Interpolação de Hermite

- Além dos valores $y_i = f(x_i)$, $i = 0, \ldots, n$, devemos conhecer os valores da derivada $z_i = f'(x_i)$. $i = 0, 1, \ldots, n$.

Teorema 6.3:

- $n \geq 0$
- $x_i \in \mathbb{R}$; $i = 0, 1, \dots, n$ distintos
- $y_i \in \mathbb{R}$; $i = 0, 1, \dots, n$
- $z_i \in \mathbb{R}$; $i = 0, 1, \dots, n$

$\Rightarrow \exists!$ polinomio interpolador $P_{2n+1} \in \mathbb{P}_{2n+1}$

tal que

$$\begin{cases} p_{2n+1}(x_i) = y_i \quad e \\ p'_{2n+1}(x_i) = z_i \, . \end{cases}$$

$$i = 0, 1, 2, \ldots, n.$$

Prova: $\boxed{n \geq 1}$

$(x^n)^2 x^1 = x^{2n+1}$

Defina:

$$H_k(x) = \left[L_k^n(x)\right]^2 \cdot \left[1 - 2L_k^n{}'(x_k) \cdot (x - x_k)\right]$$

$$K_k(x) = \left[L_k^n(x)\right]^2 \cdot (x - x_k)$$

$$L_k^n(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

$$L_k(x_i) = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$$

Defina:

$$P_{2n+1}(\underset{x_i}{x}) = \sum_{k=0}^{n} \left[\underbrace{H_k(x)}_{1} \cdot \underbrace{y_k}_{y_i} + K_k(x) \cdot z_k \right]$$

Verificando:

$P_{2n+1} \in P_{2n+1}(\mathbb{R})$, pois

H_k e $K_k \in P_{2n+1}(\mathbb{R})$.

Calculando as derivadas:

$$H'_k(x) = 2\underbrace{L_k(x)}_{1}\cdot L'_k(x)\cdot\left[1-2L'_k(x_k)\underbrace{(x-x_k)}_{=0}\right] + \underbrace{\left[L_k(x)\right]^2}_{1}\cdot\left(-2L'_k(x_k)\right)$$

$$H'_k(x_i)=0\ , \quad i=0,1,\dots,n$$

$$H_k(x_i) = \begin{cases} 0, & i \neq k \\ 1, & \underline{i = k} \end{cases}$$

$$\hat{K}_k(x) = 2 \underbrace{L_k(x)}_{1} \cdot \hat{L}_k(x) \cdot \underbrace{(x - x_k)}_{=0} + [\underbrace{L_k(x)}_{1}]^2$$

$$\hat{K}_k(x_i) = \begin{cases} 0, & \boxed{i \neq k} = \\ \underline{1}, & i = k \neq \end{cases}$$

$$K_k(x_i) = 0 \; ; \quad i = 0, 1, \dots n.$$

$$\Rightarrow P_{2n+1}(x_i) = y_i \quad \checkmark$$

$$P'_{2n+1}(x) = \sum_{k=0}^{n} \left[\underbrace{H'_k(x)}_{=0} \cdot y_k + \underbrace{K'_k(x)} \cdot z_k \right]$$

$$P'_{2n+1}(x_i) = z_i$$

Unicidade

Supor que existe $q_{2n+1} \in P_{2n+1}$ tal que:

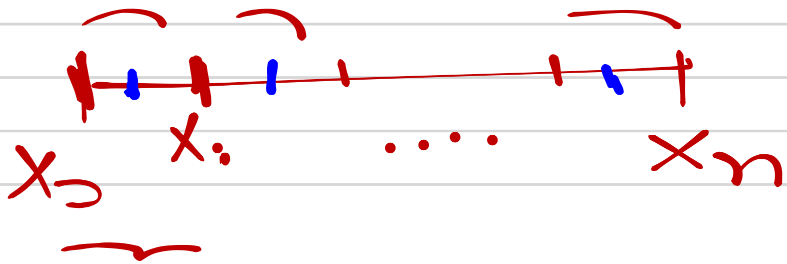
$$\begin{cases} q_{2n+1}(x_i) = y_i \\ q'_{2n+1}(x_i) = z_i \end{cases}$$

Consequentemente,

$$p_{2n+1}(x) - q_{2n+1}(x) = \overset{\in \mathbb{P}_{2n+1}}{\Pi(x)}$$

se anula em $(n+1)$ pontos distintos.

