Calculus 101

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Introduction

Calculus is the study of rates of change and the accumulation of quantities.

the least common multiple of two (or more) numbers is the smallest number that is a multiple of both (or all) of them:

- multiples of 4: 4, 8, 12, 16, 20
- multiples of 6: 6, 12, 18, 24

the greatest common factor of two (or more) numbers is the largest number that divides both (or all) of them without a remainder:

- factors of 12: 1, 2, 3, 4, 6, 12
- factors of 18: 1, 2, 3, 6, 9, 18

inequalities and absolute value:

- $[a, b] = a \leq x \leq b$
- $(a, b) = a < x < b$
- $[a, b) = a \leq x < b$
- $(a, b] = a < x \leq b$
- $(-r, r) = |x| < r$
- $(c - a, c + a) = |x - c| < a = c - a < x < c + a$

triangle inequality

$|a + b| \leq |a| + |b| \rightarrow (|a + b|)^2 \leq (|a| + |b|)^2$

if you square any real number you will get a non-negative result $((x)^2 = (-x)^2)$

1. $(a + b)^2 \leq (|a| + |b|)^2$
2. $a^2 + 2ab + b^2 \leq (|a|)^2 + 2|a||b| + (b)^2$
3. $2ab \leq 2|a||b|$
4. $ab \leq |a||b|$

eqautions for lines:

- $m = \frac{y_2 - y_1}{x_2 - x_1}$
- $y = mx + b$
- $y - y_1 = m(x - x_1)$

distance:

- $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

pythagorean:

- $c^2 = a^2 + b^2$

triangle:

- $A = \frac{1}{2}bh$
- $A = \frac{1}{2}ab\sin(\theta)$

circle:

- $C = 2\pi r$
- $A = \pi r^2$

sector of circle:

- $A = \frac{1}{2}r^2\theta$
- $S = r\theta$

sphere:

- $V = \frac{4}{3}\pi r^3$
- $A_s = 4\pi r^2$

cylinder:

- $V = \pi r^2 h$

cone:

- $V = \frac{1}{3}\pi r^2 h$
- $A_s = \pi r\sqrt{r^2 + h^2}$

cone with arbitrary base: where A is the area of the base:

- $V = \frac{1}{3}Ah$

exponents and logs:

- $x^m x^n = x^{m+n}$
- $\frac{x^m}{x^n} = x^{m-n}$
- $(x^m)^n = x^{mn}$
- $x^{-n} = \frac{1}{x^n}$
- $(xy)^n = x^n y^n$

- $(\frac{x}{y})^n = \frac{x^n}{y^n}$
- $x^{1/n} = \sqrt[n]{x}$
- $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$
- $\log_a x = y \leftrightarrow a^y = x$
- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a(a^x) = x$
- $a^{\log_a x} = x$
- $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a(x^r) = r \log_a x$
- $\log_b a = \frac{\log_c a}{\log_c b}$

factoring:

1. $x^2 - 5x + 6$
2. $(x-3)(x-2)$

factoring by grouping:

1. $3x^2 - 8x + 4$
2. $3x^2 - 6x - 2x + 4$
3. $(3x^2 - 6x) + (-2x + 4)$
4. $3x(x-2) - 2(x-2)$
5. $(3x-2)(x-2)$

completing the square:

1. $3x^2 + 7x + 4$
2. $3(x^2 + \frac{7}{3}x + 4)$
3. $3((x^2 + \frac{7}{3}x) + 4)$
4. $3((x^2 + \frac{7}{3} + \frac{49}{36}) - \frac{49}{36}) + 4$
5. $3((x + \frac{7}{6})^2 - \frac{49}{36}) + 4$
6. $3(x + \frac{7}{6})^2 - \frac{49}{12} + 4$
7. $4(x + \frac{7}{6})^2 - \frac{49}{12} + 4$
8. $3(x + \frac{7}{6})^2 - \frac{1}{12}$

obtaining solutions:

1. $ax^2 + bx + c = 0$

2. $a(x-h)^2 + k = 0$
3. $a(x-h)^2 = k'$ (if you find that k prime is negative you will have complex solutions)
4. $(x-h)^2 = \frac{k'}{a}$
5. $x - h = \pm\sqrt{\frac{k'}{a}}$
6. $x = \pm\sqrt{\frac{k'}{a}} + h$

derivation of completing the square:

1. $ax^2 + bx + c = 0$
2. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
3. $x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a} = 0$
4. $(x + \frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$
5. $(x + \frac{b}{2a})^2 = (\frac{b^2}{4a^2}) - \frac{4ac}{4a^2}$
6. $(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2}$
7. $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$
8. $x = -b \pm \frac{\sqrt{b^2-4ac}}{2a}$

rational root theorem:

If a polynomial with integer coefficients has any rational roots, then those roots must be of the form: $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading term coefficient.

consider $f(x) = x^3 - 6x^2 + 11x - 6$
so, the possible values for p: ±1, ±2, ±3, ±6 and the possible values for q: ±1
now form all possible $\frac{p}{q}$: $\pm\frac{1}{1}$, $\pm\frac{2}{1}$, $\pm\frac{3}{1}$, $\pm\frac{6}{1}$
You can test each value by plugging into the polynomial to see if it equals 0. If it does, it is a root, and you can factor out that term. Keep in mind that not all polynomials have rational roots. Some roots may be irrational (like $\sqrt{2}$) or complex (involving i)

