Calculus

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Algebra

Lines

Slope of the line through $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope-intercept equation of line with slope m and y-intercept b:

$$y = mx + b$$

Point-slope equation of line through $P_1 = (x_1, y_1)$ with slope m:

$$y - y_1 = m(x - x_1)$$

Circles

Equation of the circle with center (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

Distance and Midpoint Formulas

Distance between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint of $P_1\bar{}P_2$:

$$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$$

Laws of Exponents

$$x^m x^n = x^{m+n}$$

$$\frac{x^m}{x^n} = x^{m-n}$$

$$(x^m)^n = x^{mn}$$

$$x^{-n} = \frac{1}{x^n}$$

$$(xy)^n = x^n y^n$$

$$(\frac{x}{y})^n = \frac{x^n}{y^n}$$

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

$$\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$$

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Special Factorizations

$$x^2 - y^2 = (x+y)(x-y)$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

Binomial Theorem

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + nxy^{n-1} + y^n$$

where $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{1\cdot2\cdot3\ldots k}$

Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Inequalities and Absolute Value

If $a < b$ and $b < c$, then $a < c$.

If $a < b$, then $a + c < b + c$.

if $a < b$ and $c > 0$, then $ca < cb$.

if $a < b$ and $c < 0$, then $ca > cb$.

$|x| = x$ if $x >= 0$

$|x| = -x$ if $x <= 0$

Geometry

Formulas for area A, circumference C, and volume V
Triangle

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}ab\sin(\theta)$$

Circle

$$A = \pi r^2$$

$$C = 2\pi r$$

Sector of Circle

$$A = \frac{1}{2} r^2 \theta$$

$$s = r\theta$$

Sphere

$$V = \frac{4}{3}\pi r^3$$

$$A = 4\pi r^2$$

Cylinder

$$V = \pi r^2 h$$

Cone

$$V = \frac{1}{3}\pi r^2 h$$

$$A = \pi r \sqrt{r^2 + h^2}$$

Cone with arbitrary base

$$V = \frac{1}{3} A h$$

Trigonometry

Pythagorean Theorem: For a right trianlge with hypotenuse of length c and legs of lengths a and b, $c^2 = a^2 + b^2$.

Angle Measurement

$$\pi \text{ radians} = 180^\circ$$

$$1^\circ = \frac{\pi}{180} rad$$

$$1 \text{ rad} = \frac{180}{\pi}$$

$$s = r\theta \ (\theta \text{ in radians})$$

Right Triangle Definitions

$$\sin\theta = \frac{opp}{hyp}$$

$$\cos\theta = \frac{adj}{hyp}$$

$$\tan\theta = \frac{sin\theta}{cos\theta} = \frac{opp}{adj}$$

$$\sec\theta = \frac{1}{cos\theta} \ \csc\theta = \frac{1}{sin\theta}$$

Trigonometric Functions

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{r}{x}$$

$$\csc\theta = \frac{r}{y}$$

$$\lim_{\theta\to 0} \frac{sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0} \frac{1-cos\theta}{\theta} = 0$$

Fundamental Identities

$$sin^2\theta + cos^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + cot^2\theta = csc^2\theta$$

$$\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$$

$$\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$$

$$\tan(\frac{\pi}{2} - \theta) = \cot(\theta)$$

$$\sin(-\theta) = -sin\theta$$

$$\cos(-\theta) = cos\theta$$

$$\tan(-\theta) = -tan\theta$$

$$\sin(\theta + 2\pi) = sin\theta$$

$$\cos(\theta + 2\pi) = cos\theta$$

$$\tan(\theta + \pi) = tan\theta$$

The Law of Sines

$$\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$$

The Law of Cosines

$$a^2 = b^2 + c^2 - 2bccosA$$

Addition and Subtraction Formulas

$$\sin(x+y) = sinxcosy + cosxsiny$$

$$\sin(x-y) = sinxcosy - cosxsiny$$

$$\cos(x+y) = cosxcosy - sinxsiny$$

$$\cos(x-y) = cosxcosy + sinxsiny$$

$$tan(x+y) = \frac{tanx+tany}{1-tanxtany}$$

$$tan(x-y) = \frac{tanx-tany}{1+tanxtany}$$

Double-Angle Formulas

$$sin2x = 2sinxcosx$$

$$cos2x = cos^2x - sin^2x = 2cos^2x - 1 = 1 - 2sin^2x$$

$$tan2x = \frac{2tanx}{1-tan^2x}$$

$$sin^2x = \frac{1-cos2x}{2}$$

$$cos^2x = \frac{1+cos2x}{2}$$

Precalculus Review

78. Prove the triangle inequality by adding the two inequalities

1.) Known Inequalities

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

2.) Add and Simplify

$$(-|a|) + (-|b|) \leq a + b \leq |a| + |b|$$

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

3.) By the definition of absolute value, we know:

$$-|x| \leq x \leq |x|$$

$$-|a+b| \leq a + b \leq |a+b|$$

4.) To explain, the value $a+b$ is squeezed between $-(|a|+|b|)$ and $|a|+|b|$. By taking the absolute value on both sides, we conclude that:

