Week 11 Homework

Probability Model and Data Analysis

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Homework of Continuous RVs

Question

Continuous random variable X has $E[X] = 3$ and $Var[X] = 9$. Find the PDF, $f_X(x)$, if

(a) X is an exponential random variable,
(b) X is a continuous uniform random variable.

Solution

(a)

We know that, $E[X] = 3$ and $Var[X] = 9$.

$$
\begin{aligned}
E[X] &= \frac{1}{\lambda} \\
\frac{1}{\lambda} &= 3 \\
\therefore \lambda &= \frac{1}{3}
\end{aligned}
$$

$$
\therefore f_X(x) = \begin{cases} \dfrac{e^{-\frac{x}{3}}}{3} & x \geq 0 \\ 0 & otherwise \end{cases}
$$

(b)

We know that, $E[X] = 3$

$$\begin{aligned} E[X] &= \frac{(b+a)}{2} \\ \frac{(b+a)}{2} &= 3 \\ \therefore b+a &= 6 \end{aligned} \tag{1}$$

We know that, $Var[X] = 9$

$$\begin{aligned} V[X] &= \frac{(b-a)^2}{12} \\ \frac{(b-a)^2}{12} &= 9 \\ (b-a)^2 &= 108 \\ b-a &= \sqrt{108} = 6\sqrt{3} \end{aligned} \tag{2}$$

(1) + (2)

$$\begin{aligned} (b+a)+(b-a) &= 6+6\sqrt{3} \\ 2b &= 2(3+3\sqrt{3}) \\ \therefore b &= 3+3\sqrt{3} \\ \therefore a &= 6-b = 6-(3+3\sqrt{3}) = 3-3\sqrt{3} \end{aligned}$$

$$\therefore f_X(x) = \begin{cases} \frac{1}{6\sqrt{3}} & (3-3\sqrt{3}) \leq x \leq (3+3\sqrt{3}) \\ 0 & otherwise \end{cases}$$

Homework of Gaussian-RV

Question 1

X is the Gaussian (0, 1) random variable and Y is the Gaussian (0, 2) random variable.

(1) Sketch the PDFs $f_X(x)$ and $f_Y(y)$ in the same axes.
(2) What is $P[-1 < X \leq 1]$?
(3) What is $P[-1 < Y \leq 1]$?
(4) What is $P[X > 3.5]$?
(5) What is $P[Y > 3.5]$?

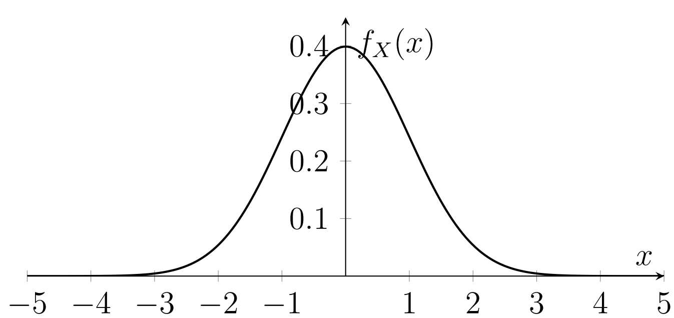
Solution

(1)

From the information provided earlier we can find the $f_X(x)$.

$$
\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \\
&= \frac{1}{\sqrt{2\pi 1^2}} e^{\frac{-(x-0)^2}{2(1)^2}} \\
\therefore f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}
\end{aligned}
$$

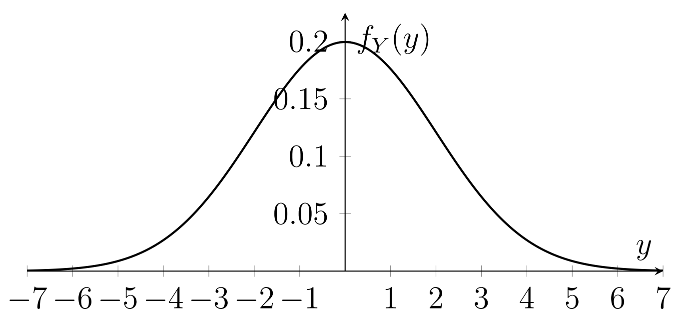
Now we can sketch the graph of $f_X(x)$.