$$\boxed{x_i;\quad i = 0, 1, \dots, n}.$$



Pelo mesmo argumento

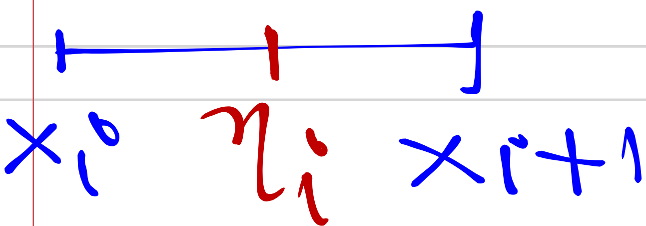
$\Pi'(x) = p'_{2n+1}(x) - q'_{2n+1}(x) \in \mathcal{P}_{2n}$

também se anula em

$(n+2)$ pontos, pois

$\Pi'(x_i) = z_i - z_i = 0.$

Obs ⊛ Π se anula em $(n+1)$ pontos, então aplicando o Teo de Rolle nos n subintervalos, vamos encontrar n pontos que anulam Π' ; $\underline{\underline{\eta_i}} \in (x_i, x_{i+1})$, $i = 0, 1, \dots, n-1$.



Desta forma acumulamos

$n + n + 1 = 2n + 1$ zeros

para $\vec{\pi}$.

mas $\vec{\pi} \in P_{2n}$, e não

podemos ter $2n+1$ zeros

para $\vec{\pi} \in P_2$, a menos que

$\pi' \equiv 0$ ou seja

$\bar{P}_{2n+1} \equiv \bar{q}_{2n+1}$

$\Rightarrow$

$$p_{2n+1} - q_{2n+1} = \overbrace{\text{constante}}$$

Como, $p_{2n+1}(x_i) - q_{2n+1}(x_i) = 0$, $\forall\, i = 0, 1, \ldots, n$; então constante $= 0$

$\boxed{\text{fim}}$

$$\boxed{n=0} \qquad \underline{k=0}$$

$$\underbrace{H_0(x)=1} \quad e \quad \underbrace{K_0(x)=(x-x_0)}$$

$$1=H_0(x)=\underbrace{[L_0(x)]}^2\cdot\left[1-\underbrace{L_0(x_0)}(x-x_0)\right]$$

para $x=x_0$

$$1=[L_0(x_0)]^2 \Rightarrow L_0(x_0)=\pm 1$$

$$\underline{L_0\in P_0} \quad \text{Lagrange} \Rightarrow L_0(x_0)=1 \Rightarrow L_0(x)=1$$

$$(x-x_0)=K_0(x)=\underbrace{[L_0(x)]^2}\cdot(x-x_0)$$

$x\neq x_0$

$$\Longrightarrow \quad L_0(x)^2 = \pm 1$$

Lagrange

$$\Longrightarrow \quad L_0(x) = 1 .$$

Exemplo 6.2.

grau 3 : $2n+1 = 3$

$\Rightarrow n = 1$

$\Rightarrow k = 0, 1$

$P_3(0) =$

$P_3(1) =$

$\overline{P}_3(0) =$

$\overline{P}_3(1) =$

$0 = x_0$ $\qquad$ $x_1 = 1$

Obs: Para $f:[a,b] \longrightarrow \mathbb{R}$ derivável em $[a,b]$ obtém-se o polinômio interpolador de Hermite de f passando por $P_{2n+1}(x_i) = f(x_i) = y_i$ e

$$P'_{2n+1}(x_i) = f'(x_i) = z_i$$

$i = 0, 1, \ldots, n.$

Erro: $p_{2n+1} \approx f$

Teo 6.4:

$$|f(x) - p_{2n+1}(x)| \leq \frac{M_{2n+2}}{(2n+2)!}\left[\Pi_{n+1}(x)\right]^2$$

$$M_{2n+2} = \max_{\eta \in [a,b]} \left| f^{(2n+1)}(\eta) \right|.$$

SPLINES

Obs: O fenômeno de Runge pode ser evitado usando os zeros de Chebyshev:

