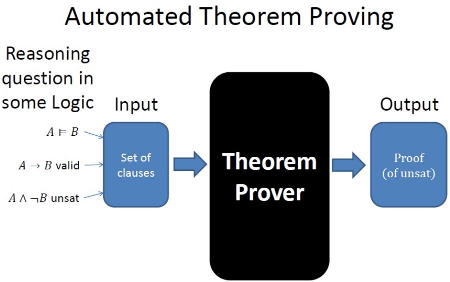
Theorem Proving



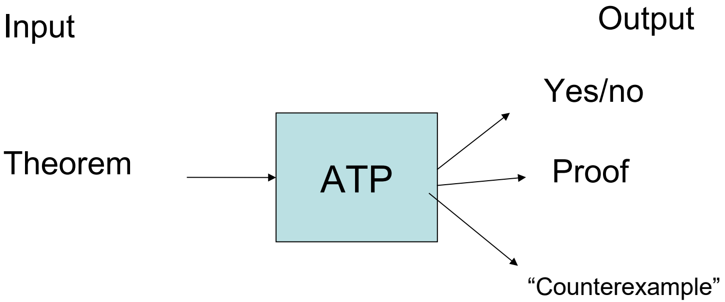
Lucien Ngalamou

Outline

A Framework for Software Verification

- Convert the informal description R of requirements for an application domain into an "equivalent" formula ϕ_R of some logic;
- Write a Program which is mean to realize ϕ_R in the programming environment supplied by your company, or wanted by the particular customer;
- Prove that the program P satisfy the formula ϕ_R.

What is an automated theorem prover?



Example theorems

- Pythagoras theorem: Given a right triangle with sides A B and C, where C is the hypotenuse, then $C^2 = A^2 + B^2$
- Fundamental theorem of arithmetic: Any whole number bigger than 1 can be represented in exactly one way as a product of primes

The model checking approach

- Create a model of the program in a decidable formalism
- Verify the model algorithmically
- Difficulties
  - Model creation is burden on programmer
  - The model might be incorrect.
- If verification fails, is the problem in the model or the program?

The axiomatic approach

- Add auxiliary specifications to the program to decompose the verification task into a set of local verification tasks
- Verify each local verification problem
- Difficulties
  - Auxiliary spec is burden on programmer
  - Auxiliary spec might be incorrect.
- If verification fails, is the problem with the auxiliary specification or the program?

Example Theorem

- The program “z = x; z = z + y;” computes the sum of ‘x’ and ‘y’ in ‘z’ according to the semantics of C
- Program-Semantics $\Rightarrow$ Specification

Theorem

- Theorem must be stated in formal logic
  - self-contained
  - no hidden assumptions
- Many different kinds of logics (propositional logic, first order logic, higher order logic, linear logic, temporal logic)
- Different from theorems as stated in math
  - theorems in math are informal
  - mathematicians find the formal details too cumbersome

Human assistance

- Some ATPs require human assistance
  - e.g.: programmer gives hints a priori, or interacts with ATP using a prompt
- Hardest theorems to prove are “mathematically interesting” theorems (eg: Fermat’s last theorem)

Output

- Can be as simple as a yes/no answer
- May include proofs and/or counterexamples
- These are formal proofs, not what mathematicians refer to as proofs
- Proofs in math are
  - informal
  - “validated” by peer review
  - meant to convey a message, an intuition of how the proof works -- for this purpose the formal details are too cumbersome

Output: meaning of the answer

- If the theorem prover says "yes" to a formula, what does that tell us?
  - Soundness: theorem prover says yes implies formula is correct
  - Subject to bugs in the Trusted Computing Base (TCB)
  - Broad defn of TCB: part the system that must be correct in order to ensure the intended guarantee
  - TCB may include the whole theorem prover
  - Or it may include only a proof checker

