IIT – JAM MATHEMATICAL STATISTICS

PREVIOUS YEAR QUESTIONS SOLVED

(2005-2023)

TOPIC-LINEAR ALGEBRA

SUBHANJAN

(1) Let $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 8 & 6 & 3 \\ 2 & 4 & 6 & 7 & 10 & 3 \\ 4 & 7 & 10 & 14 & 16 & 7 \end{pmatrix}$

Then Rank of P = ____

(a) 1 (b) 2 (c) 3 (d) 4 [JAM- 2005]

Soln:-

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 3 & 4 & 8 & 6 & 3 \\ 2 & 4 & 6 & 7 & 10 & 3 \\ 4 & 7 & 10 & 14 & 16 & 7 \end{pmatrix}$$

[Echelon Reduction]

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -2 & -2 & -4 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 \\ 0 & -1 & -2 & 0 & -4 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Rank$(P) = 3$ [$\because$ Rank = no. of non-null rows in echelon form]

SD

(2) Consider the system of equation

$$x+y+z=3$$
$$x+az=b$$
$$y+2z=3$$

This system has infinitely many solution if (a) $a=-1, b=0$ (b) $a=1, b=2$ (c) $a=0, b=1$ (d) $a=-1, b=-1$ [JAM-2005]

Soln:-
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & a \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ b \\ 3 \end{pmatrix}$$

$\Leftrightarrow A\underset{\sim}{x} = \underset{\sim}{b}$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & a \\ 0 & 1 & 2 \end{pmatrix}$, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underset{\sim}{b} = \begin{pmatrix} 3 \\ b \\ 3 \end{pmatrix}$

For the system to have ∞ly many soln

$R(A|\underset{\sim}{b}) = R(A) < n = 3$

$$(A|\underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 0 & a & b \\ 0 & 1 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-1 & b-3 \\ 0 & 1 & 2 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & a-1 & b-3 \\ 0 & 0 & a+1 & b \end{array}\right)$$

$\therefore a+1=0, b=0$

$\Rightarrow a=-1, b=0$

(3) Let $P = \begin{pmatrix} 1 & 0 & 1+x & 1+x \\ 0 & 1 & 1 & 1 \\ 1 & 1+x & 0 & 1+x \\ 1 & 1+x & 1+x & 0 \end{pmatrix}$

Then $|P| =$ —— (a) $3(x+1)^3$ (b) $3(x+1)^2$ (c) $3(x+1)$ (d) $(x+1)(2x+3)$ [JAM - 2005]

Soln:- $|P| = \begin{vmatrix} 1 & 0 & 1+x & 1+x \\ 0 & 1 & 1 & 1 \\ 1 & 1+x & 0 & 1+x \\ 1 & 1+x & 1+x & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1+x & -1-x & 0 \\ 1 & 1+x & 0 & -1-x \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1+x & -1-x & 0 \\ 1+x & 0 & -1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1+x & 2(1+x) & (1+x) \\ 1+x & (1+x) & 2(1+x) \end{vmatrix}$

$= \begin{vmatrix} 2(1+x) & (1+x) \\ (1+x) & 2(1+x) \end{vmatrix} = 4(1+x)^2 - (1+x)^2$

$= 3(1+x)^2$

SD

(4) Consider $n \times n$ matrix $P = \begin{pmatrix} \frac{2}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{2}{n+1} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{2}{n+1} \end{pmatrix}$

Let I be identity matrix of order n and let J be a $n \times n$ matrix with all elements '1'. Express $P = aI + bJ$ for suitable constants a and b. It is known that $P^{-1} = cI + dJ$ for some constant c and d. Find c, d in terms of n [JAM-2005]

Soln:- $P = \frac{1}{n+1} I + \frac{1}{n+1} \underset{\sim}{1} \underset{\sim}{1}'$, $J = \underset{\sim}{1} \underset{\sim}{1}'$

$\Rightarrow a = \frac{1}{n+1}$, $b = \frac{1}{n+1}$

$P = A + \underset{\sim}{u} \underset{\sim}{v}'$, $A = \frac{1}{n+1} I$; $\underset{\sim}{u} = \frac{1}{n+1} \underset{\sim}{1}$, $\underset{\sim}{v}' = \underset{\sim}{1}'$

$$\Rightarrow P^{-1} = A^{-1} - \frac{A^{-1} \underset{\sim}{u} \underset{\sim}{v}' A^{-1}}{1 + \underset{\sim}{v}' A^{-1} \underset{\sim}{u}}$$

$A^{-1} = (n+1) I$, $A^{-1} \underset{\sim}{u} \underset{\sim}{v}' A^{-1} = (n+1) \underset{\sim}{1} \underset{\sim}{1}'$

$\underset{\sim}{v}' A^{-1} \underset{\sim}{u} = \underset{\sim}{1}' \underset{\sim}{1} = n$

SD

$\Rightarrow P^{-1} = (n+1)I - \underset{\sim}{1}\,\underset{\sim}{1}'$

$\Rightarrow c = n+1,\ d = -1$

(5) Let $A\underset{\sim}{x} = \underset{\sim}{b}$ be a non-homogeneous system of Linear Equations. $A|\underset{\sim}{b}$ is given by

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ -1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \end{array}\right]$$

Which of following is true

(a) $R(A) = 3$ (b) The system has no solution

(c) The system has unique solution

(d) The system has ∞ly many solution

Soln:- $A|\underset{\sim}{b} = \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ -1 & 2 & 3 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \end{array}\right]$

$\sim \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 3 & 1 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 1 \\ 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$

$R(A|\underset{\sim}{b}) = 3 \neq 2 = R(A)$

$\Rightarrow$ System has no solution

(6) (a) Let A be a non-singular matrix of order $n(>1)$ with $|A| = k$. Find $|adj(A)|$

(b) Determine a, b, c so that $(1, 0, -1)$ and $(0, 1, -1)$ are eigen vectors of matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix}$$ [JAM-2006]

Soln:- (a) By Jacobis Theorem

$$|adj(A)| = |A|^{n-1} = K^{n-1}$$

(b) Let λ_1, λ_2 be eigen values corrosponding to eigen vectors $(1,0,-1)$ and $(0,1,-1)$ respectively

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ a-2 \\ 3-c \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \\ -\lambda_1 \end{bmatrix} \quad \Rightarrow \lambda_1 = 1, \quad \Rightarrow a = 2, \; c = 4$$

Similarly, $$\begin{bmatrix} 2 & 1 & 1 \\ a & 3 & 2 \\ 3 & b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 1 \\ b-c \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_2 \\ -\lambda_2 \end{bmatrix} \quad \Rightarrow \lambda_2 = 1, \quad \Rightarrow b - 4 = -1, \quad \Rightarrow b = 3$$

$$\Rightarrow (a, b, c) = (2, 3, 4)$$

SD

(7) Let A be a 4×4 non-singular matrix and B be the matrix obtained by A by adding to its third row twice the first row. Then $|2A^{-1}B| =$ ——

(a) 2 (b) 4 (c) 8 (d) 16 [JAM-2007]

Soln:—

$$\underset{4\times4}{A} = \begin{pmatrix} \underset{\sim}{a_1'} \\ \underset{\sim}{a_2'} \\ \underset{\sim}{a_3'} \\ \underset{\sim}{a_4'} \end{pmatrix}, \quad \underset{\sim}{a_i'} \in E^4, \quad |A| \neq 0$$

$$\underset{4\times4}{B} = \begin{pmatrix} \underset{\sim}{a_1'} \\ \underset{\sim}{a_2'} \\ 2\underset{\sim}{a_1'} + \underset{\sim}{a_3'} \\ \underset{\sim}{a_4'} \end{pmatrix}$$

$$|A| = \begin{vmatrix} \underset{\sim}{a_1'} \\ \underset{\sim}{a_2'} \\ \underset{\sim}{a_3'} \\ \underset{\sim}{a_4'} \end{vmatrix} = \begin{vmatrix} \underset{\sim}{a_1'} \\ \underset{\sim}{a_2'} \\ 2\underset{\sim}{a_1'} + \underset{\sim}{a_3'} \\ \underset{\sim}{a_4'} \end{vmatrix} \quad [R_3' \to R_3 + 2R_1]$$

$$= |B|$$

$$\Rightarrow |2A^{-1}B| = 2^4 |A^{-1}||B| = 2^4 |A^{-1}||A|$$

$$= 2^4 |A^{-1}A| = 2^4 |I_4| = 2^4 = 16$$

(8) If A is a 3×3 matrix $\ni A^2 = \underset{3\times3}{O}$. Then no. of non-zero eigen values of A are

(a) 0 (b) 1 (c) 2 (d) 3 [JAM-2008]

Soln :- $\underset{3\times3}{A^2} = \underset{3\times3}{0} \Rightarrow A$ is nilpotent matrix

$\Rightarrow$ All eigen values of A are zero.

(9) Let $A = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 6 & 9 & 18 \\ 1 & 2 & 6 & 12 \end{bmatrix}$. Find a basis of null space of A [JAM-2008]

Soln :- $A \sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 2 & 3 & 6 \\ 0 & 0 & 3 & 6 \end{bmatrix}$: Echelon form

Let $\underset{\sim}{x} = (x_1, x_2, x_3, x_4)' \in N(A)$

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$

$\Rightarrow$ $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$

$2x_2 + 3x_3 + 6x_4 = 0$

$3x_3 + 6x_4 = 0$

$R(A) = 3$

$\dim\{N(A)\}$

$= \dim\{V_R^{\perp}(A)\}$

$= n - \dim\{V_R(A)\}$

$= n - R(A)$

$= 4 - 3 = 1$

Let $x_4 = \alpha \Rightarrow x_3 = -2\alpha \Rightarrow x_2 = 0,$

$\Rightarrow x_1 = 0$

$\Rightarrow \underset{\sim}{x} = (x_1, x_2, x_3, x_4)' = \alpha(0, 0, -2, 1)'$

$\Rightarrow \{(0, 0, -2, 1)'\}$ forms basis of $N(A)$

(null space of A)

$\Rightarrow \dim\{N(A)\} = 1$

SD

(10) Let A be a 3×3 non-diagonal matrix with $A^{-1} = A$. S.T $Tr(A) = -|A| = \pm 1$

[JAM - 2009]

Soln :-

Let λ be a Ch-root of A

$\Rightarrow \frac{1}{\lambda} = \lambda \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

Let, Ch-roots of A : $\lambda_1, \lambda_2, \lambda_3$,

[Some $\lambda$is are +1 and some are -1] $\longleftarrow \lambda_i \in \{-1, +1\} \; \forall i = 1, 2, 3$

