Método de Newton para sistemas não-lineares

$$\boxed{x^{(k+1)} = x^{(k)} - \left[J_F(x^{(k)})\right]^{-1} \cdot F(x^{(k)})}$$

$k = 0, 1, 2, \ldots$

$$\boxed{F(X) = \vec{0}}$$

$F = (F_1, F_2, \ldots, F_n)$

$X = (x_1, x_2, \ldots, x_n)$

Na prática,

$$x^{(k+1)} = x^{(k)} - [J_F(x^{(k)})]^{-1} \cdot F(x^{(k)})$$

$$(*)\ [J_F(x^{(k)})]\underbrace{(x^{(k+1)} - x^{(k)})}_{y^{(k+1)}} = -F(x^{(k)})$$

$$(*)\ AY = b$$

$$y^{(k+1)} = x^{(k+1)} - x^{(k)}$$

$$\boxed{x^{(k+1)} = y^{(k+1)} + x^{(k)}}$$

Dado $x^{(k)}$:

1) Calcula-se $F(x^{(k)})$

2) Calcula-se $J_F(x^{(k)})$

3) Resolve-se o sistema linear $(*)$ por Eliminação Gaussiana.
Obtém-se $y^{(k+1)}$

4) $x^{(k+1)} = y^{(k+1)} + x^{(k)}$

5) Critério de parada:

- Erro Absoluto:

$$\|x^{(k+1)} - x^{(k)}\| < \varepsilon \; ; \quad \begin{cases} \varepsilon \text{ dado} \\ \varepsilon \approx 10^{-m} \end{cases}$$

- Erro Relativo

$$\frac{\|x^{(k+1)} - x^{(k)}\|}{\|x^{(k+1)}\|} < \varepsilon$$

$$\|x^{(k+1)} - x^{(k)}\| < \varepsilon \cdot \text{máx}\left\{1, \|x^{(k+1)}\|\right\}$$

Se satisfeito o critério de parada: $\xi \approx x^{(k+1)}$.

Se não, então voltamos a executar 1) - 5) $\leftarrow$

fazendo $x^{(k)} := x^{(k+1)}$

No final, pode-se obter a ordem/taxa (rate) de convergência:

$$p \approx \frac{\log\left(\frac{\bar{e}_{k+1}}{\bar{e}_k}\right)}{\log\left(\frac{\bar{e}_k}{\bar{e}_{k-1}}\right)};$$

$$\boxed{\bar{e}_{k+1} = \left\| x^{(k+1)} - x^{(k)} \right\|}$$

Exemplo: Determinar a solução para o sistema não-linear

$$F(X)=0 \longrightarrow \begin{cases} x^2+y^2 = 2 & F_1 \\ x^2-y^2 = 1 & F_2 \end{cases} (*)$$

