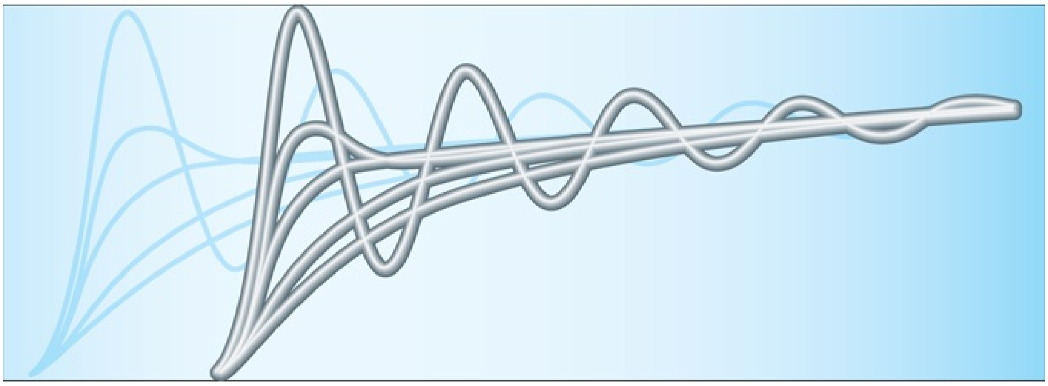
Chapter 4 Transients



Study of this chapter will enable you to:

- Solve first-order *RC* or *RL* circuits.
- Understand the concepts of transient response and steady-state response.
- Relate the transient response of first-order circuits to the time constant.
- Solve *RLC* circuits in dc steady-state conditions.
- Solve second-order circuits.
- Relate the step response of a second-order system to its natural frequency and damping ratio.
- Use the MATLAB Symbolic Toolbox to solve differential equations.

Introduction to this chapter:

*In this chapter, we study circuits that contain sources, switches, resistances, inductances, and capacitances. The time-varying currents and voltages resulting from the sudden application of sources, usually due to switching, are called **transients**.*

*In transient analysis, we start by writing circuit equations using concepts developed in **Chapter 2**, such as KCL, KVL, node-voltage analysis, and mesh-current analysis. Because the current–voltage relationships for inductances and capacitances involve integrals and derivatives, we obtain integrodifferential equations. These equations can be converted to pure differential equations by differentiating with respect to time. Thus, the study of transients requires us to solve differential equations.*

4.1 First-Order *RC* Circuits

In this section, we consider transients in circuits that contain independent dc sources, resistances, and a single capacitance.

Discharge of a Capacitance through a Resistance

As a first example, consider the circuit shown in **Figure 4.1(a)**. Prior to $t = 0,$ the capacitor is charged to an initial voltage V_i. Then, at $t = 0,$ the switch closes and current flows through the resistor, discharging the capacitor.

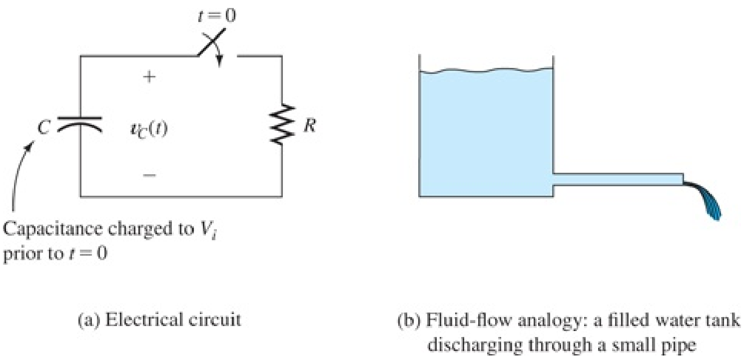


Figure 4.1
A capacitance discharging through a resistance and its fluid-flow analogy. The capacitor is charged to V_i prior to $t = 0$ (by circuitry that is not shown). At $t = 0,$ the switch closes and the capacitor discharges through the resistor.

Writing a current equation at the top node of the circuit after the switch is closed yields

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t)}{R} = 0$$

Multiplying by the resistance gives

$$RC\frac{dv_C(t)}{dt} + v_C(t) = 0 \tag{4.1}$$

As expected, we have obtained a differential equation.

Equation 4.1 indicates that the solution for $v_C(t)$ must be a function that has the same form as its first derivative. Of course, a function with this property is an exponential. Thus, we anticipate that the solution is of the form

> **Equation 4.1** indicates that the solution for $v_c(t)$ must be a function that has the same form as its first derivative. The function with this property is an exponential.

$$v_C(t) = Ke^{st} \tag{4.2}$$

in which *K* and *s* are constants to be determined.

Using **Equation 4.2** to substitute for $v_C(t)$ in **Equation 4.1**, we have

$$RCKse^{st} + Ke^{st} = 0$$

Solving for s, we obtain (4.3)

$$s = \frac{-1}{RC} \tag{4.4}$$

Substituting this into **Equation 4.2**, we see that the solution is

$$v_C(t) = Ke^{-t/RC} \tag{4.5}$$

Referring to **Figure 4.1(a)**, we reason that the voltage across the capacitor cannot change instantaneously when the switch closes. This is because the current through the capacitance is $i_C(t) = C\, dv_C/dt$. In order for the voltage to change instantaneously, the current would have to be infinite. Since the voltage is finite, the current in the resistance must be finite, and we conclude that the voltage across the capacitor must be continuous. Thus, we write

> Because the current is finite, the voltage across the capacitor cannot change instantaneously when the switch closes.

$$v_C(0+) = V_i \tag{4.6}$$

in which $v_C(0+)$ represents the voltage immediately after the switch closes. Substituting into **Equation 4.5**, we have

$$v_C(0+) = V_i = Ke^0 = K \tag{4.7}$$

Hence, we conclude that the constant *K* equals the initial voltage across the capacitor. Finally, the solution for the voltage is

$$v_C(t) = V_i e^{-t/RC} \tag{4.8}$$

A plot of the voltage is shown in **Figure 4.2**. Notice that the capacitor voltage decays exponentially to zero.

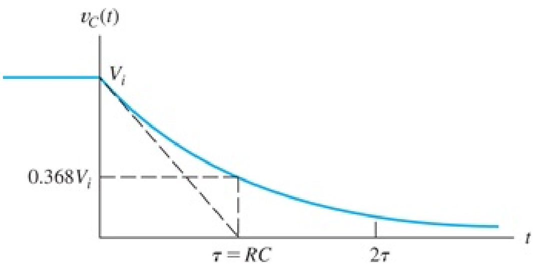


Figure 4.2
Voltage versus time for the circuit of **Figure 4.1(a)**. When the switch is closed, the voltage across the capacitor decays exponentially to zero. At one time constant, the voltage is equal to 36.8 percent of its initial value.

> The time interval $\tau = RC$ is called the time constant of the circuit.

The time interval

$$\tau = RC \tag{4.9}$$

is called the **time constant** of the circuit. In one time constant, the voltage decays by the factor $e^{-1} > 0.368$. After about five time constants, the voltage remaining on the capacitor is negligible compared with the initial value.

> At one time constant, the voltage across a capacitance discharging through a resistance is $e^{-1} \cong 0.368$ times its initial value. After about three to five time constants, the capacitance is almost totally discharged.

An analogous fluid-flow system is shown in **Figure 4.1(b)**. The tank initially filled with water is analogous to the charged capacitor. Furthermore, the small pipe is analogous to the resistor. At first, when the tank is full, the flow is large and the water level drops fast. As the tank empties, the flow decreases.

In the past, engineers have frequently applied *RC* circuits in timing applications. For example, suppose that when a garage door opens or closes, a light is to be turned on and is to remain on for 30 s. To achieve this objective, we could design a circuit consisting of (1) a capacitor that is charged to an initial voltage V_i while the door opener is energized, (2) a resistor through which the capacitor discharges, and (3) a sensing circuit that keeps the light on as long as the capacitor voltage is larger than $0.368\ V_i$. If we choose the time constant $\tau = RC$ to be 30 s, the desired operation is achieved.

(In modern designs, a typical garage door opener contains a small computer, known as a microcontroller, and software that counts seconds for timing purposes. We discuss microcontrollers in **Chapter 8**.)

Example 4.1 Capacitance Discharging Through a Resistance

The circuit of **Figure 4.1(a)** has $R = 2\ \text{M}\Omega,\ C = 3\ \mu\text{F},$ and $V_i = 100\ \text{V}$. Determine the value of time t_x for which $v_C(t) = 25\ \text{V}$.

Solution

The voltage is given by **Equation 4.8**:

$$v_C(t) = V_i e^{-t/RC} \quad \text{for } t > 0$$

in which the time constant is $\tau = RC = (2\ \text{M}\Omega) \times (3\ \mu\text{F}) = 6\ \text{s}$.

Substituting values, we have

$$v_C(t_x) = 25 = 100 e^{-t_x/6}$$

Dividing both sides by 100, we have

$$0.25 = e^{-t_x/6}$$

Then, taking the natural logarithm of both sides, we obtain:

$$\begin{aligned} \ln(0.25) &= -t_x/6 \\ t_x &= -6\ln(0.25) \\ t_x &= 8.3178\ \text{s} \end{aligned}$$

Charging a Capacitance from a DC Source through a Resistance

Next, consider the circuit shown in **Figure 4.3**. The source voltage V_s is constant–in other words, we have a dc source. The source is connected to the *RC* circuit by a switch that closes at $t = 0$. We assume that the initial voltage across the capacitor just before the switch closes is $v_C(0-) = 0$. Let us solve for the voltage across the capacitor as a function of time.

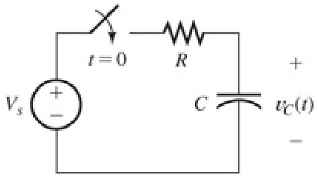


Figure 4.3
Capacitance charging through a resistance. The switch closes at $t = 0$, connecting the dc source V_s to the circuit.

We start by writing a current equation at the node that joins the resistor and the capacitor. This yields

$$C\frac{dv_C(t)}{dt} + \frac{v_C(t) - V_s}{R} = 0 \tag{4.10}$$

The first term on the left-hand side is the current referenced downward through the capacitor. The second term is the current referenced toward the left through the resistor. KCL requires that the currents leaving the node sum to zero.

> $v_c(0-)$ is the voltage across the capacitor the instant before the switch closes (at $t = 0$). Similarly, $v_c(0+)$ is the voltage across the capacitor the instant after the switch closes.

Rearranging **Equation 4.10**, we obtain

$$RC\frac{dv_C(t)}{dt} + v_C(t) = V_s \tag{4.11}$$

As expected, we have obtained a linear first-order differential equation with constant coefficients. As in the previous circuit, the voltage across the capacitance cannot change instantaneously because the voltages are finite, and thus, the current through the resistance (and therefore through the capacitance) is finite. Infinite current is required to change the voltage across a capacitance in an instant. Thus, we have

$$v_C(0+) = v_C(0-) = 0 \tag{4.12}$$

Now, we need to find a solution for $v_C(t)$ that (1) satisfies **Equation 4.11** and (2) matches the initial conditions of the circuit stated in **Equation 4.12**. Notice that **Equation 4.11** is the same as **Equation 4.1**, except for the constant on the right-hand side. Thus, we expect the solution to be the same as for **Equation 4.1**, except for an added constant term. Thus, we are led to try the solution

$$v_C(t) = K_1 + K_2 e^{st} \tag{4.13}$$

in which K_1, K_2, and s are constants to be determined.

If we use **Equation 4.13** to substitute for $v_C(t)$ in **Equation 4.11**, we obtain

$$(1 + RCs) K_2 e^{st} + K_1 = V_s \tag{4.14}$$

For equality, the coefficient of e^{st} must be zero. This leads to

$$s = \frac{-1}{RC} \tag{4.15}$$

From **Equation 4.14**, we also have

$$K_1 = V_s \tag{4.16}$$

Using **Equations 4.15** and **4.16** to substitute into **Equation 4.13**, we obtain

$$v_C(t) = V_s + K_2 e^{-t/RC} \tag{4.17}$$

in which K_2 remains to be determined.

Now, we use the initial condition (**Equation 4.12**) to find K_2. We have

$$v_C(0+) = 0 = V_s + K_2 e^0 = V_s + K_2 \tag{4.18}$$

from which we find $K_2 = -V_s$. Finally, substituting into **Equation 4.17**, we obtain the solution

$$v_C(t) = V_s - V_s e^{-t/RC} \tag{4.19}$$

The second term on the right-hand side is called the **transient response**, which eventually decays to negligible values. The first term on the right-hand side is the **steady-state response**, also called the **forced response**, which persists after the transient has decayed.

When a dc source is contained in the circuit, the total response contains two parts: forced (or steady-state) and transient.

Here again, the product of the resistance and capacitance has units of seconds and is called the time constant $\tau = RC$. Thus, the solution can be written as

$$v_C(t) = V_s - V_s e^{-t/\tau} \tag{4.20}$$

A plot of $v_C(t)$ is shown in **Figure 4.4**. Notice that $v_C(t)$ starts at 0 and approaches the final value V_s asymptotically as *t* becomes large. After one time constant, $v_C(t)$ has reached 63.2 percent of its final value. For practical purposes, $v_C(t)$ is equal to its final value V_s after about five time constants. Then, we say that the circuit has reached steady state.

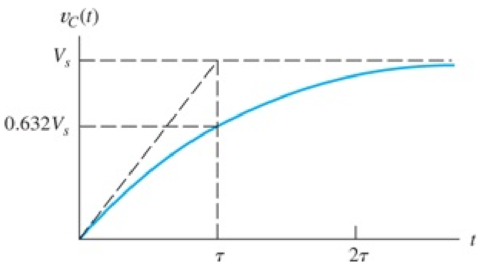


Figure 4.4
The charging transient for the *RC* circuit of **Figure 4.3**.

In the case of a capacitance charging from a dc source through a resistance, a straight line tangent to the start of the transient reaches the final value at one time constant.

It can be shown that if the initial slope of v_C is extended, it intersects the final value at one time constant as shown in **Figure 4.4**.

We have seen in this section that several time constants are needed to charge or discharge a capacitance. This is the main limitation on the speed at which digital computers can process data. In a typical computer, information is represented by voltages that nominally assume values of either $+1.8$ or 0 V, depending on the data represented. When the data change, the voltages must change. It is impossible to build circuits that do not have some capacitance that is charged or discharged when voltages change in value. Furthermore, the circuits always have nonzero resistances that limit the currents available for charging or discharging the capacitances. Therefore, a nonzero time constant is associated with each circuit in the computer, limiting its speed. We will learn more about digital computer circuits in later chapters.

RC transients are the main limitation on the speed at which computer chips can operate.

Example 4.2 First-Order RC Circuit

The switch in the circuit of **Figure 4.5(a)** has been open for a very long time prior to $t=0$ and closes at $t=0$. Find an expression for $v_C(t)$ for $t>0$.

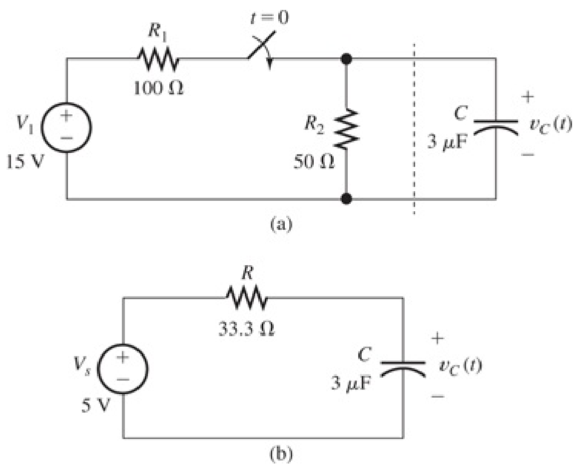


Figure 4.5
Circuit of **Example 4.2**.

Solution

While the switch is open, the capacitance discharges through R_2. Because the switch has been open for a very long time, we conclude that $v_C(0-)=0$. Furthermore, infinite current is not possible in this circuit, so the $v_C(t)$ cannot change instantly. Thus, we conclude that $v_C(0+)=0$.

We can find the Thévenin equivalent circuit for the portion of the circuit on the left hand side of the dotted line shown in **Figure 4.5(a)**. This is the circuit of **Example 2.18** on page **92**. The resulting Thévenin equivalent, with some changes in notation, is shown in **Figure 4.5(b)**.

The circuit in **Figure 4.5(b)** is the same as the circuit of **Figure 4.3**, and the voltage is given by **Equation 4.20**:

$$v_C(t)=V_s-V_s e^{(-t/RC)} \qquad \text{for} \quad t>0$$

in which the time constant is $\tau=RC=(33.3\ \Omega)\times(3\ \mu\text{F})=100\ \mu\text{s}$.

Substituting these values, we have

$$v_C(t)=5-5e^{(-10000t)} \quad \text{V} \quad \text{for} \quad t>0$$

Exercise 4.1

Suppose that $R = 5000\ \Omega$ and $C = 1\ \mu\mathrm{F}$ in the circuit of **Figure 4.1(a)**. Find the time at which the voltage across the capacitor reaches 1 percent of its initial value.

