Precalculus

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the least common multiple of two (or more) numbers is the smallest number that is a multiple of both (or all) of them:

- multiples of 4: 4, 8, 12, 16, 20
- multiples of 6: 6, 12, 18, 24

the greatest common factor of two (or more) numbers is the largest number that divides both (or all) of them without a remainder:

- factors of 12: 1, 2, 3, 4, 6, 12
- factors of 18: 1, 2, 3, 6, 9, 18

the GCD is the same as the GCF they both refer to the largest number that evenly divides two or more integers.

prime factorization: GCD(72, 27)

- $\frac{72}{2} = 36$
- $\frac{36}{2} = 18$
- $\frac{18}{2} = 9$
- $\frac{9}{3} = 3$
- $\frac{3}{3} = 1$
- $72 = 2^3 \cdot 3^2$
- $\frac{27}{3} = 9$
- $\frac{9}{3} = 3$
- $\frac{3}{3} = 1$
- $27 = 3^3$

the common factor is $3^2 = 9$

euclidean algorithm: GCD(72, 27)
keep dividing and replacing until the remainder is 0. the last non-zero remainder is the GCD.

- $\frac{72}{27} = 2$ remainder 18
- $\frac{27}{18} = 1$ remainder 9

- $\frac{18}{9} = 2$ remainder 0

GCD = 9

inequalities and absolute value:

- $[a, b] = a \leq x \leq b$
- $(a, b) = a < x < b$
- $[a, b) = a \leq x < b$
- $(a, b] = a < x \leq b$
- $(-r, r) = |x| < r$
- $(c - a, c + a) = |x - c| < a = c - a < x < c + a$

triangle inequality

$|a + b| \leq |a| + |b| \rightarrow (|a + b|)^2 \leq (|a| + |b|)^2$

if you square any real number you will get a non-negative result $((x)^2 = (-x)^2)$

1. $(a + b)^2 \leq (|a| + |b|)^2$
2. $a^2 + 2ab + b^2 \leq (|a|)^2 + 2|a||b| + (b)^2$
3. $2ab \leq 2|a||b|$
4. $ab \leq |a||b|$

lines equations:

- $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$
- $\frac{y_2-y_1}{x_2-x_1}$
- $y = mx + b$
- $y - y_1 = m(x - x_1)$
- $Ax + By = C$

eqautions forms:

- conic sections and lines: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
  - lines
  - parabolas
  - circles
  - ellipses
  - hyperbolas
- parametric form: $\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \text{for } t \in [a, b]$
- polar form: $r = f(\theta)$

- matrix form:
  - $$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
  - $$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\mathbf{b}}$$

distance:

- $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

pythagorean:

- $a^2 = b^2 + c^2$

triangle:

- $A = \frac{1}{2}bh$
- $A = \frac{1}{2}ab\sin(\theta)$

circle:

- $C = 2\pi r$
- $A = \pi r^2$

sector of circle:

- $A = \frac{1}{2}r^2\theta$
- $S = r\theta$

sphere:

- $V = \frac{4}{3}\pi r^3$
- $A_s = 4\pi r^2$

cylinder:

- $V = \pi r^2 h$

cone:

- $V = \frac{1}{3}\pi r^2 h$
- $A_s = \pi r\sqrt{r^2 + h^2}$

cone with arbitrary base: where A is the area of the base:

- $V = \frac{1}{3}Ah$

factoring:

1. $x^2 - 5x + 6$
2. $(x-3)(x-2)$

factoring by grouping:

1. $3x^2 - 8x + 4$
2. $3x^2 - 6x - 2x + 4$
3. $(3x^2 - 6x) + (-2x + 4)$
4. $3x(x-2) - 2(x-2)$
5. $(3x-2)(x-2)$

completing the square:

1. $3x^2 + 7x + 4$
2. $3(x^2 + \frac{7}{3}x + 4)$
3. $3((x^2 + \frac{7}{3}x) + 4)$
4. $3((x^2 + \frac{7}{3} + \frac{49}{36}) - \frac{49}{36}) + 4$
5. $3((x + \frac{7}{6})^2 - \frac{49}{36}) + 4$
6. $3(x + \frac{7}{6})^2 - \frac{49}{12} + 4$
7. $3(x + \frac{7}{6})^2 - \frac{49}{12} + 4$
8. $3(x + \frac{7}{6})^2 - \frac{1}{12}$

obtaining solutions:

1. $ax^2 + bx + c = 0$
2. $a(x-h)^2 + k = 0$
3. $a(x-h)^2 = k'$ (if you find that k prime is negative you will have complex solutions)
4. $(x-h)^2 = \frac{k'}{a}$
5. $x - h = \pm\sqrt{\frac{k'}{a}}$
6. $x = \pm\sqrt{\frac{k'}{a}} + h$

derivation of completing the square:

