39. $\lim_{x \to 0^-} \frac{|x|}{x}$

40. $\lim_{x \to 0^+} \frac{|x|}{x}$

41. $\lim_{x \to 0^-} \frac{1 + \cos x}{\sin x}$

42. $\lim_{x \to \infty} \frac{\sin x}{x}$

GC *In Problems 43–48, find the horizontal and vertical asymptotes for the graphs of the indicated functions. Then sketch their graphs.*

43. $f(x) = \frac{3}{x+1}$

44. $f(x) = \frac{3}{(x+1)^2}$

45. $F(x) = \frac{2x}{x-3}$

46. $F(x) = \frac{3}{9-x^2}$

47. $g(x) = \frac{14}{2x^2+7}$

48. $g(x) = \frac{2x}{\sqrt{x^2+5}}$

49. The line $y = ax + b$ is called an **oblique asymptote** to the graph of $y = f(x)$ if either $\lim_{x \to \infty} [f(x) - (ax + b)] = 0$ or $\lim_{x \to -\infty} [f(x) - (ax + b)] = 0$. Find the oblique asymptote for

$$f(x) = \frac{2x^4 + 3x^3 - 2x - 4}{x^3 - 1}$$

Hint: Begin by dividing the denominator into the numerator.

50. Find the oblique asymptote for

$$f(x) = \frac{3x^3 + 4x^2 - x + 1}{x^2 + 1}$$

51. Using the symbols M and δ, give precise definitions of each expression.

(a) $\lim_{x \to c^+} f(x) = -\infty$ (b) $\lim_{x \to c^-} f(x) = \infty$

52. Using the symbols M and N, give precise definitions of each expression.

(a) $\lim_{x \to \infty} f(x) = \infty$ (b) $\lim_{x \to -\infty} f(x) = \infty$

53. Give a rigorous proof that if $\lim_{x \to \infty} f(x) = A$ and $\lim_{x \to \infty} g(x) = B$, then

$$\lim_{x \to \infty} [f(x) + g(x)] = A + B$$

54. We have given meaning to $\lim_{x \to A} f(x)$ for $A = a, a^-, a^+, -\infty, \infty$. Moreover, in each case, this limit may be L (finite), $-\infty$, ∞, or may fail to exist in any sense. Make a table illustrating each of the 20 possible cases.

55. Find each of the following limits or indicate that it does not exist even in the infinite sense.

(a) $\lim_{x \to \infty} \sin x$ (b) $\lim_{x \to \infty} \sin \frac{1}{x}$

(c) $\lim_{x \to \infty} x \sin \frac{1}{x}$ (d) $\lim_{x \to \infty} x^{3/2} \sin \frac{1}{x}$

(e) $\lim_{x \to \infty} x^{-1/2} \sin x$ (f) $\lim_{x \to \infty} \sin\left(\frac{\pi}{6} + \frac{1}{x}\right)$

(g) $\lim_{x \to \infty} \sin\left(x + \frac{1}{x}\right)$ (h) $\lim_{x \to \infty} \left[\sin\left(x + \frac{1}{x}\right) - \sin x\right]$

56. Einstein's Special Theory of Relativity says that the mass $m(v)$ of an object is related to its velocity v by

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Here m_0 is the rest mass and c is the velocity of light. What is $\lim_{v \to c^-} m(v)$?

GC *Use a computer or a graphing calculator to find the limits in Problems 57–64. Begin by plotting the function in an appropriate window.*

57. $\lim_{x \to \infty} \frac{3x^2 + x + 1}{2x^2 - 1}$

58. $\lim_{x \to -\infty} \sqrt{\frac{2x^2 - 3x}{5x^2 + 1}}$

59. $\lim_{x \to -\infty} \left(\sqrt{2x^2 + 3x} - \sqrt{2x^2 - 5}\right)$

60. $\lim_{x \to \infty} \frac{2x + 1}{\sqrt{3x^2 + 1}}$

61. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{10}$

62. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$

63. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x^2}$

64. $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{\sin x}$

CAS *Find the one-sided limits in Problems 65–71. Begin by plotting the function in an appropriate window. Your computer may indicate that some of these limits do not exist, but, if so, you should be able to interpret the answer as either ∞ or $-\infty$.*

65. $\lim_{x \to 3^-} \frac{\sin|x - 3|}{x - 3}$

66. $\lim_{x \to 3^-} \frac{\sin|x - 3|}{\tan(x - 3)}$

67. $\lim_{x \to 3^-} \frac{\cos(x - 3)}{x - 3}$

68. $\lim_{x \to \frac{\pi}{2}^+} \frac{\cos x}{x - \pi/2}$

69. $\lim_{x \to 0^+} (1 + \sqrt{x})^{1/\sqrt{x}}$

70. $\lim_{x \to 0^+} (1 + \sqrt{x})^{1/x}$

71. $\lim_{x \to 0^+} (1 + \sqrt{x})^{x}$

Answers to Concepts Review: **1.** x increases without bound; $f(x)$ gets close to L as x increases without bound **2.** $f(x)$ increases without bound as x approaches c from the right; $f(x)$ decreases without bound as x approaches c from the left **3.** $y = 6$; horizontal **4.** $x = 6$; vertical

1.6 Continuity of Functions

In mathematics and science, we use the word *continuous* to describe a process that goes on without abrupt changes. In fact, our experience leads us to assume that this is an essential feature of many natural processes. It is this notion as it pertains to functions that we now want to make precise. In the three graphs shown in Figure 1, only the third graph exhibits continuity at c. In the first two graphs, either $\lim_{x \to c} f(x)$ does not exist, or it exists but does not equal $f(c)$. Only in the third graph does $\lim_{x \to c} f(x) = f(c)$.

A Discontinuous Machine

A good example of a discontinuous machine is the postage machine, which (in 2005) charged \$0.37 for a 1-ounce letter but \$0.60 for a letter the least little bit over 1 ounce.