Answer $t = -5\ln(0.01)\ \ \mathrm{ms} \cong 23\ \mathrm{ms}$.

Exercise 4.2

Show that if the initial slope of $v_C(t)$ is extended, it intersects the final value at one time constant, as shown in **Figure 4.4**. [The expression for $v_C(t)$ is given in **Equation 4.20**.]

4.2 DC Steady State

The transient terms in the expressions for currents and voltages in *RLC* circuits decay to zero with time.

The transient terms in the expressions for currents and voltages in *RLC* circuits decay to zero with time. (An exception is *LC* circuits having no resistance.) For dc sources, the steady-state currents and voltages are also constant.

Consider the equation for current through a capacitance:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

If the voltage $v_C(t)$ is constant, the current is zero. In other words, the capacitance behaves as an open circuit. Thus, we conclude that *for steady-state conditions with dc sources, capacitances behave as open circuits.*

Similarly, for an inductance, we have

$$v_L(t) = L\frac{di_L(t)}{dt}$$

When the current is constant, the voltage is zero. Thus, we conclude that *for steady-state conditions with dc sources, inductances behave as short circuits.*

The steps in determining the forced response for *RLC* circuits with dc sources are

1. Replace capacitances with open circuits.
2. Replace inductances with short circuits.
3. Solve the remaining circuit.

These observations give us another approach to finding the steady-state solutions to circuit equations for *RLC* circuits with constant sources. First, we replace the capacitors by open circuits and the inductors by short circuits. The circuit then consists of dc sources and resistances. Finally, we solve the equivalent circuit for the steady-state currents and voltages.

Example 4.3 Steady-State DC Analysis

Find v_x and i_x for the circuit shown in **Figure 4.6(a)** for $t >> 0$.

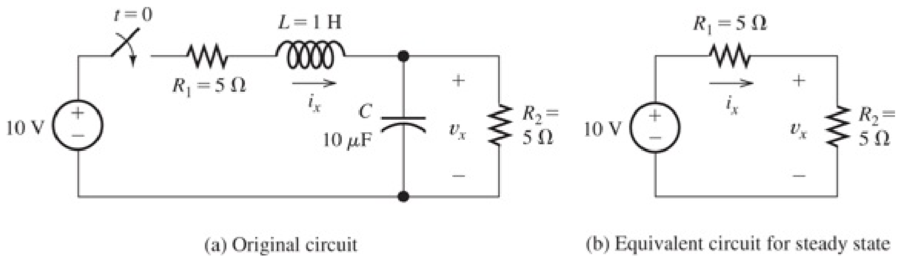


Figure 4.6

The circuit and its dc steady-state equivalent for **Example 4.3**.

Solution

After the switch has been closed a long time, we expect the transient response to have decayed to zero. Then the circuit is operating in dc steady-state conditions. We start our analysis by replacing the inductor by a short circuit and the capacitor by an open circuit. The equivalent circuit is shown in **Figure 4.6(b)**.

Steps 1 and 2.

Step 3.

This resistive circuit is readily solved. The resistances R_1 and R_2 are in series. Thus, we have

$$i_x = \frac{10}{R_1 + R_2} = 1 \text{ A}$$

and

$$v_x = R_2 i_x = 5 \text{ V}$$

Sometimes, we are only interested in the steady-state operation of circuits with dc sources. For example, in analyzing the headlight circuits in an automobile, we are concerned primarily with steady state. On the other hand, we must consider transients in analyzing the operation of the ignition system.

In other applications, we are interested in steady-state conditions with sinusoidal ac sources. For sinusoidal sources, the steady-state currents and voltages are also sinusoidal. In **Chapter 5**, we study a method for solving sinusoidal steady-state circuits that is similar to the method we have presented here for dc steady state. Instead of short and open circuits, we will replace inductances and capacitances by impedances, which are like resistances, except that impedances can have imaginary values.

Exercise 4.3

Solve for the steady-state values of the labeled currents and voltages for the circuits shown in **Figure 4.7**.

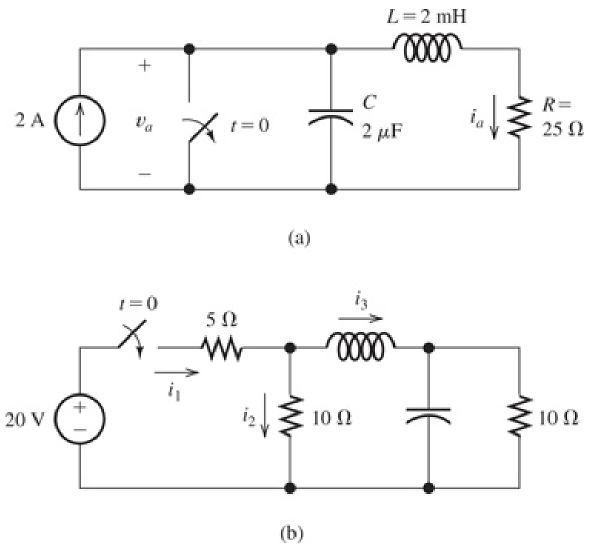


Figure 4.7
Circuits for **Exercise 4.3**.

Answer

a. $v_a = 50\ \text{V},\ i_a = 2\ \text{A};$

b. $i_1 = 2\ \text{A},\ i_2 = 1\ \text{A},\ i_3 = 1\ \text{A}.$

4.3 *RL* Circuits

In this section, we consider circuits consisting of dc sources, resistances, and a single inductance. The methods and solutions are very similar to those we studied for *RC* circuits in **Section 4.1**.

The steps involved in solving simple circuits containing dc sources, resistances, and one energy-storage element (inductance or capacitance) are as follows:

1. Apply Kirchhoff's current and voltage laws to write the circuit equation.
2. If the equation contains integrals, differentiate each term in the equation to produce a pure differential equation.
3. Assume a solution of the form $K_1 + K_2 e^{st}$.
4. Substitute the solution into the differential equation to determine the values of K_1 and *s*. (Alternatively, we can determine K_1 by solving the circuit in steady state as discussed in **Section 4.2**.)
5. Use the initial conditions to determine the value of K_2.
6. Write the final solution.

Example 4.4 *RL* Transient Analysis

Consider the circuit shown in **Figure 4.8**. Find the current *i*(*t*) and the voltage *v*(*t*).

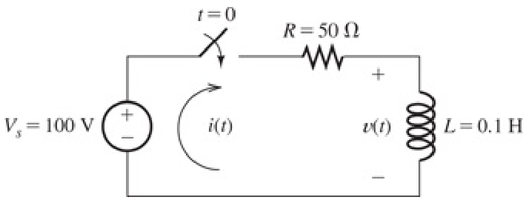


Figure 4.8
The circuit analyzed in **Example 4.4**.

Solution

First, we find the current *i*(*t*). Of course, prior to $t = 0$, the switch is open and the current is zero:

$$i(t) = 0 \qquad \text{for } t < 0 \tag{4.21}$$

After the switch is closed, the current increases in value eventually reaching a steady-state value.

Step 1.

Step 2 is not needed in this case.

Step 3.

Writing a KVL equation around the loop, we have

$$Ri(t) + L\frac{di}{dt} = V_s \tag{4.22}$$

This is very similar to **Equation 4.11**, and we are, therefore, led to try a solution of the same form as that given by **Equation 4.13**. Thus, our trial solution is

$$i(t) = K_1 + K_2 e^{st} \tag{4.23}$$

in which K_1, K_2, and s are constants that need to be determined. Following the procedure used in **Section 4.1**, we substitute the trial solution into the differential equation, resulting in

Step 4.

$$RK_1 + (RK_2 + sLK_2) e^{st} = V_s \tag{4.24}$$

from which we obtain

$$K_1 = \frac{V_s}{R} = 2 \tag{4.25}$$

and

$$s = \frac{-R}{L} \tag{4.26}$$

Substituting these values into **Equation 4.23** results in

$$i(t) = 2 + K_2 e^{-tR/L} \tag{4.27}$$

Step 5.

Next, we use the initial conditions to determine the value of K_2. The current in the inductor is zero prior to $t = 0$ because the switch is open. The applied voltage is finite, and the inductor current must be continuous (because $v_L = L\ di/dt$). Thus, immediately after the switch is closed, the current must be zero. Hence, we have

$$i(0+) = 0 = 2 + K_2 e^0 = 2 + K_2 \tag{4.28}$$

Solving, we find that $K_2 = -2$.

Substituting into **Equation 4.27**, we find that the solution for the current is

$$i(t) = 2 - 2e^{-t/\tau} \qquad \text{for } t > 0 \tag{4.29}$$

Step 6.

in which the time constant is given by

$$\tau = \frac{L}{R} \tag{4.30}$$

A plot of the current versus time is shown in **Figure 4.9(a)**. Notice that the current increases from zero to the steady-state value of 2 A. After five time constants, the current is within 99 percent of the final value. As a check, we verify that the steady-state current is 2 A. (As we saw in **Section 4.2**, this value can be obtained directly by treating the inductor as a short circuit.)

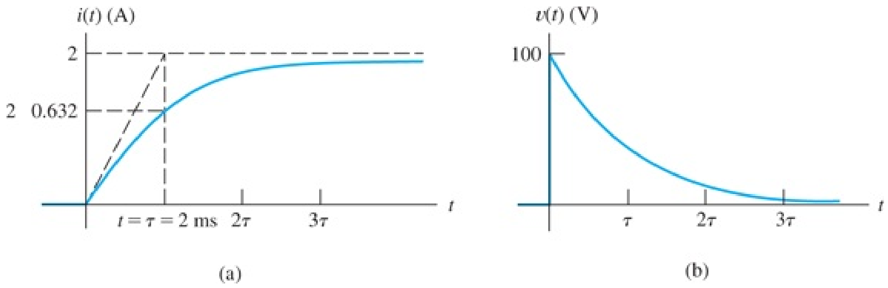


Figure 4.9
Current and voltage versus time for the circuit of **Figure 4.8**.

Now, we consider the voltage *v*(*t*). Prior to $t = 0,$ with the switch open, the voltage is zero.

$$v(t) = 0 \quad \text{for } t < 0 \tag{4.31}$$

After $t = 0,$ *v*(*t*) is equal to the source voltage minus the drop across *R*. Thus, we have

$$v(t) = 100 - 50i(t) \qquad \text{for } t > 0 \tag{4.32}$$

Substituting the expression found earlier for *i*(*t*), we obtain

$$v(t) = 100e^{-t/\tau} \tag{4.33}$$

A plot of *v*(*t*) is shown in **Figure 4.9(b)**.

At $t = 0,$ the voltage across the inductor jumps from 0 to 100 V. As the current gradually increases, the drop across the resistor increases, and the voltage across the inductor falls. In steady state, we have $v(t) = 0$ because the inductor behaves as a short circuit.

After solving several circuits with a single energy-storage element, we can use our experience to skip some of the steps listed earlier in the section. We illustrate this in the next example.

Example 4.5 *RL* Transient Analysis

Consider the circuit shown in **Figure 4.10** in which V_s is a dc source. Assume that the circuit is in steady state with the switch closed prior to $t = 0$. Find expressions for the current *i*(*t*) and the voltage *v*(*t*).

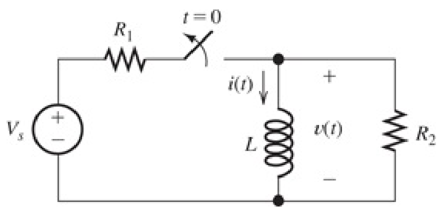


Figure 4.10
The circuit analyzed in **Example 4.5**.

First, we use dc steady-state analysis to determine the current before the switch opens.

Solution

Prior to $t = 0,$ the inductor behaves as a short circuit. Thus, we have

$$v(t) = 0 \quad \text{for } t < 0$$

and

$$i(t) = \frac{V_s}{R_1} \quad \text{for } t < 0$$

Before the switch opens, current circulates clockwise through V_s, R_1, and the inductance. When the switch opens, current continues to flow through the inductance, but the return path is through R_2. Then, a voltage appears across R_2 and the inductance, causing the current to decay.

> After the switch opens, the source is disconnected from the circuit, so the steady-state solution for $t > 0$ is zero.

Since there are no sources driving the circuit after the switch opens, the steady-state solution is zero for $t > 0$. Hence, the solution for *i*(*t*) is given by

$$i(t) = Ke^{-t/\tau} \quad \text{for } t > 0 \tag{4.34}$$

in which the time constant is

$$\tau = \frac{L}{R_2} \tag{4.35}$$

Unless an infinite voltage appears across the inductance, the current must be continuous. Recall that prior to $t = 0$, $i(t) = V_s/R_1$. Consequently, just after the switch opens, we have

$$i(0+) = \frac{V_s}{R_1} = Ke^{-0} = K$$

Substituting the value of *K* into **Equation 4.34**, we find that the current is

$$i(t) = \frac{V_s}{R_1} e^{-t/\tau} \quad \text{for } t > 0 \tag{4.36}$$

The voltage is given by

$$\begin{aligned} v(t) &= L\frac{di(t)}{dt} \\ &= 0 \quad \text{for } t < 0 \\ &= -\frac{LV_s}{R_1\tau} e^{-t/\tau} \quad \text{for } t > 0 \end{aligned}$$

Plots of the voltage and current are shown in **Figure 4.11**.

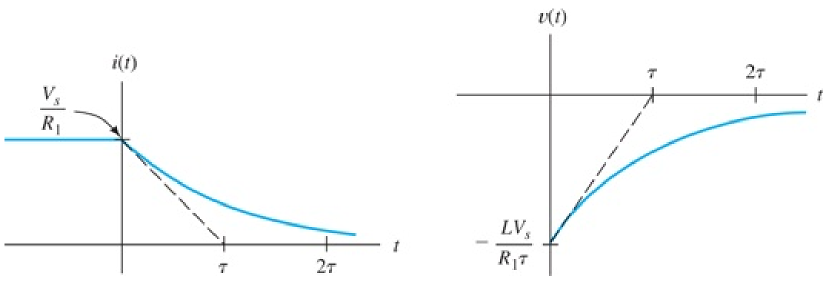


Figure 4.11
The current and voltage for the circuit of **Figure 4.10**.

Exercise 4.4

For the circuit of **Example 4.5** (**Figure 4.10**), assume that $V_s = 15\ \mathrm{V},\ R_1 = 10\ \Omega,\ R_2 = 100\ \Omega,$ and $L = 0.1\ \mathrm{H}$.

a. What is the value of the time constant (after the switch opens)?

b. What is the maximum magnitude of *v*(*t*)?

c. How does the maximum magnitude of *v*(*t*) compare to the source voltage?

d. Find the time *t* at which *v*(*t*) is one-half of its value immediately after the switch opens.

Answer

a. $\tau = 1\ \mathrm{ms};$

b. $q\,v(\ t)\ q_{\max} = 150\ \mathrm{V};$

c. the maximum magnitude of *v*(*t*) is 10 times the value of V_s;

d. $t = \tau \ln(\ 2) \ = 0.693\ \mathrm{ms}.$

Exercise 4.5

Consider the circuit shown in **Figure 4.12**, in which the switch opens at $t = 0$. Find expressions for *v*(*t*), $i_R(\ t)$, and $i_L(\ t)$ for $t > 0$. Assume that $i_L(\ t)$ is zero before the switch opens.

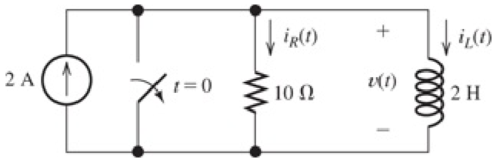


Figure 4.12
The circuit for **Exercise 4.5**.

Answer

$$v(\ t) \ = 20e^{-t/0.2},\ i_R(\ t) \ = 2e^{-t/0.2},\ i_L(\ t) \ = 2 - 2e^{-t/0.2}.$$

Exercise 4.6

Consider the circuit shown in **Figure 4.13**. Assume that the switch has been closed for a very long time prior to $t = 0$. Find expressions for *i*(*t*) and *v*(*t*).

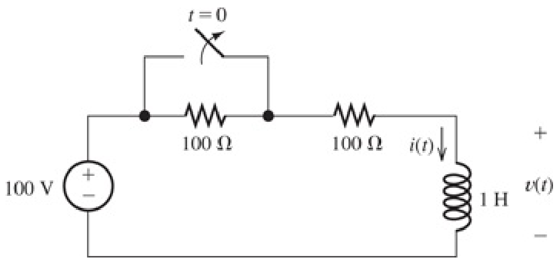


Figure 4.13

The circuit for **Exercise 4.6**.

Answer

$$
\begin{aligned}
i(t) &= 1.0 \qquad \text{for } t < 0 \\
&= 0.5 + 0.5e^{-t/\tau} \qquad \text{for } t > 0 \\
v(t) &= 0 \qquad \text{for } t < 0 \\
&= -100e^{-t/\tau} \qquad \text{for } t > 0
\end{aligned}
$$

where the time constant is $\tau = 5\ \mathrm{ms}$.

4.4 *RC* and *RL* Circuits with General Sources

Now that we have gained some familiarity with *RL* and *RC* circuits, we discuss their solution in general. In this section, we treat circuits that contain one energy-storage element, either an inductance or a capacitance.

