PRE-CALCULUS

Or Year 1 Sem 1 Calculus

Before we start...

1. This slide does not cover all topics.

2. This slide might contain misinformation.

3. Most formulars are given in exam.

4. Does anyone have no knowledge about Calculus?

5. Does everyone have something to write on?

What is limits

An approximation* of value at some point where the limit is trying to approach.

Limits symbol

Limits sign --> lim f(x) *<-- Function*

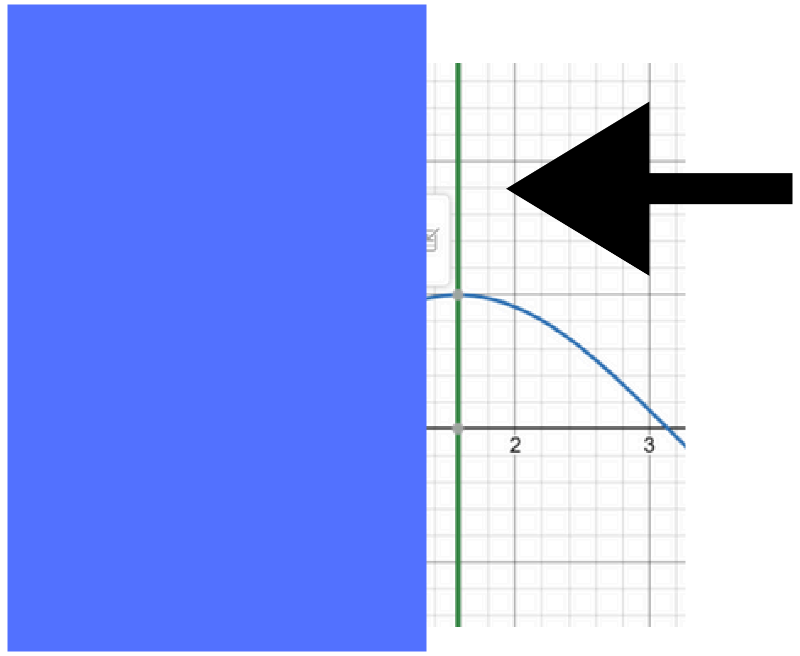
Points that we’re approaching --> X → 1

*Because some value that not exist when X is a certain point in function might have limits

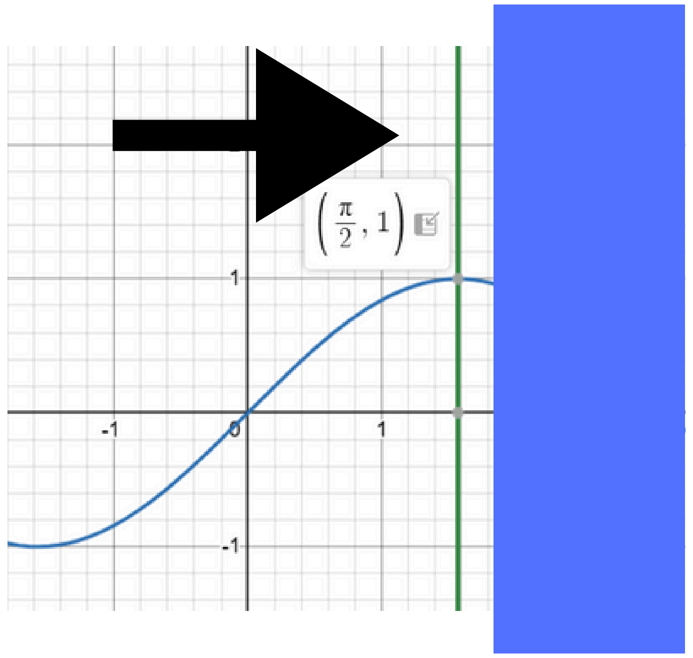
One-sided limits

Limits that approach from either left or right side of that point.

$\lim_{x \to \pi/2^{+}} f(x)$ Limits from the right

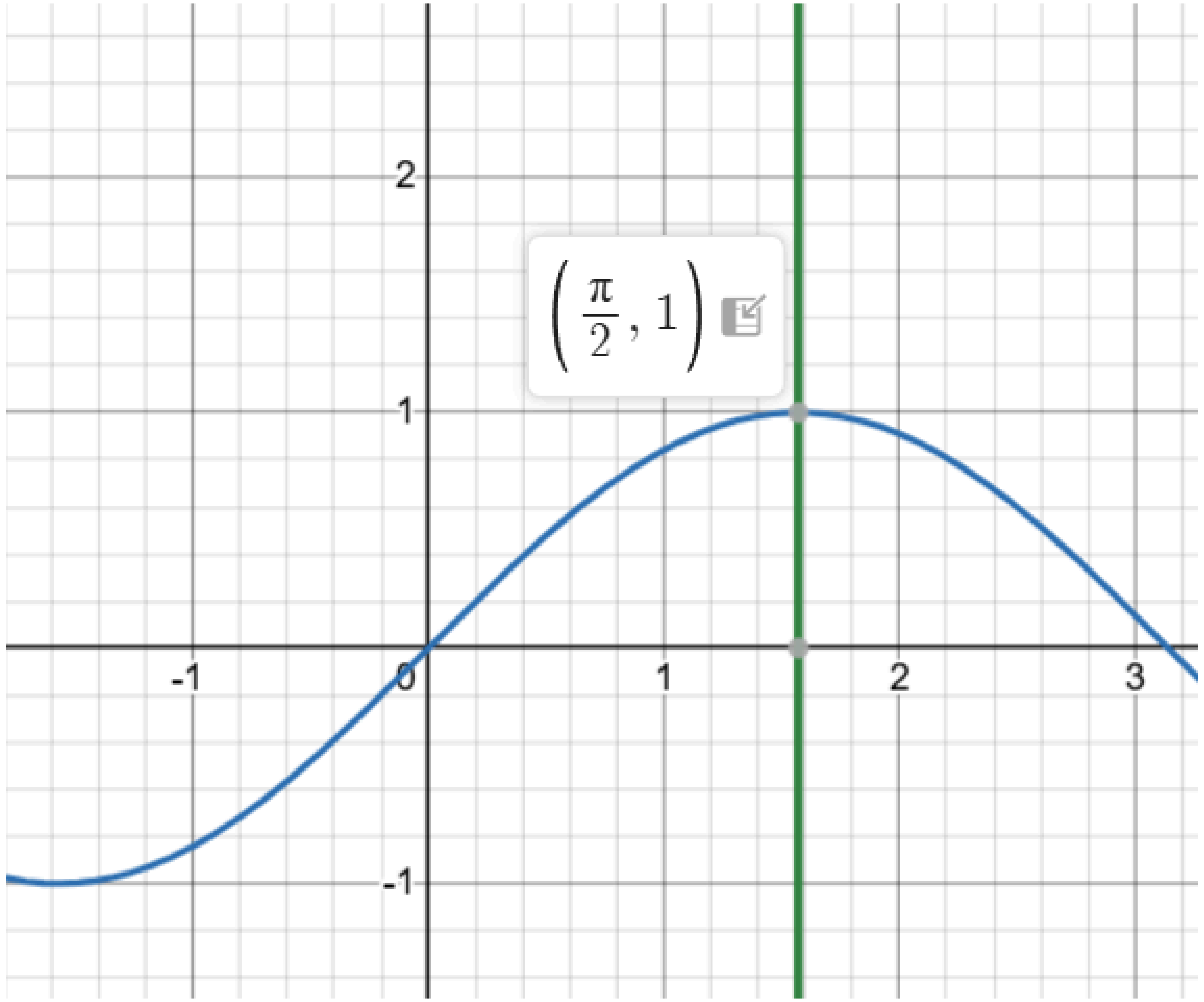


$\lim_{x \to \pi/2^{-}} f(x)$ Limits from the left



<u>When the 2 sides limits are different, the general limits DNE (does not exist).</u>