1. $ax^2 + bx + c = 0$
2. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
3. $x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$
4. $(x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$
5. $(x + \frac{b}{2a})^2 = (\frac{b^2}{4a^2}) - \frac{4ac}{4a^2}$
6. $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$

7. $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$

8. $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

rational root theorem:

If a polynomial with integer coefficients has any rational roots, then those roots must be of the form: $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading term coefficient.

consider $f(x) = x^3 - 6x^2 + 11x - 6$
so, the possible values for p: ± 1, ± 2, ± 3, ± 6 and the possible values for q: ± 1
now form all possible $\frac{p}{q}$: $\pm\frac{1}{1}$, $\pm\frac{2}{1}$, $\pm\frac{3}{1}$, $\pm\frac{6}{1}$
You can test each value by plugging into the polynomial to see if it equals 0. If it does, it is a root, and you can factor out that term. Keep in mind that not all polynomials have rational roots. Some roots may be irrational (like $\sqrt{2}$) or complex (involving i)

$f(1) = 1^3 - 6(1)^2 + 11(1) - 6 = 0$ so now we now that $(x-1)$ is a factor
we can now reduce the polynomial $\frac{x^3-6x^2+11x-6}{(x-1)}$ and now we know that $x^3 - 6x^2 + 11x - 6 = (x-1)(x^2 - 5x + 6)$ as you can see the solutions are $(x-1)(x-2)(x-3)$

complex numbers $\sqrt{-1} = i$

a complex number is made up of a real part (a) and an imaginary part (b) like so: $z = a + bi$ a complex conjugate would be written as so: $\overline{z} = a - bi$...notice that $z + \overline{z} = 2Re(z)$ you can use the complex conguate to help in divition problems with complex numbers like so:

1. $\frac{1+2i}{4-5i}$
2. $\frac{1+2i}{4-5i}\frac{4+5i}{4+5i}$
3. $\frac{4+5i+8i-10}{16-20i+20i-25i^2}$
4. $\frac{-6+13i}{41}$
5. $\frac{-6}{41} + \frac{13i}{41}$

note that multiplying a complex number by its complex conjugate will give you a real number: $z \cdot \overline{z} = (a+bi)(a-bi) = (a^2) - (bi^2) = a^2 + b^2 = (|z|)^2$

difference of squares:

- $x^2 - y^2$
- $(x+y)(x-y)$
- $x^2 + xy - xy - y^2$

sum of squares:

1. $x^2 + y^2 = x^2 - (-1y^2)$
2. $x^2 - i^2y^2 = x^2 - (iy)^2$
3. $(x+iy)(x-iy)$

example:

1. $36a^8 + 2b^6$
2. $(6a^4)^2 + (\sqrt{2}b^3)^2$
3. $(6a^4)^2 - (-1)(\sqrt{2}b^3)^2$
4. $(6a^4)^2 - (i\sqrt{2}b^3)^2$
5. $(6a^4 + i\sqrt{2}b^3)(6a^4 - i\sqrt{2}b^3)$

rectangular form: $z = a + bi$

- $\cos(\theta) = \frac{a}{r}$ and $\sin(\theta) = \frac{b}{r}$
- $r\cos(\theta) = a$ and $r\sin(\theta) = b$

polar form: $z = r(\cos(\theta) + i\sin(\theta))$

- $r = \overline{z} = \sqrt{a^2 + b^2}$
- $\theta = \tan^{-1}(\frac{b}{a})$

eulers form: $re^{i\theta}$

given $z = -1 + i\sqrt{3}$ find z^4 in both polar and rectangular form

1. $|z| = \sqrt{((-1)^2 + (\sqrt{(}3)^2))} = \sqrt{1+3} = 2$ (notice how here we take the principal square root because magnitude (distance) has no direction)
2. $\theta = \tan^{-1}(\frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}) = -60°$ (calculator will give you angles between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ because that is the period of arctan)
3. the vector is in Q2 so $-60° + 180° = 120°$
4. $z = 2(\cos(120°) + i\sin(120°))$
5. $z^2 = 4(\cos(240°) + i\sin(240°))$
6. $z^3 = 8(\cos(360°) + i\sin(350°))$
7. $z^4 = 16(\cos(120°) + i\sin(120°))$
8. $z^4 = 16(\frac{-1}{2}) + 16(\frac{\sqrt{3}}{2})i = -8 + 8\sqrt{3}i$

so there are three cube roots of one in the complex plane:

- $x^3 = 1 \rightarrow x^3 - 1 = 0$
- $1 = 1 + 0i \rightarrow 1 = 1e^{2\pi ni}$
- $x^3 = 1 \rightarrow x^3 = e^{2\pi ni}$
- $x = 1^{\frac{1}{3}} \rightarrow x = e^{\frac{2\pi n}{3}i}$

fundamental theorem of algebra:

Every non-zero polynomial equation ($f(x) \neq 0$) of degree n has exactly n complex roots including multiplicities.