Consider the circuit shown in **Figure 4.14(a)**. The circuit inside the box can be any combination of resistances and sources. The single inductance *L* is shown explicitly. Recall that we can find a Thévenin equivalent for circuits consisting of sources and resistances. The Thévenin equivalent is an independent voltage source $v_t(t)$ in series with the Thévenin resistance *R*. Thus, any circuit composed of sources, resistances, and one inductance has the equivalent circuit shown in **Figure 4.14(b)**. (Of course, we could reduce any circuit containing sources, resistances, and a single capacitance in a similar fashion.)

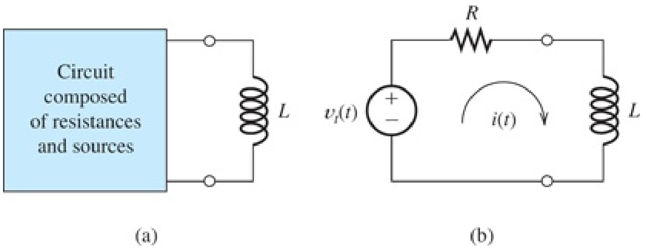


Figure 4.14
A circuit consisting of sources, resistances, and one inductance has an equivalent circuit consisting of a voltage source and a resistance in series with the inductance.

Writing a KVL equation for **Figure 4.14(b)**, we obtain

$$L\frac{di(t)}{dt} + Ri(t) = v_t(t) \tag{4.37}$$

If we divide through by the resistance *R*, we have

$$\frac{L}{R}\frac{di(t)}{dt} + i(t) = \frac{v_t(t)}{R} \tag{4.38}$$

In general, the equation for any circuit containing one inductance or one capacitance can be put into the form

$$\tau\frac{dx(t)}{dt} + x(t) = f(t), \tag{4.39}$$

in which *x*(*t*) represents the current or voltage for which we are solving. Then, we need to find solutions to **Equation 4.39** that are consistent with the initial conditions (such as the initial current in the inductance).

The constant τ (which turns out to be the time constant) is a function of only the resistances and the inductance (or capacitance). The sources result in the term *f*(*t*), which is called the **forcing function**. If we have a circuit without sources (such as **Figure 4.1**), the forcing function is zero. For dc sources, the forcing function is constant.

Equation 4.39 is called a first-order differential equation because the highest-order derivative is first order. It is a linear equation because it does not involve powers or other nonlinear functions of *x*(*t*) or its derivatives. Thus, to solve an *RL* (or *RC*) circuit, we must find the general solution of a linear first-order differential equation with constant coefficients.

Solution of the Differential Equation

An important result in differential equations states that the general solution to **Equation 4.39** consists of two parts. The first part is called the **particular solution** $x_p(t)$ and is any expression that satisfies **Equation 4.39**. Thus,

$$\tau \frac{dx_p(t)}{dt} + x_p(t) = f(t) \tag{4.40}$$

The particular solution is also called the **forced response** because it depends on the forcing function (which in turn is due to the independent sources).

> The general solution to **Equation 4.39** consists of two parts.

> The particular solution (also called the forced response) is any expression that satisfies the equation.

> In order to have a solution that satisfies the initial conditions, we must add the complementary solution to the particular solution.

Even though the particular solution satisfies the differential equation, it may not be consistent with the initial conditions, such as the initial voltage on a capacitance or current through an inductance. By adding another term, known as the complementary solution, we obtain a general solution that satisfies both the differential equation and meets the initial conditions.

For the forcing functions that we will encounter, we can often select the form of the particular solution by inspection. Usually, the particular solution includes terms with the same functional forms as the terms found in the forcing function and its derivatives.

Sinusoidal functions of time are one of the most important types of forcing functions in electrical engineering. For example, consider the forcing function

$$f(t) = 10\cos(200t)$$

Because the derivatives of sine and cosine functions are also sine and cosine functions, we would try a particular solution of the form

$$x_p(t) = A\cos(200t) + B\sin(200t)$$

where *A* and *B* are constants that must be determined. We find these constants by substituting the proposed solution into the differential equation and requiring the two sides of the equation to be identical. This leads to equations that can be solved for *A* and *B*. (In **Chapter 5**, we study shortcut methods for solving for the forced response of circuits with sinusoidal sources.)

The second part of the general solution is called the **complementary solution** $x_c(t)$ and is the solution of the **homogeneous equation**

$$\tau \frac{dx_c(t)}{dt} + x_c(t) = 0 \tag{4.41}$$

The homogeneous equation is obtained by setting the forcing function to zero.

We obtain the homogeneous equation by setting the forcing function to zero. Thus, the form of the complementary solution does not depend on the sources. It is also called the **natural response** because it depends on the passive circuit elements. The complementary solution must be added to the particular solution in order to obtain a general solution that matches the initial values of the currents and voltages.

The complementary solution (also called the natural response) is obtained by solving the homogeneous equation.

We can rearrange the homogeneous equation into this form:

$$\frac{dx_c(t)/dt}{x_c(t)} = \frac{-1}{\tau} \tag{4.42}$$

Integrating both sides of **Equation 4.42**, we have

$$\ln[x_c(t)] = \frac{-t}{\tau} + c \tag{4.43}$$

in which *c* is the constant of integration. **Equation 4.43** is equivalent to

$$x_c(t) = e^{(-t/\tau + c)} = e^c e^{-t/\tau}$$

Then, if we define $K = e^c$, we have the complementary solution

$$x_c(t) = Ke^{-t/\tau} \tag{4.44}$$

Step-by-Step Solution

Next, we summarize an approach to solving circuits containing a resistance, a source, and an inductance (or a capacitance):

1. Write the circuit equation and reduce it to a first-order differential equation.
2. Find a particular solution. The details of this step depend on the form of the forcing function. We illustrate several types of forcing functions in examples, exercises, and problems.
3. Obtain the complete solution by adding the particular solution to the complementary solution given by **Equation 4.44**, which contains the arbitrary constant *K*.
4. Use initial conditions to find the value of *K*.

We illustrate this procedure with an example.

Example 4.6 Transient Analysis of an RC Circuit with a Sinusoidal Source

Solve for the current in the circuit shown in **Figure 4.15**. The capacitor is initially charged so that $v_C(0+) = 1\ \mathrm{V}$.

Figure 4.15

A first-order *RC* circuit with a sinusoidal source. See **Example 4.6**.

Step 1: Write the circuit equation and reduce it to a first-order differential equation.

Solution

First, we write a voltage equation for $t > 0$. Traveling clockwise and summing voltages, we obtain

$$Ri(t) + \frac{1}{C}\int_0^t i(t)\ dt + v_C(0) - 2\sin(200t) = 0$$

We convert this to a differential equation by taking the derivative of each term. Of course, the derivative of the integral is simply the integrand. Because $v_C(0)$ is a constant, its derivative is zero. Thus, we have

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 400\cos(200t) \tag{4.45}$$

Multiplying by *C*, we get

$$RC\frac{di(t)}{dt} + i(t) = 400\ C\cos(200t) \tag{4.46}$$

Substituting values for *R* and *C*, we obtain

$$5\times10^{-3}\frac{di(t)}{dt} + i(t) = 400\times10^{-6}\cos(200t) \tag{4.47}$$

Step 2: Find a particular solution.

The particular solution for a sinusoidal forcing function always has the form given by **Equation 4.48**.

The second step is to find a particular solution $i_p(t)$. Often, we start by guessing at the form of $i_p(t)$, possibly including some unknown constants. Then, we substitute our guess into the differential equation and solve for the constants. In the present case, since the derivatives of sin(200*t*) and cos(200*t*) are 200 cos(200*t*) and $-200\sin(200t)$, respectively, we try a particular solution of the form

$$i_p(t) = A\cos(200t) + B\sin(200t) \tag{4.48}$$

where *A* and *B* are constants to be determined so that i_p is indeed a solution to **Equation 4.47**.

We substitute **Equation 4.48** into the differential equation, and solve for *A* and *B*.

Substituting the proposed solution into **Equation 4.47**, we obtain

$$-A\sin(200t)+B\cos(200t)+A\cos(200t)+B\sin(200t) \\ = 400\times10^{-6}\cos(200t)$$

However, the left-hand side of this equation is required to be identical to the right-hand side. Equating the coefficients of the sine functions, we have

$$-A+B=0 \tag{4.49}$$

Equating the coefficients of the cosine functions, we get

$$B+A=400\times10^{-6} \tag{4.50}$$

These equations can be readily solved, yielding

$$A=200\times10^{-6}=200\ \mu\text{A}$$

and

$$B=200\times10^{-6}=200\ \mu\text{A}$$

Substituting these values into **Equation 4.48**, we obtain the particular solution

$$i_p(t)=200\cos(200t)+200\sin(200t)\quad\mu\text{A} \tag{4.51}$$

which can also be written as

$$i_p(t)=200\sqrt{2}\cos(200t-45°)$$

(In **Chapter 5**, we will learn shortcut methods for combining sine and cosine functions.)

We obtain the homogeneous equation by substituting 0 for the forcing function in **Equation 4.46**. Thus, we have

$$RC\frac{di(t)}{dt}+i(t)=0 \tag{4.52}$$

The complementary solution is

$$i_c(t)=Ke^{-t/RC}=Ke^{-t/\tau} \tag{4.53}$$

Adding the particular solution and the complementary solution, we obtain the general solution

$$i(t)=200\cos(200t)+200\sin(200t)+Ke^{-t/RC}\ \mu\text{A} \tag{4.54}$$

Step 3: Obtain the complete solution by adding the particular solution to the complementary solution.

Step 4: Use initial conditions to find the value of *K*.

Finally, we determine the value of the constant K by using the initial conditions. The voltages and currents immediately after the switch closes are shown in **Figure 4.16**. The source voltage is 0 V and the voltage across the capacitor is $v_C(0+) = 1$. Consequently, the voltage across the resistor must be $v_R(0+) = -1\ \mathrm{V}$. Thus, we get

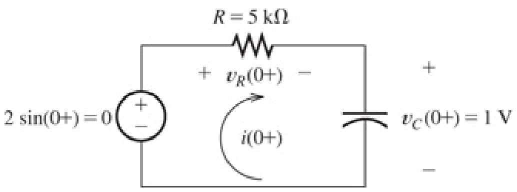


Figure 4.16

The voltages and currents for the circuit of **Figure 4.15** immediately after the switch closes.

$$i(0+) = \frac{v_R(0+)}{R} = \frac{-1}{5000} = -200\ \mu\mathrm{A}$$

Substituting $t = 0$ into **Equation 4.54**, we obtain

$$i(0+) = -200 = 200 + K\ \mu\mathrm{A} \tag{4.55}$$

Solving, we find that $K = -400\ \mu\mathrm{A}$. Substituting this into **Equation 4.54**, we have the solution

$$i(t) = 200\cos(200t) + 200\sin(200t) - 400e^{-t/RC}\ \mu\mathrm{A} \tag{4.56}$$

Plots of the particular solution and of the complementary solution are shown in **Figure 4.17**. The time constant for this circuit is $\tau = RC = 5\ \mathrm{ms}$. Notice that the natural response decays to negligible values in about 25 ms. As expected, the natural response has decayed in about five time constants. Furthermore, notice that for a sinusoidal forcing function, the forced response is also sinusoidal and persists after the natural response has decayed.

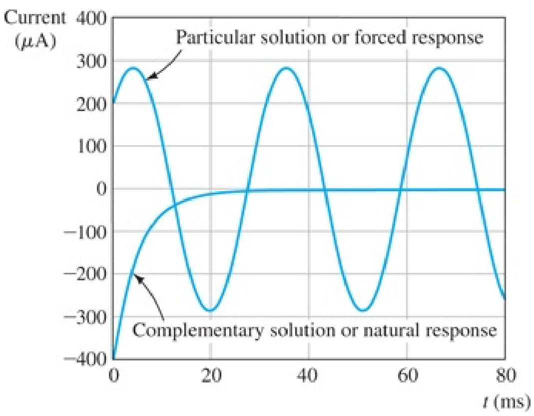


Figure 4.17

The complementary solution and the particular solution for **Example 4.6**.

> Notice that the forced response is sinusoidal for a sinusoidal forcing function.

A plot of the complete solution is shown in **Figure 4.18**.

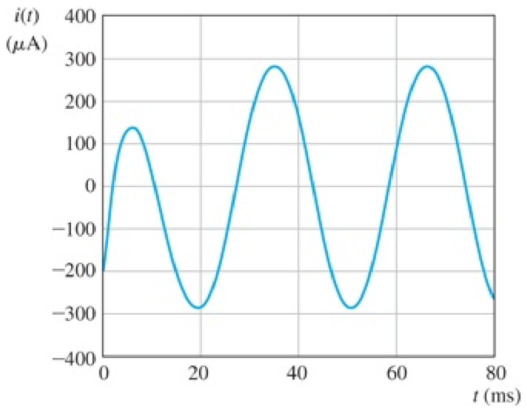


Figure 4.18
The complete solution for **Example 4.6**.

Exercise 4.7

Repeat **Example 4.6** if the source voltage is changed to $2\cos(200t)$ and the initial voltage on the capacitor is $v_C(0) = 0$. The circuit with these changes is shown in **Figure 4.19**.

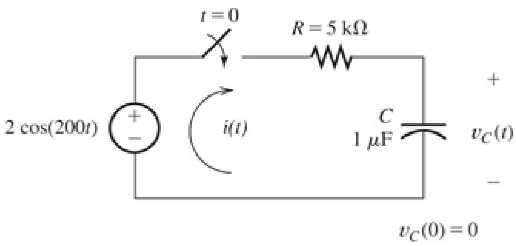


Figure 4.19
The circuit for **Exercise 4.7**.

Answer $i(t) = -200\sin(200t) + 200\cos(200t) + 200e^{-t/RC}\ \mu\text{A},$ in which $\tau = RC = 5\ \text{ms}.$

Exercise 4.8

Solve for the current in the circuit shown in **Figure 4.20** after the switch closes. [*Hint:* Try a particular solution of the form $i_p(t) = Ae^{-t}$.]

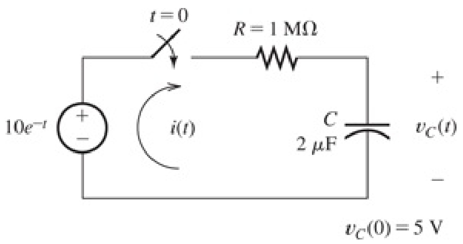


Figure 4.20
The circuit for **Exercise 4.8**.

Answer $i(t) = 20e^{-t} - 15e^{-t/2}\ \mu\text{A}.$

4.5 Second-Order Circuits

In this section, we consider circuits that contain two energy-storage elements. In particular, we look at circuits that have one inductance and one capacitance, either in series or in parallel.

Differential Equation

To derive the general form of the equations that we encounter in circuits with two energy-storage elements, consider the series circuit shown in **Figure 4.21(a)**. Writing a KVL equation, we have

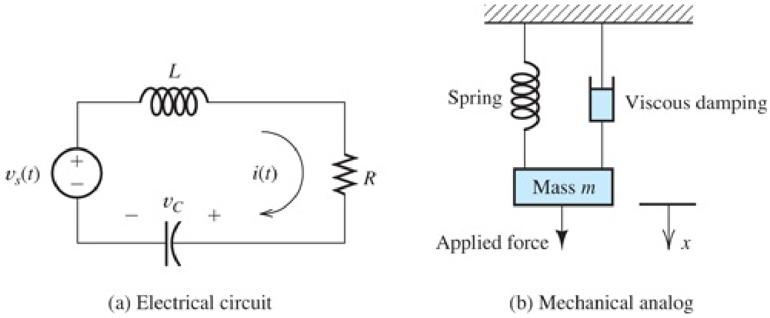


Figure 4.21
The series *RLC* circuit and its mechanical analog.

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0) = v_s(t) \tag{4.57}$$

Taking the derivative with respect to time, we get

> We convert the integrodifferential equation to a pure differential equation by differentiating with respect to time.

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv_s(t)}{dt} \tag{4.58}$$

Dividing through by *L*, we obtain

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt} \tag{4.59}$$

Now, we define the **damping coefficient** as

$$\alpha = \frac{R}{2L} \tag{4.60}$$

and the **undamped resonant frequency** as

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{4.61}$$

The **forcing function** is

$$f(t) = \frac{1}{L}\frac{dv_s(t)}{dt} \tag{4.62}$$

Using these definitions, we find that **Equation 4.59** can be written as

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = f(t) \tag{4.63}$$

This is a linear second-order differential equation with constant coefficients. Thus, we refer to circuits having two energy-storage elements as second-order circuits. (An exception occurs if we can combine the energy-storage elements in series or parallel. For example, if we have two capacitors in parallel, we can combine them into a single equivalent capacitance, and then we would have a first-order circuit.)

If a circuit contains two energy-storage elements (after substituting all possible series or parallel equivalents), the circuit equations can always be reduced to the form given by **Equation 4.63**.

