Principles of Digital Communications
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Project PDC : Theory Part

Problem 1

(a)

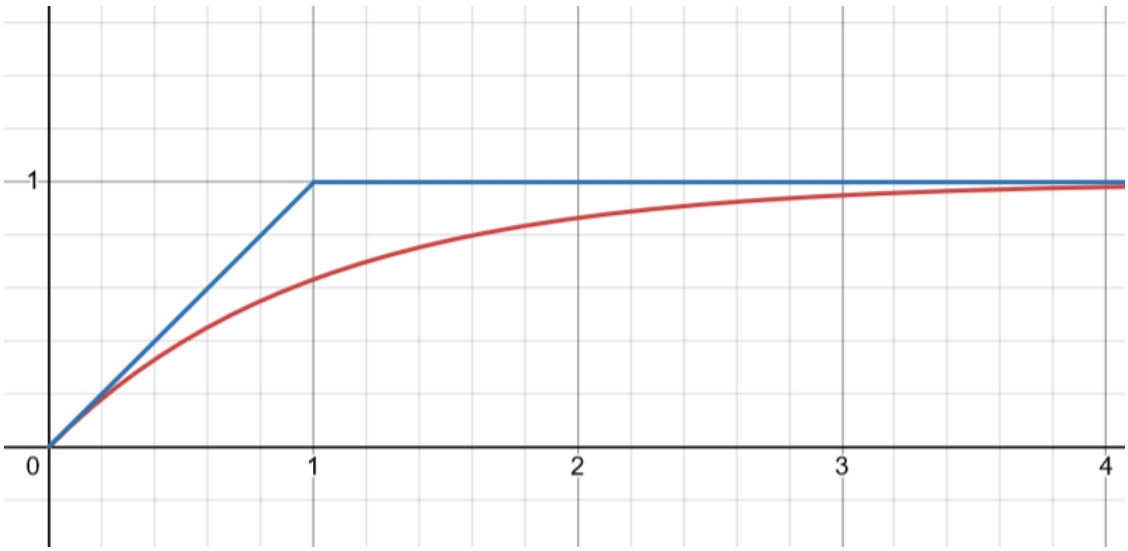
$$Pr(E^c) = Pr(\bigcap_i E_i^c) = \prod_i Pr(E_i^c) = \prod_i (1 - p_i)$$

And using the hint: $1 - p_i \leq exp(-p_i)$

$$\implies \prod_i (1 - p_i) \leq \prod_i exp(-p_i) = exp(-\sum_i p_i)$$

$$\implies Pr(E^c) \leq exp(-\sum_i p_i)$$

(b)



• Firstly, let $0 \leq s \leq 1 \implies min\{1, s\} = s$

s(1 - 1/e) is a chord with endpoints at 0 and 1 of the curve 1 - exp(-s) for $s \in [0, 1]$

Additionally, the function $f(s) = 1 - exp(-s)$ is clearly convex $\forall s \in [0, 1]$

Thus, $f(s)$ is above all its chords $\implies s(1 - \frac{1}{e}) \leq 1 - exp(-s)$

• For s > 1, $min\{1, s\} = 1 \implies (1 - \frac{1}{e})\ min\{1, s\} = 1 - \frac{1}{e}$

$1 - exp(-s)$ lowest value is $1 - \frac{1}{e}$ and $1 - exp(-s)$ is an increasing function

$$\implies 1 - exp(-s) - (1 - \frac{1}{e})\ min\{1, s\} \geq 0$$

(c)

$s = \sum_i p_i \geq 0$

$Pr(\bigcup_i E_i) = Pr(E) \leq min\{1, \sum_i p_i\} = min\{1, s\}$ by the improved union bound

$1 - Pr(E) = Pr(E^c) \leq exp(-\sum_i p_i) \implies Pr(E) \geq 1 - exp(-s)$ by part (a)

And by part (b), $1 - exp(-s) \geq (1 - \frac{1}{e})\ min\{1, s\} \implies Pr(E) \geq (1 - \frac{1}{e})\ min\{1, s\}$

Problem 2

(a) Since each bit is independent of the other, the estimator that minimizes the probability of error of the whole binary sequence is the one that minimizes each bit independently. Additionally, we have that each bit is just a AWGN channel with $N_0/2 = \sigma^2$, and therefore the minimum estimator reduces to minimum distance between transmitted signals which are v_0 and v_1. Thus, $\hat{b_i} = arg\ min_{b \in \{0,1\}} ||Y_i - v_b||$. And we can just do it for each bit, producing $(\hat{b_1},\ ...,\ \hat{b_k})$.