$f(1) = 1^3 - 6(1)^2 + 11(1) - 6 = 0$ so now we now that $(x-1)$ is a factor
we can now reduce the polynomial $\frac{x^3-6x^2+11x-6}{(x-1)}$ and now we know that $x^3-6x^2+11x-6 = (x-1)(x^2-5x+6)$ as you can see the solutions are $(x-1)(x-2)(x-3)$

complex numbers $\sqrt{-1} = i$

a complex number is made up of a real part (a) and an imaginary part (b) like so: $z = a+bi$ a complex conjugate would be written as so: $\overline{z} = a-bi$...notice that $z+\overline{z} = 2Re(z)$ you can use the complex conguate to help in divition problems with complex numbers like so:

1. $\frac{1+2i}{4-5i}$
2. $\frac{1+2i}{4-5i}\frac{4+5i}{4+5i}$

3. $\frac{4+5i+8i-10}{16-20i+20i-25i^2}$
4. $\frac{-6+13i}{41}$
5. $\frac{-6}{41}+\frac{13i}{41}$

note that multiplying a complex number by its complex conjugate will give you a real number: $z\cdot\overline{z}=(a+bi)(a-bi)=(a^2)-(bi^2)=a^2+b^2=(|z|)^2$

difference of squares:

- x^2-y^2
- $(x+y)(x-y)$
- $x^2+xy-xy-y^2$

sum of squares:

1. $x^2+y^2=x^2-(-1y^2)$
2. $x^2-i^2y^2=x^2-(iy)^2$
3. $(x+iy)(x-iy)$

example:

1. $36a^8+2b^6$
2. $(6a^4)^2+(\sqrt{2}b^3)^2$
3. $(6a^4)^2-(-1)(\sqrt{2}b^3)^2$
4. $(6a^4)^2-(i\sqrt{2}b^3)^2$
5. $(6a^4+i\sqrt{2}b^3)(6a^4-i\sqrt{2}b^3)$

rectangular form: $z=a+bi$

- $\cos(\theta)=\frac{a}{r}$ and $\sin(\theta)=\frac{b}{r}$
- $r\cos(\theta)=a$ and $r\sin(\theta)=b$

polar form: $z=r(\cos(\theta)+i\sin(\theta))$

- $r=\overline{z}=\sqrt{a^2+b^2}$
- $\theta=\tan^{-1}(\frac{b}{a})$

eulers form: $re^{i\theta}$

given $z=-1+i\sqrt{3}$ find z^4 in both polar and rectangular form

1. $|z|=\sqrt{((-1)^2+(\sqrt{(}3)^2))}=\sqrt{1+3}=2$ (notice how here we take the principal square root because magnitude (distance) has no direction)
2. $\theta=\tan^{-1}(\frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}})=-60°$ (calculator will give you angles between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ because that is the period of arctan)
3. the vector is in Q2 so $-60°+180°=120°$

4. $z = 2(\cos(120°) + i\sin(120°))$
5. $z^2 = 4(\cos(240°) + i\sin(240°))$
6. $z^3 = 8(\cos(360°) + i\sin(350°))$
7. $z^4 = 16(\cos(120°) + i\sin(120°))$
8. $z^4 = 16(\frac{-1}{2}) + 16(\frac{\sqrt{3}}{2})i = -8 + 8\sqrt{3}i$

so there are three cube roots of one in the complex plane:

- $x^3 = 1 \rightarrow x^3 - 1 = 0$
- $1 = 1 + 0i \rightarrow 1 = 1e^{2\pi ni}$
- $x^3 = 1 \rightarrow x^3 = e^{2\pi ni}$
- $x = 1^{\frac{1}{3}} \rightarrow x = e^{\frac{2\pi n}{3}i}$

fundamental theorem of algebra:

Every non-zero polynomial equation ($f(x) \neq 0$) of degree n has exactly n complex roots including multiplicities.

$p(x) = ax^n = bx^{n-1} + \ldots + k$ (n complex roots)

conic sections:

- circle:
  $(x-h)^2 + (y-k)^2 = r^2$
- parabola:
  $(y-k) = a(x-h)^2$ (opens up or down)
  $(x-h) = a(y-k)^2$ (opens left or right)
- ellipes:
  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
  $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
- hyperbolas:
  $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
  $y = \pm\frac{b}{a}x$
  $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
  $y = \pm\frac{a}{b}x$

The vertex is a point where the conic section curves the most. For the parabola, the vertex is the point where the curve changes direction. For ellipse and hyperbolas the vertices lie on the major axis and transverse axis, respectfully, and are the points closest to or farthest from the center.

The foci are special point such that the sum (ellipse) or difference (hyperbola) of distances from any point on the curve to the foci is a constant.

The major axes is the longest line that can be drawn through the center of the ellipse. It passes through the two foci and the vertices. The minor axes is the shortest line through the center, perpendicular to the major axis. It does not pass through the foci unless the ellipse is a circle.

