Week 12 Homework

Probability Model and Data Analysis

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Homework of Pairs of Random Variables-Joint PMF

Question 1

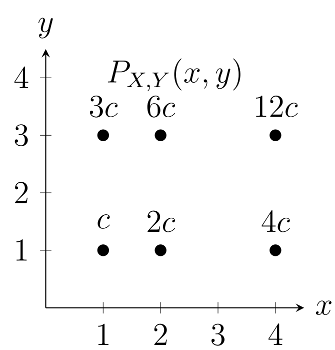
(4.2.1) Random variables X and Y have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} cxy & x = 1,2,4; \quad y = 1,3 \\ 0 & otherwise \end{cases}$$

(a) What is the value of the constant c?
(b) What is $P[Y < X]$?
(c) What is $P[Y > X]$?
(d) What is $P[Y = X]$?
(e) What is $P[Y = 3]$?

Solution

(a)



$$\therefore \sum_{x=1,2,4} \sum_{y=1,3} P_{X,Y}(x,y) = c \sum_{x=1,2,4} X \sum_{y=1,3} Y$$

$$1 = c(1(1+3) + 2(1+3) + 4(1+3)) = 28c$$

$$\therefore c = \frac{1}{28}$$

From earlier, $c = \frac{1}{28}$.

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{28}xy & x = 1,2,4; \quad\quad y = 1,3 \\ 0 & otherwise \end{cases}$$

(b)

$$\begin{aligned} P[Y < X] &= P_{X,Y}(2,1) + P_{X,Y}(3,1) + P_{X,Y}(4,3) \\ &= \frac{1}{28}(2)(1) + \frac{1}{28}(3)(1) + \frac{1}{28}(4)(3) \\ &= \frac{2}{28} + \frac{4}{28} + \frac{12}{28} \\ \therefore P[Y < X] &= \frac{18}{28} = \frac{9}{14} \end{aligned}$$

(c)

$$\begin{aligned} P[Y > X] &= P_{X,Y}(3,1) + P_{X,Y}(3,2) \\ &= \frac{1}{28}(3)(1) + \frac{1}{28}(3)(2) \\ \therefore P[Y > X] &= \frac{3}{28} + \frac{6}{28} = \frac{9}{28} \end{aligned}$$

(d)

$$\begin{aligned} P[Y = X] &= P_{X,Y}(1,1) \\ &= \frac{1}{28}(1)(1) \\ \therefore P[Y = X] &= \frac{1}{28} \end{aligned}$$

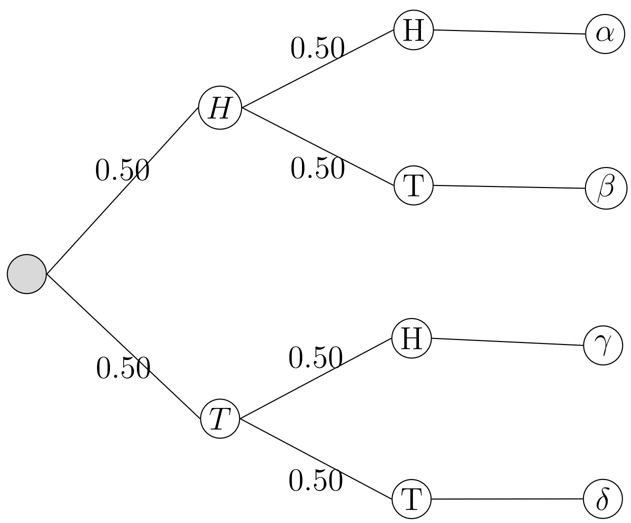
(e)

$$
\begin{aligned}
P[Y=3] &= P_{X,Y}(1,3) + P_{X,Y}(2,3) + P_{X,Y}(4,3) \\
&= \frac{1}{28}(1)(3) + \frac{1}{28}(2)(3) + \frac{1}{28}(4)(3) \\
&= \frac{3}{28} + \frac{6}{28} + \frac{12}{28} \\
\therefore P[Y=3] &= \frac{21}{28} = \frac{3}{4}
\end{aligned}
$$

Question 2

For two of fair coin, let X equal the total number of tails and let Y equal the number of heads on the last flip. Find the joint PMF $P_{X,Y}(x, y)$

Solution



$$\alpha \to X = 0, Y = 1, P_{X,Y}(0, 1) = 0.25$$
$$\beta \to X = 1, Y = 0, P_{X,Y}(1, 0) = 0.25$$
$$\gamma \to X = 1, Y = 1, P_{X,Y}(1, 1) = 0.25$$
$$\delta \to X = 2, Y = 1, P_{X,Y}(2, 1) = 0.25$$