(b) Let $Z_i \in \mathbb{R}^3$, be the noise associated to the i-th bit, and let us assume wlog. that v_1 is on the "right" of v_0

$$\begin{aligned}
Pr(\hat{b_i} \neq b_i) &= Pr(\hat{b_i} \neq b_i \mid v_{b_i} = v_0)/2 + Pr(\hat{b_i} \neq b_i | v_{b_i} = v_1)/2 \\
&= Pr(\langle Z_i\ , v_1 - v_0 \rangle / ||v_1 - v_0|| > d/2 \mid v_{b_i} = v_0)/2 \\
&\quad + Pr(\langle Z_i\ , v_0 - v_1 \rangle / ||v_0 - v_1|| > d/2 \mid v_{b_i} = v_1)/2 \quad \text{(by projecting } Z_i \text{ on } v_0 - v_1\text{)} \\
&= Pr(\langle Z_i\ , v_0 - v_1 \rangle / ||v_0 - v_1|| > d/2) \qquad \text{(by symmetry of the Gaussian noise)} \\
&= Pr(\langle Z_i\ , v_0 - v_1 \rangle > d^2/2) \\
&= Q(d/2\sigma)
\end{aligned}$$

Since each bit is send independently,

$$\begin{aligned}
Pr(error) &= 1 - Pr(error^c) \\
&= 1 - (1 - Pr(\hat{b_i} \neq b_i))^k \\
&= 1 - (1 - Q(d/2\sigma))^k
\end{aligned}$$

Let $f(k) = 1 - (1 - Q(d/2\sigma))^k$ and $g(k) = min\{1,\ kQ(d/2\sigma)\}$
$f(k)$ is of the form $b - a^x$ with $a \leq 1 \implies f(k)$ is convex $\forall k \in \mathbb{R}^+$
Thus, $f(k)$ is bigger than all its chords and for $k < 1$, $g(k)$ is one of its chords with endpoints at 0 and $Q(d/2\sigma)$
$\implies \forall k \in [0, Q(d/2\sigma)],\ \ f(k) \geq g(k)$ and $\forall k > Q(d/2\sigma),\ \ f(k) < g(k)$
$\implies Pr(error) \leq min\{1,\ kQ(d/2\sigma)\}$ only for $k \in [0, Q(d/2\sigma)]$

(c) $\mathcal{E} = \sum_{i=0}^{k} (\pm d/2)^2 = kd^2/4$, which is less than the energy of the previous system.
$\mathcal{E}_b = \mathcal{E}/k = d^2/4$, which is also less than the energy by bit of the previous encoding.
$Pr(\hat{b_i} \neq b_i)_{\pm d/2} = Pr(\hat{b_i} \neq b_i)_{v_0, v_1} = Q(d/2\sigma)$, because the distance between codewords did not change, we only used less bits.
Thus, $Pr(error)_{\pm d/2} = Pr(error)_{v_0, v_1}$

(d) If $d/2\sigma < a_2$, then $Q(d/2\sigma) > Q(a_2) \implies Pr(\hat{b_i} \neq b_i) > \frac{\alpha}{k(1-\frac{1}{e})}$, since Q is a decreasing function. Additionally question b) gives us $Pr(error) \leq k \cdot Pr(\hat{b_i} \neq b_i)$ $\implies Pr(error) \leq \frac{\alpha}{(1-\frac{1}{e})}$.
And $(1-\frac{1}{e}) > 1$, which means that we have a weaker upper bound on $Pr(error)$ than α, implying that we cannot guarantee an error probability lower than α.
If $d/2\sigma \geq a_1$, then $Q(d/2\sigma) \leq Q(a_1) \implies Pr(\hat{b_i} \neq b_i) \leq \frac{\alpha}{k}$.
And $Pr(error) \leq k \cdot Pr(\hat{b_i} \neq b_i) \implies Pr(error) \leq \alpha$. This means we always satisfy the probability requirement if $d/2\sigma \geq a_1$.

