of the form $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ where $\boldsymbol{f}\colon \mathbb{R}^n \to \mathbb{R}^n$. There, corresponding to the case of $n = 2$, we shall say more about the solution of equations of the form $f(z) = 0$ where f is a complex-valued function of a single complex variable z.

This chapter has been confined to generally applicable iterative methods for the solution of a single nonlinear equation of the form $f(x) = 0$ for a real-valued function f of a single real variable. In particular, we have not discussed specialised methods for the solution of polynomial equations or the various techniques for locating the roots of polynomials in the complex plane and on the real line (by Budan and Fourier, Descartes, Hurwitz, Lobachevskii, Newton, Schur and others), although in Chapter 5 we shall briefly touch on one such polynomial root-finding method due to Sturm.[1] For a historical survey of the solution of polynomial equations and a review of recent advances in this field, we refer to the article of Victor Pan, Solving a polynomial equation: some history and recent progress, *SIAM Rev.* **39**, 187–220, 1997.

Exercises

1.1 The iteration defined by $x_{k+1} = \frac{1}{2}(x_k^2 + c)$, where $0 < c < 1$, has two fixed points ξ_1, ξ_2, where $0 < \xi_1 < 1 < \xi_2$. Show that

$$x_{k+1} - \xi_1 = \tfrac{1}{2}(x_k + \xi_1)(x_k - \xi_1)\,, \qquad k = 0, 1, 2, \ldots\,,$$

and deduce that $\lim_{k\to\infty} x_k = \xi_1$ if $0 \le x_0 < \xi_2$. How does the iteration behave for other values of x_0?

1.2 Define the function g by $g(0) = 0$, $g(x) = -x\sin^2(1/x)$ for $0 < x \le 1$. Show that g is continuous, and that 0 is the only fixed point of g in the interval $[0, 1]$. By considering the iteration $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$, starting, first from $x_0 = 1/(k\pi)$, and then from $x_0 = 2/((2k+1)\pi)$, where k is an integer, show that according to Definition 1.3 the critical point is neither stable nor unstable.

1.3 Newton's method is applied to the solution of

$$\mathrm{e}^x - x - 2 = 0\,.$$

[1] For further details in this direction, we refer to M.A. Jenkins and J.F. Traub, A three-stage algorithm for real polynomials using quadratic iterations, *SIAM J. Numer. Anal.* **7**, 545–566, 1970, A.S. Householder, *The Numerical Treatment of a Single Nonlinear Equation,* McGraw–Hill, New York, 1970, and A. Ralston and P. Rabinowitz, *A First Course in Numerical Analysis*, Second Edition, McGraw–Hill, New York, 1978.

Show that if the starting value is positive, the iteration converges to the positive solution, and if the starting value is negative it converges to the negative solution. Obtain approximate expressions for x_1 if (i) $x_0 = 100$ and (ii) $x_0 = -100$, and describe the subsequent behaviour of the iteration. About how many iterations would be required to obtain the solution to six decimal digits in these two cases?

1.4 Consider the iteration

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)}, \qquad k = 0, 1, 2, \ldots,$$

for the solution of $f(x) = 0$. Explain the connection with Newton's method, and show that (x_k) converges quadratically if x_0 is sufficiently close to the solution. Apply this method to the same example as in Example 1.7, $f(x) = \mathrm{e}^x - x - 2$, and verify quadratic convergence beginning from $x_0 = 1$. Experiment with calculations beginning from $x_0 = 10$ and from $x_0 = -10$, and account for their behaviour.

1.5 It is sometimes said that Newton's method converges quadratically, and therefore in the successive approximations to the solution the number of correct digits doubles each time. Explain why this is not generally correct. Suppose that $f''(x)$ is defined and continuous in a neighbourhood of ξ and that x_k agrees with the solution ξ to m decimal digits; give an estimate of the number of correct decimal digits in x_{k+1}.

Illustrate your estimate by using Newton's method to determine the positive zero of $f(x) = \mathrm{e}^x - x - 1.000000005$, which is close to 0.0001; use $x_0 = 0.0005$.

1.6 Suppose that $f(\xi) = f'(\xi) = 0$, so that f has a double root at ξ, and that f'' is defined and continuous in a neighbourhood of ξ. If (x_k) is a sequence obtained by Newton's method, show that

$$\xi - x_{k+1} = -\tfrac{1}{2}\frac{(\xi - x_k)^2 f''(\eta_k)}{f'(x_k)} = \tfrac{1}{2}(\xi - x_k)\frac{f''(\eta_k)}{f''(\chi_k)},$$

where η_k and χ_k both lie between ξ and x_k. Suppose, further, that $0 < m < |f''(x)| < M$ for all x in the interval $[\xi - \delta, \xi + \delta]$ for some $\delta > 0$, where $M < 2m$; show that if x_0 lies in this interval the iteration converges to ξ, and that convergence is linear, with rate $\log_{10} 2$. Verify this conclusion by finding the solution of $e^x = 1 + x$, beginning from $x_0 = 1$.

