Theepakorn Phayonrat 67011352 — Linear Algebra Midterm Note

Lecture 1: Vector

Dot Product / Inner Product

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = v_1 w_1 + v_2 w_2$$

Note: $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ can be written as $v = (1, 2)$

Length & Unit Vector

Length of v: $\|v\| = \sqrt{v \cdot v} = (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)^{\frac{1}{2}}$

Tips: Given $v = (1, 2, 3)$,
$\|v\| = (1, 2, 0) \cdot (0, 0, 3) = \sqrt{\sqrt{1^2 + \sqrt{2}} + 3^2} = \sqrt{14}$

Unit Vector u: Vector which $\|u\| = 1$

$$u = \frac{v}{\|v\|} \text{ where } u \text{ has same direction as } v$$

Angles Between Two Vectors

For vector v and w,

1. if $v \cdot w = 0$, then the angle between them is 90°.
2. $v \cdot w = wv$

From **Pythagoras Law:** $a^2 + b^2 = c^2$,

- Perpendicular vectors:

$$\|v\|^2 + \|w\|^2 = \|v - w\|^2$$

- Pythagoras Law:

$$\begin{aligned} (v_1^2 + v_2^2) + (w_1^2 + w_2^2) &= (v_1 - w_1)^2 + (v_2 - w_2)^2 \\ 0 &= -2v_1 w_1 - 2v_2 w_2 \\ 2v_1 w_1 + 2v_2 w_2 &= 0 \\ v_1 w_1 + v_2 w_2 &= 0 \\ v \cdot w &= 0 \\ \therefore \theta &= 90° \end{aligned}$$

Note: If $v \cdot w \neq 0$, then it may take **positive** or **negative** value.

$$v \cdot w \begin{cases} < 0 & \text{, then } \theta \in \mathbb{R}^+ \\ = 0 & \text{, then } \theta = 0 \\ > 0 & \text{, then } \theta \in \mathbb{R}^- \end{cases}$$

Cosine Formula: Given nonzero v and nonzero w vectors, then

$$\frac{v \cdot w}{\|v\|\|w\|} = \cos\theta$$

Inequalities:

- **Schwarz Inequality:** $\|v \cdot w\| \leq \|v\|\|w\|$
- **Triangle Inequality:** $\|v + w\| \leq \|v\| + \|w\|$

Lecture 2: Vector (Cont.)

Matrices

Combination of vectors:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \rightarrow A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Here, A is a 32 **matrix** where $m = 3$ rows and $n = 2$ columns

Three Vectors:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

their linear combination in 3D space are $x_1 u + x_2 v + x_3 w$

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

Matrix times vector Combination of columns:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

We can compare Ax with the Dot Product of

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

System of Linear Equations

Let

$$b = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ -x_2 + x_3 \end{bmatrix}$$

Equation: $Ax = b$
Solution: $x = A^{-1}b$

Lecture 2: Vector (Cont.)

System of Linear Equations (Cont.)

From

$$Ax = b$$

We can think of

- x is the input of the system.
- b is the output of the system.
- A is the function of the system.
- **Note:** It is possible that $Ax = 0$ when $A \neq 0$ and $x0$.

The Inverse of Matrix:

$$x = A^{-1}b$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_1 + b_2 \\ b_1 + b_2 + b_3 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- For every b, there is one solution to $Ax = b$.
- The matrix A^{-1} produces $x = A^{-1}b$

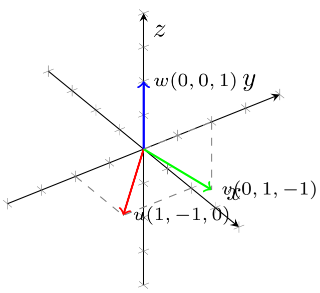
$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 + x_3 \\ -x2 + x_3 \end{bmatrix}$$

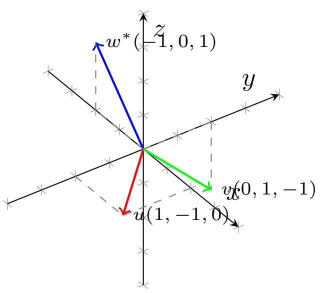
A is called the **difference matrix** because b contains differences of x.

$$Cx = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_3 \\ x2 - x_3 \end{bmatrix}$$

C is called the **cyclic difference matrix**.

