Maths

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Algebra

Lines

Slope of the line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Point-slope equation of line through $P_1 = (x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Circles

Equation of the circle with center (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

Distance and Midpoint Formulas

Distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $P_1\bar{}P_2$:

$$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Laws of Exponents

$$x^m x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$(\frac{x}{y})^n = \frac{x^n}{y^n}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Special Factorizations

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + nxy^{n-1} + y^n$$

where $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{1\cdot 2\cdot 3\ldots k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

if $a < b$ and $c > 0$, then $ca < cb$.

if $a < b$ and $c < 0$, then $ca > cb$.

$|x| = x$ if $x >= 0$

$|x| = -x$ if $x <= 0$

Geometry

Formulas for area A, circumference C, and volume V
Triangle

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}ab\sin(\theta)$$

Circle

$$A = \pi r^2$$

$$C = 2\pi r$$

Sector of Circle

$$A = \frac{1}{2}r^2\theta$$

$$s = r\theta$$

Sphere

$$V = \frac{4}{3}\pi r^3$$

$$A = 4\pi r^2$$

Cylinder

$$V = \pi r^2 h$$

Cone

$$V = \frac{1}{3}\pi r^2 h$$

$$A = \pi r\sqrt{r^2 + h^2}$$

Cone with arbitrary base

$$V = \frac{1}{3}Ah$$

Trigonometry

Pythagorean Theorem: For a right trianlge with hypotenuse of length c and legs of lengths a and b, $c^2 = a^2 + b^2$.

Angle Measurement

π radians = $180°$

$$1° = \frac{\pi}{180} rad$$

1 rad = $\frac{180}{\pi}$

$s = r\theta$ (θ in radians)

Right Triangle Definitions

$$\sin\theta = \frac{opp}{hyp}$$

$$\cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{sin\theta}{cos\theta} = \frac{opp}{adj}$$

$$\sec\theta = \frac{1}{cos\theta} \text{ csc } \theta = \frac{1}{sin\theta}$$

Trigonometric Functions

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{r}{x}$$

$$\csc\theta = \frac{r}{y}$$

$$\lim_{\theta\to 0} \frac{sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0} \frac{1-cos\theta}{\theta} = 0$$

Fundamental Identities

$$sin^2\theta + cos^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + cot^2\theta = csc^2\theta$$

$$\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$$

$$\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$$

$$\tan(\frac{\pi}{2} - \theta) = \cot(\theta)$$

$$\sin(-\theta) = -sin\theta$$

$$\cos(-\theta) = cos\theta$$

$$\tan(-\theta) = -tan\theta$$

$$\sin(\theta + 2\pi) = sin\theta$$

$$\cos(\theta + 2\pi) = cos\theta$$

$$\tan(\theta + \pi) = tan\theta$$

The Law of Sines

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

The Law of Cosines

$$a^2 = b^2 + c^2 - 2bccosA$$

Addition and Subtraction Formulas

$$\sin(x+y) = sinxcosy + cosxsiny$$

$$\sin(x-y) = sinxcosy - cosxsiny$$

$$\cos(x+y) = cosxcosy - sinxsiny$$

$$\cos(x-y) = cosxcosy + sinxsiny$$

$$tan(x+y) = \frac{tanx+tany}{1-tanxtany}$$

$$tan(x-y) = \frac{tanx-tany}{1+tanxtany}$$

Double-Angle Formulas

$$sin2x = 2sinxcosx$$

$$cos2x = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x$$

$$tan2x = \frac{2tanx}{1-tan^2x}$$

$$sin^2x = \frac{1-cos2x}{2}$$

$$cos^2x = \frac{1+cos2x}{2}$$

Precalculus Review

78. Prove the triangle inequality by adding the two inequalities

1.) Known Inequalities

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

2.) Add and Simplify

$$(-|a|) + (-|b|) \leq a+b \leq |a| + |b|$$

$$-(|a| + |b|) \leq a+b \leq |a| + |b|$$

3.) By the definition of absolute value, we know:

$$-|x| \leq x \leq |x|$$

$$-|a+b| \leq a+b \leq |a+b|$$

4.) To explain, the value $a+b$ is squeezed between $-(|a|+|b|)$ and $|a|+|b|$. By taking the absolute value on both sides, we conclude that:

$$-(|a| + |b|) \leq a+b \leq |a| + |b|$$

$$|a+b| \leq |a| + |b|$$

79. Show that if $r = \frac{a}{b}$ is a fraction in lowest terms, then r has a finite decimal expansion if and only if $b = (2^n)(5^m)$ for some n, m ≥ 0. Hint: Observe that r has a finite decimal expansion when $(10^N)(r)$ is an integer for some $N \geq 0$ (and hence b dividies 10^N).

