Limits

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December 1, 2025

when evaluating the limit of a function at a point where direct substitution yields an indeterminate form (such as $\frac{0}{0}$), this indicates a potential discontinuity. in these cases, we can often simplify the function algebraically to obtain a new function $g(x)$ that is continuous at the point $x = c$. this new function satisfies the condition for point continuity: $\lim_{x \to c} g(x) = g(c)$. the original function $f(x)$, by contrast, may not be defined at $x = c$, so it lacks this full continuity condition — though the limit $\lim_{x \to c} f(x)$ may still exist. since $f(x)$ and $g(x)$ agree for all $x \neq c$, and $g(x)$ is continuous at c, it is valid to evaluate the limit of $f(x)$ by computing $g(c)$. this reflects the idea of a removable discontinuity, where the limit exists despite the function being undefined at that point.

types of discontinuities:

- removeable: $f(x) = \frac{(x^2-1)}{x-1}$
- jump: the left-hand and right-hand limit at the point exist but are not equal $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ 5, & \text{if } x \geq 2 \end{cases}$
- infinite (essential): $f(x) = \frac{1}{x}$
- oscillatory: $f(x) = \sin(\frac{1}{x})$

indeterminate forms:

- $\frac{0}{0}$
- $\frac{\infty}{\infty}$
- $0 \times \infty$
- $\infty - \infty$
- 0^0
- ∞^0

algrbraic manipulations
limits at infty

limit laws assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, then:

- sum law: $\lim x \to c(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
- constant multiple law: for any number k, $\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x)$
- product law: $\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$
- quotient law: if $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

continuity at a point: $\lim x \to cf(x) = f(c)$

laws of continuity assume that $f(x)$ and $g(x)$ are continuous at a point $x = c$. then the following functions area lso continuous at $x = c$:

- $f(x) + g(x)$ and $f(x) - g(x)$
- $kf(x)$ for any constant k
- $f(x)g(x)$
- $\frac{f(x)}{g(x)}$ if $g(c) \neq 0$

continuity of composite functions let $F(x) = f(g(x))$ be a composite function. If g is continuous at $x = c$ and f is continuous at $x = g(c)$, then $F(x)$ is continuous at $x = c$.

squeeze theorem assume that for $x \neq c$ (in some open interval containing c), $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$ then $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = L$

$f(x) = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$

1. consider the unit circle ($r = 1$) centered at the origin. let θ be an angle with $0 < |\theta| < \frac{\pi}{2}$. we want the limit as $\theta \to 0$ from both sides.
2. $\text{area}(T_1) < \text{area}(S) < \text{area}(T_2)$
3. $\frac{1}{2}\cos(\theta)\sin(|\theta|) < \frac{1}{2}|\theta| < \frac{1}{2}\tan(|\theta|)$
4. $\cos(\theta)\sin(|\theta|) < |\theta| < \tan(|\theta|)$
5. $\cos(\theta) < \frac{|\theta|}{\sin(|\theta|)} < \frac{\tan(|\theta|)}{\sin(|\theta|)}$
6. $\cos(\theta) < \frac{\sin(\theta)}{\theta} < \frac{1}{\cos(\theta)}$
7. at $\theta = 0$, all areas equal zero, so the inequalities extend to include equality

8. $\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq \frac{1}{\cos(\theta)}$
9. $\lim_{\theta \to 0} \cos(\theta) = 1$
10. $\lim_{\theta \to 0} \frac{1}{\cos(\theta)} = 1$
11. via the squeeze theorem $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$