Independence & Dependence





Independence: w is not in the plane if u nand v.
Dependence: w^* is in the plane if u nand v.

Note: Why? $\rightarrow w^*$ is a linear combination of u and v.

Independence & Dependence (Cont.)

- u, v, w are **independent**: No combination except $0u + 0v + 0w = 0$ gives $b = 0$.
- u, v, w^* are **dependent**: Other combinations like $u + v + w^* = 0$ gives $b = 0$.

For **n n** matrix:

- Independent columns: $Ax = 0$ has one solution. A is an **invertible matrix**.
- Dependent columns: $Cx = 0$ has many solution. C is an **singular matrix**.

Vectors & Linear Equations

The main problem in Linear Algebra is to solve a system of linear equations.

Two Equations, Two Unknowns (Rows):

Given 2 equations:

$$x - 2y = 1$$
$$3x + 2y = 11$$

They can be written as:

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = b$$

Our goal is to find $\begin{bmatrix} x \\ y \end{bmatrix}$ which:

$$Ax = b \text{ where } A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$$

A is called the **coefficient matrix**.

By matrix-vector multiplication:

$$\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

We get $x = 3$ and $y = 1$.

Lecture 2: Vector (Cont.)

Three equations in Three Unknowns

Given the three unknowns are x, y, z and the linear equations are:

$$\begin{aligned} x + 2y + 3z &= 6 \\ 2x + 5y + 2z &= 4 \\ 6x - 3y + z &= 2 \end{aligned}$$

- The **row picture** shows three planes meeting at the single point.
- The **column picture** combines three columns to produce the vector $(6, 4, 2)$.

Matrix equation: $Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}$$

- **Multiplication by rows:**

$$Ax = \begin{bmatrix} (row_1) \cdot x \\ row_2) \cdot y \\ row_3) \cdot z \end{bmatrix}$$

- **Multiplication by columns:**

$$Ax = x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Identity Matrix:

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, I is the identity matrix.

Matrix Notation

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1,1) & A(1,2) \\ A(2,1) & A(2,2) \end{bmatrix}$$

For an **m n** matrix:

- the row index $i = 1..m$
- the column index $j = 1..n$
- $a_{ij} = A(i, j)$ is the component at row i column j.

The Idea of Elimination

Goal: To produce an **upper-triangular system** so that we can use the upper-triangular to eliminate the rest of the variables by process called **back substitution**.
Given equations:

$$\begin{aligned} x - 2y &= 1 \quad (E1) \\ 3x + 2y &= 11 \quad (E2) \end{aligned}$$

The Idea of Elimination (Cont.)

Here, we have:

- To eliminate x, subtract a muliple of **E1** from the **E2**.

$$(E2) - 3 \times (E1)$$

- **Pivot**: first nonzero in the row that does the elimination
- **Multiplier**: (entry to eliminate) divided by (pivot) (Here: 3/1)
- The pivot of the new $(E2)$ is 8.

$$\begin{aligned} x - 2y &= 1 \quad (E1) \\ 8y &= 8 \quad (E2) \end{aligned}$$

- To solve **n** equations, we need **n** pivots.
- The pivots are the diagonal of the triangle after the elimination.
- **Important:** Sometimes, if the pivot is 0, it is going to have **failure**.

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ 0y &= 8 \text{ No solution} \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ 0y &= 0 \text{ Infinitely many solutions (Too many solutions)} \end{aligned}$$

Lecture 3: Elimination

Elimination Using Matrices

Goal: To express all steps of elimination in the clearest possible way.
The matrix form of a linear system is $Ax = b$

- A is the coefficient (square) matrix.
- x is the vector of unknowns.
- b is the vector of RHS.