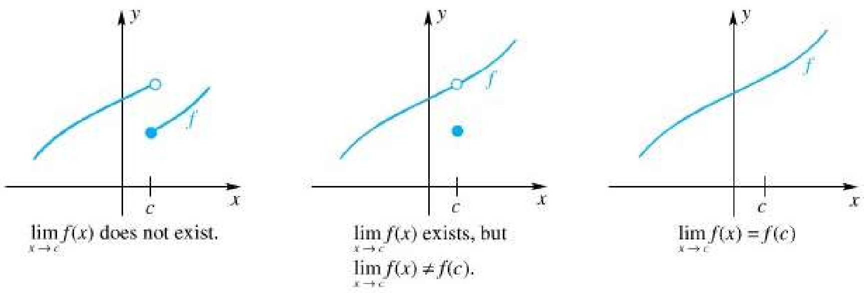


Figure 1

Here is the formal definition.

Definition **Continuity at a Point**

Let f be defined on an open interval containing c. We say that f is **continuous** at c if

$$\lim_{x \to c} f(x) = f(c)$$

We mean by this definition to require three things:

1. $\lim_{x \to c} f(x)$ exists,
2. $f(c)$ exists (i.e., c is in the domain of f), and
3. $\lim_{x \to c} f(x) = f(c)$.

If any one of these three fails, then f is **discontinuous** at c. Thus, the functions represented by the first and second graphs of Figure 1 are discontinuous at c. They do appear, however, to be continuous at other points of their domains.

EXAMPLE 1 Let $f(x) = \dfrac{x^2 - 4}{x - 2}$, $x \neq 2$. How should f be defined at $x = 2$ in order to make it continuous there?

SOLUTION

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

Therefore, we define $f(2) = 4$. The graph of the resulting function is shown in Figure 2. In fact, we see that $f(x) = x + 2$ for all x. ■

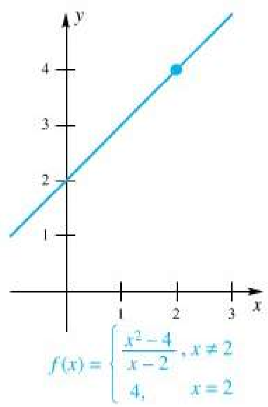


Figure 2

A point of discontinuity c is called **removable** if the function can be defined or redefined at c so as to make the function continuous. Otherwise, a point of discontinuity is called **nonremovable.** The function f in Example 1 has a removable discontinuity at 2 because we could define $f(2) = 4$ and the function would be continuous there.

Continuity of Familiar Functions Most functions that we will meet in this book are either (1) continuous everywhere or (2) continuous everywhere except at a few points. In particular, Theorem 1.3B implies the following result.

Theorem A **Continuity of Polynomial and Rational Functions**

A polynomial function is continuous at every real number c. A rational function is continuous at every real number c in its domain, that is, everywhere except where its denominator is zero.

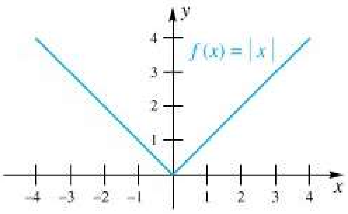


Figure 3

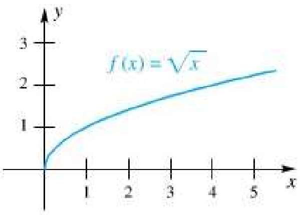


Figure 4

Recall the absolute value function $f(x) = |x|$; its graph is shown in Figure 3. For $x < 0$, $f(x) = -x$, a polynomial; for $x > 0$, $f(x) = x$, another polynomial. Thus, $|x|$ is continuous at all numbers different from 0 by Theorem A. But

$$\lim_{x \to 0} |x| = 0 = |0|$$

(see Problem 27 of Section 1.2). Therefore, $|x|$ is also continuous at 0; it is continuous everywhere.

By the Main Limit Theorem (Theorem 1.3A)

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{\lim_{x \to c} x} = \sqrt[n]{c}$$

provided $c > 0$ when n is even. This means that $f(x) = \sqrt[n]{x}$ is continuous at each point where it makes sense to talk about continuity. In particular, $f(x) = \sqrt{x}$ is continuous at each real number $c > 0$ (Figure 4). We summarize.

Theorem B Continuity of Absolute Value and *n*th Root Functions

The absolute value function is continuous at every real number c. If n is odd, the nth root function is continuous at every real number c; if n is even, the nth-root function is continuous at every positive real number c.

Continuity under Function Operations Do the standard function operations preserve continuity? Yes, according to the next theorem. In it, f and g are functions, k is a constant, and n is a positive integer.

Theorem C Continuity under Function Operations

If f and g are continuous at c, then so are kf, $f + g$, $f - g$, $f \cdot g$, f/g (provided that $g(c) \neq 0$), f^n, and $\sqrt[n]{f}$ (provided that $f(c) > 0$ if n is even).

Proof All these results are easy consequences of the corresponding facts for limits from Theorem 1.3A. For example, that theorem, combined with the fact that f and g are continuous at c, gives

$$\lim_{x \to c} f(x)g(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) = f(c)g(c)$$

This is precisely what it means to say that $f \cdot g$ is continuous at c. ■

EXAMPLE 2 At what numbers is $F(x) = (3|x| - x^2)/\left(\sqrt{x} + \sqrt[3]{x}\right)$ continuous?

SOLUTION We need not even consider nonpositive numbers, since F is not defined at such numbers. For any positive number, the functions $\sqrt{x}$, $\sqrt[3]{x}$, $|x|$, and x^2 are all continuous (Theorems A and B). It follows from Theorem C that $3|x|$, $3|x| - x^2$, $\sqrt{x} + \sqrt[3]{x}$, and finally,

$$\frac{(3|x| - x^2)}{\left(\sqrt{x} + \sqrt[3]{x}\right)}$$

are continuous at each positive number. ■

The continuity of the trigonometric functions follows from Theorem 1.4A.

Theorem D Continuity of Trigonometric Functions

The sine and cosine functions are continuous at every real number c. The functions $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous at every real number c in their domains.

