As it turns out, there's more to algebra than pure uninspiring symbol pushing. In this chapter we look at some structures and concepts in elementary algebra.

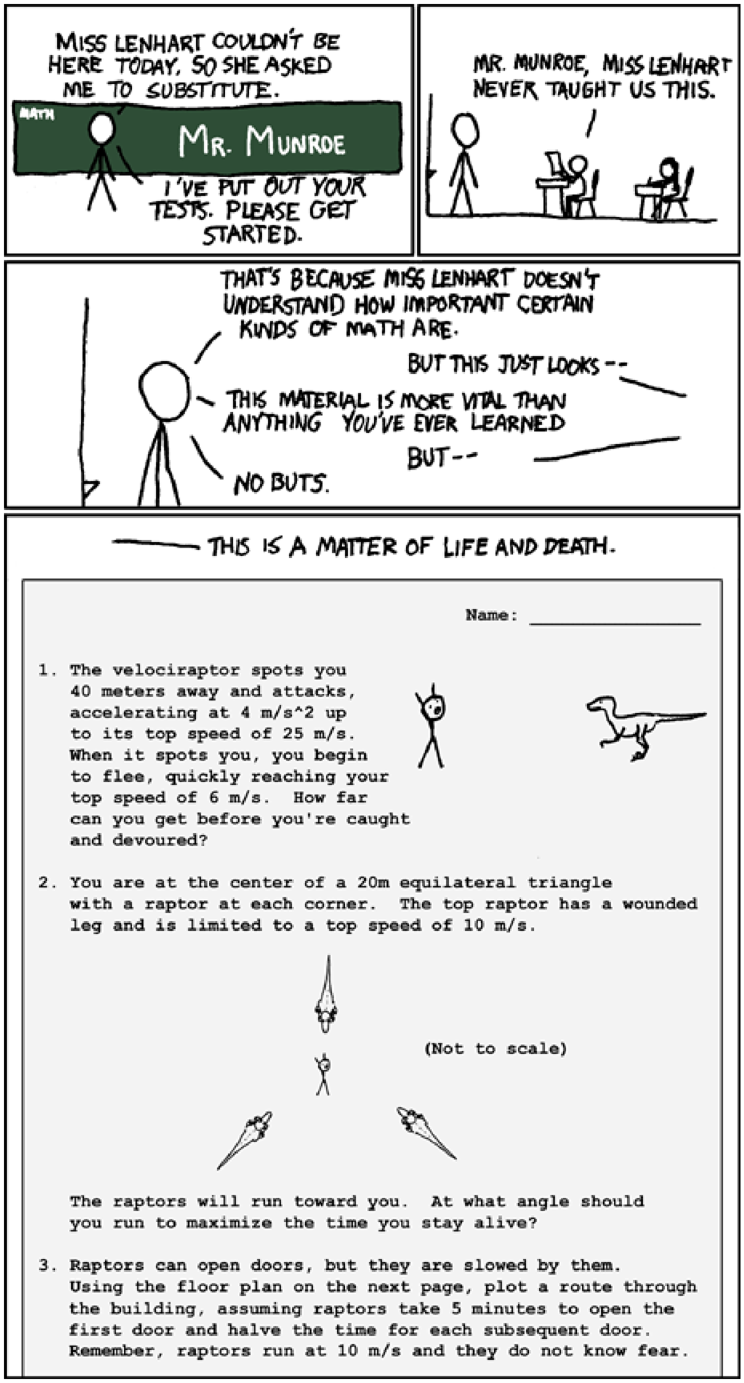

Figure 1: Comic from https://xkcd.wtf/135/

1 Many Manipulations (★)

1.1 Factorising and Re-expressing

Starting off easy, we will explore some funny-looking problems that solve themselves after some smart rearranging and rewriting.

Example 1.1 (2013 SMO(J) P15)
If $a = 1.69$, $b = 1.73$ and $c = 0.48$, find the value of

$$\frac{1}{a^2 - ac - ab + bc} + \frac{2}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab}.$$

Of course, a good calculator can do this in seconds (although typing it in may take longer...), but what if you had to do this by hand? Substituting may not be smart...

However, we note that each of the denominators are suspiciously factorisable. For example,

$$a^2 - ac - ab + bc = a(a - c) - b(a - c) = (a - b)(a - c).$$

Proof. Hence,

$$\begin{aligned}
&\frac{1}{a^2 - ac - ab + bc} + \frac{1}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab} \\
&= \frac{1}{(a-b)(a-c)} + \frac{2}{(b-c)(b-a)} + \frac{1}{(c-b)(c-a)} \\
&= \frac{(b-c) - 2(a-c) + (a-b)}{(a-b)(a-c)(b-c)} \\
&= -\frac{1}{(a-b)(b-c)} = -\frac{1}{(-0.04)(1.25)} = \boxed{20}
\end{aligned}$$

□

Example 1.2 (2013 SMO(J) P32)
If a and b are positive integers such that $a^2 + 2ab - 3b^2 - 41 = 0$, find $a^2 + b^2$.

