MATH 3961 - METRIC SPACES

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Contents

1. Metric Spaces

Definition 1.1 (Metric)**.** A metric, or distance function, on a set X is a mapping $d : X \times X \to \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We call (X, d) a **metric space**.

Definition 1.2 (Open ball)**.** Let (X, d) be a metric space. For $x \in X$ and $\epsilon > 0$, the set $B_d(x, \epsilon)$ defined by

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

is callen an **open ball** in the set X.

Definition 1.3 (Open sets in metric spaces)**.** Let (X, d) be a metric space and let U be any subset of X. Then U is called an **open set** in X if every point of U is an interior point of U; that is, for any $a \in U$, there is an open ball $B(a, \epsilon)$ such that $B(a, \epsilon) \subseteq U$.

Definition 1.4 (Properties of open sets)**.** Let (X, d) be a metric space.

- $\emptyset$ and X are open.
- The union of an arbitrary collection of open sets is open.
- The intersection of a **finite** number of open sets is open.

Definition 1.5 (Closed set)**.** A subset A of a metric space (X, d) is **closed** if it's complement $X \backslash A$ is open in X.

Definition 1.6 (Properties of closed sets)**.** Let (X, d) be a metric space.

- $\emptyset$ and X are closed.
- The union of an finite collection of closed sets is closed.
- The intersection of an arbitrary number of closed sets is closed.

Definition 1.7 (Limit point of a subset)**.** Let (X, d) be a metric space and let A be a subset of X. Then a point x in X is a **limit point** of A if every open ball $B(x, \epsilon)$ contains at least one point of A.

The set of all limit points of A is called the **derived set** A'.

Definition 1.8 (Closure of a set)**.** Let (X, d) be a metric space and let $A \subseteq X$. Then the set consisting of A and its limit points is called the **closure** of A, denoted $\overline{A}$.

$$\overline{A} = A \cup A'$$

Theorem 1.9. *The closure of a set is a closed set, and a set is closed if and only if it is equal to its closure.*

Definition 1.10 (Interior of a set)**.** Let (X, d) be a metric space and let $A \subseteq X$. A point $a \in A$ is an **interior point** of A if there exists $\epsilon > 0$ such that

$$B(a, \epsilon) \subseteq A$$

The set of interior points of A is called $\operatorname{Int} A$, the **interior** of A.

Theorem 1.11. *The set $\operatorname{Int} A$ is open, and a set A is open if and only if $\operatorname{Int} A = A$.*

Theorem 1.12 (Properties of interior and closure)**.** *The interior of a set A is the largest open subset contained in A, and the closure of A is the smallest closed set containing A.*

Definition 1.13 (Isolated point)**.** Let (X, d) be a metric space and let A be a subset of X. A point $x \in A$ is called an **isolated point** if there exists an $\epsilon > 0$ such that

$$B(x, \epsilon) \backslash \{x\} \cap A = \emptyset$$

Definition 1.14 (Boundary of a subset)**.** Let (X, d) be a metric space and $A \subseteq X$. Then the **boundary** of A is defined as

$$\partial A = \overline{A} \cap \overline{X \backslash A} = \overline{A} \backslash \operatorname{Int} A$$

Theorem 1.15 (Properties of the boundary)**.** *Let (X, d) be a metric space and $A \subset X$. Then we have*

- $\overline{A} = \operatorname{Int} A \cup \partial A$.
- *A is closed if and only if $\partial A \subseteq A$.*
- *A is open if and only if $\partial A \subseteq X \backslash A$.*
- $\partial(X \backslash A) = \partial A$.

Definition 1.16 (Diameter of a set)**.** The diameter of a subset A of a metric space (X, d), $\delta(A)$ is defined as

$$\delta(A) = \sup_{x,y \in A} d(x, y)$$

Definition 1.17 (Bounded set)**.** A subset A of a metric space (X, d) is **bounded** if its diameter is finite. Alternatively, a subset is bounded if it is contained in a large enough open set - i.e., there exists $x \in X$ and $\epsilon > 0$ such that $A \subseteq B(x, \epsilon)$

1.1. **Separable metric spaces.**

Definition 1.18 (Separable metric space)**.** Let (X, d) be a metric space. Then a subset A of X is said to be **dense** in X if $\overline{A} = X$. A metric space (X, d) is said to be **separable** if X has a countable dense subset.