$p(x) = ax^n = bx^{n-1} + \ldots + k$ (n complex roots)

conic sections:

Conic sections come from taking a three dimensional right circular double-napped cone and slicing it with a two dimensional plane. Right means the axis is perpendicular to the base. Circular means the base is a circle. Double-napped means the cone actually consists of two identical cones joined vertex to vertex.

A circle is the set of all points in a plane that are equidistant (radius) from a fixed point (center). $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \to r = \sqrt{(x-h)^2 + (y-k)^2} \to (x-h)^2 + (y-k)^2 = r^2$

$x^2 + 4x + y^2 - 6y = 23$:

1. $x^2 + 4x + y^2 - 6y = 23$
2. $x^2 + 4x + \frac{16}{4} - \frac{16}{4} + y^2 - 6y + \frac{36}{4} - \frac{36}{4} = 23$
3. $(x+2)^2 - 4 + (y-3)^2 - 9 = 23$
4. $(x+2)^2 + (y-3)^2 = 36$

An ellipse is the set of all point surrounding two foci such that $r_1 + r_2 =$ constant. The midpoint of the line segment connecting the two foci is the center. Ellipses have a major and minor axis of symmetry. At each end of the major axis we can find verticies.

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (vertices: $(h \pm a, k)$)
$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ (vertices: $(h, k \pm a)$)

The major and minor axis has a length of 2a and 2b repectively. The distance from the center to the vertices are a, and the distance from the center to the covertices are b. The distance from the center to either foci is $a^2 = b^2 + c^2 \to c^2 = a^2 - b^2$
A parabola are the set of all point that are equidistant from a fixed point (the focus) and a fixed line (the directrix).

$f(x) = -x^2 - 2x + 1$:

1. $-(x^2 + 2x - 1)$
2. $-(x^2 + 2x + \frac{4}{4} - \frac{4}{4} - 1)$
3. $-((x + \frac{2}{2})^2 - 2)$
4. $-(x+1)^2 + 2$

The definition of a hyperbola is similar to that of an ellipse, where there are two foci. While every point on a ellipse has distances to these foci that give a constant sum. Every point on a hyperbola has distances to these foci that give a constant differense. The foci sit on the outside of the branches with the two vertices between them. In between those vertices we have the center which sits on the transverse axis connecting the vertices. The a, b, and c terms from the ellipse will pretty much mean the same things. a is the distance from the center to the vertex. c is the distance from the center to a focus. b is the distance from the center to the co-vertex. The co-vertex does not lie on the hyperbola like it does for an ellipse. However, b helps define the asymptotes, which guide the shape of the branches of the hyperbola. If you form a rectangle centered at (h, k) with dimensions: width $= 2a$ and height $= 2b$. The asymptotes pass through the diagonals of this rectangle.

let us find the asymptotes for the horizontal hyperbola:

1. $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
2. lets solve for y to understand the graphs shape $-\frac{(y-k)^2}{b^2} = -\frac{(x-h)^2}{a^2} + 1$
3. $\frac{(y-k)^2}{b^2} = \frac{(x-h)^2}{a^2} - 1$
4. $(y-k)^2 = b^2(\frac{(x-h)^2}{a^2} - 1)$
5. $y - k = \pm b\sqrt{\frac{(x-h)^2}{a^2} - 1}$ consider $x \to \infty$
6. $\sqrt{\frac{(x-h)^2}{a^2} - 1} \approx \sqrt{\frac{(x-h)^2}{a^2}} = \frac{|x-h|}{a} = \frac{x-h}{a}$
7. as $x \to \infty$ then $y \approx \pm b \cdot \frac{x-h}{a} \Rightarrow y \approx k \pm \frac{b}{a}(x-h)$

the discriminant is tradtionally introduced in the context of solving quadratic equations, but it is also very powerful in analyzing conic sections. let us begin with the general second-degree equation in two variables: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ this equation represents a conic section, which could be a parabola, ellipse, hyperbola, or (in degenerate cases) a point, line, ect. the discriminant used here is: $\Delta = B^2 - 4AC$. this is not the same as the quadratic discriminant: $b^2 - 4ac$ for single-variable quadratics. it is a different discriminant, but has an analogous role. depending on the value of $\Delta = B^2 - 4AC$, the conic is:

- ellipse (or circle if $A = C$ and $B = 0$ if $\Delta < 0$
- parabola if $\Delta = 0$
- hyperbola if $\Delta > 0$

it all comes down to the quadratic part of the conic: $Ax^2 + Bxy + Cy^2$ this defines the shape of the conic, while the $Dx + Ey + F$ part just translates it. you can think of this as a sort of matrix

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

this is a quadratic form, and the matrix's properties determine the geometry. we can rotate the coordinate system (diagonalize the matrix) to eliminate the cross-term Bxy. after rotation, the conic becomes: $A'x'^2 + C'y'^2 + \text{linear terms} = 0$ now the classification is straightforward:

