Complex numbers are just as complex as the name suggests, but they are so so elegant when used properly - from circles to trigonometric expressions, complex numbers link them all.

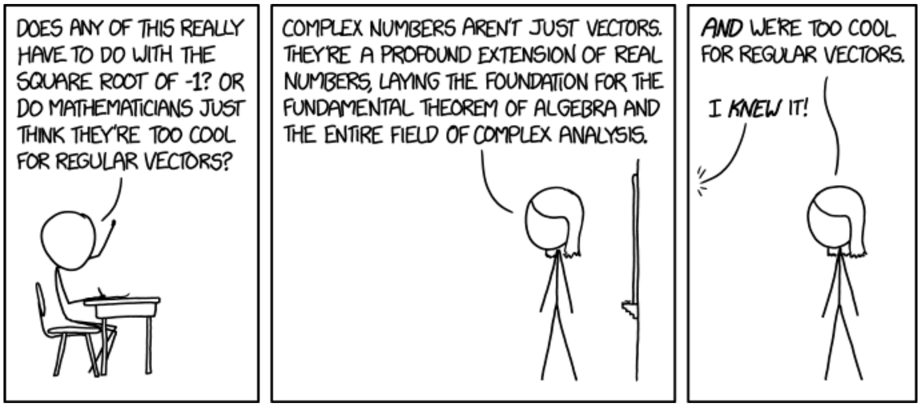


Figure 1: Comic from https://xkcd.wtf/2028/

## 0.1 Properties of Complex Numbers (★)

Complex numbers can be written in a few ways, and each of these ways have their own benefits that make information more apparent.

**Proposition 0.1** (Notations for Complex Numbers)
Let $z$ be a complex number and let $i$ be the imaginary number such that $i^2 = -1$. We write $z = a + bi$ for real $a$ and $b$.

1. The *real part* of $z$ is denoted as $Re(z)$ or $\Re(z)$. For instance, $\Re(z) = a$.
2. The *imaginary part* of $z$ is denoted as $Im(z)$ or $\Im(z)$. For instance, $\Im(z) = b$.
3. The *modulus* of $z$ is denoted as $|z|$. For instance, $|z| = \sqrt{a^2 + b^2}$.
4. The *argument* of $z$ is denoted as $\arg z$. For instance, $\arg z = \tan^{-1}\left(\frac{b}{a}\right)$.
5. The *polar form* of $z$ can be written as
$$z = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta.$$
Then, $|z| = r$ and $\arg z = \theta$.
6. The *conjugate* of $z$ is denoted as $\bar{z}$ or $z^*$. For instance,
$$\bar{z} = a - bi = re^{-i\theta} = r(\cos\theta - i\sin\theta).$$

**Remark 0.2.** Note that conjugation is distributive. For instance,

1. $\overline{z \pm w} = \overline{z} \pm \overline{w}$,
2. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$,
3. $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$.

With the different representations for complex numbers in mind, we shall briefly introduce some properties that relate to each form. As again, let

$$z = a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

**Proposition 0.3** (Absolute Square)
$z \cdot \overline{z} = |z|^2$.

*Proof.*

$$\begin{aligned} z \cdot \overline{z} &= (a+bi)(a-bi) = a^2 - (bi)^2 \\ &= a^2 + b^2 = (\sqrt{a^2+b^2})^2 \\ &= |z|^2. \end{aligned}$$

□

**Proposition 0.4** (Rotation)
Multiplying a complex number by $e^{i\theta}$ rotates the vector by $\theta$ radians counterclockwise about the origin.

## 0.2 Exercise

**Problem 0.5.** Prove that $\arg(zw) = \arg(z) + \arg(w)$ and $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$.

**Problem 0.6.** De Moivre's theorem states that

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta).$$

Prove de Moivre's theorem by induction.

## 0.3 Roots of Unity (★★)

Let $n$ be a positive integer. An $n$th *root of unity* is a complex number $z$ that satisfies the equation

$$z^n = 1.$$

Then, the $n$th roots of unity can be described by

$$z = \exp\left(\frac{2k\pi}{n}\right) = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \cdots, n-1.$$

**Proposition 0.7** (Conjugates)
If $z$ satisfies $z^n = 1$, meaning $z$ is an $n$th root of unity, then so does $\overline{z}$.

*Proof.* Recall that $z \cdot \overline{z} = |z|^2 = 1$, so $\overline{z} = \frac{1}{z} = z$. □

To start off the section, we showcase a trick on evaluating (suspiciously convenient) trigonometric sums through the lenses of the **roots of unity**.