The constant "41" seems irrelevant to the equation at this point, except that it's a prime number (this may be important!). For now, let us write the equation as $a^2 + 2ab - 3b^2 = 41$.

Proof. Notice that the coefficients of the LHS sum to $(1 + 2 - 3 =)0(!)$, so this tells us that we should factorise:

$$a^2 + 2ab - 3b^2 = a^2 - ab + 3ab - 3b^2 = a(a - b) + 3b(a - b) = (a + 3b)(a - b) = 41.$$

Now, it becomes apparent why 41 was chosen: because it is a prime number, and a, b are integers, $a + 3b$ and $a - b$ are each either 1 or 41. Moreover, $a + 3b \geq a - b$ so we have

$$\begin{cases} a + 3b &= 41 \\ a - b &= 1 \end{cases} \implies a = 11, b = 10.$$

Thus, $a^2 + b^2 = 11^2 + 10^2 = \boxed{221}$. □

Example 1.3 (2017 SMO(J) P20)
Let a, b and c be positive integers such that

$$a^2 + bc = 257 \quad \text{and} \quad ab + bc = 101.$$

Find the value of a, b and c.

Proof. Armed with the idea from the previous problem, we see again that 101 is a prime number, so $b(a+c) = 101$. Moreover, $b + c \geq a$ so that $b = 1$ and $a + c = 101$. Thus,

$$a^2 + bc = a^2 + c = a^2 + (101 - a) = 257 \implies a(a-1) = 156.$$

With some guess and check (since a is a positive integer), $a = 13$, $b = 1$ and $c = 88$. □

Example 1.4 (2016 SMO(J) P24)
If $\frac{a}{2b} = \frac{2b}{3c} = \frac{3c}{8a}$, find the value of $\frac{ac+cb}{cb-ba}$.

The given condition tells us that a, b and c are all connected, so we should express a and b in terms of c.

Proof. Let $\frac{a}{2b} = \frac{2b}{3c} = \frac{3c}{8a} = k$ for some real k. We have

$$\begin{cases} a = 2kb \\ 2b = 3kc \\ 3c = 8ka \end{cases} \implies \begin{cases} a = 3k^2c \\ 2b = 8k^2a \end{cases} \implies \begin{cases} a = 24k^5c \\ b = 12k^4c \end{cases}.$$

By the given condition, we have $\frac{2b}{3c} = \frac{24k^4c}{3c} = 8k^4 = k \implies 8k^3 = 1 \implies k = \frac{1}{2}$, $k^5 = \frac{1}{32}$

Hence,

$$\begin{aligned} \frac{ac+cb}{cb-ba} &= \frac{24k^5c^2 + 12k^4c^2}{12k^4c^2 - 288k^9c^2} \\ &= \frac{2k+1}{1-24k^5} \\ &= \frac{2 \cdot \frac{1}{2} + 1}{1 - 24 \cdot \frac{1}{32}} = \boxed{8} \end{aligned}$$

□

1.2 Homogenisation

For our first concept of the handout, we introduce the trick of counting degrees, or *homogenisation*. We only illustrate as such with a simple problem, and further applications of this idea will follow in a later.

Proposition 1.5 (Counting degrees)
We say that an expression or equation is *homogenous* (or homogenised) if there is no loose constant terms remaining. For example, $x^3 + 2022x^2y + 2021$ is non-homogenous because of the constant 2021, while $x^3 + 2022xy^3 + xy$ is homogenous because there is no loose constant.

We *want* to work with homogenous expressions because degrees cancel nicely, and expressions are usually neat.

Example 1.6

If x and y are real numbers such that $x + y = 2$ and that

$$\frac{(1-x)^2}{x} + \frac{(1-y)^2}{y} = -4,$$

find the value of xy.

Consider the term $\frac{(1-x)^2}{x}$. The numerator is a quadratic (degree 2) while the denominator is linear (degree 1), so we say that this term is of degree 1. The term $\frac{(1-y)^2}{y}$ is also similarly of degree 1, and so the LHS is of degree 1.

On the other hand, the RHS is a constant (degree 0), which makes the equation *non-homogenous*. Fortunately, we are given that $x + y = 2$.