Case-I :- Let $|A| = 1 \Rightarrow \lambda_1\lambda_2\lambda_3 = 1$

$\Rightarrow$ Two of λ_i's $= -1$ and one $\lambda_i = 1$

$\Rightarrow Tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = -1$

Case-II :- Let $|A| = -1 \Rightarrow \lambda_1\lambda_2\lambda_3 = -1$

Two of λ_i's $= +1$ and one $\lambda_i = -1$

$Tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = 1$

$\Rightarrow -|A| = + Tr(A) = \pm 1$ (Prooved)

(11) The system of equation :

$$x + 3y + 2z = K$$
$$2x + y - 4z = 4$$
$$5x - 14z = 10$$

(a) has unique solution for $K = 2$

(b) has ∞ly many solution for $K = 2$

(c) has no solution for $K = 2$

(d) has unique solution for any $K \neq 2$

[IAM-2009]

Soln:-

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -4 \\ 5 & 0 & -14 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} K \\ 4 \\ 10 \end{pmatrix}$$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$$

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} 1 & 3 & 2 & K \\ 2 & 1 & -4 & 4 \\ 5 & 0 & -14 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & K \\ 0 & -5 & -8 & 4-2K \\ 0 & -15 & -24 & 10-5K \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & K \\ 0 & -5 & -8 & 4-2K \\ 0 & 0 & 0 & K-2 \end{array}\right)$$

$$R(A) = 2, \quad R(A|\underset{\sim}{b}) = \begin{cases} 3 & \text{if } K \neq 2 \\ 2 & \text{if } K = 2 \end{cases}$$

<u>For $K=2$</u>:-

$$R(A|\underset{\sim}{b}) = R(A) = 2 < 3$$

$\Rightarrow$ The system has ∞ly many solution for $K=2$

SD

(12) Let $A = ((a_{ij}))$ be an orthogonal matrix of order n $\ni a_{1j} = \frac{1}{\sqrt{n}}$, $j = 1, 2 \cdots n$

If $\bar{a} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$, then $\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \bar{a})^2$

$=$ ——— (a) $\frac{n+1}{n}$ (b) $\frac{n-1}{n}$ (c) $\frac{n^2+1}{n}$ (d) $\frac{n^2-1}{n}$

[IAM-2009]

Soln :- [Helmarts Orthogonal matrix :

$$\begin{pmatrix} 1/\sqrt{n} & 1/\sqrt{n} & 1/\sqrt{n} & \cdots & 1/\sqrt{n} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & \cdots & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1/\sqrt{n(n-1)} & 1/\sqrt{n(n-1)} & 1/\sqrt{n(n-1)} & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{pmatrix}_{n \times n}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \bar{a})^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 - 2\bar{a} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} + n^2 \bar{a}^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 - 2n^2 \bar{a}^2 + n^2 \bar{a}^2$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 - n^2 \bar{a}^2$$

Now, A is an orthogonal matrix whose first row elements are $1/\sqrt{n}$. So A can be looked as Helmarts orthogonal matrix. $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = n$, $\bar{a} = \frac{1}{n^2} \cdot \frac{n}{\sqrt{n}} = \frac{\sqrt{n}}{n^2}$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} - \bar{a})^2 = n - n^2 \cdot \frac{n}{n^4} = n - \frac{1}{n} = \frac{n^2-1}{n} \text{ (Ans)}$$

(13) Let $P = \dfrac{\underset{\sim}{x}\,\underset{\sim}{x}'}{\underset{\sim}{x}'\,\underset{\sim}{x}}$ be a $n \times n$ matrix, $n > 1$

where $\underset{\sim}{x}$ is a non-zero coloumn vector. Then which of statement is FALSE

(a) P is idempotent (b) P is orthogonal matrix

(c) P is symmetric (d) $R(P) = 1$ [IAM-2010]

Soln :- $P' = \dfrac{\underset{\sim}{x}\,\underset{\sim}{x}'}{\underset{\sim}{x}'\,\underset{\sim}{x}} = P$: symmetric (a) ✗

$\underset{\sim}{x}'\,\underset{\sim}{x} \rightarrow$ Scaler quantity (b) ✓

$R(P) = R(\underset{\sim}{x}\,\underset{\sim}{x}') = R(\underset{\sim}{x}) = 1$ (c) ✗

$[\because \underset{\sim}{x} \neq \underset{\sim}{0}]$ (d) ✗

$P^2 = P'P = \dfrac{\underset{\sim}{x}(\underset{\sim}{x}'\,\underset{\sim}{x})\underset{\sim}{x}'}{(\underset{\sim}{x}'\,\underset{\sim}{x})^2} = \dfrac{\underset{\sim}{x}\,\underset{\sim}{x}'}{\underset{\sim}{x}'\,\underset{\sim}{x}} = P$: idempotent

but 'P' is not orthogonal

(14) Let $P = (P_{ij})$ be a 50×50 matrix where $P_{ij} = \min(i, j)$. Then Rank of $P =$ ——

(a) 1 (b) 2 (c) 25 (d) 50 [IAM-2010]

Soln :- $P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & 48 & 48 & 48 \\ 1 & 2 & 3 & \cdots & 48 & 49 & 49 \\ 1 & 2 & 3 & \cdots & 48 & 49 & 50 \end{bmatrix}$

SD

$$\sim \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 2 & \cdots & 2 & 2 & 2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & 2 & \cdots & 47 & 47 & 47 \\ 0 & 1 & 2 & \cdots & 47 & 48 & 48 \\ 0 & 1 & 2 & \cdots & 47 & 48 & 49 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

[By Successive reduction to echelon form].

$\Rightarrow R(P) = 50$ (Ans)

(15) An eigen vector of matrix $\begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$ is

(a) $(1,2)'$ (b) $(5,0)'$ (c) $(0,2)'$ (d) $(1,1)'$

[JAM-2011]

<u>Soln</u>:- Let λ be an eigen value of the matrix and corrosponding

eigen vector be $\underset{\sim}{x} = (x_1, x_2)'$

$$A\underset{\sim}{x} = \lambda \underset{\sim}{x} \Rightarrow \begin{pmatrix} x_1 + 8x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}$$

$$\Rightarrow \lambda = 1 \quad \Rightarrow x_1 + 8x_2 = x_1 \Rightarrow x_2 = 0$$

$$\Rightarrow \underset{\sim}{x} = (x_1, 0)'$$

Taking $x_1 = 5$: $\underset{\sim}{x} = (5, 0)'$

(16) (a) Consider the matrix

$$A = \begin{bmatrix} \alpha/\sqrt{11} & 1/\sqrt{66} & 1/\sqrt{6} \\ 1/\sqrt{11} & -4/\sqrt{66} & 2/\sqrt{6} \\ 1/\sqrt{11} & 7/\sqrt{66} & \beta/\sqrt{6} \end{bmatrix}$$

Find $\alpha, \beta \ni A$ becomes orthogonal matrix

Using these α, β Solve the system of eq/n

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(b) Let $A = \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 3 & 5 \\ 1 & 6 & 10 & 14 \\ 1 & 4 & 4 & \gamma \end{bmatrix}$. Find $\gamma \ni R(A) = 2$

[JAM-2011]

Soln:- (a) $A = \begin{pmatrix} \underset{\sim}{a_1'} \\ \underset{\sim}{a_2'} \\ \underset{\sim}{a_3'} \end{pmatrix}$, $\underset{\sim}{a_i} \in E^3 \ \forall i$

$$AA' = I_3 \Rightarrow \underset{\sim}{a_i'} \underset{\sim}{a_j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\Rightarrow \frac{\alpha^2}{11} + \frac{1}{66} + \frac{1}{6} = 1 = \frac{1}{11} + \frac{49}{66} + \frac{\beta^2}{6}$$

$$\Rightarrow \frac{6\alpha^2 + 1 + 11}{66} = 1 = \frac{6 + 49 + 11\beta^2}{66}$$

$$\Rightarrow \alpha^2 = 9, \quad \beta^2 = 1$$

$$\Rightarrow \alpha = \pm 3, \quad \beta = \pm 1$$

$$\underset{\sim}{a_1'} \underset{\sim}{a_2} = 0 \Rightarrow \frac{\alpha}{11} - \frac{4}{66} + \frac{2}{6} = 0$$

$$\Rightarrow 6\alpha - 4 + 22 = 0 \Rightarrow 6\alpha = -18 \Rightarrow \alpha = -3$$

$$\underset{\sim}{a_2'} \underset{\sim}{a_3} = 0 \Rightarrow \frac{1}{11} - \frac{28}{66} + \frac{2\beta}{6} = 0$$

$$\Rightarrow 22\beta - 28 + 6 = 0 \Rightarrow \beta = 1$$

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow A'A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A' \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ 1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/\sqrt{11} \\ 8/\sqrt{66} \\ 2/\sqrt{6} \end{pmatrix}$$

SD

(b) $A \sim \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \\ 1 & 4 & 4 & \gamma \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & \gamma-9 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 5 & 7 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma-4 \end{bmatrix}$

$R(A) = 2 \Rightarrow \gamma = 4$

(17) An eigen vector of matrix $M = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is (a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

[JAM-2012]

Soln :- Let λ be an eigen value of M with corrosponding eigen vector $\underset{\sim}{x} = (x, y, z)'$

$\Rightarrow M\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$\Rightarrow \begin{pmatrix} 2x+y \\ 2y+z \\ 2z \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix}$ $\Rightarrow \lambda = 2$, $\Rightarrow z = 0$, $\Rightarrow y = 0$.

$\Rightarrow \underset{\sim}{x} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$

Taking $x = 1$: $\underset{\sim}{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(18) Find Values of λ for which system of linear equations : $\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

has (a) unique solution (b) no solution

(c) ∞ly many solution [JAM - 2012]

Soln :-

$$\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$$

$$(A|\underset{\sim}{b}) = \left(\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda-\lambda^2 & 1-\lambda \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & \lambda-1 & 1-\lambda & 0 \\ 0 & 0 & -(\lambda+2)(\lambda-1) & 1-\lambda \end{array}\right)$$

(a) For unique solution $R(A|\underset{\sim}{b}) = R(A) = 3$

$\Rightarrow \lambda \neq 1, \lambda \neq -2$

(b) For no solution $R(A|\underset{\sim}{b}) \neq R(A)$

$$\Rightarrow R(A|\underset{\sim}{b}) = R(A)+1$$

$$\Rightarrow \lambda = -2$$

(c) For ∞ly many solution

$$R(A|\underset{\sim}{b}) = R(A) < 3$$

$$\Rightarrow \lambda = 1$$

(19) Let $M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. If $M^{-1} = \frac{5}{4}I + kM + \frac{1}{4}M^2$

Find the value of K. Hence or otherwise solve the system of equation

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ [JAM-2012]

Soln:- $|M - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 0 & 2-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow (\lambda-1)^2(\lambda-2) - 2 = 0$$

$$\Rightarrow (\lambda^2 - 2\lambda + 1)(\lambda - 2) - 2 = 0$$

$$\Rightarrow \lambda^3 - 4\lambda^2 + 5\lambda = 4$$

By Cayley-Hamilton Theorem :—

$$M^3 - 4M^2 + 5M = 4I$$

$$\Rightarrow M^{-1} = \frac{1}{4}M^2 - M + \frac{5}{4}I$$

$$\Rightarrow K = -1$$

Now, $M^2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 6 & 2 \\ 1 & 4 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 5/4 & 0 & 0 \\ 0 & 5/4 & 0 \\ 0 & 0 & 5/4 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ 0 & -2 & -1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$+ \begin{bmatrix} 1/4 & 6/4 & 2/4 \\ 1/4 & 4/4 & 3/4 \\ 2/4 & 2/4 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 2/4 & -2/4 & 2/4 \\ 1/4 & 1/4 & -1/4 \\ -2/4 & 2/4 & 2/4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & -1 \\ -2 & 2 & 2 \end{bmatrix}$$

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/4 \\ -1/2 \end{pmatrix}$$

(20) Let A be a 3×3 real matrix with eigen values $1, 2, 3$. Let $B = A^{-1} + A^2$. Then Trace of $B =$ ____

(a) $91/6$ (b) $95/6$ (c) $97/6$ (d) $101/6$.