Corollary 1.19. *We note that A is dense in X if and only if for any $x \in X$ and $\epsilon > 0$, there is a point $a \in A$ such that $d(x, a) < \epsilon$.*

1.2. **Subspaces.**

Definition 1.20 (Open sets in a subspace)**.** Let (X, d) be a metric space and (Y, d_Y) be a metric subspace of (X, d). Let G be a subset of Y. Then G is open in Y if and only if, for any $x \in G$, there is an open ball $B(x, \epsilon)$ in X such that

$$B(x, \epsilon) \cap Y \subseteq G$$

A subset H of Y is closed in Y if its complement $G = Y \backslash H$ of H is open in Y.

Theorem 1.21 (Open sets in a metric subspace)**.** *Let (Y, d_Y) be a metric subspace of a metric space (X, d), and let $G \subseteq Y$. Then G is open in Y if and only if there exists an open subset U in X such that $G = U \cap Y$.*

1.3. **Convergence in a Metric Space.**

Definition 1.22 (Convergence)**.** A sequence (x_n) in a metric space (X, d) is said to **converge** to a point $x \in X$ if for any $\epsilon > 0$, there exists N such that

$$n > N \quad \text{implies} \quad d(x_n, x) < \epsilon$$

The point x is called a **limit** of the sequence (x_n)

Corollary 1.23. *A sequence (x_n) in a metric space (X, d) is said to* ***converge*** *to a point $x \in X$ if any open ball $B(x, \epsilon)$ contains almost all x_n.*

Theorem 1.24 (Connection between closed sets and convergent sequences)**.** *Let (X, d) be a metric space, $A \subseteq X$ and $x \in X$. Then*

- *$x \in \overline{A}$ if and only if there is a sequence (x_n) in A such that $x_n \to x$.*
- *A is closed if and only if A contains all the limits of convergent sequences in A.*

Definition 1.25 (Uniform convergence)**.** Let (f_n) be a sequence of real-valued functions defined on a set S and let f be a function defined on S. Then we say that the sequence (f_n) converges to f uniformly if for any $\epsilon > 0$, there exists N such that

$$\sup_{x \in S} d(f_n(x), f(x)) < \epsilon$$

for all $n > N$, and where N is independent of x.

Definition 1.26 (Cauchy Sequences)**.** A sequence (x_n) in a metric space (X, d) is said to be **Cauchy** in X if for any $\epsilon > 0$, there exists N such that

$$m, n > N \Rightarrow d(x_m, x_n) < \epsilon$$

Definition 1.27 (Completeness in Metric Spaces)**.** A space X is said to be complete if every Cauchy sequence in X converges in X.

Proposition 1.28. *Every convergent sequence (x_n) in a metric space (X, d) is a Cauchy sequence.*

Corollary 1.29. *Let (X, d) be a complete metric space. Then a closed metric subspace $Y = (Y, d_Y)$ of X is complete.*

Proposition 1.30. *Let (X, d) be a metric space. If a Cauchy sequence (x_n) in (X, d) has a subsequence converging to x, then (x_n) converges to x.*

2. Continuous Mappings

Definition 2.1 (Continuous mapping between metric spaces)**.** Let (X, d) and (Y, d_Y) be two metric spaces. Then a mapping $f : X \to Y$ is said to be **continuous** at a point $a \in X$ if for any $\epsilon > 0$, there exists δ such that

$$d_X(x, a) < \delta \Rightarrow d_Y(f(x), f(a)) < \epsilon$$

Theorem 2.2 (Topological characterisations of continuity)**.** *A funtion $f : X \to Y$ is continuous at $a \in X$ if and only if for any open set W containing $f(a)$, there exists an open set G containing A such that $f(G) \subseteq W$.*

Theorem 2.3 (Sequential characterisation of continuity)**.** *A function $f : X \to Y$ is continuous at $a \in X$ if and only if for any sequence (x_n) which converges to a in X, the corresponding sequence $(f(x_n))$ converges to $f(a)$ in Y.*

Definition 2.4 (Continuous mapping)**.** A map is continuous on X if any only if it is continuous at every point in X.

