February 8

Wednesday, February 8, 2023

(b) Let $A \in \mathbb{R}^{2\times 2}$ be a symmetric matrix and let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x) = x^T Ax$. In which direction does f decrease the most at the point $(1, 1)$?

- Steepest descent direction : $-\nabla f(x) = -2Ax$
- We want to find $-\nabla f(1,1) = -2 \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} [$

$f(x) = x^T A x$. We want to find $\nabla f(x)$.

<u>Method 1:</u> $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x) \right)$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $f(x) = x^T A x = [x_1 \; x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} [$

$a_{21} = a_{12}$

$$= [x_1 \; x_2] \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}}_{Ax} = x_1 \cdot (a_{11}x$$

$$= a_{11}x_1^2 + \underline{\underline{a_{12}}} x_1 x_2 + \underline{\underline{a_{21}}} x_2 x_1 + a_{22} x$$

$$f(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2.$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x_1}(x) &= 2a_{11}x_1 + 2a_{12}x_2 = 2(a_{11}x_1 + a_{12}x_2) \\ \frac{\partial f}{\partial x_2}(x) &= 2a_{12}x_1 + 2a_{22}x_2 = 2(a_{12}x_1 + a_{22}x_2) \end{aligned}\right\} \Rightarrow$$

- If $A \in \mathbb{R}^{n\times n}$, $f(x) = x^T A x = [x_1 \, x_2 \ldots x_n] \begin{bmatrix} a \\ a \\ \vdots \end{bmatrix}$

$$Ax = \begin{bmatrix} \sum_{j=1} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{bmatrix} \quad f(x) = x' A x = x \cdot (Ax)$$

$$= x_i \cdot \sum_{j=1}^{n} a_{ij} x_j$$

$$= \sum_{i=1}^{n} x_i \Big(\sum_{j=1}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} x_i$$

$$f(x) = a_{11} x_1^2 + x_1 \underbrace{\sum_{j=2}^{n} a_{1j} x_j}_{\text{no } x_1} + x_2 a_{21} x_1 + x_2 \underbrace{\sum_{j=2}^{n} a_{2j} x_j}_{\text{no } x_1} + \dots +$$

$$\frac{\partial f}{\partial x_1}(x) = 2 a_{11} x_1 + \sum_{j=2}^{n} a_{1j} x_j + \sum_{j=2}^{n} x_j \underbrace{a_{j1}}_{= a_{1j}} = 2 a_{11} x_1 +$$

$$= 2 \sum_{j=1}^{n}$$

$$\frac{\partial f}{\partial x_i}(x) = 2 (Ax)_i \Longrightarrow \nabla f(x) = 2Ax.$$

Method 2: $A \in \mathbb{R}^{n \times n}$, $f(x) = x^T A x$, ∇f

Taylor expansion: $\quad f(x+h) \approx \underbrace{f(x) + \boxed{\nabla f(x)} \cdot h +}_{2^{nd} \text{ order}}$

$$f(x+h) = (x+h)^T A (x+h)$$

$$= (x+h)^T A x + (x+h)^T A h$$

$$= x^T A x + \underline{h^T A x} + \underline{x^T A h} + h^T A h$$

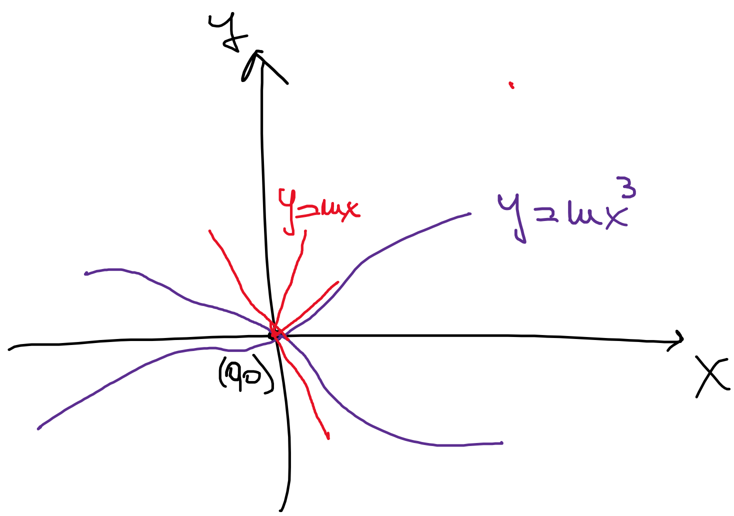
$$= f(x) + \boxed{2(Ax)} \cdot h + \frac{1}{2} h^T \boxed{2A}\, h$$

