Linear Algebra:

1) linear Spaces

let V non empty Set with 2 opers..

(1) Sum $a+b \in V \quad \forall a,b \in V$
(2) Scalar multiplication $\lambda a \in V, \forall a \in V, \lambda \in R$

it has <u>8 rules</u>:

there is zero element inside V

$$a+0=a \qquad 0 \in V$$

$a \in V$ a is a vector with two op.

R^n n-dimentional space

with Functions: let $R^I := \{ f \mid f: I \to R \}$

R^I ← Domain
← Range

$$(f+g)(x) = f(x) + g(x)$$

$$(\lambda f)(x) = \lambda f(x)$$

$\forall f,g \in R^I$

$\forall \lambda \in R$

Subspaces

let V be linear space, U is subspce iff:

① $x - y \in U \quad \forall x, y \in U$

② $\lambda x \in U \quad \forall \lambda \in \mathbb{R}$

so $U \leq V$ $\boxed{\text{subspace}}$

checking these two rules is suffient
cuz $x - x = 0 \in U$ it has zero-element

so $U = \{0_v\}$ is the smallest subspace of V

Linear span

$M \subseteq V$

$$\langle M \rangle = \bigcap V_j$$

V_j all possible subspace

linear span = all the possible linear combination

Generating Set:

let V linear span $G \leq V = \{\text{set of vector}\}$

is a generating set if $\langle G \rangle = V$

linear combination of the elements inside G can generate V

linear independency:

a set of vectors are L.I
iff
we can't generate any of them by linear combination of the other

so $a_1, a_2 \dots a_n \in V$ are called L.I

if $\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0 \iff$

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$$

$$\underbrace{\left(a_1 \mid a_2 \mid a_3\right)}_{A} \underbrace{\left(\quad \right)}_{b} = \vec{0}$$

$\boxed{b = 0}$

after G.E has to go to

$$\left(\begin{array}{ccc|c} 1 & & 0 & \\ & 1 & & 0 \\ 0 & & 1 & \end{array}\right)$$

the only solution is $\boxed{0}$

Basis

a set of vectors for R^n is called basis if the set is a generating set and linearly indepent.

let $A = \{a_1 \dots a_n\} \subseteq R^n$ be basis

any $b \in V$ can be written as unique combination of the basis

$$b = \sum_{i=1}^{n} \lambda_i a_i$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \underbrace{1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{basis for } R^3}$$

so $a_1 \dots a_n \iff$ minimal generating set
$\iff$ maximal linearly independent

so $R^2 =$ basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ dim $= 2$

$R^3 =$ basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ dim $= 3$

number of basis of V is dimension of V.

Representing Images:

in $f: X \to Y$ $\quad$ $X := \mathbb{R}^2$ for 2D image
$Y := \mathbb{R}$ for gray image
bitwise
$\mathbb{R}^3$ for RGB image.

$f \in Y^X$ linear space.

every Linear space has basis $= \{\psi_i\}$

$$f = \sum_{i=1} \lambda_i \psi_i \qquad \lambda_i \in \mathbb{R}$$

this continous representation for Computers

we use discret (Pixels)

$$f: \mathbb{R}^{m \times n} \to \mathbb{R}$$

2] Linear mapping & Matrices

linear mapping between V, V' (morphism)

$$f: V \longrightarrow V'$$

① $f(a+b) = f(a) + f(b)$ $\quad \forall a,b \in V$

② $f(\lambda a) = \lambda f(a)$ $\quad \forall \lambda \in R$

or we have isomorphism if f is bijective only one-to-one mapping.

example : $P: R^n \longrightarrow R$

$P(x) = x_i$ Projection on ith element.

for $A \in R^{m \times n}$ $\quad f: R^n \longrightarrow R^m$

$$f(x) = Ax \qquad x \in R^n, \; A \in R^{m \times n}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ 2y \end{pmatrix}$$

$$R^2 \longrightarrow R^3$$

$$f(x,y) = A x$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}}_{3 \times 2} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ 2y \end{pmatrix}$$