Mechanical Analog

The mechanical analog of the series *RLC* circuit is shown in **Figure 4.21(b)**. The displacement *x* of the mass is analogous to electrical charge, the velocity *dx/dt* is analogous to current, and force is analogous to voltage. The mass plays the role of the inductance, the spring plays the role of the capacitance, and the damper plays the role of the resistance. The equation of motion for the mechanical system can be put into the form of **Equation 4.63**.

Based on an intuitive consideration of **Figure 4.21**, we can anticipate that the sudden application of a constant force (dc voltage) can result in a displacement (current) that either approaches steady-state conditions asymptotically or oscillates before settling to the steady-state value. The type of behavior depends on the relative values of the mass, spring constant, and damping coefficient.

Solution of the Second-Order Equation

We will see that the circuit equations for currents and voltages in circuits having two energy-storage elements can always be put into the form of **Equation 4.63**. Thus, let us consider the solution of

$$\frac{d^2 x(t)}{dt^2} + 2\alpha \frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t) \tag{4.64}$$

where we have used *x*(*t*) for the variable, which could represent either a current or a voltage.

Here again, the general solution *x*(*t*) to this equation consists of two parts: a particular solution $x_p(t)$ plus the complementary solution $x_c(t)$ and is expressed as

$$x(t) = x_p(t) + x_c(t) \tag{4.65}$$

Particular Solution.

The particular solution is any expression $x_p(t)$ that satisfies the differential equation

$$\frac{d^2x_p(t)}{dt^2} + 2\alpha\frac{dx_p(t)}{dt} + \omega_0^2 x_p(t) = f(t) \tag{4.66}$$

The particular solution is also called the **forced response**. (Usually, we eliminate any terms from $x_p(t)$ that produce a zero net result when substituted into the left-hand side of **Equation 4.66**. In other words, we eliminate any terms that have the same form as the homogeneous solution.)

> For dc sources, we can find the particular solution by performing a dc steady-state analysis as discussed in **Section 4.2**.

We will be concerned primarily with either constant (dc) or sinusoidal (ac) forcing functions. For dc sources, we can find the particular solution directly from the circuit by replacing the inductances by short circuits, replacing the capacitances by open circuits, and solving. This technique was discussed in **Section 4.2**. In **Chapter 5**, we will learn efficient methods for finding the forced response due to sinusoidal sources.

Complementary Solution.

The complementary solution $x_c(t)$ is found by solving the homogeneous equation, which is obtained by substituting 0 for the forcing function *f*(*t*). Thus, the homogeneous equation is

$$\frac{d^2x_c(t)}{dt^2} + 2\alpha\frac{dx_c(t)}{dt} + \omega_0^2 x_c(t) = 0 \tag{4.67}$$

In finding the solution to the homogeneous equation, we start by substituting the trial solution $x_c(t) = Ke^{st}$. This yields

$$s^2Ke^{st} + 2\alpha sKe^{st} + \omega_0^2 Ke^{st} = 0 \tag{4.68}$$

Factoring, we obtain

$$\left(s^2 + 2\alpha s + \omega_0^2\right) Ke^{st} = 0 \tag{4.69}$$

Since we want to find a solution Ke^{st} that is nonzero, we must have

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{4.70}$$

This is called the **characteristic equation**.

The **damping ratio** is defined as

$$\zeta = \frac{\alpha}{\omega_0} \tag{4.71}$$

The form of the complementary solution depends on the value of the damping ratio. The roots of the characteristic equation are given by

> The form of the complementary solution depends on the value of the damping ratio.

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \tag{4.72}$$

and

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \tag{4.73}$$

We have three cases depending on the value of the damping ratio ζ compared with unity.

1. *Overdamped case* $(\zeta > 1)$. If $\zeta > 1$ (or equivalently, if $\alpha > \omega_0$), the roots of the characteristic equation are real and distinct. Then the complementary solution is
$$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{4.74}$$

   In this case, we say that the circuit is **overdamped**.

   > If the damping ratio is greater than unity, we say that the circuit is overdamped, the roots of the characteristic equation are real, and the complementary solution has the form given in **Equation 4.74**.

2. *Critically damped case* $(\zeta = 1)$. If $\zeta = 1$ (or equivalently, if $\alpha = \omega_0$), the roots are real and equal. Then, the complementary solution is
$$x_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t} \tag{4.75}$$

   > If the damping ratio equals unity, the circuit is critically damped, the roots of the characteristic equation are real and equal, and the complementary solution has the form given in **Equation 4.75**.

   In this case, we say that the circuit is **critically damped**.
3. *Underdamped case* $(\zeta < 1)$. Finally, if $\zeta < 1$ (or equivalently, if $\alpha < \omega_0$), the roots are complex. (By the term *complex*, we mean that the roots involve the imaginary number $\sqrt{-1}$.) In other words, the roots are of the form
$$s_1 = -\alpha + j\omega_n \quad \text{and} \quad s_2 = -\alpha - j\omega_n$$

   in which $j = \sqrt{-1}$ and the **natural frequency** is given by
$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} \tag{4.76}$$

   (In electrical engineering, we use *j* rather than *i* to stand for the imaginary number $\sqrt{-1}$ because we use *i* for current.)

> If the damping ratio is less than unity, the roots of the characteristic equation are complex conjugates, and the complementary solution has the form given in **Equation 4.77**.

For complex roots, the complementary solution is of the form

$$x_c(t) = K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t) \tag{4.77}$$

In this case, we say that the circuit is **underdamped**.

Example 4.7 Analysis of a Second-Order Circuit with a DC Source

A dc source is connected to a series *RLC* circuit by a switch that closes at $t = 0$ as shown in **Figure 4.22**. The initial conditions are $i(0) = 0$ and $v_C(0) = 0$. Write the differential equation for $v_C(t)$. Solve for $v_C(t)$ if $R = 300$, 200, and $100\ \Omega$.

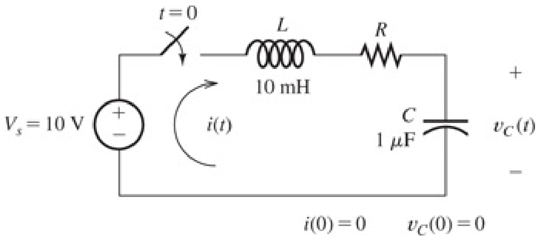


Figure 4.22
The circuit for **Example 4.7**.

Solution
First, we can write an expression for the current in terms of the voltage across the capacitance:

$$i(t) = C\frac{dv_C(t)}{dt} \tag{4.78}$$

First, we write the circuit equations and reduce them to the form given in **Equation 4.63**.

Then, we write a KVL equation for the circuit:

$$L\frac{di(t)}{dt} + Ri(t) + v_C(t) = V_s \tag{4.79}$$

Using **Equation 4.78** to substitute for *i(t)*, we get

$$LC\frac{d^2v_C(t)}{dt^2} + RC\frac{dv_C(t)}{dt} + v_C(t) = V_s \tag{4.80}$$

Dividing through by *LC*, we have

$$\frac{d^2v_C(t)}{dt^2} + \frac{R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = \frac{V_s}{LC} \tag{4.81}$$

As expected, the differential equation for $v_C(t)$ has the same form as **Equation 4.63**.

Next, we find the particular solution by solving the circuit for dc steady-state conditions.

Next, we find the particular solution. Since we have a dc source, we can find this part of the solution by replacing the inductance by a short circuit and the capacitance by an open circuit. This is shown in **Figure 4.23**. Then the current is zero, the drop across the resistance is zero, and the voltage across the capacitance (open circuit) is equal to the dc source voltage. Therefore, the particular solution is

Figure 4.23

The equivalent circuit for **Figure 4.22** under steady-state conditions. The inductor has been replaced by a short circuit and the capacitor by an open circuit.

$$v_{Cp}(t) = V_s = 10\ \text{V} \tag{4.82}$$

(It can be verified that this is a particular solution by substituting it into **Equation 4.81**.) Notice that in this circuit the particular solution for $v_C(t)$ is the same for all three values of resistance.

Next, we find the complementary solution for each value of *R*. For each resistance value, we

1. Determine the damping ratio and roots of the characteristic equation.
2. Select the appropriate form for the homogeneous solution, depending on the value of the damping ratio.
3. Add the homogeneous solution to the particular solution and determine the values of the coefficients (K_1 and K_2), based on the initial conditions.

Next, we find the homogeneous solution and general solution for each value of *R*. For all three cases, we have

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10^4 \tag{4.83}$$

Case I

$(R = 300\ \Omega)$

In this case, we get

$$\alpha = \frac{R}{2L} = 1.5 \times 10^4 \tag{4.84}$$

The damping ratio is $\zeta = \alpha/\omega_0 = 1.5$. Because we have $\zeta > 1$, this is the overdamped case. The roots of the characteristic equation are given by **Equations 4.72** and **4.73**. Substituting values, we find that

$$\begin{aligned} s_1 &= -\alpha + \sqrt{\alpha^2 - \omega_0^2} \\ &= -1.5 \times 10^4 + \sqrt{(1.5 \times 10^4)^2 - (10^4)^2} \\ &= -0.3820 \times 10^4 \end{aligned}$$

and

$$\begin{aligned} s_2 &= -\alpha - \sqrt{\alpha^2 - \omega_0^2} \\ &= -2.618 \times 10^4 \end{aligned}$$

The homogeneous solution has the form of **Equation 4.74**. Adding the particular solution given by **Equation 4.82** to the homogeneous solution, we obtain the general solution

$$v_C(t) = 10 + K_1 e^{s_1 t} + K_2 e^{s_2 t} \tag{4.85}$$

Now, we must find values of K_1 and K_2 so the solution matches the known initial conditions in the circuit. It was given that the initial voltage on the capacitance is zero. Hence,

$$v_C(0) = 0$$

Evaluating Equation 4.85 at $t = 0,$ we obtain

$$10 + K_1 + K_2 = 0 \tag{4.86}$$

Furthermore, the initial current was given as $i(0) = 0$. Since the current through the capacitance is given by

$$i(t) = C\frac{dv_C(t)}{dt}$$

we conclude that

$$\frac{dv_C(0)}{dt} = 0$$

Taking the derivative of Equation 4.85 and evaluating at $t = 0,$ we have

$$s_1 K_1 + s_2 K_2 = 0 \tag{4.87}$$

Now, we can solve Equations 4.86 and 4.87 for the values of K_1 and K_2. The results are $K_1 = -11.708$ and $K_2 = 1.708$. Substituting these values into Equation 4.85, we have the solution

$$v_C(t) = 10 - 11.708 e^{s_1 t} + 1.708 e^{s_2 t}$$

Plots of each of the terms of this equation and the complete solution are shown in Figure 4.24.

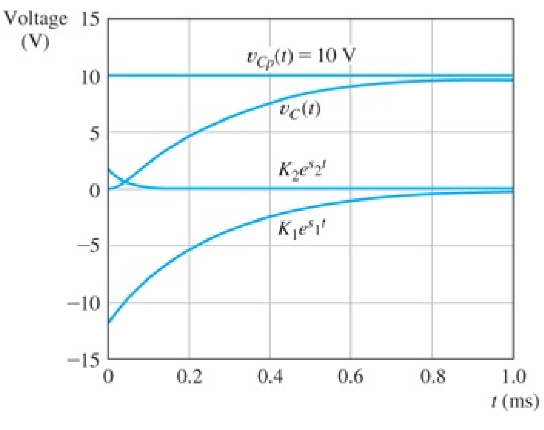


Figure 4.24

Solution for $R = 300\ \Omega$.

Case II

$(R = 200\ \Omega)$

In this case, we get

$$\alpha = \frac{R}{2L} = 10^4 \tag{4.88}$$

Now, we repeat the steps for $R = 200\,\Omega$

Because $\zeta = \alpha/\omega_0 = 1,$ this is the critically damped case. The roots of the characteristic equation are given by **Equations 4.72** and **4.73**. Substituting values, we have

$$s_1 = s_2 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha = -10^4$$

The homogeneous solution has the form of **Equation 4.75**. Adding the particular solution (**Equation 4.82**) to the homogeneous solution, we find that

$$v_C(t) = 10 + K_1 e^{s_1 t} + K_2 t e^{s_1 t} \tag{4.89}$$

As in case I, the initial conditions require $v_C(0) = 0$ and $dv_C(0)/dt = 0$. Thus, substituting $t = 0$ into **Equation 4.89**, we get

$$10 + K_1 = 0 \tag{4.90}$$

Differentiating **Equation 4.89** and substituting $t = 0$ yields

$$s_1 K_1 + K_2 = 0 \tag{4.91}$$

Solving **Equations 4.90** and **4.91** yields $K_1 = -10$ and $K_2 = -10^5.$ Thus, the solution is

$$v_C(t) = 10 - 10e^{s_1 t} - 10^5 t e^{s_1 t} \tag{4.92}$$

Plots of each of the terms of this equation and the complete solution are shown in **Figure 4.25**.

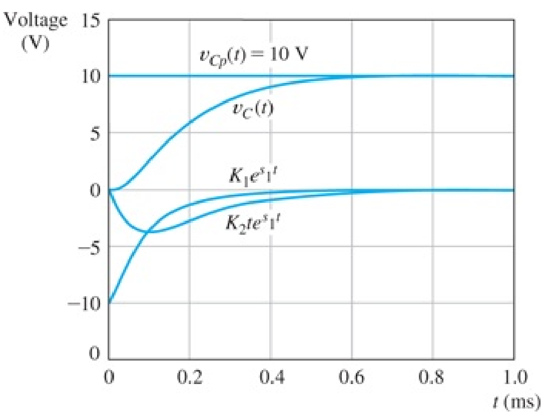


Figure 4.25
Solution for $R = 200\ \Omega$.

Case III

$(R = 100\ \Omega)$

For this value of resistance, we have

$$\alpha = \frac{R}{2L} = 5000 \tag{4.93}$$

Finally, we repeat the solution for $R = 100\,\Omega$.

Because $\zeta = \alpha / \omega_0 = 0.5,$ this is the underdamped case. Using **Equation 4.76**, we compute the natural frequency:

$$\omega_n = \sqrt{\omega_0^2 - \alpha^2} = 8660 \tag{4.94}$$

The homogeneous solution has the form of **Equation 4.77**. Adding the particular solution found earlier to the homogeneous solution, we obtain the general solution:

$$v_C(t) = 10 + K_1 e^{-\alpha t} \cos(\omega_n t) + K_2 e^{-\alpha t} \sin(\omega_n t) \tag{4.95}$$

As in the previous cases, the initial conditions are $v_C(0) = 0$ and $dv_C(0)/dt = 0$. Evaluating **Equation 4.95** at $t = 0,$ we obtain

$$10 + K_1 = 0 \tag{4.96}$$

Differentiating **Equation 4.95** and evaluating at $t = 0,$ we have

$$-\alpha K_1 + \omega_n K_2 = 0 \tag{4.97}$$

Solving **Equations 4.96** and **4.97**, we obtain $K_1 = -10$ and $K_2 = -5.774$. Thus, the complete solution is

$$v_C(t) = 10 - 10e^{-\alpha t} \cos(\omega_n t) - 5.774e^{-\alpha t} \sin(\omega_n t) \tag{4.98}$$

Plots of each of the terms of this equation and the complete solution are shown in **Figure 4.26**.

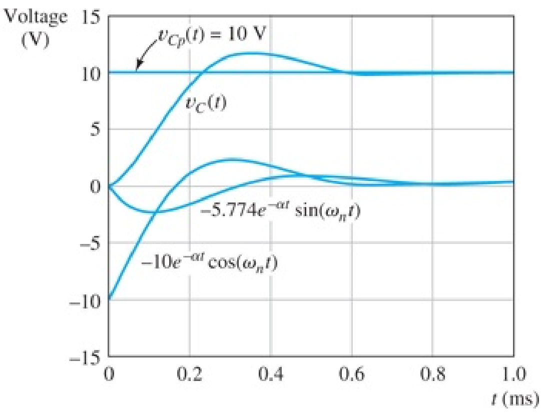


Figure 4.26
Solution for $R = 100\ \Omega$.

Figure 4.27 shows the complete response for all three values of resistance.

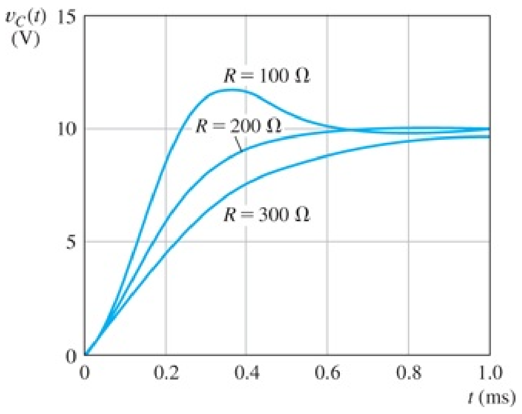


Figure 4.27

Solutions for all three resistances.

