About The Algorithm Solution Of A Planned Channel Problem

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Abstract

The paper formulates a mathematical model for the problem of bottom deformations. Based on the Petrov-Galerkin method, a method is proposed and an algorithm for solving the problem is developed. Examples of solving the problem in a rectangular region with different orientations of the shear stress fields are given.

Keywords [1]

Bed load transport model, Sediment transport mechanisms, finite element method.

1. Geometric dependencies.

Let us give exact formulas for the local characteristics of the bottom surface, not considering the slope of the bottom surface to be small. Let X, Y, Z be a fixed Cartesian coordinate system with axis Z directed vertically upward and unit vectors $\vec{i}, \vec{j}, \vec{k}$.

The bottom surface in this coordinate system is defined by Equation $Z = \zeta(X,Y)$, where ζ is a sufficiently smooth function of the variables X, Y. The normal vector $\vec{n} = (n_X, n_Y, n_Z)$ to the bottom surface has the components

$$n_X = -\frac{\partial \zeta}{\partial X}\cos\gamma, \quad n_Y = -\frac{\partial \zeta}{\partial Y}\cos\gamma, \quad n_Z = \cos\gamma, \tag{1}$$

where γ is the angle between the normal $\vec{n}$ to the bottom and the axis Z. The trigonometric functions of the angle γ have the form

$$\cos\gamma = \frac{1}{\sqrt{1+\tan^2\gamma}}, \quad \tan\gamma = \sqrt{\left(\frac{\partial \zeta}{\partial X}\right)^2 + \left(\frac{\partial \zeta}{\partial Y}\right)^2}. \tag{2}$$

The projection of the unit vector $\vec{k}$ directed vertically upwards onto the tangent plane of surface $Z = \zeta$ is called the slope vector $\vec{J}$ (Fig. 1), which can be decomposed into vectors $\vec{k}$ and $\vec{n}$: $\vec{J} = \vec{k} - \vec{n}\cos\gamma$. From here it is easy to write down its components along the axes X, Y, Z and determine the length $|\vec{J}| = J = \sin\gamma$.

Along with the absolute coordinate system, we introduce the local curvilinear orthogonal coordinate system x, y, z. The axis z is orthogonal to the tangent plane to the surface $Z = \zeta$, the axis x, y is the internal coordinates of this surface.

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In the local coordinate system, the bottom surface is defined by Equation $z=0$. The slope vector $\vec{J}$ in the local coordinate system has components $J_x=\frac{\partial\zeta}{\partial x}$, $J_y=\frac{\partial\zeta}{\partial y}$ or, in vector form, $\vec{J}=\nabla\zeta$, $|\nabla\zeta|=\sin\gamma$, $\nabla\zeta=\left(\partial/\partial x,\partial/\partial y\right)$.

The tangent plane perpendicular to the normal (1) of the bottom surface $Z=\zeta(X,Y)$, at point (X_0,Y_0,Z_0) is defined as

$$n_X\left(X-X_0\right)+n_Y\left(Y-Y_0\right)+n_Z\left(Z-Z_0\right)+D=0 \tag{3}$$

The projection of the radius of the bottom velocity vector $\vec{V}=\vec{i}V_X+\vec{j}V_Y+\vec{k}V_Z$ onto the tangent plane (3) defined by the normal (1): is determined by the expression $\vec{V}_\zeta=\vec{V}-\frac{\vec{V}\cdot\vec{n}+D}{\vec{n}\cdot\vec{n}}\vec{n}$, or

$$\vec{V}_\zeta=\vec{i}V_X+\vec{j}V_Y+\vec{k}V_Z-\left(-\vec{i}\frac{\partial\zeta}{\partial X}-\vec{j}\frac{\partial\zeta}{\partial Y}+\vec{k}\right)\left(-V_X\frac{\partial\zeta}{\partial X}-V_Y\frac{\partial\zeta}{\partial Y}+V_Z\right)\cos^2\gamma\,, \tag{4}$$

which makes it possible to obtain the direction of the unit vectors of the local coordinate system for axes x and y.

$$\vec{e}_x=\frac{\vec{V}_\zeta}{|\vec{V}_\zeta|}\qquad \vec{e}_y=\vec{n}\times\vec{e}_x=\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ -\frac{\partial\zeta}{\partial X}\cos\gamma & -\frac{\partial\zeta}{\partial Y}\cos\gamma & \cos\gamma\\ \frac{V_{\zeta X}}{|\vec{V}_\zeta|} & \frac{V_{\zeta Y}}{|\vec{V}_\zeta|} & \frac{V_{\zeta Z}}{|\vec{V}_\zeta|}\end{pmatrix}. \tag{5}$$

