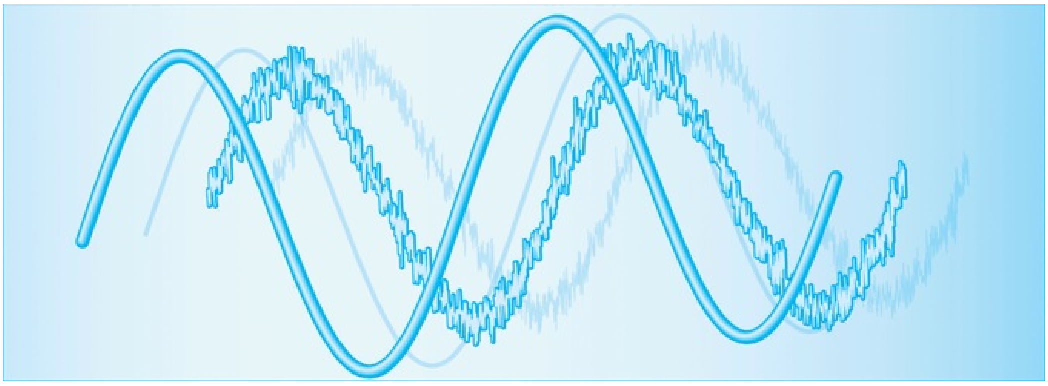
Chapter 6 Frequency Response, Bode Plots, and Resonance



Study of this chapter will enable you to:

- State the fundamental concepts of Fourier analysis.
- Use a filter's transfer function to determine its output for a given input consisting of sinusoidal components.
- Use circuit analysis to determine the transfer functions of simple circuits.
- Draw first-order lowpass or highpass filter circuits and sketch their transfer functions.
- Understand decibels, logarithmic frequency scales, and Bode plots.
- Draw the Bode plots for transfer functions of first-order filters.
- Calculate parameters for series- and parallel-resonant circuits.
- Select and design simple filter circuits.
- Use MATLAB to derive and plot network functions.
- Design simple digital signal-processing systems.

Introduction to this chapter:

*Much of electrical engineering is concerned with information-bearing currents and voltages that we call **signals**. For example, transducers on an internal combustion engine provide electrical signals that represent temperature, speed, throttle position, and the rotational position of the crankshaft. These signals are **processed** (by electrical circuits) to determine the optimum firing instant for each cylinder. Finally, electrical pulses are generated for each spark plug.*

Surveyors can measure distances by using an instrument that emits a pulse of light that is reflected by a mirror at the point of interest. The return light pulse is converted to an electrical signal that is processed by circuits to determine the round-trip time delay between the instrument and the mirror. Finally, the delay is converted to distance and displayed.

Another example of signal processing is the electrocardiogram, which is a plot of the electrical signal generated by the human heart. In a cardiac-care unit, circuits and computers are employed to extract information concerning the behavior of a patient's heart. A physician or nurse is alerted when the patient needs attention.

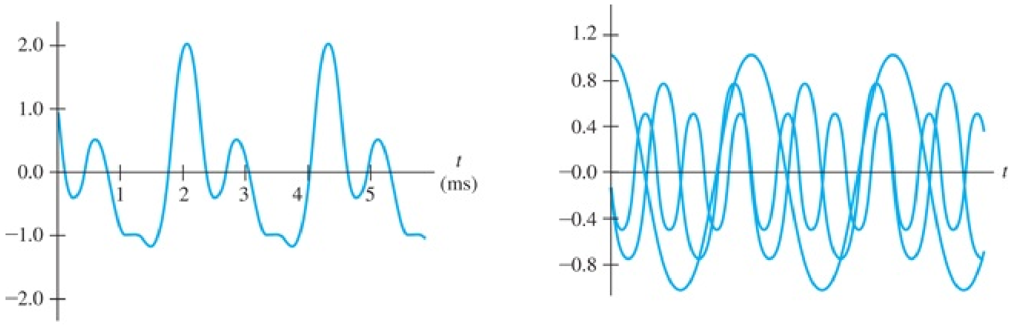
*In general, **signal processing** is concerned with manipulating signals to extract information and using that information to generate other useful electrical signals. It is an important and far-reaching subject. In this chapter, we consider several simple but, nevertheless, useful circuits from a signal-processing point of view.*

*Recall that in **Chapter 5** we learned how to analyze circuits containing sinusoidal sources, all of which have a common frequency. An important application is electrical power systems. However, most real-world information-bearing electrical signals are not sinusoidal. Nevertheless, we will see that phasor concepts can be very useful in understanding how circuits respond to nonsinusoidal signals. This is true because nonsinusoidal signals can be considered to be the sum of sinusoidal components having various frequencies, amplitudes, and phases.*

6.1 Fourier Analysis, Filters, and Transfer Functions

Fourier Analysis

As mentioned in the introduction to this chapter, most information-bearing signals are not sinusoidal. For example, the waveform produced by a microphone for speech or music is a complex nonsinusoidal waveform that is not predictable in advance. **Figure 6.1(a)** shows a (very) short segment of a music signal.



(a) Music waveform

(b) Sinusoidal components

Figure 6.1
The short segment of a music waveform shown in (a) is the sum of the sinusoidal components shown in (b).

Even though many interesting signals are not sinusoidal, it turns out that we can construct any waveform by adding sinusoids that have the proper amplitudes, frequencies, and phases. For illustration, the waveform shown in **Figure 6.1(a)** is the sum of the sinusoids shown in **Figure 6.1(b)**. The waveform shown in **Figure 6.1** is relatively simple because it is composed of only three components. Most natural signals contain thousands of components. (In theory, the number is infinite in many cases.)

When we listen to music, our ears respond differently to the various frequency components. Some combinations of amplitudes and frequencies are pleasing, whereas other combinations are not. Thus, in the design of signal-processing circuits (such as amplifiers) for audio signals, we must consider how the circuits respond to components having different frequencies.

Fourier analysis is a mathematical technique for finding the amplitudes, frequencies, and phases of the components of a given waveform. Aside from mentioning some of the results of Fourier analysis, we will not develop the theory in detail. *The important point is that all real-world signals are sums of sinusoidal components.*

> All real-world signals are sums of sinusoidal components having various frequencies, amplitudes, and phases.

The range of the frequencies of the components depends on the type of signal under consideration. The frequency ranges for several types of signals are given in **Table 6.1**. Thus, electrocardiograms are composed of numerous sinusoidal components with frequencies ranging from 0.05 Hz to 100 Hz.

Table 6.1 Frequency Ranges of Selected Signals

Electrocardiogram	0.05 to 100 Hz
Audible sounds	20 Hz to 15 kHz
AM radio broadcasting	540 to 1600 kHz
HD component video signals	Dc to 25 MHz
FM radio broadcasting	88 to 108 MHz
Cellular phone	824 to 894 MHz and 1850 to 1990 MHz
Satellite television downlinks (C-band)	3.7 to 4.2 GHz
Digital satellite television	12.2 to 12.7 GHz

Fourier Series of a Square Wave

As another example, consider the signal shown in **Figure 6.2(a)**, which is called a **square wave**. Fourier analysis shows that the square wave can be written as an infinite series of sinusoidal components,

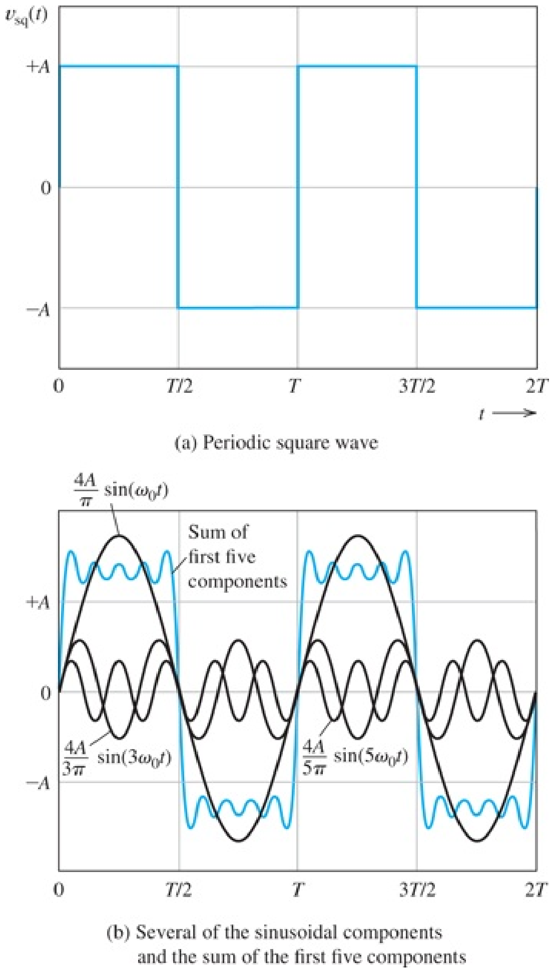


Figure 6.2
A square wave and some of its components.

$$v_{sq}(t) = \frac{4A}{\pi}\sin(\omega_0 t) + \frac{4A}{3\pi}\sin(3\omega_0 t) + \frac{4A}{5\pi}\sin(5\omega_0 t) + \ldots \quad (6.1)$$

in which $\omega_0 = 2\pi / T$ is the called the **fundamental angular frequency** of the square wave.

Figure 6.2(b) shows several of the terms in this series and the result of summing the first five terms. Clearly, even the sum of the first five terms is a fairly good approximation to the square wave, and the approximation becomes better as more components are added. Thus, the square wave is composed of an infinite number of sinusoidal components. The frequencies of the components are odd integer multiples of the fundamental frequency, the amplitudes decline with increasing frequency, and the phases of all components are $-90°$. Unlike the square wave, the components of real-world signals are confined to finite ranges of frequency, and their amplitudes are not given by simple mathematical expressions.

> The components of real-world signals are confined to finite ranges of frequency.

> Zero frequency corresponds to dc.

Sometimes a signal contains a component that has a frequency of zero. For zero frequency, a general sinusoid of the form $A\cos(\omega t + \theta)$ becomes simply $A\cos(\theta)$, which is constant for all time. Recall that we refer to constant voltages as dc, so zero frequency corresponds to dc.

In sum, the fact that all signals are composed of sinusoidal components is a fundamental idea in electrical engineering. The frequencies of the components, as well as their amplitudes and phases, for a given signal can be determined by theoretical analysis or by laboratory measurements (using an instrument called a *spectrum analyzer*). Very often, the design of a system for processing information-bearing signals is based on considerations of how the system should respond to components of various frequencies.

> . . . the fact that all signals are composed of sinusoidal components is a fundamental idea in electrical engineering.

Filters

There are many applications in which we want to retain components in a given range of frequencies and discard the components in another range. This can be accomplished by the use of electrical circuits called **filters**. (Actually, filters can take many forms, but we limit our discussion to a few relatively simple *RLC* circuits.)

Usually, filter circuits are **two-port networks**, an example of which is illustrated in **Figure 6.3**. The signal to be filtered is applied to the input port and (ideally) only the components in the frequency range of interest appear at the output port. For example, an FM radio antenna produces a voltage composed of signals from many transmitters. By using a filter that retains the components in the frequency range from 88 to 108 MHz and discards everything else, we can select the FM radio signals and reject other signals that could interfere with the process of extracting audio information.

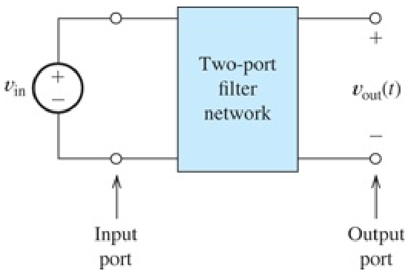


Figure 6.3
When an input signal $v_{in}(t)$ is applied to the input port of a filter, some components are passed to the output port while others are not, depending on their frequencies. Thus, $v_{out}(t)$ contains some of the components of $v_{in}(t)$, but not others. Usually, the amplitudes and phases of the components are altered in passing through the filter.

> Filters process the sinusoid components of an input signal differently depending on the frequency of each component. Often, the goal of the filter is to retain the components in certain frequency ranges and to reject components in other ranges.

As we learned in **Chapter 5**, the impedances of inductances and capacitances change with frequency. For example, the impedance of an inductance is $Z_L = \omega L\angle 90° = 2\pi f L\angle 90°$. Thus, the high-frequency components of a voltage signal applied to an inductance experience a higher impedance magnitude than do the low-frequency components. Consequently, electrical circuits can respond selectively to signal components, depending on their frequencies. Thus, *RLC* circuits provide one way to realize electrical filters. We consider several specific examples later in this chapter.

> *RLC* circuits provide one way to realize filters.

Transfer Functions

Consider the two-port network shown in **Figure 6.3**. Suppose that we apply a sinusoidal input signal having a frequency denoted as *f* and having a phasor V_{in}. In steady state, the output signal is sinusoidal and has the same frequency as the input. The output phasor is denoted as V_{out}.

The **transfer function** *H*(*f*) of the two-port filter is defined to be the ratio of the phasor output voltage to the phasor input voltage as a function of frequency:

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} \tag{6.2}$$

Because phasors are complex, the transfer function is a complex quantity having both magnitude and phase. Furthermore, both the magnitude and the phase can be functions of frequency.

> The transfer function $H(f)$ of the two-port filter is defined to be the ratio of the phasor output voltage to the phasor input voltage as a function of frequency.

The transfer-function magnitude is the ratio of the output amplitude to the input amplitude. The phase of the transfer function is the output phase minus the input phase. Thus, the magnitude of the transfer function shows how the amplitude of each frequency component is affected by the filter. Similarly, the phase of the transfer function shows how the phase of each frequency component is affected by the filter.

> The magnitude of the transfer function shows how the amplitude of each frequency component is affected by the filter. Similarly, the phase of the transfer function shows how the phase of each frequency component is affected by the filter.

Example 6.1 Using the Transfer Function to Determine the Output

The transfer function *H*(*f*) of a filter is shown in **Figure 6.4**. [Notice that the magnitude $|H(f)|$ and phase

$$\angle H(f)$$

are shown separately in the figure.] If the input signal is given by

$$v_{\text{in}}(t) = 2\cos(2000\pi t + 40°)$$

find an expression (as a function of time) for the output of the filter.

Solution

By inspection, the frequency of the input signal is $f = 1000\ \text{Hz}$. Referring to **Figure 6.4**, we see that the magnitude and phase of the transfer function are $|H(1000)| = 3$ and $\angle H(1000) = 30°$, respectively. Thus, we have

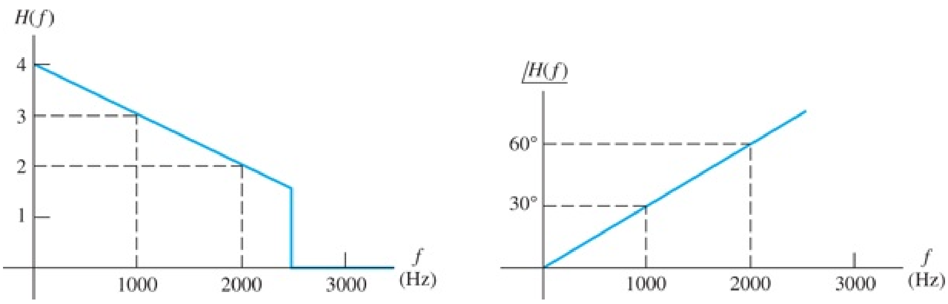


Figure 6.4
The transfer function of a filter. See **Examples 6.1** and **6.2**

$$H(1000) = 3\angle 30° = \frac{V_{\text{out}}}{V_{\text{in}}}$$

The phasor for the input signal is $V_{\text{in}} = 2\angle 40°$, and we get

$$V_{\text{out}} = H(1000) \times V_{\text{in}} = 3\angle 30° \times 2\angle 40° = 6\angle 70°$$

Thus, the output signal is

$$v_{\text{out}}(t) = 6\cos(2000\pi t + 70°)$$

In this case, the amplitude of the input is tripled by the filter. Furthermore, the signal is phase shifted by 30°. Of course, this is evident from the values shown in the plots of the transfer function at $f = 1000$.

Exercise 6.1

Repeat **Example 6.1** if the input signal is given by:

a. $v_{\text{in}}(t) = 2\cos(4000\pi t)$; and
b. $v_{\text{in}}(t) = 1\cos(6000\pi t - 20°)$.

Answer

a. $v_{\text{out}}(t) = 4\cos(4000\pi t + 60°)$;
b. b. $v_{\text{out}}(t) = 0.$

Notice that the effect of the filter on the magnitude and phase of the signal depends on signal frequency.

Example: Graphic Equalizer

You may own a stereo audio system that has a *graphic equalizer*, which is a filter that has an adjustable transfer function. Usually, the controls of the equalizer are arranged so their positions give an approximate representation of the transfer-function magnitude versus frequency. (Actually, the equalizer in a stereo system contains two filters one for the left channel and one for the right channel-and the controls are ganged together.) Users can adjust the transfer function to achieve the mix of amplitudes versus frequency that is most pleasing to them.

Input Signals with Multiple Components

If the input signal to a filter contains several frequency components, we can find the output for each input component separately and then add the output components. This is an application of the superposition principle first introduced in **Section 2.7**.

A step-by-step procedure for determining the output of a filter for an input with multiple components is as follows:

1. Determine the frequency and phasor representation for each input component.
2. Determine the (complex) value of the transfer function for each component.
3. Obtain the phasor for each output component by multiplying the phasor for each input component by the corresponding transfer-function value.
4. Convert the phasors for the output components into time functions of various frequencies. Add these time functions to produce the output.

Example 6.2 Using the Transfer Function with Several Input Components

Suppose that the input signal for the filter of **Figure 6.4** is given by

$$v_{\text{in}}(t) = 3 + 2\cos(2000\pi t) + \cos(4000\pi t - 70°)$$

Find an expression for the output signal.

Step 1.

Solution
We start by breaking the input signal into its components. The first component is

$$v_{\text{in}1}(t) = 3$$

the second component is

$$v_{\text{in}2}(t) = 2\cos(2000\pi t)$$

and the third component is

$$v_{\text{in}3}(t) = \cos(4000\pi t - 70°)$$

Step 2.

By inspection, we see that the frequencies of the components are 0, 1000, and 2000 Hz, respectively. Referring to the transfer function shown in **Figure 6.4**, we find that

$$H(0) = 4$$

$$H(1000) = 3\angle 30°$$

and

$$H(2000) = 2\angle 60°$$

The constant (dc) output term is simply *H*(*0*) times the dc input:

$$v_{\text{out1}} = H(0)\ v_{\text{in1}} = 4 \times 3 = 12$$

Step 3.

