0.1 Floors and Ceilings

In what follows, and in most modern texts, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor -1.414 \rfloor = -2$. Similarly, $\lceil x \rceil$ denotes the least integer greater than or equal to x. For example, $\lceil \pi \rceil = 4$ and $\lceil -1.414 \rceil = -1$.

Example 0.1 (2018 SMO(S) P25)

Suppose R is a real number such that

$$\left\lfloor R - \frac{1}{200} \right\rfloor + \left\lfloor R - \frac{2}{200} \right\rfloor + \cdots + \left\lfloor R - \frac{99}{200} \right\rfloor = 2018.$$

Find $\lfloor 20R \rfloor$.

Proof. By definition of the floor function,

$$\left\lfloor R - \frac{k}{200} \right\rfloor \leq R - \frac{k}{200} < \left(R - \frac{k}{200} \right) + 1,$$

thus, $\left\lfloor R - \frac{1}{200} \right\rfloor$ and $\left\lfloor R - \frac{99}{200} \right\rfloor$ differ by at most 1 (this is the key idea!).

Define $M = \max_{1 \leq k \leq 99} \left\lfloor R - \frac{k}{200} \right\rfloor$, then I claim that $M \geq 20$. Suppose otherwise, then $M < 20$. Thus,

$$\left\lfloor R - \frac{1}{200} \right\rfloor + \left\lfloor R - \frac{2}{200} \right\rfloor + \cdots + \left\lfloor R - \frac{99}{200} \right\rfloor < 99 \cdot 20 = 1980 < 2018,$$

a contradiction.

Thus, $M \geq 20$. Let a, b denote the number of $\left\lfloor R - \frac{k}{200} \right\rfloor$ attaining the values 20 and 21 respectively such that

$$\begin{cases} a + b = 99 \\ 20a + 21b = 2018 \end{cases} \implies a = 61, b = 38.$$

Hence, we seek

$$\begin{cases} R - \frac{38}{200} \geq 21 \\ R - \frac{39}{200} \leq 21 \end{cases} \implies 21.19 \leq R \leq 21.195 \implies \boxed{\lfloor 20R \rfloor = 423}$$

. $\square$

0.2 A Splurge of Inequalities

We start off this chapter with a powerful technique for proving inequalities.

Example 0.2 (Schur's Inequality)

Let a, b, c, r be positive real numbers. Prove that

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0.$$

Let $f(a, b, c) = a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b)$. In general, we say that f is symmetric if $f(a, b, c) = f(b, a, c) = f(c, b, a) = \cdots$. This means that the function remains constant even if we interchange the variables a, b and c.

Simple enough, if we interchange a and b, we have:

$$\begin{aligned} f(b,a,c) &= b^r(b-a)(b-c) + a^r(a-c)(a-b) + c^r(c-b)(c-a) \\ &= a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \\ &= f(a,b,c) \end{aligned}$$

If you are paranoid, you can manually verify this for the $3! = 6$ possible "interchanges". I'll leave that to you.

With that, we say that the function, and hence Schur's Inequality, is **symmetric** in a, b, c. So what's the big deal?

Since the value of $f(a,b,c)$ remains constant even as we interchange variables, we may impose restrictions on a, b, c that we normally cannot. In particular, we may assume **without loss of generality** that $a \geq b \geq c$ (convince yourself!). This is very useful, especially for this inequality, because we have terms in $a-b, a-c\cdots$, and the ordering of a, b, c tells us whether these terms are positive or negative.

In fact, a cursory glance tells us that if $a \geq b \geq c$, then only $b^r(b-c)(b-a)$ is negative. Hence, this motivates an enlightening rearrangement of $f(a,b,c)$:

$$\begin{aligned} f(a,b,c) &= a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \\ &= (a-b)[a^r(a-c) - b^r(b-c)] + c^r(a-c)(b-c) \end{aligned}$$

And we are done!

This technique, ironically, is known as "breaking symmetry". The inequality is essentially proven after we impose a specific ordering on the variables, but before that, the inequality may look almost intractible!

0.3 Selected Problems

Problem 0.3. Prove by contradiction that $\sqrt{2}$ is irrational.

Problem 0.4. * Prove by contradiction that $x^2 + y^2 = 3z^2$ has no positive integer solutions.

Problem 0.5. Explain why lines (**??**) and (**??**) hold in **??**. (Hint: You may want to consider the density function of the normal distribution.)

Problem 0.6. By referring to Schur's Inequality, prove that

- $a^3 + b^3 + c^3 \geq a^2(b+c) + b^2(c+a) + c^2(a+b)$,
- $$\frac{1}{a^5} + \frac{1}{b^5} + \frac{1}{c^5} + \frac{a+b+c}{a^2b^2c^2} \geq \frac{b^2+c^2}{a^3b^2c^2} + \frac{c^2+a^2}{b^3c^2a^2} + \frac{a^2+b^2}{c^3a^2b^2}.$$

Problem 0.7. (2017 USAJMO P2) Consider the equation

$$(3x^3 + xy^2)(x^2y + 3y^3) = (x-y)^7.$$

- Prove that there are infinitely many pairs of positive integers (x, y) satisfying the equation.
- Describe all pairs of positive integers (x, y) satisfying the equation.