,\lambda^*) = f(\mathbf{x}^*) + \lambda^*\, g(\mathbf{x}^*)$.
- Condition KKT 1 $\Longrightarrow$ $\nabla_{\mathbf{x}} f(\mathbf{x}^*) = -\lambda^*\,\nabla_{\mathbf{x}} g(\mathbf{x}^*)$.
- Condition KKT 3 $\Longrightarrow$ $g(\mathbf{x}^*) = 0$.
- Condition KKT 3 also $\Longrightarrow$ $\mathcal{L}(\mathbf{x}^*,\lambda^*) = f(\mathbf{x}^*)$.

忘了哪篇文献.(尴尬)

- 再谈优化问题3. 不等式约束条件

■ 局部充要

➢ 多个不等式约束

Given the optimization problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x}) \;\; \text{subject to} \;\; g_j(\mathbf{x}) \leq 0 \text{ for } j = 1, \dots, m}$$

Define the **Lagrangian** as

$$\boxed{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j \, g_j(\mathbf{x}) = f(\mathbf{x}) + \boldsymbol{\lambda}^t \, \mathbf{g}(\mathbf{x})}$$

Then $\mathbf{x}^*$ a local minimum $\iff$ there exists a unique $\boldsymbol{\lambda}^*$ s.t.

1. $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*) = \mathbf{0}$
2. $\lambda_j^* \geq 0$ for $j = 1, \dots, m$
3. $\lambda_j^* \, g(\mathbf{x}^*) = 0$ for $j = 1, \dots, m$
4. $g_j(\mathbf{x}^*) \leq 0$ for $j = 1, \dots, m$
5. Plus positive definite constraints on $\nabla_{\mathbf{xx}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)$.

忘了哪篇文献.(尴尬)

- **再谈优化问题4. 等式和不等式约束条件(规范KKT条件)**

Given the constrained optimization problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x})}$$

subject to

$$\boxed{h_i(\mathbf{x}) = 0 \text{ for } i = 1,\ldots,l \ \textbf{ and } \ g_j(\mathbf{x}) \leq 0 \text{ for } j = 1,\ldots,m}$$

Define the **Lagrangian** as

$$\boxed{\mathcal{L}(\mathbf{x},\boldsymbol{\mu},\boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\mu}^t\,\mathbf{h}(\mathbf{x}) + \boldsymbol{\lambda}^t\,\mathbf{g}(\mathbf{x})}$$

Then $\mathbf{x}^*$ a local minimum $\iff$ there exists a unique $\boldsymbol{\lambda}^*$ s.t.

1. $\nabla_{\mathbf{x}}\mathcal{L}(\mathbf{x}^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*) = \mathbf{0}$
2. $\lambda_j^* \geq 0$ for $j = 1,\ldots,m$
3. $\lambda_j^*\, g_j(\mathbf{x}^*) = 0$ for $j = 1,\ldots,m$
4. $g_j(\mathbf{x}^*) \leq 0$ for $j = 1,\ldots,m$
5. $\mathbf{h}(\mathbf{x}^*) = \mathbf{0}$
6. Plus positive definite constraints on $\nabla_{\mathbf{xx}}\,\mathcal{L}(\mathbf{x}^*,\boldsymbol{\lambda}^*)$.

- 局部充要
- 终极KKT条件

忘了哪篇文献.(尴尬)

- **再再谈优化问题1. Hamilton函数**

有的性能指标泛函结合成一个新的泛函

$$J_1 = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} \{L(x,u,t) + \lambda^T (f(x,u,t) - \dot{x})\} dt \qquad (2.6.7)$$

$J_1$ 与 $J$ 时等价的。于是问题归结为求泛函 $J_1$ 的无条件极值

为方便起见，定义一个标量函数

$$H(x,u,\lambda,t) = L(x,u,t) + \lambda^T(t) f(x,u,t)$$

常称哈密顿（Hamilton）函数

- **再再谈优化问题2. Hamilton函数的解**
  - 必要条件

**定理 2.3** 对于受控系统（2.6.2），复合型性能指标（2.6.5），$t_f$ 固定，终端状态 $x(t_f)$ 自由的问题。为使 $u^*$ 和 $x^*$ 成为极值控制和极值轨线，必存在适当选择 $\lambda(t)$，成立