Theorem 2.5 (Topological definition of continuous mapping)**.** *A mapping $f : X \to Y$ is continuous on X if and only if for any open set W in Y, the set $f^{-1}(W)$ is open in X. Alternatively, a function is continuous if the preimage of open sets are open in X.*

Theorem 2.6 (Continuity in terms of closed sets)**.** *A mapping $f : X \to Y$ is continuous on X if and only if the preimage of closed sets are closed in X.*

3. Homeomorphism and Equivalent Metrics

Definition 3.1 (Homeomorphism)**.** Let X and Y be metric spaces and let $f : X \to Y$ be a map between them. Then f is a **homeomorphism** from X to Y if we have

- f is a bijection.
- f and f^{-1} are continuous

If a homeomorphism exists between X and Y, we say that X and Y are **homeomorphic**, and that $X \simeq Y$.

Definition 3.2 (Characterisations of homeomorphism)**.** Let $f : X \to Y$ be a bijective mapping. Then the following are equivalent.

- f is a homeomorphism;
- for any $U \subseteq X$, U is open in X if and only if $f(U)$ is open in Y;
- for any $G \subseteq X$, G is closed in X if and only if $f(G)$ is closed in Y;
- for any $A \subseteq X$, $f(\overline{A}) = \overline{f(A)}$;
- for any $B \subseteq Y$, $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$
- for any $B \subseteq Y$, $f^{-1}(\operatorname{Int} B) = \operatorname{Int} f^{-1}(B)$

Definition 3.3 (Isometric mappings)**.** Let (X, d) and (Y, d_Y) be two metric spaces and $f : X \to Y$ a mapping. Then f is said to be **isometric** or an **isometry** if f preserves distances; that is, for all $x, y \in X$,

$$d_Y(f(x), f(y)) = d_X(x, y)$$

The space X is said to be isometric with the space Y if there exists a bijective isometry of X onto Y. The spaces X and Y are isometric spaces

Theorem 3.4. *Any isometric mapping from X onto Y is a homeomorphism. Moreover, if X is complete and Y is isometric with X, then Y is also complete.*

Definition 3.5 (Equivalent metrics)**.** Let (X, d_1) and (X, d_2) be two metric spaces. If the identity mapping $\mathrm{id} : (X, d_1) \to (X, d_2)$ is a homeomorphism, then the metrics d_1 and d_2 are said to be equivalent on X.

Theorem 3.6 (Characterisations of equivalent metrics)**.** *Let (X, d_1) and (X, d_2) be two metric spaces. Then the following are equivalent.*

- *The metrics d_1 and d_2 are equivalent on X;*
- *for any $U \subseteq X$, U is open in (X, d_1) if and only if U is open in (X, d_2);*
- *for any $G \subseteq X$, G is closed in (X, d_1) if and only if G is closed in (X, d_2);*
- *The sequence (x_n) converges to a in (X, d_1) if and only if it converges to a in (X, d_2).*

Theorem 3.7 (Equivalent metrics)**.** *Let (X, d_1) and (X, d_2) be two metric spaces. If there exist strictly positive numbers c and C such that*

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$$

for all $x, y \in X$, then the metrics d_1, d_2 are equivalent on X.

4. Contraction Mapping Theorem

Definition 4.1 (Uniformly continuous)**.** A function $f : X \to Y$ is uniformly continuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$$

where δ is independent of x, y.

Definition 4.2 (Contraction mapping)**.** Let f be a mapping from a metric space (X, d) to itself. Then f is called a **contraction mapping** if there exists a constant $K > 1$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq K(x, y)$$

Proposition 4.3. *If $f : X \to X$ is a contraction mapping, then f is* ***uniformly continuous****, and hence continuous, on X*

Theorem 4.4 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space and let $f : X \to X$ be a contraction mapping. Then f has a unique fixed point p in X*

Proof. **Existence:** Show that the sequence (x_n), defined as $x_n = f^n(x_0)$ for some $x_0 \in X$ is Cauchy.