From the information provided earlier we can find the $f_Y(y)$.

$$
\begin{aligned}
f_Y(y) &= \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{\frac{-(y-\mu_y)^2}{2\sigma_y^2}} \\
&= \frac{1}{\sqrt{2\pi 2^2}} e^{\frac{-(y-0)^2}{2(2)^2}} \\
\therefore f_Y(y) &= \frac{1}{\sqrt{8\pi}} e^{\frac{-y^2}{8}}
\end{aligned}
$$

Now we can sketch the graph of $f_Y(y)$.

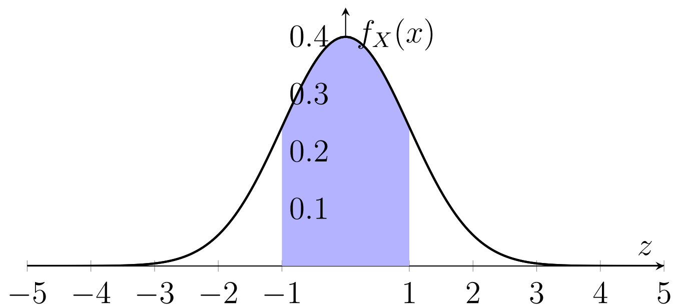


Positive Z Table

z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
+0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
+0.1	.53983	.54380	.54776	.55172	.55567	.55966	.56360	.56749	.57142	.57535
+0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
+0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
+0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
+0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
+0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
+0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
+0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
+0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
+1	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
+1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
+1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
+1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91308	.91466	.91621	.91774
+1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
+1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
+1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
+1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
+1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
+1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
+2	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
+2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
+2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
+2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
+2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
+2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
+2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
+2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
+2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
+2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
+3	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
+3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
+3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
+3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
+3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
+3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
+3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
+3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
+3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
+3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997
+4	.99997	.99997	.99997	.99997	.99997	.99997	.99998	.99998	.99998	.99998

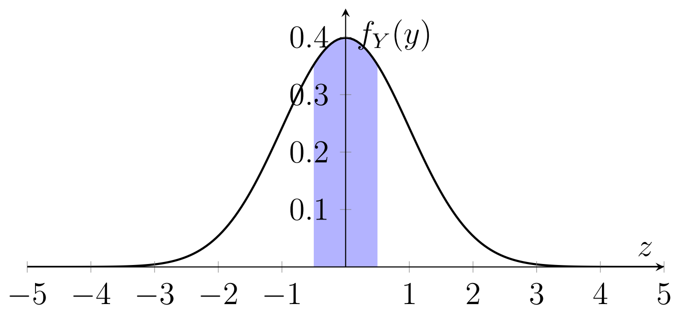
Source: `https://www.ztable.net/`

(2)



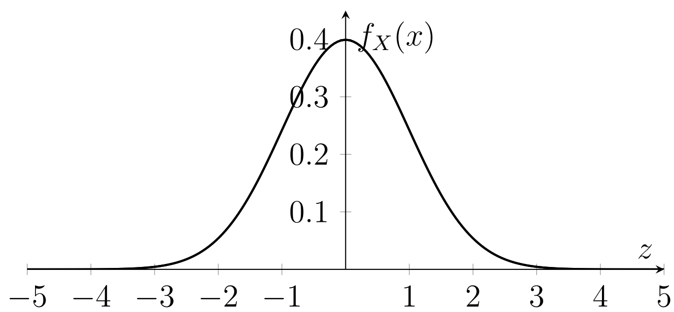
$$
\begin{aligned}
P[-1 \leq X \leq 1] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{x_2 - \mu_x}{\sigma_x}) - \Phi(\frac{x_1 - \mu_x}{\sigma_x}) \\
&= \Phi(\frac{1-0}{1}) - \Phi(\frac{-1-0}{1}) \\
&= \Phi(1) - \Phi(-1) \\
&= \Phi(1) - (1 - \Phi(1)) \\
&= 2\Phi(1) - 1 \\
&= 2(0.8413) - 1 \\
\therefore P[-1 \leq X \leq 1] &= 1.6826 - 1 = 0.6826
\end{aligned}
$$