Example $f(x) = \sin(x)$



$\lim_{x \to \pi/2^+} f(x) \approx 1$ and $\lim_{x \to \pi/2^-} f(x) \approx 1$

$\therefore \lim_{x \to \pi/2} f(x) = 1$

#

How to find limits

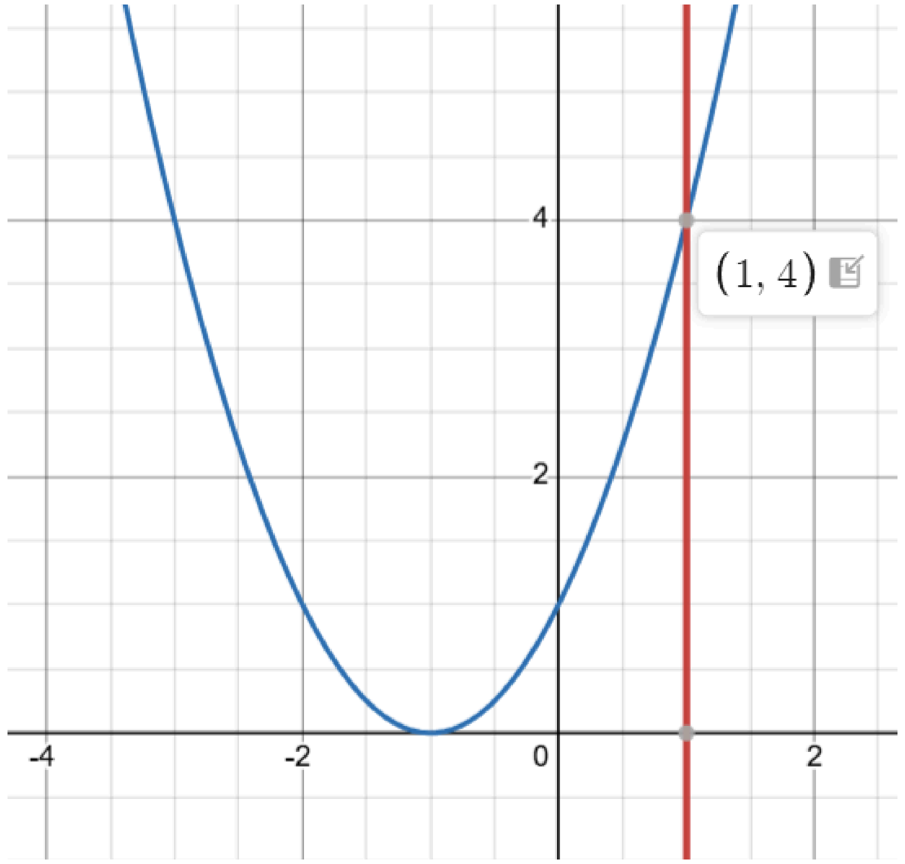
Other methods include:

1. *Direct substitution methods*
2. *Factoring methods*
3. *Rationalizing methods*
4. *L'Hôpital's rules (not learn)*

<u>Table methods</u> is the simplest methods of all. But also time consuming

How to use a table methods

$f(x) = x^2 + 2x + 1$ find $\lim_{x \to 1} f(x)$



1) Find right side limits

X	f(x)
0.5	2.25
0.9	3.61
0.99	3.9601
0.999	3.996
0.9999	3.9996

So $\lim_{x \to 1^+} f(x) \approx 4$

2) Find left side limits

X	f(x)
1.5	6.25
1.1	4.41
1.01	4.0401
1.001	4.004
1.0001	4.0004

So $\lim_{x \to 1^-} f(x) \approx 4$

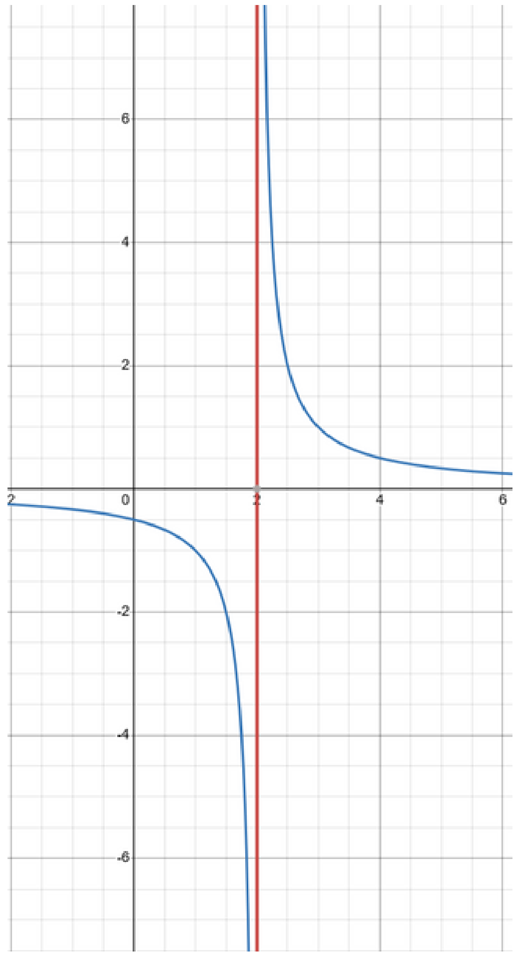
3) $\because \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) \quad \therefore \lim_{x \to 1} f(x) = 4$ #

Infinite limits & Limits Does Not Exist

A limits of a graphs that point upwards to some asymptote. When both of one-sided is not equal, that point have no limits, even if they have values; thats why it just an approximation, not exact value