Uniqueness: If p and q are fixed points of f, the we have that

$$d(p, q) = d(f(p), f(q)) \leq Kd(p, q)$$

and so $p = q$. □

Corollary 4.5 (Application of Banach fixed point theorem to ordinary differential equations)**.** *We seek to solve the differential equation*

$$\frac{dx}{dt} = f(t, x)$$

given the initial condition $x(t_0) = x_0$.

Define Y be the subspace of the set of all continuous functions on $[t_0 - \beta, t_0 + \beta]$ with the supremum metric, satisfying $d(x(t), x_0) < c\beta$ We claim that the mapping $F : Y \to Y$ defined by

$$F(x(t)) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

is a contraction mapping. As F is a contraction mapping on a complete metric space, we must have that it has a unique fixed point, which satisfies the differential equation above.

5. Completeness

We recall the definition of completeness in metric spaces.

Definition 5.1 (Completeness in Metric Spaces)**.** A space X is said to be complete if every Cauchy sequence in X converges in X.

We now state the theorem that every metric space can be completed. The space $\hat{X}$ in the theorem is called the completion of the given space X.

Theorem 5.2 (Completion of a metric spaces)**.** *Let (X, d) be a metric space. Then there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$*

Proof. TO DO □

6. Connectedness

Definition 6.1 (Disconnected)**.** Let X be a metric space (or topological space). Then X is said to be **disconnected** if there exist two non-empty subsets A_1, A_2 of X such that

$$X = A_1 \cup A_2 \text{ and } A_1 \cap \overline{A}_2, \overline{A}_1 \cap A_2 = \emptyset$$

If no two sets A_1, A_2 exist, we say that X is **connected**

Theorem 6.2 (Characterisations of connectedness)**.** *Let (X, d) be a metric space. Then the following statements are equivalent:*

- *X is disconnected;*
- *There exist two non-empty disjoint open subsets A_1, A_2 in X such that $X = A_1 \cup A_2$;*
- *There exist two non-empty disjoint closed subsets A_1, A_2 in X such that $X = A_1 \cup A_2$;*
- *There exists are proper subset of X which is both open and closed in X*

Definition 6.3 (Connected subspace)**.** Let (X, d) be a metric space and A a non-empty subset of X. Then A is said to be a connected subst of X if A is connected as a metric subspace and to be a **disconnected** subset of X if A is disconnected as a metric subspace.

Theorem 6.4 (Intervals in $\mathbb{R}$)**.** *A subset A of $\mathbb{R}$ containing at least two points is connected if and only if A is a interval.*

Theorem 6.5 (Characterisations of connectedness)**.** *Let $\mathcal{S}(2)$ be the two point discrete metric space. If A is connected then any continuous mapping $f : A \to \mathcal{S}(2)$ is a constant mapping. Alternatively, if $f : A \to \mathcal{S}(2)$ is continuous, then $f(A) = \{0\}$ or $\{1\}$.*

Theorem 6.6 (Connectedness is a topological property)**.** *Let X and Y be two metric spaces, and let $f : X \to Y$ be a continuous mapping. Then if $A \subseteq X$ is connected in X, then the image $f(A)$ is connected in Y.*

Definition 6.7 (Path-connected)**.** Let X be a metric space and A a subset of X. Then A is said to be **path-connected** if for any $a, b \in A$, there is a path joining a and b, that is, a continuous mapping $f : [0, 1] \to A$ such that $f(0) = a, f(1) = b$.

Theorem 6.8 (Path-connectedness implies connectedness)**.** *Let X be a metric space and A a subset of X. If A is path-connected, then A is connected.*

We note that the converse is not necessarily true - that is, there exist connected sets that are not path connected. However, in $\mathbb{R}^n$, we have the following.