The transverse axis is the line that connects the two vertices of a hyperbola. It lies along the direction in which the hyperbola opens. The conjugate axis is perpendicular to the transverse axis, going through the center. It does not intersect the hyperbola but is used to help define its shape.

A hyperbola has two separate branches that open outward. As the branches move farther from the center, they get closer and closer to two diagonal lines - these are the asymptotes. The equations come from setting the conic equation equal to 0, which simplifies to the lines the hyperbola approaches at infinity.

general equation: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
discriminant: $\Delta = B^2 - 4AC$

- $\Delta < 0$ (circle or ellipse)
- $\Delta = 0$ (parabola)
- $\Delta > 0$ (hyperbola)

basic classes of functions:

- polynomials: a sum of terms, where each term is made up of coefficient (a constant multiple) of a power function with a whole number exponent
- rational functions: a quotient of two polynomials
  - vertical asymptotes: vertical lines $x = a$ where the function grows without bound - usually where the denominator is zero and the numerator is not zero
  - removable discontinuity: points where the function is not defined due to a factor that cancels out from both the numerator and denominator
  - hotizontal asymptotes: horizontal lines $y = L$ where the function approaches as $x \to \infty$
    - deg $P <$ deg Q ($y = 0$)
    - deg $P =$ deg Q ($y = \frac{\text{leading coefficient of P}}{\text{leading coefficient of Q}}$)
    - deg $P >$ deq Q (no horizontal asymptote look for slant instead)
  - slant asymptote: lines $y = mx + b$ that the function approaces as $x \to \infty$, when the numerator's degree is exactly one more than the denominator's degree...to find them perform polynomial long division and the quotient is the slant asymptotoe
- algebraic functions: produced by taking sums, products and quotients of roots or polynomials and rational functions
- exponential functions: $f(x) = b^x$ where $b > 0$...the inverse of which is $f(x) = log_b x$
- trigonometric functions: built from $\sin(x)$ and $\cos(x)$ are called trigonometric functions.

constructing new functions:
If f and g are functions, we may construct new functions by forming the sum, difference, product, and quotient functions: $(f+g)(x) = f(x)+g(x)$, $(f-g)(x) = f(x)-g(x)$, $(fg)(x) = f(x)g(x)$, $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$. We can also multiply functions by constants. We call this a linear combination: $c_1 f(x) + c_2 g(x)$. Composition is another important way of constructing new functions. The composition of f and g is the function $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$.

invertable functions:
A function f is invertible if there exists another function $f^{-1}(x)$ such that: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$. This "inverse function" switches input and outputs - it reverses the effect of the original function.

A function is invertible if it is one-to-one (horizontal line test). So, this means that different input always produce different outputs.

special factorizations

- $x^2 - y^2 = (x+y)(x-y)$
- $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
- $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

binomial theorem

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x-y)^2 = x^2 - 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
- $(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + nxy^{n-1} + y^n$ where $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}$

scalars, vectors, and matrices

scalar multiplication:
$\vec{w} = (1,2)$
$3\vec{w} = (3,6)$

component form: $\vec{v} = (x,y)$
polar form: $\vec{v} = (r\cos(\theta), r\sin(\theta)) = r(\cos(\theta), \sin(\theta))$

When you graphically add two vectors you add them tail to head. Vector addition is commutative ($\vec{a} + \vec{b} = \vec{b} + \vec{a}$). Preforming both of these additions graphically will form a parallelogram and the vector that bisects this parallelogram is the solution.

A system of equations is a set of two or more equations that share the same variables. The goal is to find values for those variable that make all the equations in the system true at the same time. For example, in a system with two equations and two variables (like x and y), you are looking for a point (x,y) that satisfies both equations.

To solve a system of equations using substitution, you start by solving one of the equations for one variable in terms of the other. Then, you substitute that expression into one of the other equations. This gives you an equation with just one variable (assuming you started with two), which you can solve. Once you find that value, you plug it back into one of the original equations to find the second variable.

With elimination, the goal is to eliminate one variable by adding or subtracting the equations. You may need to multiply one or both equations first to make the coefficients of a variable match. Once a variable is eliminated, you solve the resulting equation for the remaining variable, then substitute that value back into one of the original equations to find the other variable.

You can only add or subtract matrices if they have the same number of rows and the same number of columns. This is because matrix addition and subtraction is an element-wise operation.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 12 \\ 4 & 6 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 5 & 2 & 7 \end{bmatrix}$$

The dimensions of a matrix give the number of rows and columns of the matrix in that order. Since matrix A has 2 rows and 3 columns, it is called a 2×3 matrix.