$f(x) = \frac{\cos(\theta)-1}{\theta}$

1. $\frac{\cos(\theta)-1}{\theta} \cdot \frac{\cos(\theta)+1}{\cos(\theta)+1} = \frac{\cos^2(\theta)-1}{\theta(\cos(\theta)+1)}$
2. $\sin^2(\theta) = 1 - \cos^2(\theta) \Rightarrow -\sin^2(\theta) = \cos^2(\theta) - 1$
3. $-\frac{\sin^2(\theta)}{\theta(\cos(\theta)+1)} = -\frac{\sin(\theta)}{\theta} \cdot \frac{\sin(\theta)}{\cos(\theta)+1}$
4. $\lim_{\theta \to 0} \frac{\cos(\theta)-1}{\theta} = -1 \cdot 0 \cdot \frac{1}{2} = 0$

intermediate value theorem: if $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$ such that $f(c) = M$.

existence of zeros: if $f(x)$ is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then $f(x)$ has a zero in (a, b).

we can locate zeroes of functions to arbitrary accuracy using the **bisection method**.
$f(x) = \cos^2(x) - 2\sin(\frac{x}{4})$
$f(0) = 1 > 0$, $f(2) \approx -0.786 < 0$
we can guarantee that $f(x) = 0$ has a solution in $(0, 2)$. we can locate a zero more accurately by dividing $[0, 2]$ into two intervals $[0, 1]$ and $[1, 2]$. one of these must contain a zero of $f(x)$. to determine which, we evalutate $f(x)$ at the midpoint $m = 1$. a calculator gives $f(1) \approx -0.203 < 0$, and since $f(0) = 1$, we see that $f(x)$ takes opposite signs at the endpoints of $[0, 1]$. therefore, $(0, 1)$ must contain a zero. we discard $[1, 2]$ because both $f(1)$ and $f(2)$ are negative. the bisection method consists of continuing this process until we narrow down the location of the zero to the desired accuracty.

the size of the gap:
recall that the distance from $f(x)$ to L is $|f(x) - L|$. it is convenient to refer to the quantity $|f(x) - L|$ as the gap between the value $f(x)$ and the limit L. lets reexamine the basic trigonometric limit $\lim_{x \to 0} \frac{\sin(x)}{x}$. so 1 tells us that the gap $|f(x) - 1|$ gets arbitrarily small when x is sufficiently close by not equal to 0. suppose we want the gap $|f(x) - 1|$ to be less than 0.2. how close to 0 must x be? $|f(x) - 1| < 0.2$ if $0 < |x| < 1$. if we insist instead that the gap be smaller than 0.004, we can check by zooming in $|f(x) - 1| < 0.004$ if $0 < |x| < 0.15$. it would seem that this process can be continued: by zooming in on the graph,

we can find a small interval around $c = 0$ where the gap $|f(x) - 1|$ is smaller than any prescribed positive number. to express this in a precise fasion, we follow time-honored tradition and use the greek letters ϵ (epsilon) and δ (delta to denote small numbers specifying the size of the gap and the quantity $|x - c|$, respectively. in our case, $c = 0$ and $|x - c| = |x - 0| = |x|$. The precise meaning is that for every choice of $\epsilon > 0$, there exists some δ (depending on ϵ) such that $|\frac{\sin(x)}{x} - 1| < \epsilon$ if $0 < |x| < \delta$. the number δ tells us how close is sufficiently close for a give ϵ.

formal definition of a limit:
suppose that $f(x)$ is defined for all x in an open interval containing c (but not necessarily at $x = c$). then

$$\lim_{x\to c} f(x) = L$$

if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ if } 0 < |x - c| < \delta$$

four rigorous definitions of the limit:

- $\lim_{x\to a} f(x) = L$ means $\forall\epsilon > 0, \exists\delta > 0 : 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$
- $\lim_{x\to\infty} f(x) = L$ means $\forall\epsilon > 0, \exists N > 0 : x > N \Rightarrow |f(x) - L| < \epsilon$
- $\lim_{x\to a} f(x) = \infty$ means $\forall M > 0, \exists\delta > 0 : 0 < |x - a| < \delta \Rightarrow f(x) > M$
- $\lim_{x\to\infty} f(x) = \infty$ means $\forall M > 0, \exists N > 0 : x > N \Rightarrow f(x) > M$