Exercise List: Numerical Methods for Nonlinear Equations

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Instructions

Solve the following exercises. Unless otherwise stated, use a tolerance of 10^{-5}. When asked to apply a method, organize the iterations in a table. When asked for theoretical justifications, provide clear arguments and proofs.

1. Bisection Method

(a) (Practical) Use the bisection method to approximate a root of $f(x) = x^3 - 7$ in $[1, 3]$.

(b) (Practical) Approximate $\sqrt{5}$ by solving $f(x) = x^2 - 5$ in $[2, 3]$.

(c) (Theoretical) Prove that if f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then the bisection method always converges to a root of f in $[a, b]$.

2. Fixed-Point Iteration

(a) (Practical) Solve $f(x) = e^{-x} - x = 0$ by rewriting it as $x = g(x)$ and applying fixed-point iteration.

(b) (Theoretical) State and prove the sufficient condition for convergence of fixed-point iteration in terms of $|g'(x)|$.

(c) (Theoretical) For $f(x) = x^3 + x - 1$, propose two different $g(x)$ functions and discuss which one converges faster.

3. Newton's Method

(a) (Practical) Apply Newton's method to solve $f(x) = e^x - x - 2 = 0$ starting at $x_0 = 1$.

(b) (Practical) Solve $f(x) = x^3 - 2x - 5 = 0$ starting from $x_0 = 2$.

(c) (Theoretical) Show that Newton's method converges quadratically if $f'(x^*) \neq 0$ at the root x^*, and explain why this may fail if $f'(x^*) = 0$.

(d) (Theoretical) Discuss how the choice of x_0 affects convergence, using $f(x) = \cos(x) - x$ as an example.

4. Secant Method

(a) (Practical) Apply the secant method to solve $f(x) = e^x - 3x$ with initial guesses $x_0 = 0$, $x_1 = 1$.

(b) (Practical) Solve $f(x) = \sin(x) - \frac{x}{2}$ with $x_0 = 1$, $x_1 = 2$.

(c) (Theoretical) Prove that the order of convergence of the secant method is approximately 1.618 (the golden ratio).

(d) (Theoretical) Compare the computational cost per iteration of Newton's and secant methods.

5. Comparison and Analysis

(a) (Practical) Compare the number of iterations required by bisection, Newton, and secant methods to approximate $\sqrt{7}$ with tolerance 10^{-6}.

(b) (Theoretical) Discuss the trade-off between robustness (guaranteed convergence) and speed (rate of convergence) for each method.

(c) (Theoretical) Explain why the bisection method converges linearly, Newton quadratically, and the secant method superlinearly.