Proof. We begin by writing

$$\frac{(1-x)^2}{x} + \frac{(1-y)^2}{y} = -2(x+y).$$

Now, our equation is homogenous, and as we shall see, this equation now resolves itself very cleanly. Clearing denominators (since $x, y \neq 0$), we have

$$\begin{aligned}
\frac{(1-x)^2}{x} + \frac{(1-y)^2}{y} = -2(x+y) &\implies x(1-y)^2 + y(1-x)^2 = -2xy(x+y) \\
&\implies (xy^2 - 2xy + x) + (x^2y - 2xy + y) = -2x^2y - 2xy^2 \\
&\implies 3x^2y + 3xy^2 - 4xy + (x+y) = 0 \\
&\implies 3xy(x+y) - 4xy + (x+y) = 0 \\
&\implies 2xy + 2 = 0 \implies xy = \boxed{-1}
\end{aligned}$$

□

1.3 Substitutions and Identities

Sometimes scary expressions are just expanded versions of simpler expressions! A smart substitution will help simplify things nicely. Certainly, knowing some basic identities will help reduce the problem. Here, we list the necessary few.

Proposition 1.7 (Basic Identities)

These identities can also be found in school textbooks, and familiarity is assumed.

1. $a^2 - b^2 = (a-b)(a+b)$
2. $a^2 \pm 2ab + b^2 = (a \pm b)^2$
3. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
4. $a^3 + 3a^2b + 3ab^2 + b^3 = (a+b)^3$
5. $a^3 - 3a^2b + 3ab^2 - b^3 = (a-b)^3$

To start off, we showcase an application of a nifty substitution, without which, the problem may be untractable to the unsuspecting student.

Example 1.8 (2013 SMO(J) P11)
Find the value of $\sqrt{9999^2 + 19999}$.

Certainly, we are not expected to multiply out and then take the square root of this monstrous expression! However, we do note that the choice of 9999 is completely arbitrary, so it may be wise to make a substitution for 9999 and express 19999 in terms of a.

Proof. Let $a = 9999$, then $19999 = 2 \cdot 9999 + 1 = 2a + 1$. Aha! We now know

$$\begin{aligned}\sqrt{9999^2 + 19999} &= \sqrt{a^2 + (2a+1)}\\ &= \sqrt{(a+1)^2} = a + 1\\ &= \boxed{10000}.\end{aligned}$$

How nice is that! □

Example 1.9 (2014 SMO(J) P16)
Let m and n be positive real numbers satisfying the equation

$$m + 4\sqrt{mn} - 2\sqrt{m} - 4\sqrt{n} + 4n = 3.$$

Find the value of

$$\frac{\sqrt{m} + 2\sqrt{n} + 2014}{4 - \sqrt{m} - 2\sqrt{n}}.$$

The required value is just as grizzly as the given equation. To start, we should be somewhat suspicious of the term $4\sqrt{mn}$: this is a term of degree 1, since $\sqrt{m}$ and $\sqrt{n}$ both have degree $\frac{1}{2}$. Yet, it's mixed in an expression containing terms of degree $\frac{1}{2}$ like $2\sqrt{m}$ and $4\sqrt{n}$, as well as degree 1 terms like m and $4n$.

This strongly suggests the substitutions $x = \sqrt{m}, y = \sqrt{n}$, so we want the value of

$$\frac{\sqrt{m} + 2\sqrt{n} + 2014}{4 - \sqrt{m} - 2\sqrt{n}} = \frac{x + 2y + 2014}{4 - (x + 2y)}.$$

The given equation gives

$$x^2 + 4xy - 2x - 4y + 4y^2 = 3. \implies (x+2y)^2 - 2(x+2y) = 3 \implies (x+2y)(x+2y-2) = 3.$$

Setting $z = x + 2y$, $z(z-2) = 3 \implies z^2 - 2z - 3 = 0 \implies z = 3$ only since $z = x + 2y > 0$.

Hence, the required value is $\boxed{4}$.

Example 1.10 (2015 SMO(J) P21)
Find the value of

$$\sqrt{(98 \cdot 100 + 2)(100 \cdot 102 + 2) + (100 \cdot 2)^2}.$$

Once again, this is a monstrous expression, but fortunately the individual parts in the expression are broken down slightly for us. For now, let us focus on the term $98 \cdot 100 + 2$.

It may be tempting to simply substitute $a = 98$ (or $a = 100$), but the expression $a(a+2) + 2 = a^2 + 2a + 2$ is cumbersome to work with. Perhaps we should compromise and let $a = 99$ instead!

Then, $98 \cdot 100 + 2 = (a-1)(a+1) + 2 = a^2 + 1 = 99^2 + 1$, and similarly, $100 \cdot 102 + 2 = 101^2 + 1$. The desired expression is now $\sqrt{(99^2+1)(101^2+1) + (100 \cdot 2)^2}$, and it is apparent why $100 \cdot 2$ was written instead of 200. Letting $b = 100$, we have

$$\begin{aligned}\sqrt{(99^2+1)(101^2+1)+(100\cdot 2)^2} &= \sqrt{((b-1)^2+1)((b+1)^2+1)+4b^2} \\ &= \sqrt{(b^2-2b+2)(b^2+2b+2)+4b^2} \\ &= \sqrt{((b^2+2)-2b)((b^2+2)+2b)+4b^2} \\ &= \sqrt{(b^2+2)^2-(2b)^2+4b^2} \\ &= b^2+2 = \boxed{10002}\end{aligned}$$

Example 1.11 (2018 AMC10 A/10)

Suppose that real number x satisfies

$$\sqrt{49-x^2} - \sqrt{25-x^2} = 3.$$

Find the value of $\sqrt{49-x^2} + \sqrt{25-x^2}$.

