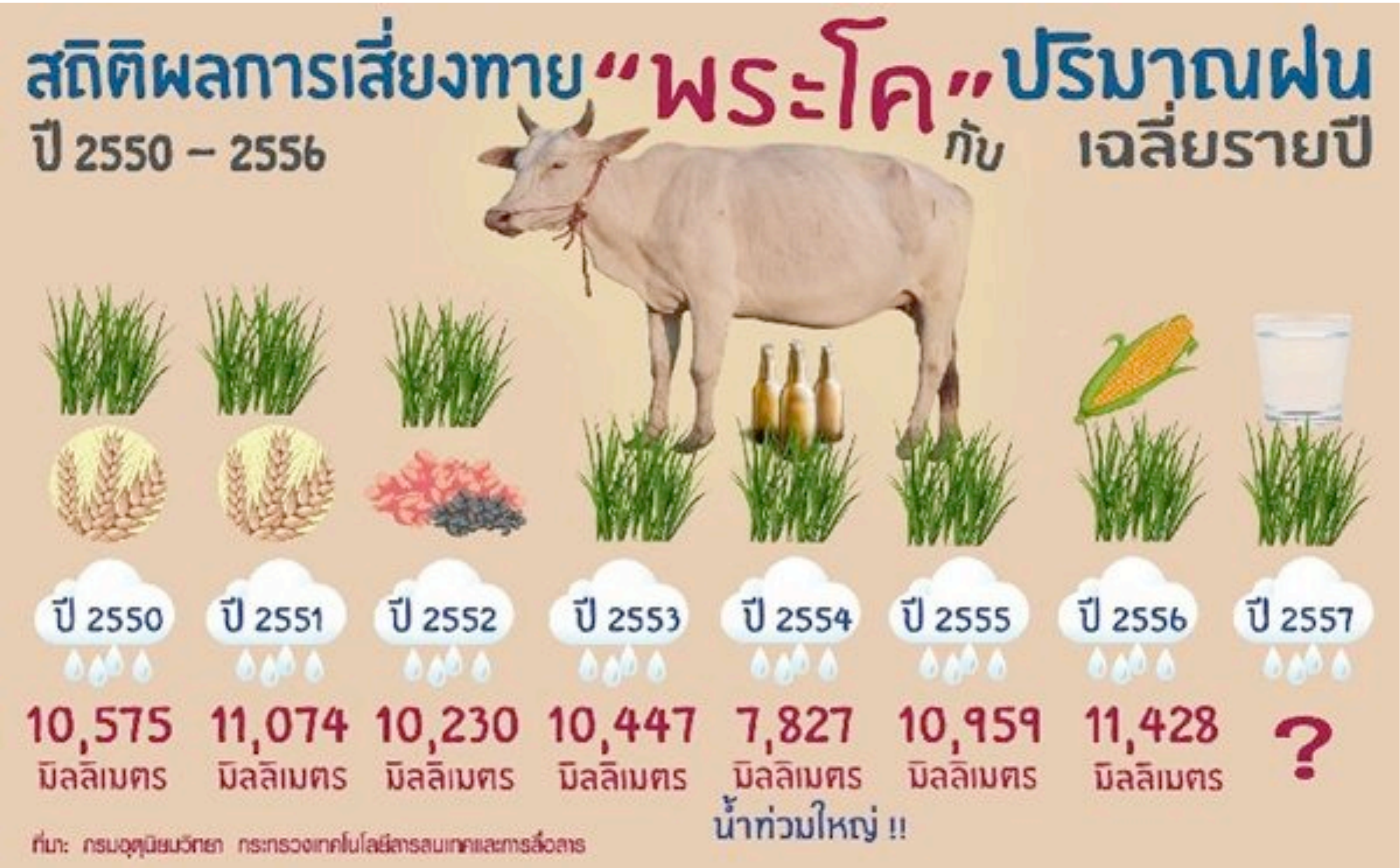
PROBABILITY REVIEW(?)

Predicting amount of rainfall



ที่มา: กรมอุตุนิยมวิทยา กระทรวงเทคโนโลยีสารสนเทศและการสื่อสาร

https://esan108.com/%E0%B8%9E
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(Linear) Regression

- $h_{\boldsymbol{\theta}}(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_4 x_4 + \theta_5 x_5$
- θs are the parameter (or weights)
- We can rewrite

Assume x_0 is always 0

$$h_\theta(x) = \Sigma_{i=0}^{n} \theta_i x_i = \theta^T \mathbf{x}$$

- Notation: vectors are bolded
- Notation: vectors are column vectors

LMS regression with gradient descent

$$\frac{\partial J}{\partial \theta_j} = -\Sigma_{i=1}^{m}(y_i - \theta^T \mathbf{x}_i)x_i^{(j)}$$

$$\theta_j \Leftarrow \theta_j + r\Sigma_{i=1}^{m}(y_i - \theta^T \mathbf{x}_i)x_i^{(j)}$$

Interpretation?

Logistic Regression

- Pass $\theta^T \mathbf{x}$ through the logistic function

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

Logistic Regression update rule

$$\theta_j \Leftarrow \theta_j - r\Sigma_{i=1}^{m}(y_i - h_\theta(x_i))x_i^{(j)}$$

Update rule for linear regression

$$\theta_j \Leftarrow \theta_j - r\Sigma_{i=1}^{m}(y_i - \theta^T \mathbf{x}_i)x_i^{(j)}$$

What is Probability?

- Frequentist

  Probability = rate of occurrence in a large number of trials

- Bayesian

  Probability = uncertainty in your knowledge of the world

Bayesian vs Frequentist

https://www.gizmodo.com.au/2017/01/how-to-find-a-lost-phone/

Bayesian vs Frequentist

- Toss a coin

- Frequentist
  - P(head) = θ, θ = #heads/#tosses

- Bayesian
  - P(head) = θ, θ~U(0.6,1.0)
  - Parameters of distributions can now have probabilities
  - Bayesian interpretation can gives prior knowledge to the phenomena – subjective view of the world
  - Prior knowledge can be updated according to the observed frequency

Bayesian statistics

- Coin with P(head) = p
- Observed frequency of heads $\hat{p}$ = #heads/#n

- In Bayesian view, we can talk about P(p | $\hat{p}$) by using Bayes's rule

$$P(p|\hat{p}) = \frac{P(\hat{p}|p)\overset{\text{Prior probability}}{P(p)}}{P(\hat{p})}$$