(3)



$$
\begin{aligned}
P[-1 \leq Y \leq 1] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{y_2 - \mu_y}{\sigma_y}) - \Phi(\frac{y_1 - \mu_y}{\sigma_y}) \\
&= \Phi(\frac{1-0}{2}) - \Phi(\frac{-1-0}{2}) \\
&= \Phi(\frac{1}{2}) - \Phi(-\frac{1}{2}) \\
&= \Phi(0.5) - (1 - \Phi(0.5)) \\
&= 2\Phi(0.5) - 1 \\
&= 2(0.6915) - 1 \\
\therefore P[-1 \leq Y \leq 1] &= 1.3830 - 1 = 0.3830
\end{aligned}
$$

(4)



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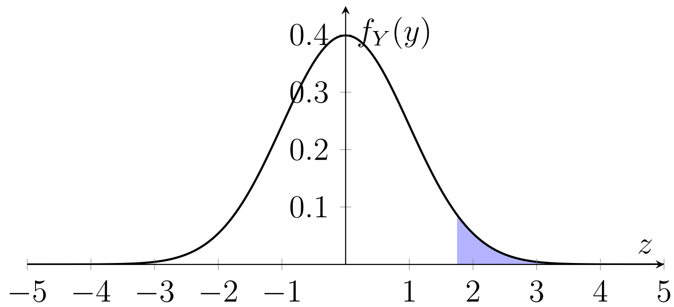
$$
\begin{aligned}
P[X \geq 3.5] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{x_2 - \mu_x}{\sigma_x}) - \Phi(\frac{x_1 - \mu_x}{\sigma_x}) \\
&= \Phi(\frac{\infty - 0}{1}) - \Phi(\frac{3.5 - 0}{1}) \\
&= \Phi(\infty) - \Phi(3.5) \\
\therefore P[X \geq 3.5] &= 1 - 0.99977 \approx 0.00023 = 2.3 \times 10^{-4}
\end{aligned}
$$

(5)



$$
\begin{aligned}
P[Y \geq 3.5] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{y_2 - \mu_y}{\sigma_y}) - \Phi(\frac{y_1 - \mu_y}{\sigma_y}) \\
&= \Phi(\frac{\infty - 0}{2}) - \Phi(\frac{3.5 - 0}{2}) \\
&= \Phi(\infty) - \Phi(1.75) \\
&= 1 - 0.95994 \\
\therefore P[Y \geq 3.5] &\approx 0.04006
\end{aligned}
$$

Question 2

The peak temperature T, in degrees Fahrenheit, on a July day in Antarctica is a Gaussian random variable with a variance of 225. With probability $\frac{1}{2}$, the temperature T exceeds 10 degrees.
(1) What is $P[T > 32]$, the probability the temperature is above freezing?
(2) What is $P[T < 0]$?
(3) What is $P[T > 60]$?

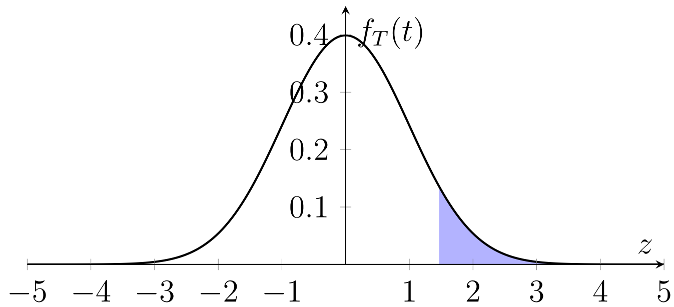
Solution

From the information provided from earlier, we know that, $Var[T] = 225$ and $P[T \geq 10] = \frac{1}{2}$.

$$\begin{aligned} \sigma_T &= \sqrt{Var[T]} \\ &= \sqrt{225} \\ \therefore \sigma_T &= 15 \end{aligned}$$