Proof Theorem 1.4A says that for every real number c in the function's domain, $\lim_{x \to c} \sin x = \sin c$, $\lim_{x \to c} \cos x = \cos c$, and so forth, for all six of the trigonometric functions. These are exactly the conditions required for these functions to be continuous at every real number in their respective domains. ■

EXAMPLE 3 Determine all points of discontinuity of $f(x) = \dfrac{\sin x}{x(1 - x)}$, $x \neq 0, 1$. Classify each point of discontinuity as removable or nonremovable.

SOLUTION By Theorem D, the numerator is continuous at every real number. The denominator is also continuous at every real number, but when $x = 0$ or $x = 1$, the denominator is 0. Thus, by Theorem C, f is continuous at every real number except $x = 0$ and $x = 1$. Since

$$\lim_{x \to 0} \frac{\sin x}{x(1 - x)} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{(1 - x)} = (1)(1) = 1$$

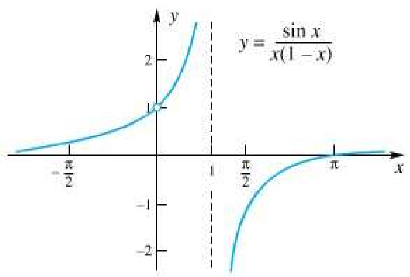


Figure 5

we could define $f(0) = 1$ and the function would continuous there. Thus, $x = 0$ is a removable discontinuity. Also, since

$$\lim_{x \to 1^+} \frac{\sin x}{x(1 - x)} = -\infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{\sin x}{x(1 - x)} = \infty$$

there is no way to define $f(1)$ to make f continuous at $x = 1$. Thus $x = 1$ is a nonremovable discontinuity. A graph of $y = f(x)$ is shown in Figure 5. ■

There is another functional operation, composition, that will be very important in later work. It, too, preserves continuity.

Theorem E **Composite Limit Theorem**

If $\lim_{x \to c} g(x) = L$ and if f is continuous at L, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

In particular, if g is continuous at c and f is continuous at $g(c)$, then the composite $f \circ g$ is continuous at c.

Proof of Theorem E (Optional)

Proof Let $\varepsilon > 0$ be given. Since f is continuous at L, there is a corresponding $\delta_1 > 0$ such that

$$|t - L| < \delta_1 \Rightarrow |f(t) - f(L)| < \varepsilon$$

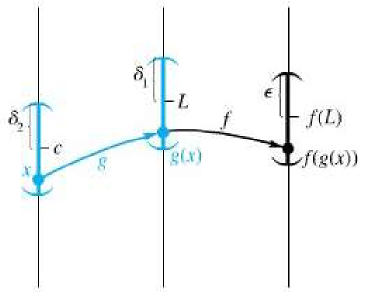


Figure 6

and so (see Figure 6)

$$|g(x) - L| < \delta_1 \Rightarrow |f(g(x)) - f(L)| < \varepsilon$$

But because $\lim_{x \to c} g(x) = L$, for a given $\delta_1 > 0$ there is a corresponding $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_2 \Rightarrow |g(x) - L| < \delta_1$$

When we put these two facts together, we have

$$0 < |x - c| < \delta_2 \Rightarrow |f(g(x)) - f(L)| < \varepsilon$$

This shows that

$$\lim_{x \to c} f(g(x)) = f(L)$$

The second statement in Theorem E follows from the observation that if g is continuous at c then $L = g(c)$. ■

EXAMPLE 4 Show that $h(x) = |x^2 - 3x + 6|$ is continuous at each real number.

SOLUTION Let $f(x) = |x|$ and $g(x) = x^2 - 3x + 6$. Both are continuous at each real number, and so their composite

$$h(x) = f(g(x)) = |x^2 - 3x + 6|$$

is also. ■

EXAMPLE 5 Show that

$$h(x) = \sin \frac{x^4 - 3x + 1}{x^2 - x - 6}$$

is continuous except at 3 and -2.

SOLUTION $x^2 - x - 6 = (x - 3)(x + 2)$. Thus, the rational function

$$g(x) = \frac{x^4 - 3x + 1}{x^2 - x - 6}$$

is continuous except at 3 and -2 (Theorem A). We know from Theorem D that the sine function is continuous at every real number. Thus, from Theorem E, we conclude that, since $h(x) = \sin(g(x))$, h is also continuous except at 3 and -2. ■

Continuity on an Interval So far, we have been discussing continuity at a point. We now wish to discuss continuity on an interval. Continuity on an interval ought to mean continuity at each point of that interval. This is exactly what it does mean for an *open* interval.

When we consider a closed interval $[a, b]$, we face a problem. It might be that f is not even defined to the left of a (e.g., this occurs for $f(x) = \sqrt{x}$ at $a = 0$), so, strictly speaking, $\lim_{x \to a} f(x)$ does not exist. We choose to get around this problem by calling f continuous on $[a, b]$ if it is continuous at each point of (a, b) and if $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$. We summarize in a formal definition.

Definition Continuity on an Interval

The function f is **right continuous** at a if $\lim_{x \to a^+} f(x) = f(a)$ and **left continuous** at b if $\lim_{x \to b^-} f(x) = f(b)$.

We say f is **continuous on an open interval** if it is continuous at each point of that interval. It is **continuous on the closed interval** $[a, b]$ if it is continuous on (a, b), right continuous at a, and left continuous at b.

For example, it is correct to say that $f(x) = 1/x$ is continuous on $(0, 1)$ and that $g(x) = \sqrt{x}$ is continuous on $[0, 1]$.

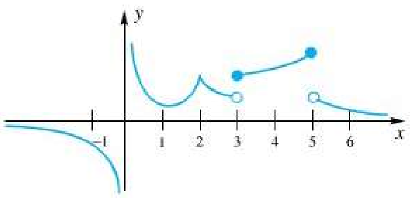


Figure 7

EXAMPLE 6 Using the definition above, describe the continuity properties of the function whose graph is sketched in Figure 7.

SOLUTION The function appears to be continuous on the open intervals $(-\infty, 0)$ $(0, 3)$, and $(5, \infty)$, and also on the closed interval $[3, 5]$. ■

EXAMPLE 7 What is the largest interval over which the function defined by $g(x) = \sqrt{4 - x^2}$ is continuous?

