

# MATH 3975 - FINANCIAL MATHEMATICS

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## 1. INTRODUCTION TO MARKETS

## 1.1. Introduction.

**Definition 1.1.1** (Time Value of Money). The **future value**  $F(0, t)$  is defined to be the value at time  $t > 0$  of \$1 invested at time 0.

The **present value** or **discount factor**  $P(0, t)$  is the amount invested at time 0 such that its value at time  $t$  is equal to \$1.

To avoid arbitrage, we must have  $P(0, t) = F^{-1}(0, t)$  for all  $t$ .

**Proposition 1.1.2.** *We must have the following relationship for  $P(t, T)$  and  $F(t, T)$ .*

$$F(t, T) = \frac{F(0, T)}{F(0, t)}$$

$$P(t, T) = \frac{P(0, T)}{P(0, t)}$$

**Definition 1.1.3** (Spot rate). The spot rate  $r(t)$  is defined as

$$F(t, t + \Delta) = 1 + r(t)\Delta$$

**Proposition 1.1.4.** *We have*

$$P(0, T) = \prod_{i=1}^n (1 + r_i)^{-1}$$

*in the discrete case, and in the continuous case, we have*

$$P(t, T) = e^{-\int_t^T r(s)ds}$$

## 1.2. Riskless Securities and Bonds.

**Theorem 1.2.1** (Fundamental Theorem of Riskless Security Pricing). *If interest rates are deterministic, the arbitrage free price of a riskless security is given by*

$$S_0 = \sum_{i=1}^n P(0, t_i) C_i$$

## 2. SINGLE PERIOD MARKET MODELS

A single period market model is the most elementary model. Only a single period is considered. At times  $t = 0$  and 1 market prices are recorded.

*Single period market models are the atoms of Financial Mathematics*

We assume that we have a finite sample space

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_k\}$$

**2.1. The most elementary market model.** Assume the sample space consists of two states,  $H$  and  $T$ , with  $\mathbb{P}(H) = p$  and  $\mathbb{P}(T) = 1 - p$ . Define the price of the stock at time 0 to be  $S_0$ , and let  $S_1$  be a random variable depending on  $H$  and  $T$ . Let  $u = \frac{S_1(H)}{S_0}$  and  $d = \frac{S_1(T)}{S_0}$ .

**Definition 2.1.1** (Trading strategy). A trading strategy  $(x, \phi)$  is a pair where  $x$  is the total initial investment at  $t = 0$ , and  $\phi$  denotes the number of shares bought at  $t = 0$ . Given a strategy  $(x, \phi)$ , the agent invests the remaining money  $x - \phi S_0$  in a money market account. We note this amount may be negative (borrowing from the money market account).

**Definition 2.1.2** (Value process). The value process of the trading strategy  $(x, \phi)$  in our elementary market model is given by  $(V_0(x, \phi), V_1(x, \phi))$  where  $V_0(x, \phi) = x$  and

$$V_1 = (x - \phi S_0)(1 + r) + \phi S_1$$

**Definition 2.1.3** (Arbitrage). An arbitrage is a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money.

More rigorously, we have a trading strategy  $(x, \phi)$  is an arbitrage if

- $x = V_0(x, \phi) = 0$ ,
- $V_1(x, \phi) \geq 0$ ,
- $\mathbb{E}[V_1(x, \phi)] > 0$ .

**Proposition 2.1.4.** *To rule out arbitrage in our model, we must have  $d < 1 + r < u$ .*

*Proof.* If this inequality is violated, consider the following strategies.

- If  $d \geq 1 + r$ , borrow  $S_0$  from the money market.
- If  $u \leq 1 + r$ , short  $S_0$  and invest in the money market.