A zero matrix is a matrix in which all of the entries are 0.

$$O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Zero matrices play a similar role in operations with matrices as the number zero plays in operations with real numbers. When we add an $m \times n$ zero matrix to any $m \times n$ matrix A, we get matrix A back. In other words, $A + O = A$ and $O + A = A$.

$$\begin{bmatrix} 4 & 1 \\ -6 & 2 \end{bmatrix} + \begin{bmatrix} -4 & -1 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

When we add any $m \times n$ matrix to its opposite, we get the $m \times n$ zero matrix. $A + (-A) = O$ and $-A + A = O$ It is also true that $A - A = O$. This is because subtracting a matrix is like adding its opposite.

matrix addition properties (A, B, and C are equal dimensions):

- commutative property of addition: $A + B = B + A$
- associative property of addition: $A + (B + C) = (A + B) + C$
- additive identity: For any matrix A, there is a unique matrix O such that $A + O = A$
- additive inverse property: For each A, there is a unique matrix $-A$ such that $A + (-A) = O$
- closure property of addition: $A + B$ is a matrix of the same dimensions A and B

matrix scalar multiplication properties (A and B are matrices of equal dimensions, c and d are scalars, and O is a zero matrix):

- associative property of multiplication: $(cd)A = c(dA)$
- distributive properties: $c(A + B) = cA + cB$
- multiplicative identity property: $1A = A$
- multiplicative properties of zero: $0 \cdot A = O$ and $c \cdot O = O$
- closure property of multiplication: cA is a matrix of the same dimensions as A

Matrices can be thought of as transformations of the plane, meaning they take points (or vectors) in a 2D space and move or change them in specific ways. Imagine every point in the plane as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$. When you multiply that vector by a 2×2 matrix, the output is a new vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that represents the transformed point.

$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

the new vector would be $M \cdot v = v'$ or you could do $v_{\text{row}} M^T = v'$

What kinds of transformations can matrices do?

- scaling: stretching or shrinking the plane along the x- and/or y-axis
- rotation: turning all points around the origin by some angle
- reflection: flipping points over a line, like the x-axis or y-axis
- shearing: slanting the shape, pushing points sideways or vertically
- projection: flattening points onto a line or plane

A column vector and a row vector can contain the same numbers, but their orientation matters a lot, especially when it comes to matrix operations like multiplication.

$$\begin{pmatrix} 2 & 5 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = (2)(4) + (5)(0) + (-1)(3) = 8 + 0 - 3 = 5$$

$$\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \begin{pmatrix} 2 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 6 \\ 20 & 0 & 15 \\ -4 & 0 & -3 \end{pmatrix}$$

The determinant of a square matrix is a special number that represents how the matrix transforms space—specifically, it measures how the matrix scales volume (or area in 2D). If the determinant is zero, it means the transformation squashes space into a lower dimension, losing information. A matrix with a zero determinant is called singular, which means it does not have an inverse. This happens when the matrix's columns (or rows) are linearly dependent, meaning one column can be expressed as a combination of others. In systems of equations, a singular (determinant-zero) matrix implies the system is either dependent—where the equations overlap and there are infinitely many solutions—or inconsistent, where the equations contradict and have no solution. A system is consistent if it has at least one solution, whether unique or infinite. Thus, the determinant and singularity give essential insight into whether a system can be solved uniquely, infinitely, or not at all.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(A) = ad - bc$

If $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, then $\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$

What does it mean to compose matrices? It means that you can apply transformations in sequence by multiplying their matrices together.

$T = BA$
$v' = B(Av) = (BA)v$
$f(x) = Ax$
$g(x) = Bx$
$g(f(x)) = B(Ax) = (BA)x$

properties of matrix multiplication (A, B, and C are n x n matrices, I is the n x n identity matrix and O is the n x n zero matrix):

- the commutative property of multiplication does not hold! $AB \neq BA$
- associative property of multiplication: $(AB)C = A(BC)$
- distributive properties: $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$
- multiplicative identity property $IA = A$ and $AI = A$
- multiplicative property of zero: $OA = O$ and $AO = O$
- dimension property: the product of an m x n matrix and an n x k matrix is an m x k matrix

Consider a system of equations like so: $3x - 2y - z = -1$
$2x + 5y + 2 = 0$
$-4x - y = 8$
$\downarrow$

$$\begin{bmatrix} 3 & -2 & -1 \\ 2 & 5 & 1 \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 8 \end{bmatrix}$$

$\uparrow$ generalized $A_{n\times n}\vec{x}_{n\times 1} = \vec{b}_{n\times 1}$

When the coefficient matrix A is not square, you cannot use the inverse to solve the system because only square matrices can have an inverse, and only non-singular square matrices (with $\det(A) \neq 0$) actually do have an inverse. In an underdetermined system (more variables than equations: $m < n$) you will usually have infintely many solutions, or sometimes none. Here you would use substitution, elimination, parametric solutions, or row-reduction (gaussian elimination). In a overdetermined system (more equations than variables: $m > n$) it might be inconsistent (no solution), or sometimes it has a best-fit solution. Here you would use gaussian elimination, least squares method, or pseudoinverses. There are other methods for square matrices like row-reduction or cramer's rule.

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

let A represent a 90° clockwise rotation and b represent a 90° counter-clockwise rotation of the unit vectors. Visually speaking, thses two rotation matrices are inverses of each other. Multiplying them in either order gives the identity matrix.