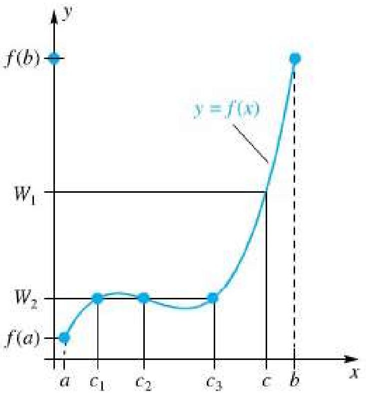


Figure 8

SOLUTION The domain of g is the interval $[-2, 2]$. If c is in the open interval $(-2, 2)$, then g is continuous at c by Theorem E; hence, g is continuous on $(-2, 2)$. The one-sided limits are

$$\lim_{x\to -2^+} \sqrt{4 - x^2} = \sqrt{4 - \left(\lim_{x\to -2^+} x\right)^2}\sqrt{4-4} = 0 = g(-2)$$

and

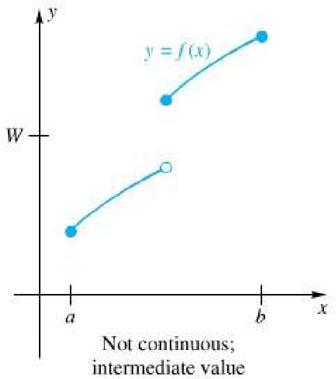
$$\lim_{x\to 2^-} \sqrt{4 - x^2} = \sqrt{4 - \left(\lim_{x\to 2^-} x\right)^2} = \sqrt{4-4} = 0 = g(2)$$

This implies that g is right continuous at -2 and left continuous at 2. Thus, g is continuous on its domain, the closed interval $[-2, 2]$. ■

Intuitively, for f to be continuous on $[a, b]$ means that the graph of f on $[a, b]$ should have no jumps, so we should be able to "draw" the graph of f from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting our pencil from the paper. Thus, the function f should take on every value between $f(a)$ and $f(b)$. This property is stated more precisely in Theorem F.

Theorem F Intermediate Value Theorem

Let f be a function defined on $[a, b]$ and let W be a number between $f(a)$ and $f(b)$. If f is continuous on $[a, b]$, then there is at least one number c between a and b such that $f(c) = W$.



Not continuous; intermediate value property fails.

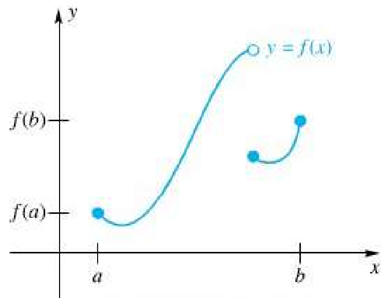
Figure 9

Figure 8 shows the graph of a function $f(x)$ that is continuous on $[a, b]$. The Intermediate Value Theorem says that for every W in $(f(a), f(b))$ there must be a c in $[a, b]$ such that $f(c) = W$. In other words, f takes on every value between $f(a)$ and $f(b)$. Continuity is needed for this theorem, for otherwise it is possible to find a function f and a number W between $f(a)$ and $f(b)$ such that there is no c in $[a, b]$ that satisfies $f(c) = W$. Figure 9 shows an example of such a function.

It seems clear that continuity is sufficient, although a formal proof of this result turns out to be difficult. We leave the proof to more advanced works.

The converse of this theorem, which is not true in general, says that if f takes on every value between $f(a)$ and $f(b)$ then f is continuous. Figures 8 and 10 show functions that take on all values between $f(a)$ and $f(b)$, but the function in Figure 10 is not continuous on $[a, b]$. Just because a function has the intermediate value property does not mean that it must be continuous.

The Intermediate Value Theorem can be used to tell us something about the solutions of equations, as the next example shows.



Not continuous, although intermediate value property holds

Figure 10

EXAMPLE 8 Use the Intermediate Value Theorem to show that the equation $x - \cos x = 0$ has a solution between $x = 0$ and $x = \pi/2$.

SOLUTION Let $f(x) = x - \cos x$, and let $W = 0$. Then $f(0) = 0 - \cos 0 = -1$ and $f(\pi/2) = \pi/2 - \cos \pi/2 = \pi/2$. Since f is continuous on $[0, \pi/2]$ and since $W = 0$ is between $f(0)$ and $f(\pi/2)$, the Intermediate Value Theorem implies the existence of a c in the interval $(0, \pi/2)$ with the property that $f(c) = 0$. Such a c is a solution to the equation $x - \cos x = 0$. Figure 11 suggests that there is exactly one such c.

We can go one step further. The midpoint of the interval $[0, \pi/2]$ is the point $x = \pi/4$. When we evaluate $f(\pi/4)$, we get

$$f(\pi/4) = \frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2} \approx 0.0782914$$

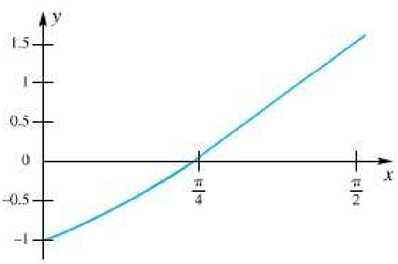


Figure 11

which is greater than 0. Thus, $f(0) < 0$ and $f(\pi/4) > 0$, so another application of the Intermediate Value Theorem tells us that there exists a c between 0 and $\pi/4$ such that $f(c) = 0$. We have thus narrowed down the interval containing the desired c from $[0, \pi/2]$ to $[0, \pi/4]$. There is nothing stopping us from selecting the midpoint of $[0, \pi/4]$ and evaluating f at that point, thereby narrowing even further the interval containing c. This process could be continued indefinitely until we find that c is in a sufficiently small interval. This method of zeroing in on a solution is called the *bisection method*, and we will study it further in Section 3.7. ■

The Intermediate Value Theorem can also lead to some surprising results.

EXAMPLE 9 Use the Intermediate Value Theorem to show that on a circular wire ring there are always two points opposite from each other with the same temperature.