[JAM-2013]

Soln:- Eigen Values of A: $1, 2, 3$

" " " A^{-1}: $1, 1/2, 1/3$

" " " A^2: $1, 4, 9$

$$\text{Trace}(B) = Tr(A^{-1}) + Tr(A^2)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + 1 + 4 + 9$$

$$= 15 + \frac{5}{6} = \frac{95}{6} \text{ (Ans)}$$

(21) Consider the system of equation

$$x + y + 2z = \alpha, \quad x + 4y + z = 4, \quad 3y - z = \beta$$

If above system always has a solution then $\alpha + \beta =$ ____ [JAM-2013]

Soln:-
$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 4 & 1 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ 4 \\ \beta \end{pmatrix}$$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$$

$$(A|\underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & \alpha \\ 1 & 4 & 1 & 4 \\ 0 & 3 & -1 & \beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & \alpha \\ 0 & 3 & -1 & 4-\alpha \\ 0 & 3 & -1 & \beta \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & \alpha \\ 0 & 3 & -1 & 4-\alpha \\ 0 & 0 & 0 & \alpha+\beta-4 \end{array}\right)$$

Now, $R(A|\underset{\sim}{b}) = R(A)$

$\Rightarrow \alpha+\beta-4=0 \Rightarrow \alpha+\beta=4$

(22) Consider the matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \alpha & \beta \\ \alpha & 0 & \beta \end{bmatrix}$

If P has eigen values $0, 3$. Find (α, β)

[JAM-2013]

Soln:- $|P-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & \alpha-\lambda & \beta \\ \alpha & 0 & \beta-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(\alpha-\lambda)(\beta-\lambda) + \alpha\beta - 1\cdot\alpha\cdot(\alpha-\lambda) = 0$

<u>If $\lambda=0$</u>:- $\alpha\beta + \alpha\beta - \alpha^2 = 0$

$\Rightarrow 2\alpha\beta - \alpha^2 = 0 \Rightarrow \alpha(-\alpha+2\beta) = 0$ — (I)

<u>If $\lambda=3$</u>:- $(1-3)(\alpha-3)(\beta-3) + \alpha\beta - \alpha(\alpha-3) = 0$ — (II)

<u>From (I)</u>:- Either $\alpha=0$ or $\alpha=2\beta$

<u>Case-I</u>:- $\alpha=0 \Rightarrow$ <u>From (II)</u>: $6(\beta-3)=0$

$\Rightarrow \beta=3$

<u>Case-II</u>:- $\alpha=2\beta \Rightarrow$ <u>From (II)</u>:

$-2\cdot(2\beta-3)(\beta-3) + 2\beta^2 - 2\beta(2\beta-3) = 0$

$\Rightarrow -2(2\beta^2 - 9\beta + 9) + 2\beta^2 - 4\beta^2 + 6\beta = 0$

$\Rightarrow -4\beta^2 + 18\beta - 18 - 2\beta^2 + 6\beta = 0$

$\Rightarrow 6\beta^2 - 24\beta + 18 = 0 \Rightarrow \beta^2 - 4\beta + 3 = 0$

$\Rightarrow \beta^2 - 3\beta - \beta + 3 = 0 \Rightarrow \beta(\beta - 3) - 1(\beta - 3) = 0$

$\Rightarrow (\beta - 1)(\beta - 3) = 0 \Rightarrow \beta = 1 \; [\because \beta \neq 3]$

$(\alpha, \beta) = \{(0,3), (2,1)\}$

(23) No. of distinct eigen values of matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$ is (a) 1 (b) 2 (c) 3 (d) 4

[IAM-2014]

Soln:- $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow [1 - \lambda + 3 \cdot 1][1 - \lambda - 1]^3 = 0$

$\Rightarrow (4 - \lambda)(-\lambda)^3 = 0 \Rightarrow \lambda = 0, 0, 0, 4$

$\therefore$ No. of distinct eigen values $= 2$

(24) In which case the system of eqn $x_1 - 2x_2 + x_3 = 3$, $2x_1 - 5x_2 + 2x_3 = 2$, $x_1 + 2x_2 + \lambda x_3 = \mu$ has ∞ly. many solution

(a) $\lambda = 1, \mu = -19$ (b) $\lambda = -1, \mu = 19$ (c) $\lambda = 2, \mu = 18$

(d) $\lambda = 1, \mu = 19$ [JAM-2014]

<u>Soln:-</u>

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -5 & 2 \\ 1 & 2 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ \mu \end{pmatrix}$$

$$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$$

$$(A|\underset{\sim}{b}) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 2 & -5 & 2 & 2 \\ 1 & 2 & \lambda & \mu \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 4 & \lambda-1 & \mu-3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & \lambda-1 & \mu-19 \end{array}\right)$$

$$R(A|\underset{\sim}{b}) = R(A) < 3 \Rightarrow \lambda = 1, \mu = 19$$

(25) The matrix $M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

(a) is an elementary matrix

(b) can be written as product of elementary matrix

(c) doesnt have linearly independent eigen vectors

(d) is a nilpotent matrix [JAM-2014]

SD

Soln:- $|M| = 24 \neq 0$

$$|M - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 1, 4, 6$

All eigen values of M are distinct

$\Rightarrow$ It has linearly independent eigen vectors

[$\because$ Distinct eigen vectors corrosponding to distinct eigen values of any square matrix are linearly independent]

As, M^{-1} exists, hence M can be written as product of elementary matrices.

(a) ✗
(b) ✓
(c) ✗
(d) ✗

(26) Let $M = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$. Which of following matrix equation does M satisfy

(a) $M^2 + 3M + 5I = 0$ (b) $M^3 - M^2 - 5M = 0$

(c) $M^3 - 3M^2 + M = 0$ (d) $M^2 - M + 5I = 0$

[JAM-2015]

Soln:- Charecteristic eqn of M is given by

$$|M - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda+1)(\lambda-2) - 3 = 0 \Rightarrow \lambda^2 - \lambda - 5 = 0$

By Cayley Hamilton Theorem

$$M^2 - M - 5I = 0 \Rightarrow M^3 - M^2 - 5M = 0$$

(27) If $|A_{n\times n}| = 0$, then (a) $R(A) \le n-2$

(b) $Tr(A) = 0$ (c) zero is an eigen value of A

(d) $\underset{\sim}{x} = \underset{\sim}{0}$ is the only soln of $A\underset{\sim}{x} = \underset{\sim}{0}$

[JAM-2015]

Soln:- $|A_{n\times n}| = 0 \Rightarrow R(A) < n \Rightarrow R(A) \le n-1$

$0 = |A| = \prod_{i=1}^{n} \lambda_i$, λ_i: ch-roots of A (eigen values)

(a) ✗
(b) ✗
(c) ✓
(d) ✗

$\Rightarrow \lambda_i = 0$ for atleast one 'i'

$\Rightarrow$ '0' is an eigen value of A

$R(A) < n \Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$ has non-trivial solution

$\Rightarrow A\underset{\sim}{x} = \underset{\sim}{0}$ for some $\underset{\sim}{x} \ne \underset{\sim}{0}$

(28) The no. of distinct real values of x for which matrix $A = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}$ is singular is (a) 1 (b) 2 (c) 3 (d) infinity

[JAM-2015]

Soln:- $|A| = 0 \Rightarrow [x + (3-1)\cdot 1][x-1]^{3-1} = 0$

$\Rightarrow (x+2)(x-1)^2 = 0 \Rightarrow x = 1, -2$

Hence matrix is singular for two distinct real values of x.

(29) The system of equations $x+y+2z=2$, $2x+3y-z=5$, $4x+7y+cz=6$ doesnt have a solution. Then c must be equal to _____ [JAM-2015]

Soln:- $A\underset{\sim}{x} = \underset{\sim}{b} \Rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 4 & 7 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 3 & -1 & 5 \\ 4 & 7 & c & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 3 & c-8 & -2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & c+7 & -5 \end{array}\right)$$

As the system has no soln $\Rightarrow R(A|\underset{\sim}{b}) = R(A)+1$

$\Rightarrow c+7=0 \Rightarrow c=-7$

(30) Let $R(A_{5\times3})=2$. Let $\underset{\sim}{b} \in R^5$ be a non-zero vector $\ni \underset{\sim}{b} \in V_c(A)$. Let $S = \{\underset{\sim}{x} \in R^3 : A\underset{\sim}{x} = \underset{\sim}{b}\}$. Define translation of subspace V is R^3 by $\underset{\sim}{x_0} \in R^3$ as the set $\underset{\sim}{x_0} + V = \{\underset{\sim}{x_0} + \underset{\sim}{v} : \underset{\sim}{v} \in V\}$. Then (a) S is empty set (b) S has only one element (c) S is translation of 1

dimensional Subspace.

(d) S is translation of 2 dimensional Subspace [JAM-2015]

Soln:— $\underset{\sim}{b} \in V_c(A) \Rightarrow R(A|\underset{\sim}{b}) = R(A)$

$\Rightarrow \underset{\sim}{b}$ is linearly dependent on coloumn vectors of A.

$\therefore A\underset{\sim}{x} = \underset{\sim}{b}$ is consistent system or it has a solution say $\underset{\sim}{x_0}$

$\Rightarrow A\underset{\sim}{x_0} = \underset{\sim}{b}$, $\underset{\sim}{x_0} \in R^3$, $\underset{\sim}{x_0} \neq \underset{\sim}{0}$

$S = \{\underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{b}\} \Rightarrow \underset{\sim}{x_0} \in S$

$N(A) = \{\underset{\sim}{x} : A\underset{\sim}{x} = \underset{\sim}{0}, \underset{\sim}{x} \in R^3\}$

$\Rightarrow \underset{\sim}{0} \in N(A)$

$\therefore S = \underset{\sim}{x_0} + N(A) = \{\underset{\sim}{x_0} + \underset{\sim}{0}, \underset{\sim}{0} \in N(A)\}$

Now $\dim V = \dim N(A) = 3 - R(A)$

$= 3 - 2 = 1$

$\Rightarrow$ S is translation of 1-dimensional Subspace.