The given equation is an equation in terms of x^2, so it would be instinctive to substitute $y = x^2$ and then squaring both sides and rearranging:

$$\begin{aligned}\sqrt{49-y} - \sqrt{25-y} = 3 &\implies 74 - 2y - 2\sqrt{(49-y)(25-y)} = 9 \\ &\implies 2\sqrt{(49-y)(25-y)} = 65 - 2y \\ &\implies 4(49-y)(25-y) = (65-2y)^2 \\ &\implies 4(y^2 - 74y + 1225) = 4y^2 - 260y + 4225 \\ &\implies -36y + 675 = 0 \implies y = \frac{75}{4},\end{aligned}$$

whence the required value is

$$\sqrt{49-x^2} + \sqrt{25-x^2} = \sqrt{49-y} + \sqrt{25-y} = \sqrt{\frac{121}{4}} + \sqrt{\frac{25}{4}} = \boxed{8}.$$

This method feels slightly disingenous though, why would the problem seek such a specific quantity? On a closer look, we notice that the given expression and the required expression are very similar. In particular, they differ only in their signs!

If we let $a = \sqrt{49-x^2}$ and $b = \sqrt{25-x^2}$, then $a - b = 3$ and we seek the value of $a + b$.

This reminds us of the *difference of squares* identity, and we finish the problem succinctly:

Proof.

$$(a-b)(a+b) = a^2 - b^2 = (49 - x^2) - (25 - x^2) = 24.$$

On the other hand,

$$3(a+b) = 24 \implies \sqrt{49-x^2} + \sqrt{25-x^2} = a + b = \frac{24}{3} = \boxed{8}.$$

□

For our ending problem, we showcase the idea of *rationalisation* for cube roots. Indeed, at the heart of rationalising surds, we rely on the difference of squares. As it turns out, this also works for cube roots!

Example 1.12 (2013 AIME II/5)
Find the real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$ in the form

$$x = \frac{\sqrt[3]{a} + \sqrt[3]{b} + 1}{c},$$

where a, b, c are positive integers.

Firstly, the coefficients $3, 3, 1$ seem vaguely familiar. In fact, $(x+1)^3 = x^3 + 3x^2 + 3x + 1$. Thus, we write

$$8x^3 - 3x^2 - 3x - 1 = 9x^3 - (x^3 + 3x^2 + 3x + 1) = 9x^3 - (x+1)^3,$$

whence $9x^3 = (x+1)^3$.

Solving for x, we have

$$x = \frac{1}{\sqrt[3]{9} - 1}.$$

We make the substitution $a = \sqrt[3]{9}$, and so $a^3 = 9$. The cube root suggests that the difference of cubes formula may be useful. Indeed,

$$(a-1)(a^2 + a + 1) = a^3 - 1,$$

and so

$$\begin{aligned} x &= \frac{1}{\sqrt[3]{9} - 1} = \frac{1}{a-1} \\ &= \frac{1}{a-1} \cdot \frac{a^2 + a + 1}{a^2 + a + 1} = \frac{a^2 + a + 1}{a^3 - 1} \\ &= \frac{\sqrt[3]{81} + \sqrt[3]{9} + 1}{8}. \end{aligned}$$

2 Solving (★)

Many questions will demand you to "find all X satisfying condition Y", the keyword being **find all**. These problems come in solving equations, functional equations, inequalities, etc. This implies that there are two parts to the problem:

1. Find the solutions and show that no other solutions exist.
2. Prove that your solutions satisfy the condition (this is part of the problem!).

It will be more instructive for us to work through a problem.

Example 2.1 (2021 SMO(O) P10)
Find all real roots to the equation

$$\sqrt[9]{x^7+30x^5} = \sqrt[7]{x^9-30x^5}.$$

As a first scout, we see that $x=0$ is a solution. Moreover, if $x=k$ is a solution, then so is $x=-k$, so we may consider only $x>0$.

Proof. Let $a=\sqrt[9]{x^7+30x^5}$, $b=\sqrt[7]{x^9-30x^5}$ so that

$$\begin{cases} a-b=0 \\ a^9+b^7=x^{16} \end{cases} \implies a^{16}=x^{16} \implies a=\pm x.$$

If $a=x$, then $a^9=a^7+30a^5$.

Aha! This gives our first solution $a=0$, corresponding to $x=0$. In what follows, we assume $x\neq 0 \implies a\neq 0$.