Example

$$f(x) = \frac{x + 2}{x^2 - 4}$$



$$\lim_{x \to 2^-} f(x) \approx -\infty \qquad \lim_{x \to 2^+} f(x) \approx \infty$$

$$\therefore \lim_{x \to 2} f(x) = \textbf{DNE}$$

Because the left side limit is not equal to right side limit

Exercise 1

1) $\lim_{x \to 4^+} (x^2 + 3)$

2) $\lim_{x \to -1^-} \frac{2}{x+1}$

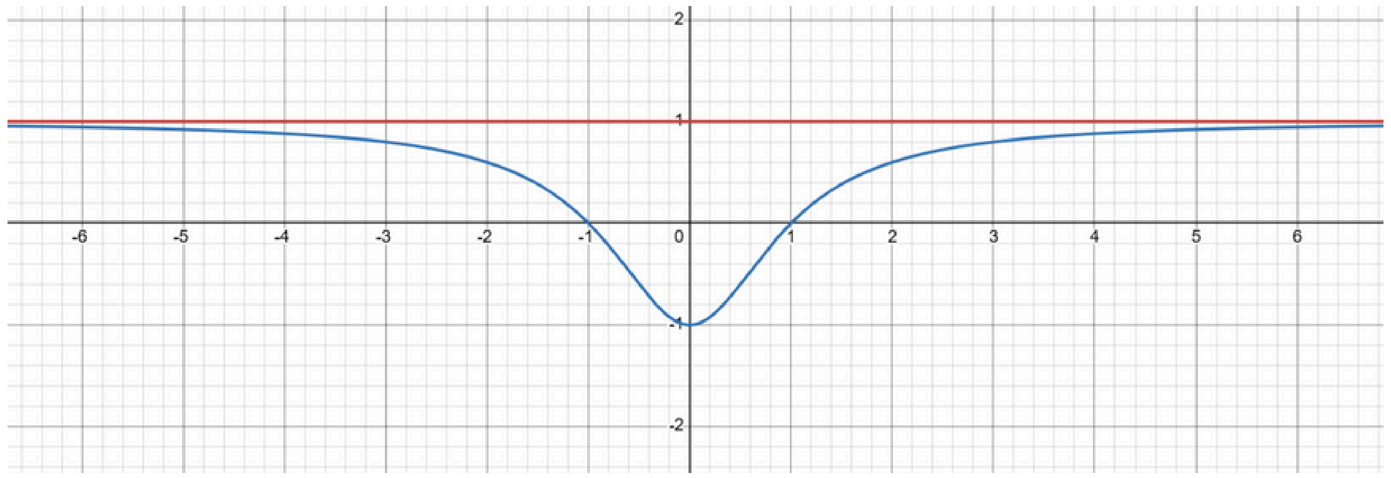
3) $\lim_{x \to \pi/2} \tan(x)$

1) 19, 2) -∞, 3) DNE

Limits at infinity

When the values we're trying to approach are $\pm\infty$

Ex. $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ find $\lim\limits_{x \to \pm\infty} f(x)$



X	f(x)
0	-1
1	0
2	0.6
5	0.923
10	0.9801
50	0.9992
100	0.9998
1000	0.9999

X	f(x)
0	-1
-1	0
-2	0.6
-5	0.923
-10	0.9801
-50	0.9992
-100	0.9998
-1000	0.9999

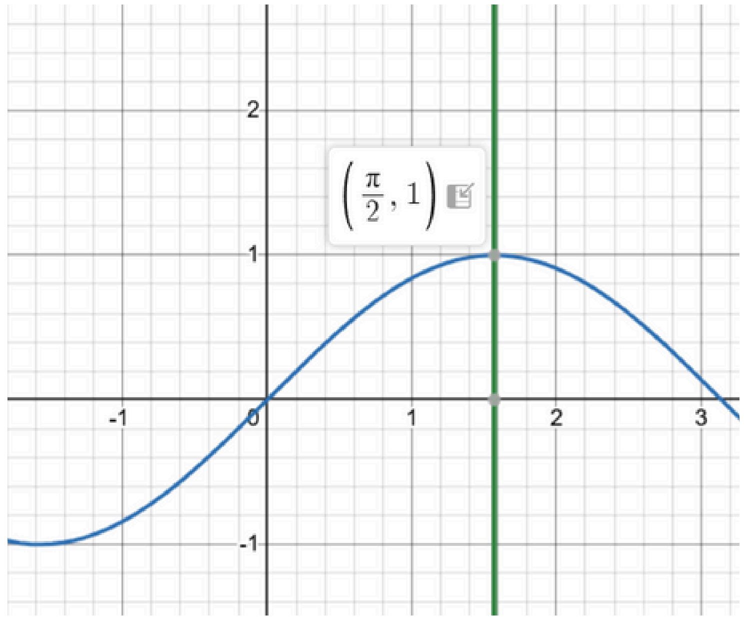
So $\lim\limits_{x \to \infty} f(x) = 1$