The Form of One Elimination Step

Given the vector:

$$b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \to b_{new} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

$\therefore$ The elimination matrix of this case is:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

To build an **elimination matrix** E

- We start with I and change one of its zeroes to to the **multiplier** $-l$

The elementary matrix or the elimination matrix E_{ij} has the extra nonzero entry $-l$ in the i, j position. Then E_{ij} subtracts a multiple l of row j from i.
Here, we want to change:

$$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & -l \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Eb = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$l = 4 \text{ by } 9 - 4(1) = 5$$

Matrix Multiplication

Given:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

Matrix multiplication agrees with elimination, and the new system of equation can be described as $EAx = Eb$.
We will get:

$$E(Ax) = Eb \text{ or } EA(x) = Eb$$

Which is true by **Associative Law**.

$$A(BC) = (AB)C$$

Note: Matrix multiplication does not follow **Commutative Law**. $\therefore$ The law is false here.

The Matrix P_{ij} for a Row Exchange

Note: Exchange means permute which we use a **permutation matrix** P_{ij}.

For example: We want to change the row 2 and 3 of any vector or matrix, we use

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{23} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$P_{23} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 3 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 5 & 2 \\ 0 & 4 & 3 \end{bmatrix}$$

The Augmented Matrix

Now we have **rectangular matrix**.

Important: If elimination does the same row operations to A and to b, then we can include b as an extra column and follow i through elimination.
Augmented Matrix:

$$\begin{bmatrix} Ab \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

Elimination then acts on the whole rows of this matrix.

By applying E_{21} in the $\begin{bmatrix} Ab \end{bmatrix}$, we have:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

Lecture 3: Elimination (Cont.)

Rules for Matrix Operations

- Matrices can be added if their shapes are the same.
- Matrices can be multiplied with any constant c.
- $-A$ comes from multiplication by $c = -1$.
- Adding A to $-A$ leaves the **zero matrix**, with all entries zero.
- To do AB, A must have n columns and B must have n rows.
- $ABC = ABC$
- Given A is mn and B is np, AB must be mp.
- A row times a column is the **dot product**. The result is a single number.
- Addition Laws:
  - $A + B = B + A$ (Commutative Law)
  - $c(A + B) = c(B + A)$ (Distributive Law)
  - $A + (B + C) = (A + B) + C$ (Associative Law)
- Multiplication Laws:
  - $C(A+B) = C(B+A)$ (Distributive Law from the Left)
  - $(A+B)C = AC+BC$ (Distributive Law from the Right)
  - $A(BC) = (AB)C$ (Associative Law for ABC)
  - In some cases, $AB \neq BA$
  - $AI = IA$
  - When A is a square matrix, then $AA = A^2$, and the matrix powers A^p follow the same rules as numbers:
    * $A^p = AAA...A(p \text{ factors})$
    * $(A^p)(A^q) = A^{p+q}$
    * $(A^p)^q) = A^{pq}$
    * $A^0 = I$
    * A^{-1} is the A **inverse**.

Block Matrices and Block Multiplication

The matrix can be cut into **blocks** (which are smaller matrices).

- The augmented matrix $\begin{bmatrix} Ab \end{bmatrix}$ is also a block matrix which has two blocks of different sizes.
- Multiplication by an elimination matrix gave $\begin{bmatrix} EA \ Eb \end{bmatrix}$ when their shapes permit.

Block Multiplication:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

Matrix Multiplication (Columns Rows): gives two full matrices.

For example:

$$\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Block Elimination:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

Note: Need to lookup more

Lecture 4: Inverse

Inverse Matrices

The matrix A is **invertible** if there exists a matrix A^{-1} that **inverts** A.

- Not all matrices have inverses.
- The first question we ask about a square matrix $\rightarrow$ Is A invertible?

Six notes about A^{-1}:

- The inverse exists $\iff$ elimination produces n pivots.
- The matrix A cannot have two different inverses $\rightarrow$ If $BA = I$ and $CA = I$, then $B = C$.
- If A is invertible, the only solution to $Ax = b$ is $x = A^{-1}b$.

Six notes about A^{-1} (Cont.)

- Suppose there is a nonzero vector x such that $Ax = 0$ $\rightarrow$ then A cannot have an inverse. (No matrix can bring 0 back to x.)
  If A is invertible, then $Ax = 0$ can only have the zero solution $x = 0$.
- If A which is a 22 matrix is invertible $\iff$ $ad - bc$ is not zero.
  Finding 222 Inverse:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Lecture 4: Inverse (Cont.)