Problem 3

(a) For the decoder, we have 2 types of errors. Error type 1 is when we miss a detection, ie when $Y_j = \sqrt{\epsilon} + Z_j$ fails to exceed the threshold $t = \alpha\sqrt{\epsilon}$. Error type 2 is when multiple Y_i's exceed the threshold.

$$\begin{aligned}
Pr\{\text{Error type 1}\} &= Pr\{|Y_j| \leq \alpha\sqrt{\epsilon}\} = Pr\{-\alpha\sqrt{\epsilon} \leq Y_j \leq \alpha\sqrt{\epsilon}\} \\
&= Q\left(\frac{-\alpha\sqrt{\epsilon} - \sqrt{\epsilon}}{\sigma}\right) - Q\left(\frac{\alpha\sqrt{\epsilon} - \sqrt{\epsilon}}{\sigma}\right) \\
&= 1 - Q\left(\frac{\alpha\sqrt{\epsilon} + \sqrt{\epsilon}}{\sigma}\right) - 1 + Q\left(\frac{\sqrt{\epsilon} - \alpha\sqrt{\epsilon}}{\sigma}\right) \\
&= Q\left(\frac{\sqrt{\epsilon} - \alpha\sqrt{\epsilon}}{\sigma}\right) - Q\left(\frac{\alpha\sqrt{\epsilon} + \sqrt{\epsilon}}{\sigma}\right) \\
&< Q\left(\frac{(1-\alpha)\sqrt{\epsilon}}{\sigma}\right) \\
Pr\{\text{Error type 2}|i \neq j\} &= Pr\{|Z_i| > \alpha\sqrt{\epsilon}\} = 2Q\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right) \\
Pr\{\text{Error type 2}\} &= \sum_{i \neq j} Pr\{\text{Error type 2}|i \neq j\} \\
&= \sum_{i \neq j} 2Q\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right) \\
&= 2(n-1)Q\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right) \\
Pr\{\text{error}\} &= Pr(\text{Error type 1} \cup \text{Error type 2}) \\
&< Pr\{\text{Error type 1}\} + Pr\{\text{Error type 2}\} \\
&< Q\left(\frac{(1-\alpha)\sqrt{\epsilon}}{\sigma}\right) + 2(n-1)Q\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right) \\
&< Q\left(\frac{(1-\alpha)\sqrt{\epsilon}}{\sigma}\right) + (2n-2)Q\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right) \\
&< Q\left(\frac{(1-\alpha)\sqrt{\epsilon}}{\sigma}\right) + (2^k-2)Q\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right) \\
&< Q\left(\frac{(1-\alpha)\sqrt{\epsilon}}{\sigma}\right) + 2^kQ\left(\frac{\alpha\sqrt{\epsilon}}{\sigma}\right)
\end{aligned}$$

(b) Using $\epsilon = \sigma^2 Ak$ and $Q(x) \leq \frac{1}{2}exp(-\frac{x^2}{2})$:

$$
\begin{aligned}
Pr\{\text{error}\} &< \frac{1}{2}exp\left(\frac{-(1-\alpha)^2(\frac{kA\sigma^2}{\sigma^2})}{2}\right) + \frac{1}{2}exp\left(ln(2^k)\right)exp\left(\frac{-\alpha^2(\frac{kA\sigma^2}{\sigma^2})}{2}\right) \\
&< \frac{1}{2}exp\left(-\frac{(1-\alpha)^2kA}{2}\right) + \frac{1}{2}exp\left(kln2 - \frac{\alpha^2kA}{2}\right)
\end{aligned}
$$