For any invertible square matrix A, the inverse A^{-1} can be calculated as:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

where:

- $\det(A)$ is the determinant of matrix A (and must be non-zero),
- $\text{adj}(A)$ is the adjugate (or classical adjoint) of A, which is the transpose of the cofactor matrix

multiplying 2x2 vs 3x3...nxn?
how to find the determinant of nxn?
how to find the minor and cofactor?
how to find the adjugate matrix?

probability and combinatorics consider a standard deck of cards (no jokers)
$P(\text{jack}) = \frac{4}{52} = \frac{1}{13}$
$P(\text{hearts}) = \frac{13}{52}$

now consider a venn diagram for the following: $P(\text{J} \cup \text{H}) = \frac{4+13-1}{52}$
since the probability of hearts overlaps with the probability of jacks (jack of hearts) youll have to subtract out the double counting...we say that that are not mutually exclusive otherwise you could just add $P(\text{J}) + P(\text{H})$ if they were mutually exclusive

addition rule:
$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$

now consider mutually exclusive events:
$P(\text{A} \cap \text{B}) = 0$
$P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B})$

multiplication rule for independent events:
if two events, A and B, are independent (meaning the occurrence of one does not affect the other), then: $P(\text{A} \cap \text{B}) = P(\text{A}) \cdot P(\text{B})$

consider a fair coin:
$P(\text{back to back heads}) = \frac{1}{2} \cdot \frac{1}{2}$
now consider dependent events say you have a bag with 3 blue and 2 red balls...what is the probability that the first pull is blue and the second is blue

$P(\text{1st blue}) \cdot P(\text{2nd blue} \mid \text{1st blue})$

$P(\text{A} \cap \text{B}) = P(\text{B}) \cdot P(\text{A} \mid \text{B}) = P(\text{A}) \cdot P(\text{B} \mid \text{A})$

permutations (order matters)
$P(n, k) = \frac{n!}{(n-k)!}$

combinations (order does not matter)
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

a probability distribution describes how likely different outcomes are for a random variable

- a random variable is something that can take on different values due to chance (like rolling a die or counting the number of heads in coin flips)
- a probability distribution assigns a probability to each possible value that the variable can take

the expected value of a random variable is the long-run average outcome you would expect if you repeated an experiment many times. think of it as the center or balance point of a probability distribution.

$E[X] = \sum_{i=1}^{n} x^i \cdot P(x_i)$

series

A sequence is an ordered list of numbers following some rule or pattern. You can get each term from the previous one by repeated addition (arithmetic sequence) or repeated multiplication (geometric sequence), or some other rule.

examples of other rules:

- fibonacci: each term is the sum of the previous two terms: $F_n = F_{n-1} + F_{n-2}$
  so the sequence starts: $0, 1, 1, 2, 3, 5, 8, 13, \ldots$
- quadtratic:
  terms might increase by a pattern that fits a quadratic formula, like n^2:
  $1, 4, 9, 16, 25 \ldots$
- alternating:
  terms might alternate between positive and negative, like:
  $1, -1, 1, -1, 1, -1, \ldots$
- recursive:
  terms can be defined using more complex rules involoving previous terms

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$
$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

The first term is an expression where the variable r is raised to the zeroth power, so the nth term is an expression that has r raised to the n-1 power.

sum of a finite geometric series ($r \neq 1$):

- $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

- $rS_n = ar + ar^2 + ar^3 + \ldots + ar^n$
- $S_n - rS_n = (a + ar + ar^2 + \ldots + ar^{n-1}) - (ar + ar^2 + ar^3 + \ldots + ar^n)$
- $S_n - rS_n = a - ar^n$
- $S_n(1-r) = a(1-r^n)$
- $S_n = a\frac{1-r^n}{1-r}$ if $r \neq 1$

summation notation
the binomial theorem
arithmetic series

Trigonometry

Angle (Degrees)	Angle (Radians)	$\cos(\theta)$	$\sin(\theta)$
$0°$	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$90°$	$\frac{\pi}{2}$	0	1
$120°$	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$135°$	$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$150°$	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$180°$	π	-1	0
$210°$	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$225°$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$240°$	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$270°$	$\frac{3\pi}{2}$	0	-1
$300°$	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$315°$	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$330°$	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$360°$	2π	1	0

$\pi = \frac{\text{circumference}}{\text{diameter}}$
$1° = \frac{\pi}{180}\text{radians}$

Trigonometric functions are special mathematical functions that originally come from studying right triangles. They relate the size of an angle in a triangle to the ratios of the lengths of the triangle's sides.

SOH-CAH-TOA is a mnemonic device that expresses the relationship between the basic trigonometric functions and the ratios of the sides in a right triangle.

The triangle definition only works for angles between 0 and $\frac{\pi}{2}$. To extend trig functions to all angles (including negative angles and angles larger then (2π or $360°$), mathematicians use the unit circle. Thus allowing use to use trig functions on the coordinate plane, enabling graphing and calculus.

trigonometric identities

- $\frac{1}{\cos(\theta)} = \sec(\theta)$
- $\frac{1}{\sin(\theta)} = \csc(\theta)$
- $\frac{1}{\tan(\theta)} = \cot(\theta)$
- $\sin(\theta + 2\pi) = \sin(\theta)$
- $\cos(\theta + 2\pi) = \cos(\theta)$
- $\tan(\theta + \pi) = \tan(\theta)$
- $\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$
- $\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$

- $\tan(\frac{\pi}{2} - \theta) = \cot(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(\theta) = -\tan$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $1 + \tan^2(\theta) = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \csc^2(\theta)$
- $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$
- $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
- $\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\cos(2\theta) = 2\cos^2(\theta) - 1$
- $\cos(2\theta) = 1 - 2\sin^2(\theta)$

law of sines and cosines

$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \frac{\sin(C)}{c}$
$a^2 = b^2 + c^2 - 2bc\cos(A)$ (SAS or SSS)