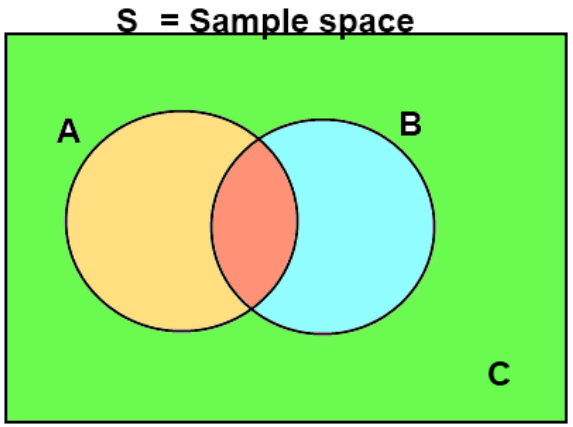
Important concepts

- Conditional probability
  - Independence
- Bayes' Rule
- Expected Value and Variance
- CDFs
- Sum of RVs
- Gaussian Random Variable
  - Multivariate Gaussian

Conditional probability

- P(A|B) probability of A given B has occurred

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$



- A student posts a facebook status after finishing Pattern Recognition homework
- P(he is happy)
- P(he is happy | the post starts with "#$@#$!@#$")

https://www.wyzant.com/resources/lessons/math/statistics_and_probability/probability/further_concepts_in_probability

Independence

- Two events are independent (statistically independent or stochastically independent) if the occurrence of one does not affect the probability of occurrence of the other.

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) \Leftrightarrow \mathrm{P}(B) = \mathrm{P}(B \mid A)$$

- P(he is happy | His friend posted a cat picture on instragram)

Bayes' Rule (Bayes's theorem or Bayes' law)

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)},$$

Usefulness: We can find P(A|B) from P(B|A) and vice versa

Expected value

- Expected value

$$\boxed{E[x]=\int_{-\infty}^{\infty} xp(x)dx} \qquad \boxed{E[g(x)]=\int_{-\infty}^{\infty} g(x)p(x)dx}$$

- Variance (σ^2) (Standard Deviation = σ)

$$Var[x]=E[(x-E[x])^2]=\sigma^2=\int_{-\infty}^{\infty}(x-E[x])^2\,p(x)dx$$

$$E[(x-E[x])^2]=E[x^2]-(E[x])^2$$

Expected Value notes

- It’s a weighted sum.
- Something can have a high expected value but low probability of occurring

- Lottery: P(win) = 10^(-20), winner gets 10^30
- P(loss) = 1-P(win), loser gets -10
- E(Lottery earnings) = 10^(-20)10^30+ (1-P(win)) (-10)
- = 10^10 – 10

- Humans are not good at gauging probability at extreme points

Expected value and Variance properties

- E[a] = a; a is a constant.
- E[aX+b] = aE[X]+b
- E[X+Y] = E[X]+E[Y]
- Var[a] = 0
- Var[aX+b] = a^{2}Var[X]

Conditional Expected Value

$$E[x \mid A] = \int_{-\infty}^{\infty} x p(x \mid A) dx$$

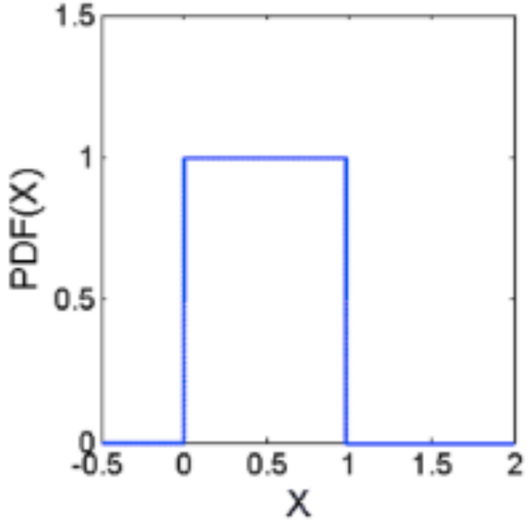
$$E[g(x) \mid A] = \int_{-\infty}^{\infty} g(x) p(x \mid A) dx$$

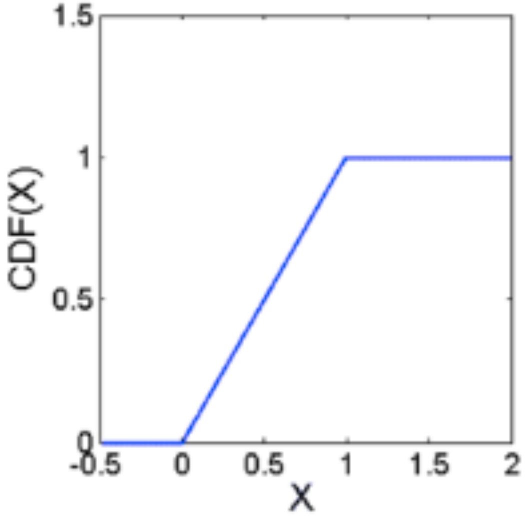
Cumulative Distribution Functions CDFs

- Probability that the RV is less than a certain amount

$$F_X(x_0) = P(X \leq x_0) = \int_{-\mathrm{inf}}^{x_0} p(x)dx$$

- CDF is the integral of PDF. Differentiating CDF wrt x gives the PDF





Joint distributions

- If we want to monitor how two events are jointly occurring, we consider the joint distribution $p_{X,Y}(x,y)$
- $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ if x and y are independent

$$\boxed{P(A) = \iint_A p_{XY}(x,y)\,dxdy}$$

$$\boxed{p_X(x) = \int_{-\infty}^{\infty} p_{XY}(x,y)\,dy}$$

$$\boxed{p_Y(y) = \int_{-\infty}^{\infty} p_{XY}(x,y)\,dx}$$

Sum of Random variables

- Z = Y + X
- What is the pdf of Z? Where Y and X continuous RVs

$$p_{X+Y}(z) = (p_X * p_Y)(z) = (p_Y * p_X)(z)$$

Central Limit Theorem (CLT)