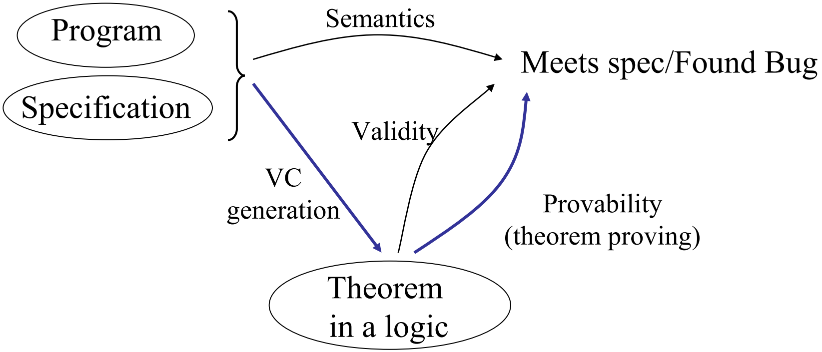
Output: meaning of the answer

- If the theorem prover says “no” to a formula, what does that tell us?
  - Completeness: formula is correct implies theorem prover says yes
  - Or, equivalently, theorem prover says no implies formula incorrect
  - Again, as before, subject to bugs in the TCB

Output: meaning of the answer

- ATPs first strive for soundness, and then for completeness if possible
- Some ATPs are incomplete: “no” answer doesn’t provide any information
- Many subtle variants
  - refutation complete
  - complete semi-algorithm

Theorem Proving and Software



- Soundness:
  - If the theorem is valid then the program meets specification
  - If the theorem is provable then it is valid

Programs ! Theorems = Axiomatic Semantics

- Consists of:
  - A language for making assertions about programs
  - Rules for establishing when assertions hold
- Typical assertions:
  - During the execution, only non-null pointers are dereferenced
  - This program terminates with x = 0
- Partial vs. total correctness assertions
  - Safety vs. liveness properties
  - Usually focus on safety (partial correctness)

Hoare Logic

C. A. R. (Tony) Hoare

The inventor of this week's logic is also famous for inventing the **Quicksort** algorithm in 1960 - when he was just **26**! A quote:

> Computer programming is an **exact science** in that all the properties of a program and all the consequences of executing it in any given environment can, in principle, be found out from the text of the program itself by means of purely **deductive reasoning**.

Hoare Logic

- A way of asserting properties of programs.
- Hoare triple: $\{A\}P\{B\}$ asserts that "If program P is started in a state satisfying condition A, if it terminates, it will terminate in a state satisfying condition B."
- A proof system for proving such assertions.
- A way of reasoning about such assertions using the notion of "Weakest Preconditions" (due to Dijkstra).

A simple programming language

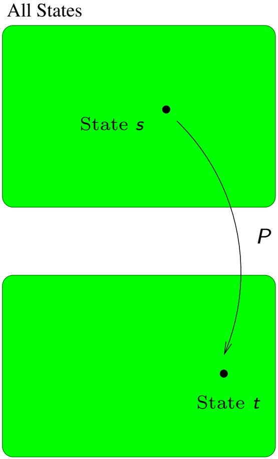
- `skip`
- `x := ` e (assignment)
- `if` b `then` S `else` T (if-then-else)
- `while` b `do` S (while)
- S `;` T (sequencing)

Example program

```
x := n;
a := 1;
while (x ≥ 1) {
   a := a * x;
   x := x - 1
 }
```

Programs as State Transformers

View program P as a partial map $[P] : \mathit{Stores} \to \mathit{Stores}$.

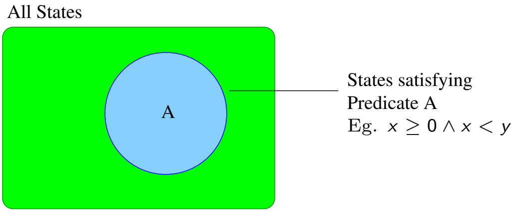


$\{x \mapsto 2,\ y \mapsto 10,\ z \mapsto 3\}$

```
y = y + 1;
z = x + y
```

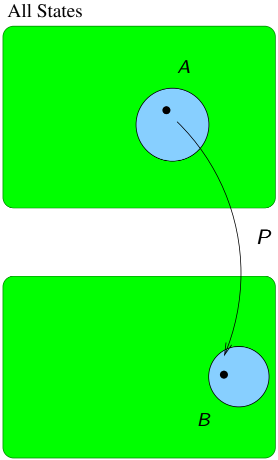
$\{x \mapsto 2,\ y \mapsto 11,\ z \mapsto 12\}$