At small slopes of the bottom

$$\left|\frac{\partial\zeta}{\partial X}\right|<<1,\ \left|\frac{\partial\zeta}{\partial Y}\right|<<1,\ \cos\gamma->1,\ \vec{n}\approx\vec{k}\ \text{and}\ V_Z\approx 0\ , \tag{6}$$

get

$$\vec{V}_\zeta\approx\vec{i}V_X+\vec{j}V_Y,\qquad \vec{e}_x=\left(\frac{V_X}{|\vec{V}|},\frac{V_X}{|\vec{V}|},0\right), \tag{7}$$

$$\vec{e}_y=\vec{n}\times\vec{e}_x=\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & 1\\ \frac{V_X}{|\vec{V}_\zeta|} & \frac{V_Y}{|\vec{V}_\zeta|} & \frac{V_Z}{|\vec{V}_\zeta|}\end{pmatrix}=\begin{pmatrix}-\frac{V_Y}{|\vec{V}_\zeta|} & \frac{V_X}{|\vec{V}_\zeta|} & 0\end{pmatrix} \tag{8}$$

Therefore, under the assumption of small slopes of the bottom (5), we obtain the following matrix transformation

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{n} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}, \qquad \cos\alpha = \frac{V_X}{|\vec{V}|}, \ \sin\alpha = \frac{V_Y}{|\vec{V}|}, \tag{9}$$

which we will use in this work.

2. Governing Equations

Let us consider the problem of the evolution of the bottom surface $\zeta = \zeta(t, x, y)$ in a channel with a sandy bottom. The mathematical model of the problem is not closed and requires the determination of the bottom shear stresses $\vec{T} = (T_X, T_Y)$ and bottom pressure P determined from the solution of the external problem of hydrodynamics.

The calculation of the change in the bottom surface is carried out using the Exner equation

$$(1-\varepsilon)\rho_s \frac{\partial \zeta}{\partial t} + \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} = 0 \tag{10}$$

and equations of the flow rate of entrained sediment [1]

$$G_x = a - b\frac{\partial \zeta}{\partial x} - c\frac{\partial p}{\partial x}, \qquad G_y = -d\left(\left(1 + \frac{1}{s_f}\right)\frac{\partial \zeta}{\partial y} + \frac{1}{s_f}\frac{\partial p}{\partial y}\right) \tag{11}$$

where ρ_s is the density of sand, ρ_w is the density of the liquid, ε is the porosity of the bottom material, d_{50} is the average diameter of bottom particles, f is the concentration of entrained particles in the active bottom layer, φ is the angle of internal friction of sediments; κ - Karman coefficient for water-soil mixture $0.2 \le \kappa \le 0.41$,

$$a = G_0 A, \quad b = G_0 B, \quad c = G_0 C, \quad d = G_0 D,$$

$$A = \max(0, 1-\Xi), \quad B = \frac{1}{\cos\gamma \tan\varphi}\left(\frac{\Xi}{2} + A\frac{1+s}{s}\right),$$

$$C = \frac{A}{s\cos\gamma \tan\varphi}, \quad D = \frac{4}{5}\frac{1}{\cos\gamma \tan\varphi},$$

$$\Xi = \sqrt{\frac{T_*}{|T|}}, \quad G_0 = G_C \frac{T^{3/2}}{\cos\gamma}, \quad G_C = \frac{4}{3}\frac{(1-\varepsilon)^{-1}}{(\rho_s - \rho_w) g \tan\varphi \sqrt{\rho_w}\,\kappa},$$

$$T_* = T_0 \max\left(0, 1 + \frac{1}{\tan\varphi}\left(\frac{\partial \zeta}{\partial X} + \frac{\partial \zeta}{\partial Y}\right)\right), \quad T_0 = \frac{9}{8}\frac{\kappa^2}{c_x}\tan\varphi\,(\rho_s - \rho_w) g\, d_{50},$$

$$s_f = f\,\rho_b, \quad \rho_b = \frac{\rho_s - \rho}{\rho}, \quad p = \frac{P}{\rho g}.$$