$P_{X,Y}(x, y)$	$y = 0$	$y = 1$
$x = 0$	0	$\frac{1}{4}$
$x = 1$	$\frac{1}{4}$	$\frac{1}{4}$
$x = 2$	$\frac{1}{4}$	0

$$P_{X,Y}(x, y) = \begin{cases} \frac{1}{4} & x = 0 \quad y = 1 \\ \frac{1}{4} & x = 1 \quad y = 0 \\ \frac{1}{4} & x = 1 \quad y = 1 \\ \frac{1}{4} & x = 2 \quad y = 1 \\ 0 & otherwise \end{cases}$$

Homework of Marginal PMF

Question 1

Given the random variables X and Y in Problem 4.2.1, find

(a) The marginal PMFs $P_X(x)$ and $P_Y(y)$
(b) The expected values $E[X]$ and $E[Y]$
(c) The standard deviation σ_X and σ_Y

Solution

(a)

$$\begin{aligned} P_X(x) &= \sum_y P_{X,Y}(x,y) \\ &= P_{X,Y}(x,1) + P_{X,Y}(x,3) \\ &= \frac{1}{28}(x)(1) + \frac{1}{28}(x)(3) \\ \therefore P_X(x) &= \frac{4x}{28} = \frac{x}{7} \end{aligned}$$

$$\therefore P_X(1) = \frac{1}{7} \qquad \therefore P_X(2) = \frac{2}{7} \qquad \therefore P_X(4) = \frac{4}{7}$$

$$\begin{aligned} P_Y(y) &= \sum_y P_{X,Y}(x,y) \\ &= P_{X,Y}(1,y) + P_{X,Y}(2,y) + P_{X,Y}(4,y) \\ &= \frac{1}{28}(1)(y) + \frac{1}{28}(2)(y) + \frac{1}{28}(4)(y) \\ \therefore P_Y(y) &= \frac{7y}{28} = \frac{y}{4} \end{aligned}$$

$$\therefore P_Y(1) = \frac{1}{4} \qquad \therefore P_Y(3) = \frac{3}{4}$$

(b)

$$\begin{aligned}
E[X] &= \sum_x x P_X(x) \\
&= 1(\frac{1}{7}) + 2(\frac{2}{7}) + 4(\frac{4}{7}) \\
&= \frac{1+4+16}{7} \\
\therefore E[X] &= \frac{21}{7} = 3
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \sum_y y P_Y(y) \\
&= 1(\frac{1}{4}) + 3(\frac{3}{4}) \\
&= \frac{1+9}{4} \\
\therefore E[Y] &= \frac{10}{4} = \frac{5}{2} = 2.5
\end{aligned}$$

(c)

To find σ_X and σ_Y, we need to find $Var[X]$ and $Var[Y]$

$$\begin{aligned}
E[X^2] &= \sum_x x^2 P_X(x) \\
&= 1^2(\frac{1}{7}) + 2^2(\frac{2}{7}) + 4^2(\frac{4}{7}) \\
&= \frac{1+8+64}{7} \\
\therefore E[X^2] &= \frac{73}{7}
\end{aligned}$$

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \frac{73}{7} - 3^2 \\
&= \frac{73}{7} - \frac{63}{7} \\
\therefore Var[X] &= \frac{10}{7} \\
\therefore \sigma_X &= \sqrt{\frac{10}{7}}
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] &= \sum_y y^2 P_Y(y) \\
&= 1^2(\frac{1}{4}) + 3^2(\frac{3}{4}) \\
&= \frac{1+27}{4} \\
\therefore E[Y^2] &= \frac{28}{4} = 7
\end{aligned}
$$

$$
\begin{aligned}
Var[Y] &= E[Y^2] - (E[Y])^2 \\
&= 7 - 2.5^2 \\
&= 7 - 6.25 \\
\therefore Var[Y] &= 0.75 = \frac{3}{4} \\
\therefore \sigma_Y &= \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}
\end{aligned}
$$

Question 2

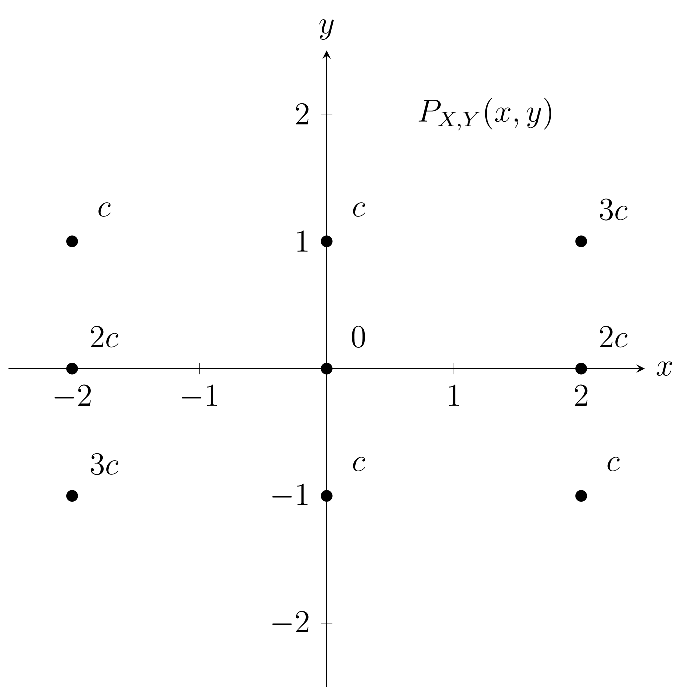
(4.2.2) Random variables X and Y have hoint PMF