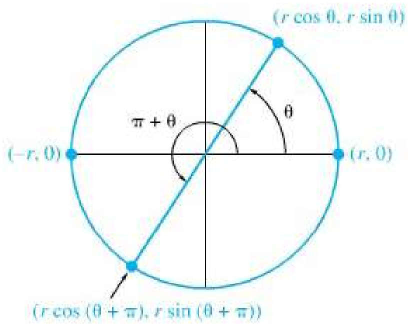


Figure 12

SOLUTION Choose coordinates for this problem so that the center of the ring is the origin, and let r be the radius of the ring. (See Figure 12.) Define $T(x, y)$ to be the temperature at the point (x, y). Consider a diameter of the circle that makes an angle θ with the x-axis, and define $f(\theta)$ to be the temperature difference between the points that make angles of θ and $\theta + \pi$; that is,

$$f(\theta) = T(r\cos\theta, r\sin\theta) - T(r\cos(\theta+\pi), r\sin(\theta+\pi))$$

With this definition

$$f(0) = T(r, 0) - T(-r, 0)$$
$$f(\pi) = T(-r, 0) - T(r, 0) = -\left[T(r, 0) - T(-r, 0)\right] = -f(0)$$

Thus, either $f(0)$ and $f(\pi)$ are both zero, or one is positive and the other is negative. If both are zero, then we have found the required two points. Otherwise, we can apply the Intermediate Value Theorem. Assuming that temperature varies continuously, we conclude that there exists a c between 0 and π such that $f(c) = 0$. Thus, for the two points at the angles c and $c + \pi$, the temperatures are the same. ■

Concepts Review

1. A function f is continuous at c if _____ $= f(c)$.

2. The function $f(x) = [\![x]\!]$ is discontinuous at _____.

3. A function f is said to be continuous on a closed interval $[a, b]$ if it is continuous at every point of (a, b) and if _____ and _____.

4. The Intermediate Value Theorem says that if a function f is continuous on $[a, b]$ and W is a number between $f(a)$ and $f(b)$, then there is a number c between _____ and _____ such that _____.

Problem Set 1.6

In Problems 1–15, state whether the indicated function is continuous at 3. If it is not continuous, tell why.

1. $f(x) = (x-3)(x-4)$

2. $g(x) = x^2 - 9$

3. $h(x) = \dfrac{3}{x-3}$

4. $g(t) = \sqrt{t-4}$

5. $h(t) = \dfrac{|t-3|}{t-3}$

6. $h(t) = \dfrac{|\sqrt{(t-3)^4}|}{t-3}$

7. $f(t) = |t|$

8. $g(t) = |t-2|$

9. $h(x) = \dfrac{x^2-9}{x-3}$

10. $f(x) = \dfrac{21-7x}{x-3}$

11. $r(t) = \begin{cases} \dfrac{t^3-27}{t-3} & \text{if } t \neq 3 \\ 27 & \text{if } t = 3 \end{cases}$

12. $r(t) = \begin{cases} \dfrac{t^3-27}{t-3} & \text{if } t \neq 3 \\ 23 & \text{if } t = 3 \end{cases}$

13. $f(t) = \begin{cases} t-3 & \text{if } t \le 3 \\ 3-t & \text{if } t > 3 \end{cases}$

14. $f(t) = \begin{cases} t^2-9 & \text{if } t \le 3 \\ (3-t)^2 & \text{if } t > 3 \end{cases}$

15. $f(x) = \begin{cases} -3x+7 & \text{if } x \le 3 \\ -2 & \text{if } x > 3 \end{cases}$

16. From the graph of g (see Figure 13), indicate the values where g is discontinuous. For each of these values state whether g is continuous from the right, left, or neither.

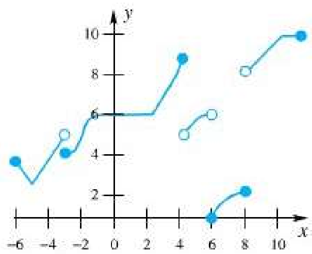


Figure 13

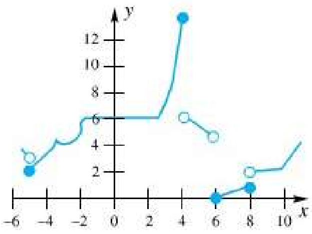


Figure 14

17. From the graph of h given in Figure 14, indicate the intervals on which h is continuous.

In Problems 18–23, the given function is not defined at a certain point. How should it be defined in order to make it continuous at that point? (See Example 1.)

18. $f(x) = \dfrac{x^2 - 49}{x - 7}$

19. $f(x) = \dfrac{2x^2 - 18}{3 - x}$

20. $g(\theta) = \dfrac{\sin\theta}{\theta}$

21. $H(t) = \dfrac{\sqrt{t} - 1}{t - 1}$

22. $\phi(x) = \dfrac{x^4 + 2x^2 - 3}{x + 1}$

23. $F(x) = \sin\dfrac{x^2 - 1}{x + 1}$

In Problems 24–35, at what points, if any, are the functions discontinuous?

24. $f(x) = \dfrac{3x + 7}{(x - 30)(x - \pi)}$

25. $f(x) = \dfrac{33 - x^2}{x\pi + 3x - 3\pi - x^2}$

26. $h(\theta) = |\sin\theta + \cos\theta|$

27. $r(\theta) = \tan\theta$

28. $f(u) = \dfrac{2u + 7}{\sqrt{u + 5}}$

29. $g(u) = \dfrac{u^2 + |u - 1|}{\sqrt[3]{u + 1}}$

30. $F(x) = \dfrac{1}{\sqrt{4 + x^2}}$

31. $G(x) = \dfrac{1}{\sqrt{4 - x^2}}$

32. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

33. $g(x) = \begin{cases} x^2 & \text{if } x < 0 \\ -x & \text{if } 0 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

34. $f(t) = [\![t]\!]$

35. $g(t) = [\![t + \frac{1}{2}]\!]$

36. Sketch the graph of a function f that satisfies all the following conditions.

(a) Its domain is $[-2, 2]$.