For $Pr\{\text{error}\}$ to go to zero, we want both the coefficients of exponential growth of the terms $exp\left(-k\frac{(1-\alpha)^2A}{2}\right)$ and $exp(k(ln2-\frac{\alpha^2A}{2}))$ to be negative. Clearly, $\frac{(1-\alpha)^2A}{2} < 0$. And for the second term :

$$
\begin{aligned}
& A > 2ln2 \\
& A > 2\epsilon^2 ln2 \text{ for some } \epsilon > 1 \text{ since the above inequality is strict} \\
\alpha^2 A > 2ln2 & \text{ with } \alpha = \frac{1}{\epsilon} \\
& 0 > ln2 - \frac{\alpha^2 A}{2}
\end{aligned}
$$

Now, we know that the upper bound of $Pr\{\text{error}\}$, $\frac{1}{2}exp(-\frac{(1-\alpha)^2kA}{2}) + \frac{1}{2}exp(kln2 - \frac{\alpha^2kA}{2})$ goes to zero as k increases, so we are done.

(c) For $\alpha = \frac{1}{2}\left(1 + \frac{2ln(2)}{A}\right)$, with $A > 2ln(2) \implies \frac{2ln(2)}{A} < 1 \implies \frac{ln(2)}{A} < \frac{1}{2}$
$\implies \alpha = \frac{1}{2} + \frac{ln(2)}{A} < 1$
Now if we replace α in the bound in (b)

$$
\begin{aligned}
(1-\alpha)^2 &= \left(1 - \frac{1}{2} - \frac{ln(2)}{A}\right)^2 \\
&= \left(\frac{1}{2} - \frac{ln(2)}{A}\right)^2 \\
&= \frac{\left(1 - \frac{2ln(2)}{A}\right)^2}{4}
\end{aligned}
$$

$$
\implies \alpha^2 = \frac{\left(1 + \frac{2ln(2)}{A}\right)^2}{4}
$$

We then have :

$$\begin{aligned}
Pr(error) &< \frac{1}{2}exp\left(-\frac{1}{8}\left(1-\frac{2ln(2)}{A}\right)^2 Ak\right) + \frac{1}{2}exp\left(-\frac{1}{8}\left(1+\frac{2ln(2)}{A}\right)^2 Ak + ln(2)k\right) \\
&= \frac{1}{2}exp\left(-\frac{1}{8}\left(1-\frac{4ln(2)}{A}+\frac{4ln(2)^2}{A^2}\right)Ak\right) \\
&\quad + \frac{1}{2}exp\left(-\frac{1}{8}\left(1+\frac{4ln(2)}{A}+\frac{4ln(2)^2}{A^2}\right)Ak + ln(2)k\right) \\
&= \frac{1}{2}exp\left(-\frac{Ak}{8}+\frac{ln(2)k}{2}-\frac{ln(2)^2k}{2A}\right) \\
&\quad + \frac{1}{2}exp\left(-\frac{Ak}{8}-\frac{ln(2)k}{2}-\frac{ln(2)^2k}{2A}+\frac{2ln(2)k}{2}\right) \\
&= exp\left(-\frac{1}{8}\left(1-\frac{2ln(2)}{A}\right)^2 Ak\right)
\end{aligned}$$

(d) We want to solve the following inequality :

$$\begin{aligned}
exp\left(-\frac{1}{8}\left(1-\frac{2ln2}{A}\right)^2 Ak\right) &< 10^{-3} \\
-\frac{1}{8}\left(1-\frac{2ln(2)}{A}\right)^2 Ak &< ln(10^{-3}) \\
k &> \frac{-8ln(10^{-3})}{\left(1-\frac{2ln2}{A}\right)^2 A}
\end{aligned}$$

And using a calculator, we get :
- $A = 4$, $k > 32.3$ so $k = 33$ bits.
- $A = 6$, $k > 15.6$ so $k = 16$ bits.
- $A = 8$, $k > 10.1$ so $k = 11$ bits.
- $A = 10$, $k > 7.4$ so $k = 8$ bits.
- $A = 12$, $k > 5.9$ so $k = 6$ bits.