com precisão de $\varepsilon = 10^{-3}$

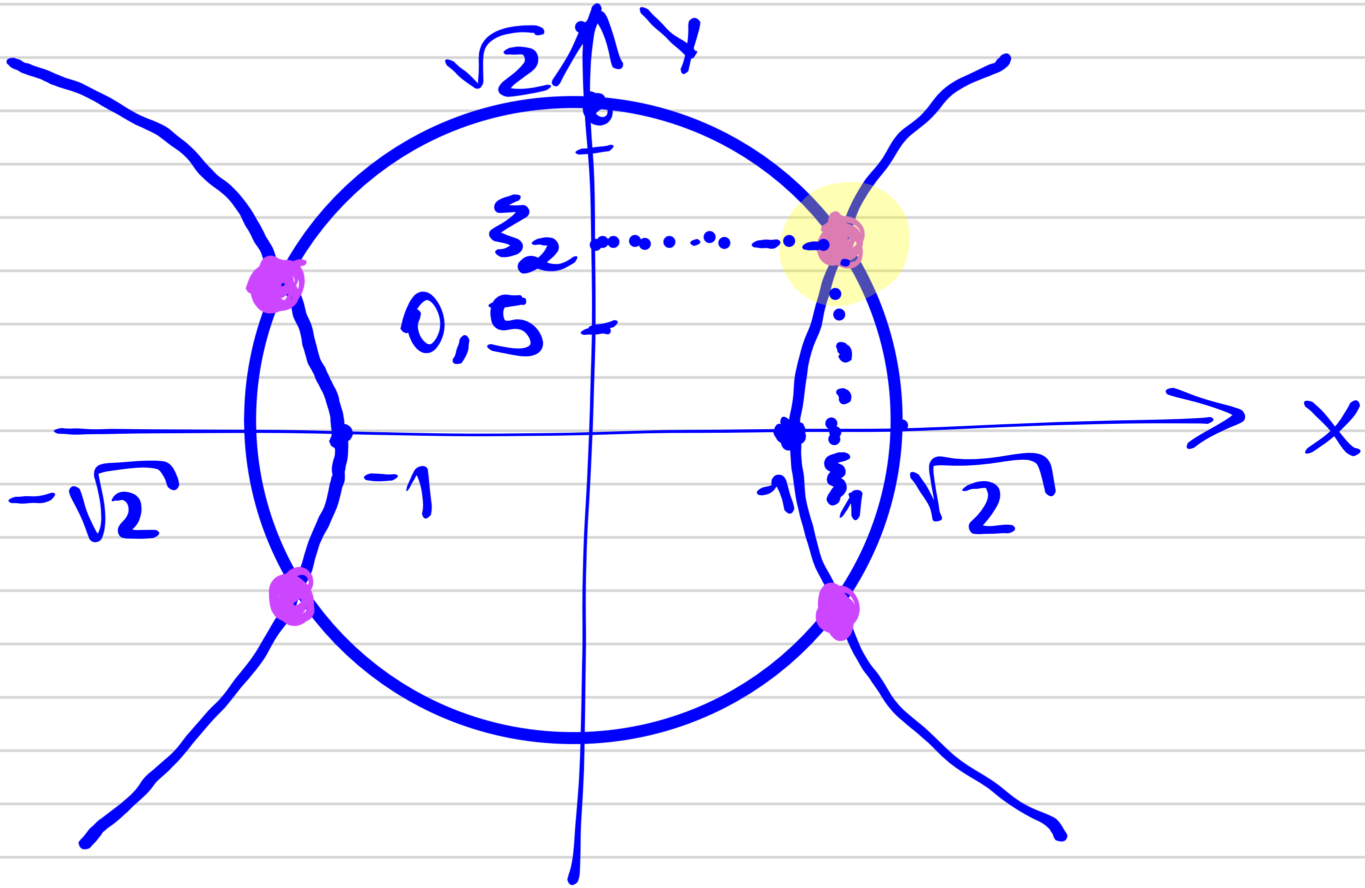
usando o Método de Newton.

$$F(X) = F(x,y) = \left(F_1(x,y), F_2(x,y)\right)$$

$$\begin{cases} F_1(x,y) = x^2 + y^2 - 2 \\ F_2(x,y) = x^2 - y^2 - 1 \end{cases}$$

$$\vec{F}(x,y) = \vec{0} \iff \begin{cases} F_1(x,y) = 0 \\ F_2(x,y) = 0 \end{cases} \quad (*)$$

Pensar no chute inicial



Considere a solução $\underline{\xi} = (\xi_1, \xi_2)$ do 1º Quadrante.

Note que, $\xi_1 \in \left]1, \sqrt{2}\right[$ e

$\xi_2 \in \left]0.5, \sqrt{2}\right[$

$$\boxed{\begin{array}{l}\text{Vamos adotar:} \\ X^{(0)} = (x^{(0)}, y^{(0)}) = (1.2, 0.7)\end{array}}$$

Calcular $J_F(x)$:

$$J_F(x) = \begin{bmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\[2ex] \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{bmatrix}$$

$$\frac{\partial F_1}{\partial x} = 2x \; ; \; \frac{\partial F_1}{\partial y} = 2y$$

$$\frac{\partial F_2}{\partial x} = 2x \quad \text{e} \quad \frac{\partial F_2}{\partial y} = -2y$$

$$F_1(x,y) = x^2 + y^2 - 2 \quad \checkmark$$

$$F_2(x,y) = x^2 - y^2 - 1 \quad \checkmark$$

$$F_1, F_2 \in C^2 \quad \checkmark$$

$$J_F(x) = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}$$

$$\det(J_F(x)) = -4xy - 4xy = -8xy$$

$$\det(J_F(x)) \neq 0 \iff x \neq 0 \text{ e } y \neq 0.$$

1ª iteração

∘ $(x^{(0)}, y^{(0)}) = (1.2, 0.7)$

1) $F(X^{(0)}) = F(1.2, 0.7) =$

$= (F_1(1.2, 0.7), F_2(1.2, 0.7)) =$

$= (-0.07, -0.05)$ ✓

$$F_1(1.2, 0.7) = (1.2)^2 + (0.7)^2 - 2 =$$

$$= -0.07$$

$$F_2(1.2, 0.7) = (1.2)^2 - (0.7)^2 - 1 =$$

$$= -0.05$$

$$F(1.2, 0.7) = \begin{bmatrix} -0.07 \\ -0.05 \end{bmatrix} \checkmark$$

2) $J_F(X^{(0)}) = J_F(1.2, 0.7) =$

$$= \begin{bmatrix} 2.4 & 1.4 \\ 2.4 & -1.4 \end{bmatrix}$$

Obs: $\det(J_F(X^{(0)})) = -6.72 \neq 0$.