$$P_{X,Y}(x,y) = \begin{cases} c|x+y| & x = -2, 0, 2; \quad\quad y = -1, 0, 1 \\ 0 & \textit{otherwise} \end{cases}$$

(a) What is the value of the constant c?
(b) What is $P[Y < X]$?
(c) What is $P[Y > X]$?
(d) What is $P[Y = X]$?
(e) What is $P[X < 1]$?

Solution

(a)



$$\therefore \sum_{x=-2,0,2} \sum_{y=-1,0,1} P_{X,Y}(x,y) = c \sum_{x=1,2,4} \sum_{y=1,3} |X+Y|$$

$\|X+Y\|$	$Y=-1$	$Y=0$	$Y=1$
$X=-2$	$\|-2+(-1)\|=3$	$\|-2+0\|=2$	$\|-2+1\|=1$
$X=0$	$\|0+(-1)\|=1\|$	$\|0+0\|=0$	$\|0+1\|=1$
$X=2$	$\|2+(-1)\|=1\|$	$\|2+0\|=2$	$\|2+1\|=1$

$$\begin{aligned}\therefore \sum_{x=-2,0,2} \sum_{y=-1,0,1} P_{X,Y}(x,y) &= c(3+2+1+1+0+1+1+2+3)\\ 1 &= c(14)\\ \therefore c &= \frac{1}{14}\end{aligned}$$

From earlier, $c = \frac{1}{14}$.

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{14}|x+y| & x=-2,0,2; \quad y=-1,0,1 \\ 0 & otherwise \end{cases}$$

(b)

$$\begin{aligned}
P[Y < X] &= P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1) \\
&= \frac{1}{14}|0+(-1)| + \frac{1}{14}|2+(-1)| + \frac{1}{14}|2+0| + \frac{1}{14}|2+1| \\
&= \frac{1}{14} + \frac{1}{14} + \frac{2}{14} + \frac{3}{14} \\
\therefore P[Y < X] &= \frac{7}{14} = \frac{1}{2}
\end{aligned}$$

(c)

$$\begin{aligned}
P[Y > X] &= P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1) \\
&= \frac{1}{14}|-2+(-1)| + \frac{1}{14}|-2+0| + \frac{1}{14}|-2+1| + \frac{1}{14}|0+1| \\
&= \frac{3}{14} + \frac{2}{14} + \frac{1}{14} + \frac{1}{14} \\
\therefore P[Y > X] &= \frac{7}{14} = \frac{1}{2}
\end{aligned}$$

(d)

$$\begin{aligned}
P[Y = X] &= P_{X,Y}(-2,-1) \\
&= \frac{1}{14}|0+0| \\
&= \frac{0}{14} \\
\therefore P[Y = X] &= 0
\end{aligned}$$

(e)

$$
\begin{aligned}
P[X<1] &= P[X=0]+P[X=-1] \\
&= \frac{1}{14}|0+-1|+\frac{1}{14}|0+0|+\frac{1}{14}|0+1|+P[Y=-1] \\
&= \frac{1}{14}+0+\frac{1}{14}+P[Y=-1] \\
&= \frac{2}{14}+P[Y=-1] \\
&= \frac{2}{14}+\frac{1}{14}|-1+(-2)|+\frac{1}{14}|-1+0|+\frac{1}{14}|-1+2| \\
&= \frac{2}{14}+\frac{3}{14}+\frac{1}{14}+\frac{1}{14} \\
\therefore P[X<1] &= \frac{7}{14}=\frac{1}{2}
\end{aligned}
$$

Question 3

Given the random variables X and Y in Problem 4.2.2, find

(a) The marginal PMFs $P_X(x)$ and $P_Y(y)$
(b) The expected values $E[X]$ and $E[Y]$
(c) The standard deviation σ_X and σ_Y