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

$$|a+b| \leq |a| + |b|$$

79. Show that if $r = \frac{a}{b}$ is a fraction in lowest terms, then r has a finite decimal expansion if and only if $b = (2^n)(5^m)$ for some n, m ≥ 0. Hint: Observe that r has a finite decimal expansion when $(10^N)(r)$ is an integer for some $N \geq 0$ (and hence b dividies 10^N).

Finite Decimal Expansion implies $b = 2^n \cdot 5^m$

1. Finite Decimal Expansion: A fraction $\frac{a}{b}$ has a finite decimal expansion if and only if $\frac{a}{b}$ can be written as $k \cdot 10^-N$ for some integer k and non-negative integer N. This is equivalent to the condition that b divides 10^N for some $N \geq 0$

2. Denominator as a Product of Powers of 2 and 5: Observe that $10^N = 2^N \cdot 5^N$. Therefore, if b divides 10^N, then b must be of the form $b = \frac{10^N}{k}$, where k is an integer that ensures b divides 10^N. This implies that b must only have the prime factors 2 and 5 because 10^N itself only contains the prime factors 2 and 5. Thus, if b divides 10^N, then b must be of the form $b = 2^n \cdot 5^m$ for some non-negative integers n and m

$b = 2^n \cdot 5^m$ **Implies Finite Decimal Expansion**

1. Form of b: Suppose $b = 2^n \cdot 5^m$. We want to show that $\frac{a}{b}$ has a finite decimal expansion. Since b can be written as $2^n \cdot 5^m$, it folows that b is a dividor of 10^N where $N = max(n, m)$.

2. Verification: To be specific, let us express $\frac{a}{b}$ in terms of 10^N:

$$\frac{a}{b} = \frac{a}{2^n \cdot 5^m}$$

We can multiply both the numerator and the denominator by 10^N, where $N = max(n, m)$. This multiplication yields:

$$\frac{a \cdot 10^N}{b \cdot 10^N} = \frac{a \cdot 10^N}{2^n \cdot 5^m \cdot 10^N} = \frac{a \cdot 10^N}{10^{N+n} \cdot 10^m} = \frac{a \cdot 10^N}{10^N}$$

Since $b \cdot 10^N = 10^{N+n} \cdot 10^m$, which simplifies to 10^N, we get that $b \cdot 10^N$ is an integer.
Hence, $\frac{a \cdot 10^N}{b \cdot 10^N}$ is an integer, implying that $\frac{a}{b}$ indeed has a finite decimal expansion.

Conclusion

We have shown that if b divides 10^N for some $N \geq 0$, then b must be of the form $2^n \cdot 5^m$. Conversely, if $b = 2^n \cdot 5^m$, then $\frac{a}{b}$ has a finite decimal expansion. Therefor, the fraction $\frac{a}{b}$ in lowest terms has a finite decimal expansion if and only if the denominator b is of the form $2^n \cdot 5^m$.

80. Let $p = p_1 \dots p_s$ be an integer with digits $p_1, \dots, p_s$. Show that $\frac{p}{10^s-1} = 0.\overline{p_1 \dots p_s}$ Use this to find the decimal expansion of $r = \frac{2}{11}$. Note that $r = \frac{2}{11} = \frac{18}{10^2-1}$

53. Show that if $f(x)$ and $g(x)$ are linear, then so is $f(x) + g(x)$. Is the same true of $f(x)g(x)$?

54. Show that if $f(x)$ and $g(x)$ are linear functions such that $f(0) = g(0)$ and $f(1) = g(1)$, then $f(x) = g(x)$.

55. Show that the ratio $\frac{\Delta y}{\Delta x}$ for the function $f(x) = x^2$ over the interval $[x_1, x_2]$ is not a constant, but depends on the interval. Determine the exact dependence of $\frac{\Delta y}{\Delta x}$ on x_1 and x_2.