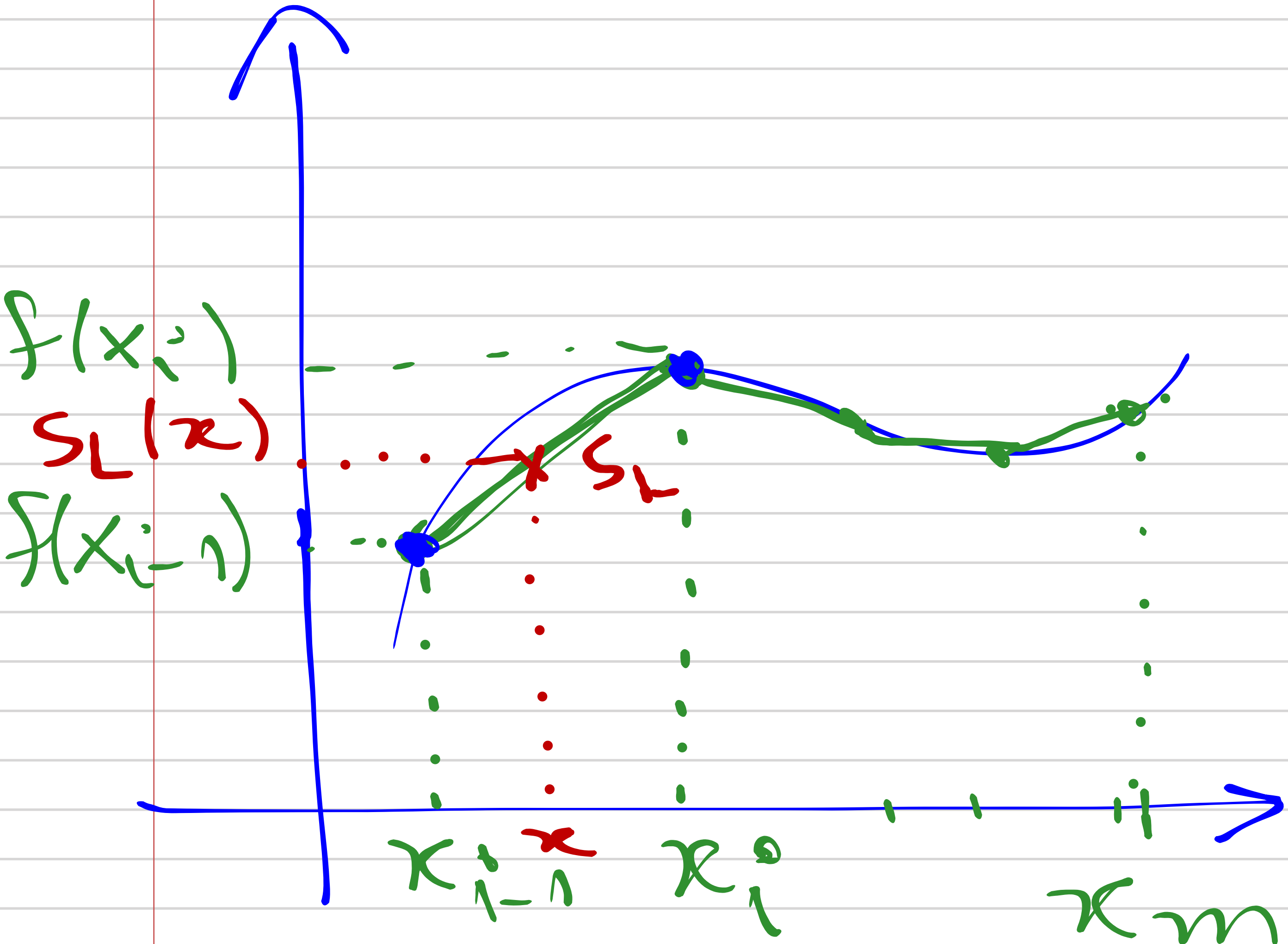
$$x_i = \frac{(a+b)}{2} + \frac{(b-a)}{2} \cdot \cos\left(\frac{i\pi}{n}\right)$$

$$i = 0, 1, \ldots, n.$$

A função f, em geral, não é conhecida nos zeros do polinômio de Chebyshev ou o conj. de dados conhecido (x_i, y_i) não corresponde aos x_i's zeros de chebyshev.

Então, o uso da <u>interpolação</u> <u>por partes</u>, as splines, são muito utilizadas.