(e) We have : $d^2/4 = \epsilon_b \implies d^2 = 4\epsilon_b \implies d = 2\sqrt{\epsilon_b}$
From problem 2, we have the message error probability :

$$\begin{aligned}
Pr(error) &= 1 - (1 - Q(\sqrt{d/2\sigma}))^k \\
&= 1 - (1 - Q(\sqrt{A}))^k
\end{aligned}$$

And using a calculator,
- $A = 4$, $k = 33$: $Pr(error) \approx 5.32 \cdot 10^{-1}$
- $A = 6$, $k = 16$: $Pr(error) \approx 1.08 \cdot 10^{-1}$

- $A = 8$, $k = 11$: $Pr(error) \approx 2.5 \cdot 10^{-2}$
- $A = 10$, $k = 8$: $Pr(error) \approx 6.0 \cdot 10^{-4}$
- $A = 12$, $k = 6$: $Pr(error) \approx 1.0 \cdot 10^{-3}$

Problem 4

(a) We will compute the ML rule for H and see if it is equivalent to $\hat{H}$. It is a binary hypothesis testing, and the likelihood ratio gives :

$$\begin{aligned}\Lambda(y) &= \frac{f_{Y|H}(y \mid (i,1))}{f_{Y|H}(y \mid (i,2))} \\ &= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\, exp\left(-\frac{\|y_1 - c_i\|^2}{2\sigma^2}\right)\ \frac{1}{\sqrt{2\pi\tau^2}}\, exp\left(-\frac{\|y_2\|^2}{2\tau^2}\right)}{\frac{1}{\sqrt{2\pi\tau^2}}\, exp\left(-\frac{\|y_1\|^2}{2\tau^2}\right)\ \frac{1}{\sqrt{2\pi\sigma^2}}\, exp\left(-\frac{\|y_2 - c_i\|^2}{2\sigma^2}\right)} \\ &= exp\left(-\frac{\|y_1 - c_i\|^2}{2\sigma^2} - \frac{\|y_2\|^2}{2\tau^2} + \frac{\|y_2 - c_i\|^2}{2\sigma^2} + \frac{\|y_1\|^2}{2\tau^2}\right)\end{aligned}$$

Taking the log gives :

$$log(\Lambda(y)) = \frac{1}{2}\left(-\frac{\|y_1 - c_i\|^2}{\sigma^2} - \frac{\|y_2\|^2}{\tau^2} + \frac{\|y_2 - c_i\|^2}{\sigma^2} + \frac{\|y_1\|^2}{\tau^2}\right)$$

The threshold is 1, thus the MAP rule can be expressed as :

$$-\frac{\|y_1 - c_i\|^2}{\sigma^2} - \frac{\|y_2\|^2}{\tau^2} \underset{H=(i,1)}{\overset{H=(i,2)}{\lessgtr}} -\frac{\|y_2 - c_i\|^2}{\sigma^2} - \frac{\|y_1\|^2}{\tau^2}$$

Which is equivalent to $\hat{H}$, and is optimal because the log of the likelihood ratio is optimal, thus : $Pr(\hat{H} \neq H)$ is minimized by this rule.

(b) To minimize $Pr(\hat{i} \neq i)$, we need to ensure that the rule for selecting $\hat{i}$ maximizes the probability of correctly identifying the transmitted codeword i.
We know that : $i = arg\ min\|y - c_i\|$ which is a sufficient statistic for the MAP rule with the vector $Y = c_i + Z$.
Therefore, in both cases of b, $i_{1,2}$ is going to be a good estimate for i, and will minimize $Pr(\hat{i} \neq i)$.