Inverse Matrices (Cont.)

Six notes about A^{-1} (Cont.)

- A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If } A = \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_{\dots} & & \\ & & & d_{n-1} & \\ & & & & d_n \end{bmatrix}$$

$$\text{, then } A^{-1} = \begin{bmatrix} 1/d_1 & & & & \\ & 1/d_2 & & & \\ & & 1/d_{\dots} & & \\ & & & 1/d_{n-1} & \\ & & & & 1/d_n \end{bmatrix}$$

The Inverse of Product AB

Note: $(AB)^{-1} = B^{-1}A^{-1}$

$$(AB)(A^{-1}B^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

$$\therefore (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

Calculating A^{-1} by Gauss-Jordan Elimination

- A^{-1} might not be explicitly needed.
- To solve $Ax = b$, we can use elimiTo solve $Ax = b$, we can use elimination which goes directly to find x.
- Elimination is also the way to calculate A^{-1}.
- The Gauss-Jordan idea is to solve for $AA^{-1} = I$, Finding each column of A^{-1}.
- A multiplies the first column of A^{-1} (call that x_1) to give the first column of I (call that e_1) $\rightarrow$ the equation is $Ax_1 = e_1 = (1, 0, 0)^t$
- Each of the column x_1, x_2, x_3 of A^{-1} is multiplied by A to produce a column of I:

  $$AA^{-1} = A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = I$$

- To invert a 33 matrix $A \rightarrow$ we have to solve three systems of equations:

  $Ax_1 = e_1 = (1, 0, 0)^t Ax_2 = e_2 = (0, 1, 0)^t Ax_3 = e_3 = (0, 0, 1)^t$

- Usually the augmented matrix $\begin{bmatrix} Ab \end{bmatrix}$ has one extra column b, but now we do the similar with I.
- $\therefore$ The augmented matrix is actually the block matrix $\begin{bmatrix} AI \end{bmatrix}$
- MORE IN GAUSS-JORDAN

Singular versus Invertible

Which matrix have inverses?
The proposed pivot test: A^{-1} exists exactly when A has a full set of n pivots.

$$\text{If } AC = I, \text{ then } CA = I \text{ and } C = A^{-1}$$

A triangular matrix is invertible $\iff$ no diagonal entries are zero.

Elimination = Factorization: $A = LU$

GO READ YOURSELF

Lecture 5: Transposes

Transposes and Permutations

- The **transposes** of A is denoted as A^T
- The columns of A^T is the rows of A.
- When A is an mn matrix, the transpose is nm.

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{, then } A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$$

- In transpose of a L becomes U (But the inverse of L is stil L.)
- The transpose of A^T is A.
- Rules for transposes:
  - $(A+B)^T = A^T + B^T$
  - $(AB)^T = B^T A^T$
  - $(A^{-1})^T = (A^T)^{-1}$
  - $(ABC)^T = C^T B^T A^T$
  - If $A = LDU$, then $A^T = U^T D^T L^T$
  - $A^T(A^{-1})^T = (A^{-1}A^T)^T = I^T = I$
  - A^T is invertible exactly when A is invertible.
  - **Inner & Outer Products:**
    - ^{T} is inside: $x^T y \to (1 \times n)(n \times 1)$
    - ^{T} is outside: $xy^T \to (n \times 1)(1 \times n)$

Permutation Matrix

- The transpose plays a special role for a permutation matrix $\to$ then this matrix P has a single I in every row and every column.
- If P^T is also a permutation matrix $\to$ then this may be the same as P or different matrix.
- Any product of P_1P_2 is again a permutation matrix.
- We can create every P from the identity matrix, by reordering the rows, of I.
- There are 6 3×3 permutation matrices:

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}$$

$$P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

Permutation Matrix (Cont.)

Note:

- There are $n!$ permutation matrices of order n.
- P^{-1} is also a permutation matrix.
- P^{-1} is also the same as P^T.

The $PA = LU$ Factorization with Row Exchanges

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Spaces of Vectors

GO READ YOURSELF