Teorema 1.7

- f função real definida e contínua em V_ξ
- $f(\xi) = 0$ (ξ é solução de $f(x) = 0$)
- f' está definida e é contínua em V_ξ ($f \in C^1(V_\xi)$)
- $f'(\xi) \neq 0$

Alguém falou
RELAXAÇÃO?

ENTÃO,

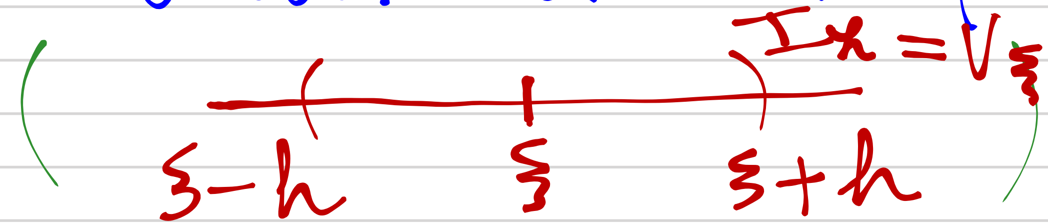
- $\exists\ \lambda \in \mathbb{R}$ e $\delta > 0$ tais que
- $x_k \to \xi, \quad \forall\ x_0 \in I_\delta = [\xi - \delta, \xi + \delta]$
  sendo $\{x_k\}$ gerada pela iteração
  $$x_{k+1} = x_k - \lambda f(x_k), \quad k = 0, 1, \ldots$$

Demonstração:

- Como $f'(\xi) \neq 0$, supor s.p.g (sem perda de generalidade),

$f'(\xi) = \alpha > 0$ (I)

• Como f' é contínua em ξ (em V_ξ), então,



$\forall \varepsilon = \frac{\alpha}{2} > 0, \ \exists \ \delta_\varepsilon > 0$ tal

$\to |f'(x) - f'(\xi)| < \varepsilon$ sempre que $|x - \xi| < \delta_\varepsilon$.

$\delta < \min\{\delta_\varepsilon, h\}$ tq $I_h = V_\xi$

Ou equivalentemente:

$$\underbrace{f'(\xi) - \varepsilon < f'(x)}_{\mathrm{I}} < f'(\xi) + \varepsilon$$

sempre que

$$\xi - \delta < x < \xi + \delta$$

$$(x \in I_\delta)$$

Die II

$$f'(x) > \underbrace{f'(\xi)}_{\alpha} - \underbrace{\varepsilon}_{\frac{\alpha}{2}} = \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}$$

$\forall x \in I_\delta$

$$\Longrightarrow \boxed{f'(x) > \frac{\alpha}{2} \; ; \; \forall x \in I_\delta} \quad (\mathrm{III})$$

• Como f' é contínua em I_δ,

$I_\delta = [\xi - \delta, \xi + \delta]$, pelo Teo.

de Weierstrass, $\exists\, M > \frac{\alpha}{2} > 0$.

tal que

IV $$\boxed{f'(x) \leq M, \; \forall x \in I_\delta}$$

De III e IV

$$\boxed{\frac{\alpha}{2} < f'(x) < M \,;\ \forall x \in I_\delta}\ .$$

V

Multiplicando (-1):

$$-\frac{\alpha}{2} > -f'(x) > -M \,;\ \forall x \in I_\delta$$

$$-M < -f'(x) < -\frac{\alpha}{2}$$

Multiplicando por $\lambda \in \mathbb{R}$; $\lambda > 0$:

$$-\lambda m < -\lambda f(x) < -\lambda \frac{\alpha}{2}$$

adicionando $(+1)$

$$\underbrace{1-\lambda m}_{-\theta} < \underbrace{1-\lambda f(x)} < \underbrace{1-\lambda\frac{\alpha}{2}}_{\theta} \quad (VI)$$

$x \in I_\delta$

$\exists\, \theta \in \mathbb{R}$, $0 < \theta < 1$ tq $\begin{cases} 1-\lambda\frac{\alpha}{2} = \theta \\ 1-\lambda m = -\theta \end{cases}$?