- if A' and C' has the same sign $\to$ ellipse
- if one is zero and the other non-zero $\to$ parabola

- if they have opposite signs $\rightarrow$ hyperbola

and it turns out: $\Delta = B^2 - 4AC$ captures exactly that information about the signs of the eigenvalues of the matrix. so:

- $\Delta < 0 \Rightarrow$ same signs $\Rightarrow$ ellipse
- $\Delta = 0 \Rightarrow$ one zero eigenvalue $\Rightarrow$ parabola
- $\Delta > 0 \Rightarrow$ opposite signs $\Rightarrow$ hyperbola

basic classes of functions:

- polynomials: a sum of terms, where each term is made up of coefficient (a constant multiple) of a power function with a whole number exponent
- rational functions: a quotient of two polynomials
  - vertical asymptotes: vertical lines $x = a$ where the function grows without bound - usually where the denominator is zero and the numerator is not zero
  - removable discontinuity: points where the function is not defined due to a factor that cancels out from both the numerator and denominator
  - hotizontal asymptotes: horizontal lines $y = L$ where the function approaches as $x \rightarrow \infty$
    - deg P < deg Q ($y = 0$)
    - deg P = deg Q ($y = \frac{\text{leading coefficient of P}}{\text{leading coefficient of Q}}$)
    - deg P > deq Q (no horizontal asymptote look for slant instead)
  - slant asymptote: lines $y = mx + b$ that the function approaces as $x \rightarrow \infty$, when the numerator's degree is exactly one more than the denominator's degree...to find them perform polynomial long division and the quotient is the slant asymptotoe
- algebraic functions: produced by taking sums, products and quotients of roots or polynomials and rational functions
- exponential functions: $f(x) = b^x$ where $b > 0$...the inverse of which is $f(x) = log_b x$
- trigonometric functions: built from $\sin(x)$ and $\cos(x)$ are called trigonometric functions.

constructing new functions:
If f and g are functions, we may construct new functions by forming the sum, difference, product, and quotient functions: $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$, $(fg)(x) = f(x)g(x)$, $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$. We can also multiply functions by constants. We call this a linear combination: $c_1 f(x) + c_2 g(x)$. Composition is another important way of constructing new functions. The composition of f and g is the function $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$.

invertable functions:
A function f is invertible if there exists another function $f^{-1}(x)$ such that: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$. This "inverse function" switches input and outputs - it reverses the effect of the original function.

A function is invertible if it is one-to-one (horizontal line test). So, this means that different input always produce different outputs.

the factor theorem
if a polynomial $f(x)$ has $f(r) = 0$, then $(x - r)$ is a factor of $f(x)$ and conversely: if $(x - r)$ is a factor of $f(x)$, then $f(r) = 0$ it connects roots (or zeros) of a polynomial to its linear factors so:

- if plugging in $x = r$ gives you 0, then $x - r$ divides the polynomial exactly
- no remainder. its a clean factor

for example $f(x) = x^3 + a^3$ if $x = -a$: $f(-a) = (-a)^3 + a^3 = 0$ so $(x + a)$ is a factor
special factorizations

- $(x + y)(x - y) = x^2 - y^2$
  1. factoring $x^2 - a^2$
  2. we are trying to factor a quadratic expression so we are looking for something of the form: $x^2 - a^2 = (x + r)(x + r)$
  3. $(x + r)(x + r) = x^2 + sx + rx + rs$
  4. $(x + r)(x + r) = x^2 + (s + r)x + rs$
  5. $r + s = 0$
  6. $rs = -a^2$
  7. $r = a$
  8. $s = -a$
  9. $x^2 - a^2 = (x + a)(x - a)$
- $(x + y)(x^2 - xy + y^2) = x^3 + y^3$
  1. $x^3 + a^3 = (x + a)(\text{quadratic})$
  2. $x^3 + a^3 = (x + a)(x^2 + bx + c)$
  3. $(x + a)(x^2 + bx + c) = x(x^2 + bx + c) + a(x^2 + bx + c)$
  4. $x(x^2 + bx + c) + a(x^2 + bx + c) = x^3 + bx^2 + cx + ax^2 + abx + ac$
  5. $x^3 + bx^2 + cx + ax^2 + abx + ac = x^3 + (b + a)x^2 + (c + ab)x + ac$
  6. we know $x^3 + a^3 = (1)x^3 + 0x^2 + 0x + (1)a^3$
  7. $b + a = 0 \Rightarrow b = -a$
  8. $c + ab = 0 \Rightarrow c = -ab \Rightarrow c = a^2$
  9. $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
- $(x - y)(x^2 + xy + y^2) = x^3 - y^3$
  1. $x^3 - a^3 = (x - a)(\text{quadratic})$
  2. $x^3 - a^3 = (x - a)(x^2 + bx + c)$
  3. $(x - a)(x^2 + bx + c) = x(x^2 + bx + c) - a(x^2 + bx + c)$
  4. $x(x^2 + bx + c) - a(x^2 + bx + c) = x^3 + bx^2 + cx - ax^2 - abx - ac$
  5. $x^3 + bx^2 + cx - ax^2 - abx - ac = x^3 + (b - a)x^2 + (c - ab)x - ac$
  6. we know $x^3 - a^3 = (1)x^3 + 0x^2 + 0x - (1)a^3$
  7. $b - a = 0 \Rightarrow b = a$