Normalized Step Response of Second-Order Systems

When we suddenly apply a constant source to a circuit, we say that the forcing function is a **step function**. A unit step function, denoted by *u*(*t*), is shown in **Figure 4.28**. By definition, we have

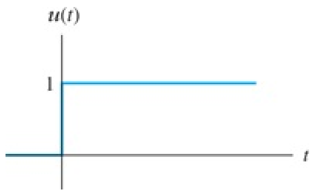


Figure 4.28

A unit step function *u*(*t*). For $t < 0,\ u(t) = 0$. For $t \geq 0,\ u(t) = 1$.

$$\begin{aligned} u(t) &= 0 \quad t < 0 \\ &= 1 \quad t \geq 0 \end{aligned}$$

For example, if we apply a dc voltage of *A* volts to a circuit by closing a switch, the applied voltage is a step function, given by

$$v(t) = Au(t)$$

This is illustrated in **Figure 4.29**.

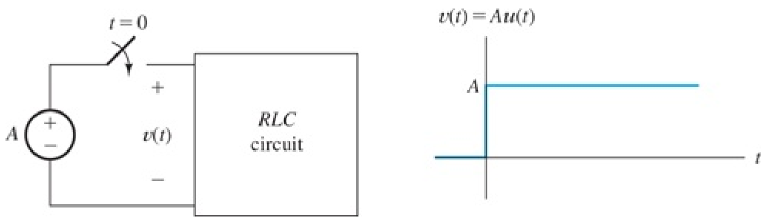


Figure 4.29

Applying a dc voltage by closing a switch results in a forcing function that is a step function.

We often encounter situations, such as **Example 4.7**, in which step forcing functions are applied to second-order systems described by a differential equation of the form

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = Au(t) \tag{4.99}$$

The differential equation is characterized by its undamped resonant frequency ω_0 and damping ratio $\zeta = \alpha/\omega_0$. [Of course, the solution for *x*(*t*) also depends on the initial conditions.] Normalized solutions are shown in **Figure 4.30** for the initial conditions $x(0) = 0$ and $x'(0) = 0$.

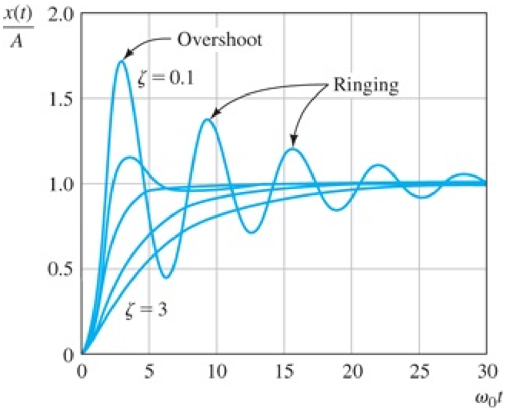


Figure 4.30

Normalized step responses for second-order systems described by **Equation 4.99** with damping ratios of $\zeta = 0.1,\ 0.5,\ 1,\ 2,$ and 3. The initial conditions are assumed to be $x(0) = 0$ and $x'(0) = 0$.

The system response for small values of the damping ratio ζ displays **overshoot** and **ringing** before settling to the steady-state value. On the other hand, if the damping ratio is large (compared to unity), the response takes a relatively long time to closely approach the final value.

Sometimes, we want to design a second-order system that quickly settles to steady state. Then we try to design for a damping ratio close to unity. For example, the control system for a robot arm could be a second-order system. When a step signal calls for the arm to move, we probably want it to achieve the final position in the minimum time without excessive overshoot and ringing.

> Frequently, electrical control systems and mechanical systems are best designed with a damping ratio close to unity. For example, when the suspension system on your automobile becomes severely underdamped, it is time for new shock absorbers.

Circuits with Parallel *L* and *C*

The solution of circuits having an inductance and capacitance in parallel is very similar to the series case. Consider the circuit shown in **Figure 4.31(a)**. The circuit inside the box is assumed to consist of sources and resistances. As we saw in **Section 2.6**, we can find a Norton equivalent circuit for any two-terminal circuit composed of resistances and sources. The equivalent circuit is shown in **Figure 4.31(b)**.

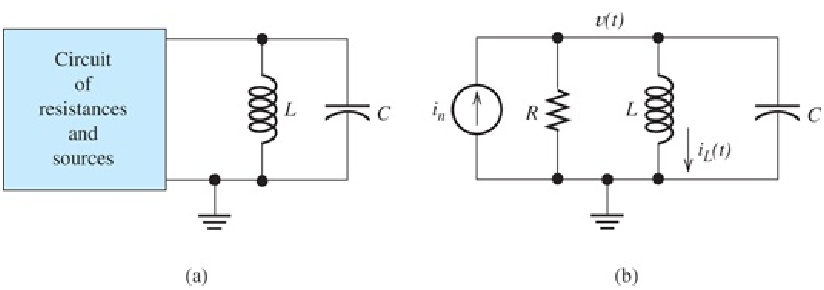


Figure 4.31

Any circuit consisting of sources, resistances, and a parallel *LC* combination can be reduced to the equivalent circuit shown in (b).

We can analyze this circuit by writing a KCL equation at the top node of **Figure 4.31(b)** which results in

$$C\frac{dv(t)}{dt}+\frac{1}{R}v(t)+\frac{1}{L}\int_0^t v(t)\,dt+i_L(0)=i_n(t) \tag{4.100}$$

This can be converted into a pure differential equation by taking the derivative with respect to time:

$$C\frac{d^2v(t)}{dt^2}+\frac{1}{R}\frac{dv(t)}{dt}+\frac{1}{L}v(t)=\frac{di_n(t)}{dt} \tag{4.101}$$

Dividing through by the capacitance, we have

$$\frac{d^2v(t)}{dt^2}+\frac{1}{RC}\frac{dv(t)}{dt}+\frac{1}{LC}v(t)=\frac{1}{C}\frac{di_n(t)}{dt} \tag{4.102}$$

Now, if we define the damping coefficient

$$\alpha=\frac{1}{2RC} \tag{4.103}$$

the undamped resonant frequency

$$\omega_0=\frac{1}{\sqrt{LC}} \tag{4.104}$$

and the forcing function

$$f(t)=\frac{1}{C}\frac{di_n(t)}{dt} \tag{4.105}$$

the differential equation can be written as

$$\frac{d^2v(t)}{dt^2}+2\alpha\frac{dv(t)}{dt}+\omega_0^2v(t)=f(t) \tag{4.106}$$

This equation has exactly the same form as **Equation 4.64**. Therefore, transient analysis of circuits with parallel *LC* elements is very similar to that of series *LC* circuits. However, notice that the equation for the damping coefficient α is different for the parallel circuit (in which $\alpha=1/2RC$) than for the series circuit (in which $\alpha=R/2L$).

Notice that the equation for the damping coefficient of the parallel *RLC circuit is different* from that for the series circuit.

Exercise 4.9

Consider the circuit shown in **Figure 4.32** with $R = 25\ \Omega$.

a. Compute the undamped resonant frequency, the damping coefficient, and the damping ratio.

b. The initial conditions are $v(0-) = 0$ and $i_L(0-) = 0$. Show that this requires that $v'(0+) = 10^6\ \mathrm{V/s}$.

c. Find the particular solution for *v*(*t*).

d. Find the general solution for *v*(*t*), including the numerical values of all parameters.

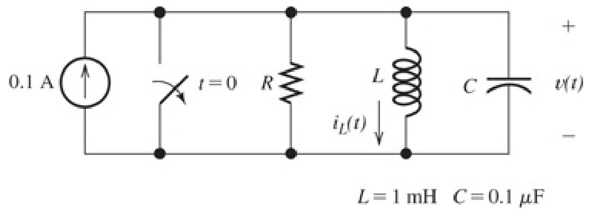


Figure 4.32

Circuit for **Exercises 4.9**, **4.10**, and **4.11**.

> $v(0-)$ and $i_L(0-)$ are the voltage and current values immediately before the switch opens.

Answer

a. $\omega_0 = 10^5,\ \alpha = 2 \times 10^5,$ and $\zeta = 2;$

b. KCL requires that $i_C(0) = 0.1\ \mathrm{A} = Cv'(0)$, thus $v'(0) = 10^6;$

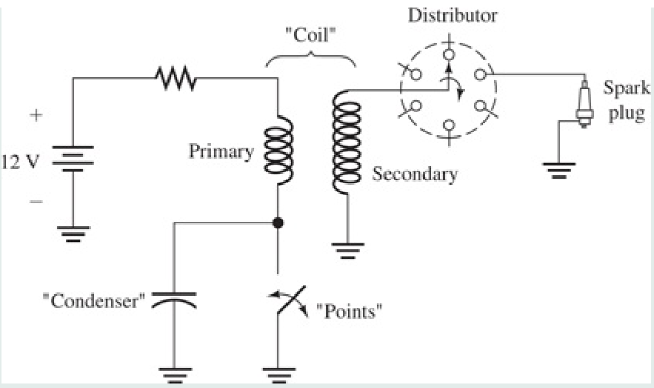
c. $v_p(t) = 0;$

d. $v(t) = 2.89\left(e^{-0.268 \times 10^5 t} - e^{-3.73 \times 10^5 t}\right)$.

PRACTICAL APPLICATION

4.1 Electronics and the Art of Automotive Maintenance

Throughout the early history of the automobile, ignition systems were designed as a straightforward application of electrical transients. The basic ignition system used for many years is shown in **Figure PA4.1**. The coil is a pair of mutually coupled inductors known as the primary and the secondary. The points form a switch that opens and closes as the engine rotates, opening at the instant that an ignition spark is needed by one of the cylinders. While the points are closed, current builds up relatively slowly in the primary winding of the coil. Then, when the points open, the current is rapidly interrupted. The resulting high rate of change of current induces a large voltage across the secondary winding, which is connected to the appropriate spark plug by the distributor. The resistance is needed to limit the current in case the engine stops with the points closed.



FIGURE

PA4.1

Classic ignition for an internal-combustion engine.

The capacitor prevents the voltage across the points from rising too rapidly when they open. (Recall that the voltage across a capacitance cannot change instantaneously.) Otherwise, arcing would occur across the points, causing them to become burned and pitted. By slowing the rise of voltage, the capacitor gives the gap between the points time to become wide enough to withstand the voltage across them. (Even so, the peak voltage across the points is many times the battery voltage.)

The primary inductance, current-limiting resistance, and capacitance form an underdamped series *RLC* circuit. Thus, an oscillatory current flows through the primary when the points open, inducing the requisite voltage in the secondary.

In its early forms, the ignition system had mechanical or vacuum systems to make adjustments to the timing, depending on engine speed and throttle setting. In more recent years, the availability of complex electronics at reasonable costs plus the desire to adjust the ignition to obtain good performance and low pollution levels with varying air temperature, fuel quality, air pressure, engine temperature, and other factors have greatly affected the design of ignition systems. The basic principles remain the same as in the days of the classic automobile, but a complex network of electrical sensors, a digital computer, and an electronic switch have replaced the points and simple vacuum advance.

The complexity of modern engineering designs has become somewhat intimidating, even to practicing engineers. In the 1960s, as a new engineering graduate, one could study the design of an ignition system, a radio, or a home appliance, readily spotting and repairing malfunctions with the aid of a few tools and standard parts. Nowadays, if my car should fail to start due to ignition malfunction, at the end of a fishing trip into the backwoods of northern Michigan, I might very well have to walk back to civilization. Nevertheless, the improvements in performance provided by modern electronics make up for its difficulty of repair.

Exercise 4.10

Repeat **Exercise 4.9** for $R = 50\ \Omega$.

Answer

a. $\omega_0 = 10^5,\ \alpha = 10^5$, and $\zeta = 1$;

b. KCL requires that $i_C(0) = 0.1\text{ A} = Cv'(0)$, thus $v'(0) = 10^6$;

c. $v_p(t) = 0$;

d. $v(t) = 10^6 te^{-10^5 t}$.

Exercise 4.11

Repeat **Exercise 4.9** for $R = 250\ \Omega$.

Answer

a. $\omega_0 = 10^5,\ \alpha = 0.2 \times 10^5,$ and $\zeta = 0.2;$

b. KCL requires that $i_C(0) = 0.1\ \mathrm{A} = Cv'(0)$, thus $v'(0) = 10^6;$

c. $v_p(t) = 0;$

d. $v(t) = 10.21 e^{-2 \times 10^4 t} \sin\left(97.98 \times 10^3 t\right)$.

4.6 Transient Analysis Using the MATLAB Symbolic Toolbox

The MATLAB Symbolic Toolbox greatly facilitates the solution of transients in electrical circuits. It makes the solution of systems of differential equations almost as easy as arithmetic using a calculator. A step-by-step process for solving a circuit in this manner is

1. Write the differential-integral equations for the mesh currents, node voltages, or other circuit variables of interest.
2. If necessary, differentiate the equations to eliminate integrals.
3. Analyze the circuit at $t = 0+$ (i.e., immediately after switches operate) to determine initial conditions for the circuit variables and their derivatives. For a first-order equation, we need the initial value of the circuit variable. For a second-order equation we need the initial values of the circuit variable and its first derivative.
4. Enter the equations and initial values into the dsolve command in MATLAB.

We illustrate with a few examples.

Example 4.8 Computer-Aided Solution of a First-Order Circuit

Solve for $v_L(t)$ in the circuit of **Figure 4.33(a)**. (Note: The argument of the cosine function is in radians.)

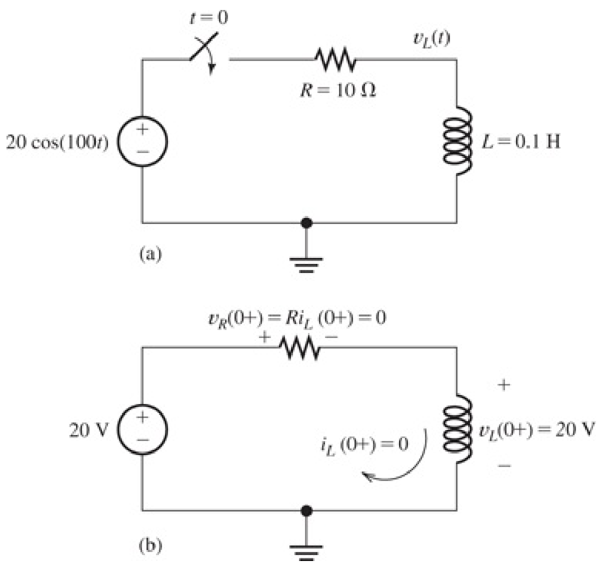


Figure 4.33

(a) Circuit of **Example 4.8**. (b) Circuit conditions at $t = 0+$.

Solution

First, we write a KCL equation at the node joining the resistance and inductance.

$$\frac{v_L(t) - 20\cos(100t)}{R} + \frac{1}{L}\int_0^t v_L(t)\, dt + i_L(0) = 0$$

Taking the derivative of the equation to eliminate the integral, multiplying each term by *R*, and substituting values, we eventually obtain

$$\frac{dv_L(t)}{dt} + 100v_L(t) = -2000\sin(100t)$$

Next, we need to determine the initial value of v_L. Because the switch is open prior to $t = 0$, the initial current in the inductance is zero prior to $t = 0$. Furthermore, the current cannot change instantaneously in this circuit. Thus, we have $i_L(0+) = 0$. Immediately after the switch closes, the voltage source has a value of 20 V, and the current flowing in the circuit is zero, resulting in zero volts across the resistor. Then KVL yields $v_L(0+) = 20\ \text{V}$. This is illustrated in **Figure 4.33(b)**.

Now, we can write the MATLAB commands. As usual, we show the commands in **boldface**, comments in regular font, and MATLAB responses in color.

```
>> clear all
>> syms VL t
>> % Enter the equation and initial value in the dsolve command.
>> % DVL represents the derivative of VL with respect to time.
>> VL = dsolve('DVL + 100*VL = -2000*sin(100*t)', 'VL(0) = 20');
>> % Print answer with 4 decimal place accuracy for the constants:
>> vpa(VL,4)
 ans =
 10.0*cos(100.0*t)-10.0*sin(100.0*t)+10.0*exp(-100.0*t)
```

In standard mathematical notation, the result becomes

$$v_L(t) = 10\cos(100t) - 10\sin(100t) + 10\exp(-100t)$$

This can be shown to be equivalent to

$$v_L(t) = 14.14\cos(100t + 0.7854) + 10\exp(-100t)$$

in which the argument of the cosine function is in radians. Some versions of MATLAB may give this result. **Keep in mind that different versions of the software may give results with different appearances that are mathematically equivalent.**

An m-file named Example_4_8 containing the commands for this example can be found in the MATLAB folder. (See **Appendix E** for information about access to this folder.)

Example 4.9 Computer-Aided Solution of a Second-Order Circuit

The switch in the circuit of **Figure 4.34(a)** is closed for a long time prior to $t = 0$. Assume that $i_L(0+) = 0$. Use MATLAB to solve for $i_L(t)$ and plot the result for $0 \le t \le 2\ \text{ms}$.