（i） $x(t)$ 和 $\lambda(t)$ 满足下列规范方程

$$\dot{x}(t)=\frac{\partial H}{\partial \lambda}=f(x,u,t)$$

$$\dot{\lambda}(t)=-\frac{\partial H}{\partial x}=-\frac{\partial L}{\partial x}-\frac{\partial f^T}{\partial x}\lambda$$

其中 $H(x,u,\lambda,t)=L(x,u,t)+\lambda^T(t)f(x,u,t)$

（ii）边值条件

$$x(t_0)=x_0$$

$$\lambda(t_f)=\frac{\partial \Phi(x(t_f),t_f)}{\partial x(t_f)}$$

（iii）极值条件

$$\frac{\partial H}{\partial u}=0$$

忘了哪篇文献.(尴尬)

- **约束**
  - 约束缩小可行域的范围，合理的约束能够简化问题(正常约束)，不合理的约束会增加求解难度(欠约束)或者直接导致无解(过约束)
  - 硬约束(hard-constraint)：在任何时候都“必须”满足的约束
  - 软约束(soft-constraint)：尽可能满足的“希望”
  - 线性方程组$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$的解问题
  - 线性方程组$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$相容，通解为$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}+(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y}$，通解中最小范数解为$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}$
  - 线性方程组$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$不相容，则最小二乘解为$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}+(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y}$，最佳最小二乘解为$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}$
  - 讨论：
  - (1)相容则有解，解的表达式中存在零空间(无穷解)，希望找到某种偏好(最小范数)，则：硬约束为$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$，软约束为某种范数$\|\boldsymbol{x}\|$最小化
  - (1)不相容则无解，希望找到某种偏好(最小二乘)，则：将硬约束$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$变成软约束，某种范数$\|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|$最小化，最佳的概念同上述最小范数

- **硬约束与软约束**

| 约束类型 | 约束描述 | 求解思路 |
| --- | --- | --- |
| 硬约束 | 要求严格满足的约束 | ➢ (1) 简化模型<br>➢ (2) 启发式算法与搜索算法<br>➢ (3) 无约束优化问题<br>➢ (4) 软约束算法 |
| 软约束 | 尽可能满足的“希望” | ➢ (1) 罚函数<br>➢ (2) 任务优先级<br>➢ (3) 部分约束 |

目标：找到一个满足所有约束条件(硬约束)并且尽可能满足优先约束条件(软约束)的解

- **硬约束1. 简化模型**

➢ 简化为凸优化问题
➢ (1) 局部最优就是全局最优解
➢ (2) 存在高效的求解算法

## Convex optimization

- Optimization problem in standard form

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) & \\
\text{subject to} \quad & f_i(x) \le 0 & i = 1, \dots, m \\
& h_i(x) = 0 & i = 1, \dots, p
\end{aligned}
$$

$x \in R^n$ is the optimization variable

$f_0 \colon R^n \to R$ is the objective function

$f_i \colon R^n \to R, i = 1, \dots, m$ are inequality constraint functions

$h_i \colon R^n \to R, i = 1, \dots, p$ are equality constraint functions

- Convex optimization problem in standard form:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) & \\
\text{subject to} \quad & f_i(x) \le 0 & i = 1, \dots, m \\
& Ax = b &
\end{aligned}
$$

where $f_0, f_1, \dots, f_m$ are convex and equality constraints are affine.

- **Local and goal optima**: any locally optimal point of a convex problem is globally optimal.
- Most problems are not convex when formulated.
- Reformulating a problem in convex form is an art, there is no systematic way.