(b) $f(-2) = f(-1) = f(1) = f(2) = 1$.

(c) It is discontinuous at -1 and 1.

(d) It is right continuous at -1 and left continuous at 1.

37. Sketch the graph of a function that has domain $[0, 2]$ and is continuous on $[0, 2)$ but not on $[0, 2]$.

38. Sketch the graph of a function that has domain $[0, 6]$ and is continuous on $[0, 2]$ and $(2, 6]$ but is not continuous on $[0, 6]$.

39. Sketch the graph of a function that has domain $[0, 6]$ and is continuous on $(0, 6)$ but not on $[0, 6]$.

40. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{cases}$$

Sketch the graph of this function as best you can and decide where it is continuous.

In Problems 41–48, determine whether the function is continuous at the given point c. If the function is not continuous, determine whether the discontinuity is removable or nonremovable.

41. $f(x) = \sin x;\ c = 0$

42. $f(x) = \dfrac{x^2 - 100}{x - 10};\ c = 10$

43. $f(x) = \dfrac{\sin x}{x};\ c = 0$

44. $f(x) = \dfrac{\cos x}{x};\ c = 0$

45. $g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

46. $F(x) = x\sin\dfrac{1}{x};\ c = 0$

47. $f(x) = \sin\dfrac{1}{x};\ c = 0$

48. $f(x) = \dfrac{4 - x}{2 - \sqrt{x}};\ c = 4$

49. A cell phone company charges \$0.12 for connecting a call plus \$0.08 per minute or any part thereof (e.g., a phone call lasting 2 minutes and 5 seconds costs \$0.12 + 3 × \$0.08). Sketch a graph of the cost of making a call as a function of the length of time t that the call lasts. Discuss the continuity of this function.

50. A rental car company charges \$20 for one day, allowing up to 200 miles. For each additional 100 miles, or any fraction thereof, the company charges \$18. Sketch a graph of the cost for renting a car for one day as a function of the miles driven. Discuss the continuity of this function.

51. A cab company charges \$2.50 for the first $\frac{1}{4}$ mile and \$0.20 for each additional $\frac{1}{8}$ mile. Sketch a graph of the cost of a cab ride as a function of the number of miles driven. Discuss the continuity of this function.

52. Use the Intermediate Value Theorem to prove that $x^3 + 3x - 2 = 0$ has a real solution between 0 and 1.

53. Use the Intermediate Value Theorem to prove that $(\cos t)t^3 + 6\sin^5 t - 3 = 0$ has a real solution between 0 and 2π.

GC **54.** Use the Intermediate Value Theorem to show that $x^3 - 7x^2 + 14x - 8 = 0$ has at least one solution in the interval $[0, 5]$. Sketch the graph of $y = x^3 - 7x^2 + 14x - 8$ over $[0, 5]$. How many solutions does this equation really have?

GC **55.** Use the Intermediate Value Theorem to show that $\sqrt{x} - \cos x = 0$ has a solution between 0 and $\pi/2$. Zoom in on the graph of $y = \sqrt{x} - \cos x$ to find an interval having length 0.1 that contains this solution.

56. Show that the equation $x^5 + 4x^3 - 7x + 14 = 0$ has at least one real solution.

57. Prove that f is continuous at c if and only if $\lim_{t \to 0} f(c + t) = f(c)$.

58. Prove that if f is continuous at c and $f(c) > 0$ there is an interval $(c - \delta, c + \delta)$ such that $f(x) > 0$ on this interval.

59. Prove that if f is continuous on $[0, 1]$ and satisfies $0 \le f(x) \le 1$ there, then f has a *fixed point*; that is, there is a number c in $[0, 1]$ such that $f(c) = c$. *Hint:* Apply the Intermediate Value Theorem to $g(x) = x - f(x)$.

60. Find the values of a and b so that the following function is continuous everywhere.

$$f(x) = \begin{cases} x + 1 & \text{if } x < 1 \\ ax + b & \text{if } 1 \le x < 2 \\ 3x & \text{if } x \ge 2 \end{cases}$$

61. A stretched elastic string covers the interval $[0, 1]$. The ends are released and the string contracts so that it covers the interval $[a, b]$, $a \ge 0$, $b \le 1$. Prove that this results in at least one point of the string being where it was originally. See Problem 59.

62. Let $f(x) = \dfrac{1}{x - 1}$. Then $f(-2) = -\dfrac{1}{3}$ and $f(2) = 1$. Does the Intermediate Value Theorem imply the existence of a number c between -2 and 2 such that $f(c) = 0$? Explain.

63. Starting at 4 A.M., a hiker slowly climbed to the top of a mountain, arriving at noon. The next day, he returned along the same path, starting at 5 A.M. and getting to the bottom at 11 A.M. Show that at some point along the path his watch showed the same time on both days.

64. Let D be a bounded, but otherwise arbitrary, region in the first quadrant. Given an angle θ, $0 \le \theta \le \pi/2$, D can be circumscribed by a rectangle whose base makes angle θ with the x-axis as shown in Figure 15. Prove that at some angle this rectangle is a square. (This means that *any* bounded region can be circumscribed by a *square*.)

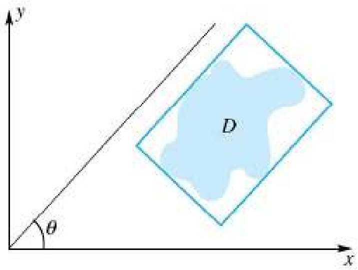


Figure 15

65. The gravitational force exerted by the earth on an object having mass m that is a distance r from the center of the earth is

$$g(r) = \begin{cases} \dfrac{GMmr}{R^3}, & \text{if } r < R \\ \dfrac{GMm}{r^2}, & \text{if } r \ge R \end{cases}$$

Here G is the gravitational constant, M is the mass of the earth, and R is the earth's radius. Is g a continuous function of r?

66. Suppose that f is continuous on $[a, b]$ and it is never zero there. Is it possible that f changes sign on $[a, b]$? Explain.