Hence $\nabla f(x) = 2Ax$, $H(x) = 2A$.

- $f : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$Df(x) = \nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_1}\right) \in \mathbb{R}^n.$$

$$H(x) = D\nabla f(x) = J(\nabla f(x)) = \begin{bmatrix} \nabla\left(\frac{\partial f}{\partial x_1}\right) \\ \vdots \\ \nabla\left(\frac{\partial f}{\partial x_n}\right) \end{bmatrix} =$$



3. Study the continuity and the differentiability (partial and total) of the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$\nabla f(0,0)$

Take $y = mx$. $f(x,y) = \frac{mx^4}{}$

$x^2+m^6x^6$

$y=\sqrt[3]{x}$

Try $x=my^3$: $f(x,y)=\dfrac{my^6}{m^2y^6+y^6}=\dfrac{m\cancel{y^6}}{\cancel{y^6}(m^2+1)}=\dfrac{m}{m...}$

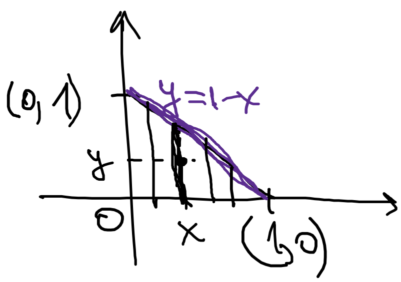
$\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y) \Rightarrow f$ is discont...

$\Rightarrow f$ is <u>not</u> dif...

Differentiable $\Longrightarrow$ Continuous.

Not Continuous $\Longrightarrow$ Not differentiable.

9. Let D be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Compute $\displaystyle\iint_D (x^2-y^2)\,dx\,dy$.

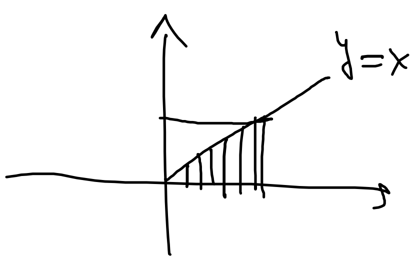


$D=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le 1-x\}$.

$$\iint_D (x^2-y^2)\,dx\,dy \overset{\text{Fubini}}{=} \int_0^1\int_0^{1-x}(x^2-y^2)\,d...$$

$y=1-x$, $(x,1-x)$

$$=\int_0^1\left[x^2(1-x)-(...\right.$$

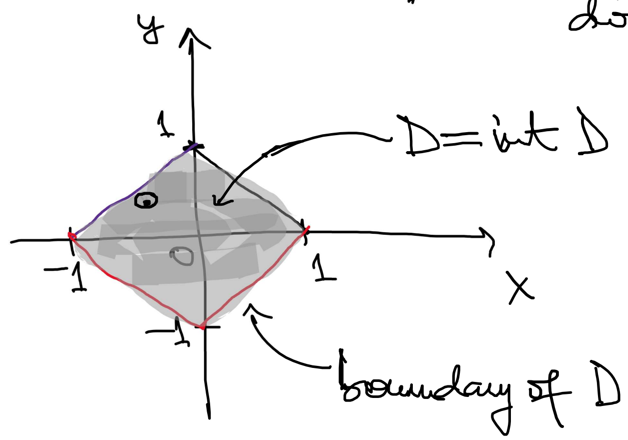


$\text{Area}(D)=\iint_D 1\,dx\,dy$

Idea: consider $|x|+|y|=1$ (boundary of the domain)