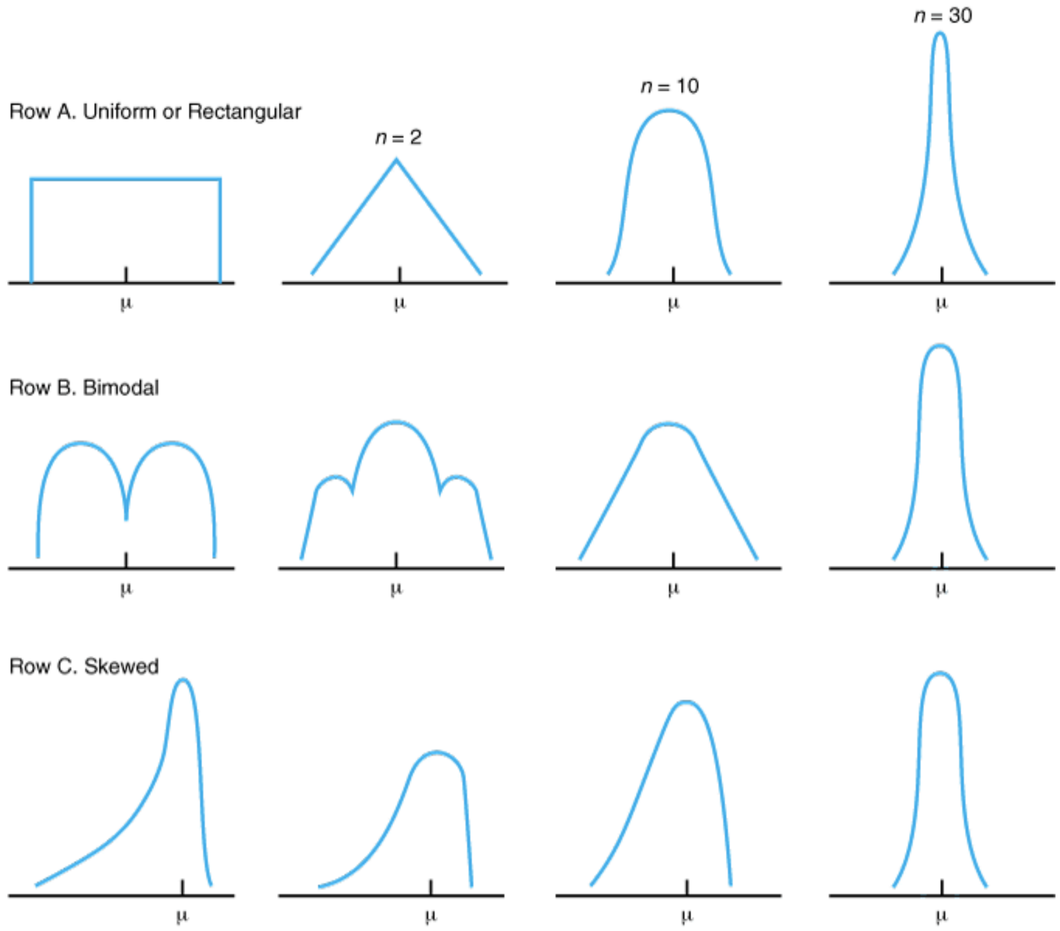
- Supposed $X_1,X_2,\ldots$ is a sequence of iid (idenpendent and identically distributed) RVs. As n approaches infinity the sum of the sequence converge in distribution to a Normal distribution

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)-\mu\right)\xrightarrow{d}N\left(0,\sigma^2\right)$$

- Other variants of CLT exists, without the dependence or identically distributed assumption

CLT implications

- A sum of RVs tends to become Normally distributed very quickly



http://www.people.vcu.edu/~albest/DENS580/DawsonTrapp/Chap4.htm

Gaussian distribution (normal distribution)

- X is normal (Gaussian): $X \sim N(\mu,\sigma^2)$

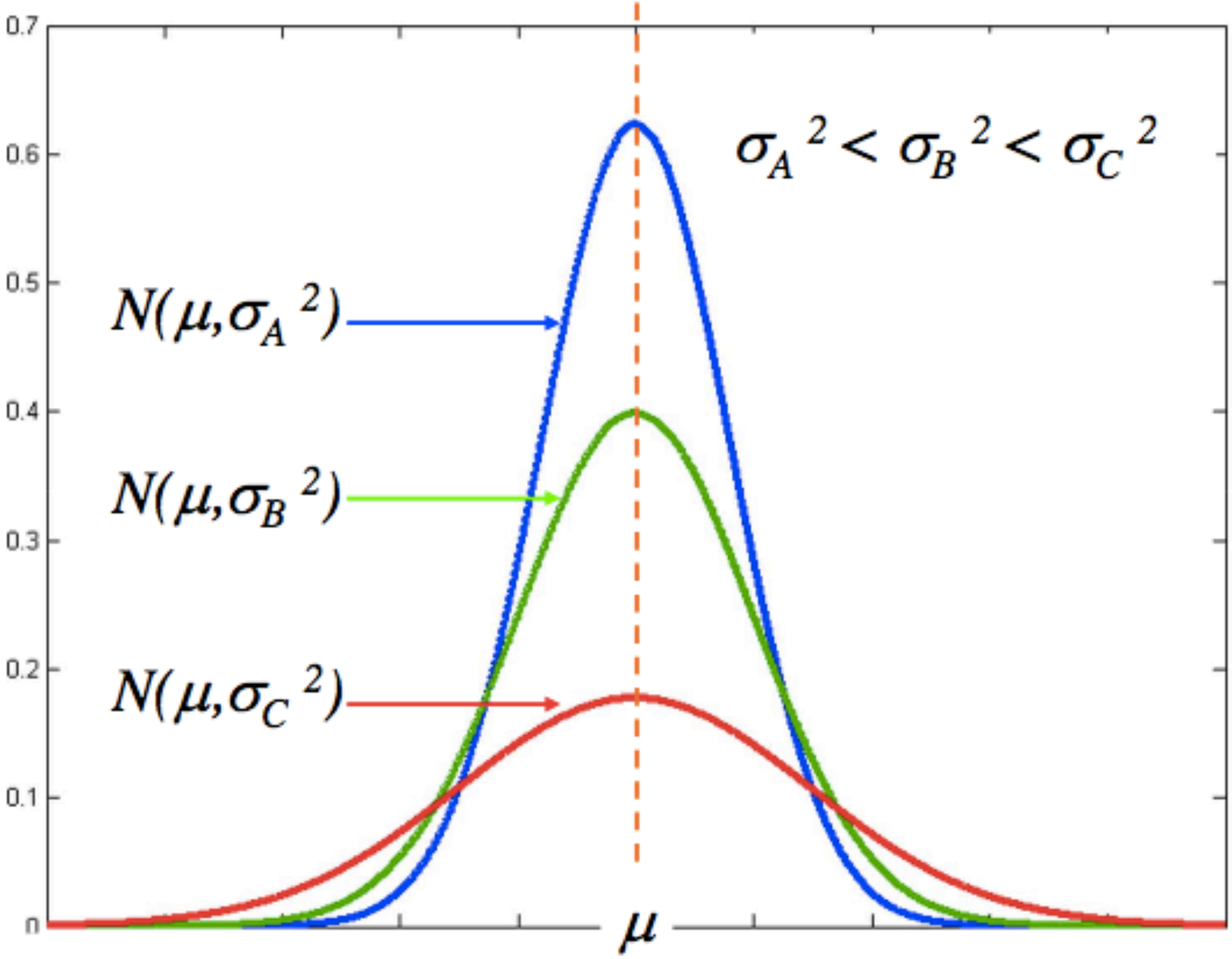
$$\boxed{p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}} \qquad \begin{aligned} E[x] &= \mu \\ Var[x] &= \sigma^2 \end{aligned}$$