These are both arbitrages and the proposition is proven. □

We have the following theorem, giving the converse of the above proposition

**Theorem 2.1.5.** *The condition  $d < 1 + r < u$  is a necessary and sufficient no arbitrage condition. That is,*

$$\text{No arbitrage} \iff d < 1 + r < u$$

**Definition 2.1.6** (Replicating strategy or hedge). A **replicating strategy** or **hedge** for the option  $h(S_1)$  in our elementary single period market model is a trading strategy  $(x, \phi)$  satisfying  $V_1(x, \phi) = h(S_1)$ . That is,

$$(x - \phi S_0)(1 + r) + \phi S_1(H) = h(S_1(H))$$

$$(x - \phi S_0)(1 + r) + \phi S_1(T) = h(S_1(T))$$

**Theorem 2.1.7.** *Let  $h(S_1)$  be an option in our market model, and let  $(x, \phi)$  be a replicating strategy for  $h(S_1)$ . Then  $x$  is the only price for the option at time  $t = 0$  which does not allow arbitrage.*

To find a replicating strategy for an arbitrary option, define

$$\phi = \frac{h(S_1(H)) - h(S_1(T))}{S_1(H) - S_1(T)}$$

$$\tilde{p} = \frac{1 + r - d}{u - d}$$

Then by solving the above two equations for  $x$ , we have

$$x = \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} h(S_1) \right] = \frac{1}{1+r} [\tilde{p}h(S_1(H)) + (1 - \tilde{p})h(S_1(T))]$$

## 2.2. A general single period market model.

**Definition 2.2.1** (Trading strategy). A **trading strategy** for an agent in our general single period market model is a pair  $(x, \phi)$ , where  $\phi = (\phi^1, \dots, \phi^n) \in \mathbb{R}^n$  specifying the initial investment in the  $i$ -th stock.

**Definition 2.2.2** (Value and gains process). Let the **value process** of the trading strategy  $(x, \phi)$  is given  $(V_0(x, \phi), V_1(x, \phi))$  where  $V_0(x, \phi) = x$  and

$$V_1(x, \phi) = (x - \sum_{i=1}^n \phi^i S_0^i)(1+r) + \sum_{i=1}^n \phi^i S_1^i$$

The **gains process** is defined as

$$G(x, \phi) = (x - \sum_{i=1}^n \phi^i S_0^i)r + \sum_{i=1}^n \phi^i \Delta S^i$$

where  $\Delta S^i$  is defined as

$$\Delta S^i = S_1^i - S_0^i$$

We have the simple result

$$V_1(x, \phi) = V_0(x, \phi) + G(x, \phi)$$

To study the prices of the stocks in relation to the money market account, we introduce the **discounted stock prices**  $\hat{S}_t^i$  defined as follows:

$$\hat{S}_0^i = S_0^i$$

$$\hat{S}_1^i = \frac{1}{1+r} S_1^i$$

**Definition 2.2.3** (Discounted value and gains process). We define the **discounted value process**  $\hat{V}(x, \phi)$  by

$$\hat{V}_0(x, \phi) = x$$

$$\hat{V}_1(x, \phi) = (x - \sum_{i=1}^n \phi^i S_0^i) + \sum_{i=1}^n \phi^i \hat{S}_1^i$$

and the discounted gains process  $\hat{G}(x, \phi)$  as

$$\hat{G}(x, \phi) = \sum_{i=1}^n \phi^i \Delta \hat{S}^i$$

with  $\Delta \hat{S}^i = \hat{S}_1^i - \hat{S}_0^i$ .

We then have the relation

$$\hat{V}_1(x, \phi) = \hat{V}_0(x, \phi) + \hat{G}(x, \phi)$$

**Definition 2.2.4** (Arbitrage). A trading strategy  $(x, \phi)$  is an arbitrage in our general single period market model if

- $x = V_0(x, \phi) = 0$ ,
- $V_1(x, \phi) \geq 0$ ,
- $\mathbb{E}[V_1(x, \phi)] > 0$ .

Alternatively, if a trading strategy satisfies the first two conditions above, it is an arbitrage if the following condition is satisfied:

$$\text{There exists } \omega \in \Omega \text{ with } V_1(x, \phi) > 0.$$

Alternatively, we can replace all references to  $V$  in the above definition with  $\hat{V}$ .