System of linear equations

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ \vdots \qquad \qquad \qquad \vdots \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn} = b_m \end{matrix}$$

$$Ax = b$$

$b \in \mathbb{R}^m$

$x \in \mathbb{R}^n$

$A \in \mathbb{R}^{m \times n}$

any linear systems has

1. no solutions
2. exactly one solutions
3. infinit many solutions

easiest way to get A^{-1}

$$A^{-1}Ax = A^{-1}b$$

$$\boxed{x = A^{-1}b}$$ solved

Inverse of matrecies:-

A of 2x2 matrice $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>exs</u> $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\det(A) = 4 - 6 = -2$

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

A, B are invertible AB is invertible

$(AB)^{-1} = B^{-1} \cdot A^{-1}$ $(AB)^{T} = B^{T} A^{T}$

if A is invertable $\Leftrightarrow$ $Ax = 0 \rightarrow x = 0$

$\Leftrightarrow$ $Ax = b \rightarrow x$ has one sol

$x = A^{-1} b$

determinate of A

$A \in \mathbb{R}^{2\times 2}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det(A) = ad - bc$

$n = 3 = \begin{pmatrix} a & e & h & a & e \\ b & f & i & b & f \\ c & g & j & c & g \end{pmatrix}$ $\det(A) = afj + eic + hbg - hfc - aig - ebj$

if A is triangular of diagonal

then $\det(A) = \prod_{i=1}^{n} a_{ii}$ multiplication of te diagonal.

Properties:

$\det(A) = \det(A^T)$
$\det(AB) = \det(A)\det(B)$

A invertible $\Leftrightarrow \det(A) \neq 0$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$ inverse determinant.

Image of $f(x)$ or A

$\operatorname{Im}(A) = \{ b \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \; f(x) = b$ or $Ax = b \}$

So liner combination of $\operatorname{col}(A)$

$x_1 \operatorname{Col}(A)_1 + x_2 \operatorname{col}(A)_2 + \dots = b$

Col space, liner Combination of columns
row space, linear combination of rows.

So $Ax = b \overset{\text{has solution}}{\Longleftrightarrow} b \in \operatorname{Im}(A)$

dim (Col space of A) = dim row space

Rank of matrices

the dimension of col or row space of A

Kernal of A:

$$Ker(A) = \{ x \in R^n \mid Ax = 0 \}$$

Proper $A \in R^{m \times n}$

$rank(A) = 0 \iff A = (1)$ $ran(A) = 1$

$rank(A) \leq min(n, m)$

$n - rank(A) = dim(Ker(A))$

Colums

$dim(Ker(A)) = 0 \iff Ker(A) = \{0\}$

if $A \in R^{n \times n}$ $\boxed{m = n}$ then $rank(A) = n$

then A has Full rank

and A is invertible.

A is invertible $\iff det(A) \neq 0$

$\iff rank(A) = n$ Full rank

$\iff$ A square $m = n$

$\iff dim\ Ker(A) = 0$

$\iff Ker(A) = \{0\}$

$\iff$ colums are L.I (basis)

$\iff$ row are L.I (basis)

ex)

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

1) $\det(A) = 4 - 6 = -2$ invertable

2) $\text{rank}(A) = 2$

3) $\dim(\text{Ker}(A)) = 0 \Rightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right)$

$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$ $\underbrace{\boxed{x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}}_{\text{Kernal}}$

4) row / col are linealy independent.

ex2)

$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\det(A) = 4 - 4 = \boxed{0}$ not inver

$\text{rank}(A) = 1$ not Full Rank.

$n - \text{rank} = 2 - 1 = 1$ dim of Kernal

$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} -2x_2 \\ x_2 \end{array}$

$\text{Ker}(A) = \left\{ x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \middle| \; x_2 \in \mathbb{R} \right\}$

$\langle \text{Ker}(A) \rangle = \mathbb{R}^1$ $\dim = 1$

A is not invertible.