56. Derivation of the Quadratic Formula

57. Let a, $c \neq 0$. Show that the roots of $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ are reciprocals of each other.

58. Complete the square to show that the parabolas $y = ax^2 + bx + c$ and $y = ax^2$ have the same shape (show that the first parabola is congruent to the second by a vertical and horizontal translation).

59. Prove Viete's Formulas, which state that the quadratic polynomial with given numbers α and β as roots is $x^2 + bx + c$, where $b = -\alpha - \beta$ and $c = \alpha\beta$.

A quadratic function is a function defined by a quadratic polynomial

$$f(x) = ax^2 + bx + c \text{ (}a\text{,}b\text{,}c\text{, constants with } a \neq 0\text{)}$$

The technique of completing the square consists of writing a quadratic polynomial as a multiple of a squareplus a constant:

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 + \frac{4ac-b^2}{4a}$$

The discriminant of f(x) is the quantity $D = b^2 - 4ac$ The roots of f(x) are given by the quadratic formula:

$$\frac{-b \pm \sqrt[2]{b^2-4ac}}{2a}$$

The general linear equation is $ax + by = c$ where a and b are not both zero. For $b = 0$, this gives the verical line $ax = c$. When $b \neq 0$, we can rewrite in slope-intercept form. For example, $-6x + 2y = 3$ can be rewritten as $y = 3x + \frac{3}{2}$

Polynomials: For any real number m, the function $f(x) = x^m$ is called the power function with exponent m. A polynomial is a sum of multiples of power functions with whole number exponents: $f(x) = x^5 - 5x^3 + 4x$, $g(t) = 7t^6 + t^3 - 3t - 1$

Thus, the function $f(x) = x + x^{-1}$ is not a polynomial because it includes a power function x^-1 with a negative exponent. The general polynomial in the variable x may be written

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 + a_0$$

The numbers $a_0, a_1, \ldots, a_n$ are called coefficients.
The degree of $P(x)$ is n (assuming that $a_n \neq 0$).
The coefficient a_n is called the leading coefficient.
The domain of $P(x)$ is $\mathbb{R}$.

Rational functions: A rational function is a quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

Every polynomialis also a rational functions with $Q(x) = 1$. The domain of a rational function $\frac{P(x)}{Q(x)}$ is th eset of numbers x such that $Q(x) \neq 0$.

Algebraic functions: An algrbraic function is produced by taking sums, products, and quotients of roots of polynomials and rational functions:

$$f(x) = \sqrt[2]{1 + 3x^2 - x^4}, \, g(t) = (\sqrt{t} - 2)^{-2}, \, h(z) = \frac{z + z^{\frac{-5}{3}}}{5z^3 - \sqrt{z}}$$

More generally, algebraic functions are defined by polynomial equations between x and y. In this case, we say that y is implicitly defined as a function of x. For example, the equation $y^4 + 2x^2y + x^4$ defines y implicitly as a function of x.

Exponential functions: The function $f(x) = b^x$, where $b > 0$, is called the exponential function with base b. The function $f(x) = b^x$ is increasing if $b > 1$ and decreasing if $b < 1$. The inverse of $f(x) = b^x$ is the logarithm function $y = log_b x$.

Tirgonometric functions: Functions built from $\sin(x)$ and $\cos(x)$ are called trigonometric functions.

If f and g are functions, we may construct new functions by forming the sum, difference, product, and quotient functions:

$$(f+g)(x) = f(x) + g(x),\ (f-g)(x) = f(x) - g(x),\ (fg)(x) = f(x)g(x),\ (\frac{f}{g})(x) = \frac{f(x)}{g(x)} \text{ (where } g(x) \neq 0\text{)}$$

We can also multiply functions by constants. A function of the form: $c_1 f(x) + c_2 g(x)$ $(c_1, c_2 constants)$ is called a linear combination of $f(x) and g(x)$.

Composition is another important way of contructing new functions. The composition of f and g is the function $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$, defined for values of x in the domain of g such that $g(x)$ lies in the domain of f.

Net functions may be produced using the operation of addition, multiplication, division, as well as composition, extraction of roots, and taking inverses. It is convenient to refer to a function constructed in this way from the basic functions listed above as an elementary function. The following functions are elementary:

$$f(x) = \sqrt{2x + \sin(x)},\ f(x) = 10^{\sqrt{x}},\ f(x) = \frac{1+x^{-1}}{1+\cos(x)}$$