

Complex Analysis:Analytic Function

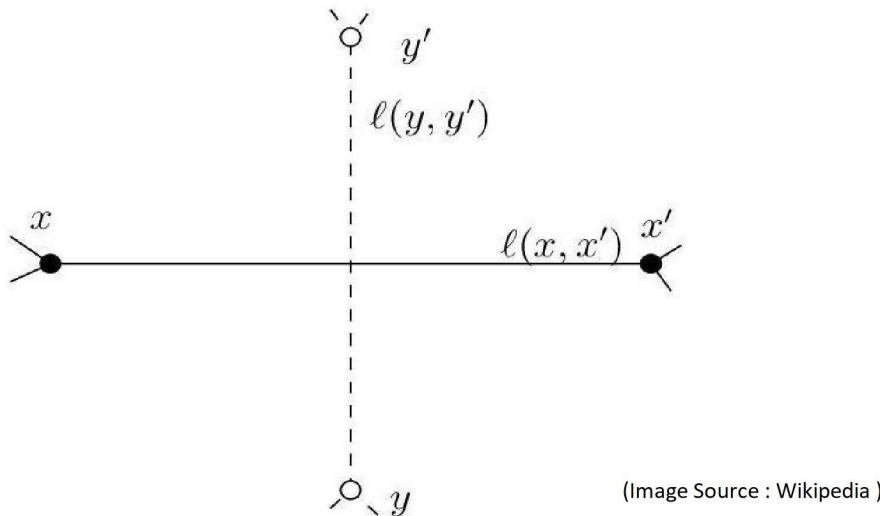
A complex number z is an ordered pair (x, y) of real numbers x and y , written, $Z = (x, y)$
 x is called the real part and y the imaginary part of z , written $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z)$.

Complex valued functions or simply complex functions which produce complex number from complex numbers.complex analysis or complex function theory is the study of complex analytic functions.

Two dimensional potential problem can be solved using analytic functions since the real and imaginary part of an analytic function are solution of two dimensional laplace's equation.

Analytic Function - A function $f(z)$ is said to be analytic at a point z_0 but at every point of some neighbourhood of z_0 .

- A function $f(z)$ is analytic in a domain if it is analytic at every point of the domain.
- An analytic function is also known as holomorphic, regular, monogenic.



(Image Source : Wikipedia)

Basic Properties of Analytic Functions

Let us recall that, for a complex function $f(z)$ with domain \mathcal{U} , the following conditions are equivalent:

1. For every closed disk D contained in \mathcal{U} , there exists a sequence of polynomials $\{p_n(z)\}$ converging uniformly to $f(z)$ over D .
2. For every closed disk D contained in \mathcal{U} , the following Cauchy's formula holds

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw, \quad (11.1)$$

where C is the boundary circle of D and z is a point in the interior of D .

3. For each point $z_0 \in \mathcal{U}$, $f(z)$ has a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (11.2)$$

(with a positive radius of convergence).

If any one of the above conditions holds, then so is the others and in that case we say that $f(z)$ is an **analytic function**. If $f(z)$ is an analytic function, then its derivatives $f^{(n)}(z)$ of all orders exist and, furthermore, the Taylor coefficients a_n of the power series in (11.2) can be determined by one of the following recipes:

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad a_n = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w) dw}{(w-z_0)^{n+1}} \quad (11.3)$$

where r is any positive number less than the radius of convergence of the power series. We use the above information to deduce a number of facts about analytic functions

1. *The limit of a uniformly convergent sequence of analytic functions is also an analytic function.*

To see this, let $\{f_n(z)\}$ be a sequence of analytic functions on \mathcal{U} converging uniformly to $f(z)$. Then there is a sequence of positive numbers $\{\varepsilon_n\}$ converging to 0 such that $|f_n(z) - f(z)| < \varepsilon_n$ for all z and for all n . Take any closed disk D contained in \mathcal{U} . Then,

for each n , there is a polynomial $p_n(z)$ such that $|f_n(z) - p_n(z)| < 1/n$ for all $z \in D$. Now, for all $z \in D$,

$$\begin{aligned} |p_n(z) - f(z)| &= |(p_n(z) - f_n(z)) + (f_n(z) - f(z))| \\ &\leq |p_n(z) - f_n(z)| + |f_n(z) - f(z)| \leq \frac{1}{n} + \varepsilon_n \end{aligned}$$

where $\frac{1}{n} + \varepsilon_n$ clearly tends to zero as $n \rightarrow \infty$. This shows that $\{p_n(z)\}$ is a sequence of polynomials converging uniformly to $f(z)$ over D . Hence $f(z)$ is analytic. Using a similar argument, we can prove

