Maths

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Algebra

Lines

Slope of the line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Point-slope equation of line through $P_1 = (x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Circles

Equation of the circle with center (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

Distance and Midpoint Formulas

Distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $\bar{P_1 P_2}$:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

Laws of Exponents

$$x^m x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$\left(\frac{x}{y} \right)^n = \frac{x^n}{y^n}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Special Factorizations

$$x^2 - y^2 = (x + y)(x - y)$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + nxy^{n-1} + y^n$$
$$\text{where } \binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{1\cdot 2\cdot 3\ldots k}$$

Quadratic Formula

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Inequalities and Absolute Value

$$\text{If } a < b \text{ and } b < c, \text{ then } a < c.$$

$$\text{If } a < b, \text{ then } a + c < b + c.$$

$$\text{if } a < b \text{ and } c > 0, \text{ then } ca < cb.$$

$$\text{if } a < b \text{ and } c < 0, \text{ then } ca > cb.$$

$$|x| = x \text{ if } x >= 0$$

$$|x| = -x \text{ if } x <= 0$$

Geometry

Formulas for area A, circumference C, and volume V
Triangle

$$A = \tfrac{1}{2}bh$$

$$A = \tfrac{1}{2}ab\sin(\theta)$$

Circle

$$A = \pi r^2$$

$$C = 2\pi r$$

Sector of Circle

$$A = \tfrac{1}{2}r^2\theta$$

$$s = r\theta$$

Sphere

$$V = \tfrac{4}{3}\pi r^3$$

$$A = 4\pi r^2$$

Cylinder

$$V = \pi r^2 h$$

Cone

$$V = \tfrac{1}{3}\pi r^2 h$$

$$A = \pi r \sqrt{r^2 + h^2}$$

Cone with arbitrary base

$$V = \tfrac{1}{3}Ah$$

Trigonometry

Pythagorean Theorem: For a right trianlge with hypotenuse of length c and legs of lengths a and b, $c^2 = a^2 + b^2$.

Angle Measurement

$$\pi \text{ radians} = 180°$$

$$1° = \tfrac{\pi}{180}rad$$

$$1 \text{ rad} = \tfrac{180}{\pi}$$

$$s = r\theta \ (\theta \text{ in radians})$$

Right Triangle Definitions

$$\sin\theta = \tfrac{opp}{hyp}$$

$$\cos\theta = \tfrac{adj}{hyp}$$

$$\tan \theta = \frac{sin\theta}{cos\theta} = \frac{opp}{adj}$$

$$\sec \theta = \frac{1}{cos\theta} \quad \csc \theta = \frac{1}{sin\theta}$$

Trigonometric Functions

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$

$$\sec \theta = \frac{r}{x}$$

$$\csc \theta = \frac{r}{y}$$

$$\lim_{\theta \to 0} \frac{sin\theta}{\theta} = 1$$

$$\lim_{\theta \to 0} \frac{1 - cos\theta}{\theta} = 0$$

Fundamental Identities

$$sin^2\theta + cos^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + cot^2\theta = csc^2\theta$$