**Definition 2.2.5** (Risk neutral measure). A measure  $\tilde{\mathbb{P}}$  on  $\Omega$  is a **risk neutral measure** if

- $\tilde{\mathbb{P}}(\omega) > 0$  for all  $\omega \in \Omega$
- $\mathbb{E}_{\tilde{\mathbb{P}}}[\Delta \hat{S}^i] = 0$  for all  $i$

**Theorem 2.2.6** (Fundamental Theorem of Asset Pricing). *In the general single period market model, there are no arbitrages if and only if there exists a risk neutral measure for the market model.*

**Definition 2.2.7** (Alternative definition of arbitrage). Define the set  $\mathbb{W}$  by the following:

$$\mathbb{W} = \{X \in \mathbb{R}^k \mid X = \hat{G}(x, \phi) \text{ for some trading strategy } (x, \phi)\}$$

Then, letting  $\mathbb{A}$  be given as

$$\mathbb{A} = \{X \in \mathbb{R}^k \mid X \geq 0, X \neq 0\}$$

Then we have a definition of arbitrage:

$$\text{no arbitrage} \iff \mathbb{W} \cap \mathbb{A} = \emptyset$$

**Definition 2.2.8** (Set of risk neutral measures). Now, consider the orthogonal component  $\mathbb{W}^\perp$ , defined as

$$\mathbb{W}^\perp = \{Y \in \mathbb{R}^k \mid \langle X, Y \rangle = 0 \text{ for all } X \in \mathbb{W}\},$$

the set of vectors in  $\mathbb{R}^k$  perpendicular to all elements of  $\mathbb{W}$ . Furthermore, defining  $\mathcal{P}^+$  as

$$\mathcal{P}^+ = \{X \in \mathbb{R}^k \mid \sum_{i=1}^k X_i = 1, X_i > 0\}$$

Then we have the following theorem.

**Theorem 2.2.9.** *A measure  $\tilde{\mathbb{P}}$  is a risk neutral measure on  $\Omega$  if and only if  $\tilde{\mathbb{P}} \in \mathcal{P}^+ \cap \mathbb{W}^\perp$ .*

We denote the set of risk neutral measures  $\mathbb{M} = \mathcal{P}^+ \cap \mathbb{W}^\perp$ .

**Definition 2.2.10** (Contingent claim). A **contingent claim** in our general single period market model is a random variable  $X$  on  $\Omega$  representing a payoff at time  $t = 1$ .

**Proposition 2.2.11.** *Let  $X$  be a contingent claim in our general single period market model, and let  $(x, \phi)$  be a hedging strategy for  $X$ , so that  $V_1(x, \phi) = X$  then the only price of  $X$  which complies with the no arbitrage principle is  $x = V_0(x, \phi)$ .*

**Definition 2.2.12** (Attainable contingent claim). A contingent claim is **attainable** if there exists a trading strategy  $(x, \phi)$  which replicates  $X$ , so that  $V_1(x, \phi) = X$ .

**Theorem 2.2.13.** *Let  $X$  be an attainable contingent claim and  $\tilde{\mathbb{P}}$  be an arbitrary risk neutral measure. Then the price  $x$  of  $X$  at time  $t = 0$  can be computed by the formula*

$$x = \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} X \right]$$

**Corollary.** *This theorem tells us that in particular, for any risk neutral measure in our model, we get the same value when taking the expectation above.*

**Definition 2.2.14.** We say that a price  $x$  for the contingent claim  $X$  **complies with the no arbitrage principle** if the extended model, which consists of the original assets  $S^1, \dots, S^n$  and an additional asset  $S^{n+1}$  which satisfies  $S_0^{n+1} = x$  and  $S_1^{n+1} = X$  is arbitrage free.

The following proposition shows that when using the risk neutral measure to price a contingent claim, one obtains a price which complies with the no-arbitrage principle.

**Proposition 2.2.15.** *Let  $X$  be a possibly unattainable contingent claim and  $\tilde{\mathbb{P}}$  any risk neutral measure for our general single period market model. Then*

$$x = \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} X \right]$$

*defines a price for the contingent claim at time  $t = 0$  which complies with the no-arbitrage principle.*

**Definition 2.2.16** (Complete market). A financial market is called **complete** if for any contingent claim  $X$  there exists a replicating strategy  $(x, \phi)$ . A model which is not complete is called **incomplete**.