(c) $\{(x,y) \in \mathbb{R}^2 \mid |x|+|y|<1\} = D$

$B(0,r) = \{x \in \mathbb{R}$
↳ open ball



$\bar{B}(0,r) = \{x \in$
↳ closed

$x>0,\ y>0 \Rightarrow x+y=1,\ y=1-x$

$x<0,\ y>0 \Rightarrow -x+y=1,\ y=1+x$

$cl(D) = D \cup \{$

$cl(D) = \{(x,y)$

Theorem 9.12 (Chain rule). Let $g : \mathbb{R}^n \to \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^p$ differentiable at x and $g(x)$, respectively. Then

$$\boxed{D(f \circ g)(x) = Df\big(g(x)\big)Dg(x).}$$

In terms of matrix dimensions: $[\quad]_{p\times n} = [\quad]_{p\times m}[\quad]_{m\times n}$.

$f(g($

$g=(g_1,g_2)$, $g : \mathbb{R}^2 \to \mathbb{R}^2$, $Dg(x) \in \mathbb{R}$

6. Consider $f : \mathbb{R}^2 \to R$ and $x = g_1(u,v)$, $y = g_2(u,v)$, i.e. $f(x,y) = (f \circ g)(u,v)$. Prove that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial u}}_{=\frac{\partial g_1}{\partial u}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial v}}_{=\frac{\partial g_1}{\partial v}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

$$\frac{\partial x}{\partial u} = \frac{\partial g_1}{\partial u}$$

$$\boxed{\begin{aligned} x &= g_1(u,v) \\ y &= g_2(u,v) \end{aligned}}$$

$(u,v) \longmapsto (x,y) \longmapsto f(x,y) = f$

$$Dg(u,v) = \begin{bmatrix} \nabla g_1(u,v) \\ \nabla g_2(u,v) \end{bmatrix} = \begin{bmatrix} \underbrace{\frac{\partial g_1}{\partial u}}_{\frac{\partial x}{\partial u}} & \frac{\partial g_1}{\partial v} \\ \underbrace{\frac{\partial g_2}{\partial u}}_{\frac{\partial y}{\partial u}} & \frac{\partial g_2}{\partial v} \end{bmatrix}_{2\times 2}$$

$$\nabla f(x,y) = \nabla f(x,y) = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right]_{1\times 2}$$

$$\nabla f(x,y)\, \nabla g(u,v) = \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = [$$

$= [$

$=$

$x > 0$

$$\sum_{n\geq 0} \frac{x^n}{n!} \qquad \text{Ratio Test:} \quad \frac{x^{n+1}}{(n+1)!} \cdot$$

$$\sum (-1)^{n+1} \frac{x^{2n+1}}{2n+1}, \qquad \sum \frac{|x|^{2n+1}}{2n+1}, \quad \text{Ra}$$

$\Rightarrow$ If $|x| < 1$ ~~then~~ the

$\Rightarrow$ Radius of converges

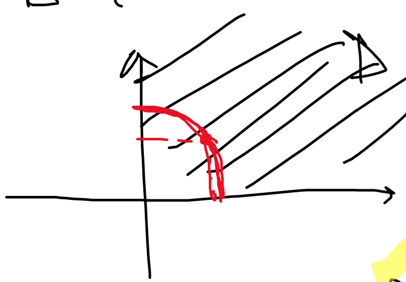
8. Compute the following integrals:

(a) $\int_0^\infty e^{-2x^2}\,dx. = \frac{1}{2}\int_{-\infty}^{\infty} e^{-2x^2} dx$ (b) $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy.$

$$I = \int_0^\infty e^{-2x^2} dx, \quad I^2 = \int_0^\infty e^{-2x^2} dx \cdot \int_0^\infty e^{-2x^2} dx = \int_0^\infty$$

$$\iint e^{-2x^2} e^{-2y^2} dx\,dy =$$

$D = \{(x,y) \in \mathbb{R}^2 \mid x, y > 0\}$ $\quad = \iint_D e^{\ldots} \ldots dx\,dy$



