9.2.3

EE24BTECH11050 - Pothuri Rahul

Question:
What is the solution for the differential equation $y' + \sin x = 0$ **Solution:**
Theoretical solution :
by rearranging the given differential equation

$$\frac{dy}{dx} = -\sin x \tag{0.1}$$

On integrating both sides w.r.to x,

$$\int \frac{dy}{dx} dx = \int (-\sin x)\, dx \tag{0.2}$$

$$y = \cos x + C \tag{0.3}$$

Solution using Trapezoidal rule :
Consider the equation (0.1),

$$\frac{dy}{dx} = -\sin x \tag{0.4}$$

$$dy = -\sin x dx \tag{0.5}$$

To apply the Trapezoidal rule, we need to convert it into definite integration. For that let us take two points on x-axis x_n and x_{n+1},which are at a small separation h and y_n, y_{n+1} be the values of respectively.

$$\int_{y_n}^{y_{n+1}} dy = -\int_{x_n}^{x_{n+1}} \sin x dx \tag{0.6}$$

By Trapezoidal rule,We can approximate it as,

$$y_{n+1} - y_n = -\frac{1}{2} \times h \left(\sin x_n + \sin x_{n+1}\right) \tag{0.7}$$

$$y_{n+1} = y_n - \frac{1}{2} \times h \left(\sin x_n + \sin x_{n+1}\right) \tag{0.8}$$

Where,

$$h = x_{n+1} - x_n \tag{0.9}$$

By taking initial conditions as $x_0 = 0, y_0 = 1$ and plotting the points resulted in this algorithm will give the approximate graph for the given differential equation (0.1)

Solution using Bilinear :
Apply laplace tranform for (0.1)

$$\frac{dy}{dx} = -\sin x \tag{0.10}$$

$$y' = -\sin x \tag{0.11}$$

$$\mathcal{L}(y') = \mathcal{L}(-\sin x) \tag{0.12}$$

Let

$$\mathcal{L}(y) = Y(S) \tag{0.13}$$

$$\tag{0.14}$$

Then

$$\mathcal{L}(y') = sY(s) - y_0 \tag{0.15}$$

From (0.12)

$$sY(s) - y_0 = -\frac{1}{s^2+1} \tag{0.16}$$

$$Y(s) = \frac{1}{s}\left(y_0 - \frac{1}{s^2+1}\right) \tag{0.17}$$

By taking initial conditions, y = 1 in (0.19)

$$Y(s) = \frac{1}{s} - \frac{1}{s\left(s^2+1\right)} \tag{0.18}$$

$$Y(s) = \frac{s}{s^2+1} \tag{0.19}$$

let us convert the equation (0.19) to Z-domain from s-domain by using the bilinear tranform technique.
For that we have to substitute

$$s = \frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}} \tag{0.20}$$

Then,

$$Y(z) = \frac{\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}}{\left(\frac{2}{h}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1} \tag{0.21}$$

On simplifying,We get

$$Y(z) = \frac{2h\left(z^2-1\right)}{z^2\left(4+h^2\right) + 2z\left(h^2-4\right) + \left(4+h^2\right)} \tag{0.22}$$

We need to find the inverse of this equation to get a difference equation,For that let us use the following properties of the z-transform

$$\mathcal{Z}\left(y\left[n+2\right]\right) = z^2 Y\left(z\right) - y\left[1\right] z - y\left[0\right] \tag{0.23}$$

$$\mathcal{Z}\left(y\left[n+1\right]\right) = zY\left(z\right) - zy\left[0\right] \tag{0.24}$$

$$\mathcal{Z}\left(\delta\left[n\right]\right) = 1,\ z \neq 0 \tag{0.25}$$

$$\tag{0.26}$$

By the time shift property

$$\mathcal{Z}\left(\delta\left[n+2\right]\right) = z^2,\ z \neq 0 \tag{0.27}$$

$$\mathcal{Z}\left(\delta\left[n+1\right]\right) = z,\ z \neq 0 \tag{0.28}$$

By rearranging the terms in equation (0.22)

$$z^2\left(4+h^2\right)Y(z) + 2z\left(h^2-4\right)Y(z) + \left(4+h^2\right)Y(z) = 2h\left(z^2-1\right) \tag{0.29}$$

$$z^2 Y(z) + 2z\frac{h^2-4}{h^2+4}Y(z) + Y(z) = \frac{2h\left(z^2-1\right)}{4+h^2} \tag{0.30}$$