**Proposition 2.2.17.** *Assume a general single period market model consisting of stocks  $S^1, \dots, S^n$  and a money market account modelled on the state space  $\Omega = \{\omega_1, \dots, \omega_k\}$  is arbitrage free. Then this model is complete if and only if the  $k \times (n+1)$  matrix  $A$  given by*

$$A = \begin{pmatrix} 1+r & S_1^1(\omega_1) & \dots & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \dots & S_1^n(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1^1(\omega_k) & \dots & S_1^n(\omega_k) \end{pmatrix}$$

has full rank, that is,  $\text{rank}(A) = k$ .

*Proof.* First, a matrix  $A$  has full rank if and only if for every  $X \in \mathbb{R}^k$ , the equation  $AZ = X$  has a solution  $Z \in \mathbb{R}^{n+1}$ .

Secondly, we have

$$\begin{pmatrix} 1+r & S_1^1(\omega_1) & \dots & S_1^n(\omega_1) \\ 1+r & S_1^1(\omega_2) & \dots & S_1^n(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1^1(\omega_k) & \dots & S_1^n(\omega_k) \end{pmatrix} \begin{pmatrix} x - \sum_{i=1}^n \phi^i S_0^i \\ \phi^1 \\ \vdots \\ \phi^n \end{pmatrix} = \begin{pmatrix} V_1(x, \phi)(\omega_1) \\ V_1(x, \phi)(\omega_2) \\ \vdots \\ V_1(x, \phi)(\omega_k) \end{pmatrix}$$

This shows that computing a replicating strategy for a contingent claim  $X$  is the same as to solve the equation  $AZ = X$ , and the proposition follows  $\square$

**Proposition 2.2.18.** *A contingent claim  $X$  is attainable, if and only if  $\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} X \right]$  takes the same value for all  $\tilde{\mathbb{P}} \in \mathbb{M}$ .*

**Theorem 2.2.19.** *Under the assumption that the model is arbitrage free, it is complete, if and only if  $\mathbb{M}$  consists of only one element - i.e., there is a unique risk measure.*

### 2.3. Single Period Investment.

**Definition 2.3.1.** A continuously differentialble function  $u : \mathbb{R}^+ \rightarrow \mathbb{R}$  is called a risk averse utility function if it has the following two properties:

- $u$  is strictly increasing - that is,  $u'(x) > 0$  for all  $x \in \mathbb{R}^+$
- $u$  is strictly concave - that is,  $u(\lambda x + (1-\lambda)y) > \lambda u(x) + (1-\lambda)u(y)$

In the case that  $u''(x)$  exists, the second condition in the above definition is equivalent to  $u''(x) < 0$  for all  $x \in \mathbb{R}^+$ . Sometimes one assumes in addition the condition:

- $\lim_{x \rightarrow 0} u'(x) = +\infty$  and  $\lim_{x \rightarrow \infty} u'(x) = 0$

**Example 2.3.2** (Utility functions). The following are all risk averse utility functions:

- (1) Logarithmic utility:  $u(x) = \log(x)$
- (2) Exponential utility:  $u(x) = 1 - e^{-\lambda x}$

- (3) Power utility:  $u(x) = \frac{1}{1-\gamma} x^{1-\gamma}$  with  $\gamma > 0, \gamma \neq 1$   
 (4) Square root utility:  $u(x) = \sqrt{x}$

**Proposition 2.3.3** (Principle of expected utility). *We assume the following axiom of agents behaviour - that of maximising expected utility.*

$$X \text{ is preferred to } Y \iff \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]$$

From Jensen's inequality, we have that for every risk averse utility function  $u$  and risky payoff  $X$ ,

$$\mathbb{E}[u(X)] \leq u(\mathbb{E}[X])$$

**Definition 2.3.4** (Certainty equivalent price). The certain value  $X_0 \in \mathbb{R}$  that makes an investor indifferent between  $X_0$  and a risky payoff  $X$  is called the **certainly equivalent price** of  $X$ , that is,

$$u(X_0) = \mathbb{E}[u(X)]$$

or equivalently,

$$X_0 = u^{-1}(\mathbb{E}[u(X)])$$

**Lemma 2.3.5.** *The certainty equivalent price of a risky payoff is invariant under a positive linear transformation of the utility function  $u(x)$ .*