- X is Standard normal (Standard Gaussian): $X \sim N(0,1)$ when $\mu=0,\ \sigma^2=1$

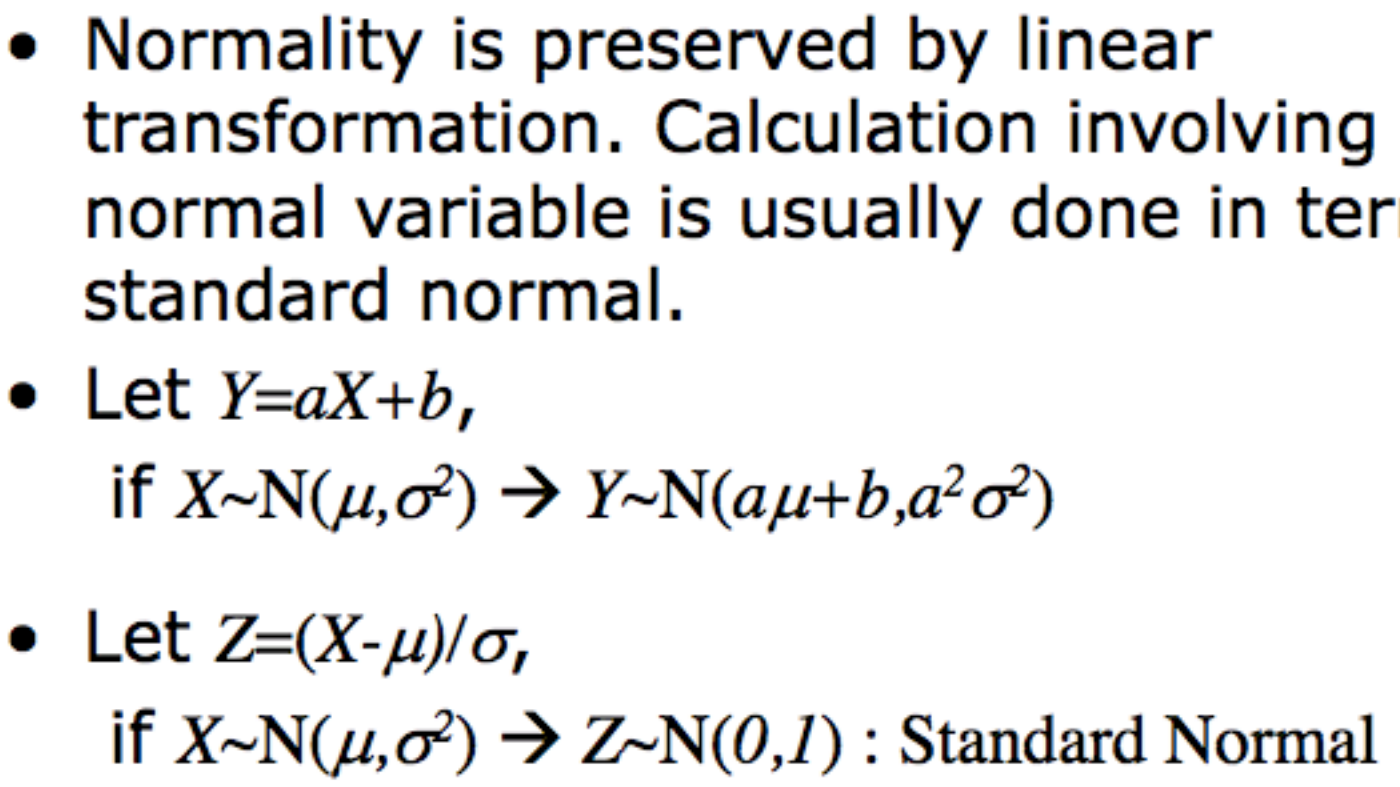
$$\boxed{p(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}} \qquad \begin{aligned} E[x] &= 0 \\ Var[x] &= 1 \end{aligned}$$

Slides from ASR lecture by Aj. Attiwong

Gaussian pdf



Linear transformation of Gaussian RV



- Normality is preserved by linear transformation. Calculation involving the normal variable is usually done in terms of standard normal.
- Let $Y=aX+b$,

  if $X\sim N(\mu,\sigma^2) \rightarrow Y\sim N(a\mu+b,a^2\sigma^2)$
- Let $Z=(X-\mu)/\sigma$,

  if $X\sim N(\mu,\sigma^2) \rightarrow Z\sim N(0,1)$: Standard Normal

Can you prove this?

Expectation of multivariate distributions

$$E[g(X_1, X_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2) p_{X_1, X_2}(x_1, x_2) dx_1 dx_2$$

$$E[g(X_1)h(X_2)] = E[g(X_1)]E[h(X_2)]$$

If X_1 and X_2 independent

Covariance of multivariate distributions

- $cov(X_1,X_2) = E[(X_1-m_1)(X_2-m_2)]$

- $cov(X_1,X_2) = E[(X_1)(X_2)] - m_1m_2$

- Covariance with itself is just the Variance
- Correlation

$$\rho = \frac{cov(X_1, X_2)}{\sqrt{V(X_1)V(X_2)}}$$

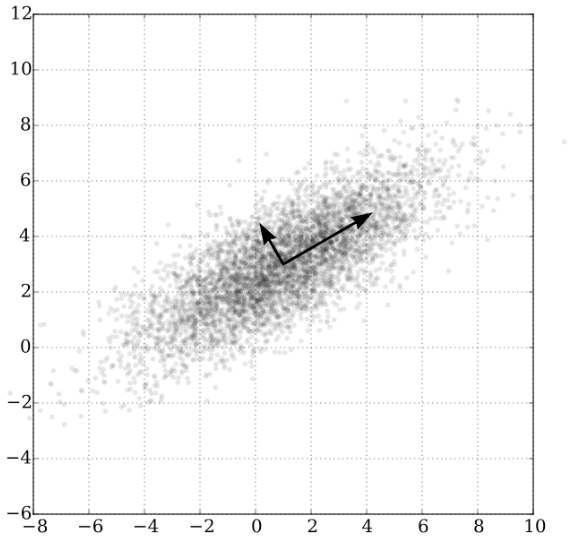
Covariance matrix

- Given a set of RVs, X_1 X_2 … X_n
- The covariance matrix is a matrix which has the covariance of the i and j RV in position (i,j)

$$\Sigma = \begin{bmatrix} \mathrm{E}[(X_1-\mu_1)(X_1-\mu_1)] & \mathrm{E}[(X_1-\mu_1)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_1-\mu_1)(X_n-\mu_n)] \\ \mathrm{E}[(X_2-\mu_2)(X_1-\mu_1)] & \mathrm{E}[(X_2-\mu_2)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_2-\mu_2)(X_n-\mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n-\mu_n)(X_1-\mu_1)] & \mathrm{E}[(X_n-\mu_n)(X_2-\mu_2)] & \cdots & \mathrm{E}[(X_n-\mu_n)(X_n-\mu_n)] \end{bmatrix}.$$

Understanding the Covariance matrix

$$= \begin{array}{cccc} A & & & B \\ & \mathrm{Cov}(X,X) & \mathrm{Cov}(Y,X) & \\ & \mathrm{Cov}(X,Y) & \mathrm{Cov}(X,Y) & \\ C & & & D \end{array}$$



Which statements are true?