8. $c - ab = 0 \Rightarrow c = a^2$
9. $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

- $x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1}$
  1. $x(x^{n-1} + x^{n-1}a + x^{n-3}a^2 + \ldots + a^{n-1})$
  2. $x^n + x^{n-1}a + x^{n-2}a^2 + \ldots + xa^{n-1}$
  3. $-a(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})$
  4. $-ax^{n-1} - a^2x^{n-2} - a^3x^{n-3} - \ldots - a^n$
  5. $(x^n + x^{n-1}a + x^{n-2}a^2 + \ldots + xa^{n-1}) + (-ax^{n-1} - a^2x^{n-2} - a^3x^{n-3} - \ldots - a^n)$
  6. $x^n - a^n$

binomial

- $(x + y)^2 = x^2 + 2xy + y^2$
- $(x - y)^2 = x^2 - 2xy + y^2$
- $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
- $(x + y)^n = \sum_{k=0}^{n} = \binom{n}{k} x^{(n-k)} y^k$

scalars, vectors, and matrices

scalar multiplication:
$\vec{w} = (1, 2)$
$3\vec{w} = (3, 6)$

component form: $\vec{v} = (x, y)$
polar form: $\vec{v} = (r\cos(\theta), r\sin(\theta)) = r(\cos(\theta), \sin(\theta))$

When you graphically add two vectors you add them tail to head. Vector addition is commutative ($\vec{a} + \vec{b} = \vec{b} + \vec{a}$). Preforming both of these additions graphically will form a parallelogram and the vector that bisects this parallelogram is the solution.

A system of equations is a set of two or more equations that share the same variables. The goal is to find values for those variable that make all the equations in the system true at the same time. For example, in a system with two equations and two variables (like x and y), you are looking for a point (x, y) that satisfies both equations.

To solve a system of equations using substitution, you start by solving one of the equations for one variable in terms of the other. Then, you substitute that expression into one of the other equations. This gives you an equation with just one variable (assuming you started with two), which you can solve. Once you find that value, you plug it back into one of the original equations to find the second variable.

With elimination, the goal is to eliminate one variable by adding or subtracting the equations. You may need to multiply one or both equations first to make the coefficients of a variable match. Once a variable is eliminated, you solve the resulting equation for the remaining variable, then substitute that value back into one of the original equations to find the other variable.

You can only add or subtract matrices if they have the same number of rows and the same number of columns. This is because matrix addition and subtraction is an element-wise operation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 4 & 6 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix}$$

The dimensions of a matrix give the number of rows and columns of the matrix in that order. Since matrix A has 2 rows and 3 columns, it is called a 2×3 matrix.

A zero matrix is a matrix in which all of the entries are 0.

$$O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrices play a similar role in operations with matrices as the number zero plays in operations with real numbers. When we add an $m \times n$ zero matrix to any $m \times n$ matrix A, we get matrix A back. In other words, $A + O = A$ and $O + A = A$.

$$\begin{bmatrix} 4 & 1 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

When we add any $m \times n$ matrix to its opposite, we get the $m \times n$ zero matrix. $A + (-A) = O$ and $-A + A = O$ It is also true that $A - A = O$. This is because subtracting a matrix is like adding its opposite.

matrix addition properties (A, B, and C are equal dimensions):

- commutative property of addition: $A + B = B + A$
- associative property of addition: $A + (B + C) = (A + B) + C$
- additive identity: For any matrix A, there is a unique matrix O such that $A + O = A$
- additive inverse property: For each A, there is a unique matrix $-A$ such that $A + (-A) = O$
- closure property of addition: $A + B$ is a matrix of the same dimensions A and B

matrix scalar multiplication properties (A and B are matrices of equal dimensions, c and d are scalars, and O is a zero matrix):

- associative property of multiplication: $(cd)A = c(dA)$
- distributive properties: $c(A + B) = cA + cB$
- multiplicative identity property: $1A = A$
- multiplicative properties of zero: $0 \cdot A = O$ and $c \cdot O = O$
- closure property of multiplication: cA is a matrix of the same dimensions as A

Matrices can be thought of as transformations of the plane, meaning they take points (or vectors) in a 2D space and move or change them in specific ways. Imagine every point in the plane as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. When you multiply that vector by a 2×2 matrix, the output is a new vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that represents the transformed point.