**Definition 2.3.6** (Risk premium). We have that  $X_0 < \mathbb{E}[X]$ , and the difference between the two is called the **risk premium**  $\rho$ , that is,

$$\rho = \mathbb{E}[X] - u^{-1}(\mathbb{E}[u(X)])$$

We can write the equation above as

$$u(\mathbb{E}[X] - \rho) = \mathbb{E}[u(X)]$$

**Definition 2.3.7** (Measures of risk aversion). We define the following risk aversion coefficients, as a measure of how risk averse the investor is. The **absolute risk aversion**  $\rho_{abs}$ , given by

$$\rho_{abs} = -\frac{u''(x)}{u'(x)}$$

and the **relative risk aversion**  $\rho_{rel}$ , given by

$$\rho_{rel} = -\frac{xu''(x)}{u'(x)}$$

We now seek to find the optimal investment in a market, which can be translated as finding a trading strategy  $(x, \phi)$  such that  $\mathbb{E}[u(V_1(x, \phi))]$  achieves an optimal value.



**Definition 2.3.8.** A trading strategy  $(x, \phi^*)$  is a solution to the optimal portfolio problem with initial investment  $x$  and utility function  $u$ , if

$$\mathbb{E}[u(V_1(x, \phi^*))] = \max_{\phi} \mathbb{E}[u(V_1(x, \phi))]$$

**Proposition 2.3.9.** *If there exists a solution to the optimal portfolio problem, then there can not exist an arbitrage in the market.*

**Proposition 2.3.10.** *Let  $(x, \phi)$  be a solution to the optimal portfolio problem with initial wealth  $x$  and utility function  $u$ , then the measure  $\mathbb{Q}$  defined by*

$$\mathbb{Q}(\omega) = \frac{\mathbb{P}(\omega)u'(V_1(x, \phi)(\omega_i))}{\mathbb{E}[u'(V_1(x, \phi))]}$$

Assume that our model is complete. In this case, there is a unique risk neutral measure which we denote by  $\tilde{\mathbb{P}}$ .

**Definition 2.3.11.** We define the set of attainable wealths from initial investment  $x > 0$  by

$$\mathbb{W}_x = \left\{ W \in \mathbb{R}^k \mid \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} W \right] = x \right\}$$

Our optimisation problem is hence:

$$\begin{aligned} & \text{maximise} && \mathbb{E}[u(W)] \\ & \text{subject to} && W \in \mathbb{W}_x \end{aligned}$$

To solve this problem, we use the Lagrange multiplier method. To do this, consider the Lagrange function

$$\mathcal{L}(W, \lambda) = \mathbb{E}[u(W)] - \lambda \left( \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} W \right] - x \right)$$

By introducing the **state price density**

$$L(\omega) = \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)},$$

we can write the Lagrange function as

$$\mathcal{L}(W, \lambda) = \sum_{i=1}^k \mathbb{P}(\omega_i) \left[ u(W(\omega_i)) - \lambda \left( L(\omega_i) \frac{1}{1+r} W(\omega_i) - x \right) \right]$$

Computing partial derivatives with respect to  $W_i = W(\omega_i)$  and setting them equal to zero, multiplying with  $\mathbb{P}(\omega_i)$  and summing over  $i$ , we deduce that

$$\lambda = \mathbb{E}[(1+r)u'(W)]$$

and (denoting the inverse function of  $u'(x)$  by  $I(x)$ )

$$W(\omega) = I \left( \lambda \frac{L(\omega)}{1+r} \right)$$

Since we have

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} W \right] = x$$

and substituting the expression from the above equation into the last equation, we obtain

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} I \left( \lambda \frac{L}{1+r} \right) \right] = x$$

### 3. MULTI PERIOD MARKET MODELS

#### 3.1. The general model.

**Definition 3.1.1** (Specification of the general model). The two most important new features of multi period market models are:

- Agents can buy and sell assets not only at the beginning of the trading period, but at any time  $t$  out of a discrete set of trading times  $t \in \{0, 1, 2, \dots, T\}$ .
- Agents can gather information over time, since they can observe prices. Hence, they can make their investment decisions at time  $t = 1$  dependent on the prices of the asset at time  $t = 1$ .

Throughout, we assume we are working on a finite state space  $\Omega$  on which there is defined a probability measure  $\mathbb{P}$ .