Predicates on States



Assertion of "Partial Correctness" $\{A\}P\{B\}$

$\{A\}P\{B\}$ asserts that "If program P is started in a state satisfying condition A, either it will not terminate, or it will terminate in a state satisfying condition B."



$\{10 \leq y\}$

```
y = y + 1;
z = x + y
```

$\{x < z\}$

Proof rules of Hoare Logic

Skip:

$$\overline{\{A\}\ \texttt{skip}\ \{A\}}$$

Assignment

$$\overline{\{A[e/x]\}\ \texttt{x := e}\ \{A\}}$$

Proof rules of Hoare Logic

If-then-else:

$$\frac{\{P \wedge b\}\ S\ \{Q\},\ \{P \wedge \neg b\}\ T\ \{Q\}}{\{P\}\ \texttt{if}\ \ b\ \ \texttt{then}\ \ S\ \ \texttt{else}\ \ T\ \{Q\}}$$

While (here P is called a *loop invariant*)

$$\frac{\{P \wedge b\}\ S\ \{P\}}{\{P\}\ \texttt{while}\ \ b\ \ \texttt{do}\ \ S\ \{P \wedge \neg b\}}$$

Sequencing:

$$\frac{\{P\}\ S\ \{Q\},\ \{Q\}\ T\ \{R\}}{\{P\}\ S\,;T\ \{R\}}$$

Weakening:

$$\frac{P \implies Q,\ \{Q\}\ S\ \{R\},\ R \implies T}{\{P\}\ S\ \{T\}}$$

Class Activity

- Download from BB (Week 5) and read the file hoare-logic.pdf (15 minutes to 20 minutes)
- Discussion

The KeY Project (Formal Methods for Components and Objects Conf. 2006)

- The KeY Tool (https://www.key-project.org) is a tool for Deductive Verification of Object-Oriented Programs
- The currently most prominent applications are:
  - Program Verification (Standalone GUI, Eclipse Integration, KeYHoare)
  - Debugging (Symbolic Execution Debugger)
  - Information Flow Analysis / Security
  - Test Case generation (KeYTestGen)

Some Buzzwords Early On

- Java as target language
- Dynamic logic as program logic
- Verification = symbolic execution + induction
- Sequent style calculus + meta variables + incremental closure
- Prover is interactive + automated
- Integration with two standard SWE tools:
  - TogetherCC, a commercial CASE tool
  - Eclipse, an open extensible IDE
- Specification languages
  - JML
  - OCL/UML
- Smart cards as main target application

Supported Specification Languages: OCL

Object Constraint Language

Part of the OMG standard UML

Scope:
Add formal constraints to UML (class) diagrams

Supported Specification Languages: JML

Java Modeling Language
Behavioral interface specification language for Java

International community effort lead by Gary T. Leavens, Iowa State
building on the Larch approach

Comes with assertion and runtime checkers

OCL and JML

both

- specify method behaviour: pre/post conditions
- specify admissible states: class invariants
- essentially full first order
- support inter-object navigation

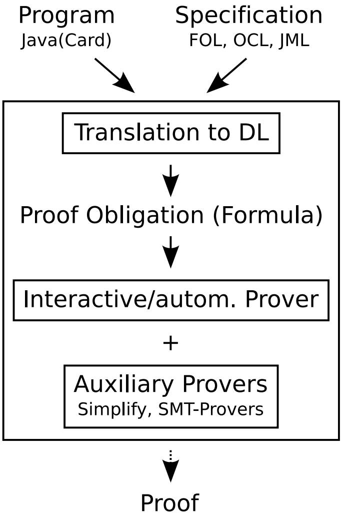
differences

- OCL model oriented:
  - attached to class diagrams
  - 'talks' UML
- JML implementation oriented:
  - attached to Java programs
  - 'talks' Java
  - specifies exceptional behaviour also
- JML only: restricting scope of side effects