Theorem 6.9 (For open sets in $\mathbb{R}^n$, path-connectedness is equivalent to connectedness)**.** *Let X be any open set in $\mathbb{R}^n$. Then X is connected if and only if X is path-connected.*

7. Compactness

Compactness in metric spaces.

Definition 7.1 (Open covering of a set)**.** Let X be a metric space (or any topological space), and let $A \subseteq X$. Then a family $\mathcal{U}$ of open sets in X, is called an **open covering** of A if

$$A \subseteq \cup_{U \in \mathcal{U}} U$$

A subset $\mathcal{V}$ of $\mathcal{U}$ is called a **finite subcovering** if $\mathcal{V}$ covers A and has a finite number of elements.

Definition 7.2 (Compact subset of a topological space)**.** let X be a metric space (or any topological space), and let $A \subseteq X$. Then A is called a **compact subset** of X if **every** open covering $\mathcal{U}$ of A has a finite subcovering $\mathcal{V}$ of A.

In metric spaces, we have the following useful theorem.

Theorem 7.3 (Implications of compactness in metric spaces)**.** *Let (X, d) be a metric space. If A is a compact subset in X, then A is closed and bounded in X.*

In $\mathbb{R}^n$, we have the following, more general, results. These are key results in characterising compact subsets of Euclidean space.

Theorem 7.4 (Heine-Borel)**.** *Every closed and bounded interval in $\mathbb{R}$ is compact.*

Theorem 7.5 (Compactness in $\mathbb{R}^n$)**.** *Let A be a subset of $\mathbb{R}^n$. Then A is compact if and only if A is closed and bounded.*

7.1. **Properties of compact sets.**

Theorem 7.6. *A subset A in $\mathbb{R}^n$ is compact if and only if every sequence in A has a convergent subsequence with limit in A.*

Definition 7.7 (Compactness is a topological property)**.** Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous mapping. If a subset A in X is compact, then the image $f(A)$ is compact in Y.

That is, continuous images of compact sets are compact.

Corollary 7.8. *The following are true in an arbitrary metric space.*

- *A continuous image of a compact subset is closed.*
- *A continuous image of a compact subset is bounded.*

Theorem 7.9. *Any closed subspace A of a compact space X is compact.*

Proof. TO DO $\square$

8. Applications to Continuous Functions $f : [a, b] \to \mathbb{R}$

Theorem 8.1 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let K be a number lying between $f(a)$ and $f(b)$. Then there exists a point $c \in [a, b]$ such that $f(c) = K$.*

Proof. Using connectedness of $[a, b]$, we have that $f([a, b])$ is connected. Thus, if $f(a) < K < f(b)$, then $K \in f([a, b])$. Hence, there exists $c \in [a, b]$ such that $f(c) = K$. $\square$

9. Topological Spaces

A topological space is defined as follows. Let X be a non-empty set. Then a family τ of subsets of X is called a topology for X if τ satisfies

- $\emptyset, X \in \tau$;
- The union of any subfamily of members of τ is in τ;
- The intersection of any **finite** subfamily of members of τ is in τ.

The pair (X, τ) is called a **topological space**, and the members of τ are called the **open sets** in (X, τ).

Definition 9.1 (Interior of a subset)**.** Let (X, τ) be any topological space and $A \subseteq X$. Then $a \in A$ is called an **interior point** of A if there exists an open set U containing a such that $U \subseteq A$. We denote by $\operatorname{Int} A$ the set of all interior points of A.

Theorem 9.2 (Properties of the interior)**.** *Let (X, τ) be any topological space and let $A \subseteq X$. Then $\operatorname{Int} A$ is the **largest** open subset contained in A.*

Definition 9.3 (Closed subsets)**.** Let A be a subset of a topological space (X, τ). Then A is called a **closed set** in X if the complement $X \backslash A$ is open in X, that is, if $X \backslash A \in \tau$.