Limits

so let us consider $f(x) = \frac{\sin(x)}{x}$ (x in radians)
the value of $f(0)$ is not defined because
$f(0) = \frac{\sin(0)}{0} = \frac{0}{0}$
however if we take $x \to 0^+$ and $x \to 0^-$ we get the impression that $f(x)$ gets closer and closer to 1 as $x \to 0$ from the left and right.

$$\lim_{x\to 0} f(x) = 1$$

recall that the distance between two numbers $|a-b|$...thus we can express the idea that $f(x)$ is close to L by saying that $|f(x) - L|$ is small.

an informal limit definition
Assume that $f(x)$ is defined for all x near c (i.e., in some open interval containing c), but not necessarily at c itself. We say that: *the limit of $f(x)$ as x approaches c is equal to L* if $|f(x) - L|$ becomes arbitarily small when x is any number sufficiently close (but not equal) to c. In this case, we write:

$$\lim_{x\to c} f(x) = L$$

we also say the $f(x)$ approaches or converges to L as $x \to c$ (and we write $f(x) \to L$)

If the values of $f(x)$ do not converge to any limit as $x \to c$, we say that $\lim_{x\to c} f(x)$ DNE. It is important to note that the value $f(c)$ itself, which may or may not be defined, plays no role in the limit. All that matters are the values $f(x)$ for x close to c. Furthermore, if $f(x)$ approaches a limit as $x \to c$, then the limiting value L is unique. We say the $\lim_{x\to c} f(x) = \infty$ if $f(x)$ increases beyond bound as x approaches c, and $\lim_{x\to c} f(x) = -\infty$ if $f(x)$ becomes arbitrarily large (in absolute value) but negative as x approaches c

The Limit Laws state that if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

- $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
- $\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x)$
- $\lim_{x\to c} f(x)g(x) = (\lim_{x\to c} f(x))(\lim_{x\to c} g(x))$
- if $\lim_{x\to c} g(x) \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$
- if $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x) DNE$, then the Limit Laws cannot be applied

Assume that $f(x)$ is defined on an open interval containing $x = c$. Then f is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$ If the limit does not exist, or if it exists but is not equal to $f(c)$, we say that f has a discontinuity (or is discontinuous) at $x = c$. A function $f(x)$ may be continuous at some points and discontinuous at others. if $f(x)$ is continuous at all points in an interval I, then $f(x)$ is said to be continuous on I. Here, if I is an interval $[a, b]$ or $[a, b)$ that includes a as a left-endpoint, we require that $\lim_{x\to a+} f(x) = f(a)$. Similarly, we require that $\lim_{x\to b-} f(x) = f(b)$ if I includes b as a right-endpoint b. If $f(x)$ is continuous at all points in its domain, then $f(x)$ is simply called continuous. Lets look at how a function can fail to be continuous...remeber that point discontinuity requires that the limit exist at that point, the value of the function exists at that point, and those two equal. If the first two conditions hold but that last one fails then we say that the function has a removable discontinuity at that point. Removable discontinuities are "mild" in the following sense: We can make f continuous at $x = c$ by redefining $f(c)$.

A "worse" type of discontinuity is a jump discontinuity, which occurs if the one-sided limits $\lim_{x\to c-} f(x)$ and $\lim_{x\to c+} f(x)$ exist but are not equal. We say that $f(x)$ has an infinite discontinuity at $x = c$ if one or both of the one-sided limits is infinite (even if $f(x)$ itself is not defined at $x = c$). We should mention that some functions have more "severe" types of discontinuity than those discussed above. For example, $f(x) = \sin(\frac{1}{x})$ oscillates infinitely often between $+1$ and -1 as $x \to 0$. Nether the left-nor the right-hand limits exist at $x = 0$, so this discontinuity is no a jump discontinuity. Although of interest from a theoretical point of view, these discontinuities rarely arise in practice.

Laws of Continuity
Assume that $f(x)$ and $g(x)$ are continuous at a point $x = c$. Then the following functions are also continuous at $x = c$:

- $f(x) + g(x)$
- $f(x) - g(x)$
- $kf(x)$ for any constant k
- $f(x)g(x)$
- $\frac{f(x)}{g(x)}$ if $g(c) \neq 0$

Continuity of Polynomial and Rational Functions
Let $P(x)$ and $Q(x)$ be polynomials. Then:

- $P(x)$ is continuous on the real line
- $\frac{P(x)}{Q(x)}$ is continuous at all values c such that $Q(x) \neq 0$

Continuity of Some Basic Functions

- $y = \sin(x)$ and $y = \cos(x)$ are continuous on the real line
- For $b > 0$, $y = b^x$ is continuous on the real line
- If n is a natural number, then $y = x^{\frac{1}{n}}$ is continuous on its domain