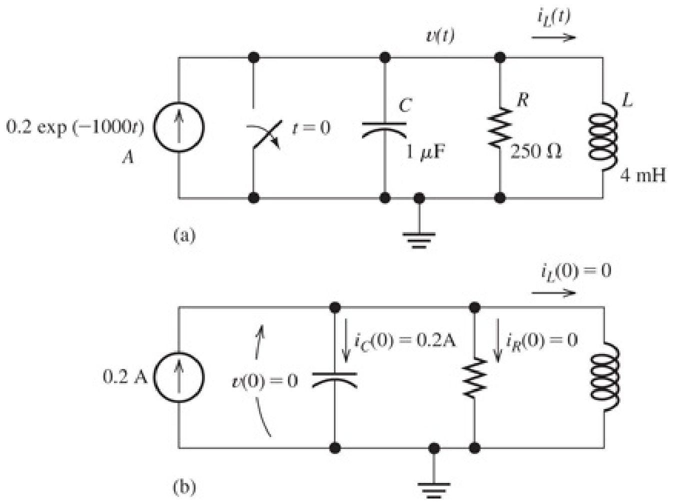


Figure 4.34

(a) Circuit of **Example 4.9**. (b) Circuit conditions at $t = 0+$.

Solution

Because this circuit contains two nodes and three meshes, node-voltage analysis is simpler than mesh analysis. We will solve for *v*(*t*) and then take 1/*L* times the integral of the voltage to obtain the current through the inductance.

We start the node-voltage analysis by writing the KCL equation at the top node of the circuit (with the switch open).

$$C\frac{dv(t)}{dt} + \frac{v(t)}{R} + \frac{1}{L}\int_0^t v(t)\ dt + i_L(0+) = 0.2\exp(-1000t)$$

Taking the derivative of the equation to eliminate the integral and substituting values, we eventually obtain

$$10^{-6}\frac{d^2v(t)}{dt^2} + 4\times 10^{-3}\frac{dv(t)}{dt} + 250v(t) = -200\exp(-1000t)$$

Because this is a second-order equation, we need the initial value for both *v*(*t*) and its first derivative. The circuit conditions at $t = 0+$ are shown in **Figure 4.34(b)**. The problem states that the initial current in the inductance is zero. The initial voltage $v(0+)$ is zero, because, with the switch closed, the capacitor is shorted. When the switch opens, the voltage remains zero, because an infinite current would be required to change the capacitor voltage instantaneously. Furthermore, the current flowing through the resistor is zero because the voltage across it is zero. Thus, the 0.2 A from the source must flow through the capacitor, and we have

$$C\frac{dv(0+)}{dt} = 0.2$$

We have established that $v(0+) = 0$ and $v'(0+) = dv(0+)/dt = 0.2\times 10^6\ \text{V/s}$.

After the voltage is found, the current is given by

$$i_L(t) = \frac{1}{L}\int_0^t v(t)\ dt = 250\int_0^t v(t)\ dt$$

We use the following MATLAB commands to obtain the solution:

```
>> clear all
>> syms IL V t
>> % Enter the equation and initial values in the dsolve command.
>> % D2V represents the second derivative of V.
>> V = dsolve('(1e-6)*D2V + (4e-3)*DV + 250*V =
-200*exp(-1000*t)', . . .
 'DV(0)=0.2e6', 'V(0)=0');
>> % Calculate the inductor current by integrating V with respect to t
>> % from 0 to t and multiplying by 1/L:
>> IL = (250)*int(V,t,0,t);
>> % Display the expression for current to 4 decimal place accuracy:
>> vpa(IL,4)
 ans =
 -(0.0008229*(246.0*cos(15688.0*t) - 246.0*exp(1000.0*t) +
 15.68*sin(15688.0*t)))/exp(2000.0*t)
>> ezplot(IL,[0 2e-3])
```

In standard mathematical notation, the result is

$$i_L(t) = -0.2024\exp(-2000t)\cos(15680t) - 0.01290\exp(-2000t)\sin(15680t) + 0.2024\exp(-1000t)$$

The plot (after some editing to dress it up) is shown in **Figure 4.35**. An m-file named Example_4_9 containing the commands for this example can be found in the MATLAB folder. (See **Appendix E** for information about accessing this folder.)

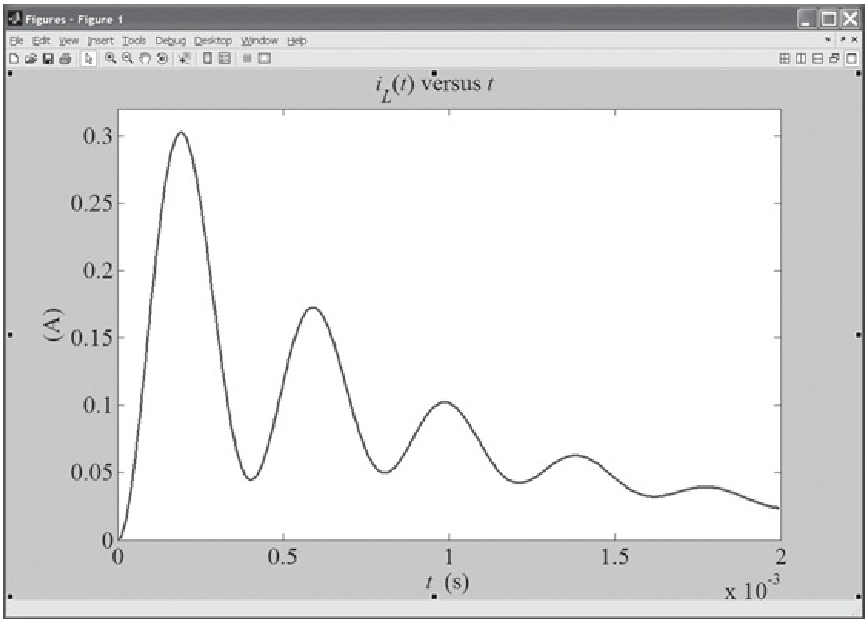


Figure 4.35
Plot of $i_L(t)$ versus *t*.

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Solving Systems of Linear Differential Equations

So far in this chapter, each of our examples has involved a single differential equation. Circuits that require two or more circuit variables (such as node voltages or mesh currents) result in systems of differential equations. While these systems can be rather formidable to solve by traditional methods, the MATLAB Symbolic Toolbox can solve them with relative ease.

Example 4.10 Computer-Aided Solution of a System of Differential Equations

Use MATLAB to solve for the node voltages in the circuit of **Figure 4.36**. The circuit has been connected for a long time prior to $t = 0$ with the switch open, so the initial values of the node voltages are zero.

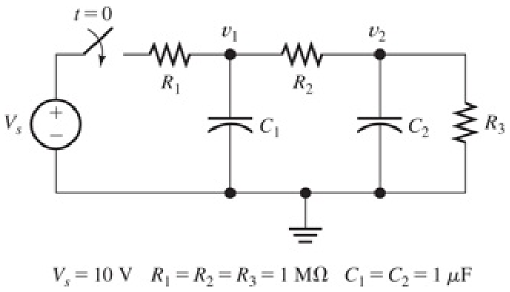

Figure 4.36
Circuit of **Example 4.10**.

Solution

First, we write the KCL equations at nodes 1 and 2.

$$C_1\frac{dv_1(t)}{dt} + \frac{v_1(t) - V_s}{R_1} + \frac{v_1(t) - v_2(t)}{R_2} = 0$$
$$C_2\frac{dv_2(t)}{dt} + \frac{v_2(t) - v_1(t)}{R_2} + \frac{v_2(t)}{R_3} = 0$$

Now substituting values, multiplying each term by 10^6, and rearranging terms, we have

$$\frac{dv_1(t)}{dt} + 2v_1(t) - v_2(t) = 10$$

$$\frac{dv_2(t)}{dt} + 2v_2(t) - v_1(t) = 0$$

The MATLAB commands and results are:

```
>> clear all
>> syms v1 v2 t
>> [v1 v2] = dsolve('Dv1 + 2*v1 - v2 = 10','Dv2 + 2*v2 -v1 = 0', . . .
 'v1(0) = 0','v2(0)= 0');
>> v1
 v1 =
 exp(-t)*(5*exp(t) - 5) + exp(-3*t)*((5*exp(3*t))/3 - 5/3)
>> v2
 v2 =
 exp(-t)*(5*exp(t) - 5) - exp(-3*t)*((5*exp(3*t))/3 - 5/3)
```

As usual, keep in mind that different versions of the software can give results different in appearance but mathematically equivalent to that shown here. In standard mathematical notation, the results can be put into the form:

$$v_1(t) = 20/3 - 5\exp(-t) - (5/3)\exp(-3t)$$

$$v_2(t) = 10/3 - 5\exp(-t) + (5/3)\exp(-3t)$$

It is always a good idea to perform a few checks on our answers. First, we can verify that the MATLAB results are both zero at $t = 0$ as required by the initial conditions. Furthermore, at $t = \infty$, the capacitors act as open circuits, and the voltage division principle yields $v_1(\infty) = 20/3\ \mathrm{V}$ and $v_2(\infty) = 10/3$. The expressions delivered by MATLAB also yield these values.

Exercise 4.12

Use the MATLAB Symbolic Toolbox to solve **Example 4.6**, obtaining the result given in **Equation 4.56** and a plot similar to **Figure 4.18** on page **186**.

Answer A sequence of commands that produces the solution and the plot is:

```
clear all
syms ix t R C vCinitial w
ix = dsolve('(R*C)*Dix + ix = (w*C)*2*cos(w*t)', 'ix(0)=-vCinitial/R');
ians = subs(ix,[R C vCinitial w],[5000 1e-6 1 200]);
vpa(ians, 4)
ezplot(ians,[0 80e-3])
```

An m-file named Exercise_4_12 containing these commands can be found in the MATLAB folder. (See **Appendix E** for information about accessing this folder.)

Exercise 4.13

Use the MATLAB Symbolic Toolbox to solve **Example 4.7** obtaining the results given in the example for $v_C(t)$ and a plot similar to **Figure 4.27** on page **195**.

Answer A list of commands that produces the solution and the plot is:

```
clear all
syms vc t
% Case I, R = 300:
vc = dsolve('(1e-8)*D2vc + (1e-6)*300*Dvc+ vc =10', 'vc(0) = 0','Dvc(0)=0');
vpa(vc,4)
ezplot(vc, [0 1e-3])
hold on % Turn hold on so all plots are on the same axes
% Case II, R = 200:
vc = dsolve('(1e-8)*D2vc + (1e-6)*200*Dvc+ vc =10', 'vc(0) = 0','Dvc(0)=0');
vpa(vc,4)
ezplot(vc, [0 1e-3])
% Case III, R = 100:
vc = dsolve('(1e-8)*D2vc + (1e-6)*100*Dvc+ vc =10', 'vc(0) = 0','Dvc(0)=0');
vpa(vc,4)
ezplot(vc, [0 1e-3])
```

An m-file named Exercise_4_13 containing these commands resides in the MATLAB folder. (See **Appendix E** for information about accessing this folder.)

Summary

1. The transient part of the response for a circuit containing sources, resistances, and a single energy-storage element (*L* or *C*) is of the form $Ke^{-t/\tau}$. The time constant is given by $\tau = RC$ or by $\tau = L/R$, where *R* is the Thévenin resistance seen looking back into the circuit from the terminals of the energy-storage element.
2. In dc steady-state conditions, inductors behave as short circuits and capacitors behave as open circuits. We can find the steady-state (forced) response for dc sources by analyzing the dc equivalent circuit.
3. To find the transient currents and voltages, we must solve linear differential equations with constant coefficients. The solutions are the sum of two parts. The particular solution, also called the forced response, depends on the sources, as well as the other circuit elements. The homogeneous solution, also called the natural response, depends on the passive elements (*R*, *L*, and *C*), but not on the sources. In circuits that contain resistances, the natural response eventually decays to zero.
4. The natural response of a second-order circuit containing a series or parallel combination of inductance and capacitance depends on the damping ratio and undamped resonant frequency.
   If the damping ratio is greater than unity, the circuit is overdamped, and the natural response is of the form
   $$x_c(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$
   If the damping ratio equals unity, the circuit is critically damped, and the natural response is of the form
   $$x_c(t) = K_1 e^{s_1 t} + K_2 t e^{s_1 t}$$
   If the damping ratio is less than unity, the circuit is underdamped, and the natural response is of the form
   $$x_c(t) = K_1 e^{-\alpha t}\cos(\omega_n t) + K_2 e^{-\alpha t}\sin(\omega_n t)$$
   The normalized step response for second-order systems is shown in **Figure 4.30** on page **196** for several values of the damping ratio.
5. The MATLAB Symbolic Toolbox is a powerful tool for solving the equations for transient circuits. A step-by-step procedure is given on page **199**.

Problems

Section 4.1: First-Order *RC* Circuits

P4.1. Suppose we have a capacitance *C* discharging through a resistance *R*. Define and give an expression for the time constant. To attain a long time constant, do we need large or small values for *R*? For *C*?

***P4.2.** The dielectric materials used in real capacitors are not perfect insulators. A resistance called a leakage resistance in parallel with the capacitance can model this imperfection. A $100-\mu\mathrm{F}$ capacitor is initially charged to 100 V. We want 90 percent of the initial energy to remain after one minute. What is the limit on the leakage resistance for this capacitor?

* Denotes that answers are contained in the Student Solutions files. See **Appendix E** for more information about accessing the Student Solutions.

***P4.3.** The initial voltage across the capacitor shown in **Figure P4.3** is $v_C(0+) = -10$ V. Find an expression for the voltage across the capacitor as a function of time. Also, determine the time t_0 at which the voltage crosses zero.

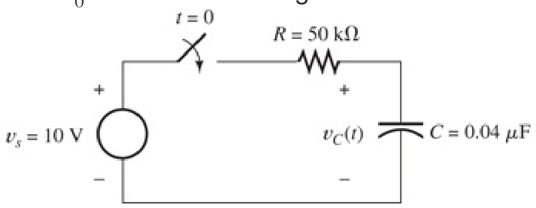

Figure P4.3

***P4.4.** A $100\{\ \mu\mathrm{F}$ capacitance is initially charged to 1000 V. At $t=0,$ it is connected to a $1-\mathrm{k}\Omega$ resistance. At what time t_2 has 50 percent of the initial energy stored in the capacitance been dissipated in the resistance?

***P4.5.** At $t=0,$ a charged $10\{\ \mu\mathrm{F}$ capacitance is connected to a voltmeter, as shown in **Figure P4.5**. The meter can be modeled as a resistance. At $t=0,$ the meter reads 50 V. At $t=30\ \mathrm{s},$ the reading is 25 V. Find the resistance of the voltmeter.

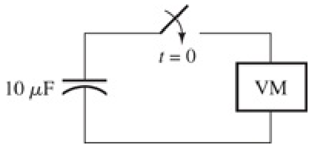

Figure P4.5

***P4.6.** At time $t_1,$ a capacitance *C* is charged to a voltage of $V_1.$ Then, the capacitance discharges through a resistance *R*. Write an expression for the voltage across the capacitance as a function of time for $t > t_1$ in terms of *R*, *C*, $V_1,$ and $t_1.$

P4.7. Given an initially charged capacitance that begins to discharge through a resistance at $t=0,$ what percentage of the initial voltage remains at two time constants? What percentage of the initial stored energy remains?

P4.8. The initial voltage across the capacitor shown in **Figure P4.3** is $v_C(0+) = 0.$ Find an expression for the voltage across the capacitor as a function of time, and sketch to scale versus time.

P4.9. In physics, the half-life is often used to characterize exponential decay of physical quantities such as radioactive substances. The half-life is the time required for the quantity to decay to half of its initial value. The time constant for the voltage on a capacitance discharging through a resistance is $\tau = RC$. Find an expression for the half-life of the voltage in terms of *R* and *C*.

P4.10. We know that a $50 - \mu\mathrm{F}$ capacitance is charged to an unknown voltage V_i at $t = 0$. The capacitance is in parallel with a $3\{\ \mathrm{k}\Omega$ resistance. At $t = 100\ \mathrm{ms},$ the voltage across the capacitance is 5 V. Determine the value of V_i.

P4.11. We know that the capacitor shown in **Figure P4.11** is charged to a voltage of 10 V prior to $t = 0$.

a. Find expressions for the voltage across the capacitor $v_C(\ t)$ and the voltage across the resistor $v_R(\ t)$ for all time.
b. Find an expression for the power delivered to the resistor.
c. Integrate the power from $t = 0$ to $t = \infty$ to find the energy delivered.
d. Show that the energy delivered to the resistor is equal to the energy stored in the capacitor prior to $t = 0$.

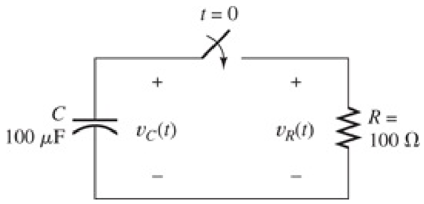

Figure P4.11

P4.12. The purchasing power *P* of a certain unit of currency declines by 3 percent per year. Determine the time constant associated with the purchasing power of this currency.

P4.13. Derive an expression for $v_C(\ t)$ in the circuit of **Figure P4.13** and sketch $v_C(\ t)$ to scale versus time.