So $\lim\limits_{x \to -\infty} f(x) = 1$

Exercise 2

1) $\lim_{x \to -\infty} \sqrt{4 - x}$

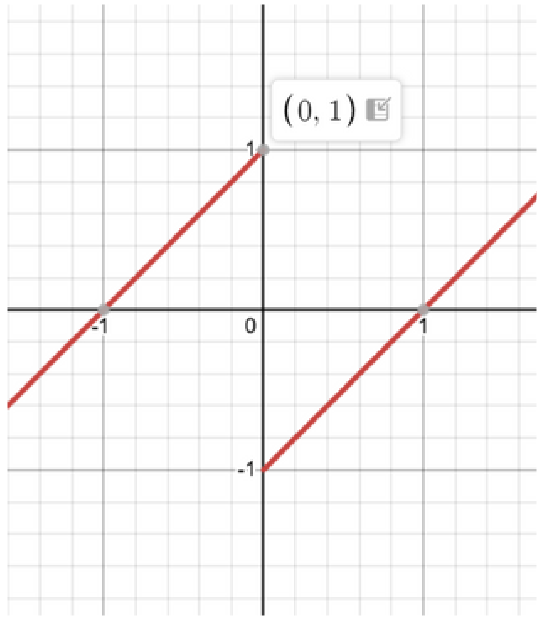
Limits with different graphs



<u>Normal-looking graph</u>

- Have limits

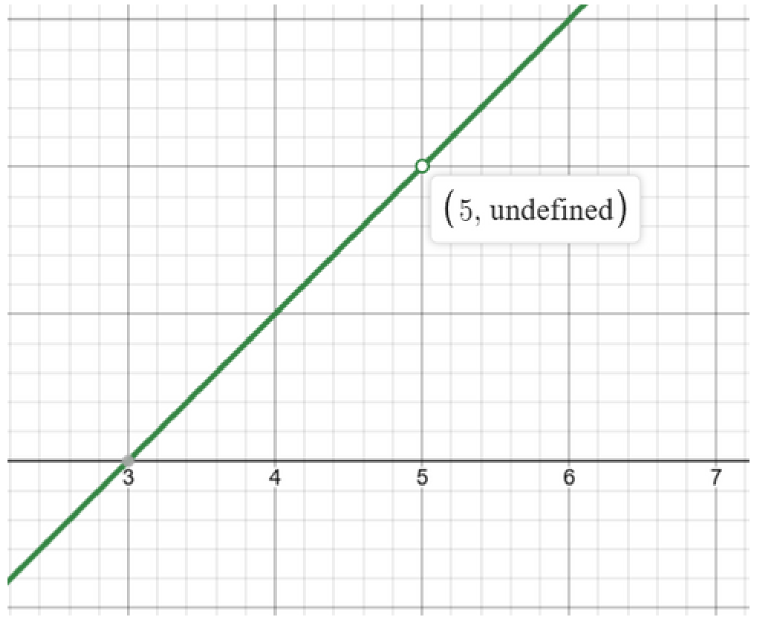
$$\sin(x)$$



<u>Jump discontinuity</u>

- Have one sided limits
- DNE general limits

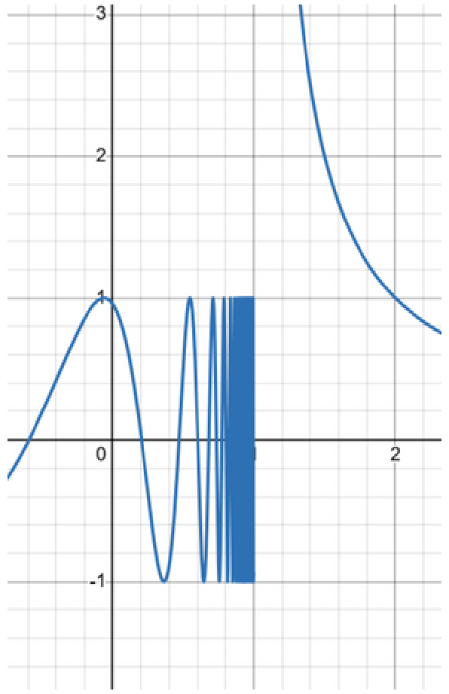
$$x \le 0 : x + 1$$
$$x > 0 : x - 1$$



<u>Removable discontinuity</u>

- Value not on graph
- Have limits

$$\frac{x^2 - 8x + 15}{x - 5}$$



<u>Essential discontinuity</u>

- One sided limits DNE in real number

$$f(x) = \begin{cases} \sin \frac{5}{x-1} & \text{for } x < 1 \\ 0 & \text{for } x = 1 \\ \frac{1}{x-1} & \text{for } x > 1. \end{cases}$$

Topic 2:

Derivatives

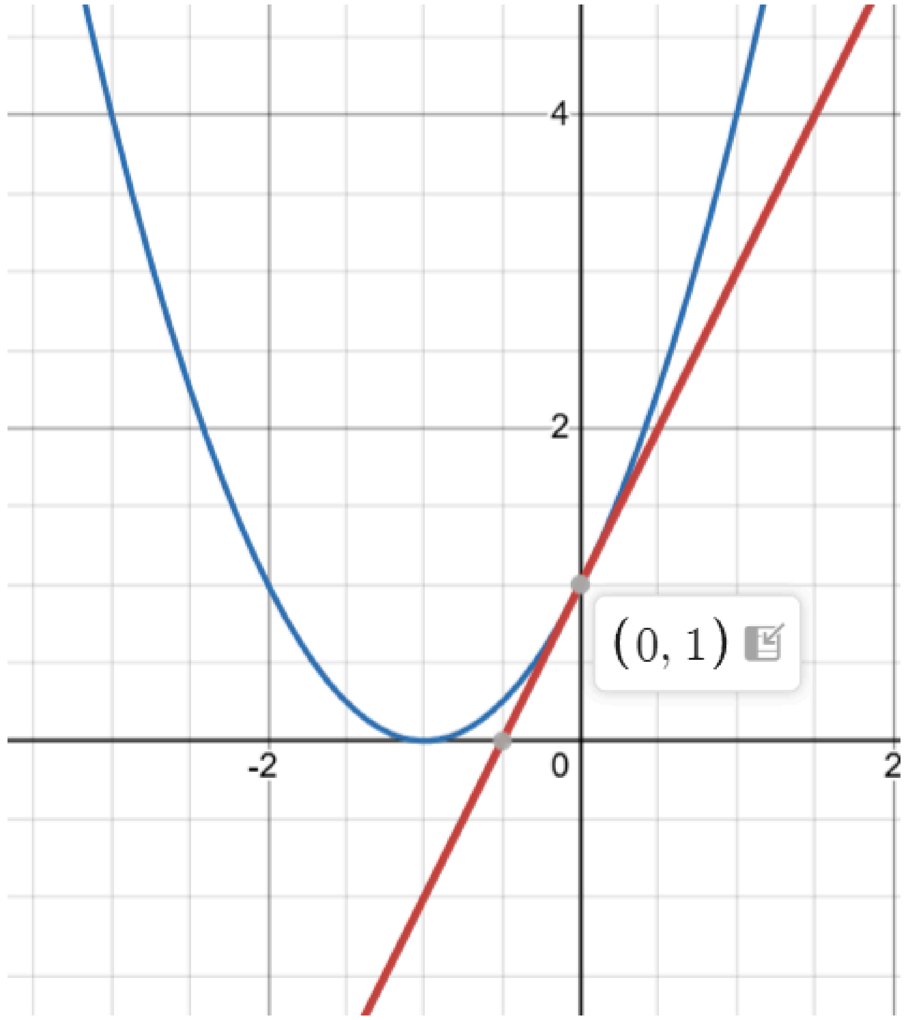
Tangent line

Line that touch the curve / have only 1 intersection point with the graph.
(Or slope of a certain point in the graph)

Example

$f(x) = x^2 + 2x + 1$

At x=0, the graph has a slope of 2



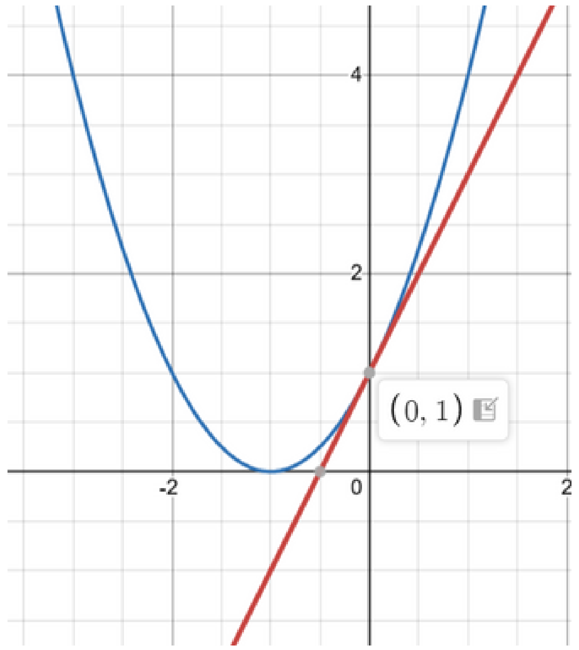
What is derivatives

f'(x) is the derivative of f(x) and is defined by: $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

Confuse? Derivatives are slope equation of a certain point in graph.

Example

$f(x) = x^2 + 2x + 1$



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + 2(x+h) + 1] - [x^2 + 2x + 1]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{2x} + 2h + \cancel{1} - \cancel{x^2} - \cancel{2x} - \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}} + 2\cancel{h}}{\cancel{h}} = \lim_{h \to 0} 2x + h + 2 = 2x + 2 \ _{\#}$$

Differentiation, but easy mode

When differentiate (act of finding derivatives), think as removing 1 unit of ‘x’ from the equation.

Everything does contain ‘x’

When x^n, we’ll make it nx^{n-1}.

Example $f(x) = x^2 + 2x + 1 \rightarrow x^2 + 2x^1 + 1x^0 \rightarrow 2x^1 + 2x^0 + 1x^0$

When x^0, we’ll remove that unit completely.