The phasor outputs for the two input sinusoids are

$$\begin{aligned} \mathbf{V}_{\text{out2}} &= H(1000) \times \mathbf{V}_{\text{in2}} = 3\angle 30° \times 2\angle 0° = 6\angle 30° \\ \mathbf{V}_{\text{out3}} &= H(2000) \times \mathbf{V}_{\text{in3}} = 2\angle 60° \times 1\angle -70° = 2\angle -10° \end{aligned}$$

Next, we can write the output components as functions of time:

Step 4.

$$\begin{aligned} v_{\text{out1}}(t) &= 12 \\ v_{\text{out2}}(t) &= 6\cos(2000\pi t + 30°) \end{aligned}$$

and

$$v_{\text{out3}}(t) = 2\cos(4000\pi t - 10°)$$

Finally, we add the output components to find the output voltage:

$$v_{\text{out}}(t) = v_{\text{out1}}(t) + v_{\text{out2}}(t) + v_{\text{out3}}(t)$$

and

$$v_{\text{out}}(t) = 12 + 6\cos(2000\pi t + 30°) + 2\cos(4000\pi t - 10°)$$

Notice that we did not add the phasors $\mathbf{V}_{\text{out2}}$ and $\mathbf{V}_{\text{out3}}$ in **Example 6.2**. The phasor concept was developed for sinusoids, all of which have the same frequency. *Hence, convert the phasors back into time-dependent signals before adding the components.*

We must convert the phasors back into time-dependent signals before adding the components.

Real-world information-bearing signals contain thousands of components. In principle, the output of a given filter for any input signal could be found by using the procedure of **Example 6.2**. However, it would usually be much too tedious to carry out. Fortunately, we will not need to do this. *It is the principle that is most important.* In summary, we can say that linear circuits (or any other systems for which the relationship between input and output can be described by linear time-invariant differential equations) behave as if they

1. Separate the input signal into components having various frequencies.
2. Alter the amplitude and phase of each component depending on its frequency.
3. Add the altered components to produce the output signal.

This process is illustrated in **Figure 6.5**.

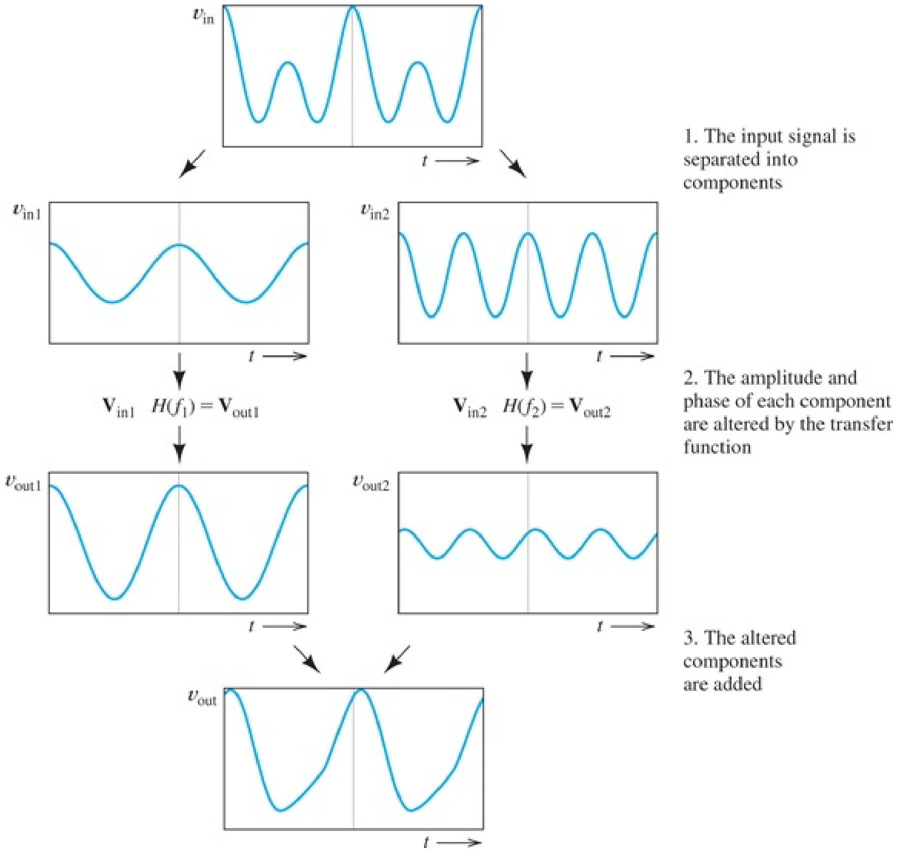


Figure 6.5
Filters behave as if they separate the input into components, modify the amplitudes and phases of the components, and add the altered components to produce the output.

The transfer function of a filter is important because it shows how the components are altered in amplitude and phase.

Experimental Determination of the Transfer Function

To determine the transfer function of a filter experimentally, we connect a sinusoidal source to the input port, measure the amplitudes and phases of both the input signal and the resulting output signal, and divide the output phasor by the input phasor. This is repeated for each frequency of interest. The experimental setup is illustrated in **Figure 6.6**. Various instruments, such as voltmeters and oscilloscopes, can be employed to measure the amplitudes and phases.

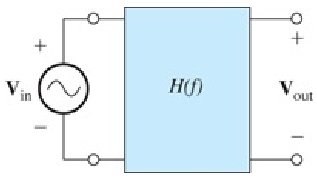


Figure 6.6
To measure the transfer function we apply a sinusoidal input signal, measure the amplitudes and phases of input and output in steady state, and then divide the phasor output by the phasor input. The procedure is repeated for each frequency of interest.

In the next few sections of this chapter, we use mathematical analysis to investigate the transfer functions of several relatively simple electrical circuits.

Exercise 6.2

Consider the transfer function shown in **Figure 6.4**. The input signal is given by

$$v_{\text{in}}(t) = 2\cos(1000\pi t + 20°) + 3\cos(3000\pi t)$$

Find an expression for the output signal.

Answer

$v_{\text{out}}(t) = 7\cos(1000\pi t + 35°) + 7.5\cos(3000\pi t + 45°)$.

Exercise 6.3

Consider the transfer function shown in **Figure 6.4**. The input signal is given by

$$v_{\text{in}}(t) = 1 + 2\cos(2000\pi t) + 3\cos(6000\pi t)$$

Find an expression for the output signal.

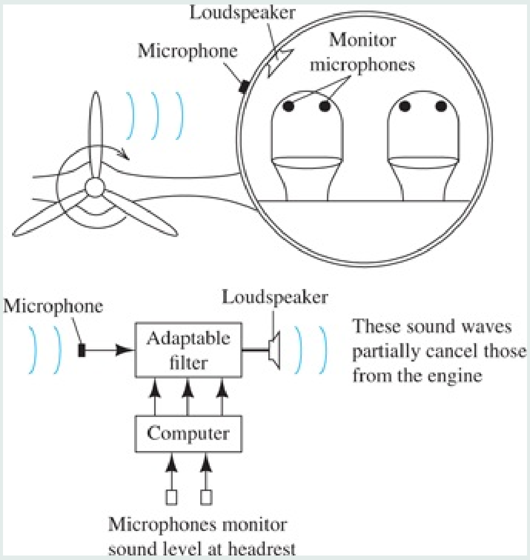
Answer

$v_{\text{out}}(t) = 4 + 6\cos(2000\pi t + 30°)$. Notice that the 3 kHz component is totally eliminated (rejected) by the filter.

PRACTICAL APPLICATION

6.1 Active Noise Cancellation

Noise and vibration are annoying to passengers in helicopters and other aircraft. Traditional sound-absorbing materials can be very effective in reducing noise levels, but are too bulky and massive for application in aircraft. An alternative approach is an electronic system that cancels noise. The diagram of such a system is shown in **Figure PA6.1**. A microphone near the sources of the noise, such as the engines, samples the noise before it enters the passenger area. The resulting electrical signal passes through a filter whose transfer function is continuously adjusted by a special-purpose computer to match the transfer function of the sound path. Finally, an inverted version of the signal is applied to loudspeakers. The sound waves from the speaker are out of phase with those from the noise source, resulting in partial cancellation. Another set of microphones on the headrest monitors the sound experienced by the passenger so that the computer can determine the filter adjustments needed to best cancel the sound.



FIGURE

PA6.1

Recently, noise-canceling systems based on these principles have appeared that contain all of the system elements in a lightweight headset. Many passengers on commercial aircraft wear these headsets to provide themselves with a quieter, more restful trip.

Active noise cancellation systems can effectively replace sound absorbing materials weighing a great deal more. As a result, active noise cancellation is very attractive for use in aircraft and automobiles. You can find many research reports and popular articles on this topic with an Internet search.

6.2 First-Order Lowpass Filters

Consider the circuit shown in **Figure 6.7**. We will see that this circuit tends to pass low-frequency components and reject high-frequency components. (In other words, for low frequencies, the output amplitude is nearly the same as the input. For high frequencies, the output amplitude is much less than the input.) In **Chapter 4**, we saw that a first-order differential equation describes this circuit. Because of these facts, the circuit is called a **first-order lowpass filter.**

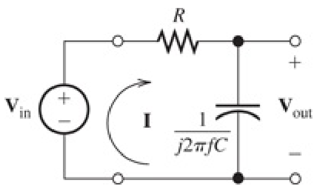


Figure 6.7
A first-order lowpass filter.

To determine the transfer function, we apply a sinusoidal input signal having a phasor V_{in}, and then we analyze the behavior of the circuit as a function of the source frequency *f*.

> We can determine the transfer functions of *RLC* circuits by using steady-state analysis with complex impedances as a function of frequency.

The phasor current is the input voltage divided by the complex impedance of the circuit. This is given by

$$I = \frac{V_{\text{in}}}{R + 1/j2\pi fC} \tag{6.3}$$

The phasor for the output voltage is the product of the phasor current and the impedance of the capacitance, illustrated by

$$V_{\text{out}} = \frac{1}{j2\pi fC} I \tag{6.4}$$

Using **Equation 6.3** to substitute for **I**, we have

$$V_{\text{out}} = \frac{1}{j2\pi fC} \times \frac{V_{\text{in}}}{R + 1/j2\pi fC} \tag{6.5}$$

Recall that the transfer function *H*(*f*) is defined to be the ratio of the output phasor to the input phasor:

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} \tag{6.6}$$

Rearranging **Equation 6.5**, we have

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{1}{1 + j2\pi fRC} \tag{6.7}$$

Next, we define the parameter:

$$f_B = \frac{1}{2\pi RC} \tag{6.8}$$

Then, the transfer function can be written as

$$H(f) = \frac{1}{1 + j(f/f_B)} \tag{6.9}$$

Magnitude and Phase Plots of the Transfer Function

As expected, the transfer function $H(f)$ is a complex quantity having a magnitude and phase angle. Referring to the expression on the right-hand side of **Equation 6.9**, the magnitude of $H(f)$ is the magnitude of the numerator (which is unity) over the magnitude of the denominator. Recall that the magnitude of a complex quantity is the square root of the sum of the real part squared and the imaginary part squared.

Thus, the magnitude is given by

$$\boxed{|H(f)| = \frac{1}{\sqrt{1+(f/f_B)^2}}} \tag{6.10}$$

Referring to the expression on the right-hand side of **Equation 6.9**, the phase angle of the transfer function is the phase of the numerator (which is zero) minus the phase of the denominator. This is given by

$$\boxed{\angle H(f) = -\arctan\left(\frac{f}{f_B}\right)} \tag{6.11}$$

Plots of the magnitude and phase of the transfer function are shown in **Figure 6.8**. For low frequencies (f approaching zero), the magnitude is approximately unity and the phase is nearly zero, which means that the amplitudes and phases of low-frequency components are affected very little by this filter. The low--frequency components are passed to the output almost unchanged in amplitude or phase.

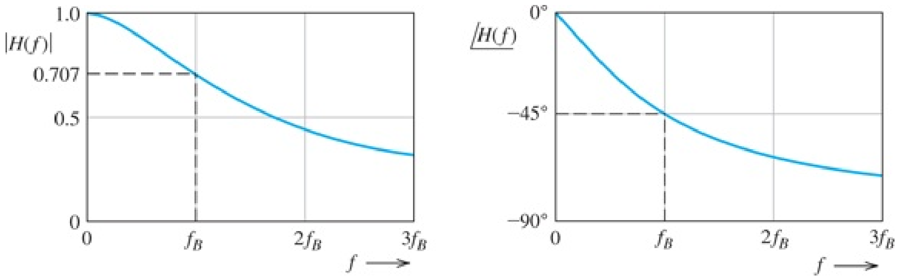


Figure 6.8
Magnitude and phase of the first-order lowpass transfer function versus frequency.

On the other hand, for high frequencies $(f >> f_B)$, the magnitude of the transfer function approaches zero. Thus, the amplitude of the output is much smaller than the amplitude of the input for the high-frequency components. We say that the high-frequency components are rejected by the filter. Furthermore, at high frequencies, the phase of the transfer function approaches $-90°$. Thus, as well as being reduced in amplitude, the high-frequency components are phase shifted.

Notice that for $f = f_B$, the magnitude of the output is $1/\sqrt{2} \cong 0.707$ times the magnitude of the input signal. When the amplitude of a voltage is multiplied by a factor of $1/\sqrt{2}$, the power that the voltage can deliver to a given resistance is multiplied by a factor of one-half (because power is proportional to voltage squared). Thus, f_B is called the **half-power frequency**.

> At the half-power frequency, the transfer-function magnitude is $1/\sqrt{2} \cong 0.707$ times its maximum value.

Applying the Transfer Function

As we saw in **Section 6.1**, if an input signal to a filter consists of several components of different frequencies, we can use the transfer function to compute the output for each component separately. Then, we can find the complete output by adding the separate components.

Example 6.3 Calculation of RC Lowpass Output

Suppose that an input signal given by

$$v_{\text{in}}(t) = 5\cos(20\pi t) + 5\cos(200\pi t) + 5\cos(2000\pi t)$$

is applied to the lowpass *RC* filter shown in **Figure 6.9**. Find an expression for the output signal.

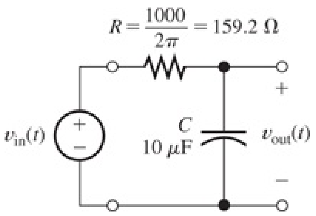


Figure 6.9
Circuit of **Example 6.3**. The resistance has been picked so the break frequency turns out to be a convenient value.

Solution

The filter has the form of the lowpass filter analyzed in this section. The half-power frequency is given by

$$f_B = \frac{1}{2\pi RC} = \frac{1}{2\pi \times (1000/2\pi) \times 10 \times 10^{-6}} = 100\ \text{Hz}$$

The first component of the input signal is

$$v_{\text{in}1}(t) = 5\cos(20\pi t)$$

For this component, the phasor is $\mathbf{V}_{\text{in}1} = 5\angle 0°$, and the angular frequency is $\omega = 20\pi$. Therefore, $f = \omega/2\pi = 10$. The transfer function of the circuit is given by **Equation 6.9**, which is repeated here for convenience:

$$H(f) = \frac{1}{1 + j(f/f_B)}$$

Evaluating the transfer function for the frequency of the first component $(f = 10)$, we have

$$H(10) = \frac{1}{1 + j(10/100)} = 0.9950\angle -5.71°$$

The output phasor for the $f = 10$ component is simply the input phasor times the transfer function. Thus, we obtain

$$\mathbf{V}_{\text{out}1} = H(10) \times \mathbf{V}_{\text{in}1} = (0.9950\angle -5.71°) \times (5\angle 0°) = 4.975\angle -5.71°$$

Hence, the output for the first component of the input signal is

$$v_{\text{out}1}(t) = 4.975\cos(20\pi t - 5.71°)$$

Similarly, the second component of the input signal is

$$v_{\text{in2}}(t) = 5\cos(200\pi t)$$

and we have

$$\boldsymbol{V}_{\text{in2}} = 5\angle 0°$$

The frequency of the second component is $f = 100$:

$$H(100) = \frac{1}{1 + j(100/100)} = 0.7071\angle -45°$$

$$\boldsymbol{V}_{\text{out2}} = H(100) \times \boldsymbol{V}_{\text{in2}} = (0.7071\angle -45°) \times (5\angle 0°) = 3.535\angle -45°$$

Therefore, the output for the second component of the input signal is

$$v_{\text{out2}}(t) = 3.535\cos(200\pi t - 45°)$$

Finally, for the third and last component, we have

$$v_{\text{in3}}(t) = 5\cos(2000\pi t)$$

$$\boldsymbol{V}_{\text{in3}} = 5\angle 0°$$

$$H(1000) = \frac{1}{1 + j(1000/100)} = 0.0995\angle -84.29°$$

$$\boldsymbol{V}_{\text{out3}} = H(1000) \times \boldsymbol{V}_{\text{in3}} = (0.0995\angle -84.29°) \times (5\angle 0°) = 0.4975\angle -84.29°$$

Consequently, the output for the third component of the input signal is

$$v_{\text{out3}}(t) = 0.4975\cos(2000\pi t - 84.29°)$$

Now, we can write an expression for the output signal by adding the output components:

$$v_{\text{out}}(t) = 4.975\cos(20\pi t - 5.71°) + 3.535\cos(200\pi t - 45°) + 0.4975\cos(2000\pi t - 84.29°)$$

Notice that each component of the input signal $v_{\text{in}}(t)$ is treated differently by this filter. The $f = 10$ component is nearly unaffected in amplitude and phase. The $f = 100$ component is reduced in amplitude by a factor of 0.7071 and phase shifted by $-45°$. The amplitude of the $f = 1000$ component is reduced by approximately an order of magnitude. Thus, the filter discriminates against the high-frequency components.

Application of the First-Order Lowpass Filter

A simple application of the first-order lowpass filter is the tone control on an old-fashion AM radio. The tone control adjusts the resistance and, therefore, the break frequency of the filter. Suppose that we are listening to an interesting news item from a distant radio station with an AM radio and lightning storms are causing electrical noise. It turns out that the components of voice signals are concentrated in the low end of the audible-frequency range. On the other hand, the noise caused by lightning has roughly equal-amplitude components at all frequencies. In this situation, we could adjust the tone control to lower the break frequency. Then, the high-frequency noise components would be rejected, while most of the voice components would be passed. In this way, we can improve the ratio of desired signal power to noise power produced by the loudspeaker and make the news more intelligible.