Thus, $a^4=a^2+30 \implies (a^2-6)(a^2-5)=0 \implies a=\pm\sqrt{5}, a=\pm\sqrt{6}$.

Now, are all of these valid solutions? We should verify so. Suppose $a=x=\sqrt{5}$ (since we have $x>0$), then

$$a^9=625\sqrt{5}=125\sqrt{5}+750\sqrt{5}=875\sqrt{5},$$

which is a contradiction. Thus $x\neq\sqrt{5}$.

Remark 2.2. Notice the use of proof by contradiction here! We assumed that $x=\sqrt{5}$ is a solution, and then derived the absurd statement that $625\sqrt{5}=875\sqrt{5}$.

Similarly, suppose $a=x=\sqrt{6}$, then $a^9=1296\sqrt{6}=216\sqrt{6}+1080\sqrt{6}$, which is consistent. Hence, $a=\sqrt{6}$ is the solution we're after. Having exhausted all cases, and verified that our solution was correct, we can now confidently say that the only solutions to the original equation are

$$\boxed{x=0,\ \sqrt{6} \text{ and } -\sqrt{6}}.$$

□

This next problem showcases the mixing of algebra and a string of divisibility arguments. Keep an eye out for these, especially if we are solving over the integers!

Example 2.3 (2018 SMO(O) P9)

Let $p(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are distinct non-zero integers. Suppose $p(a) = a^3$ and $p(b) = b^3$. Find $p(13)$.

Hmm, somehow we are apparently able to determine the cubic with only 2 given points. Perhaps the distinct and integer condition will come into play somehow. (For the astute reader, this matches the four degrees of restriction we need to determine a cubic - just as we need four points to determine a cubic as well.)

Proof. Given $p(a) = a^3$ and $p(b) = b^3$, we have

$$\begin{aligned} a^3 + ab + c &= 0 \\ (a+1)b^2 + c &= 0 \end{aligned}$$

Eliminating c, we have $(a+1)b^2 - ab - a^3 = 0$, which is a quadratic in b.

Using the quadratic formula,

$$\begin{aligned} b &= \frac{a \pm \sqrt{a^2 + 4a^3(a+1)}}{2(a+1)} \\ &= \frac{a \pm a\sqrt{4a^2 + 4a + 1}}{2(a+1)} = \frac{a \pm a(2a+1)}{2(a+1)} \\ &= a \quad \textbf{or} \quad -\frac{a^2}{a+1} \end{aligned}$$

Since a, b are distinct, we must have $b = -\frac{a^2}{a+1}$. Moreover, since b is an integer,

$$b = -\frac{a^2}{a+1} = -(a-1) - \frac{1}{a+1}$$

is also an integer. This means $a + 1 | 1 \implies a + 1 = \pm 1$, giving $a = -2$ only. Thereafter, $b = 4$ and $c = 16$ follows easily, whence the answer is

$$p(13) = 13^3 - 2 \cdot 13^2 + 4 \cdot 13 + 16 = \boxed{1927}$$

□

While more involved in nature, this next problem showcases just how powerful a string of divisibility arguments can be.

Example 2.4 (USAMO 2015 P1, JMO P2)

Solve in integers the equation

$$x^2 + xy + y^2 = \left(\frac{x+y}{3} + 1\right)^3.$$

To start off, we know LHS is an integer and so RHS must also be an integer. This means $\frac{x+y}{3}$ is also an integer, so we are motivated to write $x + y = 3k$ for some integer k.

On the other hand, LHS contains a pesky xy term. Here comes the *trick*: to kill off this nasty term, we rely on the symmetry of x and y.

Proof. Consider $a = x + y, b = x - y$:

$$xy = \frac{(a+b)(a-b)}{4} \quad \text{and} \quad x^2 + y^2 = \frac{(a+b)^2 + (a-b)^2}{4}$$

and the equation becomes

$$\frac{1}{4}\left((a+b)^2 + (a+b)(a-b) + (a-b)^2\right) = \left(\frac{a}{3} + 1\right)^3 \implies 3a^2 + b^2 = 4\left(\frac{a}{3} + 1\right)^3.$$

Letting $a = 3k$, $27k^2 + b^2 = 4(k+1)^3 \implies b^2 = 4k^3 - 15k^2 + 12k + 4$. At this point, surely the cubic must factor. Indeed, we miraculously see

$$b^2 = (k-2)^2(4k+1) \implies 4k + 1 = m^2,$$

for odd m (see the problem above for a similar reasoning).