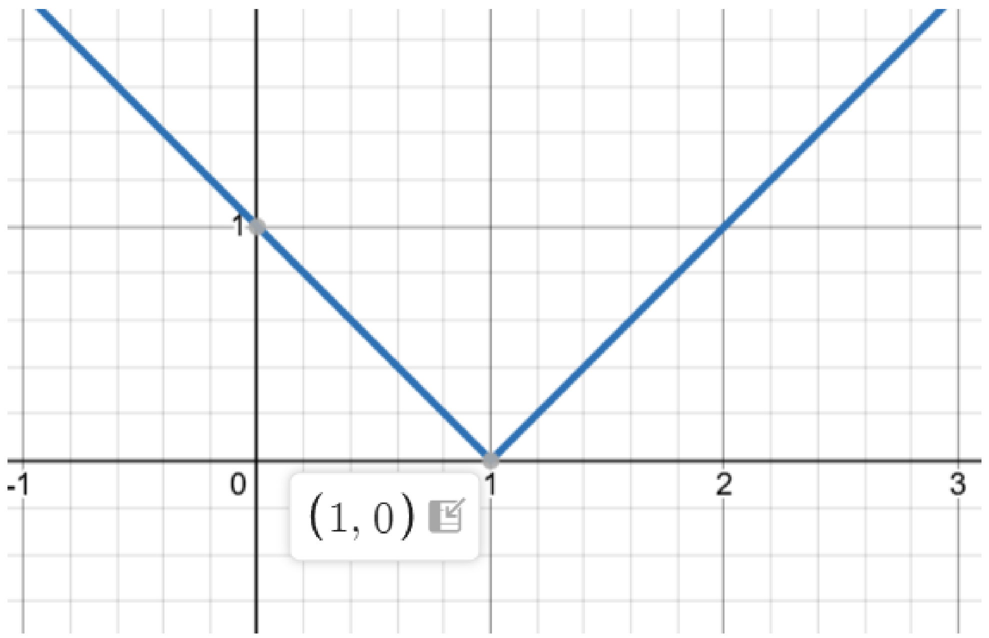
$f’(x) = 2x + 2$

$f(x) = x^2 + 2x + 1$

$\therefore f’(x) = 2x + 2$

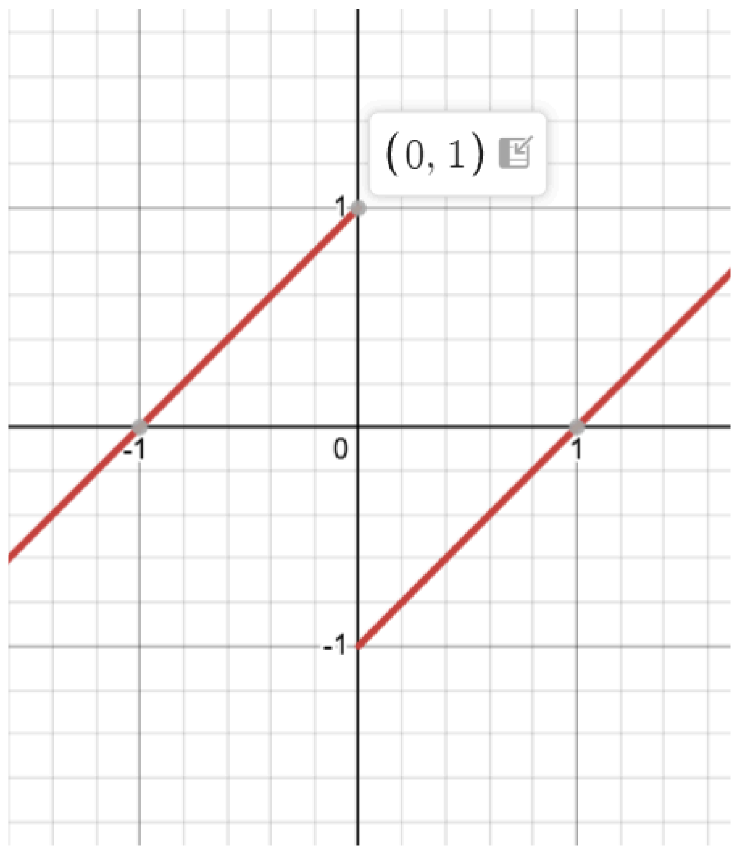
Indifferentiable point

Corner



| x + 1 |

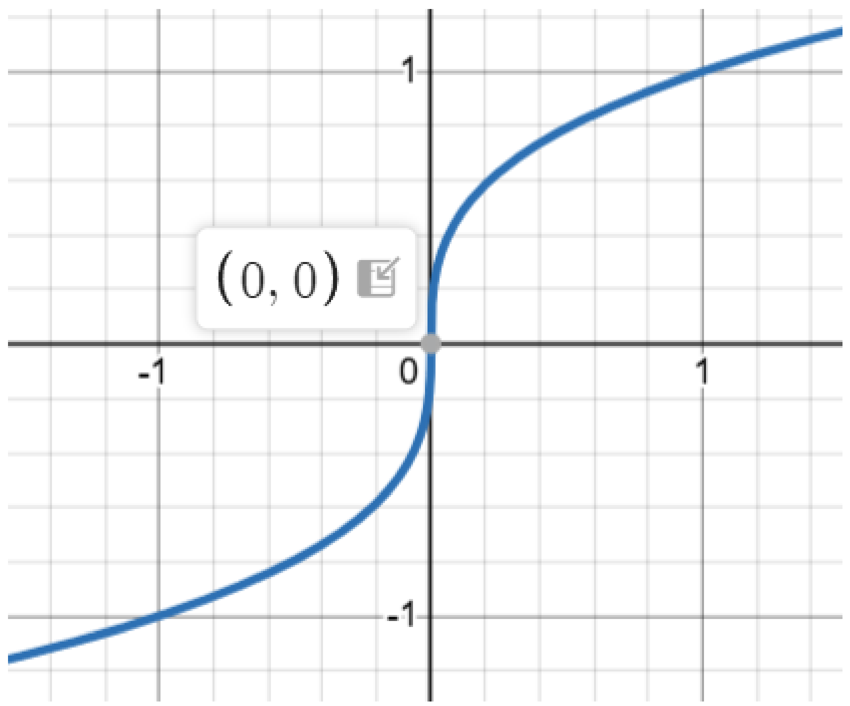
Jump discontinuity



$x \leq 0 : x + 1$

$x > 0 : x - 1$

Vertical tangent / slope



$\sqrt[3]{x}$

Differentiation rules (Given in exam, but not all)

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$ $e^v dv$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$
23. $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$
24. $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$
25. $\frac{d}{dx}[\sinh u] = (\cosh u)u'$
26. $\frac{d}{dx}[\cosh u] = (\sinh u)u'$
27. $\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u'$
28. $\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u'$
29. $\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u'$
30. $\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u'$
31. $\frac{d}{dx}[\sinh^{-1} u] = \frac{u'}{\sqrt{u^2+1}}$
32. $\frac{d}{dx}[\cosh^{-1} u] = \frac{u'}{\sqrt{u^2-1}}$
33. $\frac{d}{dx}[\tanh^{-1} u] = \frac{u'}{1-u^2}$
34. $\frac{d}{dx}[\coth^{-1} u] = \frac{u'}{1-u^2}$
35. $\frac{d}{dx}[\operatorname{sech}^{-1} u] = \frac{-u'}{u\sqrt{1-u^2}}$
36. $\frac{d}{dx}[\operatorname{csch}^{-1} u] = \frac{-u'}{|u|\sqrt{1+u^2}}$

Exercise 3

1) $g(x) = 5x^3$ find g'(x)

2) $If\ y = (x+2)(x+3)(x+4), find\ y'$

1) $15x^2$ 2) $3x^2 + 18x + 26$

Higher order differentiation

Fancy words for saying “Differentiate the equation more than 1 time”

Example

$f(x) = x^4 + 3x^3 + 5x^2 + 7x + 9$

$f'(x) = 4x^3 + 9x^2 + 10x + 7$ Differentiate 1 time.