2. *If $f(z)$ and $g(z)$ are analytic functions on \mathcal{U} , then so are their sum $f(z) + g(z)$ and their product $f(z)g(z)$.*

A consequence of the above fact is that, if $f(z)$ is an analytic function and if $p(z)$ is a polynomial, then $p(f(z))$ is also analytic. Indeed, writing $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, we have

$$p(f(z)) = a_0 + a_1 f(z) + \cdots + a_n f(z)^n,$$

which is a sum of products of analytic functions and hence is analytic as well.

3. *If $f(z)$ and $g(z)$ are analytic functions and $f(z)$ is in the domain of g for all z , then their composite $g(f(z))$ is also analytic.*

To see this, we approximate $g(z)$ by a sequence of polynomials $\{p_n(z)\}$, which is assumed to be approaching $g(z)$ uniformly. Then $p_n(f(z))$ approach to $g(f(z))$ uniformly. By the remark after 2, we know that $p_n(f(z))$ are analytic. Now our assertion follows from 1.

4. *The function $f(z) = 1/z$ ($z \neq 0$) is analytic.*

To prove this, we take any point $z_0 \neq 0$ and show that $f(z)$ has a power series expansion at z_0 . Indeed,

$$\begin{aligned} f(z) &= \frac{1}{z} = \frac{1}{z - z_0 + z_0} = \frac{1}{z_0} \frac{1}{1 - (-(z - z_0))/z_0} \\ &= \frac{1}{z_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^n} (z - z_0)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z_0^{n+1}} (z - z_0)^n \end{aligned}$$

which is the required power series expansion.

In what follows, by an **entire function** we mean an analytic function $f(z)$ which is defined for all complex numbers z ; in other words, the domain of $f(z)$ is \mathbf{C} .

5 (Liouville's theorem). *Bounded entire functions are constant functions.*

Let $f(z)$ be a bounded entire function. The boundedness of $f(z)$ means that there is a positive number M such that $|f(z)| \leq M$ for all z . Consider the power series expansion of $f(z)$ at $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Since $f(z)$ is entire, the radius of convergence is ∞ . According to (11.3), we have

$$a_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z) dz}{z^{n+1}}$$

where $r > 0$ is arbitrary. We use the parameterization $z = re^{it}$ ($0 \leq t \leq 2\pi$) for the circle $|z| = r$. Then we have

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) r^{-n} e^{-int} dt$$

and hence

$$|a_n| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})| r^{-n} |e^{-int}| dt \leq \frac{1}{2\pi} \int_0^{2\pi} Mr^{-n} dt = \frac{M}{r^n}$$

in view of $|f(re^{it})| \leq M$. Thus, for $n \geq 1$, we have $|a_n| \leq M/r^n$ and $M/r^n \rightarrow 0$ as $r \rightarrow \infty$. This shows that $a_n = 0$ for $n \geq 1$. Thus $f(z) = a_0$ for all z . Hence $f(z)$ is a constant function.

6 (the fundamental theorem of algebra) *Every nonconstant polynomial $p(z)$ has a root. That is, there exists some z_0 such that $p(z_0) = 0$.*

We prove this by contradiction. Assume that $p(z)$ is a nonconstant polynomial without roots. That is, $p(z) \neq 0$ for all $z \in \mathbf{C}$. Then the value of $p(z)$ for all z is in the domain of the analytic function $f(z) = 1/z$ (from Fact 4) for all z and hence $f(p(z)) \equiv 1/p(z)$ is analytic, according to Fact 3. So $1/p(z)$ is an entire function. Writing