- **硬约束1. 简化模型**

> 凸优化问题的特例
> (1) LP
> (2) QP
> (3) QCQP
> (4) SOCP

注意：二次型系数的对称正定性

## Disciplined convex optimization programs

**Linear Programming (LP)**

$$\begin{array}{ll}\underset{x}{\text{minimize}} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- Convex problem: affine objective and constraint functions.

**Quadratic Programming (QP)**

$$\begin{array}{ll}\underset{x}{\text{minimize}} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- Convex problem (assuming $P \in S^n \succeq 0$): convex quadratic objective and affine constraint functions.

**Quadratically Constrained QP (QCQP)**

$$\begin{array}{ll}\underset{x}{\text{minimize}} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \le 0 \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

- Convex problem (assuming $P_i \in S^n \succeq 0$): convex quadratic objective and constraint functions.

**Second-Order Cone Programming (SOCP)**

$$\begin{array}{ll}\underset{x}{\text{minimize}} & f^T x \\ \text{subject to} & \|A_i x + b_i\| \le c_i^T x + d_i \quad i = 1, \dots, m \\ & Fx = g\end{array}$$

- Convex problem: linear objective and second-order cone constraints
- For $A_i$ row vector, it reduces to an LP.
- For $c_i = 0$, it reduces to a QCQP.
- More general than QCQP and LP.

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- Convex optimization problem if $Q \succeq 0$ ($Q$ = positive semidefinite matrix)[1]
- Hard problem if $Q \not\succeq 0$ ($Q$ = indefinite matrix)
- Without loss of generality we can assume $Q = Q'$ as

$$x'Qx = x'(\tfrac{Q+Q'}{2} + \tfrac{Q-Q'}{2})x = x'\tfrac{Q+Q'}{2}x + \tfrac{1}{2}x'Qx - (\tfrac{1}{2}x'Q'x)' = x'\tfrac{Q+Q'}{2}x$$

[1] A matrix $P \in \mathbb{R}^{n \times n}$ is **positive semidefinite** ($P \succeq 0$) if $x'Px \ge 0$ for all $x$.
It is **positive definite** ($P \succ 0$) if in addition $x'Px > 0$ for all $x \neq 0$.
It is **negative (semi)definite** ($P \prec 0$, $P \preceq 0$) if $-P$ is positive (semi)definite.
It is **indefinite** otherwise.

高飞. 移动机器人中的运动规划. 深蓝学院. 2019. // Bemporad. Model Preditive Control. 2020.

# 09 硬约束 & 软约束 (2)

- **硬约束1. 简化模型**

➢ 一些优化工具

- You target is to formulate a trajectory generation problem into **Disciplined convex optimization programs** in P.41.
- Many off-the-shelf can help you solve them.
- Choose a proper solver according to your requirement.

CVX
http://cvxr.com/cvx/
Matlab wrapper. Let you write down the convex program like mathematical equations, then call other solvers to solve the problem.

Mosek
https://www.mosek.com/
Very robust convex solvers, can solve almost all kinds of convex programs. Can apply free academic license. Only library available (x86).

OOQP
http://pages.cs.wisc.edu/~swright/ooqp/
Very fast, robust QP solver. Open sourced.

GLPK
https://www.gnu.org/software/glpk/
Very fast, robust LP solver. Open sourced.