Now, we are done, since by back-substituting, we have:

$$a = 3k = \frac{3}{4}(m^2 - 1), \quad b^2 = (k-2)^2(4k+1) = \left(\frac{m^2 - 9}{4}\right)^2 m^2 \implies b = \pm\frac{m^3 - 9m}{4}.$$

Hence,

$$x = \frac{1}{8}\left(3(m^2 - 1) \pm (m^3 - 9m)\right) \quad \text{and} \quad y = \frac{1}{8}\left(3(m^2 - 1) \mp (m^3 - 9m)\right).$$

Is that all? Not quite! Don't forget that we are told to solve the given equation over the integers, so we should show also that our solutions are indeed all integers. Fortunately, since m is odd, we may let $m = 2n+1$ so that

$$\boxed{x = n^3 + 3n^2 - 1 \quad \text{and} \quad y = -n^3 + 3n + 1},$$

and permutations (note that the equation is symmetric in x and y!). □

3 Exercises

3.1 Warmup

Problem 3.1 (2015 SMO(J) P9)**.** Find the value of $\left(4\sqrt{4+2\sqrt{3}}-\sqrt{49+8\sqrt{3}}\right)^2$. **Hint:** 4

Problem 3.2 (2014 SMO(J) P13)**.** Let A be the solution of the equation

$$\frac{x-7}{x-8}-\frac{x-8}{x-9}=\frac{x-10}{x-11}-\frac{x-11}{x-12}.$$

Hint: 17

Problem 3.3 (2016 SMO(J) P11)**.** If the sum and product of two positive real numbers are both equal to 13, find the sum of the squares of these two numbers.

Problem 3.4 (2014 SMO(J) P23)**.** Let a, b, c be non-zero reals satisfying $a+2b+3c=2014$ and $2a+3b+2c=2014$, find the value of

$$\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

Problem 3.5 (2014 SMO(J) P26)**.** Let x be such that $\left(x+\frac{1}{x}\right)^2=3$. Evaluate $x^3+\frac{1}{x^3}$.

Problem 3.6 (2014/15 SDML 2A/P5)**.** Let a, b, c, d, e be five consecutive positive integers such that $a<b<c<d<e$. If $b+c+d$ is a perfect cube and $a+b+c+d+e$ is a perfect square, find the smallest possible value of e.

Problem 3.7 (2008 SMO(J) P23)**.** Evaluate

$$\frac{(2020^2-20100)(20100^2-100^2)(2000^2+20100)}{2010^6-10^6}.$$

Problem 3.8 (2014 SMO(S) P23)**.** Let n be a positive integer and let

$$x=\frac{\sqrt{n+2}-\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}, \quad y=\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}-\sqrt{n}}.$$

If $14x^2+26xy+14y^2=2014$, find the value of n. **Hint:** 19

Problem 3.9 (2014 SMO(S) P29)**.** Solve the following system in real numbers:

$$\begin{aligned} x^2 &= 4y+4 \\ y^2 &= 4z+4 \\ z^2 &= 4x+4. \end{aligned}$$

Hints: 1 2 20

Problem 3.10 (2018 SMO(J) P2)**.** Find the value of

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2018}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2017}\right)$$
$$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2018}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2017}\right).$$

Hint: 9

3.2 Problems

Problem 3.11 (2014 SMO(J) P29). Let x, y be real values with $xy < 0$, such that

$$x + y = \frac{1}{3} \quad \text{and} \quad \frac{1}{x^2} + \frac{1}{y^2} = 40.$$

Find the value of $\frac{1}{x^4} + \frac{1}{y^4}$.

Problem 3.12 (2014 SMO(S) P20). Let $x = \sqrt{37 - 20\sqrt{3}}$. Find the value of

$$\frac{x^4 - 9x^3 + 5x^2 - 7x + 68}{x^2 - 10x + 19}.$$

Problem 3.13 (2017 SMO(S) P29). Find the least positive integer n such that

$$5(3^2 + 2^2)(3^4 + 2^4) \cdots (3^{2^n} + 2^{2^n}) > 9^{256}.$$

Problem 3.14 (2005 AIME II/7). Let

$$x = \frac{4}{(\sqrt{5} + 1)(\sqrt[4]{5} + 1)(\sqrt[8]{5} + 1)(\sqrt[16]{5} + 1)}.$$

Find $(x + 1)^{48}$.

> **Remark 3.15.** This problem and the above are solved by a similar idea. What is this idea?

Problem 3.16 (2017 SMO(J) P24). Let a be an integer such that $a + 2$ and $a + 79$ are both perfect squares. Find the largest possible value of a.

Problem 3.17 (2020 SMO(J) P25). Let x be a positive integer satisfying the equation

$$\sqrt[5]{x + 76638} - \sqrt[5]{x - 76637} = 5.$$

Find the value of x.

Problem 3.18 (2014 SMO(J) P32). For $a \geq \frac{1}{8}$, define

$$g(a) = \sqrt[3]{a + \frac{a+1}{3}\sqrt{\frac{8a-1}{3}}} + \sqrt[3]{a - \frac{a+1}{3}\sqrt{\frac{8a-1}{3}}}.$$

Find the maximum value of $g(a)$.