Theorem 9.4. *In a topological space (X, τ),*

- *$\emptyset$ and X are closed;*
- *The intersection of any collection of closed sets is closed;*
- *The union of any **finite** collection of closed sets are closed.*

Definition 9.5 (Limit point of a subset)**.** Let (X, τ) be a topological space and let $A \subseteq X$. Then a point $x \in X$ is called a **limit point** or **accumulation point** of A if every open set G containing x contains a point of A different from x, i.e.,

$$G \in \tau, x \in G \Rightarrow (G \backslash \{x\}) \cap A \neq \emptyset$$

We denote by A' the set of all limit points of A, and is called the derived set of A.

Theorem 9.6 (Properties of closed sets)**.** *Let (X, τ) be a topological space and let $A \subseteq X$. Then A is closed if and only if $A' \subseteq A$.*

Definition 9.7 (Closure of a subset)**.** Let (X, τ) be a topological space and $A \subseteq X$. Then the set consisting of A together with all its limit points is called the **closure** of A, and is denoted by $\overline{A}$. Thus,

$$\overline{A} = A \cup A'$$

Proposition 9.8. *Let (X, τ) be a topological space and $A \subseteq X$. Then*

$$\overline{A} = \{x \in X \,|\, \textit{for every open set } U \textit{ containing } x,\ U \cap A \neq \emptyset\}.$$

Theorem 9.9. *Let (X, τ) be a topological space and $A \subseteq X$. Then $\overline{A}$ is the smallest closed set containing A.*

Definition 9.10 (Dense subset)**.** Let (X, τ) be a topological space and $A \subseteq X$. Then a subset A of X is said to be **dense** in X if $\overline{A} = X$.

Definition 9.11 (Nowhere dense)**.** Let (X, τ) be a topological space and $A \subseteq X$. Then A is **nowhere dense** in X if and only if the interior of the closure of A is empty. That is, $\operatorname{Int}(\overline{A}) = \emptyset$.

Alternatively, a set is nowhere dense if and only if $X \backslash \overline{A}$ is dense in X.

Definition 9.12 (Boundary of a subset)**.** Let (X, τ) be a topological space and $A \subseteq X$. Then the **boundary** of A, denoted by ∂A, is defined as

$$\partial A = \overline{A} \cap \overline{X \backslash A}$$

Theorem 9.13 (Characterisation of the boundary)**.** *Let (X, τ) be a topological space and $A \subseteq X$. Then*

$$\overline{A} = \mathit{Int}\, A \cup \partial A$$

Definition 9.14 (Convergence in topological spaces)**.** Let (X, τ) be a topological space. Then a sequence (x_n) of points in X is said to **converge** to a point $x \in X$ if for any open set U containing x, there exists a positive integer N such that

$$n > N \Rightarrow x_n \in U$$

That is, if any open set U containing x contains almost all of the terms of the sequence.

Definition 9.15 (Induced or relative topology)**.** Let (X, τ) be a topological space and $Y \subseteq X$. Let

$$\tau_Y = \{G \subseteq Y \mid G = U \cap Y \text{ for some } U \in \tau\}$$

Then τ_Y is a topology for Y, called the **induced** or **relative topology** on Y and the space (Y, τ_Y) is called a subspace of (X, τ)

Definition 9.16 (Bases for a topology)**.** Let (X, τ) be a topological space. Then a subfamily $\mathcal{B}$ of τ is called a **base** for the topology if for every open set U in τ is the union of members of $\mathcal{B}$. Equivalently, $\mathcal{B} \subseteq \tau$ is a basis for τ if and only if for any point a in an open set $U \in \tau$, there exists $V \in \mathcal{B}$ such that $a \in V \subseteq U$.

Theorem 9.17 (Characterisation of a basis for a topology)**.** *A family of nonempty subsets of a set X is a base for some topology τ on X if and only if it satisfies the following properties.*

- $X = \cup_{B \in \mathcal{B}} B$
- *For any $B_1, B_2 \in \mathcal{B}$, $B_1 \cap B_2$ is the union of members of $\mathcal{B}$. Equivalently, if $b \in B_1 \cap B_2$, then there exists $B_b \in \mathcal{B}$ such that $b \in B_b \subseteq B_1 \cap B_2$.*

Theorem 9.18 (Continuity in terms of a basis)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces and $f : X \to Y$ a mapping. Let $\mathcal{B}_Y$ be a basis for τ_Y. Then f is continuous if and only if for any $B \in \mathcal{B}_Y$, $f^{-1}(B)$ is open in X - i.e., is in τ_X.*

Theorem 9.19 (Product spaces)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces. Then the family $\mathcal{B}$ given by*

$$\mathcal{B} = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$$

is a base for a topology on $X \times Y$.