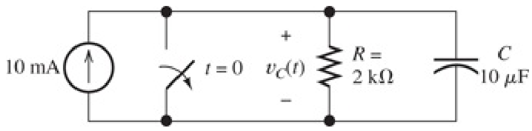

Figure P4.13

P4.14. Suppose that at $t = 0,$ we connect an uncharged $10\{\ \mu\mathrm{F}$ capacitor to a charging circuit consisting of a 2500-V voltage source in series with a $2\{\ \mathrm{M}\Omega$ resistance. At $t = 40\ \mathrm{s},$ the capacitor is disconnected from the charging circuit and connected in parallel with a $5\{\ \mathrm{M}\Omega$ resistor. Determine the voltage across the capacitor at $t = 40\ \mathrm{s}$ and at $t = 100\ \mathrm{s}$. (*Hint:* You may find it convenient to redefine the time variable to be $t' = t - 40$ for the discharge interval so that the discharge starts at $t' = 0.$)

P4.15. Suppose we have a capacitance *C* that is charged to an initial voltage V_i. Then at $t = 0,$ a resistance *R* is connected across the capacitance. Write an expression for the current. Then, integrate the current from $t = 0$ to $t = \infty,$ and show that the result is equal to the initial charge stored on the capacitance.

P4.16. A person shuffling across a dry carpet can be approximately modeled as a charged 100-pF capacitance with one end grounded. If the person touches a grounded metallic object such as a water faucet, the capacitance is discharged and the person experiences a brief shock. Typically, the capacitance may be charged to 20,000 V and the resistance (mainly of one’s finger) is $100\ \Omega$. Determine the peak current during discharge and the time constant of the shock.

P4.17. Consider the circuit of **Figure P4.17**, in which the switch instantaneously moves back and forth between contacts *A* and *B*, spending 2 seconds in each position. Thus, the capacitor repeatedly charges for 2 seconds and then discharges for 2 seconds. Assume that $v_C(\ 0) = 0$ and that the switch moves to position *A* at $t = 0$. Determine $v_C(\ 2)$, $v_C(\ 4)$, $v_C(\ 6)$, and $v_C(\ 8)$.

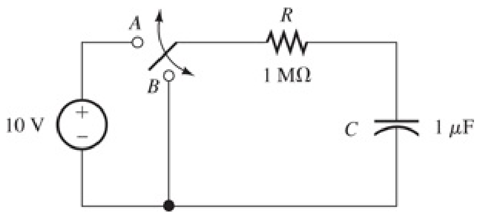

Figure P4.17

P4.18. Consider the circuit shown in **Figure P4.18**. Prior to $t = 0,\ v_1 = 100\text{ V},$ and $v_2 = 0.$

a. Immediately after the switch is closed, what is the value of the current [i.e., what is the value of $i(\ 0+)$]?
b. Write the KVL equation for the circuit in terms of the current and initial voltages. Take the derivative to obtain a differential equation.
c. What is the value of the time constant in this circuit?
d. Find an expression for the current as a function of time.
e. Find the value that v_2 approaches as *t* becomes very large.

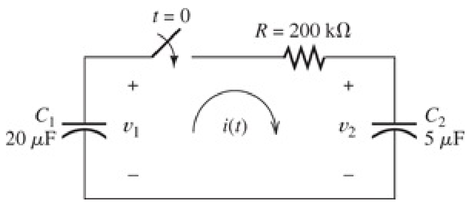

Figure P4.18

Section 4.2: DC Steady State

P4.19. List the steps for dc steady-state analysis of *RLC* circuits.

P4.20. Explain why we replace capacitances with open circuits and inductances with short circuits in dc steady-state analysis.

***P4.21.** Solve for the steady-state values of $i_1,\ i_2,$ and i_3 for the circuit shown in **Figure P4.21**.

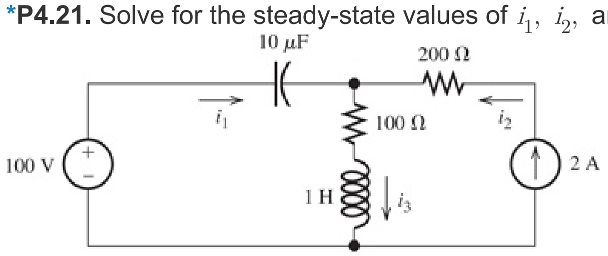

Figure P4.21

***P4.22.** Consider the circuit shown in **Figure P4.22**. What is the steady-state value of v_C after the switch opens? Determine how long it takes after the switch opens before v_C is within 1 percent of its steady-state value.

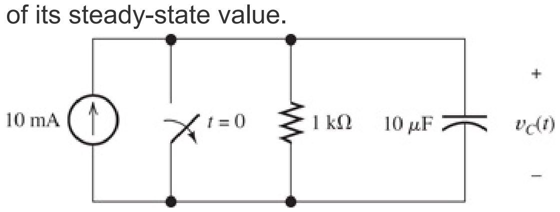

Figure P4.22

***P4.23.** In the circuit of **Figure P4.23**, the switch is in position *A* for a long time prior to $t = 0$. Find expressions for $v_R(\ t)$ and sketch it to scale for $-2 \le t \le 10\ \mathrm{s}$.

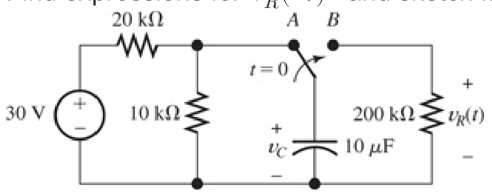

Figure P4.23

P4.24. The circuit shown in **Figure P4.24** has been set up for a long time prior to $t = 0$ with the switch closed. Find the value of v_C prior to $t = 0$. Find the steady-state value of v_C after the switch has been opened for a long time.

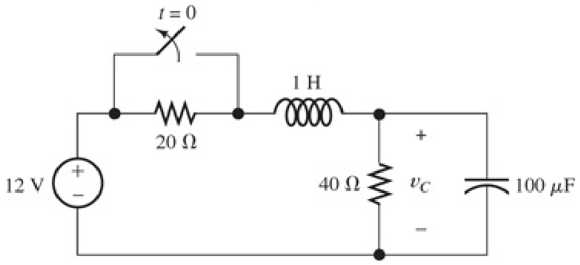

Figure P4.24

P4.25. Solve for the steady-state values of i_1, i_2, i_3, i_4, and v_C for the circuit shown in **Figure P4.25**, assuming that the switch has been closed for a long time.

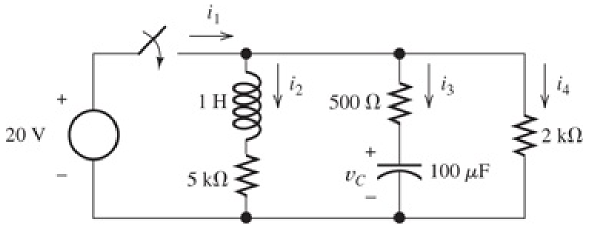

Figure P4.25

P4.26. The circuit shown in **Figure P4.26** is operating in steady state. Determine the values of i_L, v_x, and v_C.

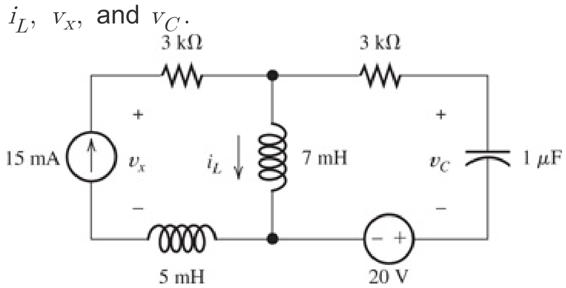

Figure P4.26

P4.27. The circuit of **Figure P4.27** has been connected for a very long time. Determine the values of v_C and i_R.

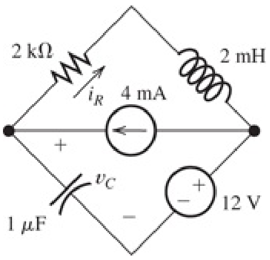

Figure P4.27

P4.28. Consider the circuit of **Figure P4.28** in which the switch has been closed for a long time prior to $t = 0$. Determine the values of $v_C(t)$ before $t = 0$ and a long time after $t = 0$. Also, determine the time constant after the switch opens and expressions for $v_C(t)$. Sketch $v_C(t)$ to scale versus time for $-0.2 \le t \le 0.5$ s.

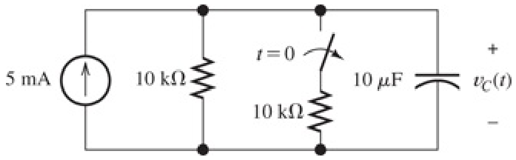

Figure P4.28

P4.29. For the circuit shown in **Figure P4.29**, the switch is closed for a long time prior to $t = 0$. Find expressions for $v_C(t)$ and sketch it to scale for $-80 \le t \le 160$ ms.

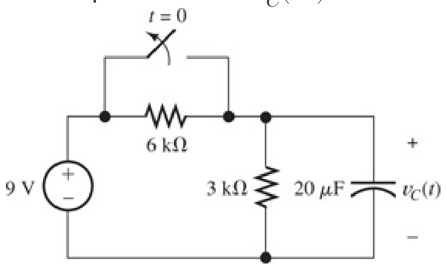

Figure P4.29

P4.30. Consider the circuit of **Figure P4.30** in which the switch has been closed for a long time prior to $t = 0$. Determine the values of $v_C(t)$ before $t = 0$ and a long time after $t = 0$. Also, determine the time constant after the switch opens and expressions for $v_C(t)$. Sketch $v_C(t)$ to scale versus time for $-4 \le t \le 16$ s.

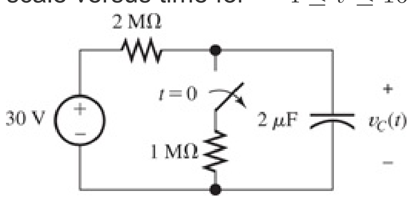

Figure P4.30

Section 4.3: *RL* Circuits

P4.31. Give the expression for the time constant of a circuit consisting of an inductance with an initial current in series with a resistance *R*. To attain a long time constant, do we need large or small values for *R*? For *L*?

P4.32. A circuit consists of switches that open or close at $t=0,$ resistances, dc sources, and a single energy storage element, either an inductance or a capacitance. We wish to solve for a current or a voltage *x*(*t*) as a function of time for $t \geq 0$. Write the general form for the solution. How is each unknown in the solution determined?

***P4.33.** The circuit shown in **Figure P4.33** is operating in steady state with the switch closed prior to $t=0$. Find *i*(*t*) for $t<0$ and for $t \geq 0$.

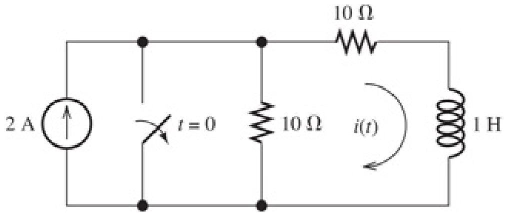

Figure P4.33

***P4.34.** Consider the circuit shown in **Figure P4.34**. The initial current in the inductor is

$$i_L(0-) = -0.2 \text{ A}.$$

Find expressions for $i_L(t)$ and *v*(*t*) for $t \geq 0$ and sketch to scale versus time.

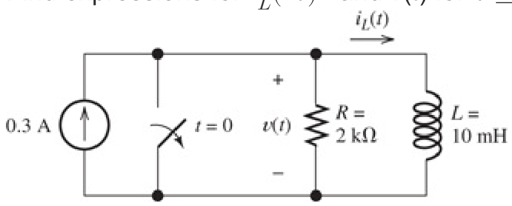

Figure P4.34

P4.35. Repeat **Problem P4.34** given $i_L(0-) = 0 \text{ A}$.

***P4.36.** Real inductors have series resistance associated with the wire used to wind the coil. Suppose that we want to store energy in a 10-H inductor. Determine the limit on the series resistance so the energy remaining after one hour is at least 75 percent of the initial energy.

P4.37. Determine expressions for and sketch $i_s(t)$ to scale versus time for $-0.2 \leq t \leq 1.0$ s for the circuit of **Figure P4.37**.

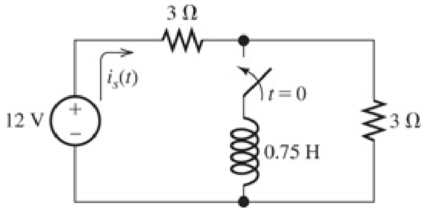

Figure P4.37

P4.38. For the circuit shown in **Figure P4.38**, find an expression for the current $i_L(t)$ and sketch it to scale versus time. Also, find an expression for $v_L(t)$ and sketch it to scale versus time.

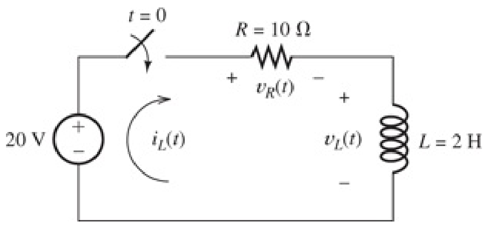

Figure P4.38

P4.39. The circuit shown in **Figure P4.39** is operating in steady state with the switch closed prior to $t = 0$. Find expressions for $i_L(t)$ for $t < 0$ and for $t \geq 0$. Sketch $i_L(t)$ to scale versus time.

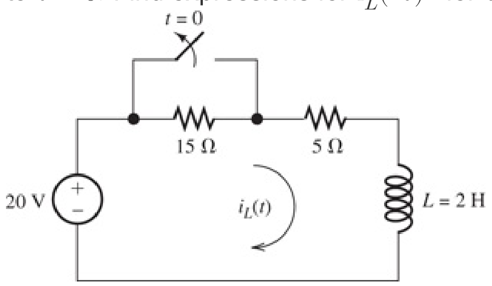

Figure P4.39

P4.40. Consider the circuit shown in **Figure P4.40**. A voltmeter (VM) is connected across the inductance. The switch has been closed for a long time. When the switch is opened, an arc appears across the switch contacts. Explain why. Assuming an ideal switch and inductor, what voltage appears across the inductor when the switch is opened? What could happen to the voltmeter when the switch opens?

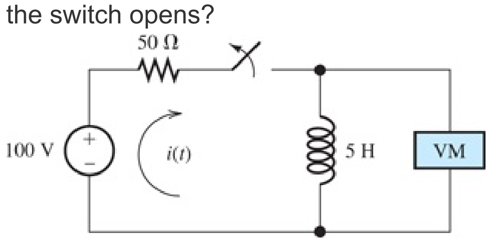

Figure P4.40

P4.41. Due to components not shown in the figure, the circuit of **Figure P4.41** has $i_L(0) = I_i$.

a. Write an expression for $i_L(t)$ for $t \geq 0$.
b. Find an expression for the power delivered to the resistance as a function of time.
c. Integrate the power delivered to the resistance from $t = 0$ to $t = \infty$, and show that the result is equal to the initial energy stored in the inductance.

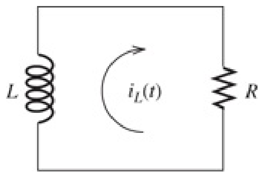

Figure P4.41

P4.42. The switch shown in **Figure P4.42** has been closed for a long time prior to $t = 0$, then it opens at $t = 0$ and closes again at $t = 1\text{ s}$. Find $i_L(t)$ for all *t*.

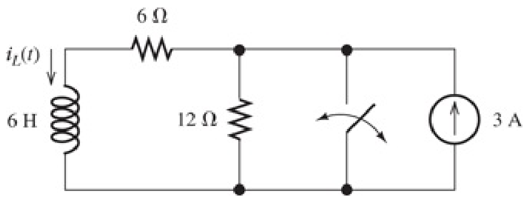

Figure P4.42

P4.43. Determine expressions for and sketch $v_R(t)$ to scale versus time for the circuit of **Figure P4.43**. The circuit is operating in steady state with the switch closed prior to $t = 0$. Consider the time interval $-1 \leq t \leq 5$ ms.

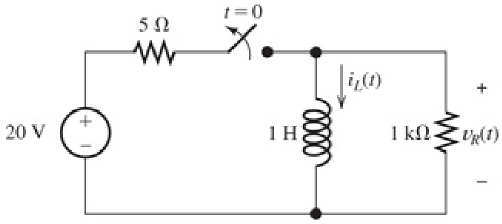

Figure P4.43

Section 4.4: *RC* and *RL* Circuits with General Sources

P4.44. What are the steps in solving a circuit having a resistance, a source, and an inductance (or capacitance)?

***P4.45.** Write the differential equation for $i_L(t)$ and find the complete solution for the circuit of **Figure P4.45**. [*Hint:* Try a particular solution of the form $i_{Lp}(t) = Ae^{-t}$.]

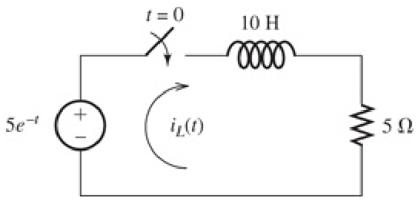

Figure P4.45

***P4.46.** Solve for $v_C(t)$ for $t > 0$ in the circuit of **Figure P4.46**. [*Hint:* Try a particular solution of the form $v_{Cp}(t) = Ae^{-3t}$.]