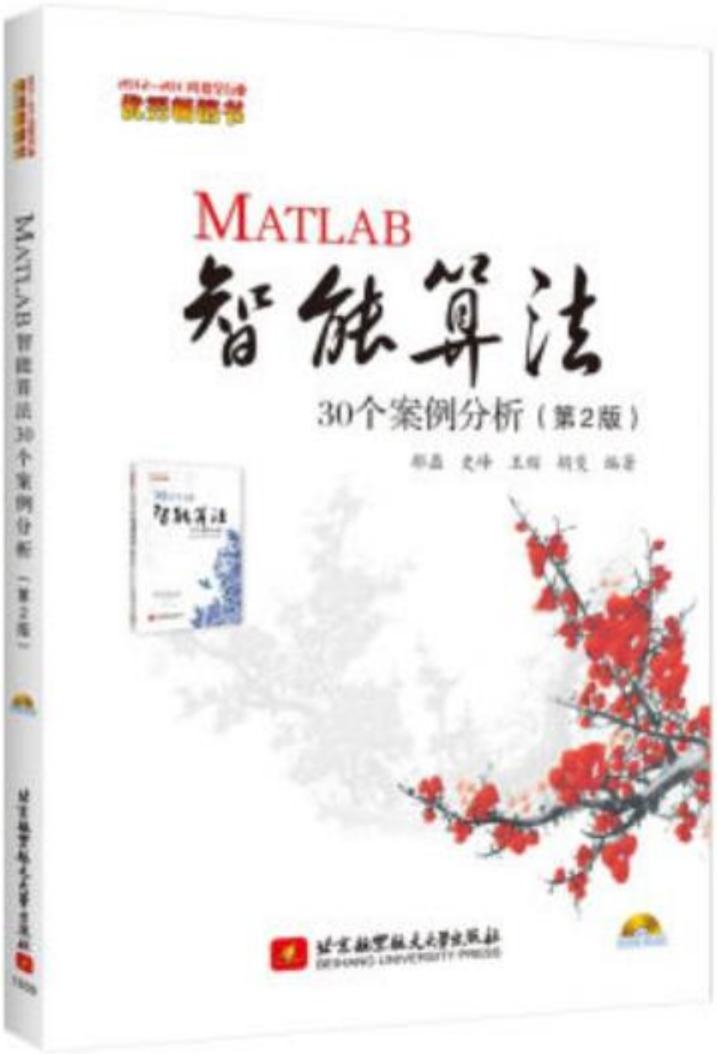
- AMPL (A Modeling Language for Mathematical Programming) most used modeling language, supports several solvers
- OPL (Optimization Programming Language), associated with commercial package IBM-CPLEX
- MOSEL associated with commercial package FICO Xpress
- GAMS (General Algebraic Modeling System) is one of the first modeling languages
- LINGO modeling language of Lindo Systems Inc.
- GNU MathProg a subset of AMPL associated with the free package GLPK (GNU Linear Programming Kit)

- YALMIP MATLAB-based modeling language
- CVX (CVXPY) Modeling language for convex problems in MATLAB (python)
- CASADI + IPOPT Nonlinear modeling + automatic differentiation, nonlinear programming solver (MATLAB, python, C++)
- Optimization Toolbox' modeling language (part of MATLAB since R2017b)
- PYOMO python-based modeling language
- PuLP An linear programming modeler for python
- JuMP A modeling language for linear, quadratic, and nonlinear constrained optimization problems embedded in julia

- **硬约束2. 启发式算法与搜索算法**

■ MATLAB智能算法30个案例分析(第2版)
➢ 遗传算法
➢ 蚁群算法
➢ 神经网络
➢ …
■ 数值最优化方法
➢ 线搜索算法
➢ 置信域算法

注意：启发式算法与搜索算法的时间效率问题




- **硬约束3. 无约束优化问题**

- 转换为无约束优化问题
- (1) Language乘子法
- (2) Hamilton乘子法

Given the constrained optimization problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x})}$$

subject to

$$\boxed{h_i(\mathbf{x}) = 0 \text{ for } i = 1, \ldots, l \ \textbf{ and } \ g_j(\mathbf{x}) \leq 0 \text{ for } j = 1, \ldots, m}$$

Define the **Lagrangian** as

$$\boxed{\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\mu}^t\, \mathbf{h}(\mathbf{x}) + \boldsymbol{\lambda}^t\, \mathbf{g}(\mathbf{x})}$$

Then $\mathbf{x}^*$ a local minimum $\Longleftrightarrow$ there exists a unique $\boldsymbol{\lambda}^*$ s.t.

