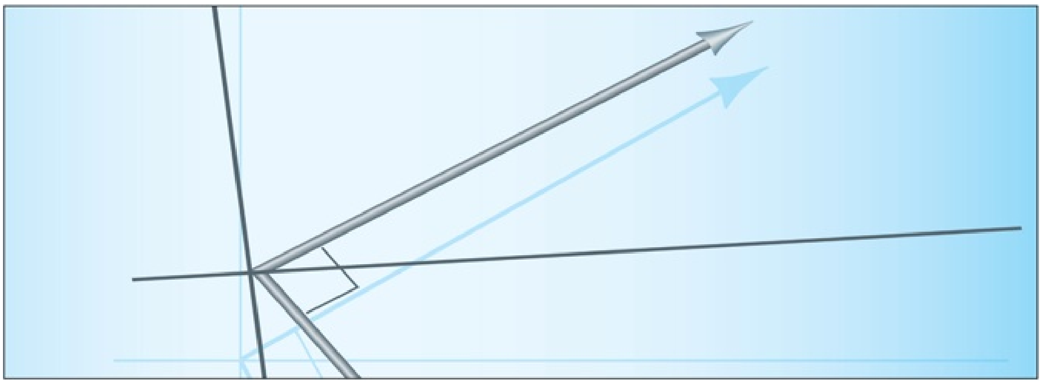
Chapter **5** Steady-State Sinusoidal Analysis



Study of this chapter will enable you to:

- Identify the frequency, angular frequency, peak value, rms value, and phase of a sinusoidal signal.
- Determine the root-mean-square (rms) value of any periodic current or voltage.
- Solve steady-state ac circuits, using phasors and complex impedances.
- Compute power for steady-state ac circuits.
- Find Thévenin and Norton equivalent circuits.
- Determine load impedances for maximum power transfer.
- Discuss the advantages of three-phase power distribution.
- Solve balanced three-phase circuits.
- Use MATLAB to facilitate ac circuit calculations.

Introduction to this chapter:

Circuits with sinusoidal sources have many important applications. For example, electric power is distributed to residences and businesses by sinusoidal currents and voltages. Furthermore, sinusoidal signals have many uses in radio communication. Finally, a branch of mathematics known as Fourier analysis shows that all signals of practical interest are composed of sinusoidal components. Thus, the study of circuits with sinusoidal sources is a central theme in electrical engineering.

*In **Chapter 4**, we saw that the response of a network has two parts: the forced response and the natural response. In most circuits, the natural response decays rapidly to zero. The forced response for sinusoidal sources persists indefinitely and, therefore, is called the steady-state response. Because the natural response quickly decays, the steady-state response is often of highest interest. In this chapter, we learn efficient methods for finding the steady-state responses for sinusoidal sources.*

We also study three-phase circuits, which are used in electric power-distribution systems. Most engineers who work in industrial settings need to understand three-phase power distribution.

5.1 Sinusoidal Currents and Voltages

A sinusoidal voltage is shown in **Figure 5.1** and is given by

$$v(t) = V_m \cos(\omega t + \theta) \tag{5.1}$$

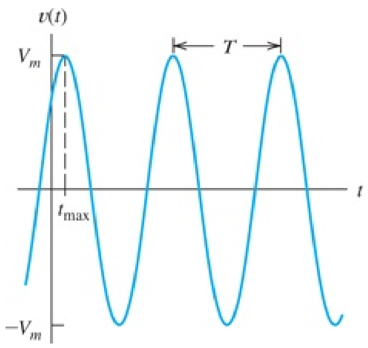


Figure 5.1
A sinusoidal voltage waveform given by $v(t) = V_m\, cos(\omega t + \theta)$. *Note:* Assuming that θ is in degrees, we have $t_{max} = \frac{-\theta}{360} \times T$. For the waveform shown, θ is $-45°$.

where V_m is the **peak value** of the voltage, ω is the **angular frequency** in radians per second, and θ is the **phase angle**.

Sinusoidal signals are periodic, repeating the same pattern of values in each **period** *T*. Because the cosine (or sine) function completes one cycle when the angle increases by 2π radians, we get

$$\omega T = 2\pi \tag{5.2}$$

The **frequency** of a periodic signal is the number of cycles completed in one second. Thus, we obtain

$$f = \frac{1}{T} \tag{5.3}$$

> We refer to ω as angular frequency with units of radians per second and *f* simply as frequency with units of hertz (Hz).

The units of frequency are hertz (Hz). (Actually, the physical units of hertz are equivalent to inverse seconds.) Solving **Equation 5.2** for the angular frequency, we have

$$\omega = \frac{2\pi}{T} \tag{5.4}$$

Using **Equation 5.3** to substitute for *T*, we find that

$$\omega = 2\pi f \tag{5.5}$$

Throughout our discussion, the argument of the cosine (or sine) function is of the form

$$\omega t + \theta$$

> Electrical engineers often write the argument of a sinusoid in mixed units: ωt is in radians and the phase angle θ is in degrees.

We assume that the angular frequency ω has units of radians per second (rad/s). However, we sometimes give the phase angle θ in degrees. Then, the argument has mixed units. If we wanted to evaluate $\cos(\omega t + \theta)$ for a particular value of time, we would have to convert θ to radians before adding the terms in the argument. Usually, we find it easier to visualize an angle expressed in degrees, and mixed units are not a problem.

For uniformity, we express sinusoidal functions by using the cosine function rather than the sine function. The functions are related by the identity

$$\sin(z) = \cos(z - 90°) \tag{5.6}$$

For example, when we want to find the phase angle of

$$v_x(t) = 10 \sin(200t + 30°)$$

we first write it as

$$\begin{aligned} v_x(t) &= 10\cos(200t + 30° - 90°) \\ &= 10\cos(200t - 60°) \end{aligned}$$

Thus, we state that the phase angle of $v_x(t)$ is $-60°$.

Root-Mean-Square Values

Consider applying a periodic voltage $v(t)$ with period T to a resistance R. The power delivered to the resistance is given by

$$p(t) = \frac{v^2(t)}{R} \tag{5.7}$$

Furthermore, the energy delivered in one period is given by

$$E_T = \int_0^T p(t)\, dt \tag{5.8}$$

The average power P_{avg} delivered to the resistance is the energy delivered in one cycle divided by the period. Thus,

$$P_{\text{avg}} = \frac{E_T}{T} = \frac{1}{T}\int_0^T p(t)\, dt \tag{5.9}$$

Using **Equation 5.7** to substitute into **Equation 5.9**, we obtain

$$P_{\text{avg}} = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}\, dt \tag{5.10}$$

This can be rearranged as

$$P_{\text{avg}} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T v^2(t)\, dt}\right]^2}{R} \tag{5.11}$$

Now, we define the **root-mean-square** (rms) value of the periodic voltage $v(t)$ as

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\, dt} \tag{5.12}$$

Using this equation to substitute into **Equation 5.11**, we get

$$P_{\text{avg}} = \frac{V_{\text{rms}}^2}{R} \tag{5.13}$$

Thus, if the rms value of a periodic voltage is known, it is relatively easy to compute the average power that the voltage can deliver to a resistance. The rms value is also called the **effective value**.

> Power calculations are facilitated by using rms values for voltage or current.

Similarly for a periodic current $i(t)$, we define the rms value as

$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\, dt} \tag{5.14}$$

and the average power delivered if $i(t)$ flows through a resistance is given by

$$P_{\text{avg}} = I_{\text{rms}}^2 R \tag{5.15}$$

RMS Value of a Sinusoid

Consider a sinusoidal voltage given by

$$v(t) = V_m \cos(\omega t + \theta) \tag{5.16}$$

To find the rms value, we substitute into **Equation 5.12**, which yields

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)\, dt} \tag{5.17}$$

Next, we use the trigonometric identity

$$\cos^2(z) = \frac{1}{2} + \frac{1}{2}\cos(2z) \tag{5.18}$$

to write **Equation 5.17** as

$$V_{\text{rms}} = \sqrt{\frac{V_m^2}{2T}\int_0^T [1 + \cos(2\omega t + 2\theta)]\, dt} \tag{5.19}$$

Integrating, we get

$$V_{\text{rms}} = \sqrt{\frac{V_m^2}{2T}\left[t + \frac{1}{2\omega}\sin(2\omega t + 2\theta)\right]_0^T} \tag{5.20}$$

Evaluating, we have

$$V_{\text{rms}} = \sqrt{\frac{V_m^2}{2T}\left[T + \frac{1}{2\omega}\sin(2\omega T + 2\theta) - \frac{1}{2\omega}\sin(2\theta)\right]} \tag{5.21}$$

Referring to **Equation 5.2**, we see that $\omega T = 2\pi$. Thus, we obtain

$$\begin{aligned}\frac{1}{2\omega}\sin(2\omega T + 2\theta) - \frac{1}{2\omega}\sin(2\theta) &= \frac{1}{2\omega}\sin(4\pi + 2\theta) - \frac{1}{2\omega}\sin(2\theta)\\ &= \frac{1}{2\omega}\sin(2\theta) - \frac{1}{2\omega}\sin(2\theta)\\ &= 0\end{aligned}$$

Therefore, **Equation 5.21** reduces to

$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}} \tag{5.22}$$

This is a useful result that we will use many times in dealing with sinusoids.

Usually in discussing sinusoids, the rms or effective value is given rather than the peak value. For example, ac power in residential wiring is distributed as a 60-Hz 115-V rms sinusoid (in the United States). Most people are aware of this, but probably few know that 115 V is the rms value and that the peak value is $V_m = V_{\text{rms}} \times \sqrt{2} = 115 \times \sqrt{2} \cong 163\ \text{V}$. (Actually, 115 V is the nominal residential distribution voltage. It can vary from approximately 105 to 130 V.)

> The rms value for a sinusoid is the peak value divided by the square root of two. This is not true for other periodic waveforms such as square waves or triangular waves.

Keep in mind that $V_{\text{rms}} = V_m/\sqrt{2}$ applies to sinusoids. To find the rms value of other periodic waveforms, we would need to employ the definition given by **Equation 5.12**.

Example 5.1 Power Delivered to a Resistance by a Sinusoidal Source

Suppose that a voltage given by $v(t) = 100\cos(100\pi t)$ V is applied to a $50-\Omega$ resistance. Sketch *v*(*t*) to scale versus time. Find the rms value of the voltage and the average power delivered to the resistance. Find the power as a function of time and sketch to scale.

Solution

By comparison of the expression given for *v*(*t*) with **Equation 5.1**, we see that $\omega = 100\pi$. Using **Equation 5.5**, we find that the frequency is $f = \omega/2\pi = 50\text{ Hz}$. Then, the period is $T = 1/f = 20\text{ ms}$. A plot of *v*(*t*) versus time is shown in **Figure 5.2(a)**.

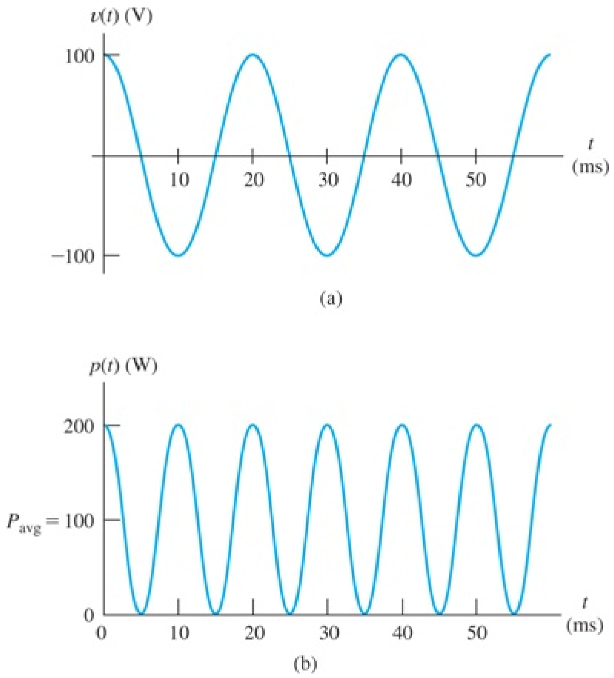


Figure 5.2
Voltage and power versus time for **Example 5.1**.

The peak value of the voltage is $V_m = 100\text{ V}$. Thus, the rms value is $V_{\text{rms}} = V_m/\sqrt{2} = 70.71\text{ V}$. Then, the average power is

$$P_{\text{avg}} = \frac{V_{\text{rms}}^2}{R} = \frac{(70.71)^2}{50} = 100\text{ W}$$

The power as a function of time is given by

$$p(t) = \frac{v^2(t)}{R} = \frac{100^2\cos^2(100\pi t)}{50} = 200\cos^2(100\pi t)\quad\text{W}$$

A plot of *p*(*t*) versus time is shown in **Figure 5.2(b)**. Notice that the power fluctuates from 0 to 200 W. However, the average power is 100 W, as we found by using the rms value.

For a sinusoidal current flowing in a resistance, power fluctuates periodically from zero to twice the average value.

RMS Values of Nonsinusoidal Voltages or Currents

Sometimes we need to determine the rms values of periodic currents or voltages that are not sinusoidal. We can accomplish this by applying the definition given by **Equation 5.12** or **5.14** directly.

Example 5.2 RMS Value of a Triangular Voltage

The voltage shown in **Figure 5.3(a)** is known as a triangular waveform. Determine its rms value.

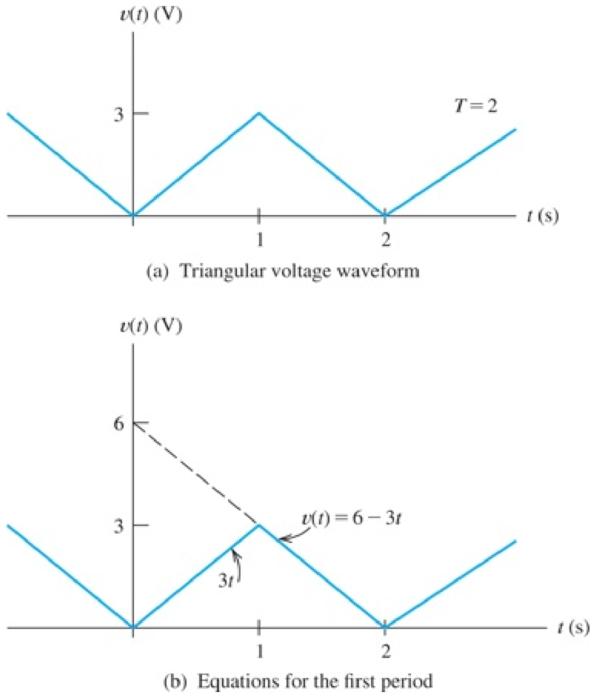


Figure 5.3
Triangular voltage waveform of **Example 5.2**.

Solution

First, we need to determine the equations describing the waveform between $t = 0$ and $t = T = 2\text{ s}$. As illustrated in **Figure 5.3(b)**, the equations for the first period of the triangular wave are

$$v(t) = \begin{cases} 3t & \text{for } 0 \le t \le 1 \\ 6 - 3t & \text{for } 1 \le t \le 2 \end{cases}$$

Equation 5.12 gives the rms value of the voltage.

$$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\, dt}$$

Dividing the interval into two parts and substituting for *v*(*t*), we have

$$V_{rms} = \sqrt{\frac{1}{2}\left[\int_0^1 9t^2 dt + \int_1^2 (6-3t)^2 dt\right]}$$

$$V_{rms} = \sqrt{\frac{1}{2}\left[3t^3\Big|_{t=0}^{t=1} + (36t - 18t^2 + 3t^3)\Big|_{t=0}^{t=1}\right]}$$

Evaluating, we find

$$V_{\text{rms}} = \sqrt{\frac{1}{2}\left[3 + (72 - 36 - 72 + 18 + 24 - 3)\right]} = \sqrt{3}\text{ V}$$

The integrals in this example are easy to carry out manually. However, when the integrals are more difficult, we can sometimes obtain answers using the MATLAB Symbolic Toolbox. Here are the MATLAB commands needed to perform the integrals in this example:

```
>> syms Vrms t
>> Vrms = sqrt((1/2)*(int(9*t^2,t,0,1) + int((6-3*t)^2,t,1,2)))
Vrms =
3^(1/2)
```

Exercise 5.1

Suppose that a sinusoidal voltage is given by

$$v(t) = 150\cos(200\pi t - 30°)\ \text{V}$$

a. Find the angular frequency, the frequency in hertz, the period, the peak value, and the rms value. Also, find the first value of time $t_{\max}$ after $t = 0$ such that *v*(*t*) attains its positive peak.

b. If this voltage is applied to a $50 - \Omega$ resistance, compute the average power delivered.

c. Sketch *v*(*t*) to scale versus time.

Answer

a. $\omega = 200\pi,\ f = 100\ \text{Hz},\ T = 10\ \text{ms},\ V_m = 150\ \text{V},\ V_{\text{rms}} = 106.1\ \text{V},\ t_{\max} = \dfrac{30°}{360°} \times T = 0.833\ \text{ms};$

b. $P_{\text{avg}} = 225\ \text{W};$

c. a plot of *v*(*t*) versus time is shown in **Figure 5.4**.

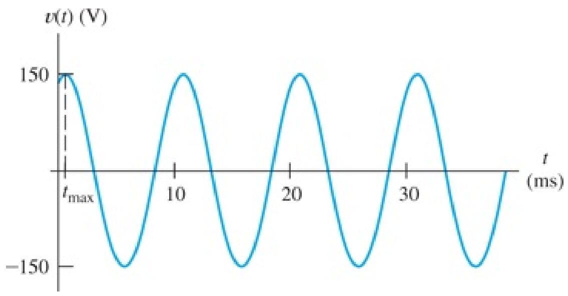


Figure 5.4
Answer for **Exercise 5.1(c)**.

Exercise 5.2

Express $v(t) = 100\sin(300\pi t + 60°)\ \text{V}$ as a cosine function.

Answer

$v(t) = 100\cos(300\pi t - 30°)\ \text{V}.$

Exercise 5.3

Suppose that the ac line voltage powering a computer has an rms value of 110 V and a frequency of 60 Hz, and the peak voltage is attained at $t = 5\ \text{ms}$. Write an expression for this ac voltage as a function of time.

Answer

$v(t) = 155.6\cos(377t - 108°)\ \text{V}.$

5.2 Phasors

In the next several sections, we will see that sinusoidal steady-state analysis is greatly facilitated if the currents and voltages are represented as vectors (called **phasors**) in the complex-number plane. In preparation for this material, you may wish to study the review of complex-number arithmetic in **Appendix A**.

We start with a study of convenient methods for adding (or subtracting) sinusoidal waveforms. We often need to do this in applying Kirchhoff's voltage law (KVL) or Kirchhoff's current law (KCL) to ac circuits. For example, in applying KVL to a network with sinusoidal voltages, we might obtain the expression

$$v(t) = 10\cos(\omega t) + 5\sin(\omega t + 60°) + 5\cos(\omega t + 90°) \tag{5.23}$$

To obtain the peak value of $v(t)$ and its phase angle, we need to put **Equation 5.23** into the form

$$v(t) = V_m \cos(\omega t + \theta) \tag{5.24}$$

This could be accomplished by repeated substitution, using standard trigonometric identities. However, that method is too tedious for routine work. Instead, we will see that we can represent each term on the right-hand side of **Equation 5.23** by a vector in the complex-number plane known as a **phasor**. Then, we can add the phasors with relative ease and convert the sum into the desired form.

Phasor Definition

For a sinusoidal voltage of the form

$$v_1(t) = V_1 \cos(\omega t + \theta_1)$$

we define the phasor as

$$\mathbf{V}_1 = V_1 \angle \theta_1$$

Thus, the phasor for a sinusoid is a complex number having a magnitude equal to the peak value and having the same phase angle as the sinusoid. We use boldface letters for phasors. (Actually, engineers are not consistent in choosing the magnitudes of phasors. In this chapter and in **Chapter 6**, we take the peak values for the magnitudes of phasors, which is the prevailing custom in circuit-analysis courses for electrical engineers. However, later in **Chapters 14** and **15**, we will take the rms values for the phasor magnitudes as power-system engineers customarily do. We will take care to label rms phasors as such when we encounter them. In this book, if phasors are not labeled as rms, you can assume that they are peak values.)

> Phasors are complex numbers that represent sinusoidal voltages or currents. The magnitude of a phasor equals the peak value and the angle equals the phase of the sinusoid (written as a cosine).

If the sinusoid is of the form

$$v_2(t) = V_2 \sin(\omega t + \theta_2)$$

we first convert to a cosine function by using the trigonometric identity

$$\sin(z) = \cos(z - 90°) \tag{5.25}$$

Thus, we have

$$v_2(t) = V_2 \cos(\omega t + \theta_2 - 90°)$$

and the phasor is

$$\mathbf{V}_2 = V_2 \angle \theta_2 - 90°$$

Phasors are obtained for sinusoidal currents in a similar fashion. Thus, for the currents

$$i_1(t) = I_1 \cos(\omega t + \theta_1)$$

and

$$i_2(t) = I_2 \sin(\omega t + \theta_2)$$

the phasors are

$$\mathbf{I}_1 = I_1 \angle \theta_1$$

and

$$\mathbf{I}_2 = I_2 \angle \theta_2 - 90°$$

respectively.

Adding Sinusoids Using Phasors

Now, we illustrate how we can use phasors to combine the terms of the right-hand side of **Equation 5.23**. In this discussion, we proceed in small logical steps to illustrate clearly why sinusoids can be added by adding their phasors. Later, we streamline the procedure for routine work.

Our first step in combining the terms in **Equation 5.23** is to write all the sinusoids as cosine functions by using **Equation 5.25**. Thus, **Equation 5.23** can be written as

$$v(t) = 10\cos(\omega t) + 5\cos(\omega t + 60^\circ - 90^\circ) + 5\cos(\omega t + 90^\circ) \tag{5.26}$$

$$v(t) = 10\cos(\omega t) + 5\cos(\omega t - 30^\circ) + 5\cos(\omega t + 90^\circ) \tag{5.27}$$

Referring to Euler's formula (**Equation A.8**) in **Appendix A**, we see that we can write

$$\cos(\theta) = \mathrm{Re}\left(e^{j\theta}\right) = \mathrm{Re}[\cos(\theta) + j\sin(\theta)] \tag{5.28}$$

where the notation Re() means that we retain only the real part of the quantity inside the parentheses. Thus, we can rewrite **Equation 5.27** as

$$v(t) = 10\,\mathrm{Re}\left[e^{j\omega t}\right] + 5\,\mathrm{Re}\left[e^{j(\omega t - 30^\circ)}\right] + 5\,\mathrm{Re}\left[e^{j(\omega t + 90^\circ)}\right] \tag{5.29}$$

When we multiply a complex number *Z* by a real number *A*, both the real and imaginary parts of *Z* are multiplied by *A*. Thus, **Equation 5.29** becomes

$$v(t) = \mathrm{Re}\left[10e^{j\omega t}\right] + \mathrm{Re}\left[5e^{j(\omega t - 30^\circ)}\right] + \mathrm{Re}\left[5e^{j(\omega t + 90^\circ)}\right] \tag{5.30}$$

Next, we can write

$$v(t) = \mathrm{Re}\left[10e^{j\omega t} + 5e^{j(\omega t - 30^\circ)} + 5e^{j(\omega t + 90^\circ)}\right] \tag{5.31}$$

because the real part of the sum of several complex quantities is equal to the sum of the real parts. If we factor out the common term $e^{j\omega t}$, **Equation 5.31** becomes

$$v(t) = \mathrm{Re}\left[\left(10 + 5e^{-j30^\circ} + 5^{j90^\circ}\right)e^{j\omega t}\right] \tag{5.32}$$

Putting the complex numbers into polar form, we have

$$v(t) = \mathrm{Re}\left[(10\angle 0^\circ + 5\angle -30^\circ + 5\angle 90^\circ)\,e^{j\omega t}\right] \tag{5.33}$$

Now, we can combine the complex numbers as

$$\begin{aligned} 10\angle 0^\circ + 5\angle -30^\circ + 5\angle 90^\circ &= 10 + 4.33 - j2.50 + j5 \\ &= 14.33 + j2.5 \\ &= 14.54\angle 9.90^\circ \\ &= 14.54e^{j9.90^\circ} \end{aligned} \tag{5.34}$$

Using this result in **Equation 5.33**, we have

$$v(t) = \mathrm{Re}\left[\left(14.54e^{j9.90^\circ}\right)e^{j\omega t}\right]$$

which can be written as

$$v(t) = \text{Re}\left[14.54e^{j(\omega t + 9.90°)}\right] \tag{5.35}$$

Now, using **Equation 5.28**, we can write this as

$$v(t) = 14.54\cos(\omega t + 9.90°) \tag{5.36}$$

Thus, we have put the original expression for $v(t)$ into the desired form. The terms on the left-hand side of **Equation 5.34** are the phasors for the terms on the right-hand side of the original expression for $v(t)$. Notice that the essential part of the work needed to combine the sinusoids is to add the phasors.

Streamlined Procedure for Adding Sinusoids

From now on, to add sinusoids, we will first write the phasor for each term in the sum, add the phasors by using complex-number arithmetic, and then write the simplified expression for the sum.

> To add sinusoids, we find the phasor for each term, add the phasors by using complex-number arithmetic, express the sum in polar form, and then write the corresponding sinusoidal time function.