**Definition 3.1.2** ( $\sigma$ -algebra). A collection  $\mathcal{F}$  of subsets of the state space  $\Omega$  is called a  $\sigma$ -algebra if the following conditions hold:

- $\Omega \in \mathcal{F}$
- If  $F \in \mathcal{F}$ , then  $F^c \in \mathcal{F}$
- If  $F_i \in \mathcal{F}$  for  $i \in \mathbb{N}$ , then  $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$ .

**Definition 3.1.3** (Partition of a  $\sigma$ -algebra). Let  $I$  denote some index set. A **partition** of a  $\sigma$ -algebra  $\mathcal{F}$  is a collection of sets  $\emptyset \neq A_i \in \mathcal{F}$  for  $i \in I$ , such that

- Every set  $F \in \mathcal{F}$  can be written as a union of some of the  $A_i$ .
- The sets  $A_i$  are pairwise disjoint.

**Definition 3.1.4.** A random variable  $X : \Omega \rightarrow \mathbb{R}$  is called  **$\mathcal{F}$ -measurable**, if for every closed interval  $[a, b] \subset \mathbb{R}$ , the preimage under  $X$  belongs to  $\mathcal{F}$ , that is,

$$X^{-1}([a, b]) \in \mathcal{F}$$

**Proposition 3.1.5.** Let  $X : \Omega \rightarrow \mathbb{R}$  be a random variable and  $(A_i)$  a partition of the  $\sigma$ -algebra  $\mathcal{F}$ , then  $X$  is  $\mathcal{F}$ -measurable if and only if  $X$  is constant on each of the sets of the partition, that is, there exist  $c_j \in \mathbb{R}$  for all  $j \in I$  such that

$$X(\omega) = c_j \text{ for all } \omega \in A_j$$

**Definition 3.1.6.** A sequence  $(\mathcal{F}_t)_{0 \leq t \leq T}$  of  $\sigma$ -algebras on  $\Omega$  is called a **filtration** if  $\mathcal{F}_s \subset \mathcal{F}_t$  whenever  $s < t$ .

**Definition 3.1.7.** A family  $(X_t)$  with  $0 \leq t \leq T$  consisting of random variables, is called a **stochastic process**. If  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is a filtration, the stochastic process  $(X_t)$  is called  **$(\mathcal{F}_t)$ -adapted** if for all  $t$  we have that  $X_t$  is  $\mathcal{F}_t$ -measurable.

**Definition 3.1.8.** Let  $(X_t)_{0 \leq t \leq T}$  be a stochastic process on  $(\Sigma, \mathcal{F}, \mathbb{P})$ . Define

$$\mathcal{F}_t^X = \sigma(X_u^{-1}([a, b]) \mid 0 \leq u \leq t, a \leq b)$$

This is the smallest  $\sigma$ -algebra which contains all the sets  $X_u^{-1}([a, b])$  where  $0 \leq u \leq t$  and  $a \leq b$ . Clearly  $(\mathcal{F}_s^X)$  is a filtration. It follows immediately from the definition that  $(X_t)$  is  $(\mathcal{F}_t^X)$  adapted.  $(\mathcal{F}_t^X)$  is called the filtration **generated** by the process  $X$ .

**Definition 3.1.9** (Value process). The **value process** corresponding to the trading strategy  $\phi = (\phi_t)_{0 \leq t \leq T}$  is the stochastic process  $(V_t(\phi))_{0 \leq t \leq T}$  where

$$V_t(\phi) = \phi_t^0 B_t + \sum_{i=1}^n \phi_t^i S_t^i$$

**Definition 3.1.10.** A trading strategy  $\phi = (\phi_t)$  is called **self financing** if for all  $t = 0, \dots, T-1$ ,

$$\phi_t^0 B_{t+1} + \sum_{i=1}^n \phi_t^i S_{t+1}^i = \phi_{t+1}^0 B_{t+1} + \sum_{i=1}^n \phi_{t+1}^i S_{t+1}^i$$

**Lemma 3.1.11.** For a self-financing trading strategy  $\phi = (\phi_t)$ , the value process can be alternatively computed via

$$V_t(\phi) = \phi_{t-1}^0 B_t + \sum_{i=1}^n \phi_{t-1}^i S_t^i$$

**Definition 3.1.12** (General multi period market model). A general multi period market model is given by the following data.