(31) Let $P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \\ 1 & 2 & -2 & -4 \end{bmatrix}$, $R(P) =$ ____

(a) 4 (b) 3 (c) 2 (d) 1 [JAM-2016]

Soln :- $P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \\ 1 & 2 & -2 & -4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow R(P) = 2$

(32) $\alpha, \beta, \gamma \in R \ni \beta \neq 0, \gamma \neq 0$. Let $P = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$ and $P^{-1} = P$. Then

(a) $\alpha = 0$ and $\beta\gamma = 1$ (b) $\alpha \neq 0$ and $\beta\gamma = 1$

(c) $\alpha = 0$ and $\beta\gamma = 2$ (d) $\alpha = 0$ and $\beta\gamma = -1$

[JAM-2016]

Soln :- $P^{-1} = P \Rightarrow \dfrac{\begin{bmatrix} 0 & -\beta \\ -\gamma & \alpha \end{bmatrix}}{-\beta\gamma} = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\beta} & \frac{-\alpha}{\beta\gamma} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix} \Rightarrow \alpha = 0,$

$\beta\gamma = 1$

(33) Let $P = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$. Then product of eigen values of P^{-1} = ____

[JAM-2016]

Soln :- Eigen values of P : $\lambda_1, \lambda_2, \lambda_3$

" " " P^{-1} : $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$

Product of eigen values of P^{-1}

$= \frac{1}{\lambda_1 \lambda_2 \lambda_3} = \frac{1}{|P|} = \frac{-1}{-18+20} = \frac{-1}{2}$

(34) Let P be a 3×3 singular matrix, such that $P\vec{v} = \vec{v}$ for non-zero vector $\vec{v}$ and $P\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ -2/5 \end{bmatrix}$. Then

(a) $P^3 = \frac{1}{5}(7P^2 - 2P)$ (b) $P^3 = \frac{1}{4}(7P^2 - 2P)$

(c) $P^3 = \frac{1}{3}(7P^2 - 2P)$ (d) $P^3 = \frac{1}{2}(7P^2 - 2P)$

[JAM-2016]

Soln:- Eigen Values of P : $0, 1, \frac{2}{5}$

$\Rightarrow \lambda(\lambda - 1)(\lambda - \frac{2}{5}) = 0$: charecteristic eqn

$\Rightarrow (\lambda^2 - \lambda)(\lambda - \frac{2}{5}) = 0 \Rightarrow \lambda^3 - \frac{7}{5}\lambda^2 + \frac{2}{5}\lambda = 0$

$\Rightarrow$ By Cayley Hamilton Theorem,

$P^3 = \frac{1}{5}(7P^2 - 2P)$

(35) For two non-zero real no's a and b consider the system of equation

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b/2 \\ a/2 \end{bmatrix}$$

Which of following is/are true

(a) IF $a=b$, the solution system lie on the line $x+y=\frac{1}{2}$

(b) IF $a=-b$, the solution system lie on the line $y-x=\frac{1}{2}$

(c) IF $a \neq \pm b$ the system has no solution

(d) IF $a \neq \pm b$ the system has unique solution

[JAM-2016]

Soln :- IF $a=b$

$\Rightarrow a(x+y) = \frac{a}{2} \Rightarrow x+y=\frac{1}{2}$ (a) ✓

(b) ✓

IF $a=-b$

$\Rightarrow a(x-y) = -\frac{a}{2} \Rightarrow x-y=-\frac{1}{2}$ (c) ✗

$\Rightarrow y-x=\frac{1}{2}$ (d) ✓

For unique soln : $R(A|\underset{\sim}{b}) = R(A) = n$

i.e $|A| \neq 0 \Rightarrow a^2 \neq b^2 \Rightarrow a \neq \pm b$

(36) The imaginary parts of the eigen values of matrix $P = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -3 & 6 \\ 0 & 0 & -3 \end{bmatrix}$ are (a) 0, 0, 0 (b) 2, −2, 0 (c) 1, −1, 0 (d) 3, −3, 0 [JAM-2017]

Soln :- $|P - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 & 5 \\ 2 & -3-\lambda & 6 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 0 & 0 & -3-\lambda \\ 3-\lambda & 2 & 5 \\ 2 & -3-\lambda & 6 \end{vmatrix} = 0$$

$$\Rightarrow (-3-\lambda)\left[(\lambda-3)(\lambda+3) - 4\right] = 0$$

$$\Rightarrow (\lambda+3)\left[\lambda^2 - 13\right] = 0 \Rightarrow \lambda = -3, \pm\sqrt{13}$$

$$\Rightarrow \lambda = \pm\sqrt{13}, -3$$

$\Rightarrow$ Imaginary part of eigen values of A are 0, 0, 0.

(37) Let $\underset{\sim}{u}, \underset{\sim}{v} \in R^4 \ni \underset{\sim}{u} = (1, 2, 3, 5)'$ and $\underset{\sim}{v} = (5, 3, 2, 1)'$. Then the equation $\underset{\sim}{u}\underset{\sim}{v}'\underset{\sim}{x} = \underset{\sim}{v}$ has (a) ∞ly many soln (b) no-soln (c) exactly one soln (d) exactly two soln [JAM-2017]

$\underset{\sim}{u}\,\underset{\sim}{v}'\,\underset{\sim}{x} = \underset{\sim}{v} \Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$

$$A = \begin{pmatrix} 5 & 3 & 2 & 1 \\ 10 & 6 & 4 & 2 \\ 15 & 9 & 6 & 3 \\ 25 & 15 & 10 & 5 \end{pmatrix}, \quad \underset{\sim}{b} = \begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}$$

$$A|\underset{\sim}{b} = \left(\begin{array}{cccc|c} 5 & 3 & 2 & 1 & 5 \\ 10 & 6 & 4 & 2 & 3 \\ 15 & 9 & 6 & 3 & 2 \\ 25 & 15 & 10 & 5 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 5 & 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & -13 \\ 0 & 0 & 0 & 0 & -24 \end{array}\right) \sim \left(\begin{array}{cccc|c} 5 & 3 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow R(A|\underset{\sim}{b}) = 2 \neq 1 = R(A)$

$\Rightarrow$ The system has no solution

(38) Let Q, A, B be real matrices of order $n \times n$ $\ni$ Q is orthogonal and A is invertible. Then eigen values of $Q'A^{-1}BQ$ are always same as that of (a) AB (b) $Q'A^{-1}B$ (c) $A^{-1}BQ'$ (d) BA^{-1}

[JAM-2017]

Soln :- Let λ be an eigen value of

$$Q'A^{-1}BQ \Rightarrow |Q'A^{-1}BQ - \lambda I| = 0$$

$$\Rightarrow |Q'|\,|A^{-1}B - \lambda I|\,|Q| = 0$$

$$[\because Q'Q = I_n]$$

$$\Rightarrow |A^{-1}B - \lambda I| = 0$$

$$\Rightarrow |A|\,|A^{-1}B - \lambda I|\,|A^{-1}| = 0 \quad [\because |A| \neq 0]$$

$$\Rightarrow |BA^{-1} - \lambda I| = 0$$

$\Rightarrow \lambda$ is an eigen value of BA^{-1}.

(39) Let $\underset{\sim}{v} \in R^K$, $\underset{\sim}{v}'\underset{\sim}{v} \neq 0$. Then Let $P = I - \frac{2\underset{\sim}{v}\underset{\sim}{v}'}{\underset{\sim}{v}'\underset{\sim}{v}}$ which of following is/are true

(a) $P^{-1} = I - P$ (b) $-1, 1$ are eigen values of P (c) $P^{-1} = P$ (d) $(I+P)\underset{\sim}{v} = \underset{\sim}{v}$

[IAM—2017]

Soln :- $P' = P$

$$P'P = P^2 = P \cdot P = \left(I - \frac{2\underset{\sim}{v}\underset{\sim}{v}'}{\underset{\sim}{v}'\underset{\sim}{v}}\right)\left(I - \frac{2\underset{\sim}{v}\underset{\sim}{v}'}{\underset{\sim}{v}'\underset{\sim}{v}}\right)$$

$= I \Rightarrow P' = P^{-1}$; P : orthogonal

$(I+P)\underset{\sim}{v} = \underset{\sim}{v} \Rightarrow P\underset{\sim}{v} = \underset{\sim}{0}$

[But '0' is not an eigen value of P and $\underset{\sim}{v} \neq \underset{\sim}{0} \Rightarrow P\underset{\sim}{v} \neq \underset{\sim}{0}$]

(a) ✗
(b) ✓
(c) ✓
(d) ✗

(40) Let a unit vector $\underset{\sim}{V} = (v_1, v_2, v_3)' \ni$

$A\underset{\sim}{V} = \underset{\sim}{0}$ where $A = \begin{bmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{bmatrix}$

Then $\sqrt{6}\left(|v_1| + |v_2| + |v_3|\right) = ____$

[JAM - 2017]

Soln:- $\|\underset{\sim}{V}\| = 1 \Rightarrow v_1^2 + v_2^2 + v_3^2 = 1$ — (*)

$A = \begin{bmatrix} 5/6 & -2/6 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -2/6 & 5/6 \end{bmatrix}, \underset{\sim}{V} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\Rightarrow 5v_1 - 2v_2 - v_3 = 0$ (I), $-v_1 + v_2 - v_3 = 0$, (II)

$-v_1 - 2v_2 + 5v_3 = 0$ (III)

(I) + 4(II) $= v_1 + 2v_2 - 5v_3 = -(-v_1 - 2v_2 + 5v_3)$

$= -$(III)

Hence main eqns for solving v_1, v_2, v_3 are (II) and (III)

(II) − (III) :- $3v_2 - 6v_3 = 0 \Rightarrow v_2 = 2v_3$

Let $v_3 = K$, $v_2 = 2K$, $v_1 = K$

From (*) $6K^2 = 1 \Rightarrow K = \pm\frac{1}{\sqrt{6}}$

$\Rightarrow \sqrt{6}\left(|v_1| + |v_2| + |v_3|\right) = \sqrt{6}\left(\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right) = 4$

(41) For $j = 1,2,3,4,5$. Let P_j be the matrix of order 5×5 obtained by replacing jth coloumn of identity matrix by vector $\underset{\sim}{v} = (5,4,3,2,1)'$

Then determinant of matrix product P_1, P_2, P_3, P_4, P_5 is ——— [JAM-2017]

Soln:- $P_1 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow |P_1| = 5$

$P_2 = \begin{bmatrix} 1 & 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow |P_2| = 4$