1. $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = \mathbf{0}$
2. $\lambda_j^* \geq 0$ for $j = 1, \ldots, m$
3. $\lambda_j^*\, g_j(\mathbf{x}^*) = 0$ for $j = 1, \ldots, m$
4. $g_j(\mathbf{x}^*) \leq 0$ for $j = 1, \ldots, m$
5. $\mathbf{h}(\mathbf{x}^*) = \mathbf{0}$
6. Plus positive definite constraints on $\nabla_{\mathbf{x}\mathbf{x}}\, \mathcal{L}(\mathbf{x}^*, \boldsymbol{\lambda}^*)$.

**定理 2.3** 对于受控系统（2.6.2），复合型性能指标（2.6.5），$t_f$ 固定，终端状态 $x(t_f)$ 自由的问题。为使 $u^*$ 和 $x^*$ 成为极值控制和极值轨线，必存在适当选择 $\lambda(t)$，成立

（i） $x(t)$ 和 $\lambda(t)$ 满足下列规范方程

$$\dot{x}(t) = \frac{\partial H}{\partial \lambda} = f(x,u,t)$$

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \frac{\partial f^T}{\partial x}\lambda$$

其中 $H(x,u,\lambda,t) = L(x,u,t) + \lambda^T(t) f(x,u,t)$

（ii）边值条件

$$x(t_0) = x_0$$

$$\lambda(t_f) = \frac{\partial \Phi(x(t_f), t_f)}{\partial x(t_f)}$$

（iii）极值条件

$$\frac{\partial H}{\partial u} = 0$$

- **软约束1. 松弛条件与部分约束**

■ 松弛条件
- 互补松弛性(Lagrange对偶定理)
- 单纯形法的不等式转等式约束

■ 部分约束(弱化约束)
- 扩大决策变量域
- 扩大约束域
- 移除约束
- 移除决策变量

- enlarge the domain of a variable (i.e., add more values),
- enlarge the domain of a constraint (i.e., add more compatible tuples),
- remove the constraint from the problem,
- remove the variable from the problem.

Bartak R. Modelling soft constraints: A survey[J]. Neural Network World, 2002, 12(5): 421-432.

- **软约束2. 罚函数**
  - 硬约束与软约束处理
  - 无约束优化问题

同样是将约束添加到目标函数中
- 硬约束是通过Lagrange乘子保证约束严格满足
- 软约束是通过权重和范数保证约束优先满足(无约束问题的病态性)

硬约束

Given the constrained optimization problem

$$\boxed{\min_{\mathbf{x}\in\mathbb{R}^2} f(\mathbf{x})}$$

subject to

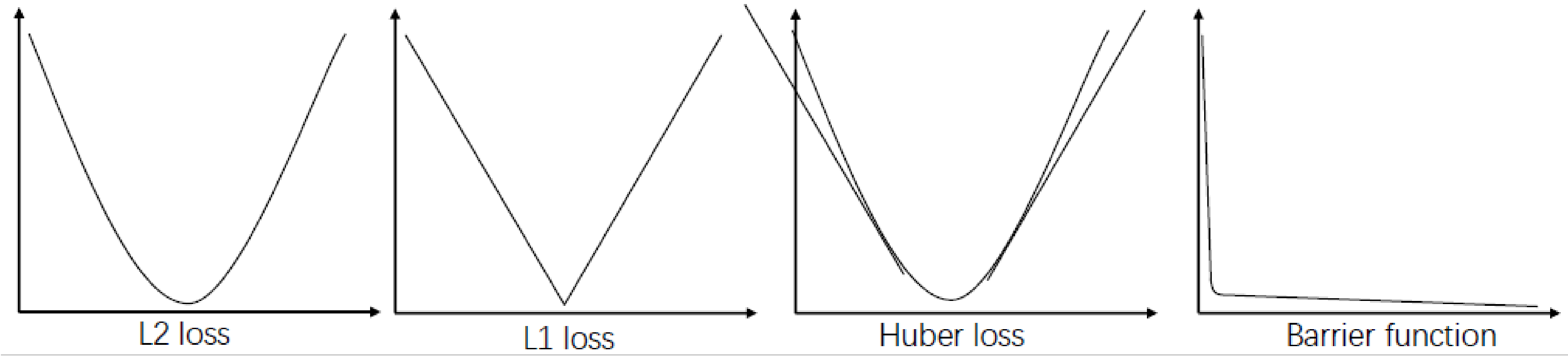
$$\boxed{h_i(\mathbf{x}) = 0 \text{ for } i = 1, \ldots, l \textbf{ and } g_j(\mathbf{x}) \leq 0 \text{ for } j = 1, \ldots, m}$$