Polar coordinates: $\quad x^2+y^2=r^2$, $x = r\cos\theta > 0$, $y = r\sin\theta > 0$, $r \in [\ldots$

$$\iint_D e^{-2(x^2+y^2)}\,dx\,dy = \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-2r^2}\, r\, d\theta\, d\theta$$

$$\left(e^{-2r^2}\right)' = -4r\,e^{-2r^2} \qquad = \int_0^\infty r\,e^{-2r^2}\,dr \cdot \underbrace{\int_0^{\frac{\pi}{2}} d\theta}_{=\frac{\pi}{2}} = \frac{\pi}{2}\cdot\left(-\frac{1}{4}\ldots\right.$$

$$I^2 = \frac{\pi}{8} \Longrightarrow I = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

• $I = \frac{1}{2}\int_{-\infty}^{\infty} e^{-2x^2}\,dx$, $\quad I^2 = \frac{1}{4}\iint_{\mathbb{R}^2} e^{-2(x^2+\ldots}$

$x = r\cos\theta$

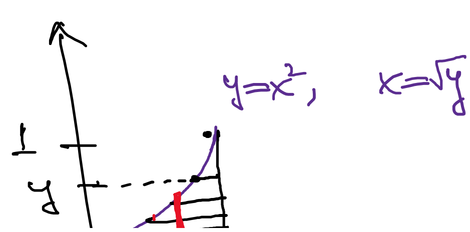
$y = r\sin\theta$

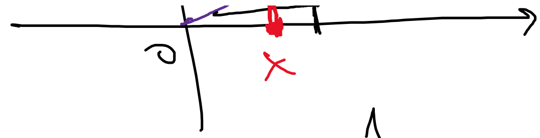
$r \in [0,\infty)$, $\theta \in [0, 2\pi]$

$$= \frac{1}{4}\int_0^\infty \int_0^{2\pi} e^{-2r^2}\, r\, d\theta\, dr$$

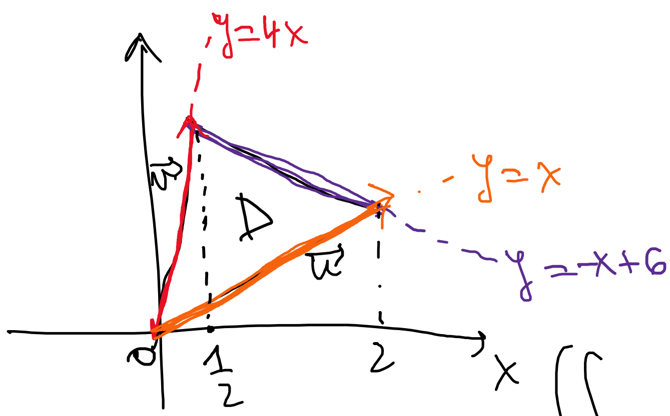
$$= \frac{\pi}{2}\cdot \int_0^\infty r\,e^{-2r^2}$$

(b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dx\,dy. = I$ $\quad D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1,\ \boxed{\sqrt{y} \le \ldots}$





$$I = \underbrace{\int_0^1 \int_0^{x^2} e^{x^3}\, dy\, dx}_{e^{x^3}\cdot x^2} = \int_0^1 e^{x^3}\cdot x^2\, dx = !$$



$0 \le x \le \frac{1}{2}$, $2x \le y \le 4x$

$\frac{1}{2} < x \le 2$, $2x \le y \le -x+6$.

$$\iint_D \ldots dx\,dy = \int_0^{\frac{1}{2}} \int_{2x}^{4} \ldots dy\, dx + \int_{\frac{1}{2}}^{2}$$