$f''(x) = 12x^2 + 18x + 10$ Differentiate 2 times.

$f'''(x) = 24x + 18$ Differentiate 3 times.

$f^4(x) = 24$ Differentiate 4 times.

When diff 4 or more time, we write as power 4 instead.

Exercise 4

1) $f(x) = 6x^3 - 12x^2 + 4$ find $f''(x)$

2) $f(x) = 6x^{10} + 10x^6$ find $f^5(x)$

1) $36x - 24$ 2) $181{,}440x^5 + 7{,}200x$

Implicit differentiation

When any side of the equations contain more than 1 variables.

Example

$$x^3 + y^3 = 6xy$$

$$3x^2 + 3y^2y' = 6y + 6xy'$$

When dif respect to x, dif x normally, and change variable y to be y' (y prime).

$$3y^2y' - 6xy' = 6y - 3x^2$$

$$y'(3y^2 - 6x) = 6y - 3x^2$$

$$y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

#

y' is equivalent to f'(x) but easier to use in equation.

Higher order implicit differentiation

Just do the implecit dif more than 1 times, and substitute the y' into the newly dif equation to get the y"

Example

$$x^2 + 4y^2 = 4$$

$$2x + 8yy' = 0$$

$$8yy' = -2x$$

$$y' = \frac{-2x}{8y}$$

$$y'' = \frac{(-2)(8y) - (-2x)(8y')}{(8y)^2}$$

$$y'' = \frac{-16y + 16xy'}{64y^2}$$

$$y'' = \frac{-16y + 16x(\frac{-2x}{8y})}{64y^2} = \frac{-x^2 - 4y^2}{16y^3}$$

#

When differentiate again and found y', substitute and continue the calculation.

Topic 3:

Extrema

What is extrema

The values highest / lowest point in the graph compare to its surrounding.

Or: The valeus that are highest / lowest in that concave area.

Types of extrema

<u>Relative extrema</u> : Max/min point between concave in **expanding** graphs.
<u>Absolute extrema</u> : Max/min point between concave in **closed-interval** graphs.

Critical value

A value that shows either maximum or minimum extrema in the graphs.

Critical value can be found by setting the derived equation equals to zero.