Continuity of Composite Functions
Let $F(x) = f(g(x))$ be a composite function. If g is continuous at $x = c$ and f is continuous at $x = g(c)$, then $F(x)$ is continuous at $x = c$. It is easy to evaluate a limit when the function in question is known to be continuous. in this case, by definition, the limit is equal to the function value. In general, we say that $f(x)$ has an indeterminate form at $x = c$ if, when $f(x)$ is evaluated at $x = c$, we obtain an indefined expression of the type $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$. Later when we study derivatives, we will be faced with limits $lim_{x\to c} f(x)$, where $f(c)$ is not defined. In such cases, substitution cannot be used directly. However, some of these limits can be evaluated using substitution, provided that we first use algebra to rewrite the formula for $f(x)$

Evaluating Limits Algebraically

- $\lim_{x\to 4} \frac{x^2-16}{x-4}$
  The function is not defined at $x = 4$ because $f(4) = \frac{4^2-16}{4-4} = \frac{0}{0}$ (arithmetically

undefined)

However, the numerator of $f(x)$ factors and
$\frac{x^2-16}{x-4} = \frac{(x+4)(x-4)}{x-4} = x+4$ (valid for $x \neq 4$)
In other words, $f(x)$ coincides with the *continuous* function $x+4$ for all $x \neq 4$. Sinve the limit depends only on the values of $f(x)$ for $x \neq 4$, we have
$\lim_{x\to 4} \frac{x^2-16}{x-4} = \lim_{x\to 4}(x+4) = 8$

In general, we say that $f(x)$ has an indeterminate form at $x = c$, we obtain an undefined expression of the type

$\frac{0}{0}, \frac{\infty}{\infty}, \infty \cdot 0, \infty - \infty$

We also say that f is indeterminate at $x = c$. Our strategy is to *transform $f(x)$ algebraically if possible into a new expression that is defined and continuous at $x = c$, and then evaluate by substitution ("plugging in").* As you study the following examples, notice that the critical step in each case is to cancel a common factor from the numerator and denominator at the appropriate moment, thereby removing the indeterminacy.

- $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$
  The function $f(x) = \frac{\sqrt{x}-2}{x-4}$ is indeterminate at $x = 4$ since

  $f(4) = \frac{\sqrt{4}-2}{4-4} = \frac{0}{0}$ (indeterminate)

  $(\frac{\sqrt{x}-2}{x-4})(\frac{\sqrt{x}+2}{\sqrt{x}+2}) = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$ (if $x \neq 4$)

  Since $\frac{1}{\sqrt{x}+2}$ is continuous at $x = 4$,

  $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

- $\lim_{x\to 2} \frac{x^2-x+5}{x-2}$
  At $x = 2$ we have

  $f(2) = \frac{2^2-2+5}{2-2} = \frac{7}{0}$ (undefined, but not an indeterminate form)
  This is *not* an indeterminate form. In fact, shows that the one-sided limits are infinte:

  $\lim_{x\to 2-} \frac{x^2-x+5}{x-2} = -\infty$, $\lim_{x\to 2+} \frac{x^2-x+5}{x-2} = \infty$

  The limit itself does not exist.

The Squeeze Theorem
In our study of the derivative, we will need to evaluate certain limits involving transcendental functions such as sine and cosine. The algebraic techniques of the previous section are often ineffective for such functions and other tools are required. One such tool is the Squeeze Theorem, which we discuss in this section and use to evaluate the trigonometric limits later. Consider a function $f(x)$ that is trapped between two functions $l(x)$ and $u(x)$ on an interval I. In other words, $l(x) \leq f(x) \leq u(x)$ for all $x \in I$.

In this case, the graph of $f(x)$ lies between the graphs of $l(x)$ and $u(x)$, with $l(x)$ as the lower and $u(x)$ as the upper function. The Squeeze Theorem applies when $f(x)$ is not just trapped, but actually squeezed at a point $x = c$ by $l(x)$ and $u(x)$. By this we mean that for all $x \neq c$ in some open interval containing c, $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. We do not require that $f(x)$ be defined at $x = c$, but it is clear graphically that $f(x)$ must approach the limit L. We state this formally: Assume that for $x \neq c$ (in some open interval containing c), $l(x) \leq f(x) \leq u(x)$ and $\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$. Then $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = L$

two important limits
$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$
$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$