2) length / distance / angels

1) Norms

let V linear space $\quad \|.\| : V \to R$

Properties

① $\|x\| = 0 \iff x = 0 \qquad \underline{\|x\| \geq 0}$

② $\|\lambda x\| = \lambda \|x\|$

③ $\|x+y\| \leq \|x\| + \|y\|$

$\forall x, y \in V \quad , \forall \lambda \in R$

$\boxed{\text{P\_norms}}$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

$\|x\|_2$ real world measurment

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \|a\| = 1$$

$\boxed{\text{so}}$ distance between $x, y \in V$

$$d(x,y) = \|y - x\|_2$$

ex) $\begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$d(x,y) = \left\| \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\|_2 = \sqrt{4+1} = \boxed{\sqrt{5}}$$

Euclidean distance.

Scalar product (dot / inner product)

$$\langle \cdot , \cdot \rangle := V \longrightarrow \mathbb{R}$$

Propirties:

1. $\langle x + x' , y \rangle = \langle x , y \rangle + \langle x' , y \rangle$

   $\langle x , \lambda y \rangle = \lambda \langle x , y \rangle$

2. $\langle x , y \rangle = \langle y , x \rangle$

3. $\langle x , x \rangle \geq 0 \qquad \langle x , x \rangle = 0 \Leftrightarrow \boxed{x = 0}$

standard scalar product is

$$\langle x , y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \odot \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 + 4 = 6$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 6 = x^T y$$

Norms induced by scalar Products

if V has scalar product

then $\|x\| = \sqrt{\langle x , x \rangle}$

Scalar Product

Cauchy-Schwarz inequality

$$\langle x,y\rangle^2 \leq \langle x,x\rangle\langle y,y\rangle \quad \forall x,y \in V$$

$\boxed{\text{Angle}}$ let V linear space. has $\langle \cdot,\cdot\rangle$ and norm induced by scalar product

$$\langle x,y\rangle = \|x\| \cdot \|y\| \cdot \cos\varphi$$

φ is angle between x, y

if $\varphi = 90^\circ \Rightarrow \cos 90^\circ = 0$

so $\langle x,y\rangle = 0 \iff x \perp y$

$$\cos\varphi = \frac{\langle x,y\rangle}{\|x\|\,\|y\|}$$

$x = \begin{pmatrix}1\\0\end{pmatrix}$ $y = \begin{pmatrix}1\\1\end{pmatrix}$

$$\cos\varphi = \frac{2}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \varphi =$$

$$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \boxed{45^\circ}$$

Orthogonal Projection:

$P_u : R^n \longrightarrow R^n$ Project $x \in R^n$ on unit vector u $\|u\| = 1$

$$P_u(x) = \langle x, u \rangle u$$

$$P_u(u) = u$$

$$P_u(x) = P_u(x_{\parallel} + x_{\perp}) = P_u(x_{\parallel}) + P_u(x_{\perp})$$

$$\varphi = 0 \qquad \varphi = 90^\circ$$

$$= \langle x_{\parallel}, u \rangle u + \langle x_{\perp}, u \rangle u$$

$$= \underbrace{\|x_{\parallel}\|} \cdot \underbrace{\|u\|}_{1} \cdot u + 0 u$$

$$= \|x_{\parallel}\| \cdot u = x_{\parallel}$$ Paraelll to u.

* ART (Algebric Reconstruction Technique)

$$Ax = b$$

$A \in R^{m \times n}$ $x \in R^n$ $b \in R^m$

a_j = i-th row of A

then $a_j x = b_j$ (affain hyper plan)

$$H_j = \{ x \in R^n \mid \langle a_j, x \rangle = b_j \}$$

* hyper plane is a subspace of dimension n-1 divides R^n in half.

So our solution $x = \bigcap_{j=1}^{m} H_j$

hyper plone is a subspace has zero element
affain hyperplain is subset does n't have 0

take $x = x_0$ any starting point $x_0 = 0$

for each iteration we project on all the affain hyper planes.

if $Ax = b$ has solution ART will converge to the solution

So to compute the solution

for each hyperplane we project x to i !