Example

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

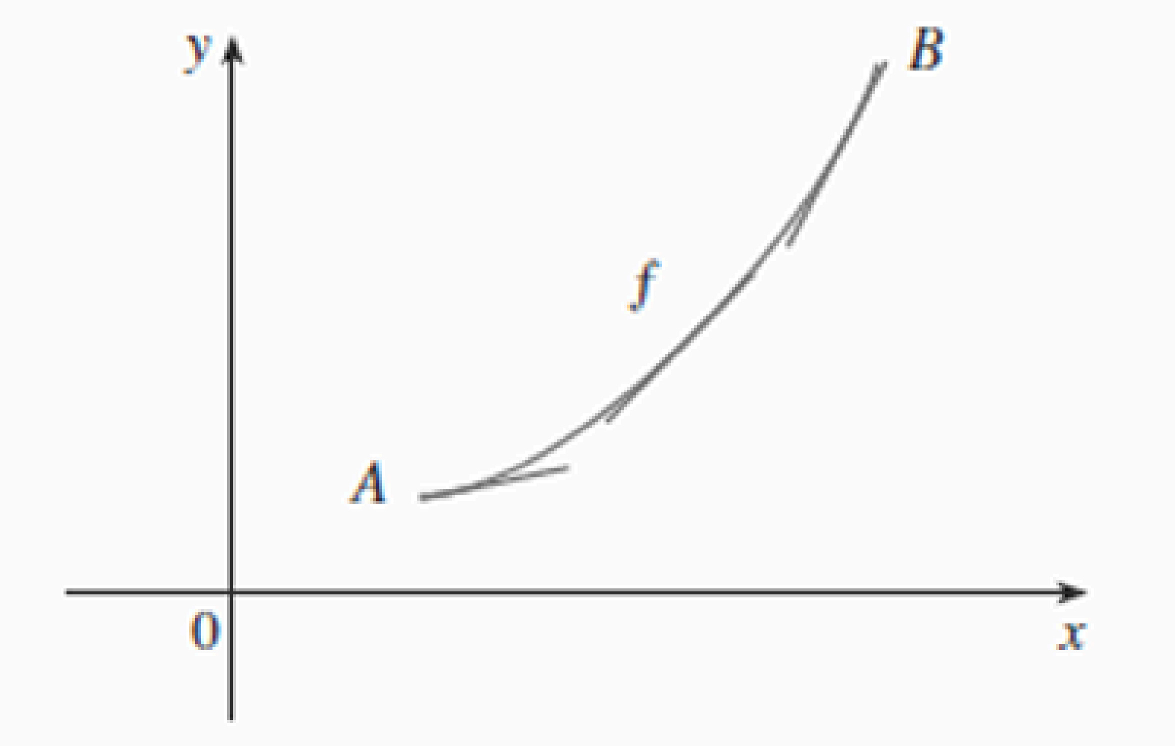
$f'(x) = 12x^3 - 12x^2 - 24x$

$0 = 12x^3 - 12x^2 - 24x$ ← Set the derived equation equals to zero

$x = -1, 0, 2$ ← Find x to get critical value

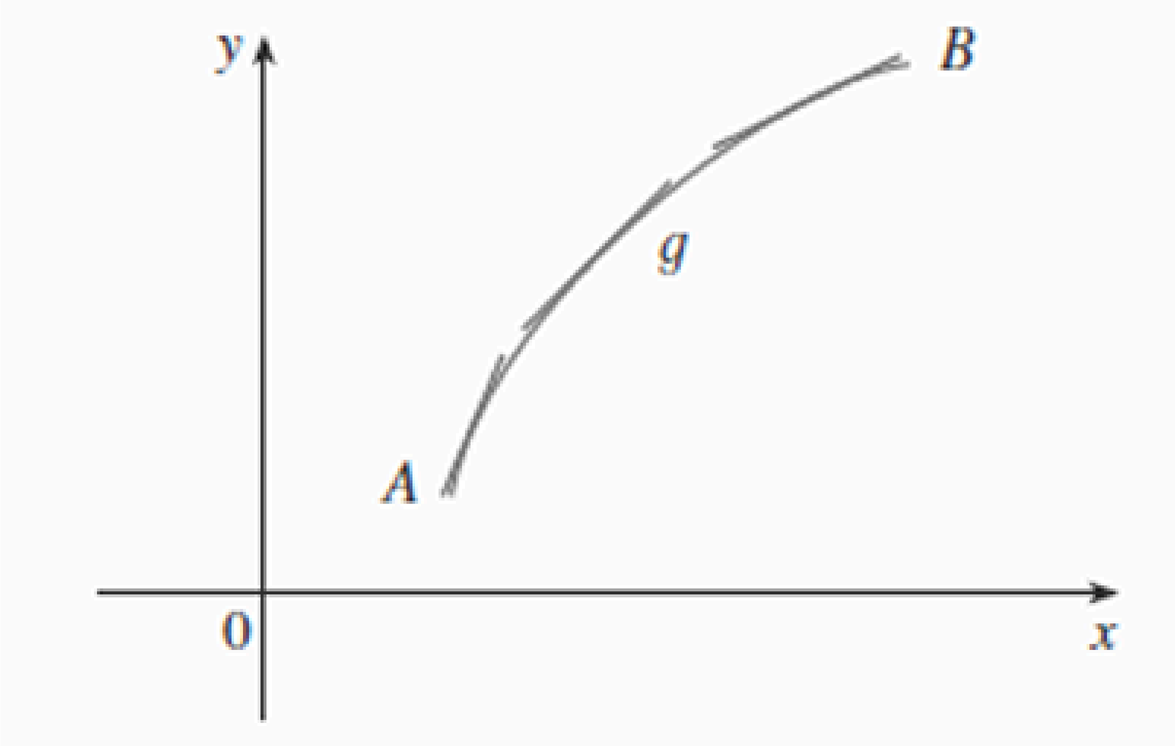
Concavity

When the slope value increase or decrease



(a) Concave upward

Slope value increase



(b) Concave downward

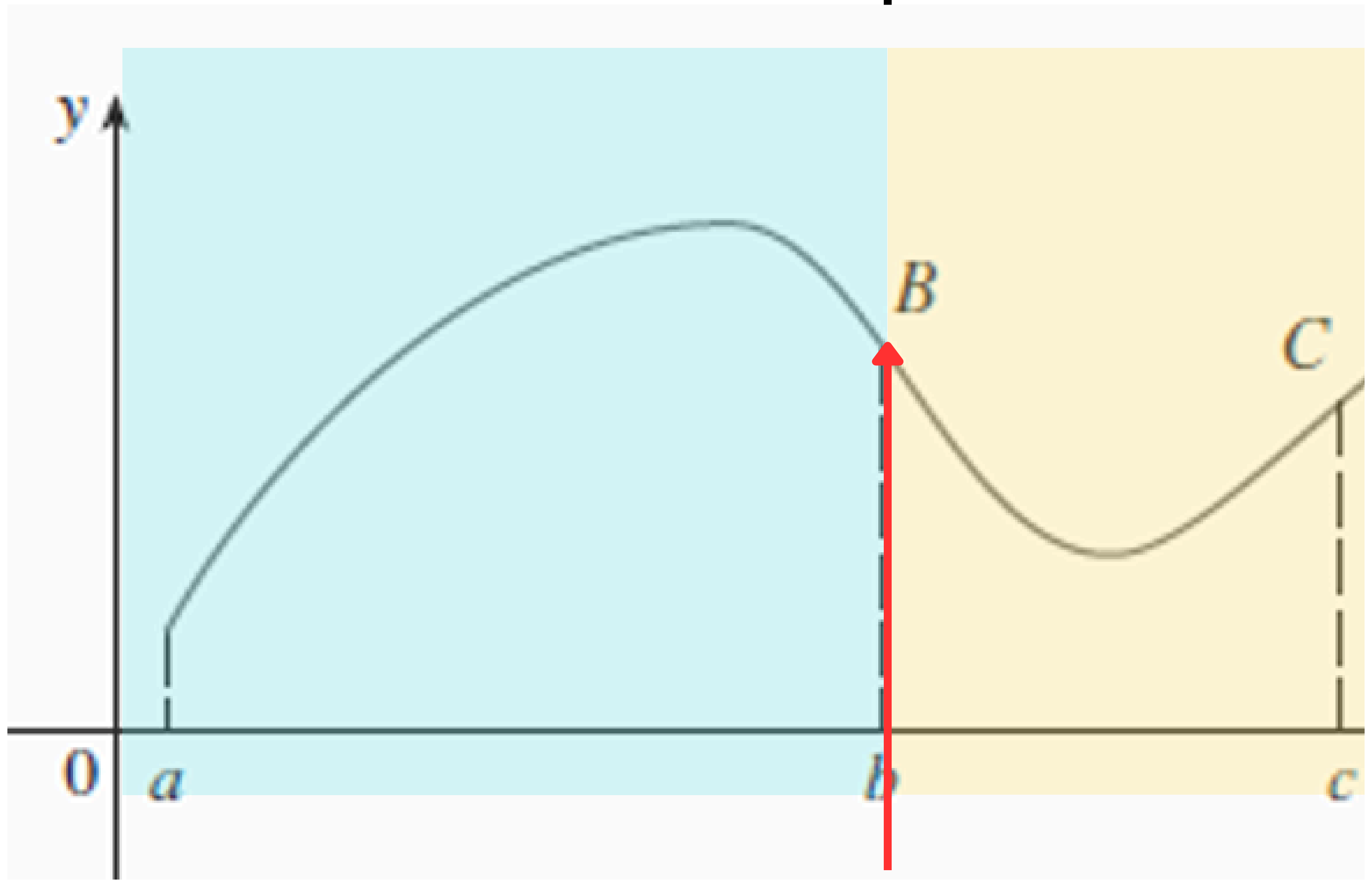
Slope value decrease

Infection point

A point that connects between 2 different concaves

Example

Concave down | Concave up



Infection point: in-between 2 concaves

How to find infection point

Find the second order derivative and set the equation equals to zero.

Example

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$0 = 4x^3 - 12x^2$ $x = 0, 3$ ⟵ Critical value

$f''(x) = 12x^2 - 24x$

$0 = 12x^2 - 24x$ $x = 0, 2$ ⟵ Infection point

Finding relative extrema

Steps to find relative extrema

1. Find a critical value.
2. Find the slope in the section between each critical value.
    a. If the critical value is between same slope direction, not extrema.
    b. If not, then do next step.
3. Substitute the critical value into the original equations to get y values.
4. Compare y values to see which one is maximum/minimum