Let us adjust the following equation (0.30) to get into standard forms,

$$z^2 Y(z) - y_1 z - y_0 + 2z\frac{h^2-4}{h^2+4}Y(z) - 2z\frac{h^2-4}{h^2+4}y_0 + Y(z) = \frac{2h\left(z^2-1\right)}{4+h^2} - y_1 z - y_0 - 2z\frac{h^2-4}{h^2+4}y_0 \tag{0.31}$$

$$\left(z^2 Y(z) - y_1 z - y_0\right) + 2\frac{h^2-4}{h^2+4}\left(zY(z) - zy_0\right) + Y(z) = z^2\left(\frac{2h}{h^2+4}\right) - z\left(y_1 + 2y_0\frac{h^2-4}{h^+4}\right) - \left(\frac{2h}{h^2+4} + y\right. \tag{0.32}$$

where $z \neq 0$

Region of convergence (ROC) is given by $z \neq 0$

by using properties,Inversing of the above equation (0.32) will result as the following ,

$$y_{n+2} + \frac{2\left(h^2-4\right)}{h^2+4}y_{n+1} + y_n = \frac{2h}{h^2+4}\delta\left[n+2\right] - \left(y_1 + 2y_0\frac{h^2-4}{h^+4}\right)\delta\left[n+1\right] - \left(\frac{2h}{h^2+4} + y_0\right)\delta\left[n\right] \tag{0.33}$$

Where δ is defined as following,

$$\delta\left[n-n_0\right] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases} \tag{0.34}$$

As $n > 0$,

$$\delta\left[n+1\right] = 0 \tag{0.35}$$

$$\delta\left[n+2\right] = 0 \tag{0.36}$$

Then (0.33) becomes as

$$y_{n+2} + \frac{2\left(h^2 - 4\right)}{h^2 + 4} y_{n+1} + y_n = -\left(\frac{2h}{h^2 + 4} + y_0\right) \delta[n] \tag{0.37}$$

To get y_1, Let us use the defination of the differentiation,

$$y' \approx \frac{y_{n+1} - y_n}{h} \tag{0.38}$$

For n = 0

$$y' = \frac{y_1 - y_0}{h} \tag{0.39}$$

$$y_1 = y_0 + hy' \tag{0.40}$$

We need to plot all these resulting points, to get the graph of thr solution of differential equation (0.1)

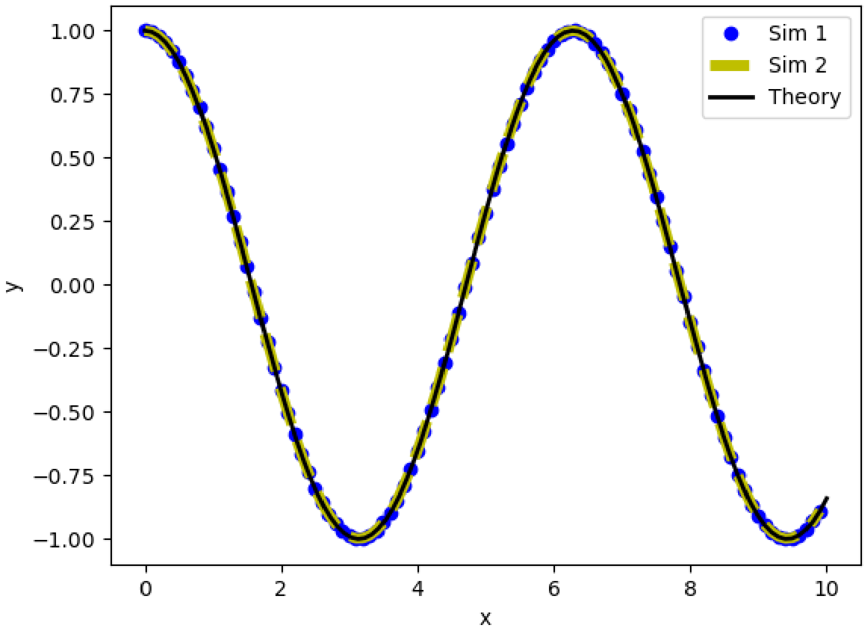

Fig. 0.1: Plot