Similarly, $|P_3| = 3$, $|P_2| = 2$, $|P_1| = 1$

$\Rightarrow |P_1 P_2 P_3 P_4 P_5| = \prod_{i=1}^{5} |P_i| = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

(42) Let $M = \begin{bmatrix} 1/4 & 3/4 \\ 3/5 & 2/5 \end{bmatrix}$, then

(a) $20M^2 - 13M + 7I = 0$ (b) $20M^2 - 13M - 7I = 0$

(c) $20M^2 + 13M + 7I = 0$ (d) $20M^2 + 13M - 7I = 0$

[JAM-2018]

Soln:- Trick:- $aM^2 + bM + cI = 0$ — (1)

Trace $= -\frac{b}{a} = \frac{1}{4} + \frac{2}{5} = \frac{13}{20}$ (Sum of roots of (1))

Determinant $= \frac{c}{a} = \frac{-7}{20}$ (Product of roots of (1))

[Think Intuitively ☺]

(a) ✗

(b) ✓

(c) ✗

(d) ✗

(43) Let $M = (\underset{\sim}{x}\ \underset{\sim}{y}\ \underset{\sim}{z})$ be an orthogonal matrix with $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \in R^3$ as its coloumn vector then $\underset{\sim}{\theta} = \underset{\sim}{x}\underset{\sim}{x}' + \underset{\sim}{y}\underset{\sim}{y}'$ is

(a) Skew-Symmetric (b) I_3 (c) Satisfy $\underset{\sim}{\theta}^2 = \underset{\sim}{\theta}$

(d) Satisfy $\underset{\sim}{\theta}\underset{\sim}{z} = \underset{\sim}{z}$ [JAM-2018]

Soln:- $M'M = I_3 \Rightarrow \underset{\sim}{x}'\underset{\sim}{x} = 1,\ \underset{\sim}{y}'\underset{\sim}{y} = 1,\ \underset{\sim}{z}'\underset{\sim}{z} = 1,$

$\underset{\sim}{x}'\underset{\sim}{y} = 0,\ \underset{\sim}{x}'\underset{\sim}{z} = 0,\ \underset{\sim}{y}'\underset{\sim}{z} = 0$ (a) ✗

$\underset{\sim}{\theta}' = \underset{\sim}{\theta}$ (b) ✗

(c) ✓

$\underset{\sim}{\theta}^2 = (\underset{\sim}{x}\underset{\sim}{x}' + \underset{\sim}{y}\underset{\sim}{y}')(\underset{\sim}{x}\underset{\sim}{x}' + \underset{\sim}{y}\underset{\sim}{y}') = \underset{\sim}{x}\underset{\sim}{x}' + \underset{\sim}{y}\underset{\sim}{y}' = \underset{\sim}{\theta}$ (d) ✗

$\underset{\sim}{\theta}\underset{\sim}{z} = \underset{\sim}{x}\underset{\sim}{x}'\underset{\sim}{z} + \underset{\sim}{y}\underset{\sim}{y}'\underset{\sim}{z} = \underset{\sim}{0}$

(44) Let M be a 3×3 matrix (non-zero), skew-symmetric then

(a) M is invertible (b) $I+M$ is invertible

(c) $\exists$ a non-zero real number $\alpha \ni \alpha I + M$ is not invertible

(d) All eigen values of M are real

[JAM-2018]

Soln:- Eigen values of M: $0, a+ib, a-ib, b\neq0$

$|M| = 0 \Rightarrow M^{-1}$ doesnt exist (a) ✗

$I+M$: $1, a+1+ib, a+1-ib, b\neq0$ (b) ✓

$|I+M| \neq 0 \Rightarrow (I+M)^{-1}$ exists (c) ✗

$\alpha I + M$: $\alpha, \alpha+a+ib, \alpha+a-ib, |\alpha I+M| \neq 0, \alpha\neq0$ (d) ✗

(45) $M = \sum_{j=1}^{4} \underset{\sim}{x_i}\, \underset{\sim}{x_i}'$, $\underset{\sim}{x_1}' = (1,-1,1,0)$,

$\underset{\sim}{x_2}' = (1,1,0,1)$, $\underset{\sim}{x_3}' = (1,3,1,0)$,

$\underset{\sim}{x_4}' = (1,1,1,0)$, Then $R(M) =$ ——

[JAM-2018]

Soln:- $\underset{\sim}{x_1}\,\underset{\sim}{x_1}' = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\underset{\sim}{x_2}\,\underset{\sim}{x_2}' = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

$\underset{\sim}{x_3}\,\underset{\sim}{x_3}' = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 3 & 9 & 3 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\underset{\sim}{x_4}\,\underset{\sim}{x_4}' = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$M = \begin{pmatrix} 4 & 4 & 3 & 1 \\ 4 & 12 & 3 & 1 \\ 3 & 3 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 4 & 3 & 1 \\ 0 & 8 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

SD

$$M \sim \begin{pmatrix} 4 & 4 & 3 & 1 \\ 0 & 8 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 4 & 4 & 3 & 1 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 4 & 4 & 3 & 1 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow R(M) = 3$

(46) Let $M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. If $V = \left\{ (x,y,0) \in R^3 : M\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

and $W = \left\{ (x,y,z) \in R^3 : M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ then

(a) $\dim(V) = 2$ (b) $\dim(W) = 2$ (c) $\dim(V) = 1$

(d) $V \cap W = \{(0,0,0)\}$ [JAM-2018]

Soln:- $M\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x - y = 0, \Rightarrow x = y = K$ (say)

$(x,y,0) = K(1,1,0) \Rightarrow \{(1,1,0)\}$ forms basis of $V \Rightarrow \dim(V) = 1$

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x - y + z = 0, \; x - y - z = 0$

$\Rightarrow z = 0, \; x = y = m$ (say)

$(x,y,z) = (m,m,0) = m(1,1,0) \Rightarrow \{(1,1,0)\}$ forms basis of $W \Rightarrow \dim(W) = 1$

$(1,1,0) \in V \cap W$,

$(0,0,0) \in V \cap W$, ...

$\Rightarrow V \cap W = \{(0,0,0), (m,m,0)\}$, $m \neq 0$ for any $\in R$

(47) If x, y, z are real no's such that $4x+2y+z=31$ and $2x+4y-z=19$, then value of $9x+7y+z$

(a) Cannot be computed (b) $\frac{281}{3}$ (c) $\frac{182}{3}$

(d) $\frac{218}{3}$ [JAM-2018]

Soln :- If possible write $9x+7y+z$ as linear combination of $(4x+2y+z)$, $(2x+4y-z)$ i.e

$$9x+7y+z = l_1(4x+2y+z) + l_2(2x+4y-z)$$

$\Rightarrow 4l_1 + 2l_2 = 9$ — (I), $2l_1 + 4l_2 = 7$ — (II)

$l_1 - l_2 = 1$ — (III)

(I) − (II) = 2(III) :- Thus only (I) and (II) will contribute to solve l_1, l_2

Solving (I) and (II), $l_2 = \frac{5}{6}$, $l_1 = \frac{11}{6}$

Clearly $(9x+7y+z)$ can be expressed as linear combination of $(4x+2y+z)$, $(2x+4y-z)$

$\Rightarrow 9x+7y+z = \frac{11}{6} \times 31 + \frac{19 \times 5}{6} = \frac{341+95}{6}$

$= \frac{436}{6} = \frac{218}{3}$ (Ans)

(48) Let P be a 3×3 non-null real matrix. If $\exists$ a 3×2 matrix Q and 2×3 matrix $R \ni$ $P = QR$ then

(a) $P\underset{\sim}{x} = \underset{\sim}{0}$ has unique soln $\underset{\sim}{0} \in R^3$

(b) $\exists\ \underset{\sim}{b} \in R^3 \ni P\underset{\sim}{x} = \underset{\sim}{b}$ has no solution

(c) $\exists$ non-zero $\underset{\sim}{b} \in R^3 \ni P\underset{\sim}{x} = \underset{\sim}{b}$

(d) $\exists$ a non-zero $\underset{\sim}{b} \in R^3 \ni P'\underset{\sim}{x} = \underset{\sim}{b}$ has a unique solution [JAM-2019]

Soln:- By rank factorization theorem

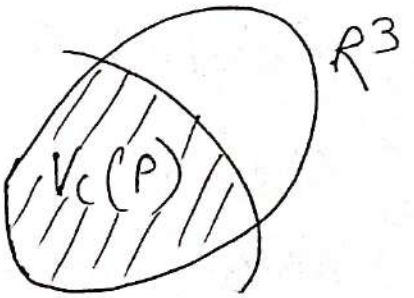
$R(P) = 2 < 3 \Rightarrow P\underset{\sim}{x} = \underset{\sim}{0}$ has non-trivial soln i.e $P\underset{\sim}{x} = \underset{\sim}{0}$, $\underset{\sim}{x} \neq \underset{\sim}{0}$

(a) ✗
(b) ✓
(c) ✗
(d) ✗

IF $\underset{\sim}{b} \in V_c(P) \Rightarrow R(P|\underset{\sim}{b}) = R(P)$
$\Rightarrow P\underset{\sim}{x} = \underset{\sim}{b}$ has a solution

IF $\underset{\sim}{b} \notin V_c(P) \Rightarrow R(P|\underset{\sim}{b}) = R(P)+1$
$\Rightarrow P\underset{\sim}{x} = \underset{\sim}{b}$ has no solution

Now $\underset{\sim}{b} \in R^3 \not\Rightarrow \underset{\sim}{b} \in V_c(P)$



(49) If $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$ and $6M^{-1} = aI_3 + bM - M^2$

then $(a, b) =$ ——— (a) $(3, 2)$ (b) $(2, 3)$

(c) $(4, 5)$ (d) $(5, 4)$ [JAM- 2019]

Soln:- $|P - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(-1-\lambda) + 1(-4+2\lambda) = 0$

$\Rightarrow -(\lambda-1)(\lambda-2)(\lambda+1) + 2\lambda - 4 = 0$

$\Rightarrow (\lambda^2-1)(\lambda-2) + 4 - 2\lambda = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 + 4 - 2\lambda = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - 3\lambda + 6 = 0$

By Cayley Hamilton Theorem,

$M^3 - 3M - 2M^2 + 6I = 0$

$\Rightarrow 6I = 3M + 2M^2 - M^3$

$\Rightarrow 6M^{-1} = 3I + 2M - M^2$

$\Rightarrow a = 3, b = 2$

(50) Let P be a 2×2 real matrix $\ni$ every non-zero vector in R^2 is an eigenvector of P. Suppose λ_1, λ_2 denotes eigen values of P and $P\begin{pmatrix} \sqrt{2} \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 \\ t \end{pmatrix}$ for some $t \in R$

Which of following is/are true

(a) $\lambda_1 \neq \lambda_2$ (b) $\lambda_1 \lambda_2 = 2$ (c) $\sqrt{2}$ is an eigenvalue of P (d) $\sqrt{3}$ is an eigen value of P [JAM-2019]