Example 5.3 Using Phasors to Add Sinusoids

Suppose that

$$v_1(t) = 20\cos(\omega t - 45°)$$

$$v_2(t) = 10\sin(\omega t + 60°)$$

Reduce the sum $v_s(t) = v_1(t) + v_2(t)$ to a single term.

In using phasors to add sinusoids, all of the terms must have the same frequency.

Solution

The phasors are

Step 1: Determine the phasor for each term.

$$\mathbf{V}_1 = 20\angle -45°$$

$$\mathbf{V}_2 = 10\angle -30°$$

Notice that we have subtracted 90° to find the phase angle for $\mathbf{V}_2$ because $v_2(t)$ is a sine function rather than a cosine function.

Next, we use complex-number arithmetic to add the phasors and convert the sum to polar form:

$$\begin{aligned}\mathbf{V}_s &= \mathbf{V}_1 + \mathbf{V}_2 \\ &= 20\angle -45° + 10\angle -30° \\ &= 14.14 - j14.14 + 8.660 - j5 \\ &= 22.80 - j19.14 \\ &= 29.77\angle -40.01°\end{aligned}$$

Step 2: Use complex arithmetic to add the phasors.

Now, we write the time function corresponding to the phasor $\mathbf{V}_s$.

Step 3: Convert the sum to polar form.

Step 4: Write the result as a time function.

$$v_s(t) = 29.77\cos(\omega t - 40.01°)$$

Exercise 5.4

Reduce the following expressions by using phasors:

$$\begin{aligned} v_1(t) &= 10 \quad \cos(\omega t) + 10 \quad \sin(\omega t) \\ i_1(t) &= 10 \quad \cos(\omega t + 30°) + 5 \quad \sin(\omega t + 30°) \\ i_2(t) &= 20 \quad \sin(\omega t + 90°) + 15 \quad \cos(\omega t - 60°) \end{aligned}$$

Answer

$$\begin{aligned} v_1(t) &= 14.14 \quad \cos(\omega t - 45°) \\ i_1(t) &= 11.18 \quad \cos(\omega t + 3.44°) \\ i_2(t) &= 30.4 \quad \cos(\omega t - 25.3°) \end{aligned}$$

Phasors as Rotating Vectors

Consider a sinusoidal voltage given by

$$v(t) = V_m \cos(\omega t + \theta)$$

In developing the phasor concept, we write

$$v(t) = \text{Re}\left[V_m e^{j(\omega t + \theta)}\right]$$

The complex quantity inside the brackets is

$$V_m e^{j(\omega t + \theta)} = V_m \angle \omega t + \theta$$

This can be visualized as a vector of length V_m that rotates counterclockwise in the complex plane with an angular velocity of ω rad/s. Furthermore, the voltage *v*(*t*) is the real part of the vector, which is illustrated in **Figure 5.5**. As the vector rotates, its projection on the real axis traces out the voltage as a function of time. The phasor is simply a "snapshot" of this rotating vector at $t = 0$.

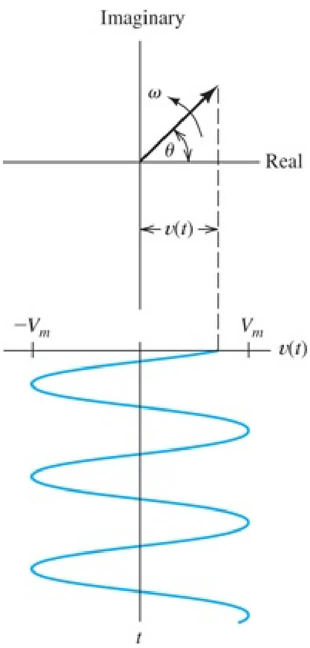


Figure 5.5
A sinusoid can be represented as the real part of a vector rotating counterclockwise in the complex plane.

> Sinusoids can be visualized as the real-axis projection of vectors rotating in the complex plane. The phasor for a sinusoid is a snapshot of the corresponding rotating vector at $t = 0$.

Phase Relationships

We will see that the phase relationships between currents and voltages are often important. Consider the voltages

$$v_1(t) = 3 \cos(\omega t + 40°)$$

and

To determine phase relationships from a phasor diagram, consider that the phasors rotate counterclockwise. Then, when standing at a fixed point, if V_1 arrives first followed by V_2 after a rotation of θ, we say that V_1 leads V_2 by θ. Alternatively, we could say that V_2 lags V_1 by θ. (Usually, we take θ as the smaller angle between the two phasors.)

$$v_2(t) = 4\cos(\omega t - 20°)$$

The corresponding phasors are

$$\mathbf{V}_1 = 3\angle 40°$$

and

$$\mathrm{V}_2 = 4\angle -20°$$

The phasor diagram is shown in **Figure 5.6**. Notice that the angle between $\mathbf{V}_1$ and $\mathbf{V}_2$ is 60°. Because the complex vectors rotate counterclockwise, we say that $\mathbf{V}_1$ *leads* $\mathbf{V}_2$ by 60°. (An alternative way to state the phase relationship is to state that $\mathbf{V}_2$ *lags* $\mathbf{V}_1$ by 60°.)

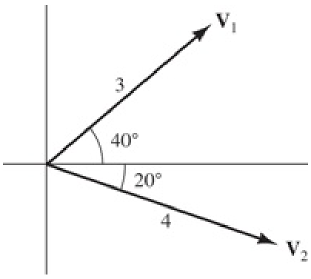


Figure 5.6
Because the vectors rotate counterclockwise, V_1 leads V_2 by 60° (or, equivalently, V_2 lags V_1 by 60°).

We have seen that the voltages versus time can be obtained by tracing the real part of the rotating vectors. The plots of $v_1(t)$ and $v_2(t)$ versus ωt are shown in **Figure 5.7**. Notice that $v_1(t)$ reaches its peak 60° earlier than $v_2(t)$. This is the meaning of the statement that $v_1(t)$ leads $v_2(t)$ by 60°.

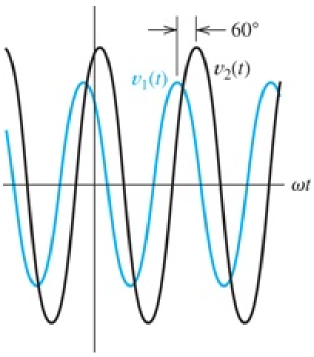


Figure 5.7
The peaks of $v_1(t)$ occur 60° before the peaks of $v_2(t)$. In other words, $v_1(t)$ leads $v_2(t)$. by 60°.

To determine phase relationships between sinusoids from their plots versus time, find the shortest time interval t_p between positive peaks of the two waveforms. Then, the phase angle is $\theta = (t_p / T) \times 360°$. If the peak of $v_1(t)$ occurs first, we say that $v_1(t)$ leads $v_2(t)$ or that $v_2(t)$ lags $v_1(t)$.

Exercise 5.5

Consider the voltages given by

$$\begin{aligned} v_1(t) &= 10 \cos(\omega t - 30°) \\ v_2(t) &= 10 \cos(\omega t + 30°) \\ v_3(t) &= 10 \sin(\omega t + 45°) \end{aligned}$$

State the phase relationship between each pair of the voltages. (*Hint:* Find the phasor for each voltage and draw the phasor diagram.)

Answer

v_1 lags v_2 by 60° (or v_2 leads v_1 by 60°)

v_1 leads v_3 by 15° (or v_3 lags v_1 by 15°)

v_2 leads v_3 by 75° (or v_3 lags v_2 by 75°)

5.3 Complex Impedances

In this section, we learn that by using phasors to represent sinusoidal voltages and currents, we can solve sinusoidal steady-state circuit problems with relative ease compared with the methods of **Chapter 4**. Except for the fact that we use complex arithmetic, sinusoidal steady-state analysis is virtually the same as the analysis of resistive circuits, which we studied in **Chapter 2**.

Inductance

Consider an inductance in which the current is a sinusoid given by

$$i_L(t) = I_m \sin(\omega t + \theta) \tag{5.37}$$

Recall that the voltage across an inductance is

$$v_L(t) = L\frac{di_L(t)}{dt} \tag{5.38}$$

Substituting **Equation 5.37** into **Equation 5.38** and reducing, we obtain

$$v_L(t) = \omega L I_m \cos(\omega t + \theta) \tag{5.39}$$

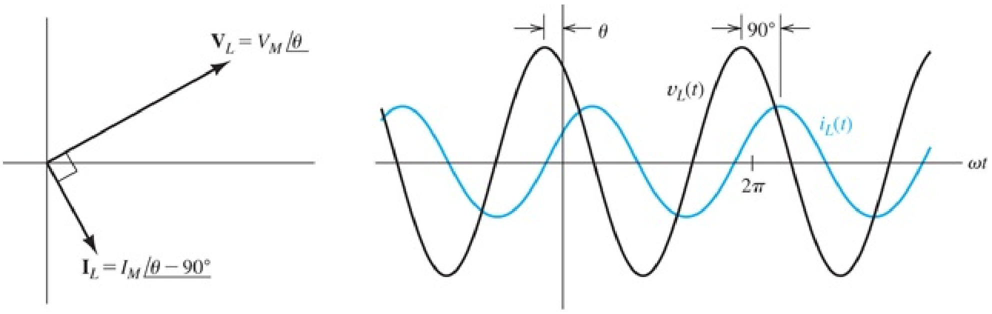
Now, the phasors for the current and voltage are

$$\mathbf{I}_L = I_m \angle \theta - 90° \tag{5.40}$$

and

$$\mathbf{V}_L = \omega L I_m \angle \theta = V_m \angle \theta \tag{5.41}$$

The phasor diagram of the current and voltage is shown in **Figure 5.8(a)**. The corresponding waveforms of current and voltage are shown in **Figure 5.8(b)**. *Notice that the current lags the voltage by* 90° *for a pure inductance.*



(a) Phasor diagram

(b) Current and voltage versus time

Figure 5.8
Current lags voltage by 90° in a pure inductance.

Current lags voltage by 90° for a pure inductance.

Equation 5.41 can be written in the form

$$V_L = (\omega L \angle 90°) \times I_m \angle \theta - 90° \quad (5.42)$$

Using **Equation 5.40** to substitute into **Equation 5.42**, we find that

$$V_L = (\omega L \angle 90°) \times I_L \quad (5.43)$$

which can also be written as

$$V_L = j\omega L \times I_L \quad (5.44)$$

We refer to the term $j\omega L = \omega L \angle 90°$ as the **impedance** of the inductance and denote it as Z_L. Thus, we have

$$Z_L = j\omega L = \omega L \angle 90° \quad (5.45)$$

and

$$V_L = Z_L I_L \quad (5.46)$$

Thus, the phasor voltage is equal to the impedance times the phasor current. This is Ohm's law in phasor form. However, for an inductance, the impedance is an imaginary number, whereas resistance is a real number. (Impedances that are pure imaginary are also called **reactances**.)

> **Equation 5.46** shows that phasor voltage and phasor current for an inductance are related in a manner analogous to Ohm's law.

Capacitance

In a similar fashion for a capacitance, we can show that if the current and voltage are sinusoidal, the phasors are related by

$$\mathbf{V}_C = Z_C \mathbf{I}_C \tag{5.47}$$

in which the impedance of the capacitance is

$$Z_C = -j\frac{1}{\omega C} = \frac{1}{j\omega C} = \frac{1}{\omega C}\angle -90^\circ \tag{5.48}$$

Notice that the impedance of a capacitance is also a pure imaginary number.

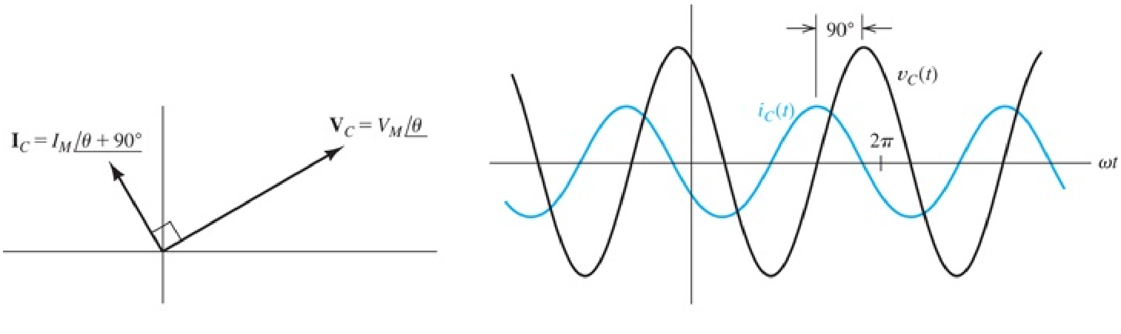
Suppose that the phasor voltage is

$$\mathbf{V}_C = V_m \angle\theta$$

Then, the phasor current is

$$\begin{aligned}\mathbf{I}_C &= \frac{\mathbf{V}_C}{Z_C} = \frac{V_m \angle\theta}{(1/\omega C) - \angle 90^\circ} = \omega C V_m \angle\theta + 90^\circ \\ \mathbf{I}_C &= I_m \angle\theta + 90^\circ\end{aligned}$$

where $I_m = \omega C V_m$. The phasor diagram for current and voltage in a pure capacitance is shown in **Figure 5.9(a)**. The corresponding plots of current and voltage versus time are shown in **Figure 5.9(b)**. Notice that the current leads the voltage by 90°. (On the other hand, current lags voltage for an inductance. This is easy to remember if you know *ELI* the *ICE* man. The letter *E* is sometimes used to stand for *electromotive force*, which is another term for voltage, *L* and *C* are used for inductance and capacitance, respectively, and *I* is used for current.)



(a) Phasor diagram

(b) Current and voltage versus time

Figure 5.9
Current leads voltage by 90° in a pure capacitance.

> Current leads voltage by 90° for a pure capacitance.

Resistance

For a resistance, the phasors are related by

$$V_R = RI_R \tag{5.49}$$

Because resistance is a real number, the current and voltage are in phase, as illustrated in **Figure 5.10**.

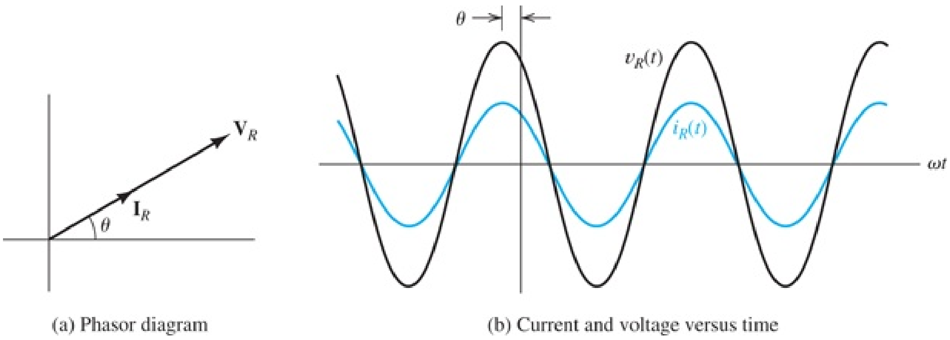


Figure 5.10
For a pure resistance, current and voltage are in phase.

> Current and voltage are in phase for a resistance.

Complex Impedances in Series and Parallel

Impedances of inductances, capacitances, and resistances are combined in series and parallel in the same manner as resistances. (Recall that we combine capacitances in series as we do resistances in parallel. However, the **impedances** of capacitances are combined in the same manner as resistances.)

Example 5.4 Combining Impedances in Series and Parallel

Determine the complex impedance between terminals shown in **Figure 5.11(a)** for $\omega = 1000 \text{ rad/s}$.

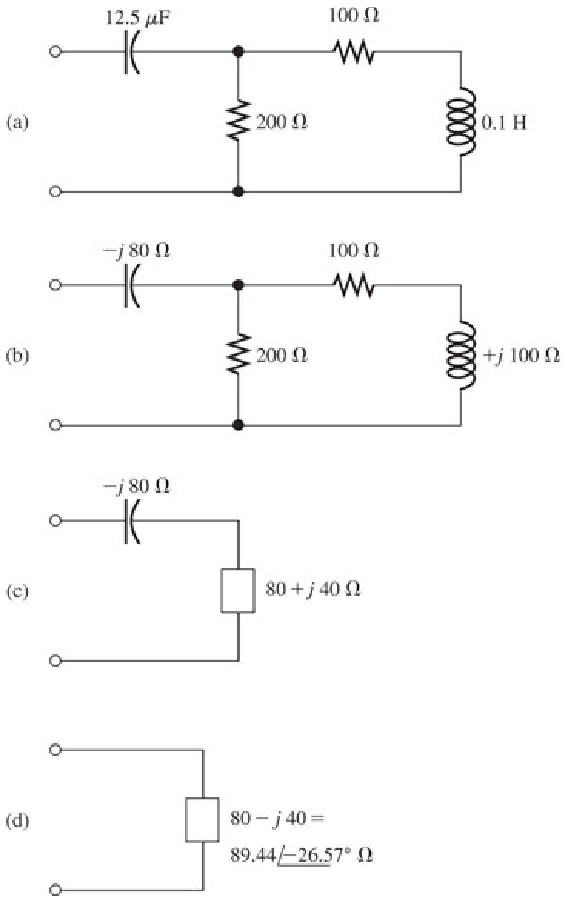


Figure 5.11
Circuit of **Example 5.4**.

Solution

First, the impedance of the inductance is $j\omega L = j100\ \Omega,$ and the impedance of the capacitance is $-j/(\ \omega C)\ \ = -j80\ \Omega.$ These values are shown in **Figure 5.11(b)**.

Next, we observe that the $200 - \Omega$ resistance is in parallel with the series impedance $100 + j100\ \Omega.$ The impedance of this parallel combination is

$$\frac{1}{1/100 + 1/(\ 100 + j100)} = 80 + j40\ \Omega$$

The resulting equivalent is shown in **Figure 5.11(c)**. (We use rectangular boxes to represent the combined impedances of dissimilar types of components.)

Then, notice that the impedances in **Figure 5.1(c)** are in series, and they are combined by adding them resulting in:

$$-j80 + 80 + j40 = 80 - j40 = 89.44 - 26.57\ \Omega$$

This is shown in **Figure 5.11(d)**.

Exercise 5.6

A voltage $v_L(t) = 100\cos(200t)$ is applied to a 0.25-H inductance. (Notice that $\omega = 200.$)

a. Find the impedance of the inductance, the phasor current, and the phasor voltage.
b. Draw the phasor diagram.

Answer

a. $Z_L = j50 = 50\angle 90°,\ I_L = 2\angle -90°,\ V_L = 100\angle 0°;$
b. the phasor diagram is shown in **Figure 5.12(a)**.

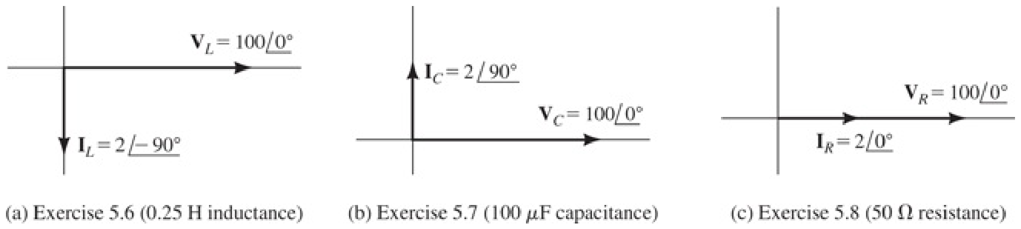


Figure 5.12
Answers for **Exercises 5.6**, **5.7**, and **5.8**. The scale has been expanded for the currents compared with the voltages so the current phasors can be easily seen.

Exercise 5.7

A voltage $v_C(t) = 100\cos(200t)$ is applied to a $100 - \mu\mathrm{F}$ capacitance.

a. Find the impedance of the capacitance, the phasor current, and the phasor voltage.
b. Draw the phasor diagram.

Answer

a. $Z_C = -j50 = 50\angle -90°,\ I_C = 2\angle 90°,\ V_C = 100\angle 0°;$
b. the phasor diagram is shown in **Figure 5.12(b)**.n

Exercise 5.8

A voltage $v_R(t) = 100\cos(200t)$ is applied to a $50 - \Omega$ resistance.

a. Find the phasor for the current and the phasor voltage.
b. Draw the phasor diagram.

Answer

a. $I_R = 2\angle 0°,\ V_R = 100\angle 0°;$
b. the phasor diagram is shown in **Figure 5.12(c)**.n

5.4 Circuit Analysis with Phasors and Complex Impedances

Kirchhoff's Laws in Phasor Form

Recall that KVL requires that the voltages sum to zero for any closed path in an electrical network. A typical KVL equation is

$$v_1(t) + v_2(t) - v_3(t) = 0 \tag{5.50}$$

If the voltages are sinusoidal, they can be represented by phasors. Then, **Equation 5.50** becomes

$$\mathbf{V}_1 + \mathbf{V}_2 - \mathbf{V}_3 = 0 \tag{5.51}$$

Thus, we can apply KVL directly to the phasors. The sum of the phasor voltages equals zero for any closed path.

Similarly, KCL can be applied to currents in phasor form. The sum of the phasor currents entering a node must equal the sum of the phasor currents leaving.

Circuit Analysis Using Phasors and Impedances

We have seen that phasor currents and voltages are related by complex impedances, and Kirchhoff's laws apply in phasor form. Except for the fact that the voltages, currents, and impedances can be complex, the equations are exactly like those of resistive circuits.

A step-by-step procedure for steady-state analysis of circuits with sinusoidal sources is

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All of the sources must have the same frequency.)
2. Replace inductances by their complex impedances $Z_L = j\omega L = \omega L \angle 90°$. Replace capacitances by their complex impedances $Z_C = 1/(j\omega C) = (1/\omega C) \angle -90°$. Resistances have impedances equal to their resistances.
3. Analyze the circuit by using any of the techniques studied in **Chapter 2**, and perform the calculations with complex arithmetic.

Example 5.5 Steady-State AC Analysis of a Series Circuit

Find the steady-state current for the circuit shown in **Figure 5.13(a)**. Also, find the phasor voltage across each element and construct a phasor diagram.

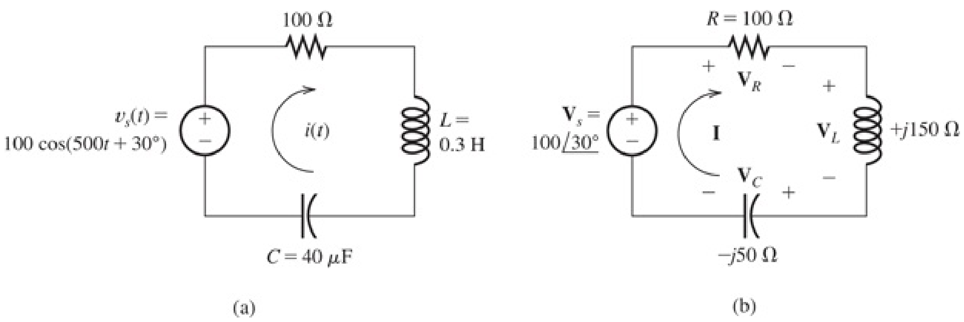


Figure 5.13
Circuit for **Example 5.5**.

Solution

From the expression given for the source voltage $v_s(t)$, we see that the peak voltage is 100 V, the angular frequency is $\omega = 500$, and the phase angle is 30°. The phasor for the voltage source is

$$V_s = 100 \angle 30°$$

Step 1: Replace the time description of the voltage source with the corresponding phasor.

The complex impedances of the inductance and capacitance are

Step 2: Replace inductances and capacitances with their complex impedances.

$$Z_L = j\omega L = j500 \times 0.3 = j150\ \Omega$$

and

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{500 \times 40 \times 10^{-6}} = -j50\ \Omega$$

The transformed circuit is shown in **Figure 5.13(b)**. All three elements are in series. Thus, we find the equivalent impedance of the circuit by adding the impedances of all three elements:

Step 3: Use complex arithmetic to analyze the circuit.

$$Z_{\text{eq}} = R + Z_L + Z_C$$

Substituting values, we have

$$Z_{\text{eq}} = 100 + j150 - j50 = 100 + j100$$

Converting to polar form, we obtain

$$Z_{\text{eq}} = 141.4 \angle 45°$$

Now, we can find the phasor current by dividing the phasor voltage by the equivalent impedance, resulting in

$$I = \frac{V_s}{Z} = \frac{100 \angle 30°}{141.4 \angle 45°} = 0.707 \angle -15°$$

As a function of time, the current is

$$i(t) = 0.707 \cos(500t - 15°)$$

Next, we can find the phasor voltage across each element by multiplying the phasor current by the respective impedance:

$$V_R = R \times I = 100 \times 0.707 \angle -15° = 70.7 \angle -15°$$

$$V_L = j\omega L \times I = \omega L \angle 90° \times I = 150 \angle 90° \times 0.707 \angle -15°$$
$$= 106.1 \angle 75°$$

$$V_C = -j\frac{1}{\omega C} \times I = \frac{1}{\omega C} \angle -90° \times I = 50 \angle -90° \times 0.707 \angle -15°$$
$$= 35.4 \angle -105°$$

The phasor diagram for the current and voltages is shown in **Figure 5.14**. Notice that the current **I** lags the source voltage V_s by 45°. As expected, the voltage V_R and current **I** are in phase for the resistance. For the inductance, the voltage V_L leads the current **I** by 90°. For the capacitance, the voltage V_C lags the current by 90°.