$$\begin{cases} 1 - \lambda m = -\theta & (1) \\ 1 - \dfrac{\lambda \alpha}{2} = \theta & (2) \end{cases}$$

$$(1) - (2): \quad \frac{\lambda \alpha}{2} - \lambda m = -2\theta$$

$$\lambda \left(\frac{\alpha}{2} - m \right) = -2\theta$$

$$\lambda = \frac{-2\theta}{\frac{\alpha}{2} - m} = \frac{-2\theta}{\frac{\alpha - 2m}{2}}$$

$$\lambda = \frac{-4\theta}{\alpha - 2m} = \frac{4\theta}{2m - \alpha}$$

$$\boxed{\lambda = \frac{4\theta}{2m-\alpha}} \quad (3)$$

$$(3) \rightarrow (2) \quad 1 - \frac{\alpha}{\cancel{2}} \cdot \frac{\cancelto{2}{4}\theta}{(2m-\alpha)} = \theta$$

$$1 - \frac{\alpha \cdot 2\theta}{2m-\alpha} = \theta \Rightarrow$$

$$\theta + \frac{2\alpha}{2m-\alpha} \cdot \theta = 1$$

$$\theta \left[1 + \frac{2\alpha}{2m-\alpha} \right] = 1$$

$$\theta = \frac{1}{1 + \frac{2\alpha}{2m-\alpha}} = \frac{1}{\frac{2m-\alpha+2\alpha}{2m-\alpha}}$$

$$\boxed{\theta = \frac{2m - \alpha > 0}{2m + \alpha > 0}}$$

4

$$m > \frac{\alpha}{2} \Rightarrow m - \frac{\alpha}{2} > 0 \Rightarrow$$

$$\frac{2m - \alpha}{2} > 0 \Rightarrow 2m - \alpha > 0$$

ainda $2m - \alpha < 2m + \alpha \Rightarrow \boxed{0 < \theta < 1}$

④ $\hookrightarrow$ ③

$$\boxed{\lambda = \frac{4}{2m+\alpha} > 0}$$

Assim, (VI) torna-se

$$-\theta < 1 - \lambda f'(x) < \theta \; ; \; \forall x \in I_S$$

$$0 < \theta < 1 \iff |1 - \lambda f'(x)| < \theta < 1 \quad \text{(VII)}$$

Escrevendo, $g(x) = x - \lambda f(x)$,

temos $g'(x) = 1 - \lambda f'(x)$ e

então,

$$|g'(x)| = |1 - \lambda f'(x)| < 1 ; \forall x \in I_S$$

(VII)

Pelo Teo 1.5.

$x_k \to \xi$ sempre $x_0 \in I_S$. $\blacksquare$

Ordem | Taxa de Convergência

$$\frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} \approx C \quad ; \quad \forall k > k^*$$

$$|x_{k+1} - x^*| \approx C \cdot |x_k - x^*|^p$$

Erro:

$$e_{k+1} = |x_{k+1} - x^*| \checkmark$$

(teórico)

$$\frac{e_{k+1}}{e_k} = \frac{\overbrace{|x_{k+1} - x^*|}}{|x_k - x^*|} \approx$$

$$\frac{e_{k+1}}{e_k} \approx \frac{\cancel{C} \cdot |x_k - x^*|^p}{\cancel{C} \cdot |x_{k-1} - x^*|^p} = \frac{e_k^{\,p}}{e_{k-1}^{\,p}}$$

$$\log\left(\frac{e_{k+1}}{e_k}\right) \approx \log\left(\frac{e_k}{e_{k-1}}\right)^p$$

$$\boxed{p \approx \frac{\log\left(\frac{e_{k+1}}{e_k}\right)}{\log\left(\frac{e_k}{e_{k-1}}\right)}}$$

Na prática, $\boxed{\bar{e}_{k+1} = |x_{k+1} - x_k|}$

$$\bar{e}_{k+1} = |x_{k+1} - x^* + x^* - x_k|$$

$$\underbrace{\bar{e}_{k+1}}_{<\varepsilon} \leq \underbrace{|x_{k+1} - x^*|}_{<\varepsilon/2} + \underbrace{|x_k - x^*|}_{<\varepsilon/2}$$

Teorema 1.8

- f e f'' funções contínuas em $I_\delta = [\xi - \delta, \xi + \delta]$ $(\delta > 0)$
- $f(\xi) = 0$ e $f''(\xi) \neq 0$
- $\exists\ A > 0$ constante tal que

$$\frac{|f''(x)|}{|f'(y)|} \leq A, \quad \forall x, y \in I_\delta \qquad (\clubsuit)$$