**Example 0.8** (2021 H2 Further Math P1/9)

Let $\omega = \cos\frac{2\pi}{11} + \sin\frac{2\pi}{11}$. Show that

$$\sum_{j=1}^{10} \omega^j = -1.$$

Simple enough, $\omega$ is an 11th roots of unity, and so it satisfies the polynomial $z^{11} - 1 = 0$. Moreover, the required sum is a geometric progression, so

$$\sum_{j=1}^{10} \omega^j = \frac{\omega^{11} - 1}{\omega - 1} - 1 = -1.$$

**Example 0.9** (cont.)

The complex numbers $\alpha$ and $\beta$ are such that

$$\alpha = w + w^3 + w^4 + w^5 + w^9 \text{ and } \beta = w^{-1} + w^{-3} + w^{-4} + w^{-5} + w^{-9}.$$

Find in its simplest form, the quadratic equation whose roots are $\alpha$ and $\beta$.

Most jarringly, the exponents for $\alpha$ don't come in order: it jumps from 1 to 3, then 5 to 9 and $\beta$ mirrors this behaviour, except with negative exponents. Fortunately, we see that $w^9 = e^{\frac{18\pi i}{11}} = e^{\frac{-4\pi i}{11}} = w^{-2}$ and that $\Re(w^{-2}) = \Re(w^2)$ (why doesn't the problem use $w^2$ and $w^{-2}$ instead?). We also make a mental note that $\alpha = \beta^*$. Maybe this will come in handy later.

For now, we have

$$\begin{aligned}
\alpha + \beta &= (w + w^{-1}) + (w^3 + w^{-3}) + (w^4 + w^{-4}) + (w^5 + w^{-5}) + (w^9 + w^{-9}) \\
&= 2\Re(w + w^3 + w^4 + w^5 + w^9) = 2\Re(w + w^2 + w^3 + w^4 + w^5) \\
&= 2\left(\cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11}\right)
\end{aligned}$$

This is usually a tough sum to evaluate, but the values here just line up way too nicely for us to ignore... With that, here comes the *trick:*

$$S = \cos\frac{2\pi}{11} + \cos\frac{4\pi}{11} + \cos\frac{6\pi}{11} + \cos\frac{8\pi}{11} + \cos\frac{10\pi}{11}.$$

We consider

$$\begin{aligned}
\frac{S \cdot 2\sin\frac{2\pi}{11}}{2\sin\frac{2\pi}{11}} &= \frac{2\sin\frac{2\pi}{11}\cos\frac{2\pi}{11} + 2\sin\frac{2\pi}{11}\cos\frac{4\pi}{11} + \cdots + 2\sin\frac{2\pi}{11}\cos\frac{10\pi}{11}}{2\sin\frac{2\pi}{11}} \\
&= \frac{\sin\frac{10\pi}{11} + \sin\frac{12\pi}{11} - \sin\frac{2\pi}{11}}{2\sin\frac{2\pi}{11}} && \text{using the product-to-sum formula} \\
&= -\frac{1}{2}
\end{aligned}$$

**Remark 0.10.** This doesn't work if we have one fewer term though. For instance if we exclude $\cos\frac{10\pi}{11}$, then the sum doesn't come out nice anymore. Hmm...

Hence, $\alpha + \beta = -1$. Furthermore, we have $w^{-i} = w^{-11+i}$ and $w^i = w^{11-i}$, so that

$$\begin{aligned}
\alpha\beta &= (w + w^3 + w^4 + w^5 + w^9)(w^{-1} + w^{-3} + w^{-4} + w^{-5} + w^{-9}) \\
&= 5 + w^{-8} + w^{-6} + w^{-5} + 2w^{-4} + w^{-3} + 2w^{-2} + 2w^{-1} + 2w + 2w^2 + w^3 + 2w^4 + w^5 + w^6 + w^8 \\
&= 5 + (w^{-10} + w^{-9} + \cdots + w^{-1}) + (w + w^2 + \cdots + w^{10}) \\
&= 5 + \sum_{j=1}^{10} \omega^j + \left( \sum_{j=1}^{10} \omega^j \right)^* = 3
\end{aligned}$$

where the final sum holds because

$$\left( \sum_{j=1}^{10} \omega^j \right) \left( \sum_{j=1}^{10} \omega^j \right)^* = \left| \left( \sum_{j=1}^{10} \omega^j \right) \right|^2 = 1$$

Hence, a possible quadratic equation is $x^2 - x + 3 = 0$.

**Example 0.11** (cont..)
Prove that $|\alpha| = \sqrt{3}$

Aha! Here's where the property that $\alpha = \beta^*$ comes in handy.

$$|\alpha|^2 = \alpha \cdot \alpha^* = \alpha\beta = 3$$

and the result follows.