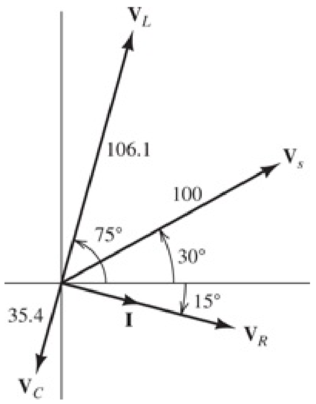


Figure 5.14
Phasor diagram for **Example 5.5**.

Example 5.6 Series and Parallel Combinations of Complex Impedances

Consider the circuit shown in **Figure 5.15(a)**. Find the voltage $v_C(t)$ in steady state. Find the phasor current through each element, and construct a phasor diagram showing the currents and the source voltage.

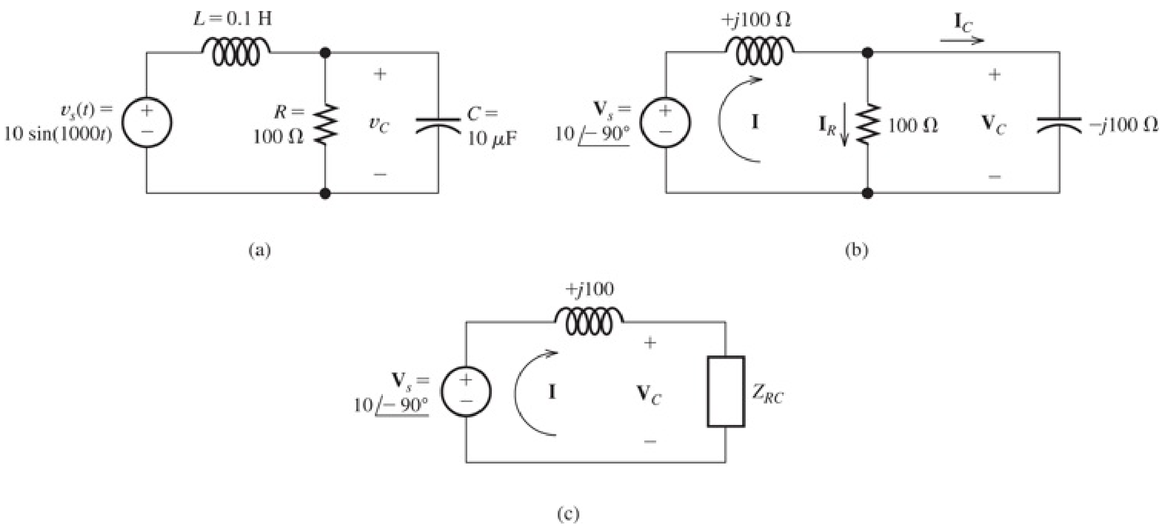


Figure 5.15
Circuit for **Example 5.6**.

Solution

The phasor for the voltage source is $V_s = 10\angle -90°$. [Notice that $v_s(t)$ is a sine function rather than a cosine function, and it is necessary to subtract 90° from the phase.] The angular frequency of the source is $\omega = 1000$. The impedances of the inductance and capacitance are

$$Z_L = j\omega L = j1000 \times 0.1 = j100\ \Omega$$

Step 1: Replace the time description of the voltage source with the corresponding phasor.

and

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{1000 \times 10 \times 10^{-6}} = -j100\ \Omega$$

Step 2: Replace inductances and capacitances with their complex impedances.

The transformed network is shown in **Figure 5.15(b)**.

Step 3: Use complex arithmetic to analyze the circuit.

To find V_C, we will first combine the resistance and the impedance of the capacitor in parallel. Then, we will use the voltage-division principle to compute the voltage across the *RC* combination. The impedance of the parallel *RC* circuit is

$$\begin{aligned} Z_{RC} &= \frac{1}{1/R + 1/Z_C} = \frac{1}{1/100 + 1/(-j100)} \\ &= \frac{1}{0.01 + j0.01} = \frac{1\angle 0°}{0.01414\angle 45°} = 70.71\angle -45° \end{aligned}$$

Converting to rectangular form, we have

$$Z_{RC} = 50 - j50$$

The equivalent network is shown in **Figure 5.15(c)**.

Now, we use the voltage-division principle to obtain

$$\begin{aligned} V_C &= V_s\frac{Z_{RC}}{Z_L + Z_{RC}} = 10\angle -90°\frac{70.71\angle -45°}{j100 + 50 - j50} \\ &= 10\angle -90°\frac{70.71\angle -45°}{50 + j50} = 10\angle -90°\frac{70.71\angle -45°}{70.71\angle 45°} \\ &= 10\angle -180° \end{aligned}$$

Converting the phasor to a time function, we have

$$v_C(t) = 10\cos(1000t - 180°) = -10\cos(1000t)$$

Next, we compute the current in each element yielding

$$
\begin{aligned}
I &= \frac{V_s}{Z_L + Z_{RC}} = \frac{10\angle -90^\circ}{j100 + 50 - j50} = \frac{10\angle -90^\circ}{50 + j50} \\
&= \frac{10\angle -90^\circ}{70.71\angle 45^\circ} = 0.1414\angle -135^\circ
\end{aligned}
$$

$$
I_R = \frac{V_C}{R} = \frac{10\angle -180^\circ}{100} = 0.1\angle -180^\circ
$$

$$
I_C = \frac{V_C}{Z_C} = \frac{10\angle -180^\circ}{-j100} = \frac{10\angle -180^\circ}{100\angle -90^\circ} = 0.1\angle -90^\circ
$$

The phasor diagram is shown in **Figure 5.16**.

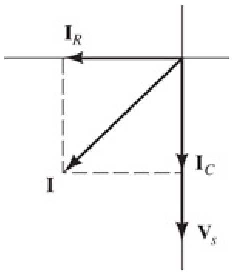


Figure 5.16
Phasor diagram for **Example 5.6**.

Node-Voltage Analysis

We can perform node-voltage analysis by using phasors in a manner parallel to that of **Chapter 2**. We illustrate with an example.

Example 5.7 Steady-State AC Node-Voltage Analysis

Use the node-voltage technique to find $v_1(t)$ in steady state for the circuit shown in **Figure 5.17(a)**.

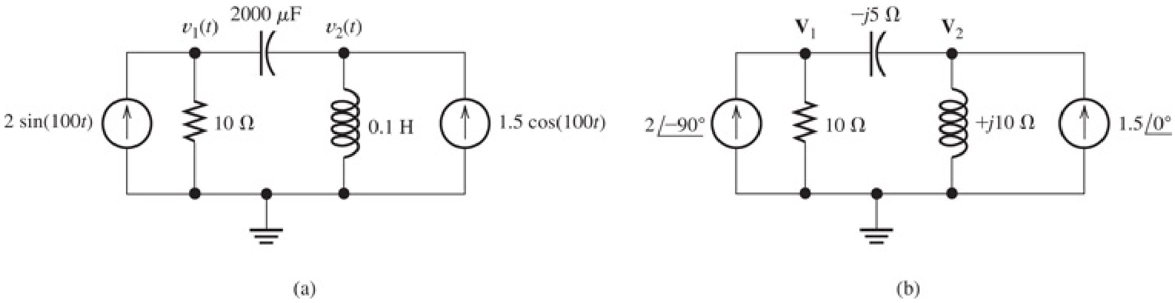


Figure 5.17
Circuit for **Example 5.7**.

Solution

The transformed network is shown in **Figure 5.17(b)**. We obtain two equations by applying KCL at node 1 and at node 2. This yields

$$\frac{V_1}{10} + \frac{V_1 - V_2}{-j5} = 2\angle -90°$$

$$\frac{V_2}{j10} + \frac{V_2 - V_1}{-j5} = 1.5\angle 0°$$

These equations can be put into the standard form

$$\begin{aligned} (0.1 + j0.2)V_1 - j0.2V_2 &= -j2 \\ -j0.2V_1 + j0.1V_2 &= 1.5 \end{aligned}$$

Now, we solve for V_1 yielding

$$V_1 = 16.1\angle 29.7°$$

Then, we convert the phasor to a time function and obtain

$$v_1(t) = 16.1\cos(100t + 29.7°)$$

Mesh-Current Analysis

In a similar fashion, you can use phasors to carry out mesh-current analysis in ac circuits.

Example 5.8 Steady-State AC Mesh-Current Analysis

Use the mesh-current technique to find $i_1(t)$ in steady state for the circuit shown in **Figure 5.18(a)**.

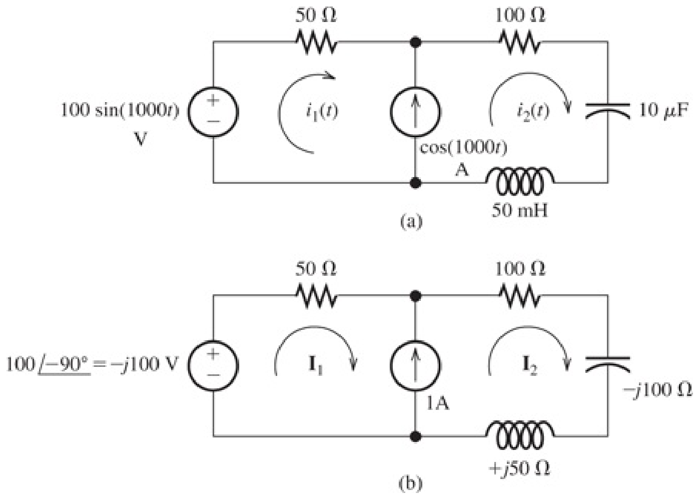


Figure 5.18
Circuit of **Example 5.8**.

Solution

First, we note that $\omega = 1000 \text{ rad/s}$ for both of the sources in this circuit. The impedance of the inductance is $j\omega L = j50\ \Omega,$ and the impedance of the capacitance is $-j/(\omega C) = -j100\ \Omega.$ The transformed network is shown in **Figure 5.18(b).**

Next, we write KVL equations. We cannot write equations around either mesh 1 or mesh 2 because we do not know the voltage across the current source. The only option is to write a KVL equation around the outside of the network, which yields:

$$j100 + 50I_1 + 100I_2 - j100\ I_2 + j50\ I_2 = 0$$

The current flowing upward through the current source is

$$I_2 - I_1 = 1$$

In standard form, these equations become:

$$50I_1 + (100 - j50)\ I_2 = -j100$$
$$-I_1 + I_2 = 1$$

Solving these equations results in:

$$I_1 = 0.7071 \angle -135° \quad \text{or} \quad i_1(t) = 0.7071 \cos(1000t - 135°) \text{ V}$$

Exercise 5.9

Consider the circuit shown in **Figure 5.19(a)**.

a. Find *i*(*t*).

b. Construct a phasor diagram showing all three voltages and the current.

c. What is the phase relationship between $v_s(t)$ and $i(t)$?

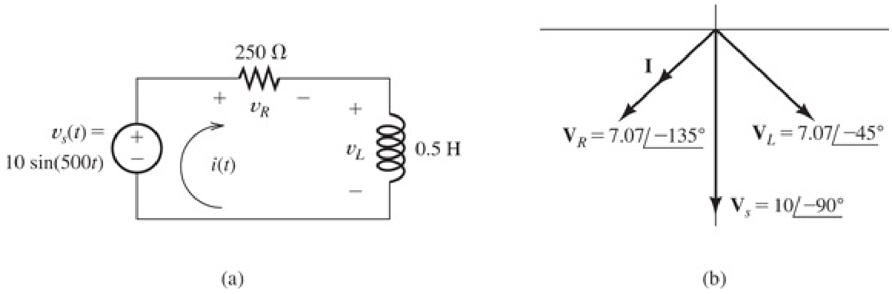


Figure 5.19
Circuit and phasor diagram for **Exercise 5.9**.

Answer

a. a. $i(t) = 0.0283 \cos(500t - 135°)$;

b. the phasor diagram is shown in **Figure 5.19(b)**;

c. *i*(*t*) lags $v_s(t)$ by 45°.

Exercise 5.10

Find the phasor voltage and the phasor current through each element in the circuit of **Figure 5.20**.

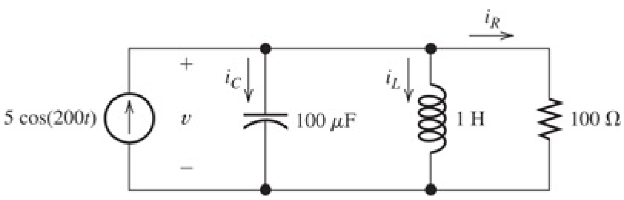


Figure 5.20
Circuit for **Exercise 5.10**.

Answer

$V = 277\ \angle -56.3°,\ I_C = 5.55\ \angle 33.7°,\ I_L = 1.39\ \angle -146.3°,\ I_R = 2.77\angle -56.3°$.

Exercise 5.11

Solve for the mesh currents shown in **Figure 5.21**.

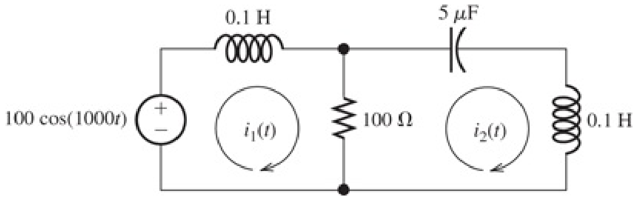


Figure 5.21
Circuit for **Exercise 5.11**.

Answer

$i_1(t) = 1.414\cos(1000t - 45°)$, $i_2(t) = \cos(1000t)$.

5.5 Power in AC Circuits

Consider the situation shown in **Figure 5.22**. A voltage $v(t) = V_m \cos(\omega t)$ is applied to a network composed of resistances, inductances, and capacitances (i.e., an *RLC* network). The phasor for the voltage source is $\boldsymbol{V} = V_m \angle 0°$, and the equivalent impedance of the network is $Z = |Z| \angle \theta = R + jX$. The phasor current is

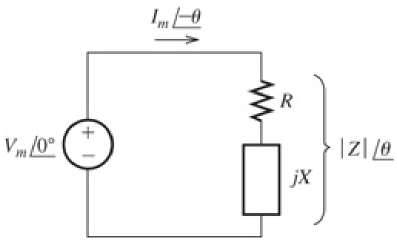


Figure 5.22

A voltage source delivering power to a load impedance $Z = R + jX$.

$$\boldsymbol{I} = \frac{\boldsymbol{V}}{Z} = \frac{V_m \angle 0°}{|Z| \angle \theta} = I_m \angle -\theta \tag{5.52}$$

where we have defined

$$I_m = \frac{V_m}{|Z|} \tag{5.53}$$

Before we consider the power delivered by the source to a general load, it is instructive to consider a pure resistive load, a pure inductive load, and a pure capacitive load.

Current, Voltage, and Power for a Resistive Load

First, consider the case in which the network is a pure resistance. Then, $\theta = 0,$ and we have

$$\begin{aligned} v(t) &= V_m \cos(\omega t) \\ i(t) &= I_m \cos(\omega t) \\ p(t) &= v(t)\, i(t) = V_m I_m \cos^2(\omega t) \end{aligned}$$

Plots of these quantities are shown in **Figure 5.23**. Notice that the current is in phase with the voltage (i.e., they both reach their peak values at the same time). Because *p(t)* is positive at all times, we conclude that energy flows continually in the direction from the source to the load (where it is converted to heat). Of course, the value of the power rises and falls with the voltage (and current) magnitude.

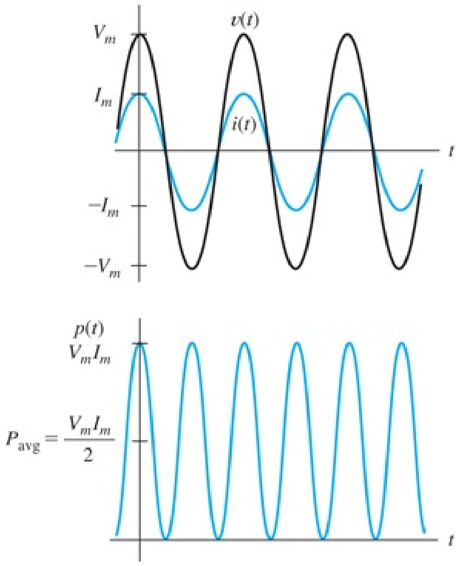


Figure 5.23
Current, voltage, and power versus time for a purely resistive load.

> Average power is absorbed by resistances in ac circuits.

Current, Voltage, and Power for an Inductive Load

Next, consider the case in which the load is a pure inductance for which $Z = \omega L \angle 90°$. Thus, $\theta = 90°$, and we get

$$\begin{aligned} v(t) &= V_m \cos(\omega t) \\ i(t) &= I_m \cos(\omega t - 90°) = I_m \sin(\omega t) \\ p(t) &= v(t)\, i(t) = V_m I_m \cos(\omega t)\, \sin(\omega t) \end{aligned}$$

Using the trigonometric identity $\cos(x)\ \sin(x) = (1/2)\ \sin(2x)$, we find that the expression for the power becomes

$$p(t) = \frac{V_m I_m}{2} \sin(2\omega t)$$

Power surges into and out of inductances in ac circuits. The average power absorbed by inductances is zero.

Plots of the current, voltage, and power are shown in **Figure 5.24(a)**. Notice that the current lags the voltage by 90°. Half of the time the power is positive, showing that energy is delivered to the inductance, where it is stored in the magnetic field. For the other half of the time, power is negative, showing that the inductance returns energy to the source. Notice that the average power is zero. In this case, we say that **reactive power** flows from the source to the load.

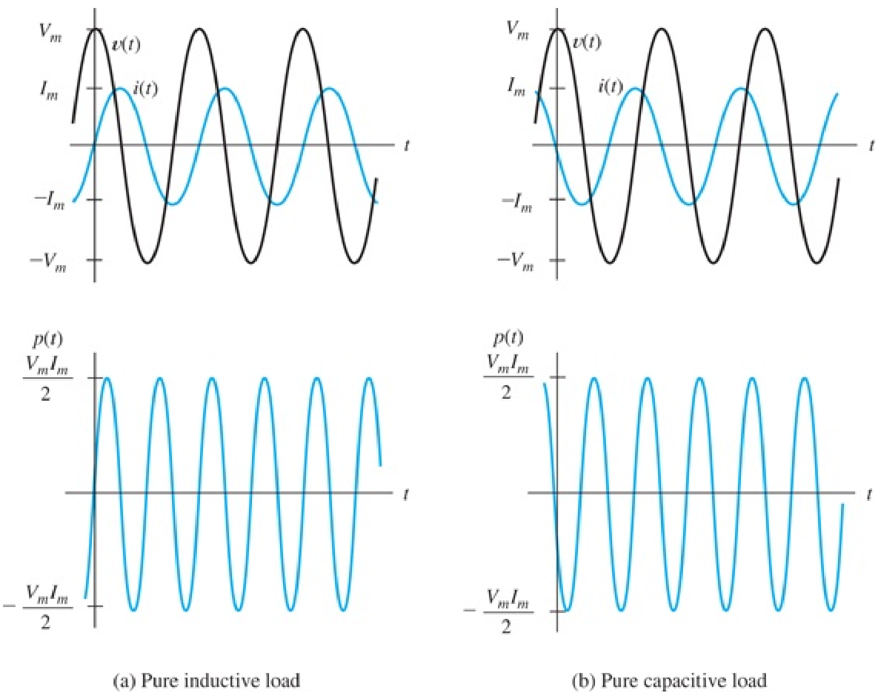


Figure 5.24

Current, voltage, and power versus time for pure energy-storage elements.

Current, Voltage, and Power for a Capacitive Load

Next, consider the case in which the load is a pure capacitance for which $Z = (1/\omega C) \angle -90°$. Then, $\theta = -90°$, and we have

$$\begin{aligned} v(t) &= V_m \cos(\omega t) \\ i(t) &= I_m \cos(\omega t + 90°) = -I_m \sin(\omega t) \\ p(t) &= v(t)\, i(t) = -V_m I_m \cos(\omega t) \sin(\omega t) \\ &= -\frac{V_m I_m}{2} \sin(2\omega t) \end{aligned}$$

Power surges into and out of capacitances in ac circuits. The average power absorbed by capacitances is zero.

Plots of the current, voltage, and power are shown in **Figure 5.24(b)**. Here again, the average power is zero, and we say that reactive power flows. Notice, however, that the power for the capacitance carries the opposite sign as that for the inductance. Thus, we say that reactive power is positive for an inductance and is negative for a capacitance. If a load contains both inductance and capacitance with reactive powers of equal magnitude, the reactive powers cancel.

Importance of Reactive Power

Even though no average power is consumed by a pure energy-storage element (inductance or capacitance), reactive power is still of concern to power-system engineers because transmission lines, transformers, fuses, and other elements must be capable of withstanding the current associated with reactive power. It is possible to have loads composed of energy-storage elements that draw large currents requiring heavy-duty wiring, even though little average power is consumed. Therefore, electric-power companies charge their industrial customers for reactive power (but at a lower rate) as well as for total energy delivered.

The power flow back and forth to inductances and capacitances is called reactive power. Reactive power flow is important because it causes power dissipation in the lines and transformers of a power distribution system.

Power Calculations for a General Load

Now, let us consider the voltage, current, and power for a general *RLC* load for which the phase θ can be any value from $-90°$ to $+90°$. We have

$$v(t) = V_m \cos(\omega t) \tag{5.54}$$

$$i(t) = I_m \cos(\omega t - \theta) \tag{5.55}$$

$$p(t) = V_m I_m \cos(\omega t) \cos(\omega t - \theta) \tag{5.56}$$

Using the trigonometric identity

$$\cos(\omega t - \theta) = \cos(\theta) \cos(\omega t) + \sin(\theta) \sin(\omega t)$$

we can put **Equation 5.56** into the form

$$p(t) = V_m I_m \cos(\theta) \cos^2(\omega t) + V_m I_m \sin(\theta) \cos(\omega t) \sin(\omega t) \tag{5.57}$$

Using the identities

$$\cos^2(\omega t) = \frac{1}{2} + \frac{1}{2} \cos(2\omega t)$$

and

$$\cos(\omega t) \sin(\omega t) = \frac{1}{2} \sin(2\omega t)$$

we find that **Equation 5.57** can be written as

$$p(t) = \frac{V_m I_m}{2} \cos(\theta) [1 + \cos(2\omega t)] + \frac{V_m I_m}{2} \sin(\theta) \sin(2\omega t) \tag{5.58}$$

Notice that the terms involving $\cos(2\omega t)$ and $\sin(2\omega t)$ have average values of zero. Thus, the average power *P* is given by

$$P = \frac{V_m I_m}{2} \cos(\theta) \tag{5.59}$$

Using the fact that $V_{\text{rms}} = V_m / \sqrt{2}$ and $I_{\text{rms}} = I_m / \sqrt{2}$, we can write the expression for average power as

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta) \tag{5.60}$$

As usual, the units of power are watts (W).

Power Factor

The term $\cos(\theta)$ is called the **power factor**:

$$PF = \cos(\theta) \tag{5.61}$$

To simplify our discussion, we assumed a voltage having zero phase. In general, the phase of the voltage may have a value other than zero. Then, θ should be taken as the phase of the voltage θ_v minus the phase of the current θ_i, or

> Power factor is the cosine of the angle θ by which the current lags the voltage. (If the current leads the voltage, the angle is negative.)

$$\theta = \theta_v - \theta_i \tag{5.62}$$

Sometimes, θ is called the **power angle**.

Often, power factor is stated as a percentage. Also, it is common to state whether the current leads (capacitive load) or lags (inductive load) the voltage. A typical power factor would be stated to be 90 percent lagging, which means that $\cos(\theta) = 0.9$ and that the current lags the voltage.

> Often, power factor is expressed as a percentage.

> If the current lags the voltage, the power factor is said to be inductive or lagging. If the current leads the voltage, the power factor is said to be capacitive or leading.

Reactive Power

In ac circuits, energy flows into and out of energy storage elements (inductances and capacitances). For example, when the voltage magnitude across a capacitance is increasing, energy flows into it, and when the voltage magnitude decreases, energy flows out. Similarly, energy flows into an inductance when the current flowing through it increases in magnitude. Although instantaneous power can be very large, the net energy transferred per cycle is zero for either an ideal capacitance or inductance.

When a capacitance and an inductance are in parallel (or series) energy flows into one, while it flows out of the other. Thus, the power flow of a capacitance tends to cancel that of an inductance at each instant in time.

The peak instantaneous power associated with the energy storage elements contained in a general load is called **reactive power** and is given by

$$Q = V_{\mathrm{rms}} I_{\mathrm{rms}} \sin(\theta) \tag{5.63}$$

where θ is the power angle given by **Equation 5.62**, V_{rms} is the effective (or rms) voltage across the load, and I_{rms} is the effective current through the load. (Notice that if we had a purely resistive load, we would have $\theta = 0$ and $Q = 0.$)

The physical units of reactive power are watts. However, to emphasize the fact that *Q* does not represent the flow of net energy, its units are usually given as *V*olt *A*mperes *R*eactive (VARs).

> The units of reactive power *Q* are VARs.

Apparent Power

Another quantity of interest is the **apparent power**, which is defined as the product of the effective voltage and the effective current, or

$$\text{apparent power} = V_{\mathrm{rms}} I_{\mathrm{rms}}$$

> Apparent power equals the product of rms current and rms voltage. The units for apparent power are stated as volt-amperes (VA).