C = B

B < 0

D < 0

A < D

A > B

https://en.wikipedia.org/wiki/Covariance_matrix

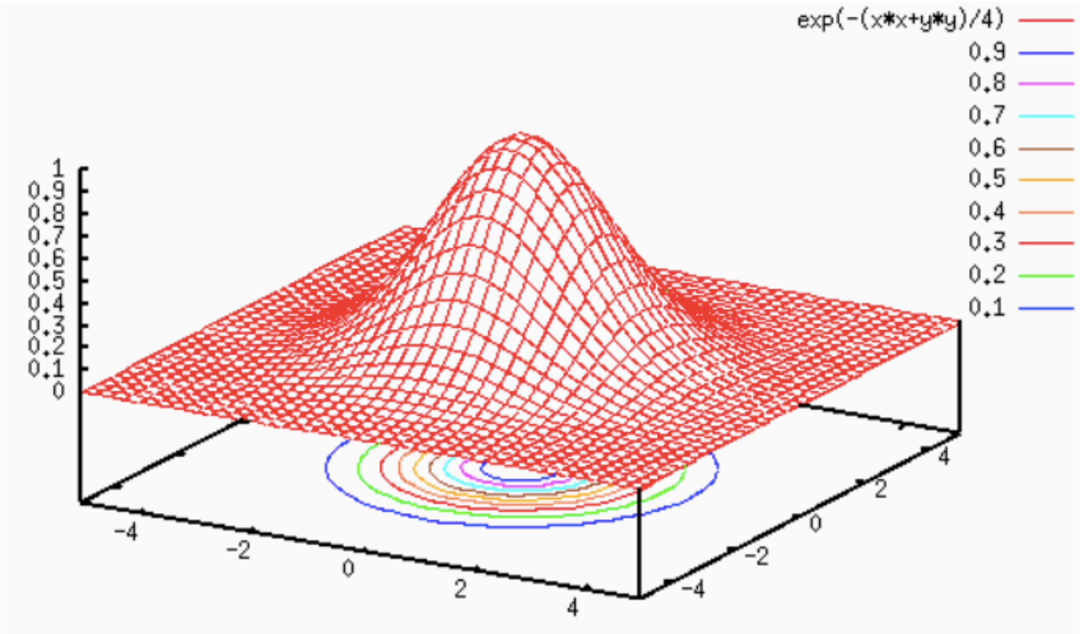
Covariance matrix observations

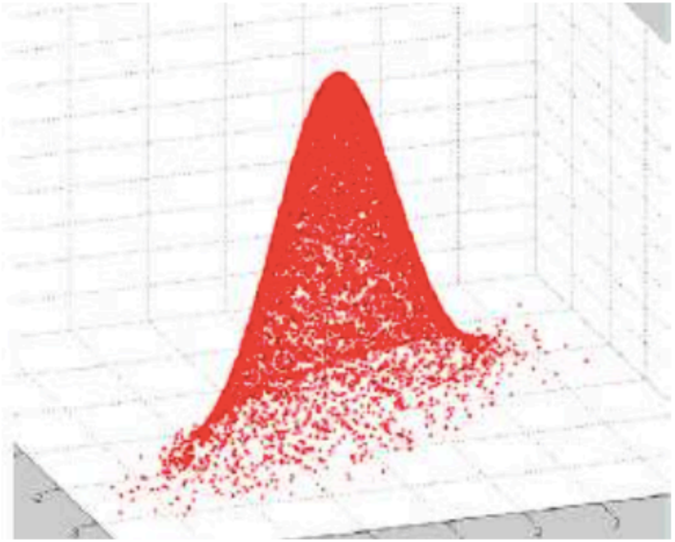
- $\Sigma = \Sigma^T$
- If the covariance matrix is diagonal, all RVs are mutually independent.
- Covariance matrix is positive-semidefinite