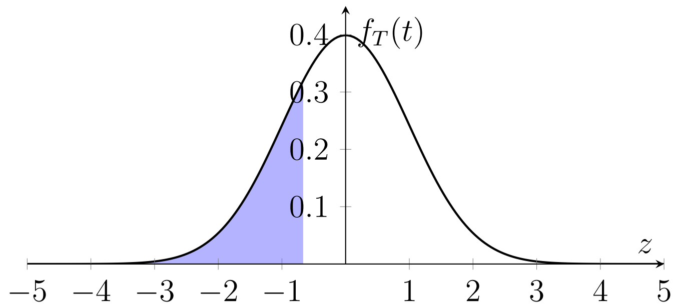
$$\begin{aligned} P[T > 10] &= \Phi(\frac{\infty - \mu_T}{\sigma_T}) - \Phi(\frac{t - \mu_T}{\sigma_T}) \\ \frac{1}{2} &= \Phi(\infty) - \Phi(\frac{t - \mu_T}{15}) \\ \frac{1}{2} &= 1 - \Phi(\frac{10 - \mu_T}{15}) \\ \Phi(\frac{10 - \mu_T}{15}) &= \frac{1}{2} \\ \Phi(\frac{10 - \mu_T}{15}) &= \Phi(0) \\ \therefore \mu_T &= 10 \end{aligned}$$

(1)



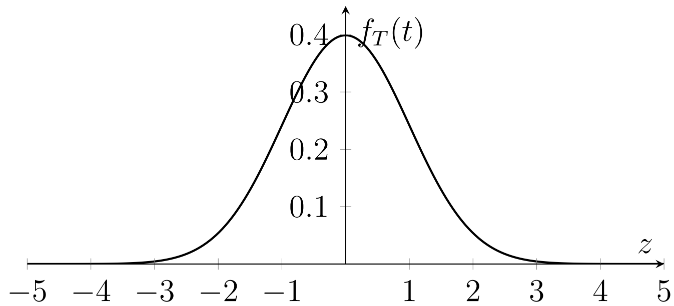
$$
\begin{aligned}
P[T > 32] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{t_2 - \mu_T}{\sigma_T}) - \Phi(\frac{t_1 - \mu_T}{\sigma_T}) \\
&= \Phi(\frac{\infty - 10}{15}) - \Phi(\frac{32 - 10}{15}) \\
&= \Phi(\infty) - \Phi(\frac{22}{15}) \\
&= \Phi(\infty) - \Phi(1.47) \\
&= 1 - 0.92922 \\
\therefore P[T > 32] &\approx 0.07078 \approx 0.0708
\end{aligned}
$$

(2)



$$
\begin{aligned}
P[T < 0] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{t_2 - \mu_T}{\sigma_T}) - \Phi(\frac{t_1 - \mu_T}{\sigma_T}) \\
&= \Phi(\frac{0 - 10}{15}) - \Phi(\frac{-\infty - 10}{15}) \\
&= \Phi(-\frac{2}{3}) - \Phi(-\infty) \\
&= \Phi(-0.667) - \Phi(-\infty) \\
&= (1 - \Phi(0.67)) - \Phi(-\infty) \\
&= (1 - 0.74857) - 0 \\
\therefore P[T < 0] &\approx 0.25143 \approx 0.2514
\end{aligned}
$$

(3)



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$$
\begin{aligned}
P[T > 60] &= \Phi(z_2) - \Phi(z_1) \\
&= \Phi(\frac{t_2 - \mu_T}{\sigma_T}) - \Phi(\frac{t_1 - \mu_T}{\sigma_T}) \\
&= \Phi(\frac{0\infty - 10}{15}) - \Phi(\frac{60 - 10}{15}) \\
&= \Phi(\infty) - \Phi(\frac{50}{15}) \\
&= \Phi(\infty) - \Phi(\frac{10}{3}) \\
&= \Phi(\infty) - \Phi(3.33) \\
&= 1 - 0.99957 \\
\therefore P[T > 60] &\approx 0.00043 = 4.3 \times 10^{-4}
\end{aligned}
$$