3) $$\begin{bmatrix} 2.4 & 1.4 \\ 2.4 & -1.4 \end{bmatrix} \cdot \begin{bmatrix} S_1^{(1)} \\ S_2^{(1)} \end{bmatrix} = -\begin{bmatrix} -0.07 \\ -0.05 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2.4 & 1.4 & 0.07 \\ 2.4 & -1.4 & 0.05 \end{array}\right] \leftarrow$$

$L_2 \leftarrow L_2 - L_1$

$$\left|\begin{array}{cc:c} 2.4 & 1.4 & 0.07 \\ 0 & -2.8 & -0.02 \end{array}\right|$$

$$L_1 \leftarrow \frac{L_1}{2.4}$$

$$\left|\begin{array}{cc:c} 1 & \frac{1.4}{2.4} & \frac{0.07}{2.4} \\ 0 & -2.8 & -0.02 \end{array}\right|$$

$$S_1 + \frac{1.4}{2.4} S_2 = \frac{0.07}{2.4}$$

$$-2.8 \cdot S_2 = -0.02$$

$$\rightarrow S_2 = \frac{-0.02}{-2.8} = 0.00714$$

$$S_1 = \frac{0.07}{2.4} - \frac{1.4}{2.4}\left(0.00714\right)$$

$$\rightarrow S_1 = +0.025$$ conferir!

$$\text{Sol} = S = (s_1^{(1)}, s_2^{(1)}) = (0.025, 0.00714)$$

$$X^{(1)} = S^{(1)} + X^{(0)}$$

$$(x^{(1)}, y^{(1)}) = (0.025, 0.00714) + (1.2, 0.7)$$

$$\boxed{(x^{(1)}, y^{(1)}) = (\underbrace{1.225}, 0.\underbrace{70714})}$$

$$4)\ \|X^{(1)} - X^{(0)}\|_\infty = \|(x^{(1)}, y^{(1)}) - (x^{(0)}, y^{(0)})\|_\infty$$

$$= \|(x^{(1)} - x^{(0)}, y^{(1)} - y^{(0)})\|_\infty =$$

$$= \max\{|x^{(1)} - x^{(0)}|, |y^{(1)} - y^{(0)}|\}.$$

$$= 0.025 > 10^{-3}$$

$$|x^{(1)} - x^{(0)}| = |1.225 - 1.2| = 0.025$$

$$|y^{(1)} - y^{(0)}| = |0.7014 - 0.7| = 0.0014$$

Repetir o processo com $(x^{(1)}, y^{(1)})$ dado para obter $(x^{(2)}, y^{(2)})$.

2ª iteração: $(x^{(2)}, y^{(2)})$

3ª iteração: $(x^{(3)}, y^{(3)})$

$$p \approx \frac{\log\left(\frac{\bar{e}_3}{\bar{e}_2}\right)}{\log\left(\frac{\bar{e}_2}{\bar{e}_1}\right)}$$

$$\bar{e}_3 = \left\| X^{(3)} - X^{(2)} \right\| \qquad \bar{e}_1 = \left\| X^{(1)} - X^{(0)} \right\|$$