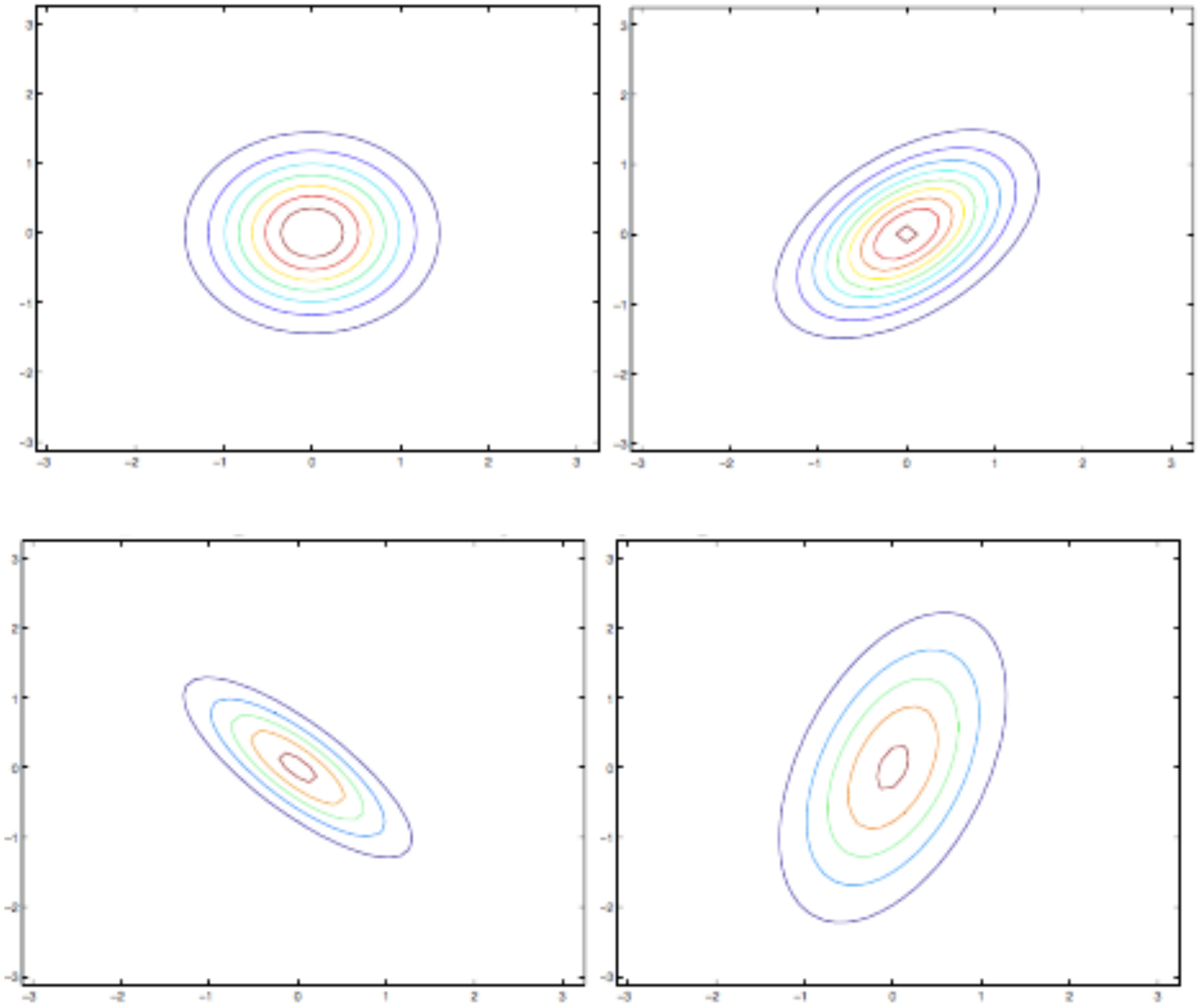
Multivariate Gaussian distribution

- Put X_1,X_2,X_3…X_n into a vector x

$$p(\mathbf{x};\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} exp[-\frac{1}{2}(\mathbf{x}-\mu)^t\Sigma^{-1}(\mathbf{x}-\mu))]$$







$$\Sigma = \begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 3 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

http://cs229.stanford.edu/notes/cs229-notes2.pdf

Affine transformation of multivariate Gaussians

- **y** = **Ax+b** , Assuming A has full rank (invertible)

$$\mathbf{x} \sim N(\mu, \Sigma)$$

$$\mathbf{y} \sim N(A\mu + b, A\Sigma A^T)$$

Important concepts

- Conditional probability
  - Independence
- Bayes' Rule
- Expected Value and Variance
- CDFs
- Sum of RVs
  - CLT
- Gaussian Random Variable
  - Multivariate Gaussian

Distribution parameter estimation

- P(head) = θ, θ = #heads/#tosses
- HHTTH

- L(θ) = P(X; θ) = P(HHTTH; θ)
- Maximum Likelihood Estimate (MLE)

Linear Regression Revisit

- $h_{\boldsymbol{\theta}}$ (x) = $\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_4 x_4 + \theta_5 x_5$
- θs are the parameter (or weights)
- We can rewrite

$$h_\theta(x) = \Sigma_{i=0}^{n} \theta_i x_i = \theta^T \mathbf{x}$$

- Notation: vectors are bolded
- Notation: vectors are column vectors

Probabilistic Interpretation of linear regression

- Real world data is our model plus some error term
  - Noise in the data
  - Something that we do not model (features we are missing)

- Let's assume the error is normally distributed with mean zero and variance σ^2
  - Why Gaussian?
  - Why saying mean is zero is a valid assumption?

$$y_i = \theta^T \mathbf{x}_i + \epsilon_i$$

Probabilistic view of Linear regression

- Find θ
- Maximize Likelihood of seeing x and y in training

- From our assumption we know that

$$y_i = \theta^T \mathbf{x}_i + \epsilon_i$$

$$p(y_i|\mathbf{x}_i;\theta) = \frac{1}{\sqrt{2\pi\sigma}} exp(-\frac{\overset{\text{error}}{\boxed{(y_i - \theta^T \mathbf{x}_i)}}^2}{2\sigma^2})$$

Error term is normally distributed with mean 0 and variance σ^2

Maximizing Likelihood

What is the assumption here?
Is it accurate?

- Max $$L(\theta) = \Pi_{i=1}^{m} p(y_i|\mathbf{x}_i;\theta)$$

- We use the log likelihood instead log(L(θ)) = l(θ)

From our previous lecture

Min $$J(\theta) = \frac{1}{m}\Sigma_{i=1}^{m}(y_i - \theta^T\mathbf{x_i})^2$$

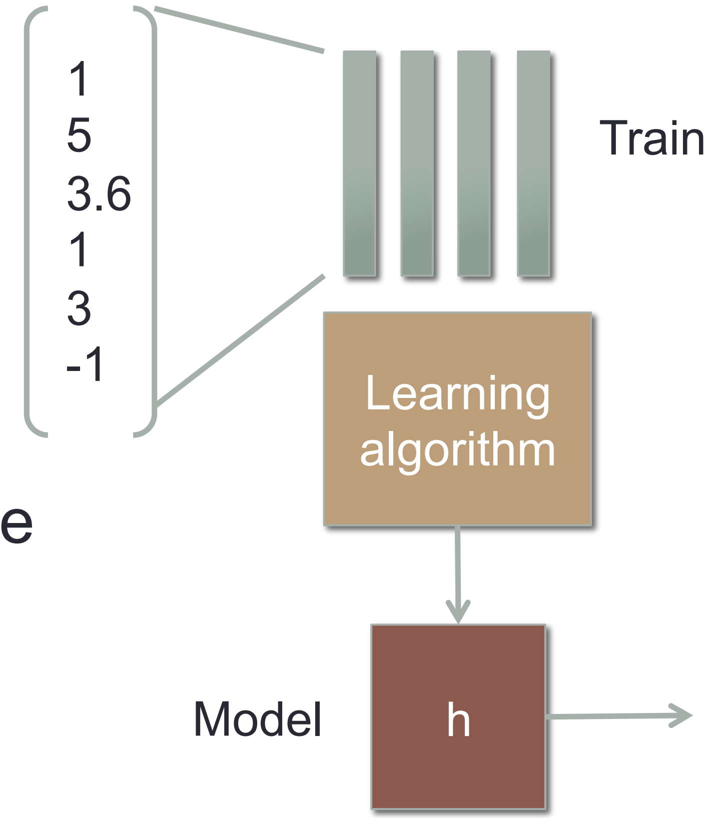
Mean square error solution and MLE solution

- Turns out MLE and MSE gets to the same solution
  - This justifies our choice of MSE as the Loss for linear regression
  - This does not mean MSE is the best Loss for regression, but you can at least justify it with a probabilistic reasoning

- Note how our choice of variance σ^2 falls out of the maximization, so this derivation is true regardless of which assumption for variance is.