Using Phasors with Components of Different Frequencies

Recall that phasors can be combined only for sinusoids with the same frequency. It is important to understand that *we should not add the phasors for components with different frequencies*. Thus, in the preceding example, we used phasors to find the output components as functions of time, which we then added.

> We should not add the phasors for components with different frequencies.

Exercise 6.4

Derive an expression for the transfer function $H(f) = \mathbf{V}_{\text{out}}/\mathbf{V}_{\text{in}}$ of the filter shown in **Figure 6.10**. Show that *H*(*f*) takes the same form as **Equation 6.9** if we define $f_B = R/2\pi L$.

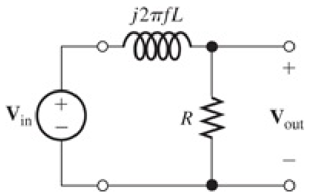


Figure 6.10
Another first-order lowpass filter; see **Exercise 6.4**.

Exercise 6.5

Suppose that the input signal for the circuit shown in **Figure 6.11** is given by

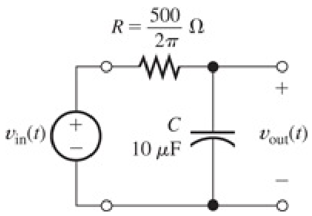


Figure 6.11
Circuit for **Exercise 6.5**.

$$v_{\text{in}}(t) = 10\cos(40\pi t) + 5\cos(1000\pi t) + 5\cos\left(2\pi 10^4 t\right)$$

Find an expression for the output signal $v_{\text{out}}(t)$.

Answer

$$\begin{aligned} v_{\text{out}}(t) &= 9.95\cos(40\pi t - 5.71°) + 1.86\cos(1000\pi t - 68.2°) \\ &\quad + 0.100\cos\left(2\pi 10^4 t - 88.9°\right) \end{aligned}$$

6.3 Decibels, the Cascade Connection, and Logarithmic Frequency Scales

In comparing the performance of various filters, it is helpful to express the magnitudes of the transfer functions in **decibels**. To convert a transfer-function magnitude to decibels, we multiply the common logarithm (base 10) of the transfer-function magnitude by 20:

$$\boxed{|H(f)|_{\mathrm{dB}} = 20\log|H(f)|} \tag{6.12}$$

(A transfer function is a ratio of voltages and is converted to decibels as 20 times the logarithm of the ratio. On the other hand, ratios of powers are converted to decibels by taking 10 times the logarithm of the ratio.)

Table 6.2 shows the decibel equivalents for selected values of transfer-function magnitude. Notice that the decibel equivalents are positive for magnitudes greater than unity, whereas the decibel equivalents are negative for magnitudes less than unity.

Table 6.2 Transfer-Function Magnitudes and Their Decibel Equivalents

$\|H(f)\|$	$\|H(f)\|_{\mathrm{dB}}$
100	40
10	20
2	6
$\sqrt{2}$	3
1	0
$1/\sqrt{2}$	-3
1/2	-6
0.1	-20
0.01	-40

In many applications, the ability of a filter to strongly reject signals in a given frequency band is of primary importance. For example, a common problem associated with audio signals is that a small amount of the ac power line voltage can inadvertently be added to the signal. When applied to a loudspeaker, this 60-Hz component produces a disagreeable hum. (Actually, this problem is rapidly becoming a thing of the past as digital technologies replace analog.)

Usually, we approach this problem by trying to eliminate the electrical path by which the power line voltage is added to the desired audio signal. However, this is sometimes not possible. Then, we could try to design a filter that rejects the 60-Hz component and passes components at other frequencies. The magnitude of a filter transfer function to accomplish this is shown in **Figure 6.12(a)**. A filter such as this, designed to eliminate components in a narrow range of frequencies, is called a **notch filter**.

Figure 6.12

Transfer-function magnitude of a notch filter used to reduce hum in audio signals.

It turns out that to reduce a loud hum (as loud as a heated conversation) to be barely audible, the transfer function must be $-80\ \mathrm{dB}$ or less for the 60-Hz component, which corresponds to $|H(f)| = 10^{-4}$ or smaller. On the other hand, the transfer-function magnitude should be close to unity for the components to be passed by the filter. We refer to the range of frequencies to be passed as the **passband**.

When we plot $|H(f)|$ without converting to decibels, it is difficult to show both values clearly on the same plot. If we choose a scale that shows the passband magnitude, we cannot see whether the magnitude is sufficiently small at 60 Hz. This is the case for the plot shown in **Figure 6.12(a)**. On the other hand, if we choose a linear scale that clearly shows the magnitude at 60 Hz, the magnitude would be way off scale at other frequencies of interest.

However, when the magnitude is converted to decibels, both parts of the magnitude are readily seen. For example, **Figure 6.12(b)** shows the decibel equivalent for the magnitude plot shown in **Figure 6.12(a)**. On this plot, we can see that the passband magnitude is approximately unity (0 dB) and that at 60 Hz, the magnitude is sufficiently small (less than $-80\ \mathrm{dB}$).

Thus, one of the advantages of converting transfer-function magnitudes to decibels before plotting is that very small and very large magnitudes can be displayed clearly on a single plot. We will see that another advantage is that decibel plots for many filter circuits can be approximated by straight lines (provided that a logarithmic scale is used for frequency). Furthermore, to understand some of the jargon used by electrical engineers, we must be familiar with decibels.

> One of the advantages of converting transfer-function magnitudes to decibels before plotting is that very small and very large magnitudes can be displayed clearly on a single plot.

Cascaded Two-Port Networks

When we connect the output terminals of one two-port circuit to the input terminals of another two-port circuit, we say that we have a **cascade** connection. This is illustrated in **Figure 6.13**. Notice that the output voltage of the first two-port network is the input voltage of the second two-port. The overall transfer function is

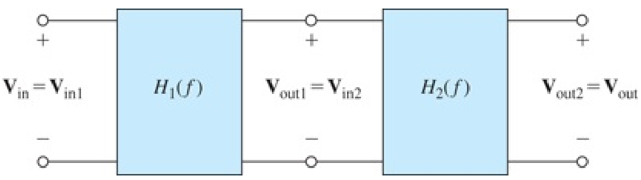


Figure 6.13

Cascade connection of two two-port circuits.

In the cascade connection, the output of one filter is connected to the input of a second filter.

$$H(f) = \frac{V_{\text{out}}}{V_{\text{in}}}$$

However, the output voltage of the cascade is the output of the second two-port (i.e., $V_{\text{out}} = V_{\text{out2}}$). Furthermore, the input to the cascade is the input to the first two-port (i.e., $V_{\text{in}} = V_{\text{in1}}$). Thus,

$$H(f) = \frac{V_{\text{out2}}}{V_{\text{in1}}}$$

Multiplying and dividing by V_{out1}, we have

$$H(f) = \frac{V_{\text{out1}}}{V_{\text{in1}}} \times \frac{V_{\text{out2}}}{V_{\text{out1}}}$$

Now, the output voltage of the first two-port is the input to the second two-port (i.e., $V_{\text{out1}} = V_{\text{in2}}$). Hence,

$$H(f) = \frac{V_{\text{out1}}}{V_{\text{in1}}} \times \frac{V_{\text{out2}}}{V_{\text{in2}}}$$

Finally, we can write

$$\boxed{H(f) = H_1(f) \times H_2(f)} \tag{6.13}$$

Thus, the transfer function of the cascade connection is the product of the transfer functions of the individual two-port networks. This fact can be extended to three or more two-ports connected in cascade.

A potential source of difficulty in applying **Equation 6.13** is that the transfer function of a two-port usually depends on what is attached to its output terminals. *Thus, in applying* ***Equation 6.13****, we must find* $H_1(f)$ *with the second two-port attached.*

In applying **Equation 6.13**, we must find $H_1(f)$ with the second two-port attached.

Taking the magnitudes of the terms on both sides of **Equation 6.13** and expressing in decibels, we have

$$20 \log|H(f)| = 20 \log[|H_1(f)| \times |H_2(f)|] \tag{6.14}$$

Using the fact that the logarithm of a product is equal to the sum of the logarithms of the terms in the product, we have

$$20 \log|H(f)| = 20 \log|H_1(f)| + 20 \log|H_2(f)| \tag{6.15}$$

which can be written as

$$\boxed{|H(f)|_{\text{dB}} = |H_1(f)|_{\text{dB}} + |H_2(f)|_{\text{dB}}} \tag{6.16}$$

Thus, in decibels, the individual transfer-function magnitudes are added to find the overall transfer-function magnitude for a cascade connection.

In decibels, the individual transfer-function magnitudes are added to find the overall transfer-function magnitude for a cascade connection.

Logarithmic Frequency Scales

We often use a **logarithmic scale** for frequency when plotting transfer functions. On a logarithmic scale, the variable is *multiplied* by a given factor for equal increments of length along the axis. (On a linear scale, equal lengths on the scale correspond to *adding* a given amount to the variable.) For example, a logarithmic frequency scale is shown in **Figure 6.14**.

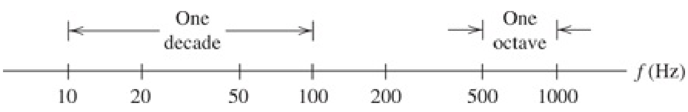


Figure 6.14
Logarithmic frequency scale.

On a logarithmic scale, the variable is multiplied by a given factor for equal increments of length along the axis.

A **decade** is a range of frequencies for which the ratio of the highest frequency to the lowest is 10. The frequency range from 2 to 20 Hz is one decade. Similarly, the range from 50 to 5000 Hz is two decades. (50 to 500 Hz is one decade, and 500 to 5000 Hz is another decade.)

An **octave** is a two-to-one change in frequency. For example, the range 10 to 20 Hz is one octave. The range 2 to 16 kHz is three octaves.

Suppose that we have two frequencies f_1 and f_2 for which $f_2 > f_1$. The number of decades between f_1 and f_2 is given by

$$\text{number of decades} = \log\left(\frac{f_2}{f_1}\right) \tag{6.17}$$

in which we assume that the logarithm is base 10. The number of octaves between the two frequencies is

$$\text{number of octaves} = \log_2\left(\frac{f_2}{f_1}\right) = \frac{\log(f_2/f_1)}{\log(2)} \tag{6.18}$$

The advantage of a logarithmic frequency scale compared with a linear scale is that the variations in the magnitude or phase of a transfer function for a low range of frequency such as 10 to 20 Hz, as well as the variations in a high range such as 10 to 20 MHz, can be clearly shown on a single plot. With a linear scale, either the low range would be severely compressed or the high range would be off scale.

Example 6.4 Decibels and Logarithmic Frequency Scales

The transfer function magnitude of a certain filter is given by

$$\left| H(f) \right| = \frac{10}{\sqrt{1 + (f/5000)^6}}$$

a. What is the value of the transfer function magnitude in decibels for very low frequencies?
b. At what frequency $f_{3\mathrm{dB}}$ is the transfer function magnitude 3 dB less than the value at very low frequencies?
c. At what frequency $f_{60\mathrm{dB}}$ is the transfer function magnitude 60 dB less than the value at very low frequencies?
d. How many decades are between $f_{3\mathrm{dB}}$ and $f_{60\mathrm{dB}}$? How many octaves?

Solution

a. Very low frequencies are those approaching zero. For $f = 0,$ we have $|H(0)| = 10.$ Then, we have $|H(0)|_{\mathrm{dB}} = 20 \log(10) = 20\ \mathrm{dB}.$
b. Because -3 dB corresponds to $1/\sqrt{2},$ we have
$$|H(f_{3\mathrm{dB}})| = \frac{10}{\sqrt{2}} = \frac{10}{\sqrt{1 + (f_{3\mathrm{dB}}/5000)^6}}$$
from which we find that $f_{3\mathrm{dB}} = 5000\ \mathrm{Hz}.$
c. Also, because $-60\ \mathrm{dB}$ corresponds to **1/1000**, we have
$$|H(f_{60\mathrm{dB}})| = \frac{10}{1000} = \frac{10}{\sqrt{1 + (f_{60\mathrm{dB}}/5000)^6}}$$
from which we find that $f_{60\mathrm{dB}} = 50\ \mathrm{kHz}.$
d. Clearly, $f_{60\mathrm{dB}} = 50\ \mathrm{kHz}$ is one decade higher than $f_{3\mathrm{dB}} = 5\ \mathrm{kHz}.$ Using **Equation 6.18**, we find that the number of octaves between the two frequencies is
$$\frac{\log(50/5)}{\log(2)} = \frac{1}{\log(2)} = 3.32$$

Exercise 6.6

Suppose that $|H(f)| = 50.$ Find the decibel equivalent.

Answer

$|H(f)|_{\mathrm{dB}} = 34\ \mathrm{dB}.$

Exercise 6.7

a. Suppose that $|H(f)|_{\mathrm{dB}} = 15\ \mathrm{dB}.$ Find $|H(f)|.$
b. Repeat for $|H(f)|_{\mathrm{dB}} = 30\ \mathrm{dB}.$

Answer

a. $|H(f)| = 5.62;$
b. $|H(f)| = 31.6.$

Exercise 6.8

a. What frequency is two octaves higher than 1000 Hz?
b. Three octaves lower?
c. Two decades higher?
d. One decade lower?

Answer

a. 4000 Hz is two octaves higher than 1000 Hz;
b. 125 Hz is three octaves lower than 1000 Hz;
c. 100 kHz is two decades higher than 1000 Hz;
d. 100 Hz is one decade lower than 1000 Hz.

Exercise 6.9

a. What frequency is halfway between 100 and 1000 Hz on a logarithmic frequency scale?
b. On a linear frequency scale?

Answer

a. 316.2 Hz is halfway between 100 and 1000 Hz on a logarithmic scale;
b. 550 Hz is halfway between 100 and 1000 Hz on a linear frequency scale.

Exercise 6.10

a. How many decades are between $f_1 = 20\ \mathrm{Hz}$ and $f_2 = 15\ \mathrm{kHz}?$ (This is the approximate range of audible frequencies.)
b. How many octaves?

Answer

a. Number of $\mathrm{decades} = \log\left(\frac{15\ \mathrm{kHz}}{20\ \mathrm{Hz}}\right) = 2.87$
b. Number of $\mathrm{octaves} = \frac{\log(\ 15000/20)}{\log(\ 2)} = 9.55$

6.4 Bode Plots

A **Bode plot** is a plot of the decibel magnitude of a network function versus frequency using a logarithmic scale for frequency. Because it can clearly illustrate very large and very small magnitudes for a wide range of frequencies on one plot, the Bode plot is particularly useful for displaying transfer functions. Furthermore, it turns out that Bode plots of network functions can often be closely approximated by straight-line segments, so they are relatively easy to draw. (Actually, we now use computers to plot functions, so this advantage is not as important as it once was.) Terminology related to these plots is frequently encountered in signal-processing literature. Finally, an understanding of Bode plots enables us to make estimates quickly when dealing with transfer functions.

> A Bode plot is a plot of the decibel magnitude of a network function versus frequency using a logarithmic scale for frequency.

To illustrate Bode plot concepts, we consider the first-order lowpass transfer function of **Equation 6.9**, repeated here for convenience:

$$H(f) = \frac{1}{1 + j(f/f_B)}$$

The magnitude of this transfer function is given by **Equation 6.10**, which is

$$\left|H(f)\right| = \frac{1}{\sqrt{1 + (f/f_B)^2}}$$

To convert the magnitude to decibels, we take 20 times the logarithm of the magnitude:

$$|H(f)|_{\text{dB}} = 20 \log |H(f)|$$

Substituting the expression for the transfer-function magnitude, we get

$$\left|H(f)\right|_{\text{dB}} = 20 \log \frac{1}{\sqrt{1 + (f/f_B)^2}}$$

Using the properties of the logarithm, we obtain

$$\left|H(f)\right|_{\text{dB}} = 20 \log(1) - 20 \log \sqrt{1 + \left(\frac{f}{f_B}\right)^2}$$

Of course, the logarithm of unity is zero. Therefore,

$$\left|H(f)\right|_{\text{dB}} = -20 \log \sqrt{1 + \left(\frac{f}{f_B}\right)^2}$$

Finally, since $\log\left(\sqrt{x}\right) = \frac{1}{2}\log(x)$, we have

$$\left| H(f) \right|_{\mathrm{dB}} = -10 \log \left[1 + \left(\frac{f}{f_B} \right)^2 \right] \tag{6.19}$$

Notice that the value given by **Equation 6.19** is approximately 0 dB for $f \ll f_B$. Thus, for low frequencies, the transfer-function magnitude is approximated by the horizontal straight line shown in **Figure 6.15**, labeled as the **low-frequency asymptote**.

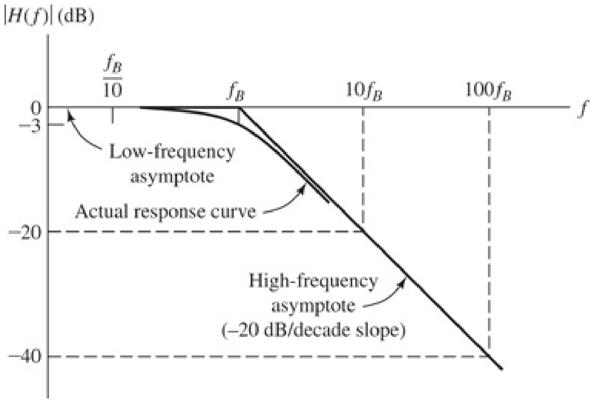


Figure 6.15
Magnitude Bode plot for the first-order lowpass filter.

> The low-frequency asymptote is constant at 0 dB.

On the other hand, for $f >> f_B$, **Equation 6.19** is approximately

$$\left| H(f) \right|_{\mathrm{dB}} \cong -20 \log \left(\frac{f}{f_B} \right) \tag{6.20}$$

Evaluating for various values of *f*, we obtain the results shown in **Table 6.3**. Plotting these values results in the straight line shown sloping downward on the right-hand side of **Figure 6.15**, labeled as the **high-frequency asymptote.** Notice that the two straight-line asymptotes intersect at the half-power frequency f_B. For this reason, f_B is also known as the **corner frequency** or as the **break frequency**.

Table 6.3 Values of the Approximate Expression (Equation 6.20) for Selected Frequencies

f	$\vert H(f) \vert_{\mathrm{dB}}$
f_B	0
$2f_B$	-6
$10f_B$	-20
$100f_B$	-40
$1000f_B$	-60

> The high-frequency asymptote slopes downward at 20 dB/decade, starting from 0 dB at f_B.

Notice that the two straight-line asymptotes intersect at the half-power frequency f_B.