Define the **Lagrangian** as

$$\boxed{\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\mu}^t\, \mathbf{h}(\mathbf{x}) + \boldsymbol{\lambda}^t\, \mathbf{g}(\mathbf{x})}$$

软约束

$$\mathcal{L}(x) = f(x) + \lambda\|h(x)\|$$






高飞. 移动机器人中的运动规划. 深蓝学院. 2019.

- **软约束2. 罚函数**

■ 外点罚函数与内点罚函数
- 都是将约束最优化问题转化为无约束最优化问题
- 外点罚函数：最优解序列从可行域外部逼近约束最优化问题的最优解
- 内点罚函数：最优解序列从可行解内部逼近约束最优化问题的最优解(适宜于不等式约束)

外点罚函数
- 等式约束
- 不等式约束
- 规范约束(等式 + 不等式)

等式约束

$$\min f(x), \tag{7.1a}$$

$$\text{s.t. } c_i(x) = 0, \quad i \in \mathcal{E}. \tag{7.1b}$$

$$P_E(x,\sigma) = f(x) + \frac{1}{2}\sigma \sum_{i \in \mathcal{E}} c_i^2(x), \tag{7.2}$$

不等式约束

$$\min f(x), \tag{7.14a}$$

$$\text{s.t. } c_i(x) \geqslant 0, \quad i \in \mathcal{I}, \tag{7.14b}$$

$$P_I(x,\sigma) = f(x) + \frac{1}{2}\sigma \sum_{i \in \mathcal{I}} \left[\min\{c_i(x), 0\}\right]^2.$$

$$\tilde{c}_i(x) = \min\{c_i(x), 0\}, \quad P_I(x,\sigma) = f(x) + \frac{1}{2}\sigma \sum_{i \in \mathcal{I}} \tilde{c}_i^2(x).$$

$$P_{EI}(x,\sigma) = f(x) + \frac{1}{2}\sigma \left[\sum_{i \in \mathcal{E}} c_i^2(x) + \sum_{i \in I} \tilde{c}_i^2(x)\right],$$

$$\bar{c}_i(x) = \begin{cases} c_i(x), & i \in \mathcal{E}, \\ \tilde{c}_i(x), & i \in \mathcal{I}, \end{cases}$$

$$P_{EI}(x,\sigma) = f(x) + \frac{1}{2}\sigma \|\bar{c}(x)\|^2.$$

- **软约束2. 罚函数**

■ 外点罚函数与内点罚函数
- 都是将约束最优化问题转化为无约束最优化问题
- 外点罚函数：最优解序列从可行域外部逼近约束最优化问题的最优解
- 内点罚函数：最优解序列从可行解内部逼近约束最优化问题的最优解(适宜于不等式约束)

内点罚函数
- 倒数障碍函数
- 对数障碍函数

$$\min f(x), \tag{7.14a}$$

$$\text{s.t. } c_i(x) \geqslant 0, \quad i \in \mathcal{I}, \tag{7.14b}$$

倒数障碍函数 $$B_{\mathrm{I}}(x,\mu) = f(x) + \mu \sum_{i\in\mathcal{I}} c_i(x)^{-1},$$

对数障碍函数 $$B_{\mathrm{L}}(x,\mu) = f(x) - \mu \sum_{i\in\mathcal{I}} \ln c_i(x),$$

有意思的软约束：
- 关节远离极限：$\min \left\| q - \frac{q_{max}+q_{min}}{2} \right\|$
- 势函数：$F = -\frac{r}{x^2}$



高立 数值最优化方法. 北京大学出版社.