Its units are volt-amperes (VA).

Using **Equations 5.60** and **5.63**, we can write

$$P^2 + Q^2 = (V_{\mathrm{rms}} I_{\mathrm{rms}})^2 \cos^2(\theta) + (V_{\mathrm{rms}} I_{\mathrm{rms}})^2 \sin^2(\theta)$$

However, $\cos^2(\theta) + \sin^2(\theta) = 1,$ so we have

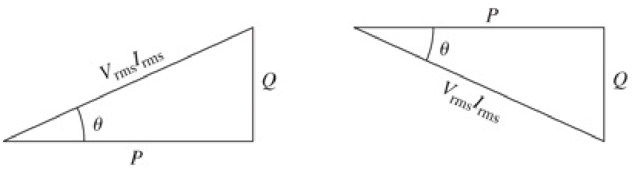
$$P^2 + Q^2 = (V_{\mathrm{rms}} I_{\mathrm{rms}})^2 \tag{5.64}$$

Units

Often, the units given for a quantity indicate whether the quantity is power (W), reactive power (VAR), or apparent power (VA). For example, if we say that we have a 5-kW load, this means that $P = 5\ \mathrm{kW}$. On the other hand, if we have a 5-kVA load, $V_{\mathrm{rms}} I_{\mathrm{rms}} = 5\ \mathrm{kVA}$. If we say that a load absorbs 5 kVAR, then $Q = 5\ \mathrm{kVAR}$.

Power Triangle

The relationships between real power *P*, reactive power *Q*, apparent power $V_{\text{rms}} I_{\text{rms}}$, and the power angle θ can be represented by the **power triangle**. The power triangle is shown in **Figure 5.25(a)** for an inductive load, in which case θ and *Q* are positive. The power triangle for a capacitive load is shown in **Figure 5.25(b)**, in which case θ and *Q* are negative.



(a) Inductive load (θ positive) (b) Capacitive load (θ negative)

Figure 5.25

Power triangles for inductive and capacitive loads.

> The power triangle is a compact way to represent ac power relationships.

Additional Power Relationships

The impedance *Z* is

$$Z = |Z| \angle\theta = R + jX$$

in which *R* is the resistance of the load and *X* is the reactance. This is illustrated in **Figure 5.26**. We can write

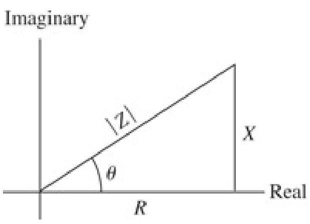


Figure 5.26

The load impedance in the complex plane.

$$\cos(\theta) = \frac{R}{|Z|} \tag{5.65}$$

and

$$\sin(\theta) = \frac{X}{|Z|} \tag{5.66}$$

Substituting **Equation 5.65** into **Equation 5.59**, we find that

$$P = \frac{V_m I_m}{2} \times \frac{R}{|Z|} \tag{5.67}$$

However, **Equation 5.53** states that $I_m = V_m / |Z|$, so we have

$$P = \frac{I_m^2}{2} R \tag{5.68}$$

In **Equation 5.69**, *R* is the real part of the impedance through which the current flows.

Using the fact that $I_{\text{rms}} = I_m/\sqrt{2}$, we get

$$P = I_{\text{rms}}^2 R \tag{5.69}$$

In **Equation 5.70**, *X* is the imaginary part (including the algebraic sign) of the impedance through which the current flows.

In a similar fashion, we can show that

$$Q = I_{rms}^2 X \tag{5.70}$$

Reactive power *Q* is positive for inductive loads and negative for capacitive loads.

In applying **Equation 5.70**, we retain the algebraic sign of *X*. For an inductive load, *X* is positive, whereas for a capacitive load, *X* is negative. This is not hard to remember if we keep in mind that *Q* is positive for inductive loads and negative for capacitive loads.

In **Equation 5.71**, V_{Rrms} is the rms voltage across the resistance.

Furthermore, in **Section 5.1**, we showed that the average power delivered to a resistance is

$$P = \frac{V_{Rrms}^2}{R} \tag{5.71}$$

where V_{Rrms} is the rms value of the voltage *across the resistance.* (Notice in **Figure 5.22** that the source voltage does not appear across the resistance, because the reactance is in series with the resistance.)

In **Equation 5.72**, V_{Xrms} is the rms voltage across the reactance.

Similarly, we have

$$Q = \frac{V_{X\,\text{rms}}^2}{X} \tag{5.72}$$

where $V_{X\,\text{rms}}$ is the rms value of the voltage *across the reactance*. Here again, *X* is positive for an inductance and negative for a capacitance.

Complex Power

Consider the portion of a circuit shown in **Figure 5.27**. The **complex power**, denoted as **S**, delivered to this circuit is defined as one half the product of the phasor voltage **V** and the complex conjugate of the phasor current IU.

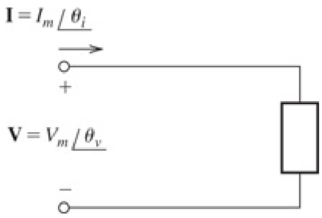


Figure 5.27

The complex power delivered to this circuit element is $S = \frac{1}{2} VI*$.

$$S = \frac{1}{2} VI* \tag{5.73}$$

The phasor voltage is $V = V_m \angle \theta_v$ in which V_m is the peak value of the voltage and θ_v is the phase angle of the voltage. Furthermore, the phasor current is $I = I_m \angle \theta_i$ where I_m is the peak value and θ_i is the phase angle of the current. Substituting into **Equation 5.73**, we have

$$S = \frac{1}{2} VI* = \frac{1}{2} \left(V_m \angle \theta_v \right) \times \left(I_m \angle -\theta_i \right) = \frac{V_m I_m}{2} \angle \theta_v - \theta_i = \frac{V_m I_m}{2} \angle \theta \tag{5.74}$$

where, as before, $\theta = \theta_v - \theta_i$ is the power angle. Expanding the right-hand term of **Equation 5.74** into real and imaginary parts, we have

$$S = \frac{V_m I_m}{2} \cos(\theta) + j \frac{V_m I_m}{2} \sin(\theta)$$

However, the first term on the right-hand side is the average power *P* delivered to the circuit and the second term is *j* times the reactive power. Thus, we can write:

$$S = \frac{1}{2} VI* = P + jQ \tag{5.75}$$

If we know the complex power **S**, then we can find the power, reactive power, and apparent power:

$$P = \mathrm{Re}(S) = \mathrm{Re}\left(\frac{1}{2} VI* \right) \tag{5.76}$$

$$Q = \mathrm{Im}(S) = \mathrm{Im}\left(\frac{1}{2} VI* \right) \tag{5.77}$$

$$\text{apparent power} = \left| \mathrm{S} \right| = \left| \frac{1}{2} \mathrm{VI}* \right| \tag{5.78}$$

where Re(**S**) denotes the real part of **S** and Im(**S**) denotes the imaginary part of **S**.

Example 5.9 AC Power Calculations

Compute the power and reactive power taken from the source for the circuit of **Example 5.6**. Also, compute the power and reactive power delivered to each element in the circuit. For convenience, the circuit and the currents that were computed in **Example 5.6** are shown in **Figure 5.28**.

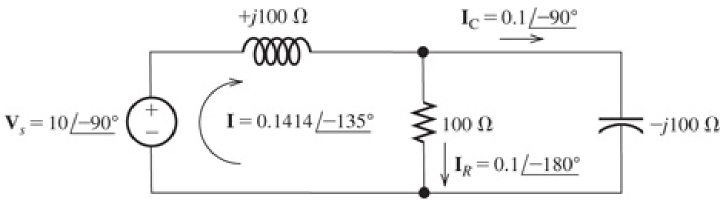


Figure 5.28

Circuit and currents for **Example 5.9**.

The circuit has two loops:

Solution

To find the power and reactive power for the source, we must first find the power angle which is given by **Equation 5.62**:

$$\theta = \theta_v - \theta_i$$

The angle of the source voltage is $\theta_v = -90°$, and the angle of the current delivered by the source is $\theta_i = -135°$. Therefore, we have

$$\theta = -90° - (-135°) = 45°$$

The rms source voltage and current are

$$V_{srms} = \frac{|V_s|}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.071 \text{ V}$$

$$I_{\text{rms}} = \frac{|I|}{\sqrt{2}} = \frac{0.1414}{\sqrt{2}} = 0.1 \text{ A}$$

Now, we use **Equations 5.60** and **5.63** to compute the power and reactive power delivered by the source:

$$\begin{aligned} P &= V_{srms} I_{\text{rms}} \cos(\theta) \\ &= 7.071 \times 0.1 \cos(45°) = 0.5 \text{ W} \\ Q &= V_{srms} I_{\text{rms}} \sin(\theta) \\ &= 7.071 \times 0.1 \sin(45°) = 0.5 \text{ VAR} \end{aligned}$$

An alternative and more compact method for computing *P* and *Q* is to first find the complex power and then take the real and imaginary parts:

$$\begin{aligned} S &= \frac{1}{2} V_s I* = \frac{1}{2}(10 \angle -90°)(0.1414 \angle 135°) = 0.707 \angle 45° = 0.5 + j0.5 \\ P &= \text{Re}(S) = 0.5 \text{ W} \\ Q &= \text{Im}(S) = 0.5 \text{ VAR} \end{aligned}$$

We can use **Equation 5.70** to compute the reactive power delivered to the inductor, yielding

$$Q_L = I_{\text{rms}}^2 X_L = (0.1)^2 (100) = 1.0 \text{ VAR}$$

For the capacitor, we have

$$Q_C = I_{C\text{rms}}^2 X_C = \left(\frac{0.1}{\sqrt{2}}\right)^2 (-100) = -0.5 \text{ VAR}$$

Notice that we have used the rms value of the current through the capacitor in this calculation. Furthermore, notice that the reactance X_C of the capacitance is negative. As expected, the reactive power is negative for a capacitance. The reactive power for the resistance is zero. As a check, we can verify that the reactive power delivered by the source is equal to the sum of the reactive powers absorbed by the inductance and capacitance. This is demonstrated by

$$Q = Q_L + Q_C$$

The power delivered to the resistance is

$$\begin{aligned} P_R &= I_{R\text{rms}}^2 R = \left(\frac{|I_R|}{\sqrt{2}}\right)^2 R = \left(\frac{0.1}{\sqrt{2}}\right)^2 100 \\ &= 0.5 \text{ W} \end{aligned}$$

The power absorbed by the capacitance and inductance is given by

$$\begin{aligned} P_L &= 0 \\ P_C &= 0 \end{aligned}$$

Thus, all of the power delivered by the source is absorbed by the resistance.

In power distribution systems, we typically encounter much larger values of power, reactive power, and apparent power than the small values of the preceding example. For example, a large power plant may generate 1000 MW. A 100-hp motor used in an industrial application absorbs approximately 85 kW of electrical power under full load.

A typical residence absorbs a *peak* power in the range of 10 to 40 kW. The *average* power for my home (which is of average size, has two residents, and does not use electrical heating) is approximately 600 W. It is interesting to keep your average power consumption and the power used by various appliances in mind because it gives you a clear picture of the economic and environmental impact of turning off lights, computers, and so on, that are not being used.

Example 5.10 Using Power Triangles

Consider the situation shown in **Figure 5.29**. Here, a voltage source delivers power to two loads connected in parallel. Find the power, reactive power, and power factor for the source. Also, find the phasor current **I**.

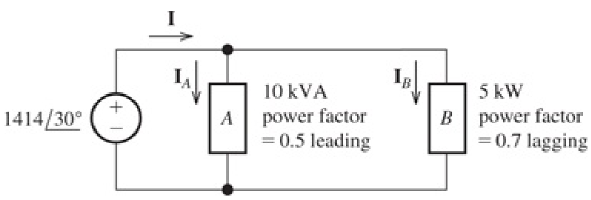


Figure 5.29
Circuit for **Example 5.10**.

Solution

By the units given in the figure, we see that load A has an *apparent power* of 10 kVA. On the other hand, the *power* for load B is specified as 5 kW.

Furthermore, load A has a power factor of 0.5 leading, which means that the current leads the voltage in load A. Another way to say this is that load A is capacitive. Similarly, load B has a power factor of 0.7 lagging (or inductive).

Our approach is to find the power and reactive power for each load. Then, we add these values to find the power and reactive power for the source. Finally, we compute the power factor for the source and then find the current.

Calculations for load *A*

Because load *A* has a leading (capacitive) power factor, we know that the reactive power Q_A and power angle θ_A are negative. The power triangle for load *A* is shown in **Figure 5.30(a)**. The power factor is

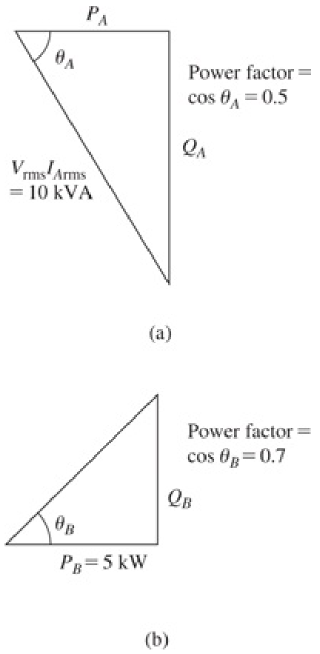


Figure 5.30
Power triangles for loads *A* and *B* of **Example 5.10**.

$$\cos(\theta_A) = 0.5$$

The power is

$$P_A = V_{\text{rms}} I_{A\text{rms}} \cos(\theta_A) = 10^4 (0.5) = 5 \text{ kW}$$

Solving **Equation 5.64** for reactive power, we have

$$\begin{aligned} Q_A &= \sqrt{(V_{\text{rms}} I_{A\text{rms}})^2 - P_A^2} \\ &= \sqrt{(10^4)^2 - (5000)^2} \\ &= -8.660 \text{ kVAR} \end{aligned}$$

Notice that we have selected the negative value for Q_A, because we know that reactive power is negative for a capacitive (leading) load.

Calculations for load *B*

The power triangle for load *B* is shown in **Figure 5.30(b)**. Since load *B* has a lagging (inductive) power factor, we know that the reactive power Q_B and power angle θ_B are positive. Thus,

$$\theta_B = \arccos(0.7) = 45.57°$$

Applying trigonometry, we can write

$$Q_B = P_B \tan(\theta_B) = 5000\tan(45.57°)$$
$$Q_B = 5.101 \text{ kVAR}$$

At this point, as shown here we can find the power and reactive power delivered by the source:

$$P = P_A + P_B = 5 + 5 = 10 \text{ kW}$$
$$Q = Q_A + Q_B = -8.660 + 5.101 = -3.559 \text{ kVAR}$$

Total power is obtained by adding the powers for the various loads. Similarly, the reactive powers are added.

Power calculations for the source.

Because *Q* is negative, we know that the power angle is negative. Thus, we have

$$\theta = \arctan\left(\frac{Q}{P}\right) = \arctan\left(\frac{-3.559}{10}\right) = -19.59°$$

The power factor is

$$\cos(\theta) = 0.9421$$

Power-system engineers frequently express power factors as percentages and would state this power factor as 94.21 percent leading.

The complex power delivered by the source is

$$S = P + jQ = 10 - j3.559 = 10.61 \angle -19.59° \text{ kVA}$$

Thus, we have

$$S = \frac{1}{2} V_s I* = \frac{1}{2}(1414 \angle 30°) I* = 10.61 \times 10^3 \angle -19.59° \text{ kVA}$$

Solving for the phasor current, we obtain:

$$I = 15.0 \angle 49.59° \text{ A}$$

The phasor diagram for the current and voltage is shown in **Figure 5.31**. Notice that the current is leading the voltage.

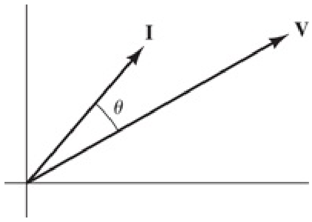


Figure 5.31
Phasor diagram for **Example 5.10**.

Power-Factor Correction

We have seen that large currents can flow in energy-storage devices (inductance and capacitance) without average power being delivered. In heavy industry, many loads are partly inductive, and large amounts of reactive power flow. This reactive power causes higher currents in the power distribution system. Consequently, the lines and transformers must have higher ratings than would be necessary to deliver the same average power to a resistive (100 percent power factor) load.

> Power-factor correction can provide a significant economic advantage for consumers of large amounts of electrical energy.

Energy rates charged to industry depend on the power factor, with higher charges for energy delivered at lower power factors. (Power factor is not taken into account for residential customers.) Therefore, it is advantageous to choose loads that operate at near unity power factor. A common approach is to place capacitors in parallel with an inductive load to increase the power factor.

Example 5.11 Power-Factor Correction

A 50-kW load operates from a 60-Hz 10-kV-rms line with a power factor of 60 percent lagging. Compute the capacitance that must be placed in parallel with the load to achieve a 90 percent lagging power factor.

Solution

First, we find the load power angle:

$$\theta_L = \arccos(0.6) = 53.13°$$

Then, we use the power-triangle concept to find the reactive power of the load. Hence,

$$Q_L = P_L \tan\left(\theta_L\right) = 66.67 \text{ kVAR}$$

After adding the capacitor, the power will still be 50 kW and the power angle will become

$$\theta_{\text{new}} = \arccos(0.9) = 25.84°$$

The new value of the reactive power will be

$$Q_{\text{new}} = P_L \tan(\theta_{\text{new}}) = 24.22 \text{ kVAR}$$

Thus, the reactive power of the capacitance must be

$$Q_C = Q_{\text{new}} - Q_L = -42.45 \text{ kVAR}$$

Now, we find that the reactance of the capacitor is

$$X_C = -\frac{V_{\text{rms}}^2}{Q_C} = \frac{\left(10^4\right)^2}{42,450} = -2356\ \Omega$$

Finally, the angular frequency is

$$\omega = 2\pi 60 = 377.0$$

and the required capacitance is

$$C = \frac{1}{\omega\left|X_C\right|} = \frac{1}{377 \times 2356} = 1.126\ \mu\text{F}$$

Exercise 5.12

a. a. A voltage source $\boldsymbol{V} = 707.1\ \angle 40°$ delivers 5 kW to a load with a power factor of 100 percent. Find the reactive power and the phasor current.

b. Repeat if the power factor is 20 percent lagging.

c. For which power factor would the current ratings of the conductors connecting the source to the load be higher? In which case could the wiring be a lower cost?

Answer

a. a. $Q = 0,\ \boldsymbol{I} = 14.14\ \angle 40°;$

b. $Q = 24.49\ \text{kVAR},\ \boldsymbol{I} = 70.7\ \angle -38.46°;$

c. The current ratings for the conductors would need to be five times higher for part (b) than for part (a). Clearly, the wiring could be a lower cost for 100 percent power factor.n

Exercise 5.13

A 1-kV-rms 60-Hz voltage source delivers power to two loads in parallel. The first load is a $10 - \mu\text{F}$ capacitor, and the second load absorbs an apparent power of 10 kVA with an 80 percent lagging power factor. Find the total power, the total reactive power, the power factor for the source, and the rms source current.

Answer

a. $P = 8\ \text{kW},\ Q = 2.23\ \text{kVAR},\ \text{PF} = 96.33$ percent lagging, $I_{\text{rms}} = 8.305\ \text{A}$.

5.6 Thévenin and Norton Equivalent Circuits

Thévenin Equivalent Circuits

In **Chapter 2**, we saw that a two-terminal network composed of sources and resistances has a Thévenin equivalent circuit consisting of a voltage source in series with a resistance. We can apply this concept to circuits composed of sinusoidal sources (all having a common frequency), resistances, inductances, and capacitances. Here, the Thévenin equivalent consists of a phasor voltage source in series with a complex impedance as shown in **Figure 5.32**. Recall that phasors and complex impedances apply only for steady-state operation; therefore, these Thévenin equivalents are valid for only steady-state operation of the circuit.

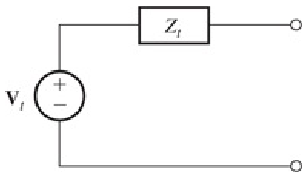


Figure 5.32

The Thévenin equivalent for an ac circuit consists of a phasor voltage source V_t in series with a complex impedance Z_t.

As in resistive circuits, the Thévenin voltage is equal to the open-circuit voltage of the two-terminal circuit. In ac circuits, we use phasors, so we can write

> The Thévenin voltage is equal to the open-circuit phasor voltage of the original circuit.

$$V_t = V_{\text{oc}} \tag{5.79}$$

The Thévenin impedance Z_t can be found by zeroing the *independent* sources and looking back into the terminals to find the equivalent impedance. (Recall that in zeroing a voltage source, we reduce its voltage to zero, and it becomes a short circuit. On the other hand, in zeroing a current source, we reduce its current to zero, and it becomes an open circuit.) Also, keep in mind that we must not zero the *dependent* sources.

> We can find the Thévenin impedance by zeroing the independent sources and determining the impedance looking into the circuit terminals.

Another approach to determining the Thévenin impedance is first to find the short-circuit phasor current I_{sc} and the open-circuit voltage V_{oc}. Then, the Thévenin impedance is given by

$$Z_t = \frac{V_{\text{oc}}}{I_{\text{sc}}} = \frac{V_t}{I_{\text{sc}}} \tag{5.80}$$

Thus, except for the use of phasors and complex impedances, the concepts and procedures for Thévenin equivalents of steady-state ac circuits are the same as for resistive circuits.

> The Thévenin impedance equals the open-circuit voltage divided by the short-circuit current.

Norton Equivalent Circuits

Another equivalent for a two-terminal steady-state ac circuit is the Norton equivalent, which consists of a phasor current source I_n in parallel with the Thévenin impedance. This is shown in **Figure 5.33**. The Norton current is equal to the short-circuit current of the original circuit:

$$I_n = I_{\mathrm{sc}} \tag{5.81}$$

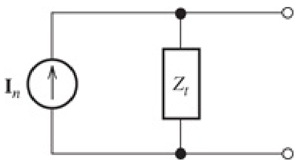


Figure 5.33
The Norton equivalent circuit consists of a phasor current source I_n in parallel with the complex impedance Z_t.

Example 5.12 Thévenin and Norton Equivalents

Find the Thévenin and Norton equivalent circuits for the circuit shown in **Figure 5.34(a)**.

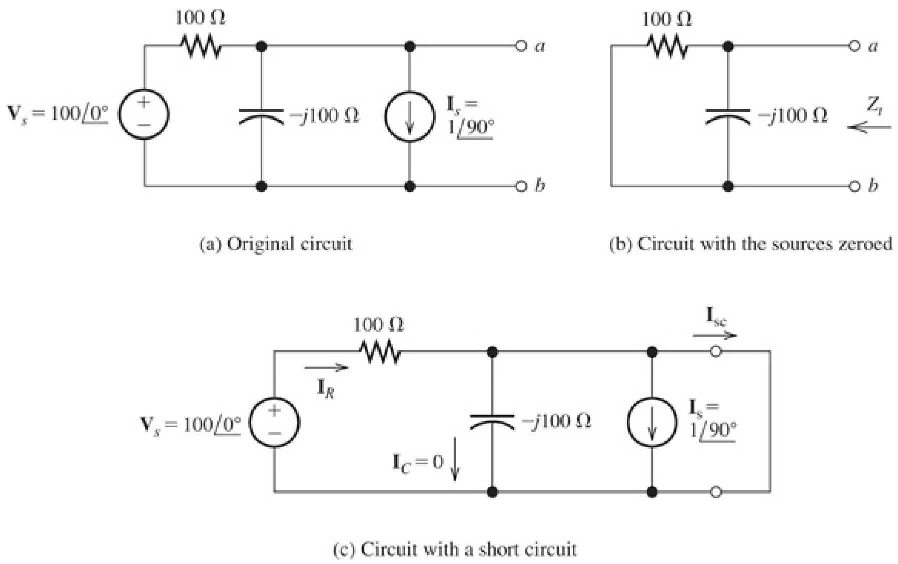


Figure 5.34
Circuit of **Example 5.12**.

First, look to see which two of the three quantities V_{oc}, I_{sc}, or Z_t. are easiest to determine.

Solution

We must find two of the three quantities: V_{oc}, I_{sc}, or Z_t. Often, it pays to look for the two that can be found with the least amount of work. In this case, we elect to start by zeroing the sources to find Z_t. After that part of the problem is finished, we will find the short-circuit current.

If we zero the sources, we obtain the circuit shown in **Figure 5.34(b)**. The Thévenin impedance is the impedance seen looking back into terminals *a*—*b*. This is the parallel combination of the resistance and the impedance of the capacitance. Thus, we have

$$\begin{aligned} Z_t &= \frac{1}{1/100 + 1/(-j100)} \\ &= \frac{1}{0.01 + j0.01} \\ &= \frac{1}{0.01414 \angle 45°} \\ &= 70.71 \angle -45° \\ &= 50 - j50\ \Omega \end{aligned}$$

Now, we apply a short circuit to terminals *a*–*b* and find the current, which is shown in **Figure 5.34(c)**. With a short circuit, the voltage across the capacitance is zero. Therefore, $I_C = 0$. Furthermore, the source voltage V_s appears across the resistance, so we have

$$I_R = \frac{V_s}{100} = \frac{100}{100} = 1 \angle 0°\ \text{A}$$

Then applying KCL, we can write

$$I_{\text{sc}} = I_R - I_s = 1 - 1 \angle 90° = 1 - j = 1.414 \angle -45°\ \text{A}$$

Next, we can solve **Equation 5.80** for the Thévenin voltage:

$$V_t = I_{\text{sc}} Z_t = 1.414 \angle -45° \times 70.71 \angle -45° = 100 \angle -90°\ \text{V}$$

Finally, we can draw the Thévenin and Norton equivalent circuits, which are shown in **Figure 5.35**.