67. Let $f(x + y) = f(x) + f(y)$ for all x and y and suppose that f is continuous at $x = 0$.

(a) Prove that f is continuous everywhere.

(b) Prove that there is a constant m such that $f(t) = mt$ for all t (see Problem 43 of Section 0.5).

68. Prove that if $f(x)$ is a continuous function on an interval then so is the function $|f(x)| = \sqrt{(f(x))^2}$.

69. Show that if $g(x) = |f(x)|$ is continuous it is not necessarily true that $f(x)$ is continuous.

70. Let $f(x) = 0$ if x is irrational and let $f(x) = 1/q$ if x is the rational number p/q in reduced form $(q > 0)$.

(a) Sketch (as best you can) the graph of f on $(0, 1)$.

(b) Show that f is continuous at each irrational number in $(0, 1)$, but is discontinuous at each rational number in $(0, 1)$.

71. A thin equilateral triangular block of side length 1 unit has its face in the vertical xy-plane with a vertex V at the origin. Under the influence of gravity, it will rotate about V until a side hits the x-axis floor (Figure 16). Let x denote the initial x-coordinate of the midpoint M of the side opposite V, and let $f(x)$ denote the final x-coordinate of this point. Assume that the block balances when M is directly above V.

(a) Determine the domain and range of f.

(b) Where on this domain is f discontinuous?

(c) Identify any fixed points of f (see Problem 59).

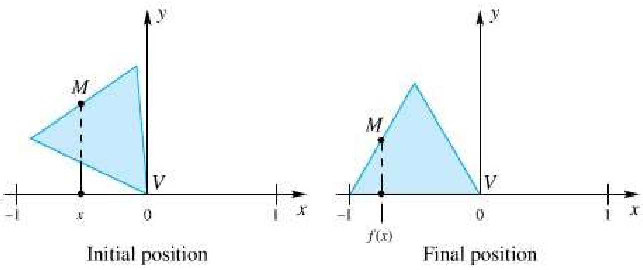


Figure 16

Answers to Concepts Review: **1.** $\lim_{x\to c} f(x)$ **2.** every integer **3.** $\lim_{x\to a^+} f(x) = f(a)$; $\lim_{x\to b^-} f(x) = f(b)$ **4.** a; b; $f(c) = W$

1.7 Chapter Review

Concepts Test

Respond with true or false to each of the following assertions. Be prepared to justify your answer.

1. If $f(c) = L$, then $\lim_{x\to c} f(x) = L$.

2. If $\lim_{x\to c} f(x) = L$, then $f(c) = L$.

3. If $\lim_{x\to c} f(x)$ exists, then $f(c)$ exists.

4. If $\lim_{x\to 0} f(x) = 0$, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < |x| < \delta$ implies $|f(x)| < \varepsilon$.

5. If $f(c)$ is undefined, then $\lim_{x\to c} f(x)$ does not exist.

6. The coordinates of the hole in the graph of $y = \dfrac{x^2 - 25}{x - 5}$ are $(5, 10)$.

7. If $p(x)$ is a polynomial, then $\lim_{x\to c} p(x) = p(c)$.

8. $\lim_{x\to 0} \dfrac{\sin x}{x}$ does not exist.

9. For every real number c, $\lim_{x\to c} \tan x = \tan c$.

10. $\tan x$ is continuous at every point of its domain.

11. The function $f(x) = 2\sin^2 x - \cos x$ is continuous at every real number.

12. If f is continuous at c, then $f(c)$ exists.

13. If f is continuous on the interval $(1, 3)$, then f is continuous at 2.

14. If f is continuous on $[0, 4]$, then $\lim_{x \to 0} f(x)$ exists.

15. If f is a continuous function such that $A \le f(x) \le B$ for all x, then $\lim_{x \to \infty} f(x)$ exists and it satisfies $A \le \lim_{x \to \infty} f(x) \le B$.

16. If f is continuous on (a, b) then $\lim_{x \to c} f(x) = f(c)$ for all c in (a, b).

17. $\lim_{x \to \infty} \frac{\sin x}{x} = 1$

18. If the line $y = 2$ is a horizontal asymptote of the graph of $y = f(x)$, then $\lim_{x \to \infty} f(x) = 2$.

19. The graph of $y = \tan x$ has many horizontal asymptotes.

20. The graph of $y = \frac{1}{x^2 - 4}$ has two vertical asymptotes.

21. $\lim_{t \to 1^+} \frac{2t}{t - 1} = \infty$.

22. If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$, then f is continuous at $x = c$.

23. If $\lim_{x \to c} f(x) = f\left(\lim_{x \to c} x\right)$, then f is continuous at $x = c$.

24. The function $f(x) = [\![x/2]\!]$ is continuous at $x = 2.3$.

25. If $\lim_{x \to 2} f(x) = f(2) > 0$, then $f(x) < 1.001f(2)$ for all x in some interval containing 2.

26. If $\lim_{x \to c} [f(x) + g(x)]$ exists, then $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exist.

27. If $0 \le f(x) \le 3x^2 + 2x^4$ for all x, then $\lim_{x \to 0} f(x) = 0$.

28. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.

29. If $f(x) \ne g(x)$ for all x, then $\lim_{x \to c} f(x) \ne \lim_{x \to c} g(x)$.

30. If $f(x) < 10$ for all x and $\lim_{x \to 2} f(x)$ exists, then $\lim_{x \to 2} f(x) < 10$.

31. If $\lim_{x \to a} f(x) = b$, then $\lim_{x \to a} |f(x)| = |b|$.

32. If f is continuous and positive on $[a, b]$, then $1/f$ must assume every value between $1/f(a)$ and $1/f(b)$.

Sample Test Problems

In Problems 1–22, find the indicated limit or state that it does not exist.