3. Bringing the equations of the problem into a Cartesian coordinate system

According to definition (9), the local coordinate system x, y, z is related to the Cartesian absolute coordinate system X, Y, Z through the velocity field of the hydrodynamic flow $\vec{V} = (V_X, V_Y, V_Z)$

$$\cos\alpha = \frac{V_X}{|\vec{V}|}, \quad \sin\alpha = \frac{V_Y}{|\vec{V}|} \tag{12}$$

Assuming that conditions $|V_Z| << |\vec{V}|$, $\vec{V} \| \vec{T}$, equations (12) are valid in the bottom layer, the velocity vector $\vec{V}$ can be replaced by the vector of bottom shear stresses $\vec{T} = (T_X, T_Y)$

$$\cos\alpha = \frac{T_X}{\sqrt{T_X^2 + T_Y^2}}, \qquad \sin\alpha = \frac{T_X}{\sqrt{T_X^2 + T_Y^2}}, \tag{13}$$

from which follows the condition $T_x = \sqrt{T_X^2 + T_Y^2}$, $T_y = 0$, which is necessary [1] for deriving equations (11). Using expressions (9), (13), we obtain

$$\begin{pmatrix} \dfrac{\partial \zeta}{\partial x} \\ \dfrac{\partial \zeta}{\partial y} \end{pmatrix} = J_k \begin{pmatrix} \dfrac{\partial \zeta}{\partial X} \\ \dfrac{\partial \zeta}{\partial Y} \end{pmatrix}, \qquad \begin{pmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \end{pmatrix} = J_k \begin{pmatrix} \dfrac{\partial p}{\partial X} \\ \dfrac{\partial p}{\partial Y} \end{pmatrix}, \quad \begin{pmatrix} G_x \\ G_y \end{pmatrix} = J_k \begin{pmatrix} G_X \\ G_Y \end{pmatrix}, \tag{14}$$

where $J_k = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ is the Jacobi matrix.

Taking into account (14), expressions (11) in the Cartesian coordinate system take the form

$$G_X = a\cos\alpha - \left(d\frac{1+s}{s}(\sin\alpha)^2 + b(\cos\alpha)^2 \right)\frac{\partial \zeta}{\partial X} + \cos\alpha\sin\alpha\left(d\frac{1+s}{s} - b \right)\frac{\partial \zeta}{\partial Y} -$$
$$- \left(c(\cos\alpha)^2 + \frac{d}{s}(\sin\alpha)^2 \right)\frac{\partial p}{\partial X} + \left(\frac{d}{s} - c \right)\cos\alpha\sin\alpha\frac{\partial p}{\partial Y},$$

$$G_Y = a\sin\alpha + \left(\cos\alpha\sin\alpha\left(d\frac{1+s}{s} - b \right) \right)\frac{\partial \zeta}{\partial X} - \left(d\frac{1+s}{s}(\cos\alpha)^2 + b(\sin\alpha)^2 \right)\frac{\partial \zeta}{\partial Y} -$$
$$+ \left(\frac{d}{s} - c \right)\cos\alpha\sin\alpha\frac{\partial p}{\partial X} - \left(c(\sin\alpha)^2 + \frac{d}{s}(\cos\alpha)^2 \right)\frac{\partial p}{\partial Y}.$$

Substituting these equations, into the Exner equation (10) we obtain the equation of bottom deformations of the form

$$
\begin{aligned}
&(1-\varepsilon)\rho_s \frac{\partial \zeta}{\partial t} + \frac{\partial (a\cos\alpha)}{\partial X} + \frac{\partial (a\sin\alpha)}{\partial X} + \\
&\frac{\partial}{\partial X}\left(S_{xx}\frac{\partial \zeta}{\partial X}\right) + \frac{\partial}{\partial X}\left(S_{xy}\frac{\partial \zeta}{\partial Y}\right) + \frac{\partial}{\partial Y}\left(S_{yx}\frac{\partial \zeta}{\partial X}\right) + \frac{\partial}{\partial Y}\left(S_{yy}\frac{\partial \zeta}{\partial Y}\right) + \\
&\frac{\partial}{\partial X}\left(H_{xx}\frac{\partial p}{\partial X}\right) + \frac{\partial}{\partial X}\left(H_{xy}\frac{\partial p}{\partial Y}\right) + \frac{\partial}{\partial Y}\left(H_{yx}\frac{\partial p}{\partial X}\right) + \frac{\partial}{\partial Y}\left(H_{yy}\frac{\partial p}{\partial Y}\right) = 0,
\end{aligned}
$$

where S_{ij}, H_{ij} - components of gravity-diffusion and pressure tensors, determined by the formulas

