9-9.5-5

EE24BTECH11050 - Pothuri Rahul

Question:
Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
Solution:

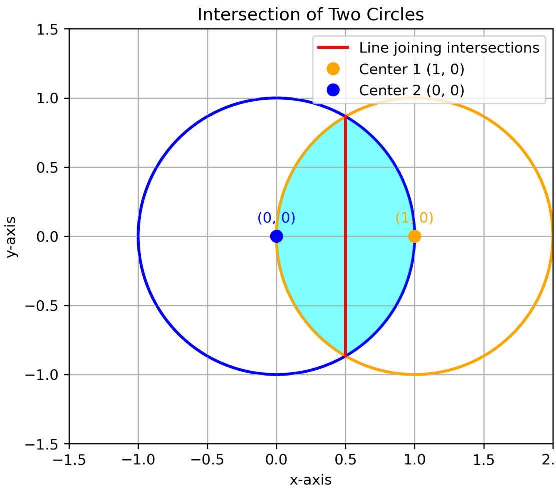

Fig. 0.1: Intersection of the circles.

The intersection of two conics with parameters V_i , u_i , f_i , $i = 1, 2$ is defined as

$$x^T (V_1 + \mu V_2) x + 2 (u_1 + \mu u_2)^T x + (f_1 + \mu f_2) = 0 \tag{0.1}$$

(0.1) represents a pair of stright lines if

$$\begin{vmatrix} V_1 + \mu V_2 & u_1 + \mu u_2 \\ (u_1 + \mu u_2)^T & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{0.2}$$

The equation of line is

$$x = h + \kappa m \tag{0.3}$$

The conic parameters for the two circles can be expressed as

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ u_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ f_1 = 0, \tag{0.4}$$

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f_2 = -1. \tag{0.5}$$

On substituting from (0.4) and (0.5) in (0.2), we obtain

$$\begin{vmatrix} 1+\mu & 0 & -1 \\ 0 & 1+\mu & 0 \\ -1 & 0 & -\mu \end{vmatrix} = 0 \tag{0.6}$$

$$(1+\mu)(1+\mu)(-\mu) + (1+\mu) = 0 \tag{0.7}$$

yielding

$$\mu = -1. \tag{0.8}$$

Substituting (0.4) and (0.5) in (0.1), we obtain

$$x^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + 2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \tag{0.9}$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \tag{0.10}$$

$$\Rightarrow (-2)(x-1) = 1 \quad \Rightarrow x = \frac{1}{2}. \tag{0.11}$$

Therefore the intersection of the two circles is a line with parameters

$$m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ h = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \tag{0.12}$$

The intersection parameters is obtained by

$$\kappa_i = \frac{1}{m^\top V m} \left(-m^\top (Vh + u) \pm \sqrt{\left[m^\top (Vh+u)\right]^2 - g(h)\left(m^\top V m\right)} \right) \tag{0.13}$$

Yielding

$$\kappa_i = \pm \frac{\sqrt{3}}{2}. \tag{0.14}$$

Hence the points of intersection are obtained from (0.2) as

$$x_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad x_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right). \tag{0.15}$$

The desired area of the region is given as

$$2\left(\int_0^{1/2} \sqrt{1-(x-1)^2} dx + \int_0^{1/2} \sqrt{1-x^2} dx \right) \tag{0.16}$$

$$= 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1) \right]_0^{1/2} + 2\left[\frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) \right]_0^{1/2} \tag{0.17}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \tag{0.18}$$

parameter	description	value
x_1	first intersection point	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
x_2	second intersection point	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
V_1	$\begin{pmatrix} 1-e_1^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
V_2	$\begin{pmatrix} 1-e_2^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
c_1	centre of first circle	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
u_1	-c_1	$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$
c_2	centre of second circle	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
u_2	-c_2	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
r_1	radius of first circle	1
f_1	$\|u_1\|^2 - r_1^2$	0
r_2	radius of second circle	1
f_2	$\|u_2\|^2 - r_2^2$	-1
m	Slope of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	Intercept of the line	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$
k_1	Parameter of first line	$\frac{\sqrt{3}}{2}$
k_2	Parameter of second line	- $\frac{\sqrt{3}}{2}$

TABLE 0: Parameters used