$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

the new vector would be $M \cdot v = v'$ or you could do $v_{\text{row}} M^T = v'$

What kinds of transformations can matrices do?

- scaling: stretching or shrinking the plane along the x- and/or y-axis
- rotation: turning all points around the origin by some angle
- reflection: flipping points over a line, like the x-axis or y-axis
- shearing: slanting the shape, pushing points sideways or vertically
- projection: flattening points onto a line or plane

A column vector and a row vector can contain the same numbers, but their orientation matters a lot, especially when it comes to matrix operations like multiplication.

$$\begin{pmatrix} 2 & 5 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = (2)(4) + (5)(0) + (-1)(3) = 8 + 0 - 3 = 5$$

$$\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 6 \\ 20 & 0 & 15 \\ -4 & 0 & -3 \end{pmatrix}$$

The determinant of a square matrix is a special number that represents how the matrix transforms space—specifically, it measures how the matrix scales volume (or area in 2D). If the determinant is zero, it means the transformation squashes space into a lower dimension, losing information. A matrix with a zero determinant is called singular, which means it does not have an inverse. This happens when the matrix's columns (or rows) are linearly dependent, meaning one column can be expressed as a combination of others. In systems of equations, a singular (determinant-zero) matrix implies the system is either dependent—where the equations overlap and there are infinitely many solutions—or inconsistent, where the equations contradict and have no solution. A system is consistent if it has at least one solution, whether unique or infinite. Thus, the determinant and singularity give essential insight into whether a system can be solved uniquely, infinitely, or not at all.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then $\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$

What does it mean to compose matrices? It means that you can apply transformations in sequence by multiplying their matrices together.

$T = BA$
$v' = B(Av) = (BA)v$
$f(x) = Ax$
$g(x) = Bx$
$g(f(x)) = B(Ax) = (BA)x$

properties of matrix multiplication (A, B, and C are n x n matrices, I is the n x n identity matrix and O is the n x n zero matrix):

- the commutative property of multiplication does not hold! $AB \neq BA$
- associative property of multiplication: $(AB)C = A(BC)$
- distributive properties: $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$
- multiplicative identity property $IA = A$ and $AI = A$
- multiplicative property of zero: $OA = O$ and $AO = O$
- dimension property: the product of an m x n matrix and an n x k matrix is an m x k matrix

Consider a system of equations like so: $3x - 2y - z = -1$
$2x + 5y + 2 = 0$
$-4x - y = 8$
$\downarrow$

$$\begin{bmatrix} 3 & -2 & -1 \\ 2 & 5 & 1 \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 8 \end{bmatrix}$$

$\uparrow$ generalized $A_{n\times n}\vec{x}_{n\times 1} = \vec{b}_{n\times 1}$

When the coefficient matrix A is not square, you cannot use the inverse to solve the system because only square matrices can have an inverse, and only non-singular square matrices (with $\det(A) \neq 0$) actually do have an inverse. In an underdetermined system (more variables than equations: $m < n$) you will usually have infintely many solutions, or sometimes none. Here you would use substitution, elimination, parametric solutions, or row-reduction (gaussian elimination). In a overdetermined system (more equations than variables: $m > n$) it might be inconsistent (no solution), or sometimes it has a best-fit solution. Here you would use gaussian elimination, least squares method, or pseudoinverses. There are other methods for square matrices like row-reduction or cramer's rule.

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

let A represent a 90° clockwise rotation and b represent a 90° counter-clockwise rotation of the unit vectors. Visually speaking, thses two rotation matrices are inverses of each other. Multiplying them in either order gives the identity matrix.

to find the $\det(A)_{n\times n}$ use cofactor expansion or row reduction (gaussian elemination). you will need to understand minors and cofactors to compute the determinant this way.

the adjugate is the transpose of its cofactor matrix $\mathrm{adj}(A) = [C_{ij}]^T$

$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$

gaussian elimination can also be used to find the inverse by augmenting the matrix

$[A \mid I] \rightarrow [I \mid A^{-1}]$ often faster for large matrices

advanced techniques: LU decomposition/row operations

probability and combinatorics consider a standard deck of cards (no jokers)
$P(\text{jack}) = \frac{4}{52} = \frac{1}{13}$
$P(\text{hearts}) = \frac{13}{52}$

now consider a venn diagram for the following: $P(\text{J} \cup \text{H}) = \frac{4+13-1}{52}$
since the probability of hearts overlaps with the probability of jacks (jack of hearts) youll have to subtract out the double counting...we say that that are not mutually exclusive otherwise you could just add $P(\text{J}) + P(\text{H})$ if they were mutually exclusive

addition rule:
$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

now consider mutually exclusive events:
$P(\text{A} \cap \text{B}) = 0$
$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B})$

multiplication rule for independent events:
if two events, A and B, are independent (meaning the occurrence of one does not affect the other), then: $P(\text{A} \cap \text{B}) = P(\text{A}) \cdot P(\text{B})$

consider a fair coin:
$P(\text{back to back heads}) = \frac{1}{2} \cdot \frac{1}{2}$
now consider dependent events say you have a bag with 3 blue and 2 red balls...what is the probability that the first pull is blue and the second is blue
$P(\text{1st blue}) \cdot P(\text{2nd blue} \mid \text{1st blue})$