JML example

```
/*@ public normal_behavior
  @ requires a != null;
  @ ensures (\forall int j; j >= 0 && j < a.length;
  @                         \result >= a[j]);
  @ ensures a.length > 0 ==>
  @         (\exists int j; j >= 0 && j < a.length;
  @                         \result == a[j]);
  @*/
public static /*@ pure @*/ int max(int[] a) {
    if ( a.length == 0 ) return 0;
    int max = a[0], i = 1;
    while ( i < a.length ) {
        if ( a[i] > max ) max = a[i];
        ++i;
    }
    return max;
}
```

KeY Architecture



Components of the Calculus

1. Non-program rules
   - first-order rules
   - rules for data-types (primarily: arithmetic)
   - rules for modalities
2. Rules for reducing/simplifying the program (symbolic execution)
   Replace the program by combination of
   - case distinctions (proof branches) and
   - sequences of updates
3. Rules for handling loops
   - rules using loop invariants
   - unwinding + induction
4. Rules for replacing a method invocations by the method's contract
5. Update simplification

Coverage of Java features

The calculus covers:

- method invocation, dynamic binding
- polymorphism
- abrupt termination
- checking for nullpointer exceptions
- object creation and initialisation
- arrays
- finiteness of integer data types
- transactions (Java Card)

By that, KeY covers the full 'Java Card' language.

Java Card

- Subset of Java, but with transaction concept
- Sun's official standard for SMART CARDS and embedded devices

Why Java Card?

Good example for real-world object-oriented language

Java Card has *no*

- garbage collection
- dynamical class loading
- multi-threading
- floating-point arithmetic

Application areas

- security critical
- financial risk
  (e.g. exchanging smart cards
  is expensive)

Implementing Rules: Taclets

Uniform language for different classes of rules

- First-order calculus
- Specific to Java DL: symbolic execution for Java
- Axioms of theories: arithmetic, lists, etc.
- Lemmas

Simple, high-level language

- Adding, modifying, and removing formulas
- Conditions restricting applicability of rules
- No complex features like loops
- Suitable both for interactive and automated systems
- Lemmas are validated wrt. base taclets

Library Case Studies

Java Collections Framework (JCF)

- Part of JCF (treating sets) specified using UML/OCL
- Some parts of reference implementation verified

Java Card API

- Most parts of Java Card API specified using UML/OCL
- Some parts of reference implementation verified

Schorr-Waite Algorithm

- Standard benchmark for verification systems
- Graph marking algorithm for garbage collection
- Java implementation: 2 classes, core algorithm 25 lines of code
- Heavy aliasing, frame problem
- Specified and verified

Security Case Studies: Java Card Software

Safety/security properties specified in dynamic logic

- 'Only certain exceptions can be thrown'
- Transactions are properly used
  (do not commit or abort a transaction that was never started,
  all started Transactions are also closed)
- Data consistency
  (also if a smartcard is "ripped out" during operation)
- Absence of overflows for integer operations

Two studies in this area (for which some critical parts were verified)

- Demoney (about 3000 lines):
  Electronic purse application provided by Trusted Logic S.A.
- SafeApplet (about 600 lines): RSA based authentication applet

Safety Case Study

Computation of Railway Speed Restrictions

- Software by DBSystems for computing schedules for train drivers: Speed restrictions, required break powers
- Software formally specified using UML/OCL (based on existing informal specification)
- Program translated from Smalltalk to Java

Avionics Software

- Java implementation of a Flight Manager module at Thales Avionics
- Comprehensive specification using JML, emphasis on class invariants
- Verification of some nested method calls using contracts

Virtual Machine for Real Time Secury Java

- Verification of some library functions of the Jamaica VM from Aicas