Also, notice that the slope of the high-frequency asymptote is $-20\ \mathrm{dB}$ per decade of frequency. (This slope can also be stated as $-6\ \mathrm{dB}$ per octave.)

If we evaluate **Equation 6.19** at $f = f_B$, we find that

$$|H(f_B)|_{\mathrm{dB}} = -3\ \mathrm{dB}$$

Thus, the asymptotes are in error by only 3 dB at the corner frequency. The actual curve for $|H(f)|_{\mathrm{dB}}$ is also shown in **Figure 6.15**.

The asymptotes are in error by only 3 dB at the corner frequency f_B.

Phase Plot

The phase of the first-order lowpass transfer function is given by **Equation 6.11**, which is repeated here for convenience:

$$\angle H(f) = -\arctan\left(\frac{f}{f_B}\right)$$

Evaluating, we find that the phase approaches zero at very low frequencies, equals $-45°$ at the break frequency, and approaches $-90°$ at high frequencies.

Figure 6.16 shows a plot of phase versus frequency. Notice that the curve can be approximated by the following straight-line segments:

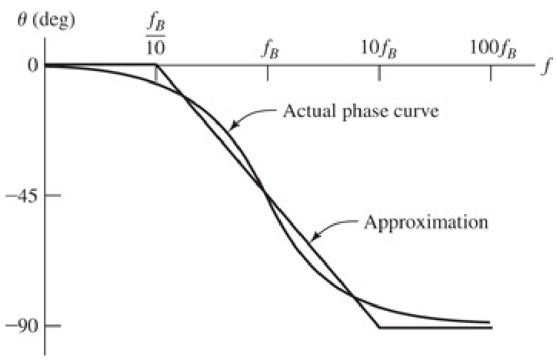


Figure 6.16
Phase Bode plot for the first-order lowpass filter.

1. A horizontal line at zero for $f < f_B/10$.
2. A sloping line from zero phase at $f_B/10$ to $-90°$ at $10f_B$.
3. A horizontal line at $-90°$ for $f > 10f_B$.

The actual phase curve departs from these straight-line approximations by less than 6°. Hence, working by hand, we could easily construct an approximate plot of phase.

Many circuit functions can be plotted by the methods we have demonstrated for the simple lowpass *RC* circuit; however, we will not try to develop your skill at this to a high degree. Bode plots of amplitude and phase for *RLC* circuits are easily produced by computer programs. We have shown the manual approach to analyzing and drawing the Bode plot for the *RC* lowpass filter mainly to present the concepts and terminology.

Exercise 6.11

Sketch the approximate straight-line Bode magnitude and phase plots to scale for the circuit shown in **Figure 6.17**.

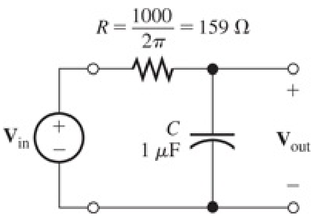


Figure 6.17
Circuit for **Exercise 6.11**.

Answer

See **Figure 6.18**.

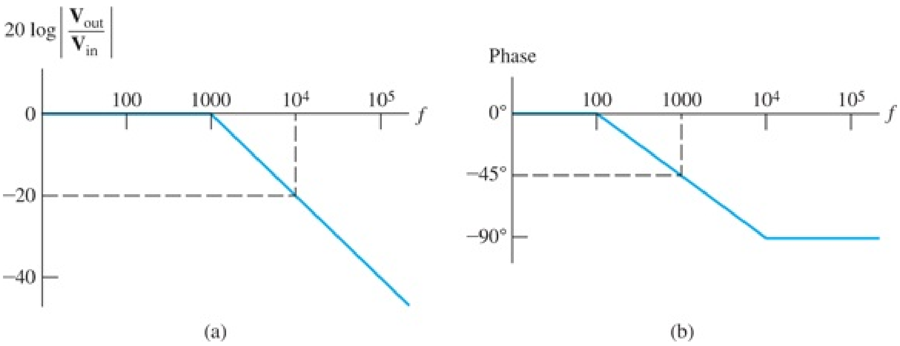


Figure 6.18
Answers for **Exercise 6.11**.

6.5 First-Order Highpass Filters

The circuit shown in **Figure 6.19** is called a **first-order highpass filter**. It can be analyzed in much the same manner as the lowpass circuit considered earlier in this chapter. The resulting transfer function is given by

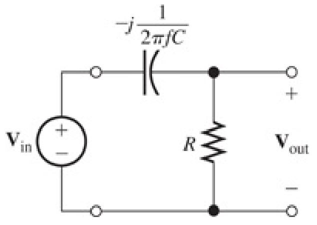


Figure 6.19
First-order highpass filter.

$$\boxed{H(f) = \frac{V_{out}}{V_{in}} = \frac{j(f/f_B)}{1+j(f/f_B)}} \tag{6.21}$$

in which

$$\boxed{f_B = \frac{1}{2\pi RC}} \tag{6.22}$$

Exercise 6.12

Use circuit analysis to derive the transfer function for the circuit of **Figure 6.19**, and show that it can be put into the form of **Equations 6.21** and **6.22**.

Magnitude and Phase of the Transfer Function

The magnitude of the transfer function is given by

$$\boxed{|H(f)| = \frac{f/f_B}{\sqrt{1+(f/f_B)^2}}} \tag{6.23}$$

This is plotted in **Figure 6.20(a)**. Notice that the transfer-function magnitude goes to zero for dc $(f=0)$. For high frequencies $(f >> f_B)$, the transfer-function magnitude approaches unity. Thus, this filter passes high-frequency components and tends to reject low-frequency components. That is why the circuit is called a highpass filter.

Figure 6.20

Magnitude and phase for the first-order highpass transfer function.

> Highpass filters are useful whenever we want to retain high-frequency components and reject low-frequency components.

Highpass filters are useful whenever we want to retain high-frequency components and reject low-frequency components. For example, suppose that we want to record warbler songs in a noisy environment. It turns out that bird calls fall in the high-frequency portion of the audible range. The audible range of frequencies is from 20 Hz to 15 kHz (approximately), and the calls of warblers fall (mainly) in the range above 2 kHz. On the other hand, the noise may be concentrated at lower frequencies. For example, heavy trucks rumbling down a bumpy road would produce strong noise components lower in frequency than 2 kHz. To record singing warblers in the vicinity of such a noise source, a highpass filter would be helpful. We would select *R* and *C* to achieve a half-power frequency f_B of approximately 2 kHz. Then, the filter would pass the songs and reject some of the noise.

Recall that if the amplitude of a component is multiplied by a factor of $1/\sqrt{2},$ the power that the component can deliver to a resistance is multiplied by a factor of 1/2. For $f = f_B,\ |H(f)| = 1/\sqrt{2} \cong 0.707,$ so that, as in the case of the lowpass filter, f_B is called the *half-power frequency*. (Here again, several alternative names are *corner frequency*, *3-dB frequency*, and *break frequency*.)

The phase of the highpass transfer function (**Equation 6.21**) is given by

$$\boxed{\angle H(f) = 90° - \arctan\left(\frac{f}{f_B}\right)} \tag{6.24}$$

A plot of the phase shift of the highpass filter is shown in **Figure 6.20(b)**.

Bode Plots for the First-Order Highpass Filter

As we have seen, a convenient way to plot transfer functions is to use the Bode plot, in which the magnitude is converted to decibels and a logarithmic frequency scale is used. In decibels, the magnitude of the highpass transfer function is

$$\left|H(f)\right|_{\mathrm{dB}} = 20\log\frac{f/f_B}{\sqrt{1+(f/f_B)^2}}$$

This can be written as

$$\left| H(f) \right|_{\text{dB}} = 20 \log \left(\frac{f}{f_B} \right) - 10 \log \left[1 + \left(\frac{f}{f_B} \right)^2 \right] \tag{6.25}$$

For $f \ll f_B,$ the second term on the right-hand side of **Equation 6.25** is approximately zero. Thus, for $f \ll f_B,$ we have

$$\left| H(f) \right|_{\text{dB}} \cong 20 \log \left(\frac{f}{f_B} \right) \qquad \text{for } f \ll f_B \tag{6.26}$$

Evaluating this for selected values of *f*, we find the values given in **Table 6.4**. Plotting these values, we obtain the low-frequency asymptote shown on the left-hand side of **Figure 6.21(a)**. Notice that the low-frequency asymptote slopes downward to the left at a rate of 20 dB per decade.

Table 6.4 Values of the Approximate Expression Given in Equation 6.26 for Selected Frequencies

f	$\lvert H(f) \rvert_{\text{dB}}$
f_B	0
$f_B/2$	-6
$f_B/10$	-20
$f_B/100$	-40

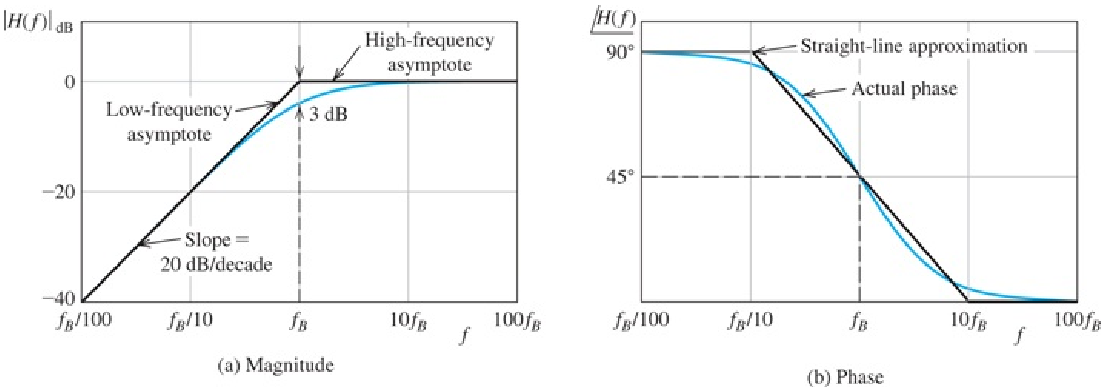


Figure 6.21

Bode plots for the first-order highpass filter.

For $f >> f_B$ the magnitude given by **Equation 6.25** is approximately 0 dB. Hence,

$$\left| H(f) \right|_{\text{dB}} \cong 0 \qquad \text{for } f >> f_B \tag{6.27}$$

This is plotted as the high-frequency asymptote in **Figure 6.21(a)**. Notice that the high-frequency asymptote and the low-frequency asymptote meet at $f = f_B.$ (That is why f_B is sometimes called the *break frequency*.)

The actual values of $\left| H(f) \right|_{\text{dB}}$ are also plotted in **Figure 6.21(a)**. Notice that the actual value at $f = f_B$ is $\left| H(f_B) \right|_{\text{dB}} = -3 \text{ dB}.$ Thus, the actual curve is only 3 dB from the asymptotes at $f = f_B.$ For other frequencies, the actual curve is closer to the asymptotes. The Bode phase plot is shown in **Figure 6.21(b)** along with straight-line approximations.

Example 6.5 Determination of the Break Frequency for a Highpass Filter

Suppose that we want a first-order highpass filter that has a transfer-function magnitude of $-30\ \mathrm{dB}$ at $f = 60\ \mathrm{Hz}$. Find the break frequency for this filter.

Solution

Recall that the low-frequency asymptote slopes at a rate of 20 dB/decade. Thus, we must select f_B to be

$$\frac{30\ \mathrm{dB}}{20\ \mathrm{dB/decade}} = 1.5\ \mathrm{decades}$$

higher than 60 Hz. Employing **Equation 6.17**, we have

$$\log\left(\frac{f_B}{60}\right) = 1.5$$

This is equivalent to

$$\frac{f_B}{60} = 10^{1.5} = 31.6$$

which yields

$$f_B \cong 1900\ \mathrm{Hz}$$

We often need a filter that greatly reduces the amplitude of a component at a given frequency, but has a negligible effect on components at nearby frequencies. The preceding example shows that to reduce the amplitude of a given component by a large factor by using a first-order filter, we must place the break frequency far from the component to be rejected. Then, components at other frequencies are also affected. This is a problem that can only be solved by using more complex (higher order) filter circuits. We consider second-order filters later in the chapter.

Exercise 6.13

Consider the circuit shown in **Figure 6.22**. Show that the transfer function of this filter is given by **Equation 6.21** if the half-power frequency is defined to be $f_B = R/2\pi L$.

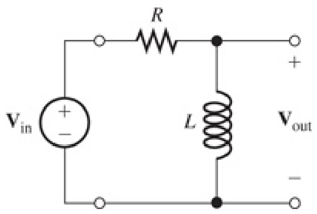


Figure 6.22
Circuit for **Exercise 6.13**.

Exercise 6.14

Suppose that we need a first-order *RC* highpass filter that reduces the amplitude of a component at a frequency of 1 kHz by 50 dB. The resistance is to be $1\ \mathrm{k\Omega}$. Find the half-power frequency and the capacitance.

Answer

$f_B = 316\ \mathrm{kHz},\ \ C = 503\ \mathrm{pF}$.

6.6 Series Resonance

In this section and the next, we consider resonant circuits. These circuits form the basis for filters that have better performance (in passing desired signals and rejecting undesired signals that are relatively close in frequency) than first-order filters. Such filters are useful in radio receivers, for example. Another application is a notch filter to remove 60-Hz interference from audio signals. Resonance is a phenomenon that can be observed in mechanical systems as well as in electrical circuits. For example, a guitar string is a resonant mechanical system.

> Resonance is a phenomenon that can be observed in mechanical systems and electrical circuits.

We will see that when a sinusoidal source of the proper frequency is applied to a resonant circuit, voltages much larger than the source voltage can appear in the circuit. The familiar story of opera singers using their voices to break wine goblets is an example of a mechanically resonant structure (the goblet) driven by an approximately sinusoidal source (the sound), resulting in vibrations in the glass of sufficient magnitude to cause fracture. Another example is the Tacoma Narrows Bridge collapse in 1940. Driven by wind forces, a resonance of the bridge structure resulted in oscillations that tore the bridge apart. Some other examples of mechanical resonant systems are the strings of musical instruments, bells, the air column in an organ pipe, and a mass suspended by a spring.

> You can find a short video clip of the bridge in motion on the internet.

Consider the series circuit shown in **Figure 6.23**. The impedance seen by the source in this circuit is given by

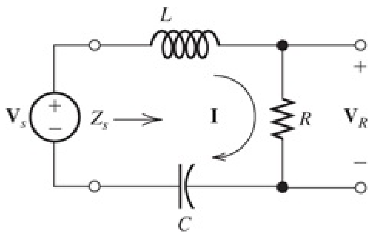


Figure 6.23
The series resonant circuit.

$$Z_s(f) = j2\pi fL + R - j\frac{1}{2\pi fC} \tag{6.28}$$

> The resonant frequency f_0 is defined to be the frequency at which the impedance is purely resistive (i.e., the total reactance is zero).

The **resonant frequency** f_0 is defined to be the frequency at which the impedance is purely resistive (i.e., the total reactance is zero). For the reactance to equal zero, the impedance of the inductance must equal the impedance of the capacitance in magnitude. Thus, we have

$$2\pi f_0 L = \frac{1}{2\pi f_0 C} \tag{6.29}$$

Solving for the resonant frequency, we get

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}} \tag{6.30}$$

The **quality factor** Q_s is defined to be the ratio of the reactance of the inductance at the resonant frequency to the resistance:

> The quality factor Q_s of a series circuit is defined to be the ratio of the reactance of the inductance at the resonant frequency to the resistance.

$$\boxed{Q_s = \frac{2\pi f_0 L}{R}} \tag{6.31}$$

Solving **Equation 6.29** for *L* and substituting into **Equation 6.31**, we obtain

$$\boxed{Q_s = \frac{1}{2\pi f_0 CR}} \tag{6.32}$$

Using **Equations 6.30** and **6.31** to substitute into **Equation 6.28**, we can eventually reduce the equation for the impedance to

$$\boxed{Z_s(f) = R\left[1 + jQ_s\left(\frac{f}{f_0} - \frac{f_0}{f}\right)\right]} \tag{6.33}$$

Thus, the series resonant circuit is characterized by its quality factor Q_s and resonant frequency f_0.

Plots of the normalized magnitude and the phase of the impedance versus normalized frequency f/f_0 are shown in **Figure 6.24**. Notice that the impedance magnitude is minimum at the resonant frequency. As the quality factor becomes larger, the minimum becomes sharper.

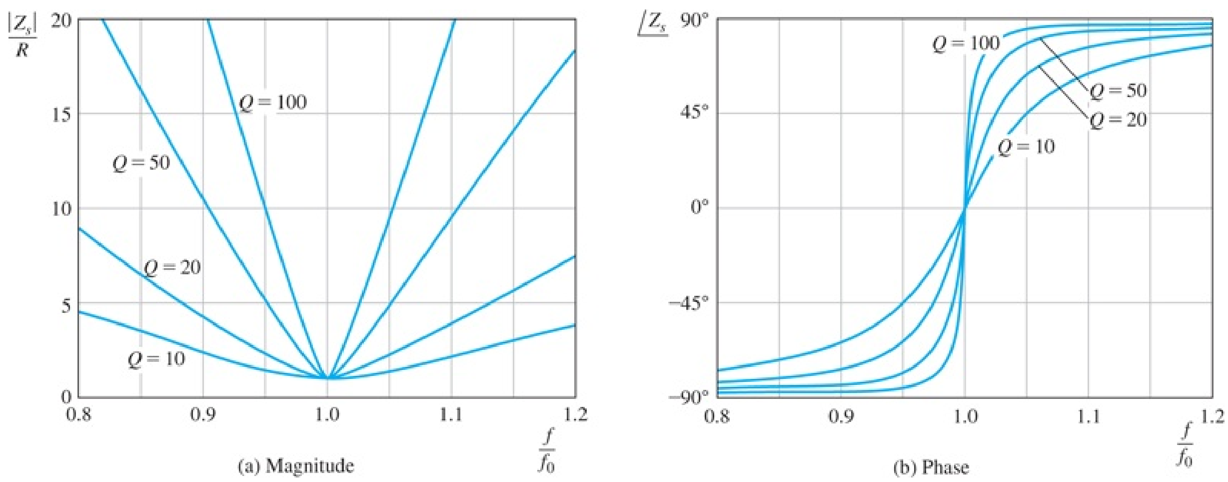


Figure 6.24
Plots of normalized magnitude and phase for the impedance of the series resonant circuit versus frequency.