- **软约束3. 任务优先级**

一个例子

- 线性方程组$\boldsymbol{Ax}=\boldsymbol{b}$相容，通解为$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}+(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y}$ (高优先级)
- 最小范数解： $\boldsymbol{y}=\mathbf{0}$ (低优先级)

- 线性方程组$\boldsymbol{Ax}=\boldsymbol{b}$不相容，最小二乘通解为$\boldsymbol{x}=\boldsymbol{A}^{+}\boldsymbol{b}+(\boldsymbol{I}-\boldsymbol{A}^{+}\boldsymbol{A})\boldsymbol{y}$ (高优先级)
- 最小范数解： $\boldsymbol{y}=\mathbf{0}$ (低优先级)

- 任务优先级原则：低任务优先级的执行不应该干扰高任务优先级的执行
- 任务优先级算法：任务优先级分层，将低优先级任务的解投影到高优先级任务的零空间

- **软约束3. 任务优先级**

举个冗余机械臂的例子来介绍:

$$\dot{x} = J\dot{q}. \tag{1}$$

$$\dot{q} = J^{\#}\dot{x} + P\dot{q}_{\mathcal{N}} \tag{2}$$

where $J^{\#}$ is the (unique) Moore–Penrose pseudoinverse [28] of the task Jacobian, $P = I - J^{\#}J$ is the $n \times n$ orthogonal projector in the Jacobian null space, and $\dot{q}_{\mathcal{N}} \in \mathbb{R}^n$ is a generic joint velocity. Equation (2) gives the joint velocity $\dot{q}$ that satisfies (1) (or minimizes the norm of the error $\dot{x} - J\dot{q}$, if $\dot{x}$ is not in the range of $J$), while minimizing in norm the distance to $\dot{q}_{\mathcal{N}}$. The solution with minimum norm is obtained for $\dot{q}_{\mathcal{N}} = 0$.

Use of redundancy can be extended to the execution of $l$ tasks in the form (1), $\dot{x}_k = J_k\dot{q}$, each of dimension $m_k$, $k = 1, \ldots, l$ (usually, with $\sum_{k=1}^{l} m_k \leq n$), that are ordered by their priority, i.e., task $i$ has higher priority than task $j$ if $i < j$. Execution of a task of lower priority should not interfere with the execution of tasks having higher priority, and this hierarchy is guaranteed by projecting the solution to the $k$th task of the stack in the null space of all higher priority tasks. This is obtained by using the recursive formula [29]

$$\dot{q}_k = \dot{q}_{k-1} + (J_k P_{A,k-1})^{\#}(\dot{x}_k - J_k\dot{q}_{k-1}) \tag{3}$$

initialized with $\dot{q}_0 = 0$ and $P_{A,0} = I$, and where $P_{A,k}$ is the projector in the null space of the augmented Jacobian of the first $k$ tasks

$$J_{A,k} = \begin{pmatrix} J_1^T & J_2^T & \ldots & J_k^T \end{pmatrix}^T. \tag{4}$$

Matrix $P_{A,k}$ can also be expressed recursively as [30]

$$P_{A,k} = P_{A,k-1} - (J_k P_{A,k-1})^{\#} J_k P_{A,k-1}. \tag{5}$$

- 运动学逆解：存在零空间，无穷解(表达最小范数解，或最小二乘解)
- k个任务分优先级，然后任务优先级先高后低来迭代求解
- 低优先级任务解投影在高优先级任务解的零空间

Flacco F, De Luca A, Khatib O. Control of redundant robots under hard joint constraints: Saturation in the null space[J]. IEEE Transactions on Robotics, 2015, 31(3): 637-654.

- **辩证思想**

  - 轨迹优化问题是在扩大可行域与缩小可行域之间寻求平衡
  - 轨迹优化问题是在目标函数与约束条件维度反向变化之间寻求平衡
  - 轨迹优化问题是在多目标优化统一解与多任务优先级解之间寻求平衡