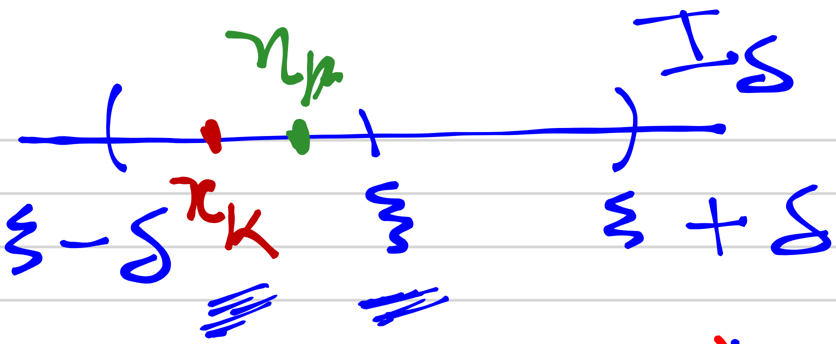
(subentende $f'(y) \neq 0$ para $y \in I_\delta$)

- $|x_0 - \xi| \leq h,\ h \leq min\{\delta, 1/A\}$

ENTÃO,

- a sequencia do Método de Newton **converge** **quadraticamente** para ξ.

Demonstração:



• Considere x_K tal que $|x_K - \xi| < h$, onde $h = \min\{\delta, 1/A\}$.

$h \le \delta$

$h \le 1/A$

$\Rightarrow x_K \in I_\delta$.

• Desenvolva f em <u>série</u> de Taylor em torno de x_k:

$$0 = f(\xi) = f(x_k) + (\xi - x_k) \cdot f'(x_k) + \frac{(\xi - x_k)^2}{2} \cdot f''(\eta_k),$$

para algum η_k entre ξ e x_k

$\Rightarrow \eta_k \in I_\xi$

$$f(x_k) + (\xi - x_k)\cdot f'(x_k) =$$

$$= -\frac{(\xi - x_k)^2}{2} \cdot f''(\eta_k)$$

$\div f'(x_k)$ [$f'(x_k) \neq 0$ pois $x_k \in I_\xi$ hipótese]

$$\frac{f(x_k)}{f'(x_k)} + (\xi - x_k) = -\frac{(\xi - x_k)^2}{2} \cdot \frac{f''(\eta_k)}{f'(x_k)}$$

Do Mét. de Newton :

$$\cancel{x_k} - x_{k+1} + \xi - \cancel{x_k} = -\frac{(\xi - x_k)^2}{2} \frac{f''(\eta_k)}{f'(x_k)}$$

$$\left| x_{k+1} - \xi \right| = \frac{1}{2} \underbrace{\left| \xi - x_k \right|^2}_{} \cdot \left| \frac{f''(\eta_k)}{f'(x_k)} \right|$$

(**)

$$\underbrace{|x_{k+1}-\xi|}_{(**)} = \frac{1}{2}\,|x_k-\xi| \cdot \underbrace{|x_k-\xi|}_{\leq h \leq \frac{1}{A}} \cdot \underbrace{\left|\frac{f''(\eta_k)}{f'(x_k)}\right|}_{\leq A}$$

$$\Rightarrow$$

$$\underbrace{|x_{k+1}-\xi|}\ \overset{(*)}{\leq}\ \frac{1}{2}\,\underbrace{|x_k-\xi|}\ ; \quad x_k \in I_\delta$$

$$\Downarrow$$

$$\Rightarrow \quad x_{k+1} \in I_\delta$$

Afirmação: $|x_k - \xi| \leq \left(\frac{1}{2}\right)^k \cdot h$.
a provar

Por indução sobre k:

(i) Verifique que vale para $k=0$

$$|x_0 - \xi| \leq \left(\frac{1}{2}\right)^0 \cdot h = h \checkmark$$

(ii) H.I: $|x_{k-1} - \xi| \leq \left(\frac{1}{2}\right)^{k-1} \cdot h$

(iii) Provar que vale para k.

$$|x_k - \xi| \leq \frac{1}{2}|x_{k-1} - \xi| \underset{H.I}{\leq}$$

$$*$$

$$\leq \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} \cdot h = \left(\frac{1}{2}\right)^{k} \cdot h$$

$$\Rightarrow \boxed{|x_k - \xi| \leq \left(\frac{1}{2}\right)^{k} \cdot h}$$

$$k \to \infty \Rightarrow \left(\frac{1}{2}\right)^{k} \to 0 \quad \therefore x_k \to \xi.$$

Falta mostrar que a convergência é quadrática:

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|^{2}} = \lim_{k\to\infty} \frac{\frac{1}{2}\cancel{|x_k-\xi|^2}\cdot\left|\frac{f''(\eta_k)}{f'(x_k)}\right|}{\cancel{|x_k-\xi|^2}}$$

$$\underbrace{\frac{1}{2}\cdot\left|\frac{f''(\xi)}{f'(\xi)}\right|}_{\leq A} \leq \frac{A}{2} = c > 0$$

$\blacksquare$