9.1. **Compactness.**

Definition 9.20 (Compactness in terms of a basis)**.** A topological space (X, τ) is compact if and only if there exists a base $\mathcal{B}$ for τ such that every open covering of X be members of $\mathcal{B}$ has a finite subcovering.

Theorem 9.21. *The product space of two compact topological spaces is compact.*

9.2. **Connectedness.**

Theorem 9.22. *Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then X is connected if and only if Y is connected.*

Theorem 9.23. *Let X be a topological space. Let $\{A_i\}$be a family of connected subsets in X and suppose that for all i, j, $A_i \cap A_j \neq \emptyset$. Then the union $A = \cup_i A_i$ is connected.*

10. Separation Properties

Definition 10.1 (T_0-spaces)**.** A topological space (X, τ) is called a T_0**-space** if for any pair of distinct points a, b of X, either there exists an open set U containing a and not b or an open set V containing b and not a.

Definition 10.2 (T_1-spaces)**.** A topological space (X, τ) is called a T_1**-space** if for any pair of distinct points a, b of X, there exists an open set U in X with $a \in U$ and $b \notin U$.

Every T_1-space is T_0, but not the reverse.

Theorem 10.3 (Characterisation of T_1-spaces)**.** *AA topological space (X, τ) is called a T_1**-space** if and only if eery singleton set $\{a\}$ of X is closed (and so every finite subset of X is closed).*

Definition 10.4 (T_2-spaces or Hausdorff spaces)**.** A topological space (X, τ) is called a T_2**-space** or a **Hausdorff space** if for any pair of distinct points a, b of X, there are disjoint open sets U and V in X such that $a \in U$ and $b \in V$.

Every T_2-space is T_1, but not the reverse.

Example 10.5. Any metric space is a T_2-space

Theorem 10.6. *Every subspace of a T_1- or T_2-space is a T_1- or T_2-space.*

Theorem 10.7. *Every product space of a T_1- or T_2-space is a T_1- or T_2-space.*

10.1. **Regular Spaces and T_3-spaces.**

Definition 10.8 (Regular space)**.** A topological space (X, τ) is called a **regular** space if for any closed set F in X and $a \in X \backslash F$, there are disjoint open sets U and V in X such that $F \subseteq U$ and $a \in V$. A regular T_1-space is called a T_3-space.

Theorem 10.9. *A topological space X is regular if and only if for any point a in X and any open set U containing A, there is an open set W containing a such that $\overline{W} \subseteq U$.*

Every T_3-space is T_2, but not the reverse.

10.2. **Normal Spaces and T_4-spaces.**

Definition 10.10 (Normal spaces and T_4-spaces)**.** A topological space (X, τ) is called a **normal space** if for any two disjoint closed sets A and B in X, there are disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$. A normal T_1-space is called a T_4-space.

Clearly every T_4-space is a T_3-space. However, a normal space may not be a T_1-space or a regular space and a regular space may not be normal.

Theorem 10.11. *Every metric space (X, d) is a T_4-space*

Theorem 10.12. *Every compact Hausdorff space is normal. Additionally, any compact subset A of a Hausdorff space X is closed.*

Our final theorem is **Urysohn's Lemma**.