So let's take x_0 init guess.

then $y = x_0$

for First H.P $y' = y + \frac{b_1 - \langle y, a_1 \rangle}{\|a_1\|_2^2} \cdot a_1$

second $=$ $y'' = y' + \frac{b_2 - \langle y', a_2 \rangle}{\|a_2\|_2^2} \cdot a_2$

third $=$ $y''' = y'' + \frac{b_3 - \langle y'', a_3 \rangle}{\|a_3\|_2^2} a_3$

$x_1 = y'''$ that's one iteration

$y = x_1$

$y' = y + \frac{b_1 - \langle y, a_1 \rangle}{\|a_1\|_2^2} a_1$

$y'' = y' + \frac{b_2 - \langle y', a_2 \rangle}{\|a_2\|_2^2} a_2$

Orthogonal

$x, y \in R^2$ are orthognal $\langle x, y \rangle = 0$

Orthomonal basis:

$B = \{ b_i \}$ basis of R^n

if all pair wise are orthogonal and len $= 1$

$$\langle b_i, b_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

for any $x \in R^n$ and b_i basis of R^n

and b_i aro ONB.

then $x = \langle x, b_1 \rangle b_1 + \cdots + \langle x, b_n \rangle b_n$

Orthonomal matrices: $\boxed{\text{Square matrix}}$

$A^{-1} = A^T$ $\quad$ $A^T A = A A^T$ orthogonal matrix

A is orthognal $\rightarrow$ $\left.\begin{array}{l} \text{rows are ONB} \\ \text{col are ONB} \end{array}\right\}$ full rank

A, B AB are orthogonal.

$|\det(A)| = 1$

A is length preserving $\boxed{\|Ax\| = \|x\|}$

Solving linear Systems:

$$Ax = b$$

Solving for x

① by $A^{-1} \Rightarrow x = A^{-1}b$ exactly one sol.
leads to numerical issues.

② decompustion

③ itereative method lik ART $y = y + \frac{b_j - \langle a_j, y \rangle}{\|a_j\|_2^2} a_j$

Decomposition:

L: lower U: upper

① Cholosky decomposition $A = LL^T$

A symmetric $A = A^T$

A positive defient . eigenvalue > 0 $\langle x, Ax \rangle > 0$ $\forall x \in \mathbb{R}^n$

$A = LL^T$

L stricktly positive diagonal matrix

② $L^T x = z$ ① $Lz = b$ $\frac{n^3}{3}$ FLOPs

numerical stapility issues.

if A positive semi defient then

stricktly positive diagonal entries drops

② LU decomposition

A is invertable and

A^T diagonally dominate for each row

$$|a_{ii}| \geq \sum_{j \neq i} a_{ij}$$

$$A = LU \qquad \frac{2n^3}{3} \text{ FLOPS}$$

$$LUx = b \Rightarrow \begin{array}{l} Ux = z \\ Lz = b \end{array}$$

if L or U has only ones as diagonal the decomposition is unique.

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③ LUP decomposition.

A is invertible: $PA = LU$

$$LUx = Pb \Rightarrow \begin{array}{l} Lz = Pb \\ Ux = z \end{array} \qquad \frac{2n^3}{3}$$

Computed in Place. numerically stabel.