Series Resonant Circuit as a Bandpass Filter

Referring to Figure 6.23, the current is given by

$$I = \frac{V_s}{Z_s(f)}$$

Using Equation 6.33 to substitute for the impedance, we have

$$I = \frac{V_s/R}{1 + jQ_s(f/f_0 - f_0/f)}$$

The voltage across the resistance is

$$V_R = RI = \frac{V_s}{1 + jQ_s(f/f_0 - f_0/f)}$$

Dividing by V_s, we obtain the transfer function

$$\boxed{\frac{V_R}{V_s} = \frac{1}{1 + jQ_s(f/f_0 - f_0/f)}}$$

Plots of the magnitude of V_R/V_s versus *f* are shown in Figure 6.25 for various values of Q_s.

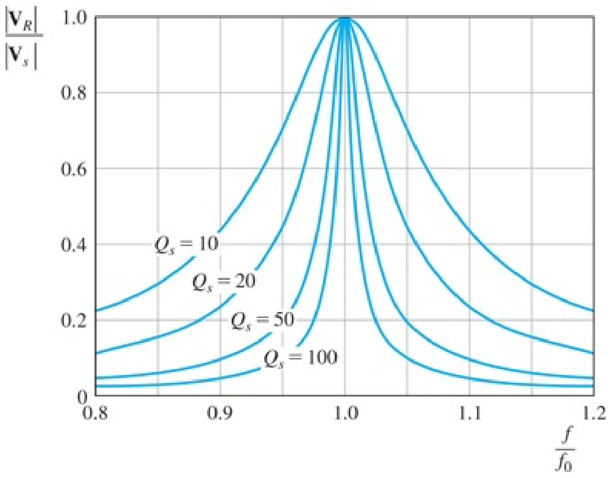


Figure 6.25
Plots of the transfer-function magnitude $|V_R/V_S|$ for the series resonant bandpass-filter circuit.

Consider a (sinusoidal) source of constant amplitude and variable frequency. At low frequencies, the impedance magnitude of the capacitance is large, the current **I** is small in magnitude, and V_R is small in magnitude (compared with V_s). At resonance, the total impedance magnitude reaches a minimum (because the reactances of the inductance and the capacitance cancel), the current magnitude is maximum, and $V_R = V_s$. At high frequencies, the impedance of the inductance is large, the current magnitude is small, and V_R is small in magnitude.

Now, suppose that we apply a source signal having components ranging in frequency about the resonant frequency. The components of the source that are close to the resonant frequency appear across the resistance with little change in amplitude. However, components that are higher or lower in frequency are significantly reduced in amplitude. Thus, a band of components centered at the resonant frequency is passed while components farther from the resonant frequency are (partly) rejected. We say that the resonant circuit behaves as a **bandpass filter**.

The resonant circuit behaves as a bandpass filter.

Recall that the half-power frequencies of a filter are the frequencies for which the transfer-function magnitude has fallen from its maximum by a factor of $1/\sqrt{2} \cong 0.707$. For the series resonant circuit, there are two half-power frequencies f_L and f_H. This is illustrated in **Figure 6.26**.

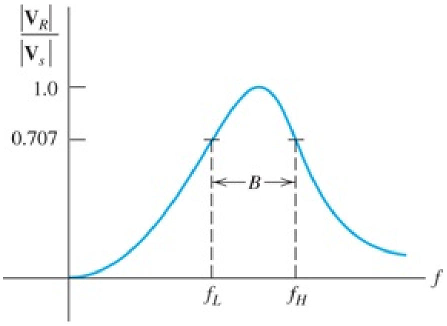


Figure 6.26
The bandwidth *B* is equal to the difference between the half-power frequencies.

The **bandwidth** *B* of this filter is the difference between the half-power frequencies:

$$\boxed{B = f_H - f_L} \tag{6.34}$$

For the series resonant circuit, it can be shown that

$$\boxed{B = \frac{f_0}{Q_s}} \tag{6.35}$$

Furthermore, for $Q_s >> 1$, the half-power frequencies are given by the approximate expressions

$$\boxed{f_H \cong f_0 + \frac{B}{2}} \tag{6.36}$$

and

$$\boxed{f_L \cong f_0 - \frac{B}{2}} \tag{6.37}$$

Example 6.6 Series Resonant Circuit

Consider the series resonant circuit shown in **Figure 6.27**. Compute the resonant frequency, the bandwidth, and the half-power frequencies. Assuming that the frequency of the source is the same as the resonant frequency, find the phasor voltages across the elements and draw a phasor diagram.

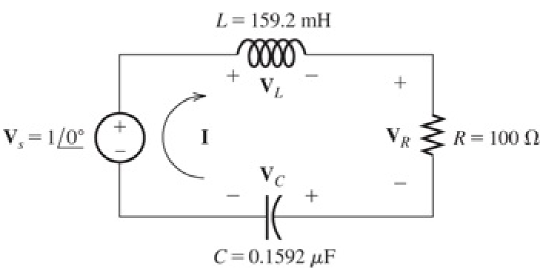


Figure 6.27
Series resonant circuit of **Example 6.6**. (The component values have been selected so the resonant frequency and Q_s turn out to be round numbers.)

Solution

First, we use **Equation 6.30** to compute the resonant frequency:

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1592 \times 0.1592 \times 10^{-6}}} = 1000 \text{ Hz}$$

The quality factor is given by **Equation 6.31**

$$Q_s = \frac{2\pi f_0 L}{R} = \frac{2\pi \times 1000 \times 0.1592}{100} = 10$$

The bandwidth is given by **Equation 6.35**

$$B = \frac{f_0}{Q_s} = \frac{1000}{10} = 100 \text{ Hz}$$

Next, we use **Equations 6.36** and **6.37** to find the approximate half-power frequencies:

$$\begin{aligned} f_H &\cong f_0 + \frac{B}{2} = 1000 + \frac{100}{2} = 1050 \text{ Hz} \\ f_L &\cong f_0 - \frac{B}{2} = 1000 - \frac{100}{2} = 950 \text{ Hz} \end{aligned}$$

At resonance, the impedance of the inductance and capacitance are

$$\begin{aligned} Z_L &= j2\pi f_0 L = j2\pi \times 1000 \times 0.1592 = j1000\ \Omega \\ Z_C &= -j\frac{1}{2\pi f_0 C} = -j\frac{1}{2\pi \times 1000 \times 0.1592 \times 10^{-6}} = -j1000\ \Omega \end{aligned}$$

As expected, the reactances are equal in magnitude at the resonant frequency. The total impedance of the circuit is

$$Z_s = R + Z_L + Z_C = 100 + j1000 - j1000 = 100\ \Omega$$

The phasor current is given by

$$\boldsymbol{I} = \frac{\boldsymbol{V}_s}{Z_s} = \frac{1\angle 0^\circ}{100} = 0.01\angle 0^\circ$$

The voltages across the elements are

$$\begin{aligned} \boldsymbol{V}_R &= R\boldsymbol{I} = 100 \times 0.01\angle 0^\circ = 1\angle 0^\circ \\ \boldsymbol{V}_L &= Z_L\boldsymbol{I} = j1000 \times 0.01\angle 0^\circ = 10\angle 90^\circ \\ \boldsymbol{V}_C &= Z_C\boldsymbol{I} = -j1000 \times 0.01\angle 0^\circ = 10\angle -90^\circ \end{aligned}$$

The phasor diagram is shown in **Figure 6.28**. Notice that the voltages across the inductance and capacitance are much larger than the source voltage in magnitude. Nevertheless, Kirchhoff's voltage law is satisfied because $\boldsymbol{V}_L$ and $\boldsymbol{V}_C$ are out of phase and cancel.

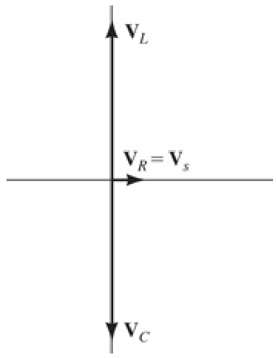


Figure 6.28

Phasor diagram for **Example 6.6**.

In **Example 6.6**, we found that the voltage magnitudes across the inductance and capacitance are Q_s times higher than the source voltage. Thus, a higher quality factor leads to higher voltage magnification. This is similar to the large vibrations that can be caused in a wine goblet by an opera singer's voice.

Exercise 6.15

Determine the *R* and *C* values for a series resonant circuit that has $L = 10\ \mu\text{H},\ f_0 = 1\ \text{MHz},$ and $Q_s = 50.$ Find the bandwidth and approximate half-power frequencies of the circuit.

Answer

$C = 2533\ \text{pF},\ R = 1.257\ \Omega,\ B = 20\ \text{kHz},\ f_L \cong 990\ \text{kHz},\ f_H \cong 1010\ \text{kHz}.$

Exercise 6.16

Suppose that a voltage $V_s = 1\angle 0°$ at a frequency of 1 MHz is applied to the circuit of **Exercise 6.15**. Find the phasor voltages across the resistance, capacitance, and inductance.

Answer

$V_R = 1\angle 0°,\ V_C = 50\angle -90°,\ V_L = 50\angle 90°.$

Exercise 6.17

Find the *R* and *L* values for a series resonant circuit that has $C = 470\ \text{pF},$ a resonant frequency of 5 MHz, and a bandwidth of 200 kHz.

Answer

$R = 2.709\ \Omega,\ L = 2.156\ \mu\text{H}.$

6.7 Parallel Resonance

Another type of resonant circuit known as a **parallel resonant circuit** is shown in **Figure 6.29**. The impedance of this circuit is given by

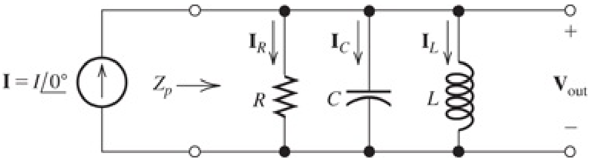


Figure 6.29
The parallel resonant circuit.

$$\boxed{Z_p = \frac{1}{1/R + j2\pi fC - j(1/2\pi fL)}} \tag{6.38}$$

As in the series resonant circuit, the **resonant frequency** f_0 is the frequency for which the impedance is purely resistive. This occurs when the imaginary parts of the denominator of **Equation 6.38** cancel. Thus, we have

$$2\pi f_0 C = \frac{1}{2\pi f_0 L} \tag{6.39}$$

Solving for the resonant frequency, we get

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}} \tag{6.40}$$

which is exactly the same as the expression for the resonant frequency of the series circuit discussed in **Section 6.6**.

For the parallel circuit, we define the quality factor Q_p as the ratio of the resistance to the reactance of the inductance at resonance, given by

$$\boxed{Q_p = \frac{R}{2\pi f_0 L}} \tag{6.41}$$

Notice that this is the reciprocal of the expression for the quality factor Q_s of the series resonant circuit. Solving **Equation 6.40** for *L* and substituting into **Equation 6.41**, we obtain another expression for the quality factor:

$$\boxed{Q_p = 2\pi f_0 CR} \tag{6.42}$$

Notice that the formula for Q_p of a parallel circuit in terms of the circuit elements is the reciprocal of the formula for Q_s of a series circuit.

If we solve **Equations 6.41** and **6.42** for *L* and *C*, respectively, and then substitute into **Equation 6.38**, we eventually obtain

$$\boxed{Z_p = \frac{R}{1 + jQ_p(f/f_0 - f_0/f)}} \tag{6.43}$$

The voltage across the parallel resonant circuit is the product of the phasor current and the impedance:

$$V_{\text{out}} = \frac{IR}{1 + jQ_p(f/f_0 - f_0/f)} \tag{6.44}$$

Suppose that we hold the current constant in magnitude and change the frequency. Then, the magnitude of the voltage is a function of frequency. A plot of voltage magnitude for the parallel resonant circuit is shown in **Figure 6.30**. Notice that the voltage magnitude reaches its maximum $V_{o\max} = RI$ at the resonant frequency. These curves have the same shape as the curves shown in **Figures 6.25** and **6.26** for the voltage transfer function of the series resonant circuit.

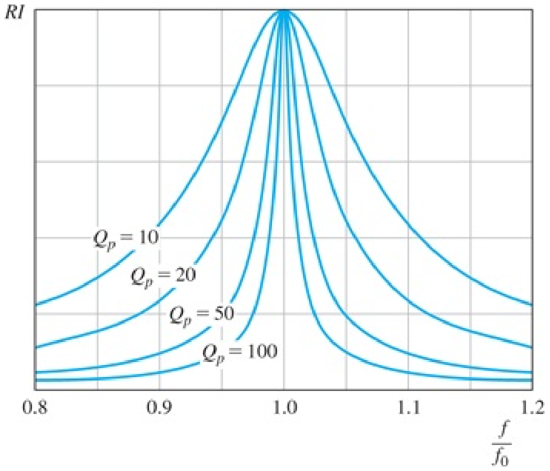


Figure 6.30
Voltage across the parallel resonant circuit for a constant-amplitude variable-frequency current source.

The half-power frequencies f_L and f_H are defined to be the frequencies at which the voltage magnitude reaches the maximum value times $1/\sqrt{2}$. The bandwidth of the circuit is given by

$$\boxed{B = f_H - f_L} \tag{6.45}$$

It can be shown that the bandwidth is related to the resonant frequency and quality factor by the expression

$$\boxed{B = \frac{f_0}{Q_p}} \tag{6.46}$$

Example 6.7 Parallel Resonant Circuit

Find the *L* and *C* values for a parallel resonant circuit that has $R = 10\ \text{k}\Omega,\ f_0 = 1\ \text{MHz},$ and $B = 100\ \text{kHz}$. If $I = 10^{-3}\angle 0°\ \text{A},$ draw the phasor diagram showing the currents through each of the elements in the circuit at resonance.

Solution

First, we compute the quality factor of the circuit. Rearranging **Equation 6.46** and substituting values, we have

$$Q_p = \frac{f_0}{B} = \frac{10^6}{10^5} = 10$$

Solving **Equation 6.41** for the inductance and substituting values, we get

$$L = \frac{R}{2\pi f_0 Q_p} = \frac{10^4}{2\pi \times 10^6 \times 10} = 159.2\ \mu\text{H}$$

Similarly, using **Equation 6.42**, we find that

$$C = \frac{Q_p}{2\pi f_0 R} = \frac{10}{2\pi \times 10^6 \times 10^4} = 159.2\ \text{pF}$$

At resonance, the voltage is given by

$$V_{\text{out}} = IR = \left(10^{-3}\angle 0°\right) \times 10^4 = 10\angle 0°\ \text{V}$$

and the currents are given by

$$\begin{aligned} I_R &= \frac{V_{\text{out}}}{R} = \frac{10\angle 0°}{10^4} = 10^{-3}\angle 0°\ \text{A} \\ I_L &= \frac{V_{\text{out}}}{j2\pi f_0 L} = \frac{10\angle 0°}{j10^3} = 10^{-2}\angle -90°\ \text{A} \\ I_C &= \frac{V_{\text{out}}}{-j/2\pi f_0 C} = \frac{10\angle 0°}{-j10^3} = 10^{-2}\angle 90°\ \text{A} \end{aligned}$$

The phasor diagram is shown in **Figure 6.31**. Notice that the currents through the inductance and capacitance are larger in magnitude than the applied source current. However, since I_C and I_L are out of phase, they cancel.

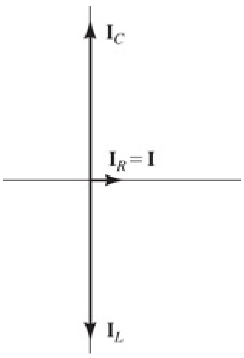


Figure 6.31
Phasor diagram for **Example 6.7**.

Exercise 6.18

A parallel resonant circuit has $R = 10\ \mathrm{k}\Omega,\ L = 100\ \mu\mathrm{H},$ and $C = 500\ \mathrm{pF}$. Find the resonant frequency, quality factor, and bandwidth.

Answer

$f_0 = 711.8\ \mathrm{kHz},\ Q_p = 22.36,\ B = 31.83\ \mathrm{kHz}$.

Exercise 6.19

A parallel resonant circuit has $f_0 = 10\ \mathrm{MHz},\ B = 200\ \mathrm{kHz},$ and $R = 1\ \mathrm{k}\Omega$. Find *L* and *C*.

Answer

$L = 0.3183\ \mu\mathrm{H},\ C = 795.8\ \mathrm{pF}$.

6.8 Ideal and Second-Order Filters

Ideal Filters

In discussing filter performance, it is helpful to consider ideal filters. An ideal filter passes components in the desired frequency range with no change in amplitude or phase and totally rejects the components in the undesired frequency range. Depending on the locations of the frequencies to be passed and rejected, we have different types of filters: lowpass, highpass, bandpass, and band reject. The transfer functions $H(f) = V_{\text{out}}/V_{\text{in}}$ of the four types of ideal filters are shown in **Figure 6.32**.

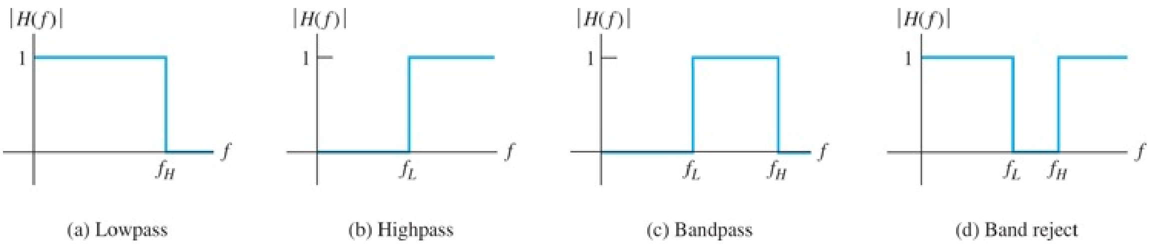


Figure 6.32
Transfer functions of ideal filters.

- An **ideal lowpass filter** [**Figure 6.32(a)**] passes components below its cutoff frequency f_H and rejects components higher in frequency than f_H.
- An **ideal highpass filter** [**Figure 6.32(b)**] passes components above its cutoff frequency f_L and rejects components lower in frequency than f_L.
- An **ideal bandpass filter** [**Figure 6.32(c)**] passes components that lie between its cutoff frequencies (f_L and f_H) and rejects components outside that range.
- An **ideal band-reject filter** [**Figure 6.32(d)**], which is also called a **notch filter**, rejects components that lie between its cutoff frequencies (f_L and f_H) and passes components outside that range.

As we have seen earlier in this chapter, filters are useful whenever a signal contains desired components in one range of frequency and undesired components in another range of frequency. For example, **Figure 6.33(a)** shows a 1-kHz sine wave that has been corrupted by high-frequency noise. By passing this noisy signal through a lowpass filter, the noise is eliminated.