Solution

(a)

$$\begin{aligned}
P_X(x) &= \sum_y P_{X,Y}(x,y) \\
&= P_{X,Y}(x,-1) + P_{X,Y}(x,0) + P_{X,Y}(x,1) \\
&= \frac{1}{14}|x+(-1)| + \frac{1}{14}|x+0| + \frac{1}{14}|x+1| \\
\therefore P_X(x) &= \frac{|x-1|+|x|+|x+1|}{14}
\end{aligned}$$

$$\begin{aligned}
P_X(-2) &= \frac{|-2-1|+|-2|+|-2+1|}{14} \\
&= \frac{|-3|+|-2|+|-1|}{14} \\
\therefore P_X(-2) &= \frac{3+2+1}{14} = \frac{6}{14} = \frac{3}{7}
\end{aligned}$$

$$\begin{aligned}
P_X(0) &= \frac{|0-1|+|0|+|0+1|}{14} \\
&= \frac{|-1|+|0|+|1|}{14} \\
\therefore P_X(0) &= \frac{1+0+1}{14} = \frac{2}{14} = \frac{1}{7}
\end{aligned}$$

$$\begin{aligned}
P_X(2) &= \frac{|2-1|+|2|+|2+1|}{14} \\
&= \frac{|1|+|2|+|3|}{14} \\
\therefore P_X(2) &= \frac{1+2+3}{14} = \frac{6}{14} = \frac{3}{7}
\end{aligned}$$

$$
\begin{aligned}
P_Y(y) &= \sum_y P_{X,Y}(x,y) \\
&= P_{X,Y}(-2,y) + P_{X,Y}(0,y) + P_{X,Y}(2,y) \\
&= \frac{1}{14}|-2+y| + \frac{1}{14}|0+y| + \frac{1}{14}|2+y| \\
\therefore P_Y(y) &= \frac{|-2+y| + |y| + |2+y|}{14}|-2+y|
\end{aligned}
$$

$$
\begin{aligned}
P_Y(-1) &= \frac{|-2-(-1)| + |-1| + |-2+-1|}{14} \\
&= \frac{|-1| + |-1| + |-3|}{14} \\
\therefore P_Y(-1) &= \frac{1+1+3}{14} = \frac{5}{14} = \frac{2.5}{7}
\end{aligned}
$$

$$
\begin{aligned}
P_Y(0) &= \frac{|-2-0| + |0| + |-2+0|}{14} \\
&= \frac{|-2| + |0| + |-2|}{14} \\
\therefore P_Y(0) &= \frac{2+0+2}{14} = \frac{4}{14} = \frac{2}{7}
\end{aligned}
$$

$$
\begin{aligned}
P_Y(1) &= \frac{|-2-1| + |1| + |-2+1|}{14} \\
&= \frac{|-3| + |1| + |-1|}{14} \\
\therefore P_Y(1) &= \frac{3+1+1}{14} = \frac{5}{14} = \frac{2.5}{7}
\end{aligned}
$$

(b)

$$\begin{aligned}
E[X] &= \sum_x x P_X(x) \\
&= -2(\frac{3}{7}) + 0(\frac{1}{7}) + 2(\frac{3}{7}) \\
&= \frac{-6+0+6}{7} \\
\therefore E[X] &= 0
\end{aligned}$$

$$\begin{aligned}
E[Y] &= \sum_y y P_Y(y) \\
&= -1(\frac{2.5}{7}) + 0(\frac{2}{7}) + 1(\frac{2.5}{7}) \\
&= \frac{-2.5+0+2.5}{7} \\
\therefore E[Y] &= 0
\end{aligned}$$

(c)

To find σ_X and σ_Y, we need to find $Var[X]$ and $Var[Y]$

$$\begin{aligned}
E[X^2] &= \sum_x x^2 P_X(x) \\
&= (-1)^2(\frac{3}{7}) + 0^2(\frac{1}{7}) + 1^2(\frac{3}{7}) \\
&= \frac{3+0+3}{7} \\
\therefore E[X^2] &= \frac{6}{7}
\end{aligned}$$

$$
\begin{aligned}
Var[X] &= E[X^2] - (E[X])^2 \\
&= \frac{6}{7} - 0^2 \\
&= \frac{6}{7} - 0 \\
\therefore Var[X] &= \frac{6}{7} \\
\therefore \sigma_X &= \sqrt{\frac{6}{7}}
\end{aligned}
$$

$$
\begin{aligned}
E[Y^2] &= \sum_y y^2 P_Y(y) \\
&= (-2)^2(\frac{2.5}{7}) + 0^2(\frac{2}{7}) + 2^2(\frac{2.5}{7}) \\
&= \frac{5+0+5}{7} \\
\therefore E[Y^2] &= \frac{10}{7}
\end{aligned}
$$

$$
\begin{aligned}
Var[Y] &= E[Y^2] - (E[Y])^2 \\
&= \frac{10}{7} - 0^2 \\
&= \frac{10}{7} - 0 \\
\therefore Var[Y] &= \frac{10}{7} \\
\therefore \sigma_Y &= \sqrt{\frac{10}{7}}
\end{aligned}
$$