Problem 3.19 (2016 SMO(J) P28). Let x, y, z be positive integers such that

$$x^2 + y - z = 124 \quad \text{and} \quad x + y^2 - z = 100.$$

Find the value of $x + y + z$.

Problem 3.20 (2014 SMO(S) R2/2). Find all positive real numbers a, b, c satisfying the system of equations:

$$a\sqrt{b} = a + c, \quad b\sqrt{c} = b + a, \quad c\sqrt{a} = c + b.$$

Hints: 15 23 6 5

Problem 3.21 (2020 SMO(J) P24)**.** Let $m > n$ be positive integers satisfying

$$(m^2 - n^2)^2 = 1 + 80n.$$

Find the smallest possible value of mn. **Hints:** 8 24 10

Problem 3.22 (2014 SMO(J) P30)**.** Find the following sum

$$\begin{aligned}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{29}\right) &+ \left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{29}\right)\\ &+ \left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{29}\right)\\ &+\cdots\\ &+\left(\frac{27}{28}+\frac{27}{29}\right)\\ &+\frac{28}{29}.\end{aligned}$$

Hints: 11 21

Problem 3.23 (2018 SMO(S) P25)**.** Suppose x, y, z are positive reals satisfying the following system of equations:

$$\begin{aligned}\frac{\sqrt{xyz}}{x+y} &= 3,\\ \frac{\sqrt{xyz}}{y+z} &= \frac{5}{2},\\ \frac{\sqrt{xyz}}{z+x} &= \frac{15}{7}.\end{aligned}$$

Find the value of $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. **Hints:** 14 3 7

3.3 Challenges

Problem 3.24 (1990 AIME P15)**.** Let a, b, x, y be real numbers satisfying

$$\begin{aligned}ax + by &= 3\\ ax^2 + by^2 &= 7\\ ax^3 + by^3 &= 16\\ ax^4 + by^4 &= 42.\end{aligned}$$

Find the value of $ax^5 + by^5$. **Hint:** 13

Problem 3.25 (2019 SMO(S) P16)**.** Suppose x, y, z are positive integers satisfying the following system of equations:

$$\begin{aligned}x^2z + y^2z + 8xy &= 200,\\ 2x^2 + 2y^2 + xyz &= 50.\end{aligned}$$

Find the maximum possible value of $x + y + z$. **Hint:** 12

Problem 3.26 (2019 SMO(S) P25)**.** Let a, b be positive integers satisfying

$$a^2 - 2b^2 = 1.$$

If $500 < a + b < 1000$, find the value of a and of b. **Hints:** 18 22 16

4 Hints to Problems

1. We really hope to bring the the same variables together.
2. Sum all three equations.
3. Find $\frac{x}{\sqrt{xyz}}$ and the like.
4. Simplify the square roots.
5. There aren't many possible values for a, b, c anymore. Find them systematically?
6. Consider equation 1 and 3. What relation should apply to b so that our assumption holds?
7. What do we know about $\frac{x}{y}$ and the like?
8. LHS has degree 4 while RHS is linear. This means that $m^2 - n^2$ can't be *too* big, otherwise LHS will outgrow RHS.
9. Find two good substitutions to make.
10. This is a quadratic in terms of some variable. Solve the quadratic in terms of this variable.
11. Write the sums in a nicer way... They really should resemble a triangular pyramid.
12. The system is linear in terms of a variable. Perhaps make this variable the subject of both equations.
13. Consider $(ax + by)(x + y)$.
14. Take the reciprocal across each equation.
15. Factorise each equation in the system.
16. Given a solution, we expect to be able to generate more solutions. Find a way to generate these solutions, and a starting point for your generation.
17. Each numerator isn't *that* different from it's denominator.
18. The bound given is very artificial. We should expect many many solutions for a and b in general.
19. What is the relationship between x and y?
20. Complete the square.
21. How many times are fractions with the same denominator summed?
22. Square both sides, then rearrange the equation in a way that resembles the form in the original equation.
23. The system is symmetric, so we may assume $a \geq b \geq c$.
24. Consider $m - n = d$.

4.1 Tangent: Pell's Equation

I'll admit, the last challenge is slightly gimmicky in the sense that it is a focus in number theory. In general, for a positive integer n such that n is not a perfect square, the equation

$$a^2 - nb^2 = 1$$

is called the **Pell's Equation**.

We focus on the equation given in the challenge:

$$a^2 - 2b^2 = 1, \tag{1}$$

and we show how we can expect to generate infinitely many solutions righteously.