$$\bar{e}_2 = \left\| X^{(2)} - X^{(1)} \right\|$$

Exemplo de solução de Equação

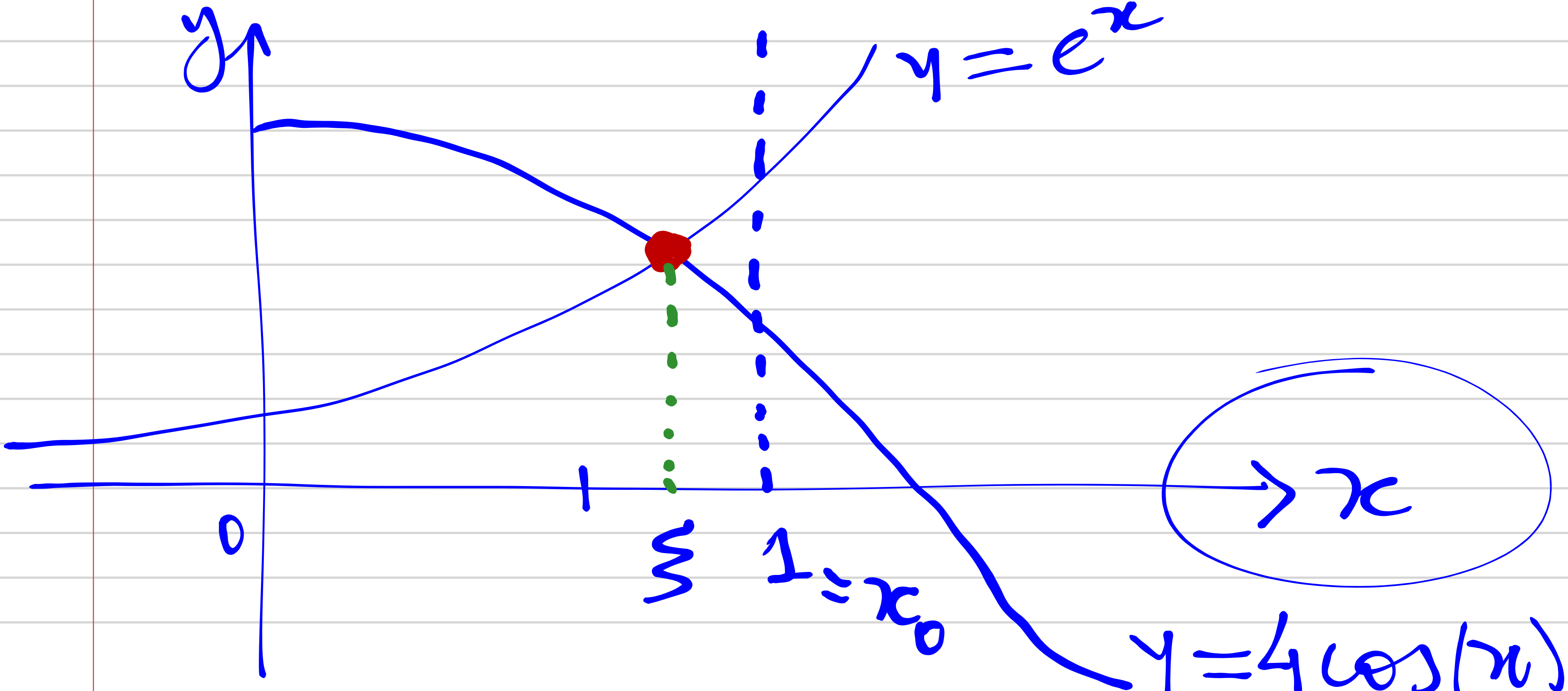
Exemplo: Encontrar a menor raiz positiva da equação:

$$\boxed{4\cos(x) - e^{x} = 0}$$

com erro inferior a 10^{-2}.

Chute inicial / initial Point.

$$\rightarrow \quad 4\cos(x) = e^{x}$$



$\boxed{\text{Chute inicial: } x_0 = 1.}$

Lembrar: Met. Newton p/ Eq:
(tangentes)

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \; ; \quad k = 0, 1, 2, \ldots$$

$$\boxed{f'(x_k) \neq 0}$$

$$f(x) = 4\cos(x) - e^{x}$$

Resolver $f(x) = 0$

$$f'(x) = -4\,\text{sen}(x) - e^{x}$$

1ª iteração: dado $x_0 = 1$

- $f(x_0) = f(1) = 4\cos(1) - e^1 =$

$$= 4.(0.54) - e \approx \underline{\underline{-0.557}}$$

- $f'(x_0) = f'(1) = -4\,\text{sen}(1) - e^1 =$

$$= -4.(0,84) - e \approx \underline{\underline{-6.084}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 1 - \frac{(-0.557)}{-6.084}$$

$$\boxed{x_1 = 0.908}$$

Critério de parada:

$$\frac{|x_1 - x_0|}{|x_1|} = \frac{|0.908 - 1|}{|0.908|} = 0.101 > 10^{-2}$$

Repetir o processo:

2ª iteração: dado $x_1 = 0.908$

- $f(x_1) = f(0.908) = 4\cos(0.908) - e^{0.908}$

$$= -0.019$$

- $f'(x_1) = f'(0.908) = -4\,\text{sen}(0.908) - e^{0.908}$

$$= -5.631$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.908 - \frac{(-0.019)}{-5.631}$$

$$\boxed{x_2 = 0.905}$$

Critério de parada:

$$\frac{|x_2 - x_1|}{|x_2|} \cong 0.0033 < 10^{-2}$$

Solução $\xi \approx x_2 = 0{,}905$

- ordem. conv. precisa de +1 iteração.