Finite Decimal Expansion implies $b = 2^n \cdot 5^m$

1. Finite Decimal Expansion: A fraction $\frac{a}{b}$ has a finite decimal expansion if and only if $\frac{a}{b}$ can be written as $k \cdot 10^- N$ for some integer k and non-negative integer N. This is equivalent to the condition that b divides 10^N for some $N \geq 0$

2. Denominator as a Product of Powers of 2 and 5: Observe that $10^N = 2^N \cdot 5^N$. Therefore, if b divides 10^N, then b must be of the form $b = \frac{10^N}{k}$, where k is an integer that ensures b divides 10^N. This implies that b must only have the prime factors 2 and 5 because 10^N itself only contains the prime factors 2 and 5. Thus, if b divides 10^N, then b must be of the form $b = 2^n \cdot 5^m$ for some non-negative integers n and m

$b = 2^n \cdot 5^m$ **Implies Finite Decimal Expansion**

1. Form of b: Suppose $b = 2^n \cdot 5^m$. We want to show that $\frac{a}{b}$ has a finite decimal expansion. Since b can be written as $2^n \cdot 5^m$, it folows that b is a dividor of 10^N where $N = max(n, m)$.

2. Verification: To be specific, let us express $\frac{a}{b}$ in terms of 10^N:

$$\frac{a}{b} = \frac{a}{2^n \cdot 5^m}$$

We can multiply both the numerator and the denominator by 10^N, where $N = max(n, m)$. This multiplication yields:

$$\frac{a \cdot 10^N}{b \cdot 10^N} = \frac{a \cdot 10^N}{2^n \cdot 5^m \cdot 10^N} = \frac{a \cdot 10^N}{10^{N+n} \cdot 10^m} = \frac{a \cdot 10^N}{10^N}$$

Since $b \cdot 10^N = 10^{N+n} \cdot 10^m$, which simplifies to 10^N, we get that $b \cdot 10^N$ is an integer.
Hence, $\frac{a \cdot 10^N}{b \cdot 10^N}$ is an integer, implying that $\frac{a}{b}$ indeed has a finite decimal expansion.

Conclusion

We have shown that if b divides 10^N for some $N \geq 0$, then b must be of the form $2^n \cdot 5^m$. Conversely, if $b = 2^n \cdot 5^m$, then $\frac{a}{b}$ has a finite decimal expansion. Therefor, the fraction $\frac{a}{b}$ in lowest terms has a finite decimal expansion if and only if the denominator b is of the form $2^n \cdot 5^m$.

80. Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$. Show that $\frac{p}{10^s-1} = 0.\overline{p_1 \dots p_s}$ Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that $r = \frac{2}{11} = \frac{18}{10^2-1}$

A quadratic function is a function defined by a quadratic polynomial

$$f(x) = ax^2 + bx + c \text{ (}a\text{,}b\text{,}c\text{, constants with } a \neq 0\text{)}$$

The technique of completing the square consists of writing a quadratic polynomial as a multiple of a squareplus a constant:

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$$

The discriminant of f(x) is the quantity $D = b^2 - 4ac$ The roots of f(x) are given by the quadratic formula:

$$\frac{-b \pm \sqrt[2]{b^2-4ac}}{2a}$$

The general linear equation is $ax + by = c$ where a and b are not both zero. For $b = 0$, this gives the verical line $ax = c$. When $b \neq 0$, we can rewrite in slope-intercept form. For example, $-6x + 2y = 3$ can be rewritten as $y = 3x + \frac{3}{2}$

Polynomials: For any real number m, the function $f(x) = x^m$ is called the power function with exponent m. A polynomial is a sum of multiples of power functions with whole number exponents: $f(x) = x^5 - 5x^3 + 4x$, $g(t) = 7t^6 + t^3 - 3t - 1$

Thus, the function $f(x) = x + x^{-1}$ is not a polynomial because it includes a power function x^-1 with a negative exponent. The general polynomial in the variable x may be written

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 + a_0$$

The numbers $a_0, a_1, \ldots, a_n$ are called coefficients.
The degree of $P(x)$ is n (assuming that $a_n \neq 0$).
The coefficient a_n is called the leading coefficient.
The domain of $P(x)$ is $\mathbb{R}$.

Rational functions: A rational function is a quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

Every polynomialis also a rational functions with $Q(x) = 1$. The domain of a rational function $\frac{P(x)}{Q(x)}$ is th eset of numbers x such that $Q(x) \neq 0$.

Algebraic functions: An algrbraic function is produced by taking sums, products, and quotients of roots of polynomials and rational functions