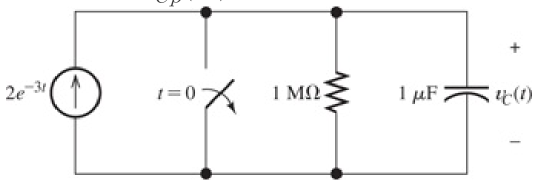

Figure P4.46

***P4.47.** Solve for *v*(*t*) for $t > 0$ in the circuit of **Figure P4.47**, given that the inductor current is zero prior to $t = 0$. [*Hint:* Try a particular solution of the form $v_p = A\cos(10t) + B\sin(10t)$.]

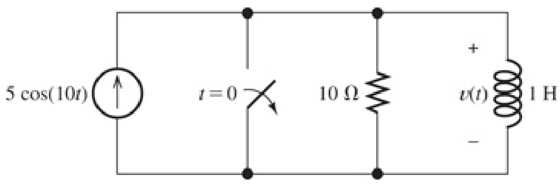

Figure P4.47

P4.48. Solve for $i_L(t)$ for $t > 0$ in the circuit of **Figure P4.48**. You will need to make an educated guess as to the form of the particular solution. [*Hint:* The particular solution includes terms with the same functional forms as the terms found in the forcing function and its derivatives.]

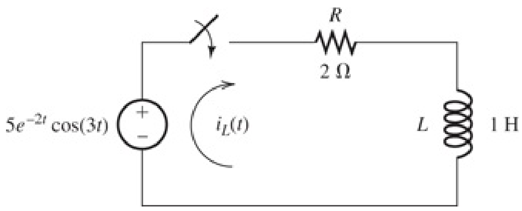

Figure P4.48

P4.49. Consider the circuit shown in **Figure P4.49**. The voltage source is known as a **ramp function,** which is defined by

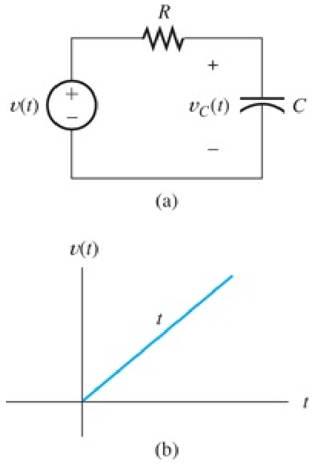

Figure P4.49

$$v(t) = \begin{cases} 0 \text{ for } t < 0 \\ t \text{ for } t \geq 0 \end{cases}$$

Assume that $v_C(0) = 0$. Derive an expression for $v_C(t)$ for $t \geq 0$. Sketch $v_C(t)$ to scale versus time. [*Hint:* Write the differential equation for $v_C(t)$ and assume a particular solution of the form $v_{Cp}(t) = A + Bt$.]

P4.50. Consider the circuit shown in **Figure P4.50**. The initial current in the inductor is $i_s(0+) = 0$. Write the differential equation for $i_s(t)$ and solve. [*Hint:* Try a particular solution of the form $i_{sp}(t) = A\cos(300t) + B\sin(300t)$.]

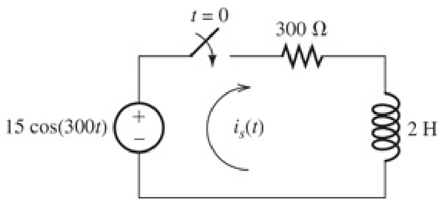

Figure P4.50

P4.51. The voltage source shown in **Figure P4.51** is called a ramp function. Assume that $i_L(0) = 0$. Write the differential equation for $i_L(t)$, and find the complete solution. [*Hint:* Try a particular solution of the form $i_p(t) = A + Bt$.]

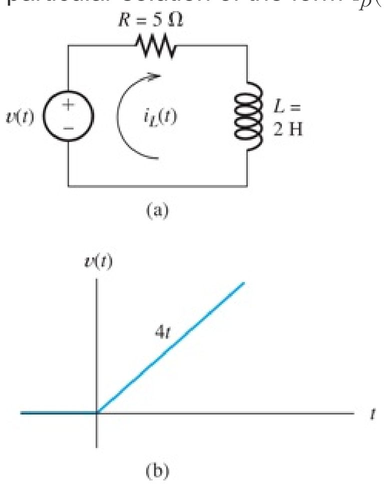

Figure P4.51

P4.52. Determine the form of the particular solution for the differential equation

$$2\frac{dv(t)}{dt} + v(t) = 5t\sin(t)$$

Then, find the particular solution. [*Hint:* The particular solution includes terms with the same functional forms as the terms found in the forcing function and its derivatives.]

P4.53. Determine the form of the particular solution for the differential equation

$$\frac{dv(t)}{dt} + 3v(t) = t^2\exp(-t)$$

Then, find the particular solution. [*Hint:* The particular solution includes terms with the same functional forms as the terms found in the forcing function and its derivatives.]

P4.54. Consider the circuit shown in **Figure P4.54**.

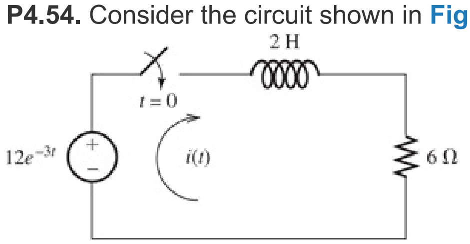

Figure P4.54

a. Write the differential equation for *i*(*t*).
b. Find the time constant and the form of the complementary solution.
c. Usually, for an exponential forcing function like this, we would try a particular solution of the form $i_p(t) = K\exp(-3t)$. Why doesn't that work in this case?

d. Find the particular solution. [*Hint:* Try a particular solution of the form $i_p(t) = Kt\exp(-3t)$.

]

e. Find the complete solution for *i*(*t*).

P4.55. Consider the circuit shown in **Figure P4.55**.

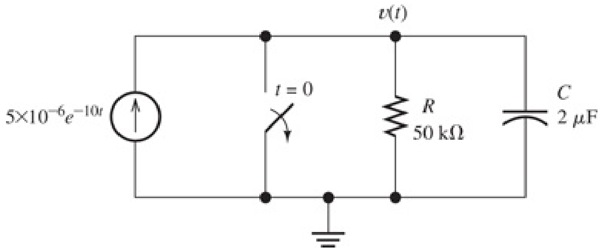

Figure P4.55

a. Write the differential equation for *v*(*t*).
b. Find the time constant and the form of the complementary solution.
c. Usually, for an exponential forcing function like this, we would try a particular solution of the form $v_p(t) = K\exp(-10t)$. Why doesn't that work in this case?
d. Find the particular solution. [*Hint:* Try a particular solution of the form $v_p(t) = Kt\exp(-10t)$.

]

e. Find the complete solution for *v*(*t*).

Section 4.5: Second-Order Circuits

P4.56. How can first- or second-order circuits be identified by inspecting the circuit diagrams?
P4.57. How can an underdamped second-order system be identified? What form does its complementary solution take? Repeat for a critically damped system and for an overdamped system.
P4.58. What is a unit step function?
P4.59. Discuss two methods that can be used to determine the particular solution of a circuit with constant dc sources.
P4.60. Sketch a step response for a second-order system that displays considerable overshoot and ringing. In what types of circuits do we find pronounced overshoot and ringing?
***P4.61.** A dc source is connected to a series *RLC* circuit by a switch that closes at $t = 0$, as shown in **Figure P4.61**. The initial conditions are $i(0+) = 0$ and $v_C(0+) = 0$. Write the differential equation for $v_C(t)$. Solve for $v_C(t)$, if $R = 80\ \Omega$.

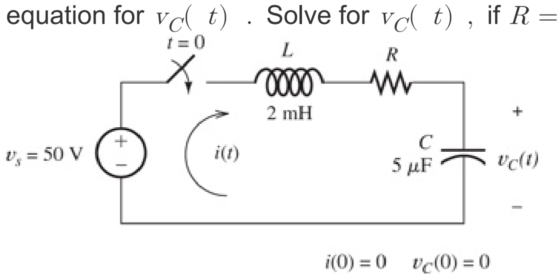

Figure P4.61

***P4.62.** Repeat **Problem P4.61** for $R = 40\ \Omega$.
***P4.63.** Repeat **Problem P4.61** for $R = 20\ \Omega$.

P4.64. Consider the circuit shown in **Figure P4.64** in which the switch has been open for a long time prior to $t = 0$ and we are given $R = 25\ \Omega$.

a. Compute the undamped resonant frequency, the damping coefficient, and the damping ratio of the circuit after the switch closes.
b. Assume that the capacitor is initially charged by a 25-V dc source not shown in the figure, so we have $v(0+) = 25\ \text{V}$. Determine the values of $i_L(0+)$ and $v'(0+)$.
c. Find the particular solution for *v*(*t*).
d. Find the general solution for *v*(*t*), including the numerical values of all parameters.

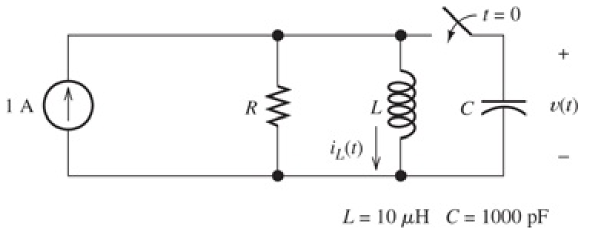

Figure P4.64

P4.65. Repeat **Problem P4.64** for $R = 50\ \Omega$.
P4.66. Repeat **Problem P4.64** for $R = 500\ \Omega$.
P4.67. Solve for *i*(*t*) for $t > 0$ in the circuit of **Figure P4.67**, with $R = 50\ \Omega,$ given that $i(0+) = 0$ and $v_C(0+) = 20\ \text{V}$. [*Hint:* Try a particular solution of the form $i_p(t) = A\cos(100t) + B\sin(100t)$.]

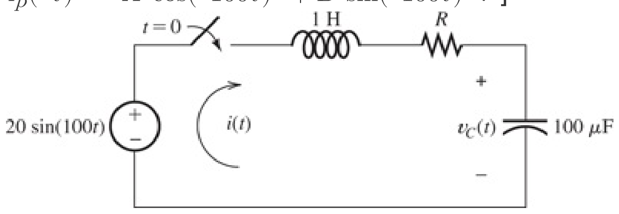

Figure P4.67

P4.68. Repeat **Problem P4.67** with $R = 200\ \Omega$.
P4.69. Repeat **Problem P4.67** with $R = 400\ \Omega$.
P4.70. Consider the circuit shown in **Figure P4.70**.

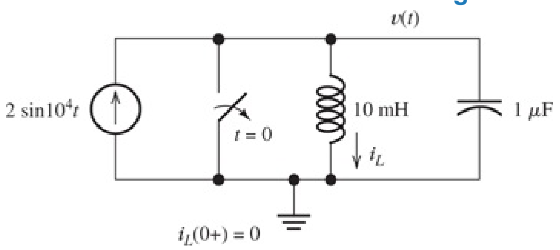

Figure P4.70

a. Write the differential equation for *v*(*t*).
b. Find the damping coefficient, the natural frequency, and the form of the complementary solution.
c. Usually, for a sinusoidal forcing function, we try a particular solution of the form $v_p(t) = A\cos\left(10^4 t\right) + B\sin\left(10^4 t\right)$. Why doesn't that work in this case?
d. Find the particular solution. [*Hint:* Try a particular solution of the form $v_p(t) = At\cos\left(10^4 t\right) + B\ t\sin\left(10^4 t\right)$.]
e. Find the complete solution for *v*(*t*).

Section 4.6: Transient Analysis Using the MATLAB Symbolic Toolbox

P4.71. Use MATLAB to derive an expression for $v_C(t)$ in the circuit of **Figure P4.13** and plot $v_C(t)$ versus time for $0 < t < 100\ \mathrm{ms}$.

P4.72. Consider the circuit shown in **Figure P4.49**. The voltage source is known as a **ramp function**, which is defined by

$$v(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t \geq 0 \end{cases}$$

Use MATLAB to derive an expression for $v_C(t)$ in terms of *R*, *C*, and *t*. Next, substitute $R = 1\ \mathrm{M}\Omega$ and $C = 1\ \mu\mathrm{F}$. Then, plot $v_C(t)$ and *v*(*t*) on the same axes for $0 < t < 5\ \mathrm{s}$.

P4.73. Consider the circuit shown in **Figure P4.50** in which the switch is open for a long time prior to $t = 0$. The initial current is $i_s(0+) = 0$. Write the differential equation for $i_s(t)$ and use MATLAB to plot $i_s(t)$ for *t* ranging from 0 to 80 ms. [*Hint:* Avoid using lowercase "i" as the first letter of the dependent variable, instead use "Is" for the current in MATLAB.]

P4.74. Consider the circuit shown in **Figure P4.64** in which the switch has been open for a long time prior to $t = 0$ and we are given $R = 25\ \Omega$.

a. Write the differential equation for *v*(*t*).
b. Assume that the capacitor is initially charged by a 50-V dc source not shown in the figure, so we have $v(0+) = 50\ \mathrm{V}$. Determine the values of $i_L(0+)$ and $v'(0+)$.
c. Use MATLAB to find the general solution for *v*(*t*).

P4.75. Consider the circuit shown in **Figure P4.70**.

a. Write the differential equation for *v*(*t*).
b. Determine the values for $v(0+)$ and $v'(0+)$.
c. Use MATLAB to find the complete solution for *v*(*t*). Then plot *v*(*t*) for $0 \leq t \leq 10\ \mathrm{ms}$.

P4.76. Use MATLAB to solve for the mesh currents in the circuit of **Figure P4.76**. The circuit has been connected for a long time prior to $t = 0$ with the switch open, so the initial values of the inductor currents are zero.

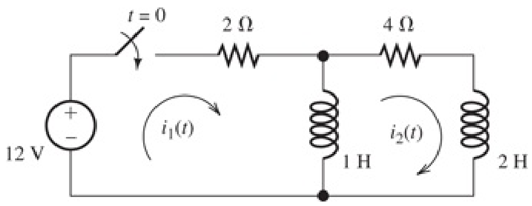

Figure P4.76

Practice Test

Here is a practice test you can use to check your comprehension of the most important concepts in this chapter. Answers can be found in **Appendix D** and complete solutions are included in the Student Solutions files. See **Appendix E** for more information about the Student Solutions.

T4.1. The switch in the circuit shown in **Figure T4.1** is closed prior to $t = 0$. The switch opens at $t = 0$. Determine the time t_x at which $v_C(t)$ reaches 15 V.

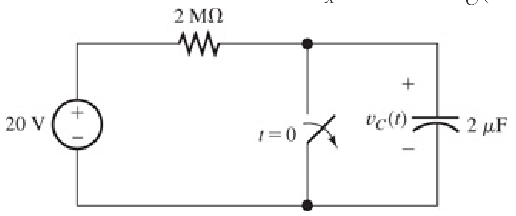

Figure T4.1

T4.2. Consider the circuit shown in **Figure T4.2**. The circuit has been operating for a long time with the switch closed prior to $t = 0$.

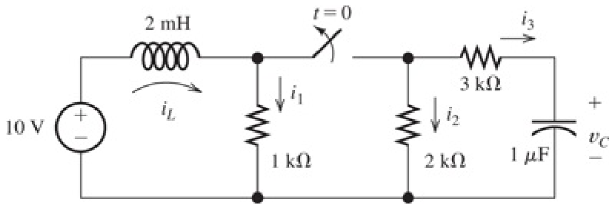

Figure T4.2

a. Determine the values of i_L, i_1, i_2, i_3, and v_C just before the switch opens.
b. Determine the values of i_L, i_1, i_2, i_3, and v_C immediately after the switch opens.
c. Find $i_L(t)$ for $t > 0$.
d. Find $v_C(t)$ for $t > 0$.

T4.3. Consider the circuit shown in **Figure T4.3**.

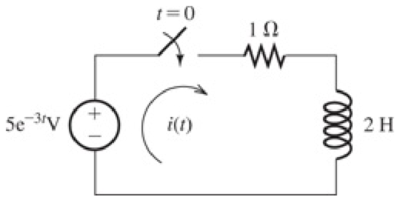

Figure T4.3

a. Write the differential equation for $i(t)$.
b. Find the time constant and the form of the complementary solution.
c. Find the particular solution.
d. Find the complete solution for $i(t)$.

T4.4. Consider the circuit shown in **Figure T4.4** in which the initial inductor current and capacitor voltage are both zero.

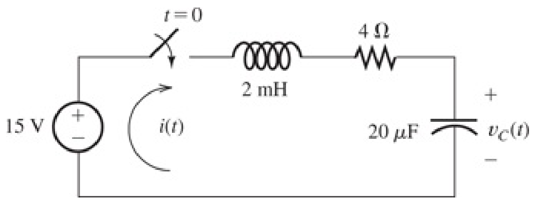

Figure T4.4

a. Write the differential equation for $v_C(t)$.
b. Find the particular solution.
c. Is this circuit overdamped, critically damped, or underdamped? Find the form of the complementary solution.
d. Find the complete solution for $v_C(t)$.

T4.5. Write the MATLAB commands to obtain the solution for the differential equation of **question T4.4** with four decimal place accuracy for the constants.