$$
S_{ij} = \begin{pmatrix} -d\dfrac{1+s}{s}(\sin\alpha)^2 - b_p(\cos\alpha)^2 & \cos\alpha\sin\alpha\left(d\dfrac{1+s}{s} - b_p\right) \\ \cos\alpha\sin\alpha\left(d\dfrac{1+s}{s} - b_p\right) & -d\dfrac{1+s}{s}(\cos\alpha)^2 - b_p(\sin\alpha)^2 \end{pmatrix},
$$

$$
H_{ij} = \begin{pmatrix} -\dfrac{d}{s}(\sin\alpha)^2 - c(\cos\alpha)^2 & \cos\alpha\sin\alpha\left(\dfrac{d}{s} - c\right) \\ \cos\alpha\sin\alpha\left(\dfrac{d}{s} - c\right) & -\dfrac{d}{s}(\cos\alpha)^2 - c(\sin\alpha)^2 \end{pmatrix}.
$$

4. Formulation of the deformable bottom problem

Assuming that the solution of the external hydrodynamic part of the problem allows us to determine the vector of bottom shear stresses $\vec{T} = (T_X, T_Y)$ and bottom pressure P acting on the bottom of the channel Ω bounded by the contour $S = S_q \bigcup S_\zeta$, we formulate the planned problem of changing the bottom surface under the action of water flowing through it, which includes :

- equation of bottom deformations

$$
\begin{aligned}
&(1-\varepsilon)\rho_s \frac{\partial \zeta}{\partial t} + \frac{\partial (a\cos\alpha)}{\partial X} + \frac{\partial (a\sin\alpha)}{\partial X} + \\
&\frac{\partial}{\partial X}\left(S_{xx}\frac{\partial \zeta}{\partial X}\right) + \frac{\partial}{\partial X}\left(S_{xy}\frac{\partial \zeta}{\partial Y}\right) + \frac{\partial}{\partial Y}\left(S_{yx}\frac{\partial \zeta}{\partial X}\right) + \frac{\partial}{\partial Y}\left(S_{yy}\frac{\partial \zeta}{\partial Y}\right) + \\
&\frac{\partial}{\partial X}\left(H_{xx}\frac{\partial p}{\partial X}\right) + \frac{\partial}{\partial X}\left(H_{xy}\frac{\partial p}{\partial Y}\right) + \frac{\partial}{\partial Y}\left(H_{yx}\frac{\partial p}{\partial X}\right) + \frac{\partial}{\partial Y}\left(H_{yy}\frac{\partial p}{\partial Y}\right) = 0
\end{aligned}
\qquad (15)
$$

initial conditions

$$
\zeta = \zeta_0, \qquad t = 0, \quad X, Y \in \Omega . \qquad (16)
$$

and boundary conditions

$$
\zeta = \zeta_g, \qquad X, Y \in S_\zeta, \qquad (17)
$$

$$n_{sX}\left(a\cos\alpha + S_{xx}\frac{\partial\zeta}{\partial X} + S_{xy}\frac{\partial\zeta}{\partial Y}\right) + n_{sY}\left(a\sin\alpha + S_{yx}\frac{\partial\zeta}{\partial X} + S_{yy}\frac{\partial\zeta}{\partial Y}\right) = q, \qquad X, Y \in S_q, \tag{18}$$

where q is the specific discharge of sediment across the border S_q. ζ_b - bottom marks at the border S_ζ, $\vec{n}_s = (n_{sX}, n_{sY})$ - normal to the border S.

5. Obtaining a discrete analog

Consider a weak variational Galerkin formulation for problem (15) - (18) with a set of test functions $\{\psi\} H^1(\Omega)$ in the following form.

Find an unknown function $\{\zeta\} \in H^1(\Omega)$ such that:

$$\begin{aligned}
&\int_\Omega \left[(1-\varepsilon)\rho_s \frac{\partial\zeta}{\partial t}\psi - \left(\frac{\partial\psi}{\partial X}a\cos\alpha + \frac{\partial\psi}{\partial Y}a\sin\alpha\right)\right]d\Omega + \int_{S_q}\psi\, q\, dS - \\
&-\int_\Omega\left(S_{xx}\frac{\partial\psi}{\partial X}\frac{\partial\zeta}{\partial X} + S_{xy}\frac{\partial\psi}{\partial X}\frac{\partial\zeta}{\partial Y} + S_{yx}\frac{\partial\psi}{\partial Y}\frac{\partial\zeta}{\partial X} + S_{yy}\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial y}\right)d\Omega \\
&-\int_{S_q}\left(H_{xx}\frac{\partial\psi}{\partial X}\frac{\partial p}{\partial X} + H_{xy}\frac{\partial\psi}{\partial X}\frac{\partial p}{\partial Y} + H_{yx}\frac{\partial\psi}{\partial Y}\frac{\partial p}{\partial X} + H_{yy}\frac{\partial\psi}{\partial y}\frac{\partial p}{\partial y}\right)d\Omega = 0
\end{aligned} \tag{19}$$

Integral identity (19) must satisfy conditions (16), (17).