- A probability space  $(\Sigma, \mathcal{F}, \mathbb{P})$  together with a filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  of  $\mathcal{F}$ .
- A money market account  $(B_t)$  which evolves according to  $B_t = (1+r)^t$ .
- A number of financial assets  $(S_t^1), \dots, (S_t^n)$  which are assumed to be  $(\mathcal{F}_t)$ -adapted stochastic processes.
- A set  $\mathcal{T}$  of self financing and  $(\mathcal{F}_t)$ -adapted trading strategies.

**Definition 3.1.13.** Assume we are given a general multi period market model as described above. The **increment process**  $(\Delta S_t^i)$  is defined as

$$\Delta S_t^i = S_t^i - S_{t-1}^i$$

and

$$\Delta B_t = B_t - B_{t-1} = r B_{t-1}$$

**Definition 3.1.14** (Gains process). Given a trading strategy  $\phi$ , the corresponding **gains process**  $(G_t(\phi))_{0 \leq t \leq T}$  is given by

$$G_t(\phi) = \sum_{s=0}^{t-1} \phi_s^0 \Delta B_{s+1} + \sum_{i=1}^n \sum_{s=0}^{t-1} \phi_s^i \Delta S_{s+1}^i$$

**Proposition 3.1.15.** *An adapted trading strategy  $\phi = (\phi)_{0 \leq t \leq T}$  is self financing, if and only if any of the two equivalent statements hold*

- $V_t(\phi) = V_0(\phi) + G_t(\phi)$
- $\hat{V}_t(\phi) = \hat{V}_0(\phi) + \hat{G}_t(\phi)$

for all  $0 \leq t \leq T$ . Here  $\hat{V}_t$  and  $\hat{G}_t$  denote the discounted value and gains process, as defined in the following definition.

**Definition 3.1.16** (Discount processes). The discounted prices are given by

$$\hat{S}_t^i = \frac{S_t^i}{B_t}$$

and discounted gains, discounted value process, and discounted gains process are all defined analogously.

**3.2. Properties of the general multi period market model.** Here, we redefine the general concepts of financial mathematics, such as arbitrage, hedging, in the context of a multi period market model.

**Definition 3.2.1** (Arbitrage). A (self-financing) trading strategy  $\phi = (\phi)_{0 \leq t \leq T}$  is called an arbitrage if

- $V_0(\phi) = 0$
- $V_T(\phi) \geq 0$
- $\mathbb{E}[V_T(\phi)] > 0$

**Definition 3.2.2** (Contingent claim). A **contingent claim** in a multi period market model is an  $\mathcal{F}_T$ -measurable random variable  $X$  on  $\Omega$  representing a payoff at terminal time  $T$ . A **hedging strategy** for  $X$  in our model is a trading strategy  $\phi \in \mathcal{T}$  such that

$$V_T(\phi) = X,$$

that is, the terminal value of the trading strategy is equal to the payoff of the contingent claim.

**Proposition 3.2.3.** *Let  $X$  be a contingent claim in a multi period market model, and let  $\phi \in \mathcal{T}$  be a hedging strategy for  $X$ , then the only price of  $X$  at time  $t$  which complies with the no arbitrage principle is  $V_t(\phi)$ . In particular, the price at the beginning of the trading period at time  $t = 0$  is the total initial investment in the hedge.*

**Definition 3.2.4** (Attainable contingent claim). A contingent claim  $X$  is called **attainable** in  $\mathcal{T}$ , if there exists a trading strategy  $\phi \in \mathcal{T}$  which replicates  $X$ , that is,  $V_T(\phi) = X$ .

**Definition 3.2.5** (Complete market). A general multi period market model is called **complete**, if and only if for any contingent claim  $X$  there exists a replicating strategy  $\phi$ . A model which is not complete is called **incomplete**.

**Definition 3.2.6** (Conditional expectation). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a finite probability space and  $X$  an  $\mathcal{F}$ -measurable random variable. Assume that  $\mathcal{G}$  is a  $\sigma$ -algebra which is contained in  $\mathcal{F}$ . Denoting the unique partition of  $\mathcal{G}$  with  $(A_i)_{i \in I}$ , the **conditional expectation**  $\mathbb{E}[X | \mathcal{G}]$  of  $X$  with respect to  $\mathcal{G}$  is defined as the random variable which satisfies

$$\mathbb{E}[X | \mathcal{G}](\omega) = \sum_x x \mathbb{P}(X = x | A_i)$$

whenever  $\omega \in A_i$ .