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

with $a_n \neq 0$, we have

$$\begin{aligned} \lim_{z \rightarrow \infty} \frac{1}{p(z)} &= \lim_{z \rightarrow \infty} \frac{1}{z^n(a_0 z^{-n} + a_1 z^{-n+1} + \cdots + z^{-1} a_{n-1} + a_n)} \\ &= \lim_{z \rightarrow \infty} \frac{1}{z^n} \lim_{z \rightarrow \infty} \frac{1}{a_0 z^{-n} + a_1 z^{-n+1} + \cdots + z^{-1} a_{n-1} + a_n} = 0 \cdot \frac{1}{a_n} = 0. \end{aligned}$$

In particular, $1/p(z)$ is bounded. Thus, $1/p(z)$ is a bounded entire function and, according to Liouville's theorem, it is a constant function. Consequently $p(z)$ is also a constant function, contradicting our assumption that $p(z)$ is nonconstant. The contradiction here tells us that $p(z)$ must have a root.

7 (the maximum modulus principle). *If $f(z)$ is an analytic function defined on \mathcal{U} , then its modulus $|f(z)|$ cannot attain its maximum in \mathcal{U} .*

To prove this fact, we suppose that $|f(z)|$ attains its maximum at some point $z_0 \in \mathcal{U}$, that is, $|f(z_0)| \geq |f(z)|$ for all z . Take any $r > 0$ such that the disk $D = \{z : |z - z_0| \leq r\}$ is contained in \mathcal{U} . We are going to show that $f(z)$ is necessarily constant in D . Let $z(t) = z_0 + re^{it}$ ($0 \leq t \leq 2\pi$) be the parameterization of the circle $|z - z_0| = r$. Let $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be the power series expansion of $f(z)$ at z_0 . Then

$$f(z_0 + re^{it}) = \sum_{n=0}^{\infty} a_n r^n e^{int} = \sum_{-\infty < n < \infty} c_n e^{int},$$

where $c_n = 0$ for $n < 0$ and $c_n = r^n a_n$. The last expression is the Fourier series of $f(z_0 + re^{it})$ considered as a function of t . Thus, by Bessel's inequality, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|^2 dt \geq \sum_{-\infty < n < \infty} |c_n|^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

On the other hand, since $|f(z)| \leq f(z_0)|$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|^2 dt \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)|^2 dt = |f(z_0)|^2 = |a_0|^2.$$

Thus we have

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq |a_0|^2,$$

or $\sum_{n=1}^{\infty} |a_n|^2 r^{2n} \leq 0$. So we must have $a_n = 0$ for $n \geq 1$. Hence the power series for $f(z)$ becomes $f(z) = a_0$, showing that $f(z)$ is a constant.

Let $f(z)$ be an analytic function. We say that a point z_0 in its domain a **zero** of $f(z)$ if $f(z_0) = 0$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \equiv a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \quad (11.4)$$

be the power series expansion of $f(z)$ at z_0 . Letting $z = z_0$ in the last identity, we see that $f(z_0) = a_0$. So z_0 is a zero of $f(z)$ means that $a_0 = 0$. Let us assume that z_0 is a zero of $f(z)$ but $f(z)$ is not identically equal to zero. Then $a_n \neq 0$ for some n . Let N be the smallest integer for which $a_N \neq 0$. Then $N > 0$ and $a_n = 0$ for all $n < N$. So the power series (11.4) becomes

$$\begin{aligned} f(z) &= \sum_{n=N}^{\infty} a_n z^n \equiv a_N(z - z_0)^N + a_{N+1}(z - z_0)^{N+1} + \dots \\ &= (z - z_0)^N (a_N + a_{N+1}(z - z_0) + a_{N+2}(z - z_0)^2 + \dots) \\ &= a_N(z - z_0)^N (1 + (z - z_0)g(z)) \end{aligned} \quad (11.5)$$

where $g(z) = a_N^{-1}a_{N+1} + a_N^{-1}a_{N+2}(z - z_0) + \dots$. Since $\lim_{z \rightarrow z_0} (z - z_0)g(z) = 0$, there exists $\delta > 0$ such that, when $|z - z_0| < \delta$, we have $|(z - z_0)g(z)| < 1$ and hence

$$1 + (z - z_0)g(z) \neq 0.$$

So, when $|z - z_0| < \delta$ and $z \neq z_0$, we have $f(z) \neq 0$. Thus z_0 is the only zero of $f(z)$ in the δ -neighborhood of z_0 . We have proved:

8. The zeros of an analytic function $f(z)$ are isolated points, unless $f(z)$ is identically zero.

In the rest of the present chapter, our purpose is to establish a general version of Cauchy's formula. We mainly focus on analytic functions $f(z)$ defined on a disk, although the statements are true for simply connected domain, in order to avoid some technicality in topology. We begin with a general fact about line integrals

9. If $F(z)$ is an analytic function and if C is a curve connecting two points z_0 and z_1 in the domain of $f(z)$, then

$$\int_C F'(z) dz = F(z_1) - F(z_0).$$

In particular, if C is a closed curve (meaning that $z_0 = z_1$) in the domain of an analytic function $F(z)$, then we have $\int_C F'(z) dz = 0$.

To see this, let $z = z(t)$ ($a \leq t \leq b$) be a parametric equation for C , with $z(a) = z_0$ and $z(b) = z_2$. Then we have

$$\begin{aligned} \int_C F'(z) dz &= \int_a^b F'(z(t)) dz(t) = \int_a^b F'(z(t)) z'(t) dt = \int_a^b \frac{d}{dt} F(z(t)) dt \\ &= F(z(b)) - F(z(a)) = F(z_1) - F(z_0), \end{aligned}$$

Here we have used the chain rule for differentiation, which tells us

$$(d/dt)F(z(t)) = F'(z(t))z'(t),$$

as well as the fundamental theorem of calculus, which says $\int_a^b f(t) dt = f(b) - f(a)$ for any continuously differentiable function $f(t)$ defined for $a \leq t \leq b$.

10. If $f(z)$ is an analytic function defined on a disk D , then there is an analytic function $F(z)$ defined on D such that $F'(z) = f(z)$, called a primitive of $f(z)$, and, as a consequence,

$$\int_C f(z) dz = 0$$

for any closed curve C in D .

To prove this, we first write down the power series expansion of $f(z)$ at the center of the disk D , say z_0 :

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \equiv a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots .$$

The radius of convergence of this series is \geq the radius of D . Consider the series

$$F(z) = a_0(z - z_0) + a_1 \frac{(z - z_0)^2}{2} + a_2 \frac{(z - z_0)^3}{3} + a_3 \frac{(z - z_0)^4}{4} + \dots$$

with the radius of convergence \geq the radius of D . Term by term differentiation gives

$$F'(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots = f(z).$$

It follows from Fact 9 that $\int_C f(z) dz = \int_C F'(z) dz = 0$ for any closed curve C .

11. If $f(z)$ is an analytic function and if z_0 is any point in the domain \mathcal{U} of $f(z)$, then the function

$$\frac{f(z) - f(z_0)}{z - z_0}$$

is analytic on \mathcal{U} as well.

Let $g(z) = (f(z) - f(z_0))/(z - z_0)$. We can regard $g(z)$ as the product of $f(z) - f(z_0)$ and $1/(z - z_0)$. Here $f(z) - f(z_0)$ is analytic on \mathcal{U} and $1/(z - z_0)$ is analytic everywhere except at z_0 . So $g(z)$ is analytic everywhere in \mathcal{U} except at z_0 . Our task is to show that $g(z)$ is analytic at z_0 as well. Let

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots$$

be the power series expansion of $f(z)$ at z_0 . Then, letting $z = z_0$, we have $f(z_0) = a_0$. Thus

$$\begin{aligned} f(z) - f(z_0) &= f(z) - a_0 = a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \dots \\ &= (z - z_0)(a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots) \end{aligned}$$

and hence

$$\frac{f(z) - f(z_0)}{z - z_0} = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

showing that $g(z)$ is analytic at z_0 .