1. $\lim_{x \to 2} \frac{x - 2}{x + 2}$

2. $\lim_{u \to 1} \frac{u^2 - 1}{u + 1}$

3. $\lim_{u \to 1} \frac{u^2 - 1}{u - 1}$

4. $\lim_{u \to 1} \frac{u + 1}{u^2 - 1}$

5. $\lim_{x \to 2} \frac{1 - 2/x}{x^2 - 4}$

6. $\lim_{z \to 2} \frac{z^2 - 4}{z^2 + z - 6}$

7. $\lim_{x \to 0} \frac{\tan x}{\sin 2x}$

8. $\lim_{y \to 1} \frac{y^3 - 1}{y^2 - 1}$

9. $\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2}$

10. $\lim_{x \to 0} \frac{\cos x}{x}$

11. $\lim_{x \to 0^-} \frac{|x|}{x}$

12. $\lim_{x \to 1/2^+} [\![4x]\!]$

13. $\lim_{t \to 2^-} ([\![t]\!] - t)$

14. $\lim_{x \to 1^-} \frac{|x - 1|}{x - 1}$

15. $\lim_{x \to 0} \frac{\sin 5x}{3x}$

16. $\lim_{x \to 0} \frac{1 - \cos 2x}{3x}$

17. $\lim_{x \to \infty} \frac{x - 1}{x + 2}$

18. $\lim_{t \to \infty} \frac{\sin t}{t}$

19. $\lim_{t \to 2} \frac{t + 2}{(t - 2)^2}$

20. $\lim_{x \to 0^+} \frac{\cos x}{x}$

21. $\lim_{x \to \pi/4^-} \tan 2x$

22. $\lim_{x \to 0^+} \frac{1 + \sin x}{x}$

23. Prove using an ε–δ argument that $\lim_{x \to 3} (2x + 1) = 7$.

24. Let $f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ x & \text{if } -1 < x < 1 \\ 1 - x & \text{if } x \ge 1 \end{cases}$

Find each value.

(a) $f(1)$ (b) $\lim_{x \to 1^+} f(x)$

(c) $\lim_{x \to 1^-} f(x)$ (d) $\lim_{x \to -1} f(x)$

25. Refer to f of Problem 24. (a) What are the values of x at which f is discontinuous? (b) How should f be defined at $x = -1$ to make it continuous there?

26. Give the ε–δ definition in each case.

(a) $\lim_{u \to a} g(u) = M$ (b) $\lim_{x \to a^-} f(x) = L$

27. If $\lim_{x \to 3} f(x) = 3$ and $\lim_{x \to 3} g(x) = -2$ and if g is continuous at $x = 3$, find each value.

(a) $\lim_{x \to 3} [2f(x) - 4g(x)]$ (b) $\lim_{x \to 3} g(x) \frac{x^2 - 9}{x - 3}$

(c) $g(3)$ (d) $\lim_{x \to 3} g(f(x))$

(e) $\lim_{x \to 3} \sqrt{f^2(x) - 8g(x)}$ (f) $\lim_{x \to 3} \frac{|g(x) - g(3)|}{f(x)}$

28. Sketch the graph of a function f that satisfies all the following conditions.

(a) Its domain is $[0, 6]$.

(b) $f(0) = f(2) = f(4) = f(6) = 2$.

(c) f is continuous except at $x = 2$.

(d) $\lim_{x \to 2^-} f(x) = 1$ and $\lim_{x \to 5^+} f(x) = 3$.

29. Let $f(x) = \begin{cases} -1 & \text{if } x \le 0 \\ ax + b & \text{if } 0 < x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

Determine a and b so that f is continuous everywhere.

30. Use the Intermediate Value Theorem to prove that the equation $x^5 - 4x^3 - 3x + 1 = 0$ has at least one solution between $x = 2$ and $x = 3$.

In Problems 31–36, find the equations of all vertical and horizontal asymptotes for the given function.

31. $f(x) = \frac{x}{x^2 + 1}$

32. $g(x) = \frac{x^2}{x^2 + 1}$

33. $F(x) = \frac{x^2}{x^2 - 1}$

34. $G(x) = \frac{x^3}{x^2 - 4}$

35. $h(x) = \tan 2x$

36. $H(x) = \frac{\sin x}{x^2}$

REVIEW & PREVIEW PROBLEMS

1. Let $f(x) = x^2$. Find and simplify each of the following.

(a) $f(2)$

(b) $f(2.1)$

(c) $f(2.1) - f(2)$

(d) $\dfrac{f(2.1) - f(2)}{2.1 - 2}$

(e) $f(a + h)$

(f) $f(a + h) - f(a)$

(g) $\dfrac{f(a + h) - f(a)}{(a + h) - a}$

(h) $\lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{(a + h) - a}$

2. Repeat (a) through (h) of Problem 1 for the function $f(x) = 1/x$.

3. Repeat (a) through (h) of Problem 1 for the function $f(x) = \sqrt{x}$.

4. Repeat (a) through (h) of Problem 1 for the function $f(x) = x^3 + 1$.

5. Write the first two terms in the expansions of the following:

(a) $(a + b)^3$

(b) $(a + b)^4$

(c) $(a + b)^5$

6. Based on your results from Problem 5, make a conjecture about the first two terms in the expansion of $(a + b)^n$ for an arbitrary n.

7. Use a trigonometric identity to write $\sin(x + h)$ in terms of $\sin x$, $\sin h$, $\cos x$, and $\cos h$.

8. Use a trigonometric identity to write $\cos(x + h)$ in terms of $\cos x$, $\cos h$, $\sin x$, and $\sin h$.

9. A wheel centered at the origin and of radius 10 centimeters is rotating counterclockwise at a rate of 4 revolutions per second. A point P on the rim of the wheel is at position $(10, 0)$ at time $t = 0$.

(a) What are the coordinates of P at times $t = 1, 2, 3$?

(b) At what time does the point P first return to the starting position $(10, 0)$?

10. Assume that a soap bubble retains its spherical shape as it expands. At time $t = 0$ the soap bubble has radius 2 centimeters. At time $t = 1$, the radius has increased to 2.5 centimeters. How much has the volume changed in this 1 second interval?

11. One airplane leaves an airport at noon flying north at 300 miles per hour. Another leaves the same airport one hour later and flies east at 400 miles per hour.

(a) What are the positions of the airplanes at 2:00 P.M.?

(b) What is the distance between the two planes at 2:00 P.M.?

(c) What is the distance between the two planes at 2:15 P.M.?