We then have the following identity:

$$\int_G X d\mathbb{P} = \int_G \mathbb{E}[X | \mathcal{G}] d\mathbb{P}$$

for any  $G \in \mathcal{G}$ .

**Proposition 3.2.7.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a finite probability space and  $X$  an  $\mathcal{F}$ -measurable random variable. Let  $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$  be sub- $\sigma$ -algebras. Assume furthermore that  $\mathcal{G}_2 \subset \mathcal{G}_1$ . Then

- **Tower property.**

$$\mathbb{E}[X | \mathcal{G}_2] = \mathbb{E}[\mathbb{E}[X | \mathcal{G}_1] | \mathcal{G}_2]$$

- **Taking out what is known.** If  $Y : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{G}$ -measurable, then

$$\mathbb{E}[YX | \mathcal{G}] = Y \mathbb{E}[X | \mathcal{G}]$$

- If  $\mathcal{G} = \{\emptyset, \Omega\}$  is the trivial  $\sigma$ -algebra, then

$$\mathbb{E}[X | \mathcal{G}] = \mathbb{E}[X]$$

**Definition 3.2.8** (Risk neutral measure). A measure  $\tilde{\mathbb{P}}$  on  $\Omega$  is called a **risk neutral measure** for a general multi period market model if

- $\tilde{\mathbb{P}}(\omega) > 0$  for all  $\omega \in \Omega$
- $\mathbb{E}_{\tilde{\mathbb{P}}}[\Delta \hat{S}_t^i | \mathcal{F}_{t-1}] = 0$  for  $i = 1, \dots, n$  and for all  $1 \leq t \leq T$ .

An alternative formulation of the second condition in the previous definition is

$$\mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{1}{1+r} S_{t+1}^i | \mathcal{F}_t \right] = S_t^i$$

**Definition 3.2.9** (Martingale). A  $\mathcal{F}_t$ -adapted process  $(X_t)$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  is called a **martingale** if for all  $s < t$ ,

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s$$

**Lemma 3.2.10.** Let  $\tilde{\mathbb{P}}$  be a risk neutral measure. Then the discounted stock prices  $(\hat{S}_t^i)$  for  $i = 1, \dots, n$  are martingales under  $\tilde{\mathbb{P}}$ .

**Proposition 3.2.11.** Let  $\phi \in \mathcal{T}$  be a trading strategy. Then the discounted value process  $(\hat{V}_t(\phi))$  and the discounted gains process  $\hat{G}_t(\phi)$  are martingales under any risk neutral measure  $\tilde{\mathbb{P}}$ .

**Theorem 3.2.12** (Fundamental Theorem of Asset Pricing). Given a general multi period market model, if there is a risk neutral measure, then there are no arbitrage strategies  $\phi \in \mathcal{T}$ . Conversely, if there are no arbitrages among self financing and adapted trading strategies, then there exists a risk neutral measure.

**Definition 3.2.13.** We say that an adapted stochastic process  $(X_t)$  is a price process for the contingent claim  $X$  which **complies with the no arbitrage principle**, if there is no adapted and self financing arbitrage strategy in the extended model, which consists of the original stocks  $(S_t^1), \dots, (S_t^n)$  and an additional asset given by  $S_t^{n+1} = X_t$  for  $0 \leq t \leq T-1$  and  $S_T^{n+1} = X$ .

**Proposition 3.2.14.** Let  $X$  be a possibly unattainable contingent claim and  $\tilde{\mathbb{P}}$  a risk neutral measure for a general multi period market model. Then

$$X_t = B_t \mathbb{E}_{\tilde{\mathbb{P}}} \left[ \frac{X}{B_T} | \mathcal{F}_t \right]$$

defines a price for the contingent claim consistent with the no arbitrage principle.

**Theorem 3.2.15.** Under the assumption that a general multi period market model is arbitrage free, it is complete, if and only if there is a unique risk neutral measure.