**Example 0.12** (cont...)
Prove that

$$\sin \frac{2\pi}{11} + \sin \frac{6\pi}{11} + \sin \frac{8\pi}{11} + \sin \frac{10\pi}{11} + \sin \frac{18\pi}{11} = \frac{\sqrt{11}}{2}.$$

Observe that this is simply the imaginary part of $\alpha$. In addition,

$$\begin{aligned}
\Im(\alpha) &= \sin \frac{2\pi}{11} + \sin \frac{6\pi}{11} + \sin \frac{8\pi}{11} + \sin \frac{10\pi}{11} + \sin \frac{18\pi}{11} \\
&= \sin \frac{2\pi}{11} + \sin \frac{6\pi}{11} + \left( \sin \frac{8\pi}{11} - \sin \frac{7\pi}{11} \right) + \sin \frac{10\pi}{11} \\
&> 0
\end{aligned}$$

since each term is positive. Thus,

$$\begin{aligned}
\Im(\alpha) &= \frac{1}{2} \cdot 2\Im(\alpha)) = \frac{1}{2} (\alpha - \alpha^*) \\
&= \frac{1}{2} (\alpha - \beta) \\
&= \frac{1}{2} \Im \left( \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \right) \\
&= \frac{\sqrt{11}}{2}
\end{aligned}$$

by taking the positive square root because $\alpha - \beta = 2\Im(\alpha) > 0$.

**Tangent: Gaussian Periods** So why didn't the question use $w^{-2}$ and $w^2$? The set $(1, 3, 4, 5, 9)$ feels too specific anyways. As it turns out, this is the set of **quadratic residues** modulo 11 (prove this!), and the sums $\alpha$ and $\beta$ are the so-called quadratic Gauss sums:

**Definition 0.13** (Quadratic Gauss Sum)**.**

$$g_a = \sum_{t=0}^{p-1} \left(\frac{t}{p}\right) \zeta^{at}$$

where $\left(\frac{t}{p}\right)$ is the Legendre's symbol

**Definition 0.14** (Legendre's symbol)**.** $\left(\frac{t}{p}\right) = \begin{cases} 1 & \text{if } t \text{ is a quadratic residue,} \\ \text{-1} & \text{if } t \text{ is a quadratic non-residue,} \\ 0 & \text{if } p|t. \end{cases}$

In what follows, we showcase a technique of *summing in two ways* that reduces this problem to a sum of roots of unity.

**Proposition 0.15** (Expressing $g_a$ in terms of $g_1$)

$$g_a = \left(\frac{a}{p}\right) g_1.$$

*Proof.* Firstly if $p|a$, then $g_a = 0$ (why?).

Henceforth, suppose $p$ does not divide $a$. Then,

$$\left(\frac{a}{p}\right) g_a = \sum_{t=0}^{p-1} \left(\frac{at}{p}\right) \zeta^{at} = \sum_{x=0}^{p-1} \left(\frac{x}{p}\right) \zeta^{x} = g_1.$$

The first equality follows because $a \nmid p \implies at \nmid p$, so that as $t$ runs over the set of residues (mod $p$), then so does $at$. Since $at$ runs over the set of residues, then we may simply let the sum run over all $x$ in the set of residues (mod $p$).

Now observe by definition that $\left(\frac{a}{p}\right)^2 = 1$, and we are done. □

What follows is the *coup de grace*:

**Proposition 0.16** (Summing in two ways)

$$g_1^2 = (-1)^{\frac{p-1}{2}} p.$$

For convenience, write $g = g_1$. According to Ireland-Rosen, the main idea is to consider the sum $\sum_{a=0}^{p-1} g_a g_{-a}$ in two ways. Again, we consider only the case when $p \nmid a$.

On one hand, we have

$$g_a g_{-a} = \left(\frac{a}{p}\right) \left(\frac{-a}{p}\right) g^2 = \left(\frac{-1}{p}\right) g^2,$$

whence it follows that

$$\sum_{a=1}^{p-1} g_a g_{-a} = \left(\frac{-1}{p}\right) (p-1) g^2.$$

On the other hand, we have

$$g_a g_{-a} = \sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)\zeta^{a(x-y)},$$

and summing both sides over $a$,

$$\begin{aligned}
\sum_{a=0}^{p-1} g_a g_{-a} &= \sum_{a=0}^{p-1}\sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)\zeta^{a(x-y)} \\
&= \sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)\sum_{a=0}^{p-1}\zeta^{a(x-y)} \\
&= \sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\left(\frac{x}{p}\right)\left(\frac{y}{p}\right)s(x,y)p \\
&= p(p-1)
\end{aligned}$$

where we have $p^{-1}\sum_{t=0}^{p-1}\zeta^{at} = s(x,y) = \begin{cases}1 & \text{if } p \mid t \\ 0 & \text{otherwise.}\end{cases}$. Putting both parts together, the result follows.