

# integration and more

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## trigonometric functions and the inverse

to obtain the inverse of a function, the function must be one-to-one, meaning that each input corresponds to exactly one unique output and no two different inputs share the same output value. however, the trigonometric functions such as sine, cosine, and tangent are not one-to-one over their entire domains because they are periodic and repeat their values infinitely many times. therefore, to define their inverses  $\arcsin(x)$ ,  $\arccos(x)$ , and  $\arctan(x)$  we must restrict the domain of each trigonometric function to an interval where it passes the horizontal line test. this ensures that each inverse function is well-defined and produces a single, unique output for every input within its range.

- $y = \sin(x)$ , domain:  $x \in (-\infty, \infty)$ , range:  $y \in [-1, 1]$ , period:  $2\pi$   
 $y = \arcsin(x)$ , domain:  $x \in [-1, 1]$ , range:  $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $y = \cos(x)$ , domain:  $x \in (-\infty, \infty)$ , range:  $y \in [-1, 1]$ , period:  $2\pi$   
 $y = \arccos(x)$ , domain:  $x \in [-1, 1]$ , range:  $y \in [0, \pi]$
- $y = \tan(x) = \frac{\sin(x)}{\cos(x)}$ , domain:  $x \neq \frac{\pi}{2}, n \in \mathbb{Z}$ , range:  $y \in (-\infty, \infty)$ , period:  $\pi$   
 $y = \arctan(x)$ , domain:  $x \in (-\infty, \infty)$ , range:  $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$

## FTC

- if  $f$  is a continuous function on an interval  $[a, b]$ , and  $F(x) = \int_a^x f(t) dt$  is defined for  $x \in [a, b]$ , then  $F'(x) = f(x)$
- if  $f$  is continuous function on  $[a, b]$ , and  $F$  is any antiderivative of  $f$ , meaning  $F'(x) = f(x)$ , then  $\int_a^b f(x) dx = F(b) - F(a)$

## integration power rule

- $\frac{d}{dx} [\frac{x^{n+1}}{n+1}] = x^n \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1} + C$

## u-substitution

- $\int f(g(x))g'(x) dx \Rightarrow \int f(u) du$
- $\int_a^b f(g(x))g'(x) dx = \int_{u(a)}^{u(b)} f(u) du$

## integration by parts

- $(uv)'(x) = u'(x)v(x) + u(x)v'(x) \Rightarrow \int u dv = uv - \int v du$

### trigonometric substitution

1.  $\sqrt{a^2 - b^2 x^2} = \sqrt{a^2(1 - \frac{b^2 x^2}{a^2})} = \sqrt{a^2(1 - (\frac{bx}{a})^2)} = \sqrt{a^2(1 - (\frac{b}{a}x)^2)} = a\sqrt{1 - (\frac{b}{a}x)^2}$
2.  $\sin^2(\theta) = \cos^2(\theta) = 1 \Leftrightarrow \cos(\theta) = \sqrt{1 - (\sin(\theta))^2}$
3.  $\frac{b}{a}x = \sin(\theta) \Leftrightarrow x = \frac{a}{b} \sin(\theta)$
4.  $a \cos(\theta) = \sqrt{a^2 - b^2 x^2}$

when you do a trig substitution, you often let something like  $x = \sin(\theta)$ . that automatically restricts  $\theta$  to the range of the inverse sine function, which is  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ . within this range,  $\sin(\theta)$  can be positive or negative, but  $\cos(\theta)$  is always nonnegative. now, when you use the pythagorean identity  $\sin^2(\theta) + \cos^2(\theta) = 1$ , and solve for  $\cos(\theta)$ , you get  $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$ . mathematically, both the positive and negative square roots are valid - but you have to choose which sign is correct for the range of  $\theta$  you are working in. because we restricted  $\theta$  to  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  when we said  $x = \sin(\theta)$ , we know  $\cos(\theta) \geq 0$  there. Therefore, we choose the positive square root:  $\cos(\theta) = +\sqrt{1 - \sin^2(\theta)}$ . if instead we had chosen a substitution involving  $\cos(\theta)$ , we would pick a domain where  $\sin(\theta)$  has a definite sign and make the corresponding choice.

### integration by partial fraction decomposition

- distinct linear factors  $\frac{A}{x-a} + \frac{B}{x-b}$
- repeated linear factors  $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$
- irreducible quadratic factors  $\frac{Ax+B}{x^2+bx+c}$
- repeated quadratic factors  $\frac{A_1x+B_1}{x^2+bx+c} + \dots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$

a rational function is of the form  $\frac{P(x)}{Q(x)}$  where  $P(x)$  and  $Q(x)$  are polynomials. with a proper fraction ( $\deg(P) < \deg(Q)$ ) you can go straight to the decomposition, but for something improper ( $\deg(P) > \deg(Q)$ ), you will need to use polynomial division ( $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$ ) where  $S(x)$  is a polynomial (the quotient), and  $\frac{R(x)}{Q(x)}$  is now a proper fraction (remainder over the denominator).

### riemann sums

- given a definite integral like so:  $\int_a^b f(x) dx$  you can approximate it by breaking  $[a, b]$  into smaller subintervals with a width of  $\Delta x = \frac{b-a}{n}$ . you now have subintervals like so:  $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$  where  $x_i = a + i\Delta x$ .

- $L_n = \sum_{i=0}^{n-1} f(x_i) \cdot \Delta x$
- $R_n = \sum_{i=1}^n f(x_i) \cdot \Delta x$
- $M_n = \sum_{i=1}^n f(m_i) \cdot \Delta x$  where  $m_i = \frac{x_{i-1} + x_i}{2}$
- $T_n = \sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} \cdot \Delta x$

## review/rewriting (always ask why?)

1. signed area
2. areas between two curves
3. change of variables
4. volume as the integral of cross-sectional area
5. average value of a function
6. mean value theorem for integrals
7. volume of a solid of revolution: disk method, shell method
8. arc lengths
9. review properties of logs and exponents
10. defining the number e
11. derivative of the inverse
12. derivative of  $\ln(x)$  and the antiderivative of  $\frac{1}{x}$
13. compound interest,  $P(t) = P_0 e^{kt}$ , doubling/halving time, defining e as a limit vs a sum
14. L'Hopital's rule
15. complex numbers, hyperbolic trig functions
16. numeric integration techniques, simpsons rule, bounds of error
17. improper integrals
18. taylor polynomials
19. (some) differential equations
20. sequences, convergence, power series, taylor series
21. polar coords