Using the finite element method in the formulation of Petrov - Galerkin. We divide the computational domain Ω into finite elements Ω_e, $\Omega = \bigcup_e \Omega_e$. We introduce an approximation of the bottom surface ζ on a finite element

$$\zeta = N_\alpha \zeta_\alpha, \quad \alpha = 1,..,C, \tag{20}$$

where N_α are the shape functions of the finite element, $\zeta_\alpha = \zeta_\alpha(t)$ are the bottom elevation values at the nodes of the finite element, C is the number of nodes on the finite element. We assume that approximation $\zeta_\alpha = \zeta_\alpha(t)$ on time layers (n) is linear and for it the relations

$$\zeta_\alpha = \theta\zeta_\alpha^{n+1} + (1-\theta)\zeta_\alpha^n, \qquad \frac{d\,\zeta_\alpha}{dt} \approx \frac{\zeta_\alpha^{n+1} - \zeta_\alpha^n}{\Delta t}, \quad \theta \approx 0.5. \tag{21}$$

Choosing as weight functions ψ, on a finite element, functions of the form N_α, taking into account (20) - (21), from integral identity (19), a finite element recurrent discrete analogue of problem (19), (16), (17)

$$\left[\frac{M_{\alpha\beta}}{\Delta t} + \theta K_{\alpha\beta}\right]^n \zeta_\alpha^{n+1} = \left[\frac{M_{\alpha\beta}}{\Delta t} - (1-\theta)K_{\alpha\beta}\right]^n \zeta_\alpha^n + \theta F_\alpha^{n+1} + (1-\theta)F_\alpha^n, \tag{22}$$

Where

$$M_{\alpha\beta} = (1-\varepsilon)\rho_s \int_V \left[N_\alpha N_\beta\right] dV, \tag{23}$$

$$K_{\alpha\beta}=\int_{V_e}\left(S_{xx}\frac{\partial N_\alpha}{\partial X}\frac{\partial N_\beta}{\partial X}+S_{xy}\frac{\partial N_\alpha}{\partial X}\frac{\partial N_\beta}{\partial Y}+S_{yx}\frac{\partial N_\alpha}{\partial Y}\frac{\partial N_\beta}{\partial X}+S_{yy}\frac{\partial N_\alpha}{\partial Y}\frac{\partial N_\beta}{\partial Y}\right)dV \tag{24}$$

$$F_\alpha=-\int_{V_e}\left(\frac{\partial N_\alpha}{\partial x}a\cos\alpha+\frac{\partial N_\alpha}{\partial y}a\sin\alpha\right)dV+\int_{S_{qe}}N_\alpha q\,dS-F_\alpha^p . \tag{25}$$

$$F_\alpha^p=\int_{V_e}\left(H_{xx}\frac{\partial N_\alpha}{\partial X}\frac{\partial N_\beta}{\partial X}+H_{xy}\frac{\partial N_\alpha}{\partial X}\frac{\partial N_\beta}{\partial Y}+H_{yx}\frac{\partial N_\alpha}{\partial Y}\frac{\partial N_\beta}{\partial X}+H_{yy}\frac{\partial N_\alpha}{\partial Y}\frac{\partial N_\beta}{\partial Y}\right)p_\beta\,dV .$$