Flood or no flood

- What would be the output?
- y = 0 if not flooded
- y = 1 if flooded
- Anything in between is a score for how likely it is to flood



Training phase

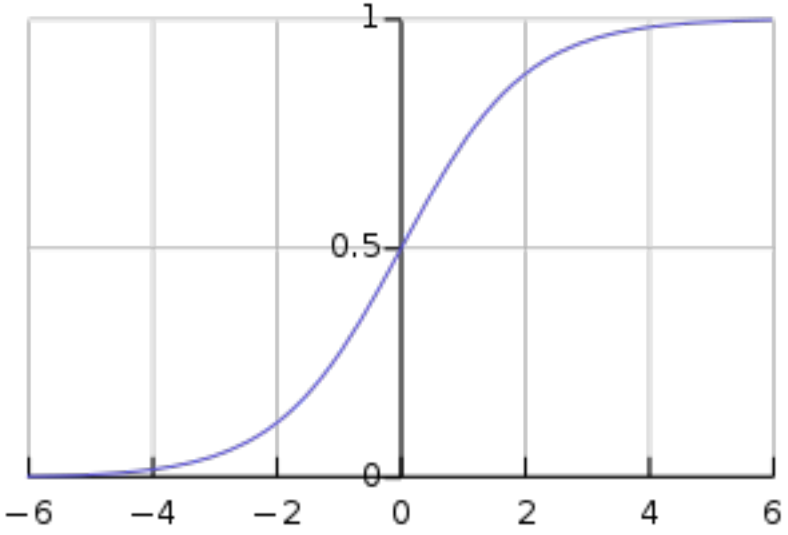
Can we use regression?

- Yes
- $h_{\boldsymbol{\theta}}(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 + \theta_4 x_4 + \theta_5 x_5$

- But
- What does it mean when h is higher than 1?
- Can h be negative? What does it mean to have a negative flood value?

Logistic function

- Let's force h to be between 0 and 1 somehow
- Introducing the logistic function (sigmoid function)



$$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$

https://en.wikipedia.org/wiki/Logistic_function

Logistic Regression

- Pass $\theta^T \mathbf{x}$ through the logistic function

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

Loss function?

- MSE error no longer a good candidate

- Let’s turn to use probabilistic argument for logistic regression

Logistic Function derivative

The derivative has a nice property by design.
This is also why many algorithm we'll learn later in class also uses the logistic function

$$
\begin{aligned}
g'(z) &= \frac{d}{dz}\frac{1}{1+e^{-z}} \\
&= \frac{1}{(1+e^{-z})^2}\left(e^{-z}\right) \\
&= \frac{1}{(1+e^{-z})}\cdot\left(1-\frac{1}{(1+e^{-z})}\right) \\
&= g(z)(1-g(z)).
\end{aligned}
$$

Probabilistic view of Logistic Regression

- Let’s assume, we’ll classify as 1 with probability in accordance to the output of

$$h_\theta(x) = \frac{1}{1 + e^{-\theta^T x}}$$

$$\begin{aligned} P(y=1 \mid x; \theta) &= h_\theta(x) \\ P(y=0 \mid x; \theta) &= 1 - h_\theta(x) \end{aligned}$$

or

$$p(y \mid x; \theta) = (h_\theta(x))^y \, (1 - h_\theta(x))^{1-y}$$

Maximizing log likelihood

$$p(y \mid x; \theta) = (h_\theta(x))^y \, (1 - h_\theta(x))^{1-y}$$

$$\begin{aligned} g'(z) &= \frac{d}{dz}\frac{1}{1+e^{-z}} \\ &= \frac{1}{(1+e^{-z})^2}\left(e^{-z}\right) \\ &= \frac{1}{(1+e^{-z})} \cdot \left(1 - \frac{1}{(1+e^{-z})}\right) \\ &= g(z)(1-g(z)). \end{aligned}$$

$$\theta_j \Leftarrow \theta_j + r\Sigma_{i=1}^{m}(y_i - h_\theta(x_i))x_i^{(j)}$$

Logistic Regression update rule

$$\theta_j \Leftarrow \theta_j + r\Sigma_{i=1}^{m}(y_i - h_\theta(x_i))x_i^{(j)}$$

Update rule for linear regression

$$\theta_j \Leftarrow \theta_j + r\Sigma_{i=1}^{m}(y_i - \theta^T\mathbf{x}_i)x_i^{(j)}$$

Loose ends from previous lecture

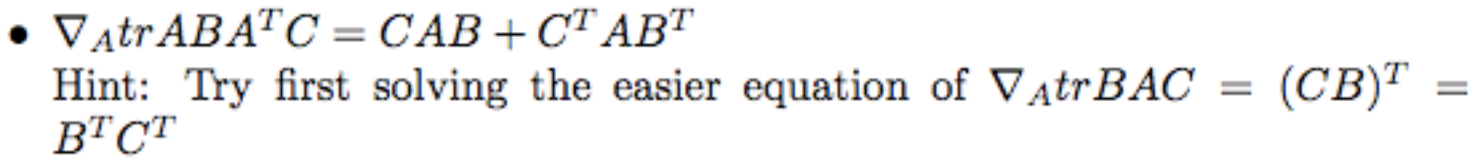
$$\theta_j \Leftarrow \theta_j + r\Sigma_{i=1}^{m}(y_i - \theta^T \mathbf{x}_i)x_i^{(j)}$$

$$\theta = (X^T X)^{-1} X^T y$$

$$X = \left[\begin{array}{ccc} - & x_1^T & - \\ - & x_2^T & - \\ - & x_m^T & - \end{array} \right]$$

X is mxn   The inverse is on nxn matrix

Homework



- $\nabla_A trABA^TC = CAB + C^TAB^T$
  Hint: Try first solving the easier equation of $\nabla_A trBAC = (CB)^T = B^TC^T$