Spline linear



$$(x, S_L(x)) = (x_{i-1}, f(x_{i-1})) + \alpha\left[(x_i, f(x_i)) - (x_{i-1}, f(x_{i-1}))\right]$$

$$\begin{cases} x = x_{i-1} + \alpha(x_i - x_{i-1}) & (1) \\ S_L(x) = f(x_{i-1}) + \alpha(f(x_i) - f(x_{i-1})) & (2) \end{cases}$$

$L\delta\alpha$ ①

$$\alpha = \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \rightarrow ②$$

$$S_L(x) = f(x_{i-1}) + \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \cdot \left(f(x_i) - f(x_{i-1})\right)$$

$\Rightarrow$

$$S_1(x) = \frac{(x_i - x)}{(x_i - x_{i-1})} \cdot f(x_{i-1}) + \frac{(x - x_{i-1})}{(x_i - x_{i-1})} \cdot f(x_i) .$$

$$\boxed{x \in [x_{i-1}, x_i] .}$$

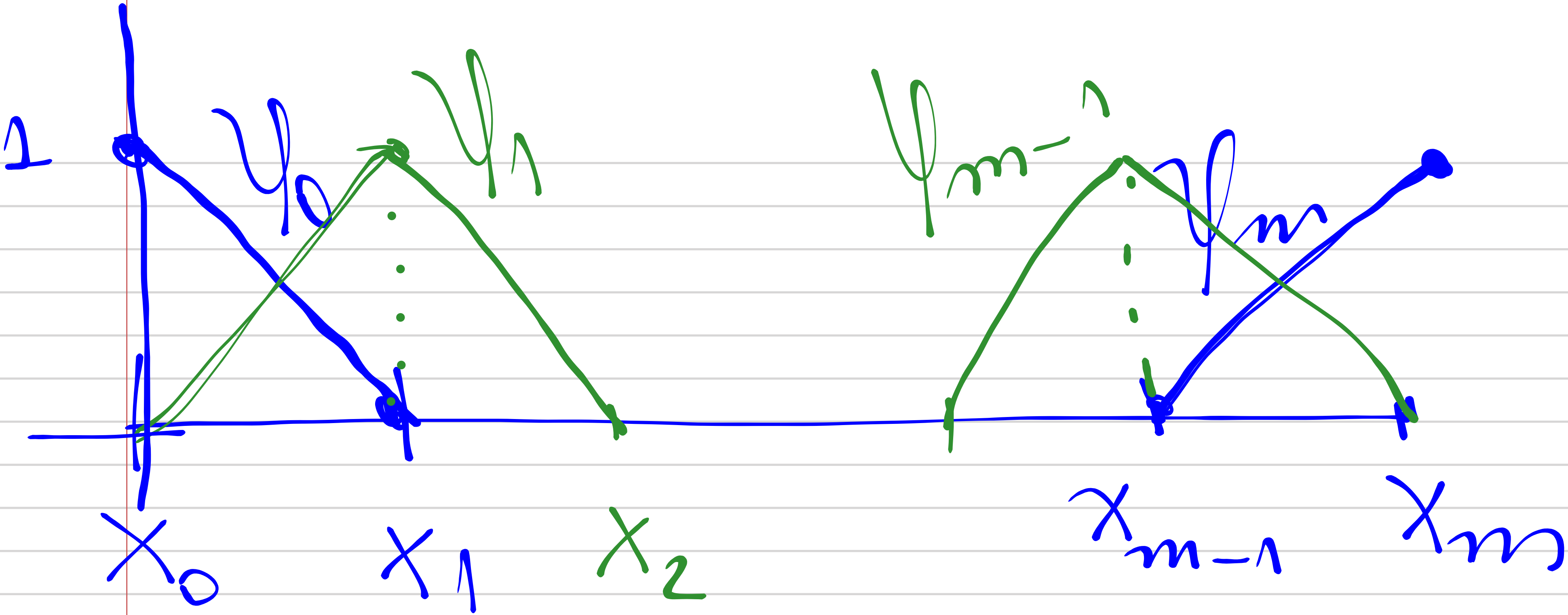
Para escrever uma spline para todo $x \in [a,b]$, podemos usar uma base de funções e escrever

$$\boxed{\begin{aligned} S(x) = \varphi_0(x) \cdot f(x_0) + \varphi_1(x) \cdot f(x_1) + \cdots \\ + \cdots + \varphi_m(x) \cdot f(x_m). \end{aligned}}$$

onde

$$\varphi_0(x) = \begin{cases} \dfrac{x_1 - x}{x_1 - x_0} & ; \; x \in [x_0, x_1] \\ 0 & ; \; x > x_1 \end{cases}$$

$$\varphi_m(x) = \begin{cases} \dfrac{x - x_{m-1}}{x_m - x_{m-1}} & ; \; x \in [x_{m-1}, x_m] \\ 0 & ; \; x < x_{m-1} \end{cases}$$