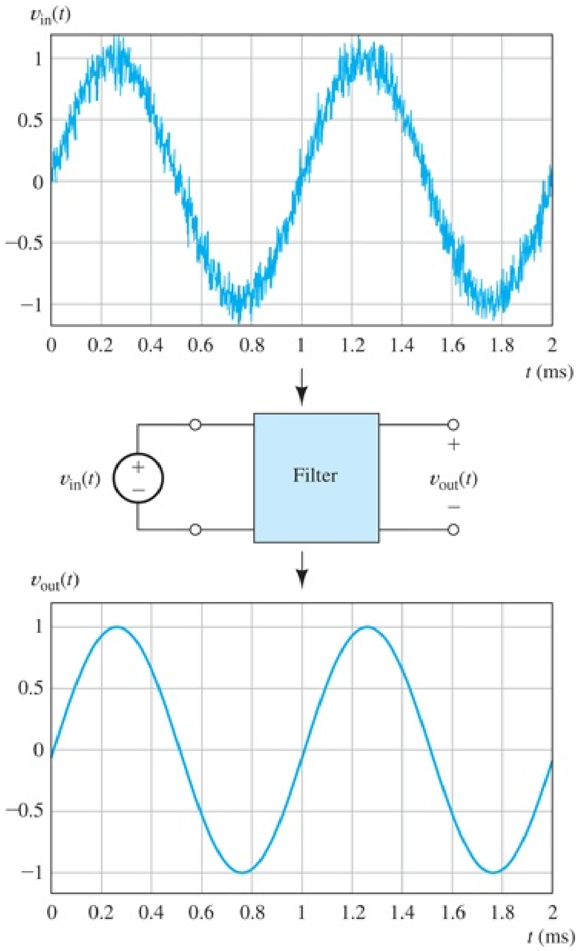


Figure 6.33

The input signal v_{in} consists of a 1-kHz sine wave plus high-frequency noise. By passing v_{in} through an ideal lowpass filter with the proper cutoff frequency, the sine wave is passed and the noise is rejected, resulting in a clean output signal.

Example 6.8 Cascaded Ideal Filters

Electrocardiographic (ECG) signals are voltages between electrodes placed on the torso, arms, or legs of a medical patent. ECG signals are used by cardiologists to help diagnose various types of heart disease.

Unfortunately, the voltages between the electrodes can contain undesirable noises (called “artifacts” in medical jargon). The undesirable components are dc and frequency components below 0.5 Hz known as “baseline wander”, a large 60-Hz sinewave due to power-line interference, and “muscle noise” with components above about 100 Hz caused by muscle movement, such as when the patient is on a treadmill. The part of the ECG signal of interest to cardiologists lies between about 0.5 Hz and 100 Hz.

We wish to design a cascade connection of ideal filters to eliminate the noise and preserve the ECG signal components of interest.

Solution

First, we can use an ideal highpass filter having a transfer function magnitude $|H_1(f)|$ as shown in **Figure 6.34(a)** to eliminate the dc and baseline wander with components below 0.5 Hz. Notice that we have used logarithmic frequency scales in **Figure 6.34** to show low and high frequencies more clearly than could be accomplished with a linear frequency scale.

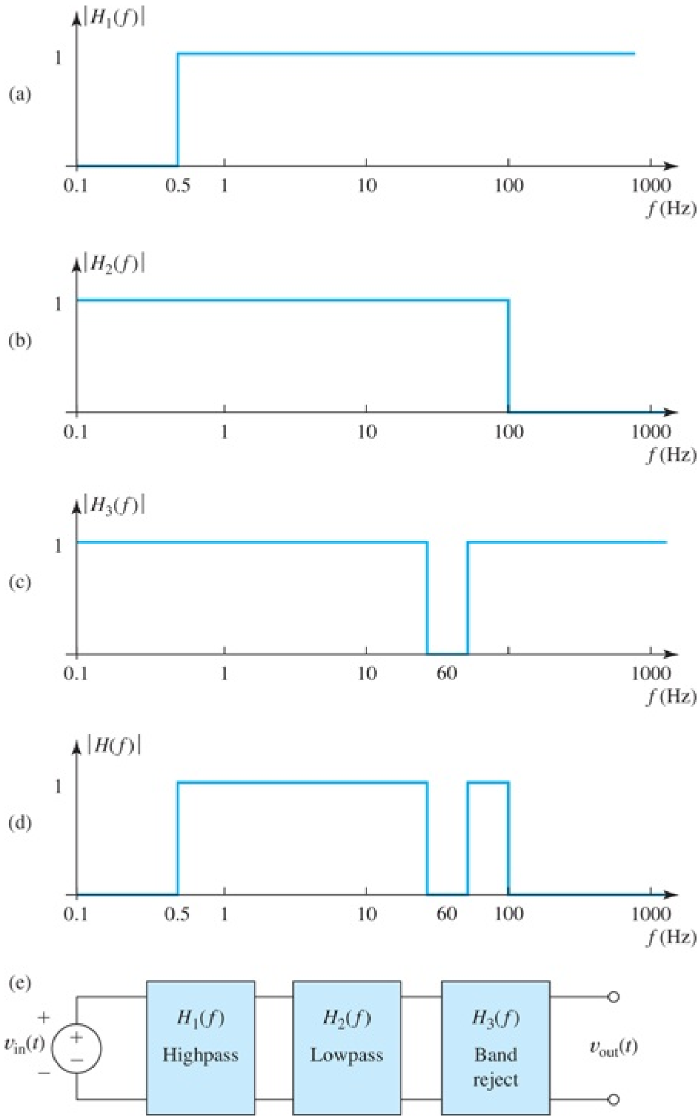


Figure 6.34

Cascaded filters of **Example 6.8**.

Next, we can use an ideal lowpass filter having a transfer function magnitude $|H_2(f)|$ as shown in **Figure 6.34(b)** to eliminate the muscle noise components above 100 Hz.

Finally, we employ a band-reject filter having a transfer function magnitude $|H_3(f)|$ as shown in **Figure 6.34(c)** with cutoff frequencies slightly above 60 Hz and slightly below 60 Hz to eliminate the power line interference. We should strive to keep the cutoff frequencies of the band-reject filter very close to 60 Hz to avoid removing too many components of the ECG.

The overall transfer function magnitude $|H(f)| = |H_1(f)| \times |H_2(f)| \times |H_3(f)|$ is shown in **Figure 6.34(d)** and the cascaded filters are shown in **Figure 6.34(e)**.

Unfortunately, it is not possible to construct ideal filters—they can only be approximated by real circuits. As the circuits are allowed to increase in complexity, it is possible to design filters that do a better job of rejecting unwanted components and retaining the desired components. Thus, we will see that second-order circuits perform better (i.e., closer to ideal) than the first-order circuits considered earlier in this chapter.

Second-Order Lowpass Filter

Figure 6.35(a) shows a second-order lowpass filter based on the series resonant circuit of **Section 6.6**. The filter is characterized by its resonant frequency f_0 and quality factor Q_s, which are given by **Equations 6.30** and **6.31**. It can be shown that the transfer function for this circuit is given by

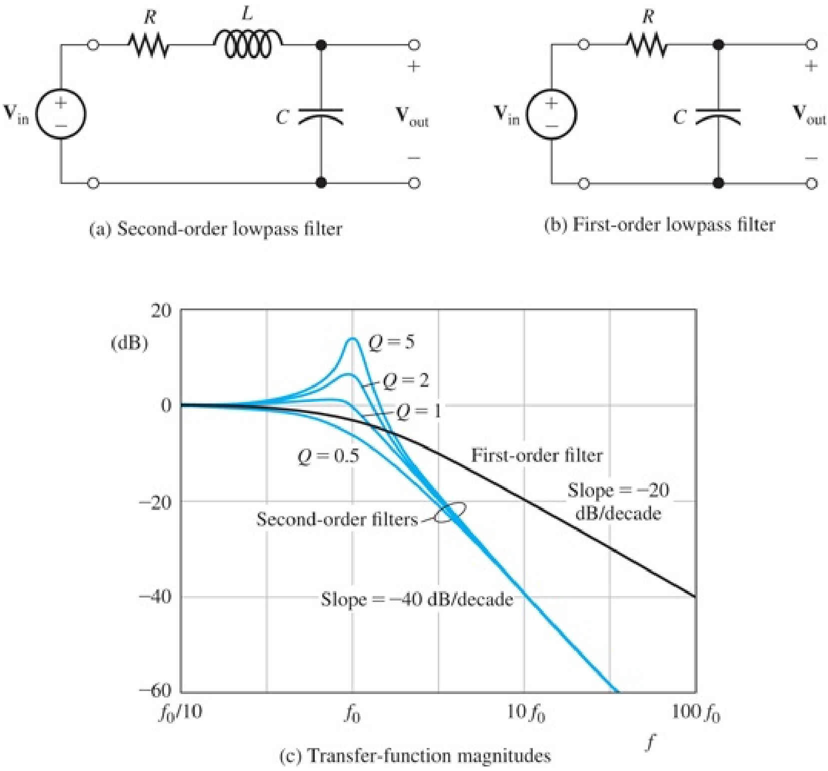


Figure 6.35
Lowpass filter circuits and their transfer-function magnitudes versus frequency.

$$H(f) = \frac{V_{out}}{V_{in}} = \frac{-jQ_s(f_0/f)}{1 + jQ_s(f/f_0 - f_0/f)} \tag{6.47}$$

Bode plots of the transfer-function magnitude are shown in **Figure 6.35(c)**. Notice that for $Q_s >> 1,$ the transfer-function magnitude reaches a high peak in the vicinity of the resonant frequency. Usually, in designing a filter, we want the gain to be approximately constant in the passband, and we select $Q_s \cong 1.$ (Actually, $Q_s = 0.707$ is the highest value for which the transfer-function magnitude does not display an increase before rolling off. The transfer function for this value of Q_s is said to be *maximally flat*, is also known as a *Butterworth function*, and is often used for lowpass filters.)

Comparison of First- and Second-Order Filters

For comparison, a first-order lowpass filter is shown in **Figure 6.35(b)**, and the Bode plot of its transfer function is shown in **Figure 6.35(c)**. The first-order circuit is characterized by its half-power frequency $f_B = 1/(2\pi RC)$. (We have selected $f_B = f_0$ in making the comparison.) Notice that above f_0 the magnitude of the transfer function falls more rapidly for the second-order filter than for the first-order filter ($-40\ \mathrm{dB/decade}$ versus $-20\ \mathrm{dB/decade}$).

> The transfer-function magnitude of a second-order lowpass filter declines 40 dB per decade well above the break frequency, whereas the transfer-function magnitude for the first-order filter declines at only 20 dB per decade. Thus, the second-order filter is a better approximation to an ideal lowpass filter.

Second-Order Highpass Filter

A second-order highpass filter is shown in **Figure 6.36(a)**, and its magnitude Bode plot is shown in **Figure 6.36(b)**. Here again, we usually want the magnitude to be as nearly constant as possible in the passband, so we select $Q_s \cong 1$. (In other words, we usually want to design the filter to approximate an ideal filter as closely as possible.)

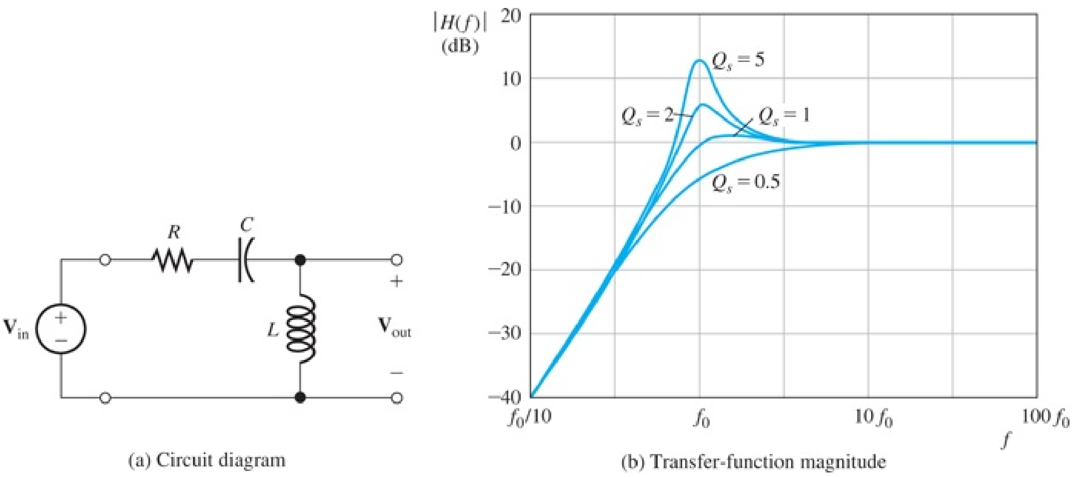


Figure 6.36
Second-order highpass filter and its transfer-function magnitude versus frequency for several values of Q_s.

Second-Order Bandpass Filter

A second-order bandpass filter is shown in **Figure 6.37(a)**, and its magnitude Bode plot is shown in **Figure 6.37(b)**. The half-power bandwidth *B* is given by **Equations 6.34** and **6.35**, which state that

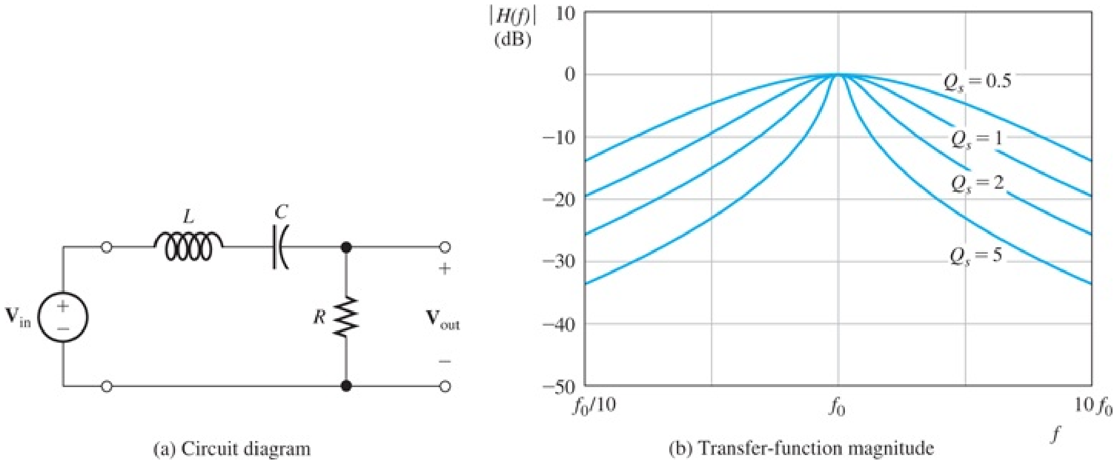


Figure 6.37

Second-order bandpass filter and its transfer-function magnitude versus frequency for several values of Q_s.

$$B = f_H - f_L$$

and

$$B = \frac{f_0}{Q_s}$$

Second-Order Band-Reject (Notch) Filter

A second-order band-reject filter is shown in **Figure 6.38(a)** and its magnitude Bode plot is shown in **Figure 6.38(b)**. In theory, the magnitude of the transfer function is zero for $f = f_0$. [In decibels, this corresponds to $|H(f_0)| = -\infty$ dB.] However, real inductors contain series resistance, so rejection of the f_0 component is not perfect for actual circuits.

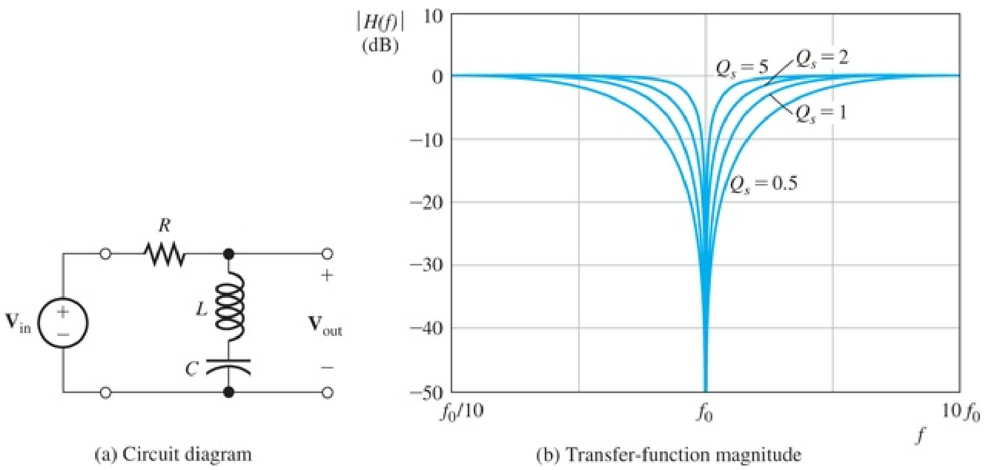


Figure 6.38

Second-order band-reject filter and its transfer-function magnitude versus frequency for several values of Q_s.

Example 6.9 Filter Design

Suppose that we need a filter that passes components higher in frequency than 1 kHz and rejects components lower than 1 kHz. Select a suitable second-order circuit configuration, choose $L = 50\ \mathrm{mH},$ and specify the values required for the other components.

Solution

We need to pass high-frequency components and reject low-frequency components. Therefore, we need a highpass filter. The circuit diagram for a second-order highpass filter is shown in **Figure 6.36(a)**, and the corresponding transfer-function magnitude plots are shown in **Figure 6.36(b)**. Usually, we want the transfer function to be approximately constant in the passband. Thus, we choose $Q_s \cong 1.$ We select $f_0 \cong 1\ \mathrm{kHz},$ so the components above 1 kHz are passed, while lower-frequency components are (at least partly) rejected. Solving **Equation 6.30**, for the capacitance and substituting values, we have

$$\begin{aligned} C &= \frac{1}{(2\pi)^2 f_0^2 L} = \frac{1}{(2\pi)^2 \times 10^6 \times 50 \times 10^{-3}} \\ &= 0.507\ \mu\mathrm{F} \end{aligned}$$

Solving **Equation 6.31** for the resistance and substituting values, we get

$$R = \frac{2\pi f_0 L}{Q_s} = \frac{2\pi \times 1000 \times 50 \times 10^{-3}}{1} = 314.1\ \Omega$$

The circuit and values are shown in **Figure 6.39**.

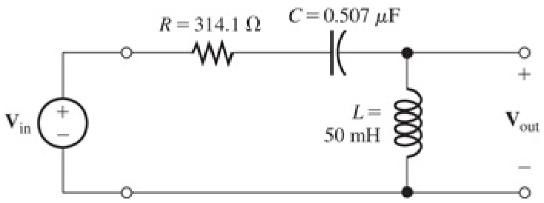


Figure 6.39
Filter designed in **Example 6.9**.