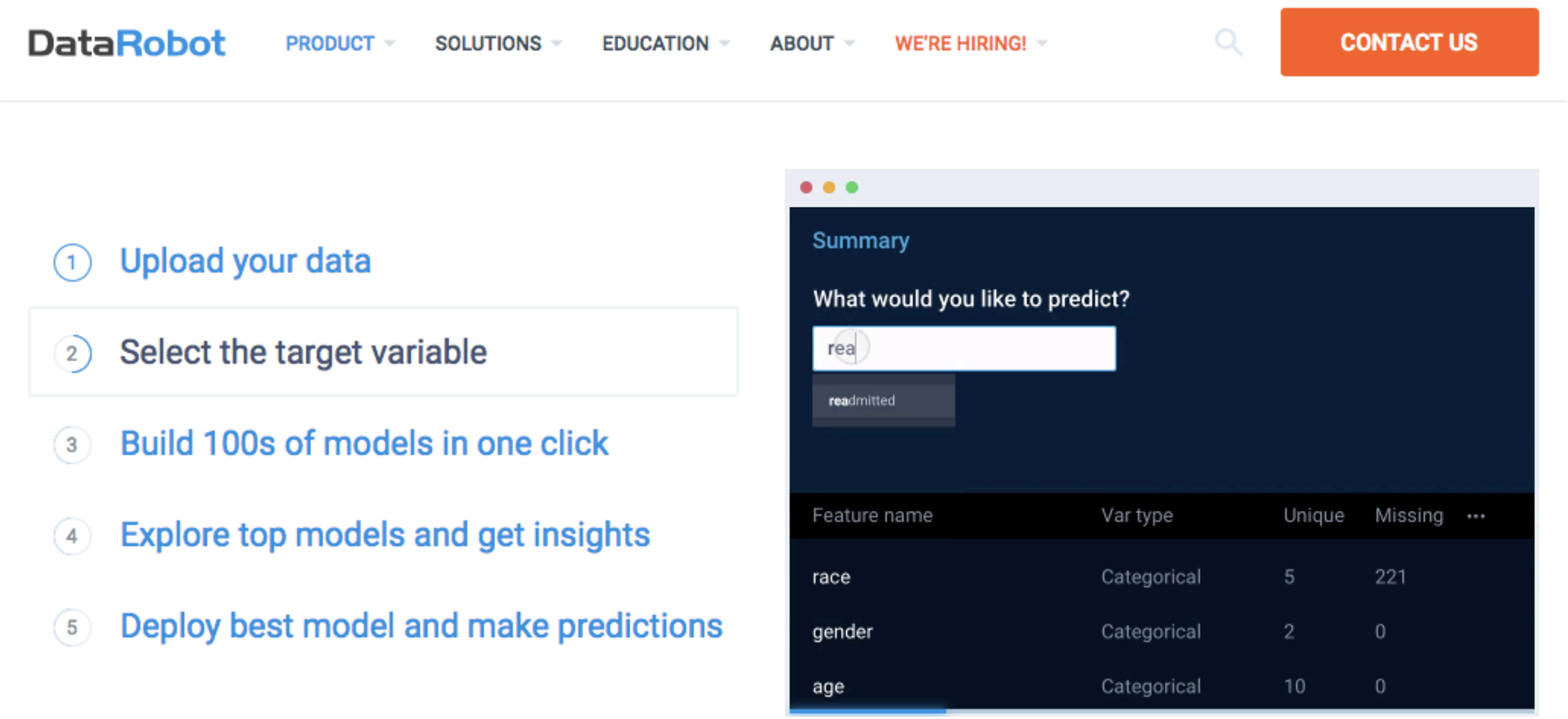
Homework II

Can I just use scikit-learn?

No

One button machines

- Machine learning as a tool for non-experts
- Can a non-expert just provide the data and let the machine decides how to proceed



1. Upload your data
2. Select the target variable
3. Build 100s of models in one click
4. Explore top models and get insights
5. Deploy best model and make predictions

Summary

What would you like to predict?

rea

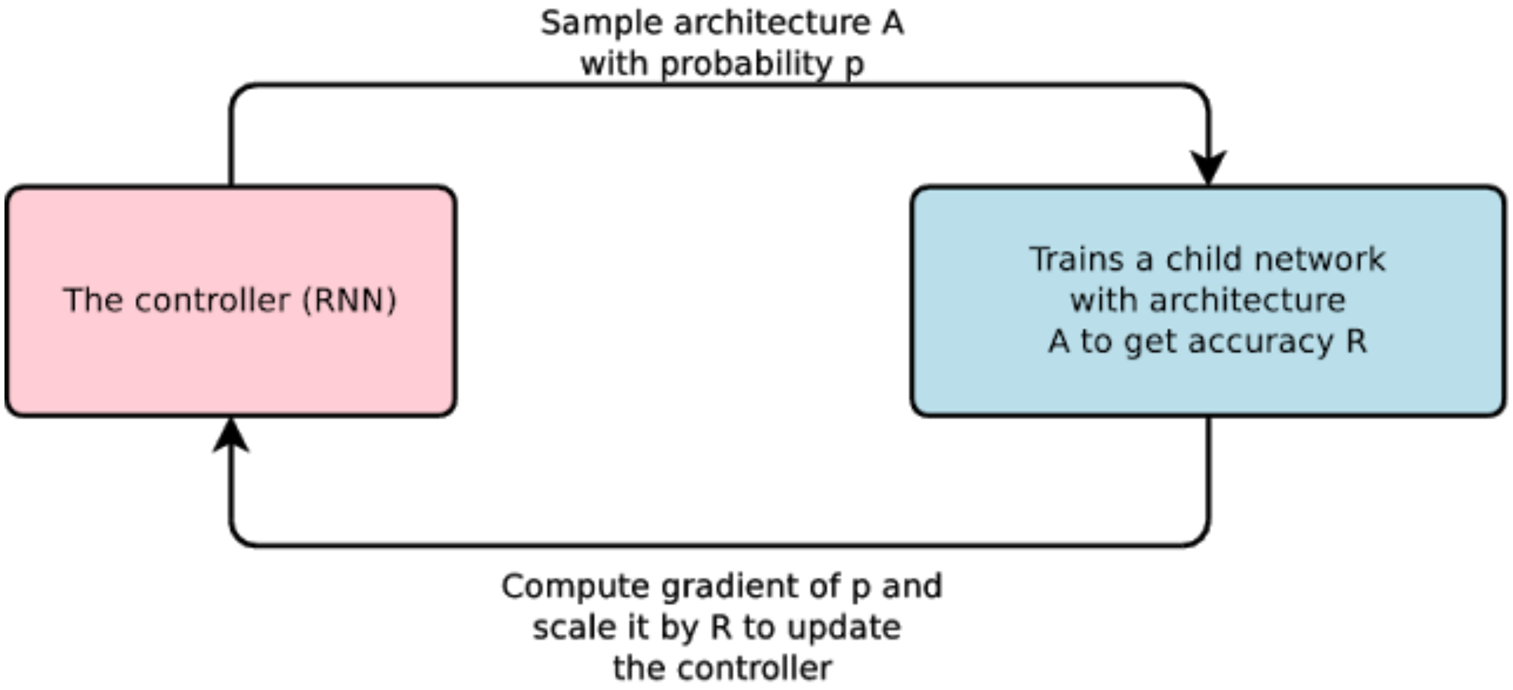
readmitted

Feature name	Var type	Unique	Missing
race	Categorical	5	221
gender	Categorical	2	0
age	Categorical	10	0

https://www.datarobot.com

Reinforcement Learning for Model Selection

- Tuning a network takes time
- Let machine learning learns how to tune a network
- Matches or outperforms ML experts performance



https://research.googleblog.com/2017/05/using-machine-learning-to-explore.html