Soln :- $P\begin{pmatrix}\sqrt{2}\\ \sqrt{3}\end{pmatrix} = \sqrt{2}\begin{pmatrix}\sqrt{2}\\ \sqrt{3}\end{pmatrix}$

$\Rightarrow t = \sqrt{6}$ and $\lambda_1 = \lambda_2 = \sqrt{2}$

$\lambda_1\lambda_2 = 2$

(a) ✗
(b) ✓
(c) ✓
(d) ✗

(51) Let P be a $n\times n$ non-null skew symmetric matrix, $n \in$ even. Which of following is/are always true

(a) $P\underset{\sim}{x} = \underset{\sim}{0}$ has ∞ly many soln

(b) $P\underset{\sim}{x} = \lambda \underset{\sim}{x}$ has unique soln for every non-zero $\lambda \in R$

(c) If $Q = (I+P)(I-P)^{-1}$ then $Q'Q = I$

(d) The sum of eigen values of $P = 0$

[IAM- 2019]

Soln:- $|P| \neq 0 \Rightarrow R(P) = n$

$\Rightarrow P\underset{\sim}{x} = \underset{\sim}{0}$ has only one soln i.e $\underset{\sim}{x} = \underset{\sim}{0}$

Eigen values of P : $a+ib, a-ib, c+id, c-id, \ldots$

Sum of eigen value of $P = 2(a+c+\cdots)$
$= \text{Trace}(P) = 0$

(a) ✗
(b) ✓
(c) ✓
(d) ✓

$Q'Q = (I+P)^{-1}(I-P)(I+P)(I-P)^{-1}$

$= (I+P)^{-1}(I+P)(I-P)(I-P)^{-1}$ → commutative

$= I_n$

$(P-\lambda I)\underset{\sim}{x} = \underset{\sim}{0}$ has unique soln

if $\underset{\sim}{x} = \underset{\sim}{0}$ is the only soln

i.e if $R(P-\lambda I) = n$ i.e $|P-\lambda I| \neq 0$

Now $|P-\lambda I| \neq 0 \ \forall \ \lambda (\neq 0) \in R$

(52) If $\begin{bmatrix} \sqrt{5}/3 & -2/3 & c \\ 2/3 & \sqrt{5}/3 & d \\ a & b & 1 \end{bmatrix}$ is real orthogonal matrix then $a^2+b^2+c^2+d^2 =$ ___

[JAM-2019]

Soln :- For an orthogonal matrix sum of squares of elements of a particular row/coloumn $= 1$

$$\Rightarrow a^2+b^2+1 = 1 = c^2+d^2+1$$

$$\Rightarrow a^2+b^2 = 0 = c^2+d^2 \Rightarrow a^2+b^2+c^2+d^2 = 0$$

(53) If $ad-bc = 2$ and $ps-qr = 1$, then

$$\begin{vmatrix} a & b & 0 & 0 \\ 3 & 10 & 2p & q \\ c & d & 0 & 0 \\ 2 & 7 & 2r & s \end{vmatrix} = \text{———}$$

[JAM-2019]

Soln :-

$$\begin{vmatrix} a & b & 0 & 0 \\ 3 & 10 & 2p & q \\ c & d & 0 & 0 \\ 2 & 7 & 2r & s \end{vmatrix} = a \begin{vmatrix} 10 & 2p & q \\ d & 0 & 0 \\ 7 & 2r & s \end{vmatrix} - b \begin{vmatrix} 3 & 2p & q \\ c & 0 & 0 \\ 2 & 2r & s \end{vmatrix}$$

$$= -a \begin{vmatrix} d & 0 & 0 \\ 10 & 2p & q \\ 7 & 2r & s \end{vmatrix} + b \begin{vmatrix} c & 0 & 0 \\ 3 & 2p & q \\ 2 & 2r & s \end{vmatrix}$$

$$= -ad(2ps-2qr) + bc(2ps-2qr) = 2(bc-da)(ps-qr)$$

$$= 2 \times (-2) \times 1 = -4 \text{ (Ans)}$$

(54) Consider the system of Linear equations

$ax+2y+z=0,\ y+5z=1,\ by-5z=-1$

Which of following is true

(a) The system has unique soln for $a=1, b=-1$

(b) " " " " " for $a=-1, b=1$

(c) " " " no " for $a=1, b=0$

(d) " " " ∞ly many " for $a=0, b=0$

[JAM-2020]

Soln :- $\begin{pmatrix} a & 2 & 1 \\ 0 & 1 & 5 \\ 0 & b & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} a & 2 & 1 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & b & -5 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} a & 2 & 1 & 0 \\ 0 & 1 & 5 & 1 \\ 0 & 0 & -5-5b & -1-b \end{array}\right)$$

For ∞ly many soln $R(A|\underset{\sim}{b}) = R(A) < n = 3$

$-5-5b = 0 = -1-b \Rightarrow b=-1$ (a) ✗

(b) ✓

For unique soln, $R(A|\underset{\sim}{b}) = R(A) = 3$ (c) ✗

$\Rightarrow -5-5b \neq 0,\ -1-b \neq 0,\ a \neq 0$ (d) ✗

$\Rightarrow b \neq -1,\ a \neq 0$

$\Rightarrow$ The system has unique solution for $a=-1, b=1$

(55) Let M be a $n \times n$ non-null skew symmetric matrix. Then matrix $(I-M)(I+M)^{-1} = P$ is (a) Singular (b) Symmetric (c) orthogonal (d) idempotent [JAM-2020]

Soln :- $P' = (I-M)^{-1}(I+M) \neq P$ (a) ✗ (b) ✗

$P'P = (I-M)^{-1}(I+M)(I-M)(I+M)^{-1}$ (c) ✓

$= (I-M)^{-1}(I-M)(I+M)(I+M)^{-1}$ (d) ✗

$= I_n$

$|P| = \pm 1 \neq 0$

(56) For real constants a, b, let $M = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ a & b \end{bmatrix}$ be an orthogonal matrix then which of following is/are true

(a) $a+b=0$ (b) $b = \sqrt{1-a^2}$ (c) $ab = -1/2$

(d) $M^2 = I_2$ [JAM-2020]

Soln :- $M'M = I \Rightarrow \begin{bmatrix} 1/\sqrt{2} & a \\ 1/\sqrt{2} & b \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ a & b \end{bmatrix} = I_2$ (a) ✓ (b) ✗

$\Rightarrow \begin{bmatrix} a^2 + \frac{1}{2} & \frac{1}{2} + ab \\ \frac{1}{2} + ab & \frac{1}{2} + b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (c) ✓ (d) ✗

$\Rightarrow ab = -\frac{1}{2}$, $a^2 = \frac{1}{2}$, $b^2 = \frac{1}{2}$, $a^2 + b^2 = 1$

$(a+b)^2 = a^2 + b^2 + 2ab = 1 - 1 = 0 \Rightarrow a+b = 0 \Rightarrow b = \pm\sqrt{1-a^2}$

$M' \neq M \Rightarrow M^2 \neq I_2$

(57) The rank of matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix}$

= ____ [JAM-2020]

Soln:- $A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 3 & 4 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow R(A) = 3$

(58) Consider the matrix $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$

Let P be a non-singular matrix $\ni$ $P^{-1}MP$ is a diagonal matrix. Then Trace $(P^{-1}M^3P) =$ ____ [JAM-2020]

Soln:- $|M - \lambda I| = 0$, 'M' has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ Let.

$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 2 \\ 0 & 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)[(3-\lambda)(4-\lambda) - 2] = 0$

$\Rightarrow (1-\lambda)[(\lambda-3)(\lambda-4) - 2] = 0$

$\Rightarrow (\lambda-1)[\lambda^2 - 7\lambda + 10] = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 10\lambda - \lambda^2 + 7\lambda - 10 = 0$

$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0 \Rightarrow (\lambda-1)[\lambda-2](\lambda-5) = 0$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 5$

$Tr(P^{-1}M^3P) = Tr(M^3)$ $[\because Tr(AB) = Tr(BA)]$

$= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 1 + 8 + 125 = 134$

(59) Let P be a 3×3 matrix having charecteristic roots $\lambda_1 = -2/3$, $\lambda_2 = 0$, $\lambda_3 = 1$. Define, $Q = 3P^3 - P^2 - P + I_3$ and $R = 3P^3 - 2P$. If $\alpha = |Q|$ and $\beta = Tr(R)$ Then $\alpha + \beta =$ — [JAM- 2020]

Soln:- Eigen Values of P: $-\frac{2}{3}, 0, 1$

" " " R: $+\frac{4}{9}, 0, 1$

" " " Q: $\frac{1}{3}, 1, 2$

$\alpha = |Q| = \frac{2}{3}$, $\beta = Tr(R) = \frac{13}{9}$

$\alpha + \beta = \frac{2}{3} + \frac{13}{9} = \frac{19}{9} = 2.11$ (Ans)

(60) If α, β, γ are eigen values of $M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 3 \\ -1 & 2 & 2 \end{bmatrix}$. If $\gamma = 1$ and $\alpha > \beta$

Then $2\alpha + 3\beta =$ — [JAM - 2021]

Soln:- $Tr(M) = 1 + \alpha + \beta = 5 \Rightarrow \alpha + \beta = 4$ — (I)

$|M| = \alpha\beta = -1(5) = -5$ — (II)

$\Rightarrow \alpha(4 - \alpha) = -5 \Rightarrow \alpha^2 - 4\alpha - 5 = 0 \Rightarrow (\alpha+1)(\alpha-5) = 0$

$\Rightarrow \alpha = -1, +5$

$\alpha = -1, \beta = 5$ ✗

$\alpha = 5, \beta = -1$ ✓ $[\because \alpha > \beta]$

$\Rightarrow 2\alpha + 3\beta = 10 - 3 = 7$

(61) Let $M = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ be a 2×2 matrix

If $\alpha = |M^4 - 6I_2|$, then $\alpha^2 =$ ____

[JAM-2021]

Soln:- $|M - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & -6 \\ 3 & -4-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda+4)(\lambda-5) + 18 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \lambda^2 + 2\lambda + \lambda - 2 = 0 \Rightarrow (\lambda+1)(\lambda-2) = 0$

Eigen Values of M: $-1, 2$ $\Rightarrow \lambda = -1, 2$

" " " M^4: $1, 16$

$\alpha = |M^4 - 6I_2| = (1-6) \times (16-6) = -50$

$\Rightarrow \alpha^2 = 2500$ (Ans)

(62) Let A be a 3×3 real matrix such that $A \neq I_3$ and sum of entries in each row $= 1$. Then which of following is/are necessarily true

(a) $A - I_3$ is invertible matrix

(b) The set $\{\underset{\sim}{x} \in R^3 : (A - I_3)\underset{\sim}{x} = \underset{\sim}{0}\}$ has atleast two elements

(c) Charecteristic polynomial $P(\lambda)$ of

$A + 2A^2 + A^3$ has $\lambda - 4$ as a factor.