There are several reasons why we might not use the exact values that we calculated for the components in the last example. First, fixed-value capacitors and resistors are readily available only in certain standard values. Furthermore, the design called for a filter to reject components lower than 1 kHz and pass components higher than 1 kHz. We arbitrarily selected $f_0 = 1\ \mathrm{kHz}.$ Depending on whether it is more important to reject the low frequencies or to pass the high frequencies without change in amplitude, a slightly higher or lower value for f_0 could be better. Finally, our choice of Q_s was somewhat arbitrary. In practice, we could choose variable components by using the calculations as a starting point. Then, we would adjust the filter experimentally for the most satisfactory performance.

Exercise 6.20

Suppose that we need a filter that passes components lower in frequency than 5 kHz and rejects components higher than 5 kHz. Select a suitable second-order circuit configuration, choose $L = 5\ \text{mH}$, and specify the values required for the other components.

Answer

See **Figure 6.40**.

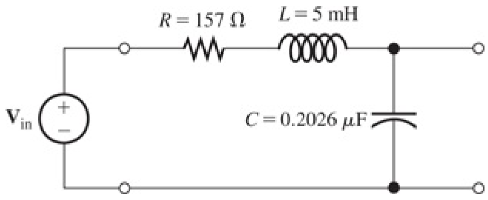


Figure 6.40
Answer for **Exercise 6.20**.

Exercise 6.21

Suppose that we want a filter that passes components between $f_L = 45\ \text{kHz}$ and $f_H = 55\ \text{kHz}$. Higher and lower frequencies are to be rejected. Design a circuit using a 1-mH inductance.

Answer

We need a bandpass filter with $f_0 \cong 50\ \text{kHz}$ and $Q_s = 5$. The resulting circuit is shown in **Figure 6.41**.

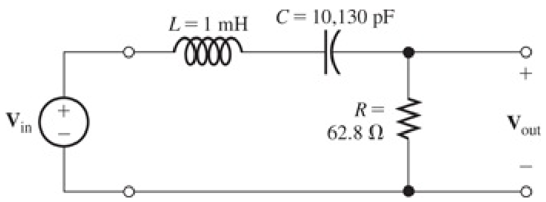


Figure 6.41
Answer for **Exercise 6.21**.

6.9 Bode Plots with MATLAB

So far in this chapter, we have used manual methods to illustrate Bode-plot concepts for simple filters. While manual methods can be extended to more complex circuits, it is often quicker and more accurate to use computer software to produce Bode plots.

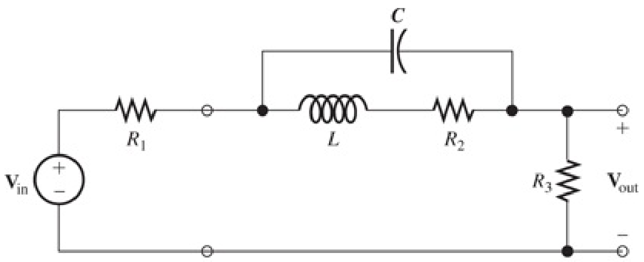
Because subtle programming errors can result in grossly erroneous results, it is good practice to employ independent checks on computer-generated Bode plots. For example, a complex circuit can often be readily analyzed manually at very high and at very low frequencies. At very low frequencies, the inductances behave as short circuits and the capacitances behave as open circuits, as we discussed in **Section 4.2**. Thus, we can replace the inductances by shorts and the capacitances by opens and analyze the simplified circuit to determine the value of the transfer function at low frequencies, providing an independent check on the plots produced by a computer.

> Manual analysis at dc and very high frequencies often provides some easy checks on computer-aided Bode plots.

Similarly, at very high frequencies, the inductances become open circuits, and the capacitances become shorts. Next, we illustrate this approach with an example.

Example 6.10 Computer-Generated Bode Plot

The circuit of **Figure 6.42** is a notch filter. Use MATLAB to generate a magnitude Bode plot of the transfer function $H(f) = V_{out}/V_{in}$ with frequency ranging from 10 Hz to 100 kHz. Then, analyze the circuit manually at very high and very low frequencies to provide checks on the plot. Use the plot to determine the frequency of maximum attenuation and the value of the transfer function at that frequency.



$R_1 = 90\ \Omega$ $\quad L = 100$ mH

$R_2 = 10\ \Omega$ $\quad C = 0.1\ \mu$F

$R_3 = 100\ \Omega$

Figure 6.42
Filter of **Example 6.10**.

Solution

Using the voltage-divider principle, we can write the transfer function for the filter as

$$H(f) = \frac{V_{out}}{V_{in}} = \frac{R_3}{R_1 + R_3 + 1/[j\omega C + 1/(R_2 + j\omega L)]}$$

A MATLAB m-file that produces the Bode plot is:

```
clear
% Enter the component values:
R1 = 90; R2 = 10; R3 = 100;
L = 0.1; C = 1e-7;
% The following command generates 1000 frequency values
% per decade, evenly spaced from 10^1 to 10^5 Hz
% on a logarithmic scale:
f = logspace(1,5,4000);
w = 2*pi*f;
% Evaluate the transfer function for each frequency.
% As usual, we are using i in place of j:
H = R3./(R1+R3+1./(i*w*C + 1./(R2 + i*w*L)));
% Convert the magnitude values to decibels and plot:
semilogx(f,20*log10(abs(H)))
```

The resulting plot is shown in **Figure 6.43**. This circuit is called a notch filter because it strongly rejects components in the vicinity of 1591 Hz while passing higher and lower frequencies. The maximum attenuation is 60 dB.

The m-file is named Example_6_10 and appears in the MATLAB folder, and if you have access to MATLAB, you can run it to see the result. (See Appendix E for information on how to access the MATLAB folder.) Then, you can use the toolbar on the figure screen to magnify a portion of the plot and obtain the notch frequency and maximum attenuation with excellent accuracy.

The command

```
f = logspace(1,5,4000)
```

generates an array of 4000 frequency values, starting at 10^1 Hz and ending at 10^5 Hz, evenly spaced on a logarithmic scale with 1000 points per decade. (Typically, we might start with 100 points per decade, but this transfer function changes very rapidly in the vicinity of 1590 Hz, so we increased the number of points to more accurately determine the location and depth of the notch.)

As a partial check on our analysis and program, we analyze the circuit at $f = 0$ (dc) to determine the transfer function at very low frequencies. To do so, we replace the inductance by a short and the capacitance by an open circuit. Then, the circuit becomes a simple resistive voltage divider consisting of R_1, R_2, and R_3. Therefore, we have

$$H(0) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{R_3}{R_1 + R_2 + R_3} = 0.5$$

In decibels, this becomes

$$H_{\text{dB}}(0) = 20\log(0.5) = -6\text{ dB}$$

which agrees very well with the plotted value at 10 Hz.

For a second check, we replace the capacitance by a short circuit and the inductance by an open circuit to determine the value of the transfer function at very high frequencies. Then, the circuit again becomes a simple resistive voltage divider consisting of R_1 and R_3. Thus, we have

$$H(\infty) = \frac{R_3}{R_1 + R_3} = 0.5263$$

In decibels, this becomes

$$H_{\mathrm{dB}}(\infty) = 20\log(0.5263) = -5.575\ \mathrm{dB}$$

which agrees very closely with the value plotted at 100 kHz.

Exercise 6.22

If you have access to MATLAB, run the m-file Example_6_10 that is contained in the MATLAB folder.

Answer

The resulting plot should be very similar to **Figure 6.43**.

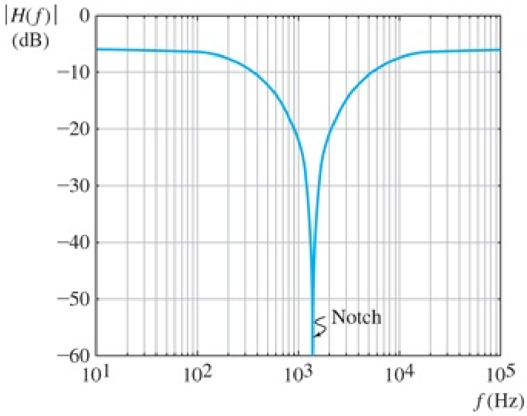


Figure 6.43
Bode plot for **Example 6.10** produced using MATLAB.

6.10 Digital Signal Processing

So far, we have introduced the concepts related to filters in the context of *RLC* circuits. However, many modern systems make use of a more sophisticated technology called **digital signal processing** (DSP). In using DSP to filter a signal, the analog input signal $x(t)$ is converted to digital form (a sequence of numbers) by an **analog-to-digital converter** (ADC). A digital computer then uses the digitized input signal to compute a sequence of values for the output signal. Finally, if desired, the computed values are converted to analog form by a **digital-to-analog converter** (DAC) to produce the output signal $y(t)$. The generic block diagram of a DSP system is shown in **Figure 6.44**.

Figure 6.44
Generic block diagram of a digital signal-processing (DSP) system.

Besides filtering, many other operations, such as speech recognition, can be performed by DSP systems. DSP was used in the early days of the Space Telescope to focus blurry images resulting from an error in the telescope's design. High-definition televisions, digital cell phones, and MP3 music players are examples of products that have been made possible by DSP technology.

DSP is a large and rapidly evolving field that will continue to produce novel products. We discuss digital filters very briefly to give you a glimpse of this exciting field.

Conversion of Signals from Analog to Digital Form

Analog signals are converted to digital form by a DAC in a two-step process. First, the analog signal is sampled (i.e., measured) at periodic points in time. Then, a code word is assigned to represent the approximate value of each sample. Usually, the code words consist of binary symbols. This process is illustrated in **Figure 6.45**, in which each sample value is represented by a three-bit code word corresponding to the amplitude zone into which the sample falls. Thus, each sample value is converted into a code word, which in turn can be represented by a digital waveform as shown in the figure.

Figure 6.45

An analog signal is converted to an approximate digital equivalent by sampling. Each sample value is represented by a three-bit code word. (Practical converters use longer code words, and the width Δ of each amplitude zone is much smaller.)

The rate f_s at which a signal must be sampled depends on the frequencies of the signal components. We have seen that all real signals can be considered to consist of sinusoidal components having various frequencies, amplitudes, and phases. If a signal contains no components with frequencies higher than f_H, the signal can (in theory) be exactly reconstructed from its samples, provided that the sampling frequency f_s is selected to be more than twice f_H:

$$\boxed{f_s > 2f_H} \tag{6.48}$$

> If a signal contains no components with frequencies higher than f_H, the signal can be exactly reconstructed from its samples, provided that the sampling rate f_s is selected to be more than twice f_H.

For example, high-fidelity audio signals have a highest frequency of about 15 kHz. Therefore, the minimum sampling rate that should be used for audio signals is 30 kHz. Practical considerations dictate a sampling frequency somewhat higher than the theoretical minimum. For instance, audio compact-disc technology converts audio signals to digital form with a sampling rate of 44.1 kHz. Naturally, it is desirable to use the lowest practical sampling rate to minimize the amount of data (in the form of code words) that must be stored or manipulated by the DSP system.

Of course, the interval between samples T is the reciprocal of the sampling rate:

$$T = \frac{1}{f_s} \tag{6.49}$$

A second consideration important in converting analog signals to digital form is the number of amplitude zones to be used. Exact signal amplitudes cannot be represented, because all amplitudes falling into a given zone have the same code word. Thus, when a DAC converts the code words to recreate the original analog waveform, it is possible to reconstruct only an approximation to the original signal with the reconstructed voltage in the middle of each zone, which is illustrated in Figure 6.46. Thus, some

quantization error exists between the original signal and the reconstruction. This error can be reduced by using a larger number of zones, which requires longer code words. The number *N* of amplitude zones is related to the number of bits *k* in a code word by

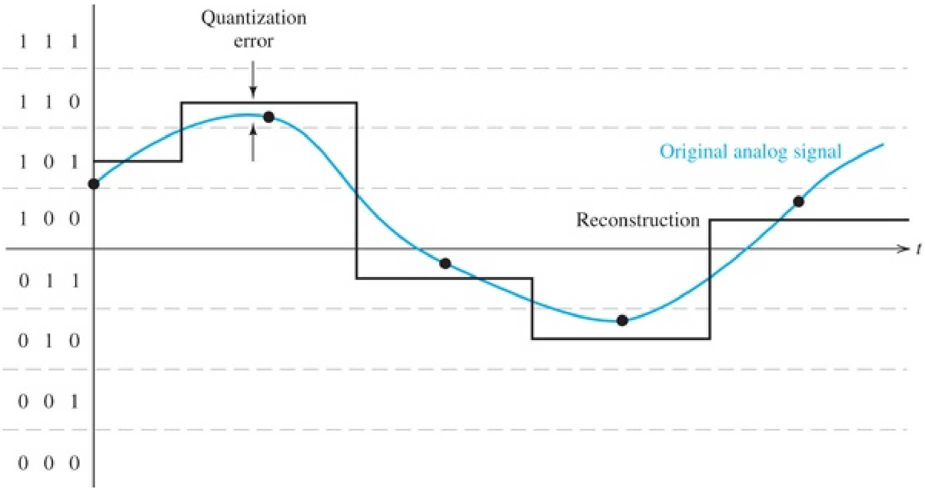


Figure 6.46
Quantization error occurs when an analog signal is reconstructed from its digital form.

$$\boxed{N = 2^k} \tag{6.50}$$

Hence, if we are using an 8-bit $(k = 8)$ ADC, there are $N = 2^8 = 256$ amplitude zones. In compact-disc technology, 16-bit words are used to represent sample values. With this number of bits, it is very difficult for a listener to detect the effects of quantization error on the reconstructed audio signal.

Often, in engineering instrumentation, we need to determine the DAC specifications needed for converting sensor signals to digital form. For example, suppose that we need to digitize a signal that ranges from -1 to $+1$ V with a resolution of at most $\Delta = 0.5\ \mathrm{mV}$. (Δ is illustrated in the upper right-hand corner of **Figure 6.45**.) Then, the minimum number of zones is the total signal range (2 V) divided by Δ, which yields $N = 4000$. However, *N* must be an integer power of two. Thus, we require $k = 12$. (In other words, a 12-bit ADC is needed.)

In the remainder of this section, we will ignore quantization error and assume that the exact sample values are available to the digital computer.

Digital Filters

We have seen that ADCs convert analog signals into sequences of code words that can accurately represent the amplitudes of the signals at the sampling instants. Although the computer actually manipulates code words that represent signal amplitudes, it is convenient to focus on the numbers that the code words represent. Conceptually, the signal $x(t)$ is converted into a list of values $x(nT)$ in which T is the interval between samples and n is a variable that takes on integer values. Often, we omit the sampling period from our notation and write the input and output samples simply as $x(n)$ and $y(n)$, respectively.

Digital Lowpass Filter

Digital filters can be designed to mimic the *RLC* filters that we discussed earlier in this chapter. For example, consider the first-order *RC* lowpass filter shown in **Figure 6.47**, in which we have denoted the input voltage as $x(t)$ and the output voltage as $y(t)$. Writing a Kirchhoff's current equation at the top node of the capacitance, we have

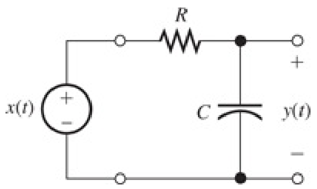


Figure 6.47
First-order *RC* lowpass filter.

$$\frac{y(t) - x(t)}{R} + C\frac{dy(t)}{dt} = 0 \tag{6.51}$$

Multiplying each term by *R* and using the fact that the time constant is $\tau = RC,$ we find that

$$y(t) - x(t) + \tau\frac{dy(t)}{dt} = 0 \tag{6.52}$$

We can approximate the derivative as

$$\frac{dy(t)}{dt} \cong \frac{\Delta y}{\Delta t} = \frac{y(n) - y(n-1)}{T} \tag{6.53}$$

and write the approximate equivalent to the differential equation

$$y(n) - x(n) + \tau\frac{y(n) - y(n-1)}{T} = 0 \tag{6.54}$$

This type of equation is sometimes called a **difference equation** because it involves differences between successive samples. Solving for the *n*th output value, we have

$$y(n) = ay(n-1) + (1-a)x(n) \tag{6.55}$$

in which we have defined the parameter

$$a = \frac{\tau / T}{1 + \tau / T} \tag{6.56}$$

Equation 6.55 defines the calculations that need to be carried out to perform lowpass filtering of the input *x*(*n*). For each sample point, the output is *a* times the previous output value plus $(1-a)$ times the present input value. Usually, we have $\tau >> T$ and *a* is slightly less than unity.

Example 6.11 Step Response of a First-Order Digital Lowpass Filter

Compute and plot the input and output samples for $n = 0$ to 20, given $a = 0.9$. The input is a step function defined by

$$\begin{aligned} x(n) &= 0 \text{ for } n < 0 \\ &= 1 \text{ for } n \geq 0 \end{aligned}$$

Assume that $y(n) = 0$ for $n < 0$.

Solution

We have

$$\begin{aligned} y(0) &= ay(-1) + (1-a)x(0) = 0.9 \times 0 + 0.1 \times 1 = 0.1 \\ y(1) &= ay(0) + (1-a)x(1) = 0.19 \\ y(2) &= ay(1) + (1-a)x(2) = 0.271 \\ &\cdots \\ y(20) &= 0.8906 \end{aligned}$$

Plots of $x(n)$ and $y(n)$ are shown in **Figure 6.48**. Notice that the response of the digital filter to a step input is very similar to that of the *RC* filter shown in **Figure 4.4** on page **172**.

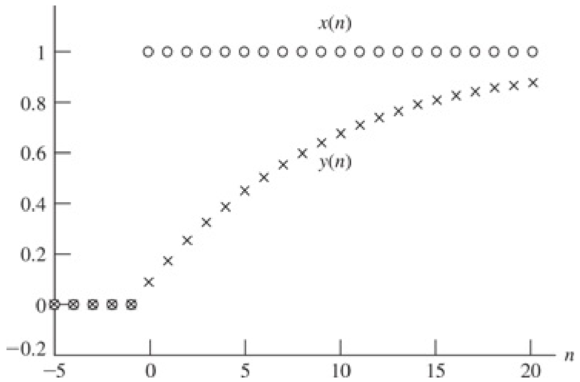


Figure 6.48
Step input and corresponding output of a first-order digital lowpass filter.