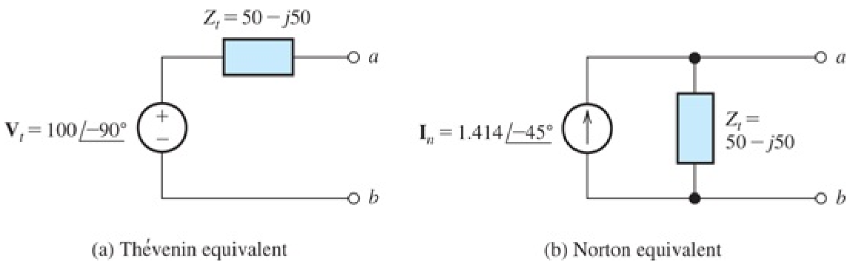


Figure 5.35
Thévenin and Norton equivalents for the circuit of **Figure 5.34(a)**.

Maximum Average Power Transfer

Sometimes we are faced with the problem of adjusting a load impedance to extract the maximum average power from a two-terminal circuit. This situation is shown in **Figure 5.36**, in which we have represented the two-terminal circuit by its Thévenin equivalent. Of course, the power delivered to the load depends on the load impedance. A short-circuit load receives no power because the voltage across it is zero. Similarly, an open-circuit load receives no power because the current through it is zero. Furthermore, a pure reactive load (inductance or capacitance) receives no power because the load power factor is zero.

Figure 5.36
The Thévenin equivalent of a two-terminal circuit delivering power to a load impedance.

Two situations are of interest. First, suppose that the load impedance can take any complex value. Then, it turns out that the load impedance for maximum-power transfer is the complex conjugate of the Thévenin impedance:

$$Z_{\text{load}} = Z_t{}^*$$

Let us consider why this is true. Suppose that the Thévenin impedance is

> If the load can take on any complex value, maximum-power transfer is attained for a load impedance equal to the complex conjugate of the Thévenin impedance.

$$Z_t = R_t + jX_t$$

Then, the load impedance for maximum-power transfer is

$$Z_{\text{load}} = Z_t{}^* = R_t - jX_t$$

Of course, the total impedance seen by the Thévenin source is the sum of the Thévenin impedance and the load impedance:

$$\begin{aligned} Z_{\text{total}} &= Z_t + Z_{\text{load}} \\ &= R_t + jX_t + R_t - jX_t \\ &= 2R_t \end{aligned}$$

Thus, the reactance of the load cancels the internal reactance of the two-terminal circuit. Maximum power is transferred to a given load resistance by maximizing the current. For given resistances, maximum current is achieved by choosing the reactance to minimize the total impedance magnitude. Of course, for fixed resistances, the minimum impedance magnitude occurs for zero total reactance.

Having established the fact that the total reactance should be zero, we have a resistive circuit. We considered this resistive circuit in **Chapter 2**, where we showed that maximum power is transferred for. $R_{\text{load}} = R_t$.

> If the load is required to be a pure resistance, maximum-power transfer is attained for a load resistance equal to the magnitude of the Thévenin impedance.

The second case of interest is a load that is constrained to be a pure resistance. In this case, it can be shown that the load resistance for maximum-power transfer is equal to the magnitude of the Thévenin impedance:

$$Z_{\text{load}} = R_{\text{load}} = |\, Z_t|$$

Example 5.13 Maximum Power Transfer

Determine the maximum power that can be delivered to a load by the two-terminal circuit of **Figure 5.34(a)** if

a. the load can have any complex value and
b. the load must be a pure resistance.

Solution

In **Example 5.12**, we found that the circuit has the Thévenin equivalent shown in **Figure 5.35(a)**. The Thévenin impedance is

$$Z_t = 50 - j50\ \Omega$$

a. The complex load impedance that maximizes power transfer is

$$Z_{\text{load}} = Z_t^* = 50 + j50$$

The Thévenin equivalent with this load attached is shown in **Figure 5.37(a)**. The current is

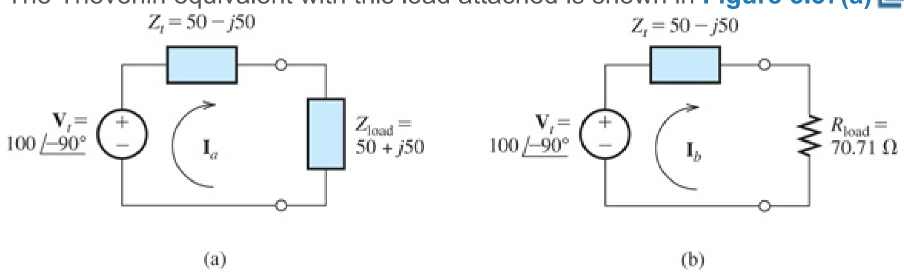


Figure 5.37
Thévenin equivalent circuit and loads of **Example 5.13**.

$$\begin{aligned} I_a &= \frac{V_t}{Z_t + Z_{\text{load}}} \\ &= \frac{100\ \angle -90^\circ}{50 - j50 + 50 + j50} \\ &= 1\ \angle -90^\circ\ \text{A} \end{aligned}$$

The rms load current is $I_{arms} = 1/\sqrt{2}$. Finally, the power delivered to the load is

$$P = I_{arms}^2 R_{\text{load}} = \left(\frac{1}{\sqrt{2}}\right)^2 (50) = 25\ \text{W}$$

b. The purely resistive load for maximum power transfer is

$$\begin{aligned} R_{\text{load}} &= |Z_t| \\ &= |50 - j50| \\ &= \sqrt{50^2 + (-50)^2} \\ &= 70.71\ \Omega \end{aligned}$$

The Thévenin equivalent with this load attached is shown in **Figure 5.37(b)**. The current is

$$\begin{aligned} I_b &= \frac{V_t}{Z_t + Z_{\text{load}}} \\ &= \frac{100\ \angle -90^\circ}{50 - j50 + 70.71} \\ &= \frac{100\ \angle -90^\circ}{130.66\ \angle -22.50^\circ} \\ &= 0.7654\ \angle -67.50^\circ\ \text{A} \end{aligned}$$

The power delivered to this load is

$$\begin{aligned} P &= I_{brms}^2 R_{\text{load}} \\ &= \left(\frac{0.7653}{\sqrt{2}}\right)^2 70.71 \\ &= 20.71 \text{ W} \end{aligned}$$

Notice that the power available to a purely resistive load is less than that for a complex load.

Exercise 5.14

Find the Thévenin impedance, the Thévenin voltage, and the Norton current for the circuit shown in **Figure 5.38**.

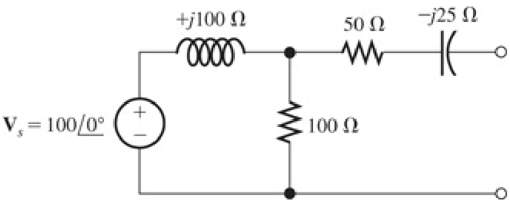


Figure 5.38
Circuit of **Exercises 5.14** and **5.15**.

Answer $Z_t = 100 + j25\ \Omega,\ V_t = 70.71\ \angle - 45°,\ I_n = 0.686\ \angle - 59.0°$.

Exercise 5.15

Determine the maximum power that can be delivered to a load by the two-terminal circuit of **Figure 5.38** if

a. the load can have any complex value and

b. the load must be a pure resistance.

Answer

a. 6.25 W;

b. 6.16 W.

5.7 Balanced Three-Phase Circuits

We will see that there are important advantages in generating and distributing power with multiple ac voltages having different phases. We consider the most common case: three equal-amplitude ac voltages having phases that are 120° apart. This is known as a **balanced three-phase source**, an example of which is illustrated in **Figure 5.39**. [Recall that in double-subscript notation for voltages the first subscript is the positive reference. Thus, $v_{an}(t)$ is the voltage between nodes *a* and *n* with the positive reference at node *a*.] In **Chapter 16**, we will learn how three-phase voltages are generated.

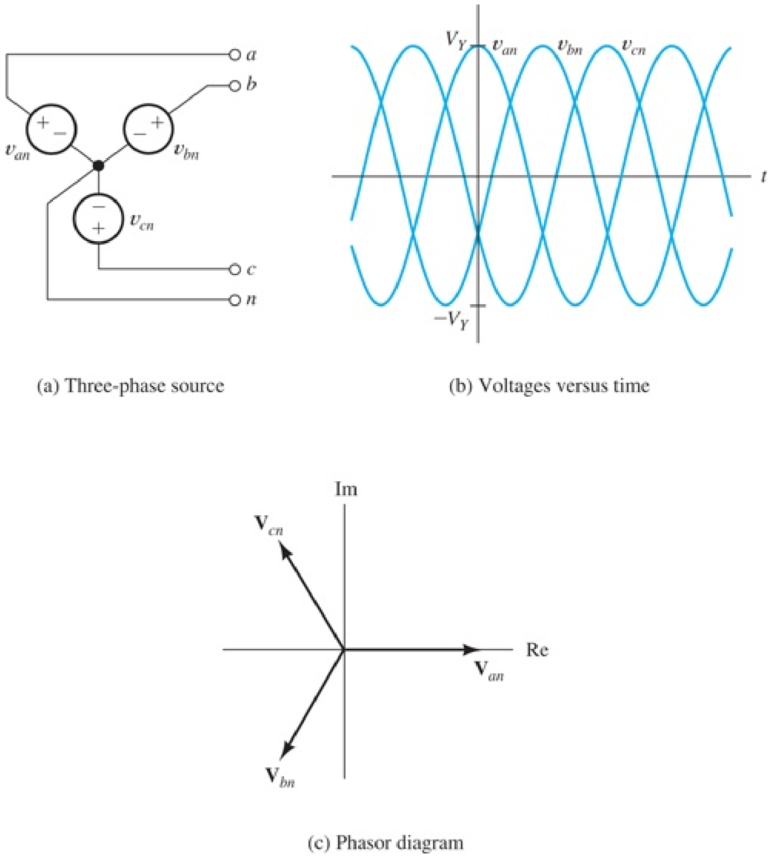


Figure 5.39
A balanced three-phase voltage source.

Much of the power used by business and industry is supplied by three-phase distribution systems. Plant engineers need to be familiar with three-phase power.

The source shown in **Figure 5.39(a)** is said to be **wye connected (Y connected)**. Later in this chapter, we consider another configuration, known as the delta (Δ) connection.

The three voltages shown in **Figure 5.39(b)** are given by

$$v_{an}(t) = V_Y \cos(\omega t) \tag{5.82}$$

$$v_{bn}(t) = V_Y \cos(\omega t - 120°) \tag{5.83}$$

$$v_{cn}(t) = V_Y \cos(\omega t + 120°) \tag{5.84}$$

where V_Y is the magnitude of each source in the wye-connected configuration. The corresponding phasors are

$$\mathbf{V}_{an} = V_Y \angle 0° \tag{5.85}$$

$$\mathbf{V}_{bn} = V_Y \angle -120° \tag{5.86}$$

$$\mathbf{V}_{cn} = V_Y \angle 120° \tag{5.87}$$

The phasor diagram is shown in **Figure 5.39(c)**.

Phase Sequence

This set of voltages is said to have a **positive phase sequence** because the voltages reach their peak values in the order *abc*. Refer to **Figure 5.39(c)** and notice that v_{an} leads v_{bn}, which in turn leads v_{cn}. (Recall that we think of the phasors as rotating counterclockwise in determining phase relationships.) If we interchanged *b* and *c*, we would have a **negative phase sequence**, in which the order is *acb*.

> Three-phase sources can have either a positive or negative phase sequence.

Phase sequence can be important. For example, if we have a three-phase induction motor, the direction of rotation is opposite for the two phase sequences. To reverse the direction of rotation of such a motor, we would interchange the *b* and *c* connections. (You may find this piece of information useful if you ever work with three-phase motors, which are very common in industry.) Because circuit analysis is very similar for both phase sequences, we consider only the positive phase sequence in most of the discussion that follows.

> We will see later in the book that the direction of rotation of certain three-phase motors can be reversed by changing the phase sequence.

Wye–Wye Connection

Consider the three-phase source connected to a balanced three-phase load shown in **Figure 5.40**. The wires $a - A$, $b - B$, and $c - C$ are called **lines**, and the wire *n*–*N* is called the **neutral**. This configuration is called a wye–wye (Y–Y) connection with neutral. By the term *balanced load*, we mean that the three load impedances are equal. (In this book, we consider only balanced loads.)

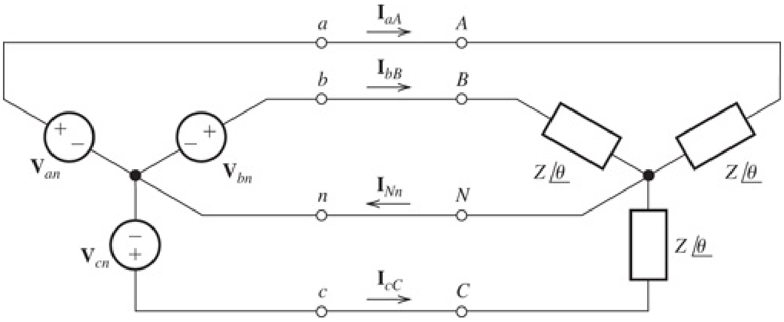


Figure 5.40
A three-phase wye–wye connection with neutral.

Three-phase sources and loads can be connected either in a wye configuration or in a delta configuration.

Later, we will see that other configurations are useful. For example, the neutral wire $n-N$ can be omitted. Furthermore, the source and load can be connected in the form of a delta. We will see that currents, voltages, and power can be computed for these other configurations by finding an equivalent wye–wye circuit. Thus, the key to understanding three-phase circuits is a careful examination of the wye–wye circuit.

The key to understanding the various three-phase configurations is a careful examination of the wye–wye circuit.

In **Chapters 5** and **6**, we take the magnitude of a phasor to be the peak value. Power-systems engineers often use the rms value as the magnitude for phasors, which we do in **Chapters 14** and **15**. We will label rms phasors as rms.

Often, we use the term *phase* to refer to part of the source or the load. Thus, phase *A* of the source is $v_{an}(t)$, and phase *A* of the load is the impedance connected between *A* and *N*. We refer to V_Y as the **phase voltage** or as the **line-to-neutral voltage** of the wye-connected source. (Power-systems engineers usually specify rms values rather than peak magnitudes. Unless stated otherwise, we use phasors having magnitudes equal to the peak values rather than the rms values.) Furthermore, $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$ are called **line currents**. (Recall that in the double-subscript notation for currents, the reference direction is from the first subscript to the second. Thus, $\mathbf{I}_{aA}$ is the current referenced from node *a* to node *A*, as illustrated in **Figure 5.38**.)

The current in phase *A* of the load is given by

$$I_{aA} = \frac{\mathbf{V}_{an}}{Z\angle\theta} = \frac{V_Y\angle 0°}{Z\angle\theta} = I_L\angle -\theta$$

where $I_L = V_Y/Z$ is the magnitude of the line current. Because the load impedances are equal, all of the line currents are the same, except for phase. Thus, the currents are given by

$$i_{aA}(t) = I_L\cos(\omega t-\theta) \tag{5.88}$$

$$i_{bB}(t) = I_L\cos(\omega t-120°-\theta) \tag{5.89}$$

$$i_{cC}(t) = I_L\cos(\omega t+120°-\theta) \tag{5.90}$$

The neutral current in **Figure 5.40** is given by

$$i_{Nn}(t) = i_{aA}(t) + i_{bB}(t) + i_{cC}(t)$$

In terms of phasors, this is

$$\begin{aligned}\mathbf{I}_{Nn} &= \mathbf{I}_{aA}+\mathbf{I}_{bB}+\mathbf{I}_{cC}\\ &= I_L\angle -\theta + I_L\angle -120° - \theta + I_L\angle 120° - \theta\\ &= I_L\angle -\theta \times (1 + 1\angle -120° + 1\angle 120°)\\ &= I_L\angle -\theta \times (1 - 0.5 - j0.866 - 0.5 + j0.866)\\ &= 0\end{aligned}$$

Thus, the sum of three phasors with equal magnitudes and 120° (We make use of this fact again later in this section.)

> The sum of three equal magnitude phasors 120° apart in phase is zero.

We have shown that the neutral current is zero in a balanced three-phase system. Consequently, the neutral wire can be eliminated without changing any of the voltages or currents. Then, the three source voltages are delivered to the three load impedances with three wires.

> The neutral current is zero in a balanced wye–wye system. Thus in theory, the neutral wire can be inserted or removed without affecting load currents or voltages. This is *not* true if the load is unbalanced, which is often the case in real power distribution systems.

An important advantage of three-phase systems compared with single phase is that the wiring for connecting the sources to the loads is less expensive. As shown in **Figure 5.41**, it would take six wires to connect three single-phase sources to three loads separately, whereas only three wires (four if the neutral wire is used) are needed for the three-phase connection to achieve the same power transfer.

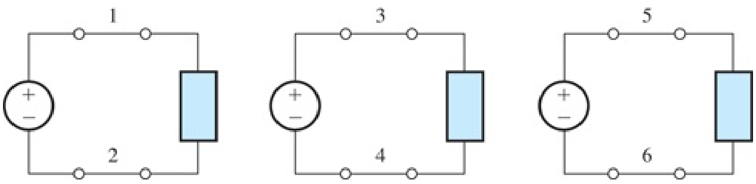


Figure 5.41
Six wires are needed to connect three single-phase sources to three loads. In a three-phase system, the same power transfer can be accomplished with three wires.

Power

Another advantage of balanced three-phase systems, compared with single-phase systems, is that the total power is constant (as a function of time) rather than pulsating. (Refer to **Figure 5.2** on page **220** to see that power pulsates in the single-phase case.) To show that the power is constant for the balanced wye–wye connection shown in **Figure 5.40**, we write an expression for the total power. The power delivered to phase *A* of the load is $v_{an}(t)\, i_{aA}(t)$. Similarly, the power for each of the other phases of the load is the product of the voltage and the current. Thus, the total power is

$$p(t) = v_{an}(t)\, i_{aA}(t) + v_{bn}(t)\, i_{bB}(t) + v_{cn}(t)\, i_{cC}(t) \tag{5.91}$$

Using **Equations 5.82**, **5.83**, and **5.84** to substitute for the voltages and **Equations 5.88**, **5.89**, and **5.90** to substitute for the currents, we obtain

$$\begin{aligned} p(t) = \; & V_Y \cos(\omega t)\, I_L \cos(\omega t - \theta) \\ & + V_Y \cos(\omega t - 120°)\, I_L \cos(\omega t - \theta - 120°) \\ & + V_Y \cos(\omega t + 120°)\, I_L \cos(\omega t - \theta + 120°) \end{aligned} \tag{5.92}$$

Using the trigonometric identity

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

we find that **Equation 5.92** can be written as

$$p(t) = 3\frac{V_Y I_L}{2}\cos(\theta) + \frac{V_Y I_L}{2}[\cos(2\omega t - \theta) + \cos(2\omega t - \theta - 240°) + \cos(2\omega t - \theta + 480°)] \tag{5.93}$$

However, the term in brackets is

$$\begin{aligned}&\cos(2\omega t - \theta) + \cos(2\omega t - \theta - 240°) + \cos(2\omega t - \theta + 480°)\\ &\quad = \cos(2\omega t - \theta) + \cos(2\omega t - \theta + 120°) + \cos(2\omega t - \theta - 120°)\\ &\quad = 0\end{aligned}$$

(Here, we have used the fact, established earlier, that the sum is zero for three sine waves of equal amplitude and 120° apart in phase.) Thus, the expression for power becomes

$$p(t) = 3\frac{V_Y I_L}{2}\cos(\theta) \tag{5.94}$$

Notice that the total power is constant with respect to time. A consequence of this fact is that the torque required to drive a three-phase generator connected to a balanced load is constant, and vibration is lessened. Similarly, the torque produced by a three-phase motor is constant rather than pulsating as it is for a single-phase motor.

In balanced three-phase systems, total power flow is constant with respect to time.

The rms voltage from each line to neutral is

$$V_{Y\mathrm{rms}} = \frac{V_Y}{\sqrt{2}} \tag{5.95}$$

Similarly, the rms value of the line current is

$$I_{L\mathrm{rms}} = \frac{I_L}{\sqrt{2}} \tag{5.96}$$

Using **Equations 5.95** and **5.96** to substitute into **Equation 5.94**, we find that

In **Equations 5.97** and **5.98**, $V_{Y\mathrm{rms}}$ is the rms line-to-neutral voltage, $I_{L\mathrm{rms}}$ is the rms line current, and θ is the angle of the load impedances.

$$P_{\mathrm{avg}} = p(t) = 3V_{Y\mathrm{rms}}I_{L\mathrm{rms}}\cos(\theta) \tag{5.97}$$

Reactive Power

As in single-phase circuits, power flows back and forth between the sources and energy-storage elements contained in a three-phase load. This power is called *reactive power*. The higher currents that result because of the presence of reactive power require wiring and other power-distribution components having higher ratings. The reactive power delivered to a balanced three-phase load is given by

$$Q = 3\frac{V_Y I_Y}{2}\sin(\theta) = 3V_{Y\,\mathrm{rms}}I_{L\mathrm{rms}}\sin(\theta) \tag{5.98}$$

Line-to-Line Voltages

As we have mentioned earlier, the voltages between terminals *a*, *b*, or *c* and the neutral point *n* are called **line-to-neutral voltages**. On the other hand, voltages between *a* and *b*, *b* and *c*, or *a* and *c* are called **line-to-line voltages** or, more simply, **line voltages**. Thus $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$ are line-to-neutral voltages, whereas $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ are line-to-line voltages. (For consistency, we choose the subscripts cyclically in the order *abcabc*.) Let us consider the relationships between line-to-line voltages and line-to-neutral voltages.

We can obtain the following relationship by applying KVL to **Figure 5.40**:

$$\mathbf{V}_{ab} = \mathbf{V}_{an} - \mathbf{V}_{bn}$$

Using **Equations 5.85** and **5.86** to substitute for $\mathbf{V}_{an}$ and $\mathbf{V}_{bn}$, we obtain

$$\mathbf{V}_{ab} = V_Y \angle 0° - V_Y \angle -120° \tag{5.99}$$

which is equivalent to

$$\mathbf{V}_{ab} = V_Y \angle 0° + V_Y \angle 60° \tag{5.100}$$

This relationship is illustrated in **Figure 5.42**. It can be shown that **Equation 5.100** reduces to

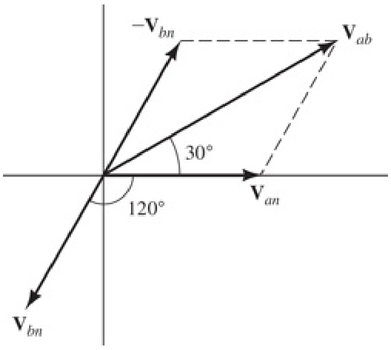


Figure 5.42
Phasor diagram showing the relationship between the line-to-line voltage V_{ab} and the line-to-neutral voltages V_{an} and V_{bn}.

$$\mathbf{V}_{ab} = \sqrt{3} V_Y \angle 30° \tag{5.101}$$

We denote the magnitude of the line-to-line voltage as V_L. The magnitude of the line-to-line voltage is $\sqrt{3}$ times the magnitude of the line-to-neutral voltage:

$$V_L = \sqrt{3} V_Y \tag{5.102}$$

Thus, the relationship between the line-to-line voltage $\mathbf{V}_{ab}$ and the line-to-neutral voltage $\mathbf{V}_{an}$ is

$$\mathbf{V}_{ab} = \mathbf{V}_{an} \times \sqrt{3} \angle 30° \tag{5.103}$$

Similarly, it can be shown that

$$\mathbf{V}_{bc} = \mathbf{V}_{bn} \times \sqrt{3} \angle 30° \tag{5.104}$$

and

$$\mathbf{V}_{ca} = \mathbf{V}_{cn} \times \sqrt{3} \angle 30° \tag{5.105}$$

These voltages are shown in **Figure 5.43**.

Figure 5.43
Phasor diagram showing line-to-line voltages and line-to-neutral voltages

Figure 5.43(b) provides a convenient way to remember the phase relationships between line-to-line and line-to-neutral voltages.

Example 5.14 Analysis of a Wye–Wye System

A balanced positive-sequence wye-connected 60-Hz three-phase source has line-to-neutral voltages of $V_Y = 1000\ \mathrm{V}$. This source is connected to a balanced wye-connected load. Each phase of the load consists of a 0.1-H inductance in series with a $50 - \Omega$ resistance. Find the line currents, the line-to-line voltages, the power, and the reactive power delivered to the load. Draw a phasor diagram showing the line-to-neutral voltages, the line-to-line voltages, and the line currents. Assume that the phase angle of V_{an} is zero.

