1. Motivation

1.1. Votrex in descrete order parameter space

1.1.1. Manifold vs Graph

1.1.2. Fractional charge

1.2. Vortex

1.2.1. Vortex in $SO(2)$

topologically protected

1.2.2. Vortex in $SO(3)$

topologically unprotected

2. Formalism

2.1. Domain wall network

2.1.1. Domain wall network(DWN)

$$D := (P, \Phi, c)$$

where P is a *base graph*, mathematically plane graph (V, E, F).
We'll use Bourbaki's definition of graph about V, E and
here, F is a face cycles defined in clock-wise order.
$f = \langle e_1, e_2, \cdots, e_n \rangle$ where $\langle\rangle$ is a cyclically ordered sequence.
V, E, F are called *domains, domain walls, domin vortices* respectively.

Φ is a *order parameter set*, mathematically set,
c is a *coloring*, mathematically function $c : V(P) \to \Phi$ whose incidence is differnet.

2.1.2. Symmetry breaking, Degenerate ground states, Phase

2.1.2.1. Symmetry breaking

$$G_{broken} < G_{origin}$$

2.1.2.2. Degenerate ground states

$$\Phi = G_{origin}/G_{broken}$$

2.1.2.3. Phase

If $G_{broken} \lhd G_{origin}$, then Φ is a group and called *phase group*. and for domain $v \in V(P)$, $c(v) \in \Phi$ is called *phase of domain*.

2.1.3. Phase domain wall network(pDWN)

2.1.3.1. Definition

Domain wall network D is a pDWN if Φ is a phase group.

2.1.3.2. Phase shift of domain wall

shift map is defined by

$$c_{dw} : E(P) \to \Phi \quad \phi \mapsto c(o(e))^{-1} * c(t(e))$$

$c_{dw}(e)$ is called *phase shift of domain wall e*.
$c_{dw}(e) \neq 1_\Phi$

2.1.3.3. Phase loop of vortex

$$f \in F(P) \quad f = \langle e_1, e_2, \cdots, e_n \rangle$$

$$c_{vor}(f) := \langle c_{dw}(e_1), c_{dw}(e_2), \cdots, c_{dw}(e_n) \rangle$$

$c_{vor}(f)$ is called *phase loop of vortex f*.
$c_{dw}(e_1) * c_{dw}(e_2) * \cdots * c_{dw}(e_n) = 1_\Phi$

2.1.3.4. Phase equivalence

Order parameter space

degenerate ground states

possible dw

possible vor

If $c(v_1) = c(v_2)$, then two domains v_1,v_2 are called *phase equivalent*.
If $c_{dw}(e_1) = c_{dw}(e_2)$, then two domain walls e_1,e_2 are called *phase equivalent*.
If $c_{vor}(f_1) = c_{vor}(f_2)$, then two vortices f_1,f_2 are called *phase equivalent*.

2.1.3.5. Guage invariance

Boundary decomposition

3. Examples

3.1. $Z_2 \times Z_2$

3.1.1. all dw

3.1.1.1. vortex, anti-vortex, abab-vortex *

3.1.2. 01,10 dw

3.1.2.1. abab vortex

3.2. $Z_2 \times Z_2 \times Z_2$

3.3. $Z_3 \times Z_3$

3.3.1. z_{12}, z_{23}