(c) c_0 is due to the fact that $\hat{i_0}$ is the MAP estimator with side information b so it is upper bounded by the probability of error without the side information.
c_1 is due to the fact that i is inside H and incorrectly guessing H would mean incorrectly guessing i but the converse is not true, thus $Pr(\hat{i} \neq i) \leq Pr(\hat{H} \neq H)$.
c_2 is simply a decomposition of the event into 2 disjoint events. Not guessing correctly H means not guessing b or guessing b but not guessing i.
c_3 is due to the fact that in the case we know b, $\hat{i}$ is equal to i with the side information meaning $\hat{i} = \hat{i_0}$.
c_4 is just due to the fact $Pr(A \cap B) \leq Pr(B)$, since intersection can only reduce the number of possibilities.

(d) $H = (i,1) \implies b = 1 \implies Y = (Y_1, Y_2) = (c_i + Z, \tilde{Z})$

$$\begin{aligned} d_1 &= \frac{\|y_1 - c_i\|^2}{\sigma^2} + \frac{\|y_1\|^2}{\tau^2} \\ &= \frac{\|Z\|^2}{\sigma^2} + \frac{\|\tilde{Z}\|^2}{\tau^2} \end{aligned}$$

And, for some i'

$$\begin{aligned} d_2 &= \frac{\|y_2 - c_{i'}\|^2}{\sigma^2} + \frac{\|y_2\|^2}{\tau^2} \\ &= \frac{\|\tilde{Z} - c_{i'}\|^2}{\sigma^2} + \frac{\|Z + c_i\|^2}{\tau^2} \end{aligned}$$

If $\frac{\|Z\|^2}{\sigma^2} + \frac{\|\tilde{Z}\|^2}{\tau^2} > \frac{\|\tilde{Z} - c_{i'}\|^2}{\sigma^2} + \frac{\|Z + c_i\|^2}{\tau^2} \equiv d_1 > d_2$, then the rule returns $\hat{b} \neq 1$.

(e) For $\sigma = \tau$, and under $H = (i,1)$, the equation just above becomes for some i' :

$$\begin{aligned} &\|\tilde{Z} - c_{i'}\|^2 + \|c_i + Z\|^2 - \|Z\|^2 - \|\tilde{Z}\|^2 \leq 0 \\ &\|\tilde{Z} - c_{i'}\|^2 - \|\tilde{Z}\|^2 \leq -\|c_i + Z\|^2 + \|Z\|^2 \\ &\|c_{i'}\|^2 - 2\langle \tilde{Z}, c_{i'} \rangle \leq -\|c_i\|^2 - 2\langle c_i, Z \rangle \\ &\|c_{i'}\|^2 + \|c_i\|^2 \leq 2\langle \tilde{Z}, c_{i'} \rangle - 2\langle c_i, Z \rangle \\ &\|c_{i'}\|^2 + \|c_i\|^2 \leq 2\langle Z, c_{i'} - c_i \rangle \end{aligned}$$

$$\begin{aligned} \Pr(\hat{b} \neq 1 \mid H = (i,1)) &= Pr(\frac{\|Z\|^2}{\sigma^2} + \frac{\|\tilde{Z}\|^2}{\tau^2} > \frac{\|\tilde{Z} - c_{i'}\|^2}{\sigma^2} + \frac{\|Z + c_i\|^2}{\tau^2},\ \forall i') \\ &= \Pr\left(\bigcup_{i' \in \{1,\dots,m\}} \left(\|c_{i'}\|^2 + \|c_i\|^2 \leq 2\langle Z, c_{i'} - c_i \rangle \right) \right) \\ &\leq \sum_{i' \in \{1,\dots,m\}} \Pr\left(\|c_{i'}\|^2 + \|c_i\|^2 \leq 2\langle Z, c_{i'} - c_i \rangle \right) \quad \text{(By the union bound)} \end{aligned}$$

And, for each i', since $\langle c_i, Z \rangle$ is normally distributed with mean zero and variance $\sigma^2 \|c_{i'} - c_i\|^2$, we can use the Q-function to bound the probability.