Solution

First, by computing the complex impedance of each phase of the load, we find that

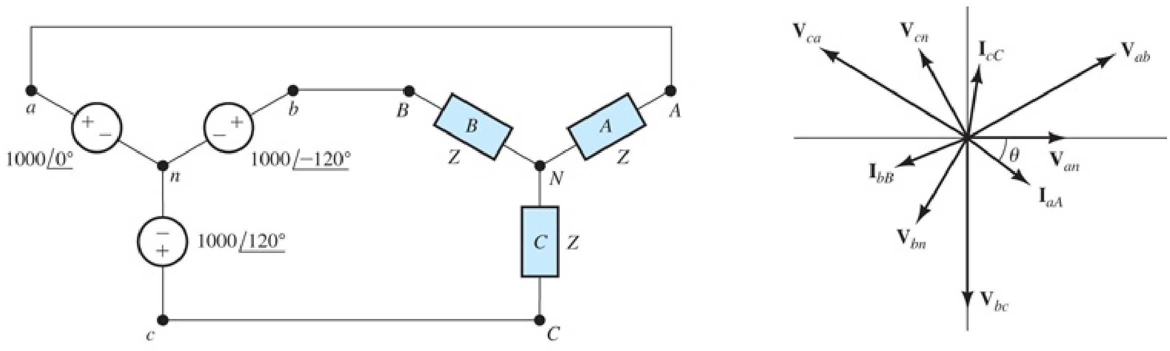
$$\begin{aligned} Z &= R + j\omega L = 50 + j2\pi(60)(0.1) = 50 + j37.70 \\ &= 62.62\angle 37.02^\circ \end{aligned}$$

Next, we draw the circuit as shown in **Figure 5.44(a)**. In balanced wye–wye calculations, we can assume that *n* and *N* are connected. (The currents and voltages are the same whether or not the neutral connection actually exists.) Thus, V_{an} appears across phase *A* of the load, and we can write

$$I_{aA} = \frac{V_{an}}{Z} = \frac{1000\angle 0^\circ}{62.62\angle 37.02^\circ} = 15.97\angle -37.02^\circ$$

Similarly,

$$\begin{aligned} I_{bB} &= \frac{V_{bn}}{Z} = \frac{1000\angle -120^\circ}{62.62\angle 37.02^\circ} = 15.97\angle -157.02^\circ \\ I_{cC} &= \frac{V_{cn}}{Z} = \frac{1000\angle 120^\circ}{62.62\angle 37.02^\circ} = 15.97\angle 82.98^\circ \end{aligned}$$



(a) Circuit diagram

(b) Phasor diagram

Figure 5.44

Circuit and phasor diagram for **Example 5.14**.

We use **Equations 5.103**, **5.104**, and **5.105** to find the line-to-line phasors:

$$\begin{aligned} V_{ab} &= V_{an} \times \sqrt{3}\angle 30^\circ = 1732\angle 30^\circ \\ V_{bc} &= V_{bn} \times \sqrt{3}\angle 30^\circ = 1732\angle -90^\circ \\ V_{ca} &= V_{cn} \times \sqrt{3}\angle 30^\circ = 1732\angle 150^\circ \end{aligned}$$

The power delivered to the load is given by **Equation 5.94**:

$$P = 3\frac{V_Y I_L}{2}\cos(\theta) = 3\left(\frac{1000 \times 15.97}{2}\right)\cos(37.02^\circ) = 19.13 \text{ kW}$$

The reactive power is given by **Equation 5.98**:

$$Q = 3\frac{V_Y I_L}{2}\sin(\theta) = 3\left(\frac{1000 \times 15.97}{2}\right)\sin(37.02^\circ) = 14.42 \text{ kVAR}$$

The phasor diagram is shown in **Figure 5.44(b)**. As usual, we have chosen a different scale for the currents than for the voltages.

Exercise 5.16

A balanced positive-sequence wye-connected 60-Hz three-phase source has line-to-line voltages of $V_L = 1000\ \mathrm{V}$. This source is connected to a balanced wye-connected load. Each phase of the load consists of a 0.2-H inductance in series with a $100 - \Omega$ resistance. Find the line-to-neutral voltages, the line currents, the power, and the reactive power delivered to the load. Assume that the phase of $\boldsymbol{V}_{an}$ is zero.

Answer $\boldsymbol{V}_{an} = 577.4\ \angle 0°$, $\boldsymbol{V}_{bn} = 577.4\ \angle -120°$, $\boldsymbol{V}_{cn} = 577.4\ \angle 120°$; $\boldsymbol{I}_{aA} = 4.61\ \angle -37°$, $\boldsymbol{I}_{bB} = 4.61\ \angle -157°$, $\boldsymbol{I}_{cC} = 4.61\ \angle 83°$; $P = 3.19\ \mathrm{kW}$; $Q = 2.40\ \mathrm{kVAR}$.

Delta-Connected Sources

A set of balanced three-phase voltage sources can be connected in the form of a delta, as shown in **Figure 5.45**. Ordinarily, we avoid connecting voltage sources in closed loops. However, in this case, it turns out that the sum of the voltages is zero:

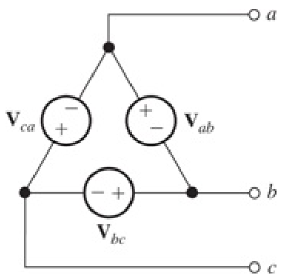


Figure 5.45
Delta-connected three-phase source.

$$\boldsymbol{V}_{ab} + \boldsymbol{V}_{bc} + \boldsymbol{V}_{ca} = 0$$

Thus, the current circulating in the delta is zero. (Actually, this is a first approximation. There are many subtleties of power distribution systems that are beyond the scope of our discussion. For example, the voltages in actual power distribution systems are not exactly sinusoidal; instead, they are the sum of several harmonic components. The behavior of harmonic components is an important factor in making a choice between wye- and delta-connected sources or loads.)

For a given delta-connected source, we can find an equivalent wye-connected source (or vice versa) by using **Equations 5.103** through **5.105**. Clearly, a delta-connected source has no neutral point, so a four-wire connection is possible for only a wye-connected source.

Wye- and Delta-Connected Loads

Load impedances can be either wye connected or delta connected, as shown in **Figure 5.46**. It can be shown that the two loads are equivalent if

$$Z_\Delta = 3Z_Y \tag{5.106}$$

Thus, we can convert a delta-connected load to an equivalent wye-connected load, or vice versa.

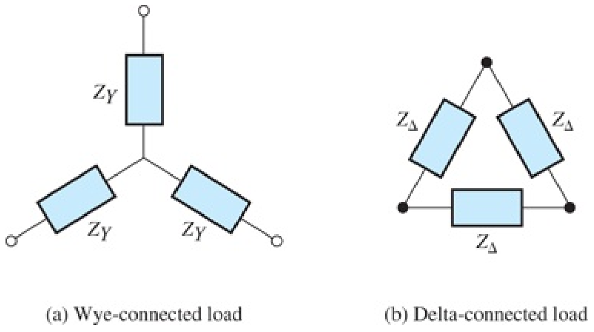


Figure 5.46
Loads can be either wye connected or delta connected.

Delta–Delta Connection

Figure 5.47 shows a delta-connected source delivering power to a delta-connected load. We assume that the source voltages are given by

$$V_{ab} = V_L \angle 30° \tag{5.107}$$

$$V_{bc} = V_L \angle -90° \tag{5.108}$$

$$V_{ca} = V_L \angle -150° \tag{5.109}$$

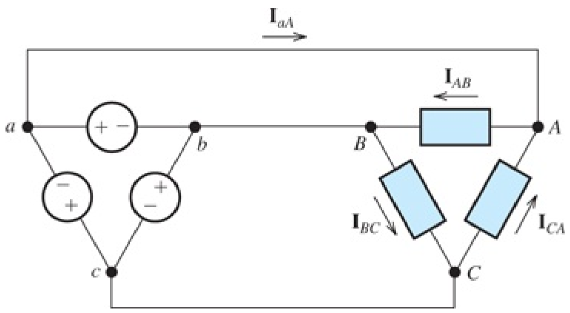


Figure 5.47
A delta-connected source delivering power to a delta-connected load.

These phasors are shown in **Figure 5.43**. (We have chosen the phase angles of the delta-connected source to be consistent with our earlier discussion.)

If the impedances of the connecting wires are zero, the line-to-line voltages at the load are equal to those at the source. Thus $V_{AB} = V_{ab}$, $V_{BC} = V_{bc}$, and $V_{CA} = V_{ca}$. We assume that the impedance of each phase of the load is $Z_\Delta \angle \theta$. Then, the load current for phase *AB* is

$$I_{AB} = \frac{V_{AB}}{Z_\Delta \angle \theta} = \frac{V_{ab}}{Z_\Delta \angle \theta} = \frac{V_L \angle 30°}{Z_\Delta \angle \theta} = \frac{V_L}{Z_\Delta} \angle 30° - \theta$$

We define the magnitude of the current as

$$I_\Delta = \frac{V_L}{Z_\Delta} \tag{5.110}$$

Hence,

$$\boldsymbol{I}_{AB} = I_\Delta \angle 30° - \theta \tag{5.111}$$

Similarly,

$$\boldsymbol{I}_{BC} = I_\Delta \angle -90° - \theta \tag{5.112}$$

$$\boldsymbol{I}_{CA} = I_\Delta \angle 150° - \theta \tag{5.113}$$

The current in line $a - A$ is

$$\begin{aligned} \boldsymbol{I}_{aA} &= \boldsymbol{I}_{AB} - \boldsymbol{I}_{CA} \\ &= I_\Delta \angle 30° - \theta - I_\Delta \angle 150° - \theta \\ &= \left(I_\Delta \angle 30° - \theta \right) \times (1 - 1 \angle 120°) \\ &= \left(I_\Delta \angle 30° - \theta \right) \times (1.5 - j0.8660) \\ &= \left(I_\Delta \angle 30° - \theta \right) \times (\sqrt{3} \angle -30°) \\ &= \boldsymbol{I}_{AB} \times \sqrt{3} \angle -30° \end{aligned}$$

The magnitude of the line current is

$$I_L = \sqrt{3} I_\Delta \tag{5.114}$$

For a balanced delta-connected load, the line-current magnitude is equal to the square root of three times the current magnitude in any arm of the delta.

Example 5.15 Analysis of a Balanced Delta–Delta System

Consider the circuit shown in **Figure 5.48(a)**. A delta-connected source supplies power to a delta-connected load through wires having impedances of $Z_{\text{line}} = 0.3 + j0.4\ \Omega$. The load impedances are $Z_\Delta = 30 + j6$. The source voltages are

$$\begin{aligned} \boldsymbol{V}_{ab} &= 1000 \angle 30° \\ \boldsymbol{V}_{bc} &= 1000 \angle -90° \\ \boldsymbol{V}_{ca} &= 1000 \angle 150° \end{aligned}$$

Figure 5.48

Circuit of **Example 5.15**.

Find the line current, the line-to-line voltage at the load, the current in each phase of the load, the power delivered to the load, and the power dissipated in the line.

Solution

First, we find the wye-connected equivalents for the source and the load. (Actually, we only need to work with one third of the circuit because the other two thirds are the same except for phase angles.) We choose to work with the *A* phase of the wye-equivalent circuit. Solving **Equation 5.103** for V_{an}, we find that

$$V_{an} = \frac{V_{ab}}{\sqrt{3}\ \angle 30^\circ} = \frac{1000\ \angle 30^\circ}{\sqrt{3}\ \angle 30^\circ} = 577.4\ \angle 0^\circ$$

Often, it is convenient to start an analysis by finding the wye–wye equivalent of a system.

Using **Equation 5.106**, we have

$$Z_Y = \frac{Z_\Delta}{3} = \frac{30 + j6}{3} = 10 + j2$$

Now, we can draw the wye-equivalent circuit, which is shown in **Figure 5.48(b)**.

In a balanced wye–wye system, we can consider the neutral points to be connected together as shown by the dashed line in **Figure 5.48(b)**. This reduces the three-phase circuit to three single-phase circuits. For phase *A* of **Figure 5.48(b)**, we can write

$$V_{an} = \left(Z_{\text{line}} + Z_Y \right) I_{aA}$$

Therefore,

$$\begin{aligned} \boldsymbol{I}_{aA} &= \frac{\boldsymbol{V}_{an}}{Z_{\text{line}} + Z_Y} = \frac{577.4\ \angle 0°}{0.3 + j0.4 + 10 + j2} \\ &= \frac{577.4\ \angle 0°}{10.3 + j2.4} = \frac{577.4\ \angle 0°}{10.58\ \angle 13.12°} \\ &= 54.60\ \angle -13.12° \end{aligned}$$

To find the line-to-neutral voltage at the load, we write

$$\begin{aligned} \boldsymbol{V}_{An} &= \boldsymbol{I}_{Aa} Z_Y = 54.60\ \angle -13.12° \times (\ 10 + j2) \\ &= 54.60\ \angle -13.12° \times 10.20\ \angle 11.31° \\ &= 556.9\ \angle -1.81° \end{aligned}$$

Now, we compute the line-to-line voltage at the load:

$$\begin{aligned} \boldsymbol{V}_{AB} &= \boldsymbol{V}_{An} \times \sqrt{3}\ \angle 30° = 556.9\ \angle -1.81° \times \sqrt{3}\ \angle 30° \\ &= 964.6\ \angle 28.19° \end{aligned}$$

The current through phase *AB* of the load is

$$\begin{aligned} \boldsymbol{I}_{AB} &= \frac{\boldsymbol{V}_{AB}}{Z_\Delta} = \frac{964.6\ \angle 28.19°}{30 + j6} = \frac{964.6\ \angle 28.19°}{30.59\ \angle 11.31°} \\ &= 31.53\ \angle 16.88° \end{aligned}$$

The power delivered to phase *AB* of the load is the rms current squared times the resistance:

$$P_{AB} = I^2_{AB\text{rms}} R = \left(\frac{31.53}{\sqrt{2}}\right)^2 (\ 30) \ = 14.91\ \text{kW}$$

The powers delivered to the other two phases of the load are the same, so the total power is

$$P = 3P_{AB} = 44.73\ \text{kW}$$

The power lost in line *A* is

$$P_{\text{line}A} = I^2_{aA\text{rms}} R_{\text{line}} = \left(\frac{54.60}{\sqrt{2}}\right)^2 (\ 0.3) \ = 0.447\ \text{kW}$$

The power lost in the other two lines is the same, so the total line loss is

$$P_{\text{line}} = 3 \times P_{\text{line}A} = 1.341\ \text{kW}$$

Exercise 5.17

A delta-connected source has voltages given by

$$\begin{aligned} \boldsymbol{V}_{ab} &= 1000\ \angle 30° \\ \boldsymbol{V}_{bc} &= 1000\ \angle -90° \\ \boldsymbol{V}_{ca} &= 1000\ \angle 150° \end{aligned}$$

This source is connected to a delta-connected load consisting of $50 - \Omega$ resistances. Find the line currents and the power delivered to the load.

Answer $\boldsymbol{I}_{aA} = 34.6\ \angle 0°,\ \boldsymbol{I}_{bB} = 34.6\ \angle - 120°,\ \boldsymbol{I}_{cC} = 34.6\ \angle 120°;\ P = 30\ \text{kW}$.

5.8 AC Analysis Using MATLAB

In this section, we will illustrate how MATLAB can greatly facilitate the analysis of complicated ac circuits. In fact, a practicing engineer working at a computer might have little use for a calculator, as it is easy to keep a MATLAB window open for all sorts of engineering calculations. Of course, you will probably need to use calculators for course exams and when you take the Professional Engineer (PE) exams. The PE exams allow only fairly simple scientific calculators, and you should practice with one of those allowed before attempting the exams.

Complex Data in MATLAB

By default, MATLAB assumes that $i = j = \sqrt{-1}$. However, I have encountered at least one bug in the software attributable to using *j* instead of *i*, and therefore I recommend using *i* in MATLAB and the Symbolic Toolbox. We need to be careful to avoid using *i* for other purposes when using MATLAB to analyze ac circuits. For example, if we were to use *i* as the name of a current or other variable, we would later experience errors if we also used *i* for the imaginary unit without reassigning its value.

Complex numbers are represented in rectangular form (such as $3 + 4\mathrm{i}$ or alternatively $3 + \mathrm{i} * 4$) in MATLAB.

We can use the fact that $M \angle\theta = M \exp(j\theta)$ to enter polar data. In MATLAB, angles are assumed to be in radians, so we need to multiply angles that are expressed in degrees by $\pi/180$ to convert to radians before entering them. For example, we use the following command to enter the voltage $V_s = 5\sqrt{2}\angle 45°$:

```
>> Vs = 5*sqrt(2)*exp(i*45*pi/180)
Vs =
5.0000 + 5.0000i
```

We can readily verify that MATLAB has correctly computed the rectangular form of $5\sqrt{2}\angle 45°$.

Alternatively, we could use Euler's formula

$$M \angle\theta = M \exp(j\theta) = M \cos(\theta) + jM \sin(\theta)$$

to enter polar data, again with angles in radians. For example, $V_s = 5\sqrt{2}\angle 45°$ can be entered as:

```
>> Vs = 5*sqrt(2)*cos(45*pi/180) + i*5*sqrt(2)*sin(45*pi/180)
Vs =
5.0000 + 5.0000i
```

Values that are already in rectangular form can be entered directly. For example, to enter $Z = 3 + j4$, we use the command:

```
>> Z = 3 + i*4
Z =
3.0000 + 4.0000i
```

Then, if we enter

```
>> Ix = Vs/Z
Ix =
1.4000 - 0.2000i
```

MATLAB performs the complex arithmetic and gives the answer in rectangular form.

Finding the Polar Form of MATLAB Results

Frequently, we need the polar form of a complex value calculated by MATLAB. We can find the magnitude using the abs command and the angle in radians using the angle command. To obtain the angle in degrees, we must convert the angle from radians by multiplying by $180/\pi$. Thus, to obtain the magnitude and angle in degrees for Vs, we would enter the following commands:

```
>> abs(Vs) % Find the magnitude of Vs.
ans =
7.0711
>> (180/pi)*angle(Vs) % Find the angle of Vs in degrees.
ans =
45.0000
```

Adding New Functions to MATLAB

Because we often want to enter values or see results in polar form with the angles in degrees, it is convenient to add two new functions to MATLAB. Thus, we write an m-file, named pin.m, containing the commands to convert from polar to rectangular form, and store it in our working MATLAB folder. The commands in the m-file are:

```
function z = pin(magnitude, angleindegrees)
z = magnitude*exp(i*angleindegrees*pi/180)
```

Then, we can enter $\mathrm{Vs} = 5\sqrt{2}\ \angle 45°$ simply by typing the command:

```
>> Vs = pin(5*sqrt(2),45)
Vs =
5.0000 + 5.0000i
```

We have chosen pin as the name of this new function to suggest "polar input." This file is included in the MATLAB folder. (See **Appendix E** for information about accessing this folder.)

Similarly, to obtain the polar form of an answer, we create a new function, named pout (to suggest "polar out"), with the commands:

```
function [y] = pout(x);
magnitude = abs(x);
angleindegrees = (180/pi)*angle(x);
y = [magnitude angleindegrees];
```

which are stored in the m-file named pout.m. Then, to find the polar form of a result, we can use the new function. For example,

```
>> pout(Vs)
ans =
7.0711    45.0000
```

Here is another simple example:

```
>> pout(i*200)
ans =
200 90
```

Solving Network Equations with MATLAB

We can readily solve node voltage or mesh equations and perform other calculations for ac circuits in MATLAB. The steps are:

1. Write the mesh current or node voltage equations.
2. Put the equations into matrix form, which is $ZI = V$ for mesh currents, in which **Z** is the coefficient matrix, **I** is the column vector of mesh current variables to be found, and **V** is the column vector of constant terms. For node voltages, the matrix equations take the form $YV = I$ in which **Y** is the

coefficient matrix, **V** is the column vector of node voltage variables to be determined, and **I** is the column vector of constants.

3. Enter the matrices into MATLAB and compute the mesh currents or node voltages using the inverse matrix approach. $I = \mathrm{inv}(Z) \times V$ for mesh currents or $V = \mathrm{inv}(Y) \times I$ for node voltages, where inv denotes the matrix inverse.
4. Use the results to compute any other quantities of interest.

Example 5.16 Phasor Mesh-Current Analysis with MATLAB

Determine the values for the mesh currents, the real power supplied by V_1, and the reactive power supplied by V_1 in the circuit of **Figure 5.49**.

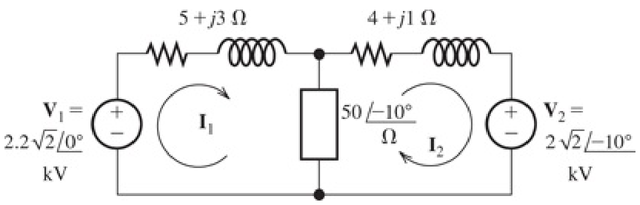


Figure 5.49
Circuit for **Example 5.16**.

Solution

First, we apply KVL to each loop obtaining the mesh-current equations:

$$(5 + j3)I_1 + (50\angle{-10°})(I_1 - I_2) = 2200\sqrt{2}$$
$$(50\angle{-10°})(I_2 - I_1) + (4 + j)I_2 + 2000\sqrt{2}\angle 30 = 0$$

In matrix form, these equations become

$$\begin{bmatrix} (5 + j3 + 50\angle{-10°}) & -50\angle{-10°} \\ -50\angle{-10°} & (4 + j + 50\angle{-10°}) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 2200\sqrt{2} \\ -2000\sqrt{2}\angle{-10°} \end{bmatrix}$$

We will solve these equations for I_1 and I_2. Then, we will compute the complex power delivered by V_1

$$S_1 = \frac{1}{2}V_1 I_1^*$$

Finally, the power is the real part of S_1 and the reactive power is the imaginary part.

We enter the coefficient matrix **Z** and the voltage matrix **V** into MATLAB, making use of our new pin function to enter polar values. Then, we calculate the current matrix.

```
>> Z = [(5 + i*3 + pin(50,-10)) (-pin(50,-10));...
(-pin(50,-10)) (4 + i + pin(50,-10))];
>> V = [2200*sqrt(2); -pin(2000*sqrt(2),-10)];
>> I = inv(Z)*V
I =
74.1634 + 29.0852i
17.1906 + 26.5112i
```

This has given us the values of the mesh currents in rectangular form. Next, we obtain the polar form for the mesh currents, making use of our new pout function:

```
>> pout(I(1))
ans =
79.6628    21.4140
>> pout(I(2))
ans =
31.5968    57.0394
```

Thus, the currents are $I_1 = 79.66\ \angle 21.41°\ \mathrm{A}$ and $I_2 = 31.60\ \angle 57.04°\ \mathrm{A},$ rounded to two decimal places. Next, we compute the complex power, real power, and reactive power for the first source.

$$\mathrm{S}_1 = \frac{1}{2}\mathrm{V}_1\mathrm{I}_1^*$$

```
>> S1 = (1/2)*(2200*sqrt(2))*conj(I(1));
>> P1 = real(S1)
P1 =
1.1537e + 005
>> Q1 = imag(S1)
Q1 =
-4.5246e + 004
```

Thus, the power supplied by V_1 is 115.37 kW and the reactive power is $-45.25\ \mathrm{kVAR}$. The commands for this example appear in the m-file named Example_5_16.

Exercise 5.18

Use MATLAB to solve for the phasor node voltages in polar form for the circuit of **Figure 5.50**.

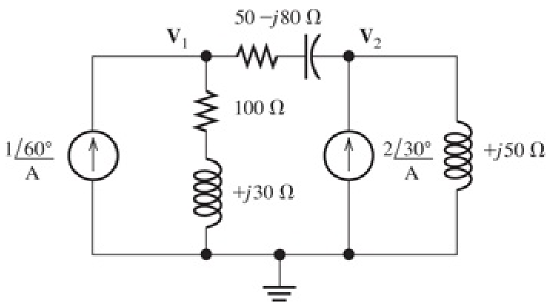


Figure 5.50
Circuit for **Exercise 5.18**.

Answer The MATLAB commands are:

```
clear all
Y = [(1/(100+i*30)+1/(50-i*80)) (-1/(50-i*80));...
(-1/(50-i*80)) (1/(i*50)+1/(50-i*80))];
I = [pin(1,60); pin(2,30)];
V = inv(Y)*I;
pout(V(1))
pout(V(2))
```

and the results are $V_1 = 79.98\ \angle 106.21°$ and $V_2 = 124.13\ \angle 116.30°$.