$$\sin(\tfrac{\pi}{2} - \theta) = \cos(\theta)$$

$$\cos(\tfrac{\pi}{2} - \theta) = \sin(\theta)$$

$$\tan(\tfrac{\pi}{2} - \theta) = \cot(\theta)$$

$$\sin(-\theta) = -sin\theta$$

$$\cos(-\theta) = cos\theta$$

$$\tan(-\theta) = -tan\theta$$

$$\sin(\theta + 2\pi) = sin\theta$$

$$\cos(\theta + 2\pi) = cos\theta$$

$$\tan(\theta + \pi) = tan\theta$$

The Law of Sines

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

The Law of Cosines

$$a^2 = b^2 + c^2 - 2bccosA$$

Addition and Subtraction Formulas

$$\sin(x + y) = sinxcosy + cosxsiny$$

$$\sin(x - y) = sinxcosy - cosxsiny$$

$$\cos(x + y) = cosxcosy - sinxsiny$$

$$\cos(x - y) = cosxcosy + sinxsiny$$

$$tan(x + y) = \frac{tanx + tany}{1 - tanxtany}$$

$$tan(x - y) = \frac{tanx - tany}{1 + tanxtany}$$

Double-Angle Formulas

$$sin2x = 2sinxcosx$$

$$cos2x = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x$$

$$tan2x = \frac{2tanx}{1 - tan^2x}$$

$$sin^2x = \frac{1 - cos2x}{2}$$

$$cos^2x = \frac{1 + cos2x}{2}$$

Precalculus Review

78. Prove the triangle inequality by adding the two inequalities

1.) Known Inequalities
$-|a| \le a \le |a|$
$-|b| \le b \le |b|$

2.) Add and Simplify
$(-|a|) + (-|b|) \le a + b \le |a| + |b|$
$-(|a| + |b|) \le a + b \le |a| + |b|$

3.) By the definition of absolute value, we know:
$-|x| \le x \le |x|$
$-|a + b| \le a + b \le |a + b|$

4.) To explain, the value $a + b$ is squeezed between $-(|a| + |b|)$ and $|a| + |b|$. By taking the absolute value on both sides, we conclude that:
$-(|a| + |b|) \le a + b \le |a| + |b|$
$|a + b| \le |a| + |b|$

79. Show that if $r = \frac{a}{b}$ is a fraction in lowest terms, then r has a finite decimal expansion if and only if $b = (2^n)(5^m)$ for some n, m ≥ 0. Hint: Observe that r has a finite decimal expansion when $(10^N)(r)$ is an integer for some $N \geq 0$ (and hence b dividies 10^N).

Finite Decimal Expansion implies $b = 2^n \cdot 5^m$

1. Finite Decimal Expansion: A fraction $\frac{a}{b}$ has a finite decimal expansion if and only if $\frac{a}{b}$ can be written as $k \cdot 10^{-}N$ for some integer k and non-negative integer N. This is equivalent to the condition that b divides 10^N for some $N \geq 0$

2. Denominator as a Product of Powers of 2 and 5: Observe that $10^N = 2^N \cdot 5^N$. Therefore, if b divides 10^N, then b must be of the form $b = \frac{10^N}{k}$, where k is an integer that ensures b divides 10^N. This implies that b must only have the prime factors 2 and 5 because 10^N itself only contains the prime factors 2 and 5. Thus, if b divides 10^N, then b must be of the form $b = 2^n \cdot 5^m$ for some non-negative integers n and m

$b = 2^n \cdot 5^m$ **Implies Finite Decimal Expansion**

1. Form of b: Suppose $b = 2^n \cdot 5^m$. We want to show that $\frac{a}{b}$ has a finite decimal expansion. Since b can be written as $2^n \cdot 5^m$, it folows that b is a dividor of 10^N where $N = max(n, m)$.

2. Verification: To be specific, let us express $\frac{a}{b}$ in terms of 10^N:

$\frac{a}{b} = \frac{a}{2^n \cdot 5^m}$

We can multiply both the numerator and the denominator by 10^N, where $N = max(n, m)$. This multiplication yields:

$\frac{a \cdot 10^N}{b \cdot 10^N} = \frac{a \cdot 10^N}{2^n \cdot 5^m \cdot 10^N} = \frac{a \cdot 10^N}{10^{N+n} \cdot 10^m} = \frac{a \cdot 10^N}{10^N}$

Since $b \cdot 10^N = 10^{N+n} \cdot 10^m$, which simplifies to 10^N, we get that $b \cdot 10^N$ is an integer.
Hence, $\frac{a \cdot 10^N}{b \cdot 10^N}$ is an integer, implying that $\frac{a}{b}$ indeed has a finite decimal expansion.

Conclusion

We have shown that if b divides 10^N for some $N \geq 0$, then b must be of the form $2^n \cdot 5^m$. Conversely, if $b = 2^n \cdot 5^m$, then $\frac{a}{b}$ has a finite decimal expansion. Therefor, the fraction $\frac{a}{b}$ in lowest terms has a finite decimal expansion if and only if the denominator b is of the form $2^n \cdot 5^m$.

80. Let $p = p_1 \ldots p_s$ be an integer with digits $p_1, \ldots, p_s$. Show that $\frac{p}{10^s - 1} = 0.\overline{p_1 \ldots p_s}$ Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that $r = \frac{2}{11} = \frac{18}{10^2 - 1}$