Linear Algebra

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NOTE: For these notes, I will be using *Linear Algrebra with Applications Seventh Edition* by Gareth Williams and the internet.

1 Linear Equation, Vectors, and Matrices

An equation in the variables x and y that can be written in the form $ax + by = c$, where a, b, and c are real constants (a and b not both zero), is called a linear equation. The graph of such an equation is a straight line in the xy plane. Consider the system of two linear equations,

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

A pair of values of x and y that satisfies both equations is called a solution. It can be seen by substitution that $x = 3$, $y = 2$ is a solution to this system. A solution to such a system will be a point at which the graphs of the two equations intersect. The following examples illustrate that three possibilities can arise for such systems of equations. There can be a unique solution, no solution, or many solutions. We use the point/slope form $y = mx + b$, where m is the slope and b is the y-intercept, to graph these lines.

Unique solution: these have different slopes

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

No solution: these have the same slopes and different y-intercepts

$$\begin{aligned} -2x + y &= 3 \\ -4x + 2y &= 2 \end{aligned}$$

Many solutions: these have the same slope and the same y-intercepts

$$\begin{aligned} 4x - 2y &= 6 \\ 6x - 3y &= 9 \end{aligned}$$

You can think of the number of equations in a system as analogous to the number of constraints or objects you have in a geometric space. Each equation typically represents a constraint that reduces the dimensionality of the solution space.

1 Variable: An equation in one variable (e.g., $x = a$) represents a point on a number line, which is zero-dimensional.
2 Variables: An equation in two variables (e.g., $y = mx + b$) represents a line in a two-dimensional space, which is one-dimensional.
3 Variables: An equation in three variables (e.g., $z = ax + by + c$) represents a plane in three-dimensional space, which is two-dimensional.

n Variables: More generally, an equation involving n variables defines an n-1 dimensional hyperplane in n-dimensional space.

1. Understanding Dimensions and Equations

Dimensions: The dimensionality of a space refers to the number of coordinates needed to describe a point within that space. For example:

- A point in 0 dimensions (0D).
- A line in 1 dimension (1D), which can be described by one variable (like x).
- A plane in 2 dimensions (2D), described by two variables (like x and y).
- A volume in 3 dimensions (3D), described by three variables (like x, y, and z).

2. Equations as Constraints

When you introduce equations, they impose constraints on the variables:

- One Equation:
  - In 2D, an equation like $y = mx + b$ represents a line. This line is one-dimensional, meaning you can move along it using a single parameter (for example, x).
- Two Equations:
  - If you have two equations, such as:

$$y = mx + b_1$$
$$y = mx + b_2$$

    In 2D, these represent two lines, If they intersect, they define a unique solution-a single point, which is zero-dimensional

3. Generalizing to Higher Dimensions

Three Equations:

- In 3D, three equations can define a point or a line, dpending on how they intersect. For instance:

$$z = ax + by + c$$
$$z = dx + ey + f$$
$$z = gx + hy + i$$

If these planes intersect at a single point, the solution is zero-dimensional (a specific point)

4. Conclusion

- The number of equations can be thought of as the number of constraints or "objects" you are using to limit the solution space.
- Generally, for n variables, if you have k independent equations, you can expect the solution space to be $n - k$ dimensions:
  - if $k = n$, you have a unique solution (0D).
  - if $k < n$, the solution space remains $n - k$ dimensional, meaning you have more freedom in choosing solutions.

Our aim in this chapter is to analyze larger systems of linear equations. A linear equation in n variables x_1, x_2, x_3, ..., x_n is one that can be written in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

where the coefficients a_1, a_2, ..., a_n and b are constants. The following is an example of a system of three linear equations.

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 + x_2 + 2x_3 &= -6 \end{aligned}$$