

9-9.5-5

EE24BTECH11050 - Pothuri Rahul

Question:

Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.

Solution: The intersection of two conics with parameters V_i , u_i , f_i , $i = 1, 2$ is defined as

$$x^T (V_1 + \mu V_2) x + 2(u_1 + \mu u_2)^T x + (f_1 + \mu f_2) = 0 \quad (0.1)$$

(0.1) represents a pair of straight lines if

$$\begin{vmatrix} V_1 + \mu V_2 & u_1 + \mu u_2 \\ (u_1 + \mu u_2)^T & f_1 + \mu f_2 \end{vmatrix} = 0 \quad (0.2)$$

The equation of line is

$$x = h + km \quad (0.3)$$

The conic parameters for the two circles can be expressed as

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, f_1 = 0, \quad (0.4)$$

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_2 = -1. \quad (0.5)$$

On substituting from (0.4) and (0.5) in (0.2), we obtain

$$\begin{vmatrix} 1 + \mu & 0 & -1 \\ 0 & 1 + \mu & 0 \\ -1 & 0 & -\mu \end{vmatrix} = 0 \quad (0.6)$$

$$(1 + \mu)(1 + \mu)(-\mu) + (1 + \mu) = 0 \quad (0.7)$$

yielding

$$\mu = -1. \quad (0.8)$$

Substituting (0.4) and (0.5) in (0.1), we obtain

$$x^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + 2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \quad (0.9)$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \quad (0.10)$$

$$\Rightarrow (-2)(x-1) = 1 \Rightarrow x = \frac{1}{2}. \quad (0.11)$$

Therefore the intersection of the two circles is a line with parameters

$$m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \quad (0.12)$$

The intersection parameters of the chord with the first circle is obtained by substituting line equation in the conic equation.

$$\kappa_i = \pm \frac{\sqrt{3}}{2}. \quad (0.13)$$

Hence the points of intersection are obtained from (0.2) as

$$x_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad x_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right). \quad (0.14)$$

The desired area of the region is given as

$$2 \left(\int_0^{1/2} \sqrt{1 - (x-1)^2} dx + \int_0^{1/2} \sqrt{1 - x^2} dx \right) \quad (0.15)$$

$$= 2 \left[\frac{1}{2} (x-1) \sqrt{1 - (x-1)^2} + \frac{1}{2} \sin^{-1}(x-1) \right]_0^{1/2} + 2 \left[\frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1}(x) \right]_0^{1/2} \quad (0.16)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \quad (0.17)$$

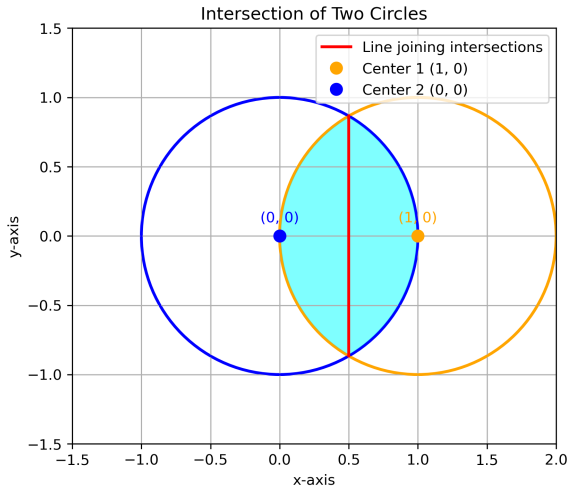


Fig. 0.1: Intersection of the circles.

parameter	discription	value
x_1	first intersection point	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
x_2	second intersection point	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
V_1	$\begin{pmatrix} 1 - e_1^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
V_2	$\begin{pmatrix} 1 - e_2^2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
u_1	-(centre of first circle)	$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$
u_2	-(centre of second circle)	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
f_1	$\ u_1\ ^2 - (\text{radius of first circle})^2$	0
f_2	$\ u_2\ ^2 - (\text{radius of second circle})^2$	-1
m	Slope of the line	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	Intercept of the line	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$

TABLE 0: Parameters used