9-9.5-5

EE24BTECH11050 - Pothuri Rahul

Question:
Find the area bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$.
Solution:

variable	discription
x_1	first intersection point
x_2	second intersection point
V, u, f	Parameters of conic
h, m	Parameters of line

TABLE 0: Variables Used

The intersection of two conics with parameters V_i , u_i , f_i , $i = 1, 2$ is defined as

$$x^T (V_1 + \mu V_2) x + 2 (u_1 + \mu u_2)^T x + (f_1 + \mu f_2) = 0 \tag{0.1}$$

The equation of line is

$$x = h + \kappa m \tag{0.2}$$

The conic parameters for the two circles can be expressed as

$$V_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ u_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ f_1 = 0, \tag{0.3}$$

$$V_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ f_2 = -1. \tag{0.4}$$

On substituting from (0.3) in (9.1.4.1), we obtain

$$\begin{pmatrix} 1 + \mu & 0 & -1 \\ 0 & 1 + \mu & 0 \\ -1 & 0 & -\mu \end{pmatrix} = 0 \tag{0.5}$$

yielding

$$\mu = -1. \tag{0.6}$$

Substituting (0.3) in (9.1.3.1), we obtain

$$x^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + 2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}^T \begin{pmatrix} x \\ y \end{pmatrix} + 1 = 0 \tag{0.7}$$

$$\Rightarrow (-2)(x - 1) = 1 \quad \Rightarrow x = \frac{1}{2}. \tag{0.8}$$

Therefore the intersection of the two circles is a line with parameters

$$m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \tag{0.9}$$

The intersection parameters of the chord with the first circle in is obtained by substituting line equation in the conic equation.

$$\kappa_i = \pm \frac{\sqrt{3}}{2}. \tag{0.10}$$

Hence the points of intersection are obtained from (0.2) as

$$x_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad x_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2} \right). \tag{0.11}$$

The desired area of the region is given as

$$2 \left(\int_0^{1/2} \sqrt{1 - (x - 1)^2} dx + \int_0^{1/2} \sqrt{1 - x^2} dx \right) \tag{0.12}$$

$$= 2 \left[\frac{1}{2} (x - 1) \sqrt{1 - (x - 1)^2} + \frac{1}{2} \sin^{-1}(x - 1) \right]_0^{1/2} + 2 \left[\frac{1}{2} x \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1}(x) \right]_0^{1/2} \tag{0.13}$$
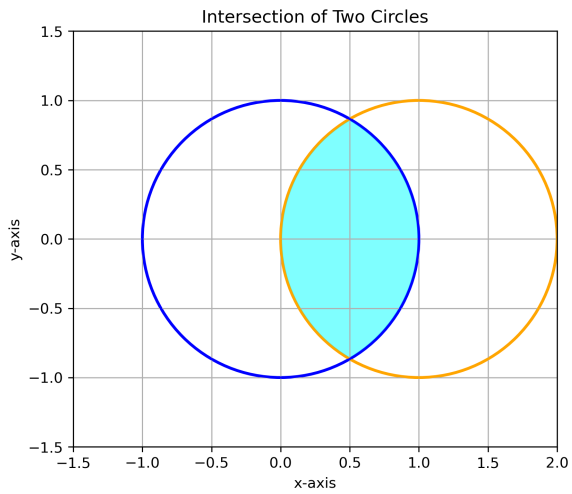
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \tag{0.14}$$

Fig. 0.1: Intersection of the circles.