Summary

1. A sinusoidal voltage is given by $v(t) = V_m \cos(\omega t + \theta)$, where V_m is the peak value of the voltage, ω is the angular frequency in radians per second, and θ is the phase angle. The frequency in hertz is $f = 1/T$, where *T* is the period. Furthermore, $\omega = 2\pi f$.
2. For uniformity, we express sinusoidal voltages in terms of the cosine function. A sine function can be converted to a cosine function by use of the identity $\sin(z) = \cos(z - 90°)$.
3. The root-mean-square (rms) value (or effective value) of a periodic voltage *v*(*t*) is

   $$V_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\ dt}$$

   The average power delivered to a resistance by *v*(*t*) is

   $$P_{\text{avg}} = \frac{V^2_{\text{rms}}}{R}$$

   Similarly, for a current *i*(*t*), we have

   $$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\ dt}$$

   and the average power delivered if *i*(*t*) flows through a resistance is

   $$P_{\text{avg}} = I^2_{\text{rms}} R$$

   For a sinusoid, the rms value is the peak value divided by $\sqrt{2}$.
4. We can represent sinusoids with phasors. The magnitude of the phasor is the peak value of the sinusoid. The phase angle of the phasor is the phase angle of the sinusoid (assuming that we have written the sinusoid in terms of a cosine function).
5. We can add (or subtract) sinusoids by adding (or subtracting) their phasors.
6. The phasor voltage for a passive circuit is the phasor current times the complex impedance of the circuit. For a resistance, $\mathbf{V}_R = R\mathbf{I}_R$, and the voltage is in phase with the current. For an inductance, $\mathbf{V}_L = j\omega L\mathbf{I}_L$, and the voltage leads the current by 90°. For a capacitance, $\mathbf{V}_C = -j(1/\omega C)\mathbf{I}_C$, and the voltage lags the current by 90°.
7. Many techniques learned in **Chapter 2** for resistive circuits can be applied directly to sinusoidal circuits if the currents and voltages are replaced by phasors and the passive circuit elements are replaced by their complex impedances. For example, complex impedances can be combined in series or parallel in the same way as resistances (except that complex arithmetic must be used). Node voltages, the current-division principle, and the voltage-division principle also apply to ac circuits.
8. When a sinusoidal current flows through a sinusoidal voltage, the average power delivered is $P = V_{\text{rms}} I_{\text{rms}} \cos(\theta)$, where θ is the power angle, which is found by subtracting the phase angle of the current from the phase angle of the voltage (i.e., $\theta = \theta_v - \theta_i$). The power factor is $\cos(\theta)$.
9. Reactive power is the flow of energy back and forth between the source and energy-storage elements (*L* and *C*). We define reactive power to be positive for an inductance and negative for a capacitance. The net energy transferred per cycle by reactive power flow is zero. Reactive power is important because a power distribution system must have higher current ratings if reactive power flows than would be required for zero reactive power.
10. Apparent power is the product of rms voltage and rms current. Many useful relationships between power, reactive power, apparent power, and the power angle can be obtained from the power triangles shown in **Figure 5.25** on page **244**.
11. In steady state, a network composed of resistances, inductances, capacitances, and sinusoidal sources (all of the same frequency) has a Thévenin equivalent consisting of a phasor voltage

source in series with a complex impedance. The Norton equivalent consists of a phasor current source in parallel with the Thévenin impedance.

12. For maximum-power transfer from a two-terminal ac circuit to a load, the load impedance is selected to be the complex conjugate of the Thévenin impedance. If the load is constrained to be a pure resistance, the value for maximum power transfer is equal to the magnitude of the Thévenin impedance.
13. Because of savings in wiring, three-phase power distribution is more economical than single phase. The power flow in balanced three-phase systems is smooth, whereas power pulsates in single-phase systems. Thus, three-phase motors generally have the advantage of producing less vibration than single-phase motors.

Problems

Section 5.1: Sinusoidal Currents and Voltages

P5.1. Consider the plot of the sinusoidal voltage $v(t) = V_m \cos(\omega t + \theta)$ shown in **Figure 5.1** on page **216** and the following statements:

1. Stretches the sinusoidal curve vertically.
2. Compresses the sinusoidal curve vertically.
3. Stretches the sinusoidal curve horizontally.
4. Compresses the sinusoidal curve horizontally.
5. Translates the sinusoidal curve to the right.
6. Translates the sinusoidal curve to the left.

Which statement best describes

a. Increasing the peak amplitude V_m?
b. Increasing the frequency *f*?
c. Decreasing θ?
d. Decreasing the angular frequency ω?
e. Increasing the period?

P5.2. What are the units for angular frequency ω? For frequency *f*? What is the relationship between them?

***P5.3.** A voltage is given by $v(t) = 10\sin(1000\pi t + 30°)$. First, use a cosine function to express *v*(*t*). Then, find the angular frequency, the frequency in hertz, the phase angle, the period, and the rms value. Find the power that this voltage delivers to a $50\text{-}\Omega$ resistance. Find the first value of time after $t = 0$ that *v*(*t*) reaches its peak value. Sketch *v*(*t*) to scale versus time.

* Denotes that answers are contained in the Student Solutions files. See See **Appendix E** for more information about accessing the Student Solutions.

P5.4. Repeat **Problem P5.3** for $v(t) = 50\sin(500\pi t + 120°)$.

***P5.5.** A sinusoidal voltage *v*(*t*) has an rms value of 20 V, has a period of $100\ \mu\text{s}$, and reaches a positive peak at $t = 20\ \mu\text{s}$. Write an expression for *v*(*t*).

P5.6. A sinusoidal voltage has a peak value of 15 V, has a frequency of 125 Hz, and crosses zero with positive slope at $t = 1\ \text{ms}$. Write an expression for the voltage.

P5.7. A current $i(t) = 10\cos(2000\pi t)$ flows through a $100\text{-}\Omega$ resistance. Sketch *i*(*t*) and *p*(*t*) to scale versus time. Find the average power delivered to the resistance.

P5.8. We have a voltage $v(t) = 1000\sin(500\pi t)$ across a $500\text{-}\Omega$ resistance. Sketch *v*(*t*) and *p*(*t*) to scale versus time. Find the average power delivered to the resistance.

P5.9. Suppose we have a sinusoidal current *i*(*t*) that has an rms value of 5 A, has a period of 10 ms, and reaches a positive peak at $t = 3\ \text{ms}$. Write an expression for *i*(*t*).

P5.10. A **Lissajous figure** results if one sinusoid is plotted versus another. Consider $x(t) = \cos(\omega_x t)$ and $y(t) = \cos\left(\omega_y t + \theta\right)$. Use a computer program of your choice to generate values of *x* and *y* for 20 seconds at 100 points per second and obtain a plot of *y* versus *x* for

a. $\omega_x = \omega_y = 2\pi$ and $\theta = 90°$;
b. $\omega_x = \omega_y = 2\pi$ and $\theta = 45°$;
c. $\omega_x = \omega_y = 2\pi$ and $\theta = 0°$;
d. $\omega_x = 2\pi,\ \omega_y = 4\pi,$ and $\theta = 0°$.

*__P5.11.__ Find the rms value of the voltage waveform shown in **Figure P5.11**.

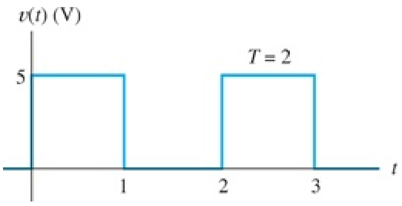


Figure P5.11

*__P5.12.__ Calculate the rms value of the half-wave rectified sine wave shown in **Figure P5.12**.

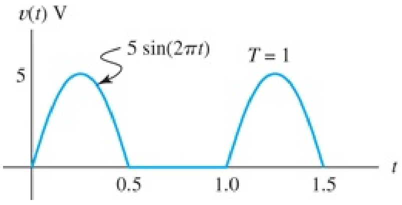


Figure P5.12

*__P5.13.__ Find the rms value of the current waveform shown in **Figure P5.13**.

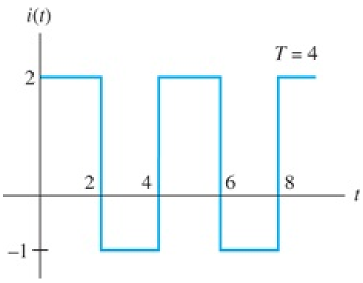


Figure P5.13

P5.14. Determine the rms value of $v(t) = A\cos(2\pi t) + B\sin(2\pi t)$.

P5.15. Determine the rms value of $v(t) = 5 + 10\cos(20\pi t)$.

P5.16. Compute the rms value of the periodic waveform shown in **Figure P5.16**.

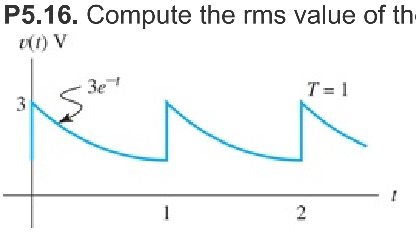


Figure P5.16

P5.17. Find the rms value of the voltage waveform shown in **Figure P5.17**.

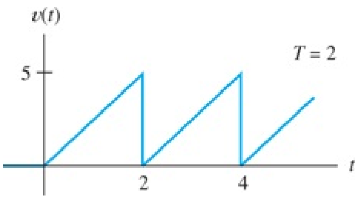


Figure P5.17

P5.18. Is the rms value of a periodic waveform always equal to the peak value divided by the square root of two? When is it?

Section 5.2: Phasors

P5.19. What steps do we follow in adding sinusoidal currents or voltages? What must be true of the sinusoids?

P5.20. Describe two methods to determine the phase relationship between two sinusoids of the same frequency.

***P5.21.** Suppose that $v_1(t) = 100\cos(\omega t)$ and $v_2(t) = 100\sin(\omega t)$. Use phasors to reduce the sum $v_s(t) = v_1(t) + v_2(t)$ to a single term of the form $V_m \cos(\omega t + \theta)$. Draw a phasor diagram, showing $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_s$. State the phase relationships between each pair of these phasors.

***P5.22.** Consider the phasors shown in **Figure P5.22**. The frequency of each signal is $f = 200\text{ Hz}$. Write a time-domain expression for each voltage in the form $V_m \cos(\omega t + \theta)$. State the phase relationships between pairs of these phasors.

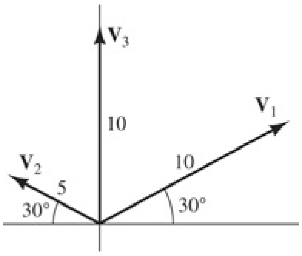


Figure P5.22

***P5.23.** Reduce $5\cos(\omega t + 75°) - 3\cos(\omega t - 75°) + 4\sin(\omega t)$ to the form $V_m \cos(\omega t + \theta)$.

P5.24. Two sinusoidal voltages of the same frequency have rms values of 8 V and 3 V. What is the smallest rms value that the sum of these voltages could have? The largest? Justify your answers.

P5.25. Suppose that $v_1(t) = 100\cos(\omega t + 45°)$ and $v_2(t) = 150\sin(\omega t + 60°)$. Use phasors to reduce the sum $v_s(t) = v_1(t) + v_2(t)$ to a single term of the form $V_m \cos(\omega t + \theta)$. Draw a phasor diagram showing $\mathbf{V}_1$, $\mathbf{V}_2$, and $\mathbf{V}_s$. State the phase relationships between each pair of these phasors.

P5.26. Write an expression for the sinusoid shown in **Figure P5.26** of the form $v(t) = V_m \cos(\omega t + \theta)$, giving the numerical values of V_m, ω, and θ. Also, determine the phasor and the rms value of *v*(*t*).

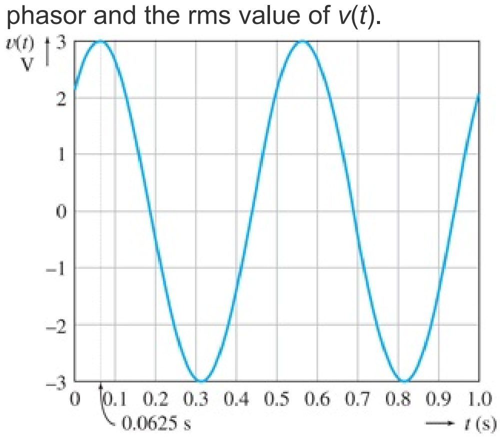


Figure P5.26

P5.27. We have $v_1(t) = 10\cos(\omega t + 30°)$. The current $i_1(t)$ has an rms value of 5 A and leads $v_1(t)$ by 20°. (The current and the voltage have the same frequency.) Draw a phasor diagram showing the phasors. Write an expression for $i_1(t)$ of the form $I_m \cos(\omega t + \theta)$.
P5.28. Reduce $5\sin(\omega t) + 5\cos(\omega t + 30°) + 5\cos(\omega t + 150°)$ to the form $V_m \cos(\omega t + \theta)$.
P5.29. Using a computer program of your choice, obtain a plot of $v(t) = \cos(19\pi t) + \cos(21\pi t)$ for *t* ranging from 0 to 2 s in 0.01-s increments. (Notice that because the terms have different frequencies, they cannot be combined by using phasors.) Then, considering that the two terms can be represented as the real projection of the sum of two vectors rotating (at different speeds) in the complex plane, comment on the plot.
P5.30. A sinusoidal current $i_1(t)$ has a phase angle of 30°. Furthermore, $i_1(t)$ attains its positive peak 2 ms earlier than current $i_2(t)$ does. Both currents have a frequency of 200 Hz. Determine the phase angle of $i_2(t)$.

Section 5.3: Complex Impedances

P5.31. Write the relationship between the phasor voltage and phasor current for an inductance. Repeat for capacitance.
P5.32. State the phase relationship between current and voltage for a resistance, for an inductance, and for a capacitance.
***P5.33.** A voltage $v_L(t) = 10\cos(2000\pi t)$ is applied to a 100-mH inductance. Find the complex impedance of the inductance. Find the phasor voltage and current, and construct a phasor diagram. Write the current as a function of time. Sketch the voltage and current to scale versus time. State the phase relationship between the current and voltage.
***P5.34.** A voltage $v_C(t) = 10\cos(2000\pi t)$ is applied to a $10\text{-}\mu\mathrm{F}$ capacitance. Find the complex impedance of the capacitance. Find the phasor voltage and current, and construct a phasor diagram. Write the current as a function of time. Sketch the voltage and current to scale versus time. State the phase relationship between the current and voltage.
P5.35. A certain circuit element is known to be a resistance, an inductance, or a capacitance. Determine the type and value (in ohms, henrys, or farads) of the element if the voltage and current for the element are given by

a. $v(t) = 100\sin(200t + 30°)\ \mathrm{V}, i(t) = \cos(200t + 30°)\ \mathrm{A}$;
b. $v(t) = 500\cos(100t + 50°)\ \mathrm{V}, i(t) = 2\cos(100t + 50°)\ \mathrm{A}$;
c. $v(t) = 100\cos(400t + 30°)\ \mathrm{V}, i(t) = \sin(400t + 30°)\ \mathrm{A}$.

P5.36. Sketch plots of the magnitudes of the impedances of a 10-mH inductance, a $10\text{-}\mu\mathrm{F}$ capacitance, and a $50\text{-}\Omega$ resistance to scale versus frequency for the range from zero to 1000 Hz.
P5.37.

a. A certain element has a phasor voltage of $\boldsymbol{V} = 100\angle 30°\ \mathrm{V}$ and current of $\boldsymbol{I} = 5\angle 120°\ \mathrm{A}$. The angular frequency is 500 rad/s. Determine the nature and value of the element.
b. Repeat for $\boldsymbol{V} = 20\angle -45°\ \mathrm{V}$ and current of $\boldsymbol{I} = 5\angle -135°\ \mathrm{A}$.
c. Repeat for $\boldsymbol{V} = 5\angle 45°\ \mathrm{V}$ and current of $\boldsymbol{I} = 5\angle 45°\ \mathrm{A}$.

P5.38.

a. The current and voltage for a certain circuit element are shown in **Figure P5.38(a)**. Determine the nature and value of the element.

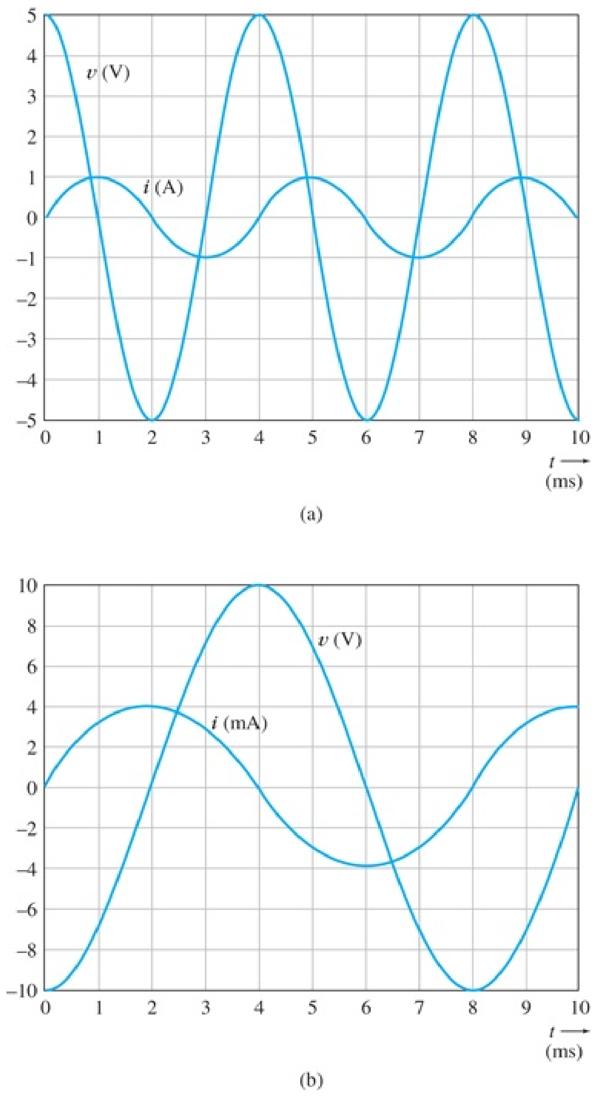


Figure P5.38

b. Repeat for **Figure P5.38(b)**.

Section 5.4: Circuit Analysis with Phasors and Complex Impedances

P5.39. Give a step-by-step procedure for steady-state analysis of circuits with sinusoidal sources. What condition must be true of the sources?

***P5.40.** Find the complex impedance in polar form of the network shown in **Figure P5.40** for $\omega = 500$. Repeat for $\omega = 1000$ and $\omega = 2000$.

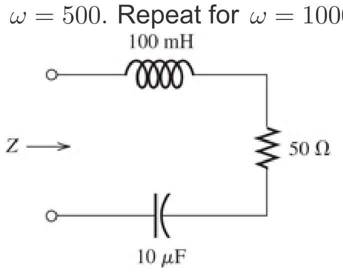


Figure P5.40

***P5.41.** Find the phasors for the current and for the voltages of the circuit shown in **Figure P5.41**. Construct a phasor diagram showing V_s, I, V_R, and V_L. What is the phase relationship between V_s and **I**?

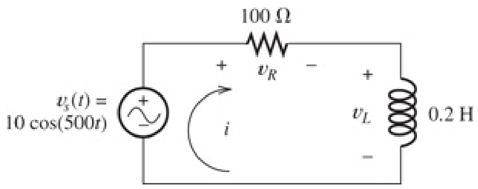


Figure P5.41

P5.42. Change the inductance to 0.1 H, and repeat Problem P5.41.
P5.43. Find the complex impedance of the network shown in **Figure P5.43** for $\omega = 500$. Repeat for $\omega = 1000$ and $\omega = 2000$.

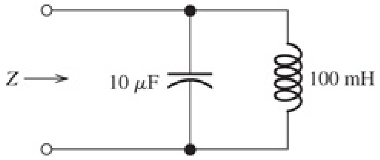


Figure P5.43

P5.44. A 10-mH inductance, a $100\text{-}\Omega$ resistance, and a $100\text{-}\mu\mathrm{F}$ capacitance are connected in parallel. Calculate the impedance of the combination for angular frequencies of 500, 1000, and 2000 radians per second. For each frequency, state whether the impedance is inductive, purely resistive, or capacitive.
P5.45. Find the phasors for the current and the voltages for the circuit shown in **Figure P5.45**. Construct a phasor diagram showing V_s, I, V_R, and V_C. What is the phase relationship between V_s and **I**?

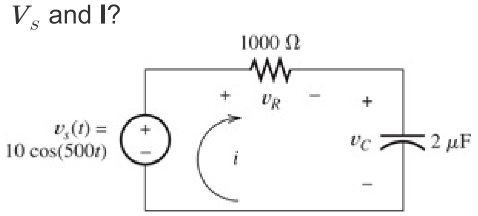


Figure P5.45

***P5.46.** Repeat Problem P5.45, changing the capacitance value to $1\ \mu\mathrm{F}$.
P5.47. Find the phasors for the voltage and the currents of the circuit shown in **Figure P5.47**. Construct a phasor diagram showing I_s, V, I_R, and I_L. What is the phase relationship between **V** and I_s?

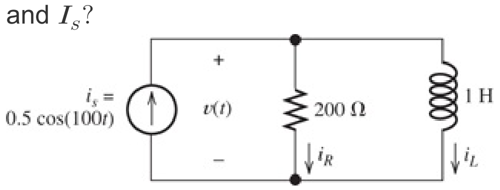


Figure P5.47

***P5.48.** Find the phasors for the voltage and the currents for the circuit shown in **Figure P5.48**. Construct a phasor diagram showing I_s, V, I_R, and I_C. What is the phase relationship between **V** and I_s?

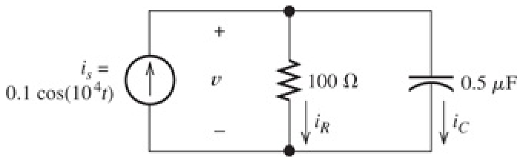


Figure P5.48

***P5.49.** Consider the circuit shown in **Figure P5.49**. Find the phasors I_s, V, I_R, I_L, and I_C. Compare the peak value of $i_L(t)$ with the peak value of $i_s(t)$. Do you find the answer surprising? Explain.

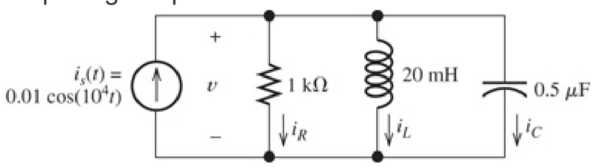


Figure P5.49

P5.50. Consider the circuit shown in **Figure P5.50**. Find the phasors V_s, I, V_L, V_R, and V_C. Compare the peak value of $v_L(t)$ with the peak value of $v_s(t)$. Do you find the answer surprising? Explain.

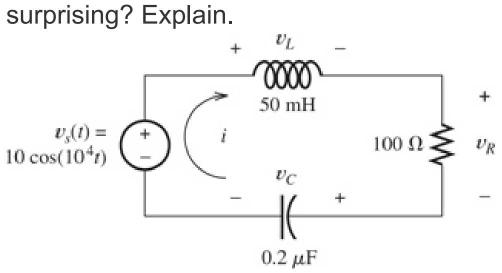


Figure P5.50

P5.51. Consider the circuit shown in **Figure P5.51**. Find the phasors V_1, V_2, V_R, V_L, and **I**. Draw the phasor diagram to scale. What is the phase relationship between **I** and V_1? Between **I** and V_L?

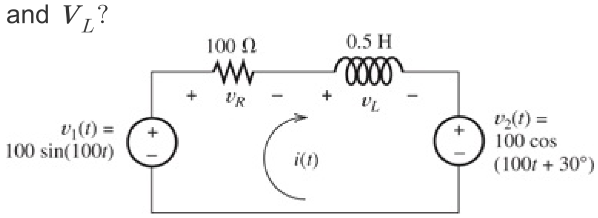


Figure P5.51

P5.52. Consider the circuit shown in **Figure P5.52**. Find the phasors I, I_R, and I_C. Construct the phasor diagram.

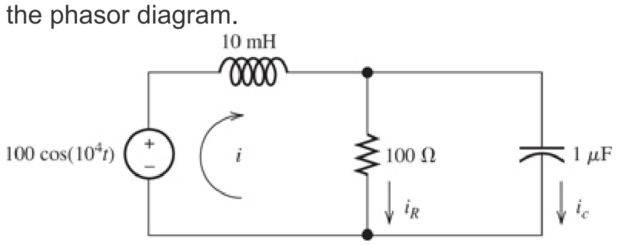


Figure P5.52

*P5.53. Solve for the node voltages shown in **Figure P5.53**.

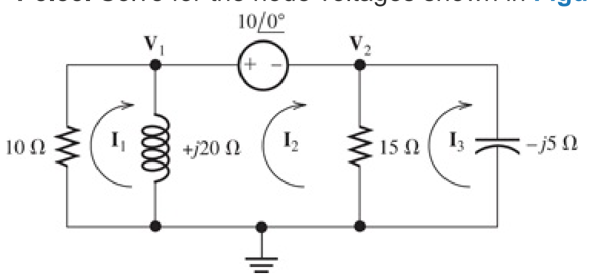


Figure P5.53

P5.54. Solve for the node voltage shown in **Figure P5.54**.

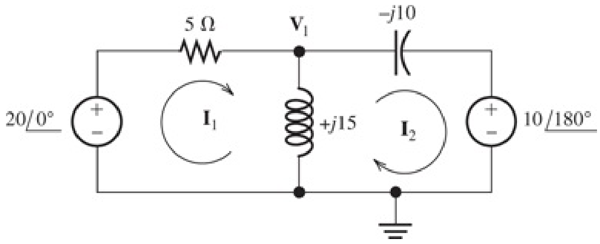


Figure P5.54

P5.55. Solve for the node voltage shown in **Figure P5.55**.

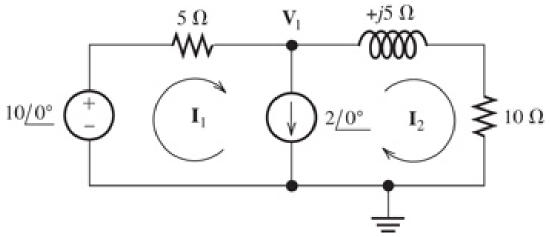


Figure P5.55

P5.56. Solve for the node voltages shown in **Figure P5.56**.

