is said to be in the Mandelbrot set. If the sequence diverges from the origin, then the point z_0 is not in the set.

A standard reference for theoretical results concerning the convergence of Newton's method in complete normed linear spaces is

- L.V. KANTOROVICH AND G.P. AKILOV, *Functional Analysis*, Second edition, Pergamon Press, Oxford, New York, 1982.

A further significant book in the area of iterative solution of systems of nonlinear equations is the text by

- J.M. ORTEGA AND W.C. RHEINBOLDT, *Iterative Solution of Nonlinear Equations in Several Variables*, Reprint of the 1970 original, Classics in Applied Mathematics, 30, SIAM, Philadelphia, 2000.

It gives a comprehensive treatment of the numerical solution of n nonlinear equations in n unknowns, covering asymptotic convergence results for a number of algorithms, including Newton's method, as well as existence theorems for solutions of nonlinear equations based on the use of topological degree theory and Brouwer's Fixed Point Theorem.

Exercises

4.1 Suppose that the function $\boldsymbol{g}$ is a contraction in the ∞-norm, as in (4.5). Use the fact that

$$\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\|_p \leq n^{1/p}\|\boldsymbol{g}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{y})\|_\infty$$

to show that $\boldsymbol{g}$ is a contraction in the p-norm if $L < n^{-1/p}$.

4.2 Show that the simultaneous equations $\boldsymbol{f}(x_1, x_2) = \mathbf{0}$, where $\boldsymbol{f} = (f_1, f_2)^{\mathrm{T}}$, with

$$f_1(x_1, x_2) = x_1^2 + x_2^2 - 25\,, \qquad f_2(x_1, x_2) = x_1 - 7x_2 - 25\,,$$

have two solutions, one of which is $x_1 = 4$, $x_2 = -3$, and find the other. Show that the function $\boldsymbol{f}$ does not satisfy the conditions of Theorem 4.3 at either of these solutions, but that if the sign of f_2 is changed the conditions are satisfied at one solution, and that if $\boldsymbol{f}$ is replaced by $\boldsymbol{f}^* = (f_2 - f_1, -f_2)^{\mathrm{T}}$, then the conditions are satisfied at the other. In each case, give a value of the relaxation parameter λ which will lead to convergence.

4.3 The complex-valued function $z \mapsto g(z)$ of the complex variable z is holomorphic in a convex region Ω containing the point ζ, at which $g(\zeta) = \zeta$. By applying the Mean Value Theorem (Theorem A.3) to the function φ of the real variable t defined by $\varphi(t) = g((1-t)u + tv)$ show that if u and v lie in Ω, then there is a complex number η in Ω such that

$$g(u) - g(v) = (u - v)g'(\eta)\,.$$

Hence show that if $|g'(\zeta)| < 1$, then the complex iteration defined by $z_{k+1} = g(z_k)$, $k = 0, 1, 2, \ldots$, converges to ζ provided that z_0 is sufficiently close to ζ.

4.4 Suppose that in Exercise 3 the real and imaginary parts of g are u and v, so that $g(x + \imath y) = u(x, y) + \imath v(x, y)$, $\imath = \sqrt{-1}$. Show that the iteration defined by $\boldsymbol{x}^{(k+1)} = \boldsymbol{g}^*(\boldsymbol{x}^{(k)})$, $k = 0, 1, 2, \ldots$, where $\boldsymbol{g}^*(\boldsymbol{x}) = (u(x_1, x_2), v(x_1, x_2))^{\mathrm{T}}$, generates the real and imaginary parts of the sequence defined in Exercise 3. Compare the condition for convergence given in that exercise with the sufficient condition given by Theorem 4.2.

4.5 Verify that the iteration $\boldsymbol{x}^{(k+1)} = \boldsymbol{g}(\boldsymbol{x}^{(k)})$, $k = 0, 1, 2, \ldots$, where $\boldsymbol{g} = (g_1, g_2)^{\mathrm{T}}$ and g_1 and g_2 are functions of two variables defined by

$$g_1(x_1, x_2) = \tfrac{1}{3}(x_1^2 - x_2^2 + 3)\,, \quad g_2(x_1, x_2) = \tfrac{1}{3}(2x_1x_2 + 1)\,,$$

has the fixed point $\boldsymbol{x} = (1, 1)^{\mathrm{T}}$. Show that the function $\boldsymbol{g}$ does not satisfy the conditions of Theorem 4.3. By applying the results of Exercises 3 and 4 to the complex function g defined by

$$g(z) = \tfrac{1}{3}(z^2 + 3 + \imath)\,, \qquad z \in \mathbb{C}\,, \quad \imath = \sqrt{-1}\,,$$

show that the iteration, nevertheless, converges.