③ QR decomposition:

A invertable. $A = QR$

Q is orthogonal

R Upper triangular with positive diagonal entries

$$QRx = b \qquad \frac{2n^3}{3} \text{ FLOPs.}$$

$$Q^TQRx = Q^Tb$$

$$Rx = Q^Tb \qquad \text{numerically stable.}$$

if $A \in \mathbb{R}^{m \times n}$ $m > n$

Thin QR decombosition

$Q \in \mathbb{R}^{m \times m}$ orthognal

$R \in \mathbb{R}^{m \times n}$ $= \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$ $R_1 \in \mathbb{R}^{n \times n}$

$$A = (Q_1 \; Q_2)\begin{pmatrix} R_1 \\ 0 \end{pmatrix} = Q_1 R_1$$

$Q_1 \in \mathbb{R}^{m \times n}$

$$A = \underset{n \times n}{Q_1} \underset{n \times n}{R_1}$$

if rank$(A) = n$ and R_1 has positive diagonal

then $Q_1 R_1$ are unique.

$$\boxed{R_1 x = Q_1^T b}$$

④ SVD

$A \in R^{m \times n}$ singular value decomposition

$m \geq n$

$$A = U \Sigma V^T$$

$U \in R^{m \times n}$ orthognal

$V \in R^{n \times n}$ orthognal

$\Sigma \in R^{n \times n}$ diagonal matrix.

$= diag(\sigma_1, \sigma_2 --- \sigma_n)$ Called singular values

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4 \geq ---\geq \sigma_n \geq 0$

for $m < n$

$U \in R^{m \times m}$ orthogonal

$\Sigma \in R^{m \times m}$ diagonal

$V \in R^{n \times m}$

let $A = U \Sigma V^T$

$$A^T A = (U \Sigma V^T)^T (U \Sigma V^T)$$

$n \times m$ $n \times n$ nn

$$= V \Sigma \underbrace{U^T U}_{I_n} \Sigma V^T$$

$=$

$$= V \Sigma^2 V^T$$

singular values of A are sequare root of eigen values of $A^T A$

$Ax = b \Rightarrow U\Sigma V^T x = b$

$x = \underbrace{V\Sigma^{-1}U^T}\, b$ does not exisit

$\Sigma^{-1} = diag(\sigma_1^{-1}, \ldots\ \sigma_n^{-1})$

using u_i Col(U) v_i col(V)

$$x = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i$$

if $\sigma_i = 0$ leave them out

$$x^* = \sum_{i=1}^{ran(A)} \frac{u_i^T b}{\sigma_i} v_i$$

this is solution least sequere problem and pesudo inverse

$$x^* = \min_x \|Ax - b\|$$

pesudo inverse

$$A^+ = \sum_{i=1}^{ran(A)} v_i \sigma_i^{-1} u_i^T$$

if A is ill conditioned

we use

Truncated SVD to stableise the x^* we approximate it

$\boxed{\text{very small } \sigma_i}$

we stop at K $\quad x^* = \sum_{i=1}^{K} \frac{u_i^T b}{\sigma_i} v_i$

Computing SVD costly and need lots of memory.

Least sequare problem:

$A \in R^{m\times n}$ $\quad b \in R^{m}$ $\quad x \in R^{n}$

we need to $\min\limits_{x} \|Ax-b\|$ or $\min\limits_{x} \|Ax-b\|^2$

in case of over determinate system (no solution)

we need to find x that min the error

thus $\min\limits_{x} \|Ax-b\|$

Orthognal Complement

let $U \leq V$ so

$$U^{\perp} := \{x \in R^n \mid \langle x,u\rangle = 0 \ \forall u \in U\}$$

to $\min \|\underbrace{b-\overset{Im(A)}{Ax}}_{\text{vector}}\| \iff \langle b-Ax, \underbrace{Ax'}_{Im(A)}\rangle \ \forall x' \in \mathbb{R}^n$

$\iff \langle A^T(b-Ax), x'\rangle = 0$

So

$$A^T(b-Ax) = 0$$

$$A^TAx = A^Tx$$

has uniq solution

if A^TA is invertible so $\boxed{x = (A^TA)^{-1}A^Tb}$

$\text{rank}(A^TA) = n$

Pesodu inverse

$$(A^TA)^{-1}A = A^+ \Rightarrow \boxed{x = A^+ b}$$

if A is invertible then $\boxed{A^+ = A^{-1}}$

SVD & Pesudo inverse

$$A^+ = \sum_{i=1}^{\text{rank}(A)} v_i \sigma_i^{-1} u_i^T$$

Golub method to find least sequare problem:

let $A \in R^{m \times n}$ & $m \geq n$ $\underline{\text{rank}(A) = n}$

$Q \in R^{m \times m}$ orthogonal

$$Q^T A = \begin{pmatrix} R \\ 0 \end{pmatrix} \quad \text{and} \quad Q^T b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} R \\ 0 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