$$\begin{aligned} &\Pr\left(\|c_{i'}\|^2 + \|c_i\|^2 \leq 2\langle Z, c_{i'} - c_i \rangle \right) \\ &\quad = Q\left(\frac{\|c_{i'}\|^2 + \|c_i\|^2}{2\sigma \|c_{i'} - c_i\|} \right) \\ &\quad \leq Q\left(\frac{\|c_{i'}\|^2 + \| - c_i\|^2}{2\sigma(\sqrt{\|c_{i'}\|^2 + \| - c_i\|^2})} \right) \text{ using Q decreasing and the triangle inequality.} \\ &\quad \leq Q\left(\sqrt{\frac{(\|c_{i'}\|^2 + \| - c_i\|^2)^2}{4\sigma^2(\|c_{i'}\|^2 + \| - c_i\|^2)}} \right) \\ &\quad \leq Q\left(\sqrt{\frac{\|c_{i'}\|^2 + \|c_i\|^2}{4\sigma^2}} \right) \end{aligned}$$

Thus, the upper bound for $\Pr(\hat{b} \neq 1 \mid H = (i,1))$ using the union bound is:

$$\Pr(\hat{b} \neq 1 \mid H = (i,1)) \leq \sum_{i'=1}^{m} Q\left(\sqrt{\frac{\|c_i\|^2 + \|c_{i'}\|^2}{4\sigma^2}}\right)$$

(f) From (e), $\Pr(\hat{b} \neq 1 \mid H = (i,1))$ using the union bound:

$$\Pr(\hat{b} \neq 1 \mid H = (i,1)) \leq \sum_{i'=1}^{m} Q\left(\sqrt{\frac{\|c_i\|^2 + \|c_{i'}\|^2}{4\sigma^2}}\right)$$

And, given $\|c_i\| = \sqrt{E}$ for all $i \in \{1, \ldots, m\}$, this simplifies to:

$$\Pr(\hat{b} \neq 1 \mid H = (i,1)) \leq mQ\left(\sqrt{\frac{2E}{4\sigma^2}}\right) = mQ\left(\sqrt{\frac{E}{2\sigma^2}}\right)$$

We also know that :

$$Q(x) \approx \frac{1}{2}e^{-x^2/2} \quad \text{for large } x$$

With $E_b = \frac{E}{\log_2(m)}$, and $\frac{E_b}{\sigma^2} > 4\ln 2$, we have:

$$\frac{E}{\sigma^2} > 4\ln 2 \log_2(m) \quad \Rightarrow \quad \frac{E}{2\sigma^2} > 2\ln 2 \log_2(m)$$

Hence:

$$Q\left(\sqrt{\frac{E}{2\sigma^2}}\right) \approx \frac{1}{2}e^{-\frac{E}{4\sigma^2}} \leq \frac{1}{2}e^{-2\ln 2 \log_2(m)} = \frac{1}{2}\left(2^{-\log_2(m)}\right)^2 = \frac{1}{2}\left(\frac{1}{m}\right)^2 = \frac{1}{2m^2}$$

Thus:

$$\Pr(\hat{b} \neq 1 \mid H = (i,1)) \leq m \cdot \frac{1}{2m^2} = \frac{1}{2m}$$

And as m grows:

$$\Pr(\hat{b} \neq 1 \mid H = (i,1)) \leq \frac{1}{2m} \to 0 \quad \text{as } m \to \infty$$

From inequality (c_4):

$$\Pr(\hat{i}_o \neq i) \leq \Pr(\hat{i} \neq i) \leq \Pr(\hat{b} \neq b) + \Pr(\hat{i}_o \neq i)$$

Therefore, with $\Pr(\hat{b} \neq b) \to 0$ as $m \to \infty$, we get:

$$\Pr(\hat{i} \neq i) \leq \Pr(\hat{b} \neq b) + \Pr(\hat{i}_o \neq i) \to \Pr(\hat{i}_o \neq i) \quad \text{as } m \to \infty$$

Which implies, $\Pr(\hat{i} \neq i) - \Pr(\hat{i}_o \neq i)$ approaches 0 as m grows.