1
φ_0
φ_1
φ_{m-1}
φ_m
x_0
x_1
x_2
x_{m-1}
x_m

$$\varphi_k(x) = \begin{cases} \dfrac{x - x_{k-1}}{x_k - x_{k-1}} \; ; & x \in [x_{k-1}, x_k] \\[2ex] \dfrac{x_{k+1} - x}{x_{k+1} - x_k} \; ; & x \in [x_k, x_{k+1}] \\[2ex] 0 \; ; & \begin{matrix} x < x_{k-1} \\ x > x_{k+1} \end{matrix} \end{cases}$$

Teo 11.1.. $f \in C^2[a,b]$

• S_1 interpola f en

$a = x_0 < x_1 < \dots < x_m = b$

$$\Rightarrow \|f - S_1(x)\|_\infty \leq \frac{1}{8} h^2 \cdot \|f''\|_\infty$$

$$h = \max_{1 \leq i \leq m} h_i \; ; \; h_i = |x_i - x_{i-1}|$$

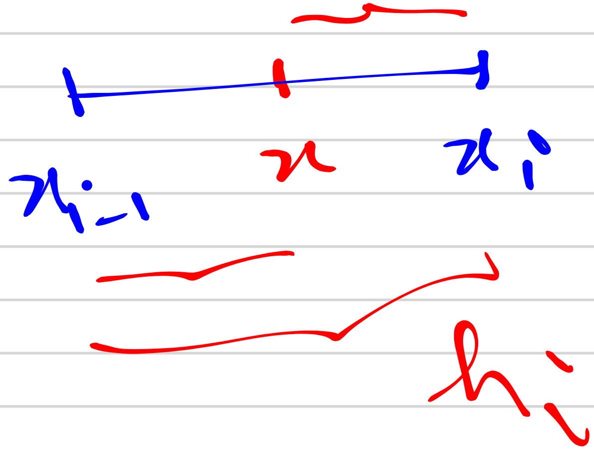
Demonstração:

Aplicar Teo 6.2. em cada subintervalo com $n=1$

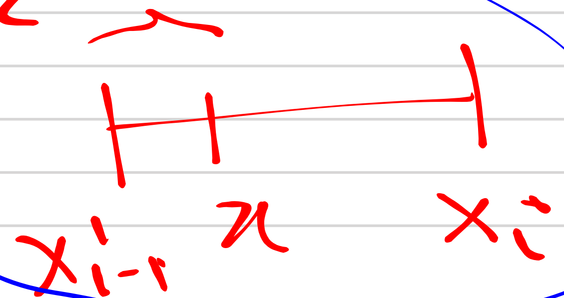
$(k=0,1)$ $\quad x_{i-1}$ ——— x_i

$$f(x)-S_L(x) \overset{\text{I}}{=} \frac{f''(\xi)}{2}\cdot(x-x_{i-1})\cdot(x-x_i)$$

$$|f(x) - S_L(x)| = \frac{1}{2}\,|f''(\xi)| \cdot \underbrace{|x - x_{i-1}|}_{\le \frac{h_i}{2}} \cdot \underbrace{|x - x_i|}_{\le \frac{h_i}{2}}$$



$\le \frac{h_i}{2}$



$$\|f - S_L\|_\infty = \max_{x \in [x_{i-1}, x_i]} |f(x) - S_L(x)| =$$

$$= \frac{1}{2} |f''(\xi)| \cdot \max_{x \in [x_{i-1}, x_i]} \overbrace{|x - x_i|}^{\le \frac{1}{2} h_i} \cdot \overbrace{|x - x_{i-1}|}^{\le \frac{h_i}{2}}$$

$$= \frac{1}{8} \underbrace{|f''(\xi)|} \cdot h^2 \le \frac{1}{8} h^2 \cdot \underbrace{\|f''\|_\infty}$$

$$\|f - S_L\|_\infty \leq \frac{1}{8} h^2 \cdot \|f''\|_\infty \leq$$

$$\leq \frac{1}{4} h^2 \|f''\|_\infty .$$