$P(\text{A} \cap \text{B}) = P(\text{B}) \cdot P(\text{A} \mid \text{B}) = P(\text{A}) \cdot P(\text{B} \mid \text{A})$

permutations (order matters)
$P(n, k) = \frac{n!}{(n-k)!}$

combinations (order does not matter)
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

a probability distribution describes how likely different outcomes are for a random variable

- a random variable is something that can take on different values due to chance (like rolling a die or counting the number of heads in coin flips)
- a probability distribution assigns a probability to each possible value that the variable can take

the expected value of a random variable is the long-run average outcome you would expect if you repeated an experiment many times. think of it as the center or balance point of a probability distribution.

$E[X] = \sum_{i=1}^{n} x^i \cdot P(x_i)$

series

A sequence is an ordered list of numbers following some rule or pattern. You can get each term from the previous one by repeated addition (arithmetic sequence) or repeated multiplication (geometric sequence), or some other rule.

examples of other rules:

- fibonacci: each term is the sum of the previous two terms: $F_n = F_{n-1} + F_{n-2}$ so the sequence starts: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$
- quadtratic:
  terms might increase by a pattern that fits a quadratic formula, like n^2:
  $1, 4, 9, 16, 25 \ldots$
- alternating:
  terms might alternate between positive and negative, like:
  $1, -1, 1, -1, 1, -1, \ldots$
- recursive:
  terms can be defined using more complex rules involoving previous terms

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$
$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

The first term is an expression where the variable r is raised to the zeroth power, so the nth term is an expression that has r raised to the n-1 power.

sum of a finite geometric series ($r \neq 1$):

1. $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$
2. $rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$
3. $S_n - rS_n = (a + ar + ar^2 + \ldots + ar^{n-1}) - (ar + ar^2 + ar^3 + \ldots + ar^n)$
4. $S_n - rS_n = a - ar^n$
5. $S_n(1 - r) = a(1 - r^n)$
6. $S_n = a\frac{1-r^n}{1-r}$ if $r \neq 1$

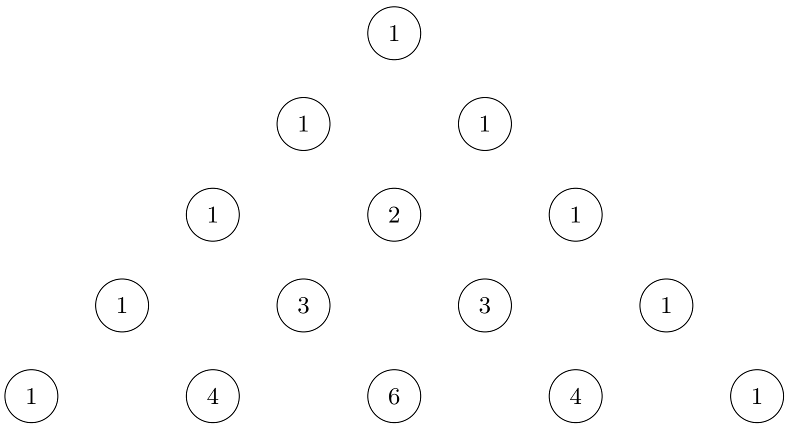
$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$

The binomial theorem gives a formula for expanding powers of a binomial

$(a+b)^n = \sum_{k=0}^{n} = \binom{n}{k} a^{(n-k)} b^k$

Each term in the expansion represents on way to choose how many a's and how many b's you take from those n factors

Pascal's triangle gives you the binomial coeffiecients used in binomial expansion



$(a+b)^4 = 1a^4b^0 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1a^0b^4$

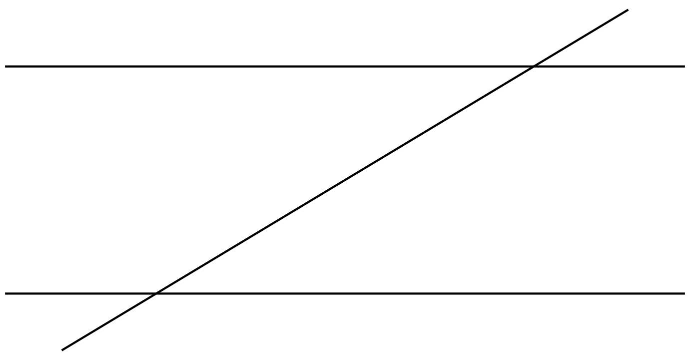
Each number $\binom{n}{k}$ counts the number of ways to chose k b's (and thus n-k a's) from n total factors of (a+b)