So $Rx = b_1$ $\underline{\text{backward sub}}$

R is invertible

r: residual $:= b - Ax$

$$\boxed{\|r\| = \|b_2\|}$$

Eigenvalues / Eigenvectors

let $A \in \mathbb{R}^{n \times n}$, a vector $x \in \mathbb{C}^n$, $x \neq 0$

is called an eingenvector if

$$Ax = \lambda x$$

for some $\lambda \in \mathbb{C}$ where λ is called eigenvalue.

if $A = A^T$ then all eigenvalues of A are real.

Calculate eigenvalues using Characteristic polynomial.

$$\det(\lambda I_n - A) = 0$$

then calculate eigenvector of each eigenvalue

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$\underline{(A - \lambda I_n)x} = 0$$

if A is triangular or diagonal then eigenvalues are the diagonal entries.

A invertible $\Leftrightarrow \det(A) \neq 0$
$\Leftrightarrow$ non of eigenvalues $= 0$

Multiplicity:

① Algerbraic multiplicity:

the numer of fators of each eigenvalu.
if $\lambda_1 = 1$ $\lambda_2 = 1$ $\lambda_3 = 2$

then λ_1 has $A.M = 2$
λ_3 has $A.M = 1$

② Geometric multiplicity:

for eigenvalue we find eigenvector and the dim of eigenspan is G.M

Similarity:

A & B are similar if $\exists P \in \mathbb{R}^{n \times n}$ invertible

$$\boxed{A = P^{-1} B P}$$

this is only chang in basis.

A and B has the same
① eigenvalues
② eigenvectors
③ rank
④ det
⑤ G.M + A.M

Diagonalizable:

A is diagonalizable if $\exists P \in \mathbb{R}^{n \times n}$ invertibl

wher $P^{-1}AP$ = $\overset{\text{similar}}{\text{diagonal matrix}}$ ← diag entries

this is a way to find eigenvalues easily.

for $A \in \mathbb{R}^{n \times n}$

for all eigenvalues $G.M \leq A.M$

if $GM < A.M$ A is defective.

A is diagonizable ⇔ A has n L.I eigenv.
⇔ non defective

if $A = A^T$ → then all eingenvalues are real
→ A is diagonizable
→ P is orthognal $dia = P^T A P$

A is normal → $(A^T A = AA^T)$
⇔ A diagonizable with P orthogonal

Practical Approaches to find eigenvalues/vectors

let $A = A^T$ symmetric.

if we know x is eigenvector
then we min

$$\| Ax - \lambda x \| = \min_{\lambda}$$

So $$x^T x \lambda = x^T A x \quad \text{(normal equation}$$

then $$\lambda = \frac{x^T A x}{x^T x} \Rightarrow r(x)$$

$r(x)$ is called Rayleigh Quotient.

$$x^T x = \langle x, x \rangle = \|x\|_2^2$$

So in the iterative algo to avoid dividing
we unify the $\frac{x_0}{\|x_0\|}$ so

$$r(x_0) = x_0^T A x_0$$

QR Algoritm:

let $A \in \mathbb{R}^{n \times n}$

the $A^0 = A$

for $K = 1, 2 \ldots$

$$Q^K R^K = A^{K-1}$$

$$A^K = R^K Q^K$$

A^K is similere to A^{K-1}

$$A^K = R^K Q^K = Q^{K^T} Q^K R^K Q^K$$

$$\underbrace{A^K = Q^{K^T} A^{K-1}}_{\text{similar}} \underbrace{Q^K}_{\text{orthrongonl}}$$

so same eigenvalues / vectors.