Theorem 10.13 (Urysohn's Lemma)**.** *Let X be a normal space. Then, for any disjoint closed sets A and B in X, there exists a continuous function $f : X \to [0, 1]$ such that $F(A) = \{0\}$ and $F(B) = \{1\}$.*

Theorem 10.14 (Tietze Extension Theorem)**.** *Let X be a normal space, A a closed subset of X, and $f : A \to \mathbb{R}$ continuous. Then there is a continuous function $g : X \to R$ such that $g|_A = f$ - that is, g restricted to A is f.*

11. Hilbert Spaces

Let V be a vector space over a field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 11.1 (Inner product space)**.** A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is called an inner product if

- $\langle u, v\rangle = \overline{\langle u, v\rangle}$ for all $u, v \in V$.
- $\langle u, u\rangle \geq 0$ for all $u \in V$ with equality if and only if $u = 0$.
- $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$.

We say that V equipped with $\langle \cdot, \cdot \rangle$ is an **inner product space**.

Definition 11.2 (Induced norm)**.** If V is an inner product space with $\langle \cdot, \cdot \rangle$, we define

$$\|u\| := \sqrt{\langle u, u\rangle}$$

for all $u \in E$. The operation $\| \cdot \|$ is the **induced norm**.

Theorem 11.3 (Cauchy-Schwarz inequality)**.** *Let V be an inner product space. The*

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

Definition 11.4 (Hilbert space)**.** An inner product space which is complete with respect to the induced norm is called a **Hilbert space**.

Proposition 11.5 (Continuity of the inner product)**.** *Let V be an inner product space. Then the inner product is continuous with respect to the induced norm.*

Proposition 11.6 (Parallelogram identity)**.** *Let V be an inner product space and $\| \cdot \|$ the induced norm. Then*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$$

for all $u, v \in E$

11.1. **Projections and orthogonal complements.**

Definition 11.7 (Projection)**.** Let V be a normed space and M a non-empty closed subset of V. We define the **set of projections of** x **onto** M by

$$P_M(x) = \{m \in M \mid \|x - m\| = d(x, M)\}$$

This set is non-empty as M is closed.

Definition 11.8 (Orthogonal complement)**.** For an arbitrary non-empty subset M of an inner product space H we set

$$M^\perp := \{x \in H \mid \langle x, m\rangle = 0 \text{ for all } m \in M\}$$

We call $M^\perp$ the orthogonal complement of M in H.

Lemma 11.9 (Properties of the orthogonal complement)**.** *Suppose M is a non-empty subset of the inner product space H. Then $M^\perp$ is a closed subspace of H and $M^\perp = \overline{M}^\perp = (\text{span } M)^\perp = (\text{span } \overline{M}) \perp$*

Theorem 11.10 (Key properties of the orthogonal complement)**.** *Suppose that M is a closed subspace of a Hilbert space H. Then*

- $H = M \oplus M^\perp$

Corollary 11.11 (Dense subspace of a Hilbert space)**.** *A subspace M of a Hilbert space H is dense in H if and only if $M^\perp = \{0\}$.*

11.2. **Orthogonal systems.**

Definition 11.12 (Orthogonal systems)**.** Let H be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $M \subset H$ be a non-empty subset.

- M is called an **orthogonal system** if $\langle u, v\rangle = 0$ for all $u, v \in M$ with $u \neq v$.
- M is called an **orthonomal system** if it is orthogonal and $\|u\| = 1$.
- M is called a **complete orthonormal sytem** or **orthonormal basis** of H if it is an orthonormal system and $\overline{\text{span } M} = H$.

Theorem 11.13 (Pythagoras's Theorem)**.** *Suppose that H is an inner product space and M an orthogonal system in H. Then the following assertions are true:*

- *$M\backslash\{0\}$ is linearly independent.*
- *If (x_n) is a sequence in M with distinct terms and H is complete, then $\sum x_k$ converges if and only if $\sum \|x_k\|^2$ converges. In that case,*

$$\left\|\sum x_k\right\|^2 = \sum \|x_k\|^2$$

Theorem 11.14 (Bessel's Inequality)**.** *Let H be an inner product space and M an orthonormal system in H. Then*

$$\sum_{m \in M} |\langle x, m \rangle|^2 \leq \|x\|^2$$

for all $x \in H$. Moreover, the set $\{m \in M \mid \langle x, m \rangle \neq 0\}$ is at most countable for all $x \in H$.