$a, a+d, a+2d, \ldots, a+(n-1)d$

sum of a finite arithmetic sequence:

1. $S_n = a + (a+d) + (a+2d) + \ldots + [a+(n-1)d]$
2. $S_n = [a+(n-1)d] + [a+(n-2)d] + [a+(n-3)d] + \ldots + a$
3. $2S_n = n[2a+(n-1)d]$
4. $S_n = \frac{n}{2}[2a+(n-1)d]$

$\sum_{k=1}^{n}[a+(k-1)d] = \frac{n}{2}[2a+(n-1)d]$

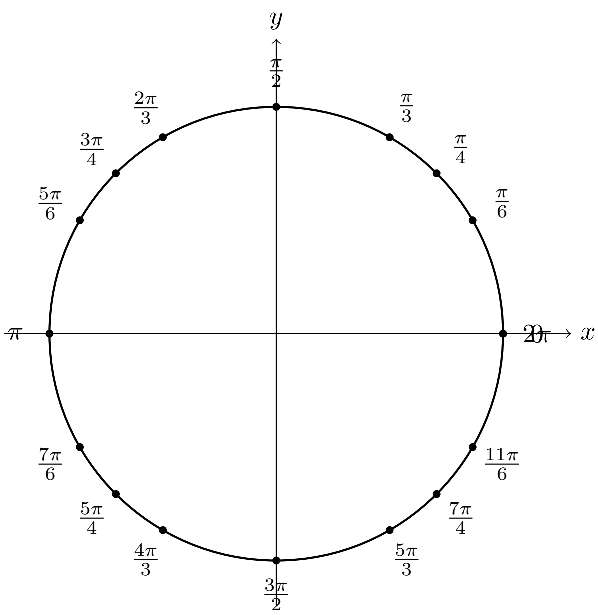


angle relationships formed when a transversal intersects parallel lines:

- complementary angles: two angles whose measures add up to 90°
- supplementary angles: two angles whose measures add up to 180°
- vertical angles: pairs of opposite angles formed when two lines intersect. they are always equal in measure.
- corresponding angles: when a transversal crosses two lines, corresponding angles are pairs located on the same side of the transversal and in matching corners (one interior, one exterior). If the lines are parallel, corresponding angles are equal.
- alternate interior angles: angles located between the two lines but on opposite sides of the transversal. if the lines are parallel, these angles are equal.
- alternate exterior angles: angles located outside the two lines and on opposite sides of the transversal. if the lines are parallel, these angles are equal.
- same-side (consecutive) interior angles: angles that lie between the two lines and on the same side of the transversal. if the lines are parallel, these angles are supplementary (add up to 180°)

circle angle theorems:

- inscribed angles subtended by the same arch are equal
- an angle insribed in a semi-circle equals 90°
- the angle subtendted by an arc at the center is double the angle subtended by the same arc at any point on the circumference



One radian is the angle made at the center of a circle when the arc length is equal to the radius of the circle.

θ radians $= \frac{s}{r} \Rightarrow s = r\theta$

$\pi = \frac{\text{circumference}}{\text{diameter}}$

$1^\circ = \frac{\pi}{180}$ radians

Trigonometric functions are special mathematical functions that originally come from studying right triangles. They relate the size of an angle in a triangle to the ratios of the lengths of the triangle's sides.

SOH-CAH-TOA is a mnemonic device that expresses the relationship between the basic trigonometric functions and the ratios of the sides in a right triangle.

The triangle definition only works for angles between 0 and $\frac{\pi}{2}$. To extend trig functions to all angles (including negative angles and angles larger then (2π or 360°), mathematicians use the unit circle. Thus allowing use to use trig functions on the coordinate plane, enabling graphing and calculus.

trigonometric identities:

- $\frac{1}{\cos(\theta)} = \sec(\theta)$
- $\frac{1}{\sin(\theta)} = \csc(\theta)$
- $\frac{1}{\tan(\theta)} = \cot(\theta)$
- $\sin(\theta + 2\pi) = \sin(\theta)$
- $\cos(\theta + 2\pi) = \cos(\theta)$
- $\tan(\theta + \pi) = \tan(\theta)$
- $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$
- $\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$
- $\tan(\frac{\pi}{2} - \theta) = \cot(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(\theta) = -\tan$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \tan^2(\theta) = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
- $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
- $\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\cos(2\theta) = 2\cos^2(\theta) - 1$
- $\cos(2\theta) = 1 - 2\sin^2(\theta)$
- $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$
- $\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x}$
- $\tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x}$
- $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$
- $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$
- $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$
- $\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

law of sines and cosines

$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \frac{\sin(C)}{c}$

$a^2 = b^2 + c^2 - 2bc\cos(A)$ (SAS or SSS)