(d) A cannot be orthogonal matrix

[JAM-2021]

Soln:- $A\underset{\sim}{1} = \underset{\sim}{1} \Rightarrow$ '1' is an eigen value of A

$A - I_3$ has a eigen value $= 0$

$\Rightarrow |A - I_3| = 0 \Rightarrow R(A - I_3) < n = 3$

$\Rightarrow (A - I_3)\underset{\sim}{x} = \underset{\sim}{0}$ has non-trivial solution

i.e $\{\underset{\sim}{x} : (A - I_3)\underset{\sim}{x} = \underset{\sim}{0}\}$ has atleast two elements

'1' is an eigen value of A

$\Rightarrow 1 + 2\cdot 1^2 + 1^3 = 4$ is an eigen value of $A + 2A^2 + A^3 \Rightarrow \lambda - 4$ is a factor of Ch-polynomial $P(\lambda)$ of $A + 2A^2 + A^3$

(a) ✗
(b) ✓
(c) ✓
(d) ✗

(63) Let M be a 3x3 real matrix. Let $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\\alpha\end{pmatrix}$ be eigen vectors corrospond to distinct eigen values of M, $\alpha \in R$. Then which of following is not possible value of α

(a) 0 (b) 1 (c) -2 (d) 2 [JAM-2021]

Soln:- Distinct eigen vectors corrospond to distinct eigen values of a matrix are linearly independent.

$$-\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\-1\\-2\end{pmatrix} \Rightarrow \alpha \neq -2$$

(a) ✗
(b) ✗
(c) ✓
(d) ✗

(64) Consider the linear system $\underset{m\times n}{A}\,\underset{n\times 1}{\underset{\sim}{x}} = \underset{m\times 1}{\underset{\sim}{b}}$

Further suppose $\exists$ an $m\times 1$ vector $\underset{\sim}{c} \ni$ $A\underset{\sim}{x} = \underset{\sim}{c}$ has no solution. Then which of following is/are necessarily true

[JAM 2021]

(a) If $m \leq n$ and $\underset{\sim}{d}$ is the first Coloumn vector of A then $A\underset{\sim}{x} = \underset{\sim}{d}$ has unique solution

(b) If $m \geq n$, $R(A) < n$.

(c) $R(A) < m$

(d) If $m \geq n$ then the system $A\underset{\sim}{x} = \underset{\sim}{0}$ has a solution other than $\underset{\sim}{x} = \underset{\sim}{0}$

Soln :- $\underset{\sim}{d} \in V_c(A) \Rightarrow R(A|\underset{\sim}{d}) = R(A) \leq m \leq n$

(a) ✗
(b) ✗
(c) ✓
(d) ✗

$R(A) \leq m$, $R(A) \leq n$

$\Rightarrow R(A) \leq \min\{m, n\}$

$\Rightarrow R(A) \leq n$ if $m \geq n$

$\underset{\sim}{c} \in R^m$ but $\underset{\sim}{c} \notin V_c(A) \Rightarrow V_c(A) \subset R^m$

$\Rightarrow \dim V_c(A) < \dim R^m \Rightarrow R(A) < m$

$R(A) \leq n$ if $m \geq n$

(65) For $a \in R$, Consider the system of eqn

$$ax + ay = a+2 \quad \text{--- (I)}$$

$$x + ay + (a-1)z = a-4 \quad \text{--- (II)}$$

$$ax + ay + (a-2)z = -8 \quad \text{--- (III)}$$

in unknowns x, y, z. Then which of following is true

(a) System has unique solution for $a = 1$

(b) System has ∞ly many solution for $a = 2$

(c) System has unique solution for $a = -2$

(d) System has ∞ly many solution for $a = -2$

[JAM-2021]

Soln:- $\begin{pmatrix} a & a & 0 \\ 1 & a & a-1 \\ a & a & a-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+2 \\ a-4 \\ -8 \end{pmatrix}$

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$A|b = \left(\begin{array}{ccc|c} a & a & 0 & a+2 \\ 1 & a & a-1 & a-4 \\ a & a & a-2 & -8 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} a & a & 0 & a+2 \\ 0 & a^2-a & a^2-a & a^2-5a-2 \\ 0 & 0 & a-2 & -10-a \end{array}\right)$$

(a) ✗

(b) ✗

(c) ✓

(d) ✗

For $a=1$:- $A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & -1 & -11 \end{array}\right)$

$R(A|\underset{\sim}{b}) = 3 = R(A)+1$

$\Rightarrow$ System has no Solution

For $a=2$:-

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} 2 & 2 & 0 & 4 \\ 0 & 2 & 2 & -8 \\ 0 & 0 & 0 & -12 \end{array}\right)$$

$\Rightarrow R(A|\underset{\sim}{b}) = 3 = R(A)+1$

$\Rightarrow$ System has no solution

For $a=-2$:-

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & 6 & 6 & 12 \\ 0 & 0 & -4 & -8 \end{array}\right)$$

$R(A|\underset{\sim}{b}) = R(A) = 3$

$\Rightarrow$ System has unique solution

(66) Let the system of equations

$$x+ay+z=1$$
$$2x+4y+z=-b$$
$$3x+y+2z=b+2$$

has ∞ly many solutions where $a, b \in R$ are constants. Then $2a+8b =$ _____

(a) -11 (b) -10 (c) -13 (d) -14 [JAM-2022]

Soln:- $\begin{pmatrix} 1 & a & 1 \\ 2 & 4 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -b \\ b+2 \end{pmatrix} \Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 2 & 4 & 1 & -b \\ 3 & 1 & 2 & b+2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 4-2a & -1 & -b-2 \\ 0 & 1-3a & -1 & b-1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & a & 1 & 1 \\ 0 & 4-2a & -1 & -b-2 \\ 0 & a+3 & 0 & -1-2b \end{array}\right).$$

$R(A|\underset{\sim}{b}) = R(A) < n = 3$ (for ∞ly many solution)

$\Rightarrow a+3=0, \quad -1-2b=0$

$\Rightarrow a=-3, \quad b=-\frac{1}{2}$

$2a+8b = -6-4 = -10$ (Ans)

(67) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then sum of all elements of $A^{100} =$ ____

(a) 101 (b) 103 (c) 102 (d) 100 [JAM-2022]

Soln :-

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} = (A^2)^2$$

SD

$$A^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & 1 \end{pmatrix}$$

$$= (A^2)^3$$

Similarly $A^{100} = (A^2)^{50} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 50 & 50 & 1 \end{pmatrix}$

Sum of all elements of $A^{100} = 50+50+1+1+1$
$= 103$ (Ans)

(68) If $n \geq 2$. Which of following is/are true.

(a) If A, B are $n \times n$ real orthogonal matrix $\ni$ $|A|+|B| = 0$ then, the matrix $A+B$ is singular.

(b) If A is an $n \times n$ real matrix $\ni$ $I+A$ is non-singular then $I + (I+A)^{-1}(I-A)$ is singular matrix.

(c) If A is an $n \times n$ real skew symmetric matrix then $I - A^2$ is non-singular.

(d) If A is an $n \times n$ real orthogonal matrix, then $|A - \lambda I_n| \neq 0$ $\forall$ $\lambda \in \{x \in R : x \neq \pm 1\}$ [JAM-2022]

SD

Soln :- (a) $A'(A+B)B' = A'+B'$

$\Rightarrow |A'(A+B)B'| = |(A+B)'|$

$\Rightarrow |A|\,|A+B|\,|B| = |A+B|$

$\Rightarrow |A+B|\left[|A||B|-1\right] = 0$

$\Rightarrow |A+B|\left[-|A|^2-1\right] = 0 \quad \left[\because |A|+|B| = 0\right]$

$\Rightarrow |A+B| = 0 \quad \left[\because |A|^2 = 1 \text{ as } A \text{ is an orthogonal matrix}\right]$

(a) ✓ (b) ✗ (c) ✓ (d) ✓

(b) $|I+A| \neq 0$

$I + (I+A)^{-1}(I-A) = (I+A)^{-1}\left[I+A+I-A\right]$

$= 2I(I+A)^{-1} = 2(I+A)^{-1}$

$\Rightarrow |I + (I+A)^{-1}(I-A)| = |2(I+A)^{-1}| \neq 0$

(c) Eigen values of $A \rightarrow$ all complex conjugate if $n \in$ even

$\rightarrow$ one zero, rest complex conjugates if $n \in$ odd

$\Rightarrow$ No eigen values of A equals ± 1

$\Rightarrow$ No eigen value of $I - A^2$ equals 0

$\Rightarrow |I - A^2| \neq 0$

(d) $A \rightarrow$ orthogonal matrix $\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \lambda = \pm 1$ i.e. $|A - \lambda I| \neq 0 \; \forall \lambda \in \{x \in R : x \neq \pm 1\}$

(69) Let A be a 3×3 matrix $\ni$

$|A| = 6$, and $\operatorname{adj}(A) = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

Then $\operatorname{Tr}(A) =$ _____ [JAM-2022]

Soln:- $A^{-1} = \dfrac{\operatorname{adj}(A)}{|A|} = \dfrac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

$$0 = |A^{-1} - \lambda I| = \left| \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 6\lambda & 0 & 0 \\ 0 & 6\lambda & 0 \\ 0 & 0 & 6\lambda \end{pmatrix} \right|$$

$$\Rightarrow \begin{vmatrix} 1-6\lambda & -1 & 2 \\ 5 & 7-6\lambda & 1 \\ -1 & 1 & 1-6\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-6\lambda & -1 & 0 \\ 5 & 7-6\lambda & 15-12\lambda \\ -1 & 1 & 3-6\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-6\lambda)\left[(7-6\lambda)(3-6\lambda) - 15 + 12\lambda\right] + 1\left[15 - 30\lambda + 15 - 12\lambda\right] = 0$$

$$\Rightarrow (1-6\lambda)\left[(6\lambda-3)(6\lambda-7) - 15 + 12\lambda\right] + (30-42\lambda) = 0$$

$$\Rightarrow (6\lambda-1)\left[(6\lambda-3)(6\lambda-7) - 15 + 12\lambda\right] + 42\lambda - 30 = 0$$

$$\Rightarrow (6\lambda-1)\left[36\lambda^2 - 60\lambda + 21 - 15 + 12\lambda\right] + 42\lambda - 30 = 0$$

$\Rightarrow (6\lambda-1)\left[36\lambda^2-48\lambda+6\right]+42\lambda-30=0$

$\Rightarrow (6\lambda-1)\left[6\lambda^2-8\lambda+1\right]+7\lambda-5=0$

$\Rightarrow 36\lambda^3-48\lambda^2+6\lambda-6\lambda^2+8\lambda-1+7\lambda-5=0$

$\Rightarrow 36\lambda^3-54\lambda^2+21\lambda-6=0$ — (1)