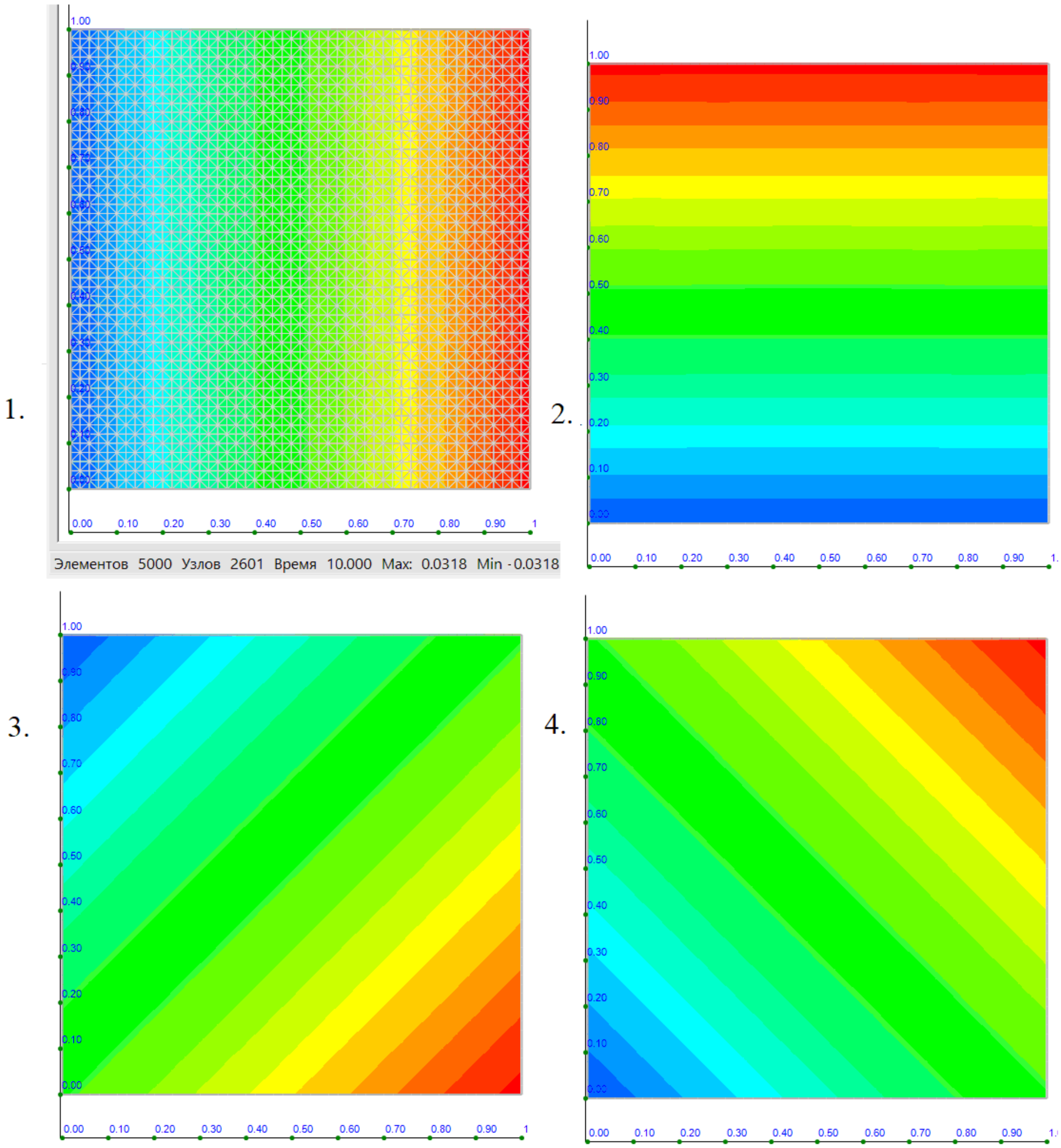

Fig. 1. Bottom surface erosion schemes

6. Examples of solving test problems.

Let us consider examples of calculating bottom changes in a square area with edges of unit length. Homogeneous boundary conditions are specified on the contour of the region $\partial\zeta / \partial\vec{n} = 0$. Figure 1 shows the results of bottom deformations obtained at time $t = 10\ s$ for the following variants of stress fields

$$1: \vec{T} = (2T_0,\ 0),\ 2: \vec{T} = (0,\ 2T_0),\ 3: \vec{T} = (2T_0, -2T_0),\ 4: \vec{T} = (2T_0,\ 2T_0).$$

From the results obtained (fields 1-4), one can see the spatial isotropy of the obtained solutions. The calculations performed in the quasi-one-dimensional approximation show good agreement of the obtained solutions with the known experimental data [1] from which one can draw a conclusion about the applicability of the proposed method and calculation algorithm for modeling arbitrary planned tasks of channel deformations.

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Literature

[1]. Petrov A.G., Potapov I.I. Selected Sections of Channel Dynamics. - Moscow.: Lenand, (2019). 244 p.