Intermediate Value Theorem
This is a basic result which states that a continuous function on an interval cannot skip values. If $f(x)$ is continuous on a closed interval $[a, b]$ and $f(a) \neq f(b)$, then for every value M between $f(a)$ and $f(b)$, there exists at least one value $c \in (a, b)$ such that $f(c) = M$. Prove that the equation $\sin(x) = 0.3$ has at least one solution. We may apply the IVT since $\sin(x)$ is continuous. We choose an interval where we suspect that a solution exists. The desired value 0.3 lies between the two function value $\sin(0) = 0$ and $\sin(\frac{\pi}{2} = 1$. so the interval $[0, \frac{\pi}{2}]$ will work. The IVT tells us that $\sin(x) = 0.3$ has at least one solution in $(0, \frac{\pi}{2})$. Since $\sin(x)$ is periodic, $\sin(x)$ actually has infinitely many solutions. The IVT can be used to show the existence of zeros of functions. If $f(x)$ is continuous and takes on both positive and negative values, say, $f(a) < 0$ and $f(b) > 0$, then the IVT guarantees that $f(c) = 0$ for some c between a and b. If $f(x)$ is continuous on $[a, b]$ and if $f(a)$ and $f(b)$ are nonzero and have opposite signs, then $f(x)$ has a zero in (a, b). We can locate zeros of functions to arbitrary accuracy using the Bisection Method. Show that $f(x) = \cos^2(x) - 2\sin(\frac{x}{4})$ has a zero in $(0, 2)$. Then locate the zero more accurately using the Bisecition Method. Using a calculator, we find that $f(0)$ and $f(2)$ have opposite signs: $f(0) = 1 > 0$, $f(2) \approx -0.786 < 0$. We can locate a zero more accurately by dividing $[0, 2]$ into two intervals $[0, 1]$ and $[1, 2]$. One of these must contain a zero of $f(x)$. To determine which, we evaluate $f(x)$ at the midpoint $m = 1$. A calculator gives $f(1) \approx -0.203$, and since $f(0) = 1$, we see that $f(x)$ takes on opposite signs at the endpoints of $[0, 1]$. Therefore, $(0, 1)$ must contain a zero. We discard the $[1, 2]$ because both $f(1)$ and $f(2)$ are negative. The Bisection Method consists of continuing this process until we narrow down the location of the zero to the desired accuracy. The IVT seems to state the obvious, namely that a continuous function cannot skip values. Yet its proof is quite subtle because it depends on the completeness property of the real numbers. To highlight the subtlety observe that IVT is false for functions defined only on the rational numbers. For example, $f(x) = x^2$ does not have the intermediate value property if we restrict its domain to the rational numbers. Indeed, $f(0) = 0$ and $f(2) = 4$ but $f(c) = 2$ has no solution for c rational. The solution $c = \sqrt{2}$ is "missing" from the set of rational numbers because it is irrational. From the beginnings of calculus, the IVT was surely regarded as obvious. However, it was not possible to give a genuinely rigorous proof until the completeness property was clarified in the second half of the ninteenth century.

The Size of the Gap
Recall that the distance from $f(x)$ to L is $|f(x) - L|$. It is convenient to refer to that quantity $|f(x) - L|$ as the *gap* between the value $f(x)$ and the limit L. Let us reexamine the basic trigonometric limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. In this example, $f(x) = \frac{\sin(x)}{x}$ and $L = 1$, so (1) tells us that the gap $|f(x) - 1|$ gets arbitrarily small when x is sufficiently

close but not equal to 0. Suppose we want the gap $|f(x)-1|$ to be less than 0.2. How close to 0 must x be? The following statement is true: $|\frac{\sin(x)}{x}-1| < 0.2$ if $0 < |x| < 1$. If we insist instead that the gap be smaller than 0.004... $|\frac{\sin(x)}{x}-1| < 0.004$ if $0 < |x|$. It would seem that this process can be continued: By zooming in on the graph, we can find a small interval around $c = 0$ where the gap $|f(x)-1|$ is smaller than any prescribed positive number.To express this in a precise fashion, we follow time-honored tradition and use the Greek letter ϵ and δ to denote small numbers specifying the size of the gap and the quantity $|x-c|$, respectively. In our case, $c = 0$ and $|x-c| = |x-0| = |x|$. The precise meaning is that for every choise of $\epsilon > 0$, there exists some δ (depending on ϵ) such that $|\frac{\sin(x)}{x}-1| < \epsilon$ if $0 < |x| < \delta$. The number δ tells us how close is sufficiently close for a given ϵ. With this motivation, we are ready to state the formal definition of the limit.

FORMAL DEFINITION OF A LIMIT

Suppose the $f(x)$ is defined for all x in an open interval containing c (but no necessarily at $x = c$). Then

$$\lim_{x\to c} f(x) = L$$

if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x)-L| < \epsilon \text{ if } 0 < |x-c| < \delta$$

The condition $0 < |x-c| < \delta$ in this definition excludes $x = c$. As in our previous informal definition, we formulate it this way so that the limit depends only on values of $f(x)$ near c but no on $f(c)$ itself. As we have seen, in many cases the limit exists even when $f(c)$ is not defined.

differentiation

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$y - f(a) = f'(a)(x-a)$$

$\frac{d}{dx}(c) = 0$
$\frac{d}{dx}x = 1$
$\frac{d}{dx}(x^n) = nx^{-1}$ (power rule)
$\frac{d}{dx}[cf(x)] = cf'(x)$
$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$
$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$ (product rule)
$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$ (quotient rule)
$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (chain rule)
$\frac{d}{dx}f(x)^n = nf(x)^{n-1}f'(x)$ (general power rule)
$\frac{d}{dx}\sin(x) = \cos(x)$
$\frac{d}{dx}\cos(x) = -\sin(x)$
$\frac{d}{dx}\tan(x) = \sec^2(x)$ $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
$\frac{d}{dx}\cot(x) = -\csc^2(x)$
$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{(1-x^2)}}$
$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{(1-x^2)}}$
$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(a^x) = (\ln a)a^x$
$\frac{d}{dx}\ln \mid x \mid = \frac{1}{x}$
$\frac{d}{dx}\log_a x = \frac{1}{(\ln a)x}$

implicit differentiation
EVT
rolle's theorem
MVT
optimization
newtons method

integration