Let eigen values of λ : $\underbrace{\lambda_1, \lambda_2, \lambda_3}_{\text{roots of (1)}}$

$Tr(A) = \frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}$

$= \frac{\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3} = \frac{21/36}{6/36}$

$= \frac{21}{6} = \frac{7}{2} = 3.5$

(70) Let $A=\begin{pmatrix}1 & -1 & 2\\ -1 & 0 & 1\\ 2 & 1 & 1\end{pmatrix}$, let $\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$ be an eigen vector corrosponding to smallest eigen value of A satisfying $x_1^2+x_2^2+x_3^2=1$. Then value of $|x_1|+|x_2|+|x_3| =$ ——— [JAM-2022]

Soln:- $|A-\lambda I|=0$

SD

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 2 \\ -1 & -\lambda & 1 \\ 2 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & -\lambda & 1-2\lambda \\ 2 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda(\lambda-3)+2\lambda-1\right]+1\left[\lambda-3-2+4\lambda\right]=0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2-\lambda-1\right]+\left[5\lambda-5\right]=0$$

$$\Rightarrow (\lambda-1)\left[\lambda^2-\lambda-1\right]-5(\lambda-1)=0$$

$$\Rightarrow (\lambda-1)(\lambda^2-\lambda-6)=0$$

$$\Rightarrow (\lambda-1)(\lambda+2)(\lambda-3)=0$$

$$\Rightarrow \lambda=-2,1,3$$

$$\Rightarrow A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow x_1-x_2+2x_3=-2x_1$$

$$\Rightarrow -x_1+x_3=-2x_2$$

$$\Rightarrow 2x_1+x_2+x_3=-2x_3$$

SD

$\Rightarrow 3x_1 - x_2 + 2x_3 = 0$ — (I)

$\Rightarrow x_1 - 2x_2 - x_3 = 0$ — (II)

$\Rightarrow 2x_1 + x_2 + 3x_3 = 0$ — (III).

(II) + (III) = (I)

$\Rightarrow$ Only '2' eqn's will contribute to soln of (x_1, x_2, x_3)

Let $x_3 = K$

$3x_1 - x_2 = -2K$ — (I) $\Rightarrow$ $6x_1 - 2x_2 = -4K$ (IV)

$x_1 - 2x_2 = K$ — (II)

$\Rightarrow 5x_1 = -5K$ (by (IV) − (II))

$\Rightarrow x_1 = -K, \; x_2 = -K, \; x_3 = K$

$x_1^2 + x_2^2 + x_3^2 = 1 \Rightarrow 3K^2 = 1 \Rightarrow K = \pm\frac{1}{\sqrt{3}}$

$\Rightarrow |x_1| + |x_2| + |x_3| = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$

(71) For real Constants α, β. Suppose the system of linear equations $x + 2y + 3z = 6$, $x + y + \alpha z = 3$, $2y + z = \beta$ has infinitely many solutions. Then $4\alpha + 3\beta =$ ____ (a) 18 (b) 23 (c) 28 (d) 32

[JAM-2023]

Soln:- $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & \alpha \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ \beta \end{pmatrix} \Rightarrow A\underset{\sim}{x} = \underset{\sim}{b}$

SD

$$A|\underset{\sim}{b} = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 1 & \alpha & 3 \\ 0 & 2 & 1 & \beta \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & \alpha-3 & -3 \\ 0 & 2 & 1 & \beta \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & \alpha-3 & -3 \\ 0 & 0 & 2\alpha-5 & \beta-6 \end{array}\right)$$

For ∞ly many solutions $R(A|\underset{\sim}{b}) = R(A) < n = 3$

$\Rightarrow 2\alpha - 5 = 0 = \beta - 6 \Rightarrow \alpha = \frac{5}{2}, \beta = 6$

$4\alpha + 3\beta = 10 + 18 = 28$ (Ans)

(72) Let M be a 3×3 real matrix. If $P = M + M'$, $Q = M - M'$. Then which of following is/are always true

(a) $|P^2 Q^3| = 0$ (b) $Tr(Q + Q^2) = 0$ (c) $\underset{\sim}{x}' Q^2 \underset{\sim}{x} = 0 \; \forall \underset{\sim}{x} \in R^3$

(d) $\underset{\sim}{x}' P \underset{\sim}{x} = 2\underset{\sim}{x}' M \underset{\sim}{x} \; \forall \underset{\sim}{x} \in R^3$ [JAM-2023]

Soln:- $\underset{3\times3}{P}$: symmetric matrix

$\underset{3\times3}{Q}$: skew symmetric matrix

(a) ✓

(b) ✗

(c) ✗

(d) ✓

$Tr(Q) = 0, \; |Q| = 0$

$\Rightarrow |P^2 Q^3| = |P|^2 |Q|^3 = 0$

$(Q^2)' = (-Q)^2 = Q^2$: symmetric

$Tr(Q + Q^2) = Tr(Q^2) \neq 0$

$\underset{\sim}{x}' Q^2 \underset{\sim}{x} \neq 0$ as Q^2 is symmetric.

$$\underset{\sim}{x}' P \underset{\sim}{x} = \underbrace{\underset{\sim}{x}' M \underset{\sim}{x}}_{\text{scaler}} + \underbrace{\underset{\sim}{x}' M' \underset{\sim}{x}} = \underset{\sim}{x}' M \underset{\sim}{x} + (\underset{\sim}{x}' M' \underset{\sim}{x})'$$

$$= \underset{\sim}{x}' M \underset{\sim}{x} + \underset{\sim}{x}' M \underset{\sim}{x} = 2\, \underset{\sim}{x}' M \underset{\sim}{x} \;\forall\, \underset{\sim}{x} \in R^3$$

(73) Let P be a 3×3 matrix having eigen values $1,1,2$. Let $(1,-1,2)'$ be the only linearly independent eigen vector corrosponding to eigen value 1. If $\text{adj}(2P) = Q$ then which of following is/are true. (a) $Tr(Q) = 20$ (b) $|Q| = 64$ (c) $(2,-2,4)'$ is an eigen vector of matrix Q (d) $Q^3 = 20Q^2 - 124Q + 256 I_3$ [JAM-2023]

Soln:- $Q = (2P)^{-1} |2P|$. $\left[\because A^{-1} = \frac{\text{adj}(A)}{|A|} \Rightarrow \text{adj}(A) = A^{-1}|A|\right]$

Eigen values of P: $1,1,2$

" " " $2P$: $2,2,4$.

" " " $(2P)^{-1}$: $\frac{1}{2}, \frac{1}{2}, \frac{1}{4}$ $\Rightarrow |2P| = 16$

$\therefore$ Eigen Values of Q: $8,8,4$

$Tr(Q) = 8+8+4 = 20$, $|Q| = 8\times8\times4 = 256$

Charecteristic eqn of Q:-

$$(\lambda-8)(\lambda-8)(\lambda-4)=0$$

$$\Rightarrow (\lambda^2-16\lambda+64)(\lambda-4)=0$$

$$\Rightarrow \lambda^3-4\lambda^2-16\lambda^2+64\lambda+64\lambda-256=0$$

$$\Rightarrow \lambda^3-20\lambda^2+128\lambda-256=0$$

By Cayley Hamilton Theorem:-

$$Q^3-20Q^2+128Q-256I=0$$

$$\Rightarrow Q^3=20Q^2-128Q+256I_3$$

$$Q=(2P)^{-1}|2P|=(2P)^{-1}\times 16=8P^{-1}$$

$$P(1,-1,2)'=1\cdot(1,-1,2)'$$

$$\Rightarrow 8P^{-1}(1,-1,2)'=8P^{-1}P(1,-1,2)'$$

$$\Rightarrow 8P^{-1}(1,-1,2)'=8(1,-1,2)'$$

$$\Rightarrow Q(1,-1,2)'=8(1,-1,2)'$$

So $(1,-1,2)'$ is eigen vector corrospond to eigen value '8' and is the only linearly independent eigen vector

$\Rightarrow (2,-2,4)'$ cannot be any eigenvector of Q.

(a) ✓
(b) ✗
(c) ✗
(d) ✗

(74) Let $M = M_1 M_2$ where M_1, M_2 are two distinct matrices of order 3×3
Consider two statements

(I) Rows of M are linear combination of rows of M_2

(II) Coloumns of M are linear combination of coloumns of M_1

Then (a) (I) is true (b) (II) is true

(c) Both True (d) None True

[JAM - 2023]

Soln :- $V_C(M) = V_C(M_1 M_2) \subset V_C(M_1)$

$\Rightarrow \underset{\sim}{x} \in V_C(M) \Rightarrow \underset{\sim}{x} \in V_C(M_1)$

$\Rightarrow \underset{\sim}{x}$ can be written as linear combination of Coloumns of M_1

Similarly, $V_R(M) = V_R(M_1 M_2) \subset V_R(M_2)$

$\Rightarrow y' \in V_R(M) \Rightarrow y' \in V_R(M_2)$

$\Rightarrow y'$ can be written as linear Combination of rows of M_2

So both (I), (II) are true

(75) Let $M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & +1 \end{pmatrix}$. If a non-zero vector $\underset{\sim}{x} = (x, y, z)' \in R^3$ satisfies $M^6 \underset{\sim}{x} = \underset{\sim}{x}$ then subspace of R^3 that contains vector $\underset{\sim}{x}$ is

(a) $\{\underset{\sim}{x} \in R^3 : x = 0, y + z = 0\}$

(b) $\{\underset{\sim}{x} \in R^3 : y = 0, x + z = 0\}$ [JAM-2023]

(c) $\{\underset{\sim}{x} \in R^3 : z = 0, x + y = 0\}$

(d) $\{\underset{\sim}{x} \in R^3 : x = 0, y - z = 0\}$

SD

Soln :- $|M-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\left[(\lambda-1)(\lambda-2)-1\right] + 1\left[\lambda-1\right] = 0$

$\Rightarrow (\lambda-1)\left[(\lambda-1)(\lambda-2)-1-1\right] = 0$

SD

$\Rightarrow (\lambda-1)[\lambda^2-3\lambda+2-2]=0$

$\Rightarrow \lambda(\lambda-1)(\lambda-3)=0$

$\Rightarrow \lambda=0,1,3$

Eigen values of M: $0,1,3$

" " " M^6: $0,1,$ 729

$M^6 \underset{\sim}{x} = \underset{\sim}{x} \Rightarrow \underset{\sim}{x}$ is eigen vector of M^6 (corrosponding to eigen value '1')

$\Rightarrow M\underset{\sim}{x} = \underset{\sim}{x} \Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x-y \\ -x+2y-z \\ -y+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow y=0,$

$-x-z=0$

$\Rightarrow x+z=0$

Subspace of R^3 that contains $\underset{\sim}{x}$ is $\{(x,y,z)' \in R^3 : y=0,\ x+z=0\}$

SD