Now let us take any analytic function defined on a disk D and let z_0 be any point in the interior of D , not necessarily the center of D . Let C be any closed curve in D . Assume that z_0 is not a point on C . According to Fact 11, the function $(f(z) - f(z_0))/(z - z_0)$ on D is analytic. Hence, according to Fact 10, we have

$$\int_C \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

We can rearrange terms to rewrite the above identity as

$$\int_C \frac{f(z_0)}{z - z_0} dz = \int_C \frac{f(z)}{z - z_0} dz.$$

Since $f(z_0)$ is a constant, we have

$$f(z_0) \int_C \frac{dz}{z - z_0} = \int_C \frac{f(z)}{z - z_0} dz. \quad (11.7)$$

It turns out that, as we will see in the future, the following expression

$$W(C; z_0) = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0} \quad (11.8)$$

is always an integer, called the **winding number** of C around z_0 , (or the index of C at z_0). Dividing both sides of (11.7) by $2\pi i$ and using (11.8), we arrive at

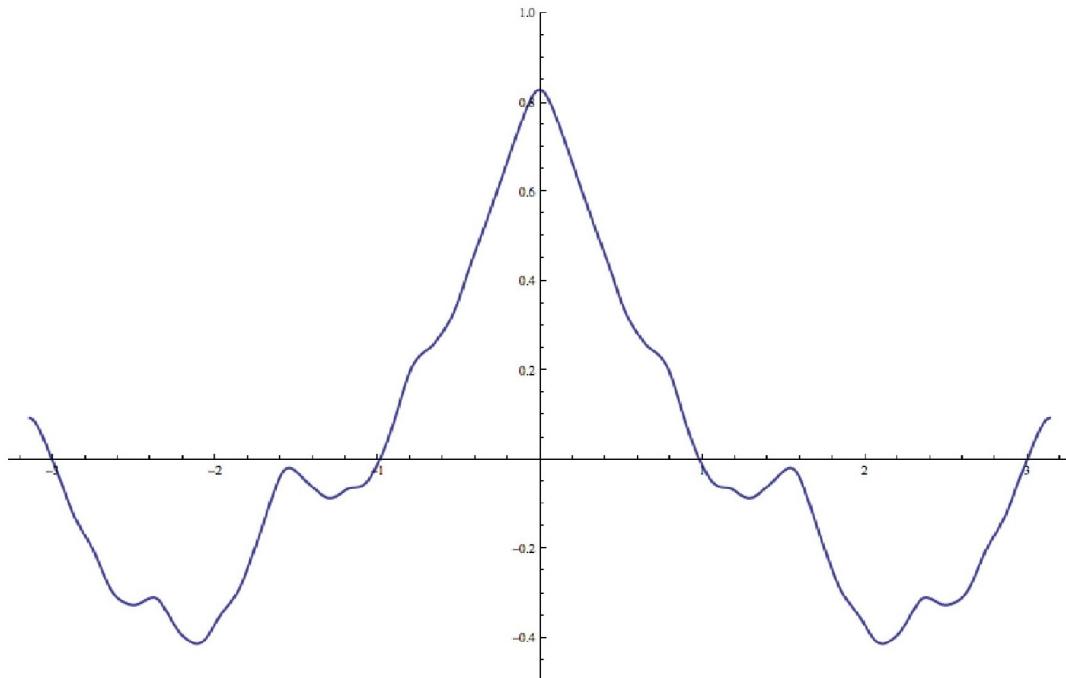
$$W(C; z_0)f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

We conclude

12 (Cauchy's formula, the general case). *If $f(z)$ is an analytic function on a disk D , z_0 is a point in the interior of D , C is a closed curve not passing through z_0 , then*

$$W(C; z_0)f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

where $W(C; z_0)$ is the winding number of C around z_0 .



(Image Source : Wikipedia)

Cauchy-Riemann Equations : Cauchy-Riemann Equations are used to determine whether a complex function is analytic or not.

They provide a criterion (a test) for the analyticity of a complex function
 $W = f(z) = u(x,y) + iv(x,y)$.

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two Cauchy-Riemann equations.

$$U_x = V_y, \quad U_y = -V_x$$

everywhere in D ; here and (and similarly for v) are the usual notations for partial derivatives.

If we use the polar form and set $f(z) = u(r,0) + iv(r,0)$, then the Cauchy-Riemann equations are,

$$U_r = 1/r V_\theta, \quad V_r = -1/r U_\theta$$