Squaring (1),

$$\begin{aligned} a^2 - 2b^2 = 1 &\implies (a^2 - 2b^2)^2 = 1 \\ &\implies (a^2 + 2b^2)^2 - 8a^2b^2 = 1 \\ &\implies (a^2 + 2b^2)^2 - 2(2ab)^2 = 1. \end{aligned}$$

Hence, if (a, b) is a solution of (1), then so is $(a^2 + 2b^2, 2ab)$. We note that $(a, b) = (3, 2)$ is the smallest positive solution to (1), and so we recursively generate the others:

$$(3, 2) \to (17, 12) \to (577, 408) \cdots .$$

Thus, $(a, b) = (577, 408)$ is our solution.

Right, so there's that. But how do we know these are *all* the solutions? In fact, they are not! The user can readily verify that $(99, 70)$ is indeed a solution that is omitted through our recursion. Why did our "generator" miss this?

We first show something more general:

Lemma 4.1 (Generating Solutions to Pell's Equation)
For a general Pell's Equation

$$x^2 - Dy^2 = 1, \tag{2}$$

where D is not a square, then (2) has solutions generated by

$$(u_{n+1}, v_{n+1}) = (u_1u_n + Dv_1v_n, v_1u_n + u_1v_n). \tag{3}$$

Here, (u_1, v_1) is the *fundamental solution*, so to say, the solution with the smallest $v_1 > 0$.

Proof. We work by induction. By definition, (u_1, v_1) is a solution to the equation. Now, suppose (u_k, v_k) is a solution to (2) for some $k \geq 1$. Then, we see that

$$\begin{aligned} u_{n+1}^2 - Dv_{n+1}^2 &= (u_1u_n + Dv_1v_n)^2 - D(v_1u_n + u_1v_n)^2 \\ &= (u_1^2 - Dv_1^2)(u_n^2 - Dv_n^2) \\ &= 1 \cdot 1 = 1. \end{aligned}$$

Hence, (u_{n+1}, v_{n+1}) is also a solution to (2). □

Now, onto the chase for the missing solution. We *squared* the equation when moving from one solution to the next. However, we missed the solutions generated by "odd powers". To wit, observe that we have, by considering the difference of squares, that

$$3^2 - 2 \cdot 2^2 = 1$$

$$(3+2\sqrt{2})(3-2\sqrt{2})=1.$$

Squaring this initial relation, we get

$$\begin{aligned}(3+2\sqrt{2})^2(3-2\sqrt{2})^2&=1\\(17+12\sqrt{2})(17-12\sqrt{2})&=1\\17^2-2\cdot 12^2&=1.\end{aligned}$$

This agrees with our initial recurrence that $(17,12)$ is indeed a solution. Now, cubing the initial relation gives

$$\begin{aligned}(3+2\sqrt{2})^3(3-2\sqrt{2})^3&=1\\(27+54\sqrt{2}+72+16\sqrt{2})(27-54\sqrt{2}+72-16\sqrt{2})&=1\\(99+70\sqrt{2})(99-70\sqrt{2})&=1\\99^2-2\cdot 70^2&=1,\end{aligned}$$

and out comes $(99,70)$ as our solution.

It now remains to show that the recurrence given by (u_{n+1},v_{n+1}) generates all the solutions to (2)

Lemma 4.2
All solutions to (2) are given by (u_{n+1},v_{n+1}).

Proof. By another induction argument, it is not hard to see that

$$u_n+v_n\sqrt{D}=(u_1+v_1\sqrt{D})^n.$$

We leave this to the reader.

Now, define $z_n\equiv u_n+v_n\sqrt{D}=(u_1+v_1\sqrt{D})^n$ for nonnegative n. Then, we crucially have

$$z_0<z_1<z_2<\cdots<z_n<\cdots.$$

Suppose for the sake of contradiction that (2) had a solution $z=u+v\sqrt{D}$ *not* of the form in (u_{n+1},v_{n+1}). Then, there exists an integer m such that $z_m<z<z_{m+1}$ so that

$$\begin{aligned}&(u_m+v_m\sqrt{D})<(u+v\sqrt{D})<(u_{m+1}+v_{m+1}\sqrt{D})\\\implies &1<(u+v\sqrt{D})(u_m-v_m\sqrt{D})<\frac{u_{m+1}+v_{m+1}\sqrt{D}}{u_m+v_m}<u_1+v_1\sqrt{D},\end{aligned}$$

and thus

$$1<(uu_m-Dvv_m)+(u_mv-uv_m)\sqrt{D}<u_1+v_1\sqrt{D}.$$

However,

$$(uu_m-Dvv_m)^2-D(u_mv-uv_m)^2=(u^2-Dv^2)(u_m^2-Dv_m^2)=1,$$

so $(x,y)=(uu_m-Dvv_m,u_mv-uv_m)$ is a solution to (2), and is smaller than (u_1,v_1). This contradicts our assumption that (u_1,v_1) is the fundamental (or smallest) solution, and this means that there are no other unexpected solutions. The proof is now complete. ☐

Question: How do we know that there exists a fundamental solution to (2)?