Example

$y = 2x^3 - 9x^2 + 12x$

$y' = 6x^2 - 18x + 12$

$0 = 6x^2 - 18x + 12$

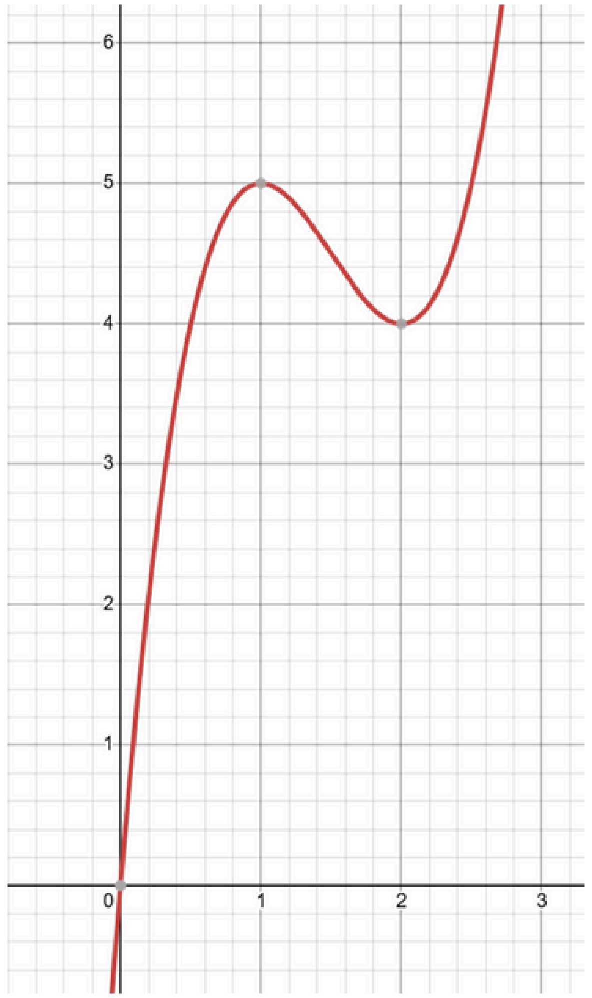
$0 = 6\underline{(x - 1)(x - 2)}$

$x = 1, 2$

Important, always make it into this form

Combining sign

		<1<		<2<	
(x - 1)	-		+		+
(x - 2)	-		-		+
y'	+		-		+
y (slope)	↗		↘		↗



$y(1) = 5$ ← Relative max at (1, 5)

$y(2) = 4$ ← Relative min at (2, 4)

#

Exercise 5

Find relative extrema of $f(x) = x^4 - 4x^3$, answer in (x, y)

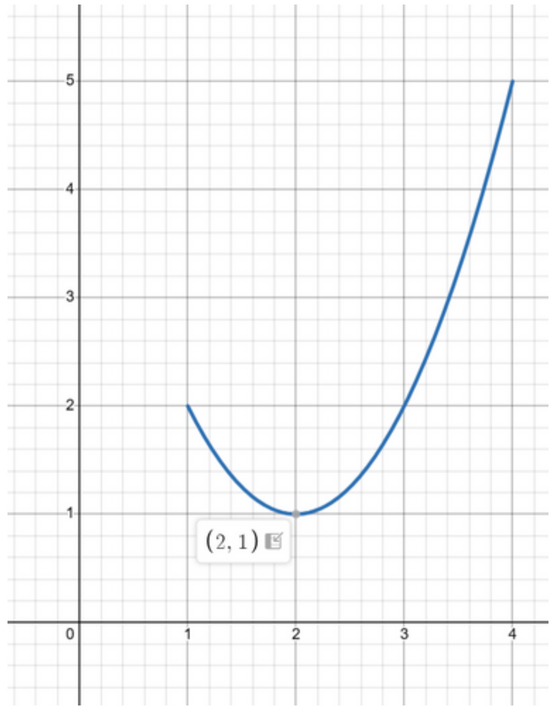
no max, (3, -27)

Finding absolute extrema in closed interval graph

Similar to relative extrema, but the edge of the interval also count as max/min.

Example

$y = x^2 - 4x + 5$ in a closed interval [1, 4]



$y' = 2x - 4$

$0 = 2x - 4$

$x = 2$ ← Critical value

Edge of interval

$y(1) = (1)^2 - 4(1) + 5 = 2 \rightarrow (1, 2)$

$y(2) = (2)^2 - 4(2) + 5 = 1 \rightarrow (2, 1)$ ← Absolute min

$y(4) = (4)^2 - 4(4) + 5 = 5 \rightarrow (4, 5)$ ← Absolute max

#

Second derivative test

Faster way to determine the extrema, but it can only tell whether the critical value is max/min. It does not compare between the critical value and the interval for absolute extrema. (It gives max/min, but not gives value)

Example $f(x) = 2x^3 - 3x^2 - 36x + 17$

$f(x) = 2x^3 - 3x^2 - 36x + 17$
$f'(x) = 6x^2 - 6x - 36$
$0 = 6x^2 - 6x - 36$
$x = -2, 3$

$f''(x) = 12x - 6$ Not y value of the critical points
$f''(-2) = -30\ (<0)$ ← Relative max
$f''(3) = 30\ (>0)$ ← Relative min

#

When the test result in negative, its maxima. If its positive, its minima

Exercise 6

$f(x) = 2x^3 - 3x^2 - 36x + 17$ in closed interval [-5, 5], answer in (x, y)

abs min at (-5, -128), abs max at (-2, 61)

Topic 4:

Integration

What is integration

If derivative is to find the slope equation from a graphs equation, then integrate is to find a graph equations from that slope equation.

Example

F(x) is the result of integrating f(x)

$$F(x) = x^3 + x^2 + x + C$$

Differentiation $\downarrow$ $\quad f(x) = 3x^2 + 2x + 1 + C \quad$ $\uparrow$ Integration

$$f'(x) = 6x + 2$$

When integrate, don't forget +C (constant)

Initial conditions

Use to determine the +C values. (Not count the integral interval $\int_b^a$)

Example

$f(x) = 3x^2 + 2x + 1$ where $F(1) = 5$

$\int f(x)dx = x^3 + x^2 + x + C$

$F(1) = (1)^3 + (1)^2 + 1 + C$

$5 = 3 + C$

$C = 2$

$\therefore F(x) = x^3 + x^2 + x + 2$ #

Integration rules

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

5. $\int \dfrac{du}{u} = \underset{u'}{\underline{\ln|u|}} + C$

6. $\int e^u\,du = \underset{u'}{\underline{e^u}} + C$

7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$

19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a} \arctan \dfrac{u}{a} + C$

20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a} \operatorname{arcsec} \dfrac{|u|}{a} + C$

Exercise 7

1) $\int 7x dx$

2) $\int (x^2 + 2x) dx$

3) $y' = 8x - 4; y(2) = 5$, find y

1) $7x^2/2 + C$, 2) $x^3/3 + x^2 + C$, 3) $y = 4x^2 - 4x - 3$

Integration by part

When 2 equations are join by multiplying each other. $\int u\,dv = uv - \int v\,du$

Example

$\int xe^x dx$

$$\int u\,dv = uv - \int v\,du$$

u = x dv = e^xdx

du = dx v = e^x

Pick x as u because its easier

= (x)(e^x) - ∫(e^x)(dx)

= xe^x - e^x + C #

Tips: 'u' priorities

1. Logarithm
2. Inverse Trig
3. Algebraic
4. Trigonometry
5. Exponential

remember as LIATE

Exercise 8

1) $\int \ln x \, dx$

xln(x) - x + C

Excluded topics

1. Economy applications
    a. Marginal cost & rate pf change
    b. Maximizing profit & minimizing cost
    c. Elasticity of demands
    d. Producer & consumer surplus
2. Sketching graphs using extrema
3. Integration of rational function by partial fraction
4. Integrate: Improper integral
5. Finding area between 2 graphs using integration
6. Sequence and series (not learn)