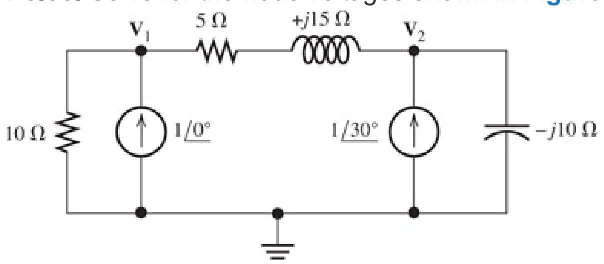


Figure P5.56

*P5.57. Solve for the mesh currents shown in **Figure P5.54**.

P5.58. Solve for the mesh currents shown in **Figure P5.55**.

P5.59. Solve for the mesh currents shown in **Figure P5.53**.

P5.60.

a. A 20-mH inductance is in series with a $50\text{-}\mu\text{F}$ capacitance. Sketch or use the computer program of your choice to produce a plot of the impedance magnitude versus angular frequency. Allow ω to range from zero to 2000 rad/s and the vertical axis to range from 0 to $100\ \Omega$.

b. Repeat with the inductance and capacitance in parallel.

P5.61.

a. A 20-mH inductance is in series with a $50\text{-}\Omega$ resistance. Sketch or use the computer program of your choice to produce a plot of the impedance magnitude versus angular frequency. Allow ω to range from zero to 5000 rad/s.
b. Repeat with the inductance and resistance in parallel.

Section 5.5: Power in AC Circuits

P5.62. What are the customary units for real power? For reactive power? For apparent power?
P5.63. How are power factor and power angle related?
P5.64. Assuming that a nonzero ac source is applied, state whether the power and reactive power are positive, negative, or zero for

a. a pure resistance;
b. a pure inductance;
c. a pure capacitance.

P5.65. A load is said to have a leading power factor. Is it capacitive or inductive? Is the reactive power positive or negative? Repeat for a load with lagging power factor.
P5.66.

a. Sketch a power triangle for an inductive load, label the sides, and show the power angle.
b. Repeat for a capacitive load.

P5.67. Discuss why power plant and distribution system engineers are concerned

a. with the real power absorbed by a load;
b. with the reactive power.

P5.68. Define what we mean by "power-factor correction." For power-factor correction of an inductive load, what type of element should we place in parallel with the load?
***P5.69.** Consider a load that has an impedance given by $Z = 100 - j50\ \Omega$. The current flowing through this load is $I = 15\sqrt{2}\ \angle 30°\ \text{A}$. Is the load inductive or capacitive? Determine the power factor, power, reactive power, and apparent power delivered to the load.
P5.70. We have a load with an impedance given by $Z = 30 + j40\ \Omega$. The voltage across this load is $V = 1500\sqrt{2}\ \angle 30°\ \text{V}$. Is the load inductive or capacitive? Determine the power factor, power, reactive power, and apparent power delivered to the load.
P5.71. The phasor voltage across a certain load is $V = 1000\sqrt{2}\ \angle 30°\ \text{V}$, and the phasor current through it is $I = 15\sqrt{2}\ \angle 60°\ \text{A}$. Determine the power factor, power, reactive power, apparent power, and impedance. Is the power factor leading or lagging?
P5.72. The voltage across a load is $v(t) = 10^4\sqrt{2}\cos(\omega t + 10°)\ \text{V}$, and the current through the load is $i(t) = 20\sqrt{2}\cos(\omega t - 20°)\ \text{A}$. The reference direction for the current points into the positive reference for the voltage. Determine the power factor, the power, the reactive power, and the apparent power for the load. Is this load inductive or capacitive?
P5.73. Assuming that a nonzero ac voltage source is applied, state whether the power and reactive power are positive, negative, or zero for

a. a resistance in series with an inductance;
b. a resistance in series with a capacitance. (Assume that the resistances, inductance, and capacitance are nonzero and finite in value.)

P5.74. Assuming that a nonzero ac voltage source is applied, what can you say about whether the power and reactive power are positive, negative, or zero for a pure capacitance in series with a pure inductance? Consider cases in which the impedance magnitude of the capacitance is greater than, equal to, or less than the impedance magnitude of the inductance.
P5.75. Repeat Problem P5.74 for the inductance and capacitance in parallel.

P5.76. Determine the power for each source shown in **Figure P5.76**. Also, state whether each source is delivering or absorbing energy.

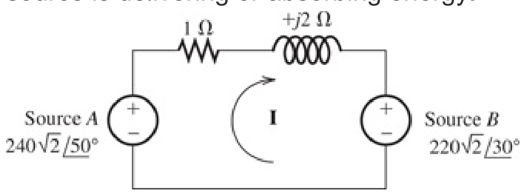


Figure P5.76

P5.77. Determine the power for each source shown in **Figure P5.77**. Also, state whether each source is delivering or absorbing energy.

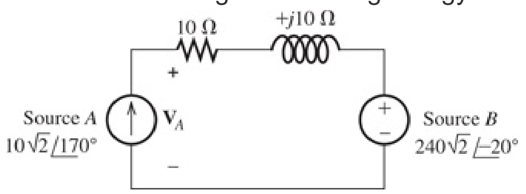


Figure P5.77

P5.78. A 60-Hz 220-V-rms source supplies power to a load consisting of a resistance in series with an inductance. The real power is 1500 W, and the apparent power is 2500 VAR. Determine the value of the resistance and the value of the inductance.

P5.79. Consider the circuit shown in **Figure P5.79**. Find the phasor current I. Find the power, reactive power, and apparent power delivered by the source. Find the power factor and state whether it is lagging or leading.

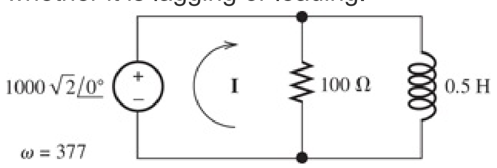


Figure P5.79

***P5.80.** Repeat Problem P5.79, replacing the inductance by a $10\text{-}\mu\mathrm{F}$ capacitance.

***P5.81.** Two loads, *A* and *B*, are connected in parallel across a 1-kV-rms 60-Hz line, as shown in **Figure P5.81**. Load *A* consumes 10 kW with a 90 percent lagging power factor. Load *B* has an apparent power of 15 kVA with an 80 percent lagging power factor. Find the power, reactive power, and apparent power delivered by the source. What is the power factor seen by the source?

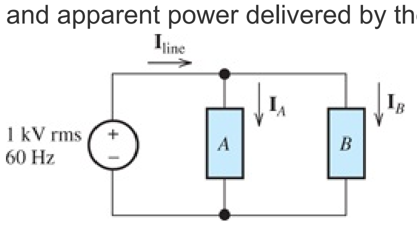


Figure P5.81

P5.82. Repeat Problem P5.81 if load *A* consumes 5 kW with a 90 percent lagging power factor and load *B* consumes 10 kW with an 80 percent leading power factor.

P5.83. Find the power, reactive power, and apparent power delivered by the source in **Figure P5.83**. Find the power factor and state whether it is leading or lagging.

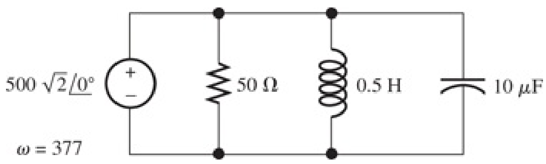


Figure P5.83

P5.84. Repeat Problem P5.83 with the resistance, inductance, and capacitance connected in series rather than in parallel.

***P5.85.** Consider the situation shown in **Figure P5.85**. A 1000-V-rms source delivers power to a load. The load consumes 100 kW with a power factor of 25 percent lagging.

a. Find the phasor **I**, assuming that the capacitor is not connected to the circuit.
b. Find the value of the capacitance that must be connected in parallel with the load to achieve a power factor of 100 percent. Usually, power-systems engineers rate capacitances used for power-factor correction in terms of their reactive power rating. What is the rating of this capacitance in kVAR? Assuming that this capacitance is connected, find the new value for the phasor **I**.
c. Suppose that the source is connected to the load by a long distance. What are the potential advantages and disadvantages of connecting the capacitance across the load?

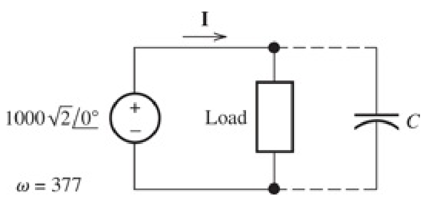


Figure P5.85

Section 5.6: Thévenin and Norton Equivalent Circuits

P5.86. Of what does an ac steady-state Thévenin equivalent circuit consist? A Norton equivalent circuit? How are the values of the parameters of these circuits determined?

P5.87. To attain maximum power delivered to a load, what value of load impedance is required if

a. the load can have any complex value;
b. the load must be pure resistance?

P5.88. For an ac circuit consisting of a load connected to a Thévenin circuit, is it possible for the load voltage to exceed the Thévenin voltage in magnitude? If not, why not? If so, under what conditions is it possible? Explain.

***P5.89.**

a. Find the Thévenin and Norton equivalent circuits for the circuit shown in **Figure P5.89**.

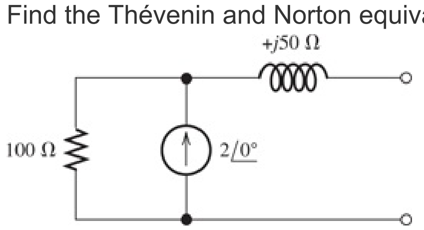


Figure P5.89

b. Find the maximum power that this circuit can deliver to a load if the load can have any complex impedance.

c. Repeat if the load is purely resistive.

P5.90.

a. Find the Thévenin and Norton equivalent circuits for the circuit shown in **Figure P5.90**.

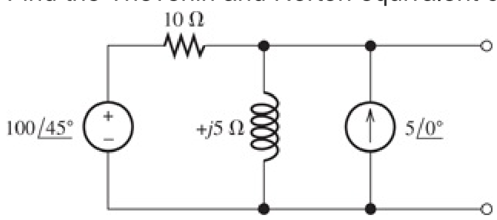


Figure P5.90

b. Find the maximum power that this circuit can deliver to a load if the load can have any complex impedance.
c. Repeat if the load must be purely resistive.

P5.91. Draw the Thévenin and Norton equivalent circuits for **Figure P5.91**, labeling the elements and terminals.

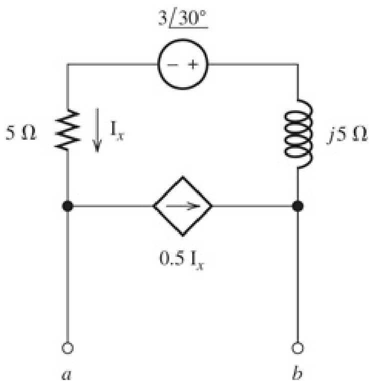


Figure P5.91

P5.92. Draw the Thévenin and Norton equivalent circuits for **Figure P5.92**, labeling the elements and terminals.

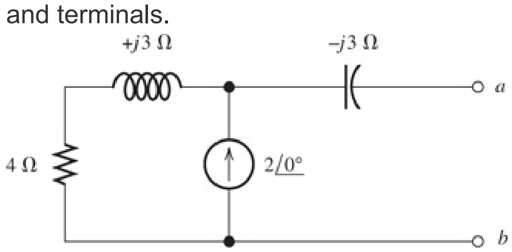


Figure P5.92

P5.93. The Thévenin equivalent of a two-terminal network is shown in **Figure P5.93**. The frequency is $f = 60\ \mathrm{Hz}$. We wish to connect a load across terminals $a - b$ that consists of a resistance and a capacitance in series such that the power delivered to the resistance is maximized. Find the value of the resistance and the value of the capacitance.

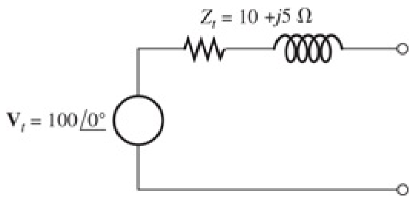


Figure P5.93

***P5.94.** Repeat Problem P5.93 with the load required to consist of a resistance and a capacitance in parallel.

Section 5.7: Balanced Three-Phase Circuits

P5.95. A balanced positive-sequence three-phase source has

$$v_{an}(t) = 100\cos(377t + 90°)\ \text{V}$$

a. Find the frequency of this source in Hz.
b. Give expressions for $v_{bn}(t)$ and $v_{cn}(t)$.
c. Repeat part (b) for a negative-sequence source.

P5.96. A three-phase source has

$$v_{an}(t) = 100\cos(\omega t - 60°)$$
$$v_{bn}(t) = 100\cos(\omega t + 60°)$$
$$v_{cn}(t) = -100\cos(\omega t)$$

Is this a positive-sequence or a negative-sequence source? Find time-domain expressions for $v_{ab}(t)$, $v_{bc}(t)$, and $v_{ca}(t)$.

***P5.97.** A balanced wye-connected three-phase source has line-to-neutral voltages of 440 V rms. Find the rms line-to-line voltage magnitude. If this source is applied to a wye-connected load composed of three $30\text{-}\Omega$ resistances, find the rms line-current magnitude and the total power delivered.

***P5.98.** Each phase of a wye-connected load consists of a $50\text{-}\Omega$ resistance in parallel with a $100\text{-}\mu\text{F}$ capacitance. Find the impedance of each phase of an equivalent delta-connected load. The frequency of operation is 60 Hz.

P5.99. What can you say about the flow of power as a function of time between a balanced three-phase source and a balanced load? Is this true of a single-phase source and a load? How is this a potential advantage for the three-phase system? What is another advantage of three-phase power distribution compared with single-phase?

P5.100. A delta-connected source delivers power to a delta-connected load, as shown in Figure P5.100. The rms line-to-line voltage at the source is $V_{ab\text{rms}} = 440\ \text{V}$. The load impedance is $Z_\Delta = 10 - j2$. Find I_{aA}, V_{AB}, I_{AB}, the total power delivered to the load, and the power lost in the line.

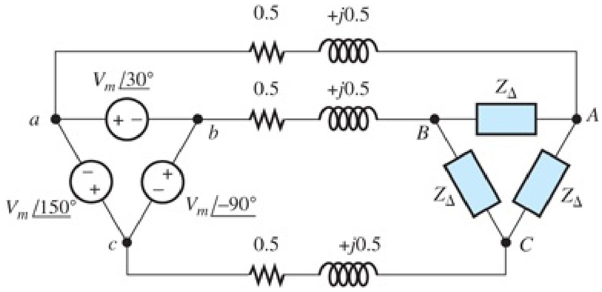


Figure P5.100

*P5.101.** Repeat Problem P5.100, with $Z_\Delta = 5 - j2$.

P5.102. A negative-sequence wye-connected source has line-to-neutral voltages $\mathbf{V}_{an} = V_Y\angle 0°$, $\mathbf{V}_{bn} = V_Y\angle 120°$, and $\mathbf{V}_{cn} = V_Y\angle -120°$. Find the line-to-line voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$. Construct a phasor diagram showing both sets of voltages and compare with **Figure 5.41** on page **261**.

P5.103. A balanced positive-sequence wye-connected 60-Hz three-phase source has line-to-line voltages of $V_L = 440 \text{ V rms}$. This source is connected to a balanced wye-connected load. Each phase of the load consists of a 0.3-H inductance in series with a $50\text{-}\Omega$ resistance. Find the line-to-neutral voltage phasors, the line-to-line voltage phasors, the line-current phasors, the power, and the reactive power delivered to the load. Assume that the phase of $\mathbf{V}_{an}$ is zero.

P5.104. A balanced wye-connected three-phase source has line-to-neutral voltages of 240 V rms. Find the rms line-to-line voltage. This source is applied to a delta-connected load, each arm of which consists of a $10\text{-}\Omega$ resistance in parallel with a $+j5\text{-}\Omega$ reactance. Determine the rms line current magnitude, the power factor, and the total power delivered.

P5.105. In this chapter, we have considered balanced loads only. However, it is possible to determine an equivalent wye for an unbalanced delta, and vice versa. Consider the equivalent circuits shown in **Figure P5.105**. Derive formulas for the impedances of the wye in terms of the impedances of the delta. [*Hint*: Equate the impedances between corresponding pairs of terminals of the two circuits with the third terminal open. Then, solve the equations for Z_a, Z_b, and Z_c in terms of Z_A, Z_B, and Z_C. Take care in distinguishing between upper- and lowercase subscripts.]

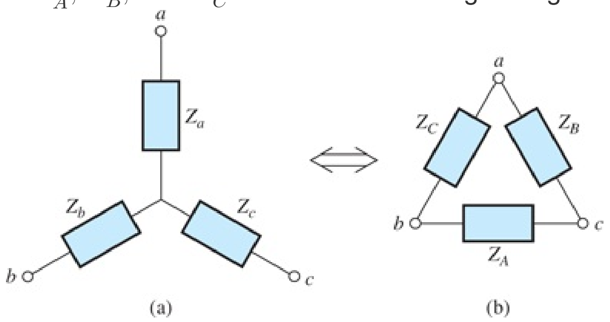


Figure P5.105

P5.106. Repeat Problem P5.105, but solve for the impedances of the delta in terms of those of the wye. [*Hint*: Start by working in terms of the admittances of the delta (Y_A, Y_B, and Y_C) and the impedances of the wye (Z_a, Z_b, and Z_c). Short terminals *b* and *c* for each circuit. Then equate the admittances between terminal *a* and the shorted terminals for the two circuits. Repeat this twice more with shorts between the remaining two pairs of terminals. Solve the equations to determine Y_A, Y_B, and Y_C in terms of Z_a, Z_b, and Z_c. Finally, invert the equations for Y_A, Y_B, and Y_C to obtain equations relating the impedances. Take care in distinguishing between upper- and lowercase subscripts.]

Section 5.8: AC Analysis Using MATLAB

*P5.107** Use MATLAB to solve for the node voltages shown in **Figure P5.107.**

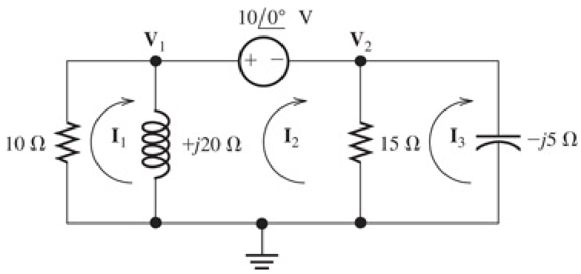


Figure P5.107

P5.108 Use MATLAB to solve for the mesh currents shown in **Figure P5.107.**

***P5.109** Use MATLAB to solve for the mesh currents shown in **Figure P5.109.**

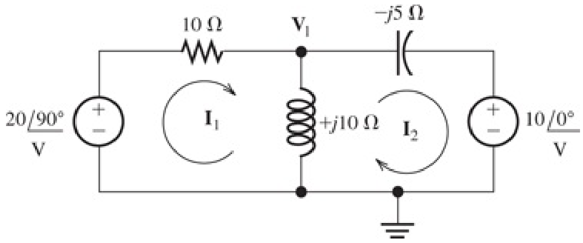


Figure P5.109

P5.110 Use MATLAB to solve for the mesh currents shown in **Figure P5.110.**

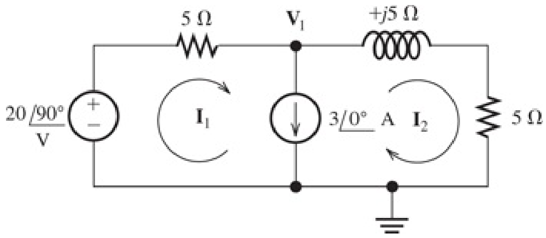


Figure P5.110

P5.111 Use MATLAB to solve for the node voltages shown in **Figure P5.111.**

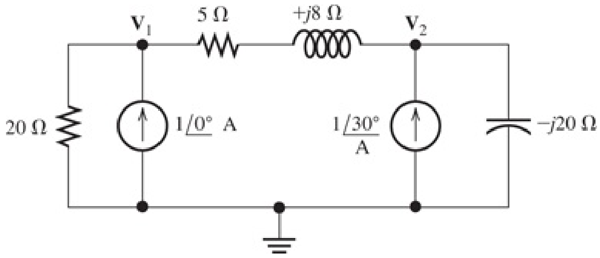


Figure P5.111

P5.112 Use the MATLAB Symbolic Toolbox to determine the rms value of *v*(*t*) which has a period of 1 s and is given by $v(t) = 10\exp(-5t)\sin(20\pi t)$ V for $0 \le t \le 1\text{ s}$.

Practice Test

Here is a practice test you can use to check your comprehension of the most important concepts in this chapter. Answers can be found in **Appendix D** and complete solutions are included in the Student Solutions files. See **Appendix E** for more information about the Student Solutions.

T5.1. Determine the rms value of the current shown in **Figure T5.1** and the average power delivered to the $50\text{-}\Omega$ resistance.

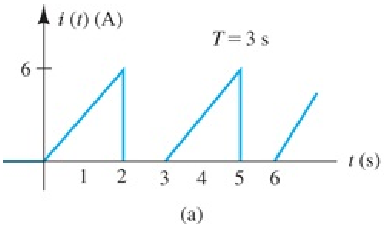


Figure T5.1

T5.2. Reduce the expression

$$v(t) = 5\sin(\omega t + 45°) + 5\cos(\omega t - 30°)$$

to the form $V_m \cos(\omega t + \theta)$.

T5.3. We have two voltages $v_1(t) = 15\sin(400\pi t + 45°)$ V and $v_2(t) = 5\cos(400\pi t - 30°)$ V. Determine (including units):

a. the rms value of $v_1(t)$;
b. the frequency of the voltages;
c. the angular frequency of the voltages;
d. the period of the voltages;
e. the phase relationship between $v_1(t)$ and $v_2(t)$.

T5.4. Find the phasor values of V_R, V_L, and V_C in polar form for the circuit of **Figure T5.4**.

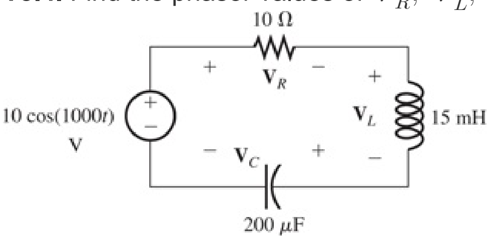


Figure T5.4

T5.5. Use the node-voltage approach to solve for $v_1(t)$ under steady-state conditions in the circuit of **Figure T5.5**.

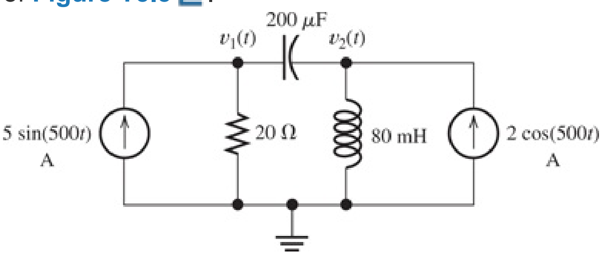


Figure T5.5

T5.6. Determine the complex power, power, reactive power, and apparent power absorbed by the load in **Figure T5.6**. Also, determine the power factor for the load.

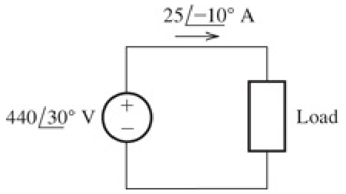


Figure T5.6

T5.7. Determine the line current I_{aA} in polar form for the circuit of **Figure T5.7**. This is a positive-sequence, balanced, three-phase system with $V_{an} = 208\ \angle 30°\text{V}$ and $Z_{\Delta} = 6 + j8\ \Omega$.

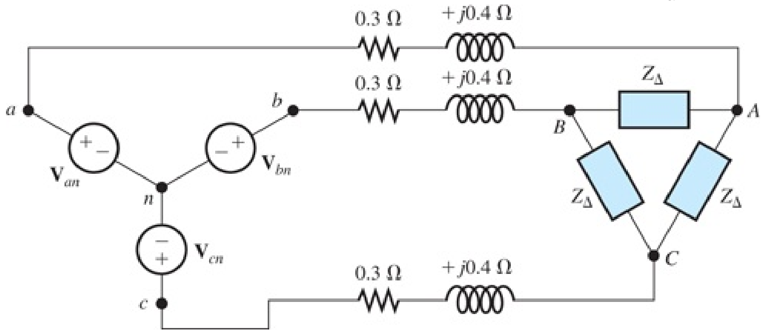


Figure T5.7

T5.8. Write the MATLAB commands to obtain the values of the mesh currents of **Figure T5.8** in polar form. You may use the pin and pout functions defined in this chapter if you wish.

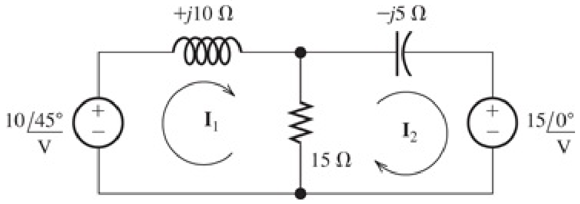


Figure T5.8