1.7 Extend the result of the previous exercise to a case where f has a triple root at ξ, so that $f(\xi) = f'(\xi) = f''(\xi) = 0$.

1.8 Suppose that the function f has a continuous second derivative, that $f(\xi) = 0$, and that in the interval $[X, \xi]$, with $X < \xi$, $f'(x) > 0$ and $f''(x) < 0$. Show that the Newton iteration, starting from any x_0 in $[X, \xi]$, converges to ξ.

1.9 The secant method is used to determine solutions of the equation $x^2 - 1 = 0$. Starting from $x_0 = 1 + \varepsilon$, $x_1 = -1 + \varepsilon$, show that $x_2 = \frac{1}{2}\varepsilon + \mathcal{O}(\varepsilon^2)$, and determine x_3, x_4 and x_5, neglecting terms of order $\mathcal{O}(\varepsilon^2)$. Explain why, at least for sufficiently small values of ε, the sequence (x_k) converges to the solution -1.

Repeat the calculation with x_0 and x_1 interchanged, so that $x_0 = -1 + \varepsilon$ and $x_1 = 1 + \varepsilon$, and show that the sequence now converges to the solution 1.

1.10 Write the secant iteration in the form

$$x_{k+1} = \frac{x_k\, f(x_{k-1}) - x_{k-1}\, f(x_k)}{f(x_{k-1}) - f(x_k)}\,, \qquad k = 1, 2, 3, \ldots\,.$$

Supposing that f has a continuous second derivative in a neighbourhood of the solution ξ of $f(x) = 0$, and that $f'(\xi) > 0$ and $f''(\xi) > 0$, define

$$\varphi(x_k, x_{k-1}) = \frac{x_{k+1} - \xi}{(x_k - \xi)(x_{k-1} - \xi)}\,,$$

where x_{k+1} has been expressed in terms of x_k and x_{k-1}. Find an expression for

$$\psi(x_{k-1}) = \lim_{x_k \to \xi} \varphi(x_k, x_{k-1})\,,$$

and then determine $\lim_{x_{k-1} \to \xi} \psi(x_{k-1})$. Deduce that

$$\lim_{x_k, x_{k-1} \to \xi} \varphi(x_k, x_{k-1}) = f''(\xi)/2f'(\xi)\,.$$

Now assume that

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = A\,.$$

Show that $q - 1 - 1/q = 0$, and hence that $q = \frac{1}{2}(1 + \sqrt{5})$.

Deduce finally that

$$\lim_{k\to\infty}\frac{|x_{k+1}-\xi|}{|x_k-\xi|^q}=\left(\frac{f''(\xi)}{2f'(\xi)}\right)^{q/(1+q)}.$$

1.11 A variant of the secant method defines two sequences u_k and v_k such that all the values $f(u_k)$, $k=0,1,2,\ldots$, have one sign, and all the values $f(v_k)$, $k=0,1,2,\ldots$, have the opposite sign. From the numbers u_k and v_k the secant formula is used to define

$$w_k=\frac{u_k f(v_k)-v_k f(u_k)}{f(v_k)-f(u_k)}, \qquad k=0,1,2,\ldots;$$

we define $u_{k+1}=w_k$, $v_{k+1}=v_k$ if $f(w_k)$ has the same sign as $f(u_k)$, and otherwise $u_{k+1}=u_k$, $v_{k+1}=w_k$. Suppose that f'' is defined and continuous on the interval $[u_0,v_0]$, and that, for some K, f'' has constant sign in $[u_K,v_K]$. Explain, graphically or otherwise, why either $u_k=u_K$ for all $k\geq K$, or $v_k=v_K$ for all $k\geq K$. Deduce that the method converges linearly, and determine the asymptotic rate of convergence; explain clearly what you mean by convergence of this method. What advantages, if any, do you think this method has compared with the secant method of Definition 1.8?

1.12 A *two-cycle* of the iteration defined by the function g is a pair of distinct numbers a,b such that $b=g(a)$ and $a=g(b)$. Use the fact that a and b are fixed points of the iteration defined by the function $h(x)=g(g(x))$ to give a definition of *stability* for a two-cycle. Show that if $|g'(a)\,g'(b)|<1$, then the two-cycle is stable, and that if $|g'(a)\,g'(b)|>1$ the two-cycle is not stable.

Show that if a,b is a two-cycle for Newton's method for the function f, and if $|f(a)f(b)f''(a)f''(b)|<[f'(a)f'(b)]^2$, then the two-cycle is stable.

Show that Newton's method for the solution of $f(x)=0$ with

$$f\colon x\mapsto x(x^2-1)$$

has a two-cycle of the form $a,-a$, and find the value of a; is this two-cycle stable?