4.6 Suppose that all the second-order partial derivatives of the function $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ are defined and continuous in a neighbourhood of the point $\boldsymbol{\xi}$ in $\mathbb{R}^n$, at which $\boldsymbol{f}(\boldsymbol{\xi}) = \boldsymbol{0}$. Assume also that the Jacobian matrix, $J_f(\boldsymbol{x})$, of $\boldsymbol{f}$ is nonsingular at $\boldsymbol{x} = \boldsymbol{\xi}$, and denote its inverse by $K(\boldsymbol{x})$ at all $\boldsymbol{x}$ for which it exists. Defining the Newton iteration by $\boldsymbol{x}^{(k+1)} = \boldsymbol{g}(\boldsymbol{x}^{(k)})$, $k = 0, 1, 2, \ldots$, with $\boldsymbol{x}_0$ given, where $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x} - K(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$, show that the (i, j)-entry of the Jacobian matrix $J_g(\boldsymbol{x}) \in \mathbb{R}^{n\times n}$ of $\boldsymbol{g}$ is

$$\delta_{ij} - \sum_{r=1}^{k} \frac{\partial K_{ir}}{\partial x_j} f_r - \sum_{r=1}^{k} K_{ir} J_{rj}\,, \qquad i,j = 1,\ldots,n\,,$$

where J_{rj} is the (r,j)-entry of $J_f(\boldsymbol{x})$. Deduce that all the elements of this matrix vanish at the point $\boldsymbol{\xi}$.

4.7 The vector function $\boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x})$ of two variables is defined by

$$f_1(x_1,x_2) = x_1^2 + x_2^2 - 2\,, \qquad f_2(x_1,x_2) = x_1 - x_2\,.$$

Verify that the equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ has two solutions, $x_1 = x_2 = 1$ and $x_1 = x_2 = -1$. Show that one iteration of Newton's method for the solution of this system gives $\boldsymbol{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})^{\mathrm{T}}$, with

$$x_1^{(1)} = x_2^{(1)} = \frac{\left(x_1^{(0)}\right)^2 + \left(x_2^{(0)}\right)^2 + 2}{2\left(x_1^{(0)} + x_2^{(0)}\right)}\,.$$

Deduce that the iteration converges to $(1,1)^{\mathrm{T}}$ if $x_1^{(0)} + x_2^{(0)}$ is positive, and, if $x_1^{(0)} + x_2^{(0)}$ is negative, the iteration converges to the other solution. Verify that convergence is quadratic.

4.8 Suppose that $\boldsymbol{\xi} = \lim_{k\to\infty} \boldsymbol{x}^{(k)}$ in $\mathbb{R}^n$. Following Definition 1.4, explain what is meant by saying that *the sequence* $(\boldsymbol{x}^{(k)})$ *converges to* $\boldsymbol{\xi}$ *linearly, with asymptotic rate* $-\log_{10}\mu$, where $0 < \mu < 1$.

Given the vector function $\boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x})$ of two real variables x_1 and x_2 defined by

$$f_1(x_1,x_2) = x_1^2 + x_2^2 - 2\,, \qquad f_2(x_1,x_2) = x_1 + x_2 - 2\,,$$

show that $\boldsymbol{f}(\boldsymbol{\xi}) = \boldsymbol{0}$ when $\boldsymbol{\xi} = (1,1)^{\mathrm{T}}$. Suppose that $x_1^{(0)} \neq x_2^{(0)}$; show that one iteration of Newton's method for the solution of $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ with starting value $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)})^{\mathrm{T}}$ then gives $\boldsymbol{x}^{(1)} = (x_1^{(1)}, x_2^{(1)})^{\mathrm{T}}$ such that $x_1^{(1)} + x_2^{(1)} = 2$. Determine $\boldsymbol{x}^{(1)}$ when

$$x_1^{(0)} = 1 + \alpha\,,\; x_2^{(0)} = 1 - \alpha\,,$$

where $\alpha \neq 0$. Assuming that $x_1^{(0)} \neq x_2^{(0)}$, deduce that Newton's method converges linearly to $(1,1)^{\mathrm{T}}$, with asymptotic rate of convergence $\log_{10} 2$. Why is the convergence not quadratic?

4.9 Suppose that the equation $e^z = z + 2$, $z \in \mathbb{C}$, has a solution

$$z = (2m + \tfrac{1}{2})\imath\pi + \ln[(2m + \tfrac{1}{2})\pi] + \eta\,,$$

where m is a positive integer and $\imath = \sqrt{-1}$. Show that

$$\eta = \ln[1 - \imath(\ln(2m + \tfrac{1}{2})\pi + \eta + 2)/(2m + \tfrac{1}{2}\pi)]$$

and deduce that $\eta = \mathcal{O}(\ln\, m/m)$ for large m.
(Note that $|\ln(1 + \imath t)| < |t|$ for all $t \in \mathbb{R} \setminus \{0\}$.)