Exercise 6.23

a. Determine the value of the time constant τ, in terms of the sampling interval *T* corresponding to $a = 0.9$.

b. b. Recall that the time constant is the time required for the step response to reach $1 - \exp(-1) = 0.632$ times its final value. Estimate the value of the time constant for the response shown in **Figure 6.48**.

Answer

a. $\tau = 9T$;

b. $\tau \cong 9T$.

Other Digital Filters

We could develop digital bandpass, notch, or highpass filters that mimic the behavior of the *RLC* filters discussed earlier in this chapter. Furthermore, high-order digital filters are possible. In general, the equations defining such filters are of the form

$$y(n) = [C] \sum_{\ell=1}^{N} a_{\ell} y(n-\ell) + \sum_{k=0}^{M} b_k x(n-k) \tag{6.57}$$

The type of filter and its performance depend on the values selected for the coefficients a_{ℓ} and b_k. For the first-order lowpass filter considered in **Example 6.11**, the coefficients are $a_1 = 0.9,\ b_0 = 0.1,$ and all of the other coefficients are zero.

Exercise 6.24

Consider the *RC* highpass filter shown in **Figure 6.49**. Apply the method that we used for the lowpass filter to find an equation having the form of **Equation 6.57** for the highpass filter. Give expressions for the coefficients in terms of the time constant $\tau = RC$ and the sampling interval *T*.

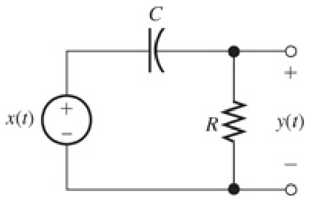


Figure 6.49
RC highpass filter. See **Exercise 6.24**.

Answer

$y(n) = a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$ in which

$$a_1 = b_0 = -b_1 = \frac{\tau / T}{1 + \tau / T}$$

A Simple Notch Filter

A simple way to obtain a notch filter is to select $a_\ell = 0$ for all ℓ, $b_0 = 0.5$, $b_d = 0.5$, and to set the remaining b_k coefficients to zero. Then, the output of the digital filter is given by

$$y(n) = 0.5x(n) + 0.5x(n-d) = 0.5[x(n) + x(n-d)]$$

Thus, each input sample is delayed in time by *Td* and added to the current sample. Finally, the sum of the input and its delayed version is multiplied by 0.5. To see that this results in a notch filter, consider a sinewave delayed by an interval *Td*. We can write

$$A\cos[\omega(t - Td)] = A\cos(\omega t - \omega Td) = A\cos(\omega t - \theta)$$

Hence, a time delay of *Td* amounts to a phase shift of ωTd radians or $fTd \times 360°$. (Keep in mind that, in this discussion, *T* represents the interval between samples, *not the period of the sinewave*.) For low frequencies, the phase shift is small, so the low-frequency components of *x*(*n*) add nearly in phase with those of $x(n-d)$. On the other hand, for the frequency

$$f_{\text{notch}} = \frac{1}{2Td} = \frac{f_s}{2d} \tag{6.58}$$

the phase shift is 180°. Of course, when we phase shift a sinewave by 180° and add it to the original, the sum is zero. Thus, any input component having the frequency f_{notch} does not appear in the output. The first-order lowpass filter and this simple notch filter are just two of many possible digital filters that can be realized by selection of the coefficient values in **Equation 6.57.**

Exercise 6.25

Suppose that the sampling frequency is $f_s = 10\text{ kHz}$, and we want to eliminate the 500-Hz component with a simple notch filter.

a. Determine the value needed for *d*.

b. What difficulty would be encountered if we wanted to eliminate the 300-Hz component?

Answer

a. $d = 10$;

b. **Equation 6.58** yields $d = 16.67$, but *d* is required to be an integer value.

Digital Filter Demonstration

Next, we will use MATLAB to demonstrate the operation of a digital filter. First, we will create samples of a virtual signal including noise and interference. The signal of interest consists of a 1-Hz sinewave and is representative of many types of real world signals such as delta waves contained in the electroencephalogram (EEG) of an individual in deep sleep, or the output of a pressure sensor submerged in the ocean with waves passing over. Part of the interference consists of a 60-Hz sinewave, which is a common real-world problem due to coupling between the ac power line and the signal sensor. The other part of the interference is random noise, which is also common in real-world data.

The MATLAB code that we use to create our simulated data is

```
t = 0:1/6000:2;
signal = cos(2*pi*t);
interference = cos(120*pi*t);
white_noise = randn(size(t));
noise = zeros(size(t));
for n = 2:12001
noise(n) = 0.25*(white_noise(n) - white_noise(n - 1));
end
x = signal + interference + noise; % This is the simulated data.
```

The first command generates a 12,001-element row vector containing the sample times for a two-second interval with a sampling frequency of $f_s = 6000\ \mathrm{Hz}$. The second and third commands set up row matrices containing samples of the signal and the 60-Hz interference. In the next line, the random-number generator feature of MATLAB generates "white noise" that contains components of equal amplitudes (on average) at all frequencies up to half of the sampling frequency. The white noise is then manipulated by the commands in the `for-end` loop, producing noise with components from dc to 3000 Hz peaking around 1500 Hz. Then, the signal, interference and noise are added to produce the simulated data $x(n)$. (Of course, in a real-world application, the data are obtained by applying the outputs of sensors, such as EEG electrodes, to analog-to-digital converters.)

Next, we use MATLAB to plot the signal, interference, noise, and the simulated data.

```
subplot(2,2,1)
plot(t, signal)
axis([0 2 -2 2])
subplot(2,2,2)
plot(t, interference)
axis([0 2 -2 2])
subplot(2,2,3)
plot(t,noise)
axis([0 2 -2 2])
subplot(2,2,4)
plot(t,x)
axis([0 2 -3 3])
```

The resulting plots are shown in **Figure 6.50**. The simulated data is typical of what is often obtained from sensors in real-world experiments. In a biomedical setting, for example, an electrocardiograph produces data that is the sum of the heart signal, 60-Hz power-line interference, and noise from muscle contractions, especially when the subject is moving, as in a stress test.

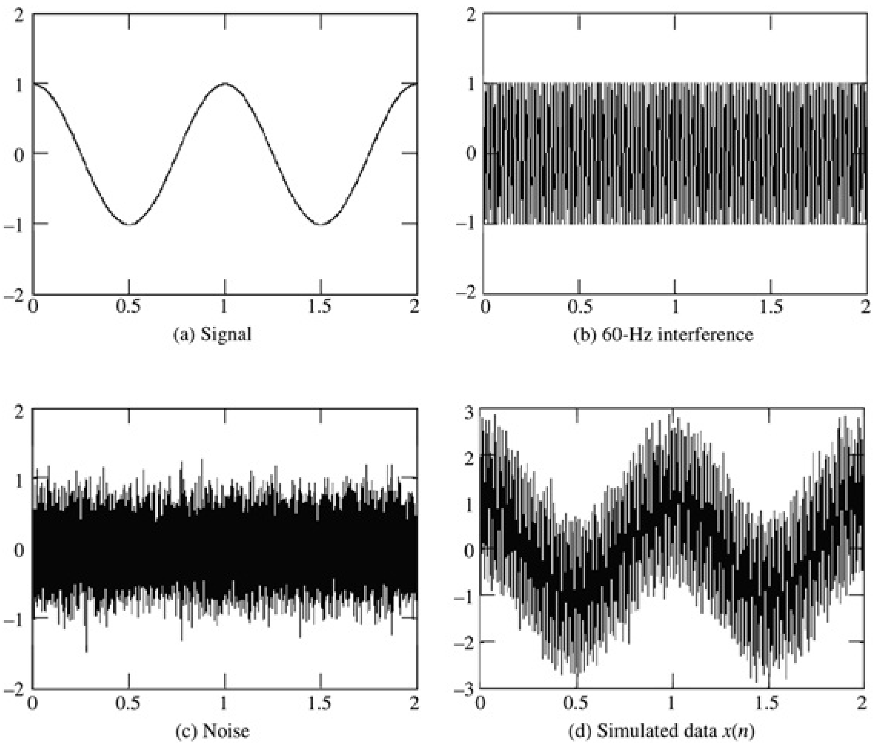


Figure 6.50

Simulated pressure-sensor output and its components.

Actually, the plot of the 60-Hz interference appears a little uneven in **Figure 6.50(b)** because of finite screen resolution for the display. This is a form of distortion called aliasing that occurs when the sampling rate is too low. If you run the commands on your own computer and use the zoom tool to expand the display horizontally, you will see a smooth plot of the 60-Hz sinewave interference. An m-file named DSPdemo that contains the commands used in this demonstration of a digital filter appears in the MATLAB folder.

What we need is a digital filter that processes the data $x(n)$ of **Figure 6.50(d)** and produces an output closely matching the signal in **Figure 6.50(a)**. This filter should pass the signal (1-Hz sinewave), reject the 60-Hz interference, and reject the noise, which has its largest components in the vicinity of 1500 Hz.

To achieve this, we will use a digital notch filter to remove the 60-Hz sinewave interference cascaded with a lowpass filter to remove most of the noise. The conceptual diagram of the digital filter is shown in **Figure 6.51**.

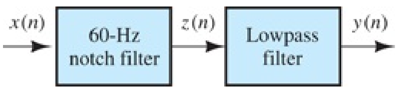


Figure 6.51

Digital filter.

Equation 6.58 reveals that by using $d = 50$ and $f_s = 6000\ \text{Hz},$ we can realize a notch filter with zero gain at precisely 60 Hz. (If 60-Hz interference is a problem, it is a good idea to pick the sampling frequency to be an even integer multiple of 60 Hz, which is one reason we picked the sampling frequency to be 6000 Hz.) The output $z(n)$ of the notch filter is given in terms of the input data $x(n)$ as

$$z(n) = \frac{1}{2}\left[x(n) + x(n-50)\right]$$

Also, we need a lowpass filter to eliminate the noise. We decide to use the first-order lowpass filter discussed earlier in this section. Because we do not want the lowpass filter to disturb the signal, we choose its break frequency to much higher than 1 Hz, say $f_B = 50 \text{ Hz}$. For an *RC* lowpass filter, the break frequency is

$$f_B = \frac{1}{2\pi RC}$$

Solving for the time constant and substituting values, we have

$$\tau = RC = \frac{1}{2\pi f_B} = \frac{1}{2\pi(50)} = 3.183 \text{ ms}$$

The gain constant for the (approximately) equivalent digital filter is given by **Equation 6.56** in which $T = 1/f_s = 1/6000 \text{ s}$ is the sampling interval. We then have

$$a = \frac{\tau/T}{1+\tau/T} = 0.9503$$

Substituting this value into **Equation 6.55** yields the equation for the present *y*(*n*) output of the lowpass filter in terms of its input *z*(*n*) and previous output $y(n-1)$.

$$y(n) = 0.9503y(n-1) + 0.0497z(n)$$

The MATLAB commands to filter the simulated data *x*(*n*) and plot the output *y*(*n*) are:

```
for n = 51:12001
z(n) = (x(n) + x(n - 50))/2; % This is the notch filter.
end
y = zeros(size(z));
for n = 2:12001
y(n) = 0.9503*y(n-1) + 0.0497*z(n); % This is the lowpass filter.
end
figure
plot(t,y)
```

The resulting plot is shown in **Figure 6.52**. As desired, the output is nearly identical to the 1-Hz sinewave signal. This relatively simple digital filter has done a very good job of eliminating the noise and interference because most of the noise and the interference have frequencies much higher than does the signal. When the frequencies of the signal are nearer to those of the noise and interference, we would need to resort to higher-order filters.

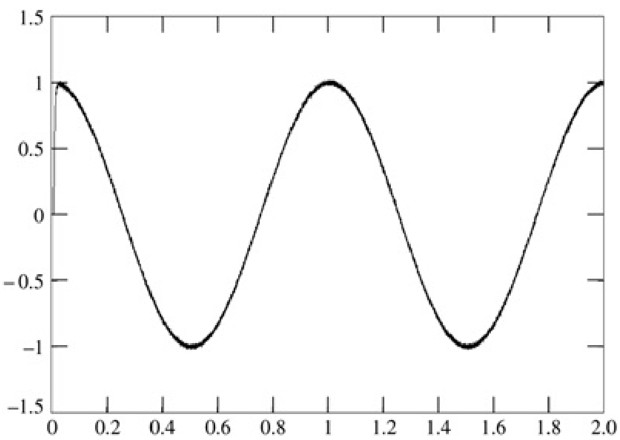


Figure 6.52
Output signal.

Comparison of Filter Technologies

We have discussed two ways to filter signals: *RLC* circuits and digital filters. There are a number of other filter types, such as **active filters** that are composed of resistances, capacitances and **operational amplifiers**, or **op amps** (which we discuss in **Chapter 14**). Other filters are based on mechanical resonances in piezoelectric crystals, surface acoustic waves, the propagation of electric fields in wave guides, switched capacitor networks, and transmission lines.

In all cases, the objective of a filter is to separate a desired signal from noise and interference. Radio amateurs operating in the frequency band between 28 and 29.7 MHz often need to place a band reject filter between the transmitter and antenna to eliminate second-harmonic frequency components from reaching the antenna. If they are not removed, second-harmonic components can cause some very annoying interference on their neighbor's television screens. In this application, an *RLC* filter would be the technology of choice because of the large currents and voltages involved.

> The objective of a filter is to separate a desired signal from noise and interference.

On the other hand, a sleep researcher may wish to filter brain waves to separate delta waves that appear at frequencies of 4 Hz or less from higher frequency brain waves. In this case, a digital filter is appropriate.

In summary, there are many applications for filters and many technologies for implementing filters. Most of the principles we have introduced in our discussion of *RLC* circuits and digital filters apply to filters based on other technologies.

Summary

1. The fundamental concept of Fourier theory is that we can construct any signal by adding sinusoids with the proper amplitudes, frequencies, and phases.
2. In effect, a filter decomposes the input signal into its sinusoidal components, adjusts the amplitude and phase of each component, depending on its frequency, and sums the adjusted components to produce the output signal. Often, we need a filter that passes components in a given frequency range to the output, without change in amplitude or phase, and that rejects components at other frequencies.
3. The transfer function of a filter circuit is the phasor output divided by the phasor input as a function of frequency. The transfer function is a complex quantity that shows how the amplitudes and phases of input components are affected when passing through the filter.
4. We can use circuit analysis with phasors and complex impedances to determine the transfer function of a given circuit.
5. A first-order filter is characterized by its half-power frequency f_B.
6. A transfer-function magnitude is converted to decibels by taking 20 times the common logarithm of the magnitude.
7. Two-port filters are cascaded by connecting the output of the first to the input of the second. The overall transfer function of the cascade is the product of the transfer functions of the individual filters. If the transfer functions are converted to decibels, they are added for a cascade connection.
8. On a logarithmic frequency scale, frequency is multiplied by a given factor for equal increments of length along the axis. A decade is a range of frequencies for which the ratio of the highest frequency to the lowest is 10. An octave is a two-to-one change in frequency.
9. A Bode plot shows the magnitude of a network function in decibels versus frequency, using a logarithmic scale for frequency.
10. The Bode plots for first-order filters can be closely approximated by straight-line asymptotes. In the case of a first-order lowpass filter, the transfer-function magnitude slopes downward at 20 dB/decade for frequencies that are higher than the half-power frequency. For a first-order highpass filter, the transfer-function magnitude slopes at 20 dB/decade below the break frequency.
11. At low frequencies, inductances behave as short circuits, and capacitances behave as open circuits. At high frequencies, inductances behave as open circuits, and capacitances behave as short circuits. Often, *RLC* filters can be readily analyzed at low- or high-frequencies, providing checks on computer-generated Bode plots.
12. The key parameters of series and parallel resonant circuits are the resonant frequency and quality factor. The impedance of either type of circuit is purely resistive at the resonant frequency. High-quality-factor circuits can have responses that are much larger in magnitude than the driving source.
13. Filters may be classified as lowpass, highpass, bandpass, and band-reject filters. Ideal filters have constant (nonzero) gain (transfer-function magnitude) in the passband and zero gain in the stopband.
14. The series resonant circuit can be used to form any of the four filter types.
15. A second-order filter is characterized by its resonant frequency and quality factor.
16. MATLAB is useful in deriving and plotting network functions of complex *RLC* filters.
17. In using DSP to filter a signal, the analog input signal *x*(*t*) is converted to digital form (a sequence of numbers) by an ADC. A digital computer uses the digitized input signal to compute a sequence of values for the output signal, and, finally, (if desired) the computed values are converted to analog form by a DAC to produce the output signal *y*(*t*).
18. If a signal contains no components with frequencies higher than $f_H,$ the signal can be exactly reconstructed from its samples, provided that the sampling rate f_s is selected to be more than twice

f_H.

19. Approximately equivalent digital filters can be found for *RLC* filters.

Problems

Section 6.1: Fourier Analysis, Filters, and Transfer Functions

P6.1. What is the fundamental concept of Fourier theory?

P6.2. The triangular waveform shown in **Figure P6.2** can be written as the infinite sum

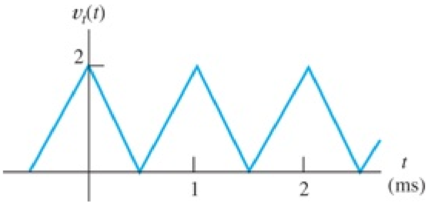


Figure P6.2

$$
\begin{aligned}
v_t(t) = 1 &+ \frac{8}{\pi^2}\cos(2000\pi t) \\
&+ \frac{8}{(3\pi)^2}\cos(6000\pi t) + \cdots \\
&+ \frac{8}{(n\pi)^2}\cos(2000n\pi t) + \cdots
\end{aligned}
$$

in which *n* takes odd integer values only. Use MATLAB to compute and plot the sum through $n = 19$ for $0 \leq t \leq 2\ \text{ms}$. Compare your plot with the waveform shown in **Figure P6.2**.

P6.3. The full-wave rectified cosine wave shown in **Figure P6.3** can be written as

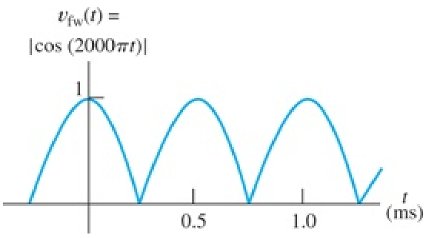


Figure P6.3

$$
\begin{aligned}
v_{fw} &= \frac{2}{\pi} + \frac{4}{\pi(1)(3)}\cos(4000\pi t) \\
&- \frac{4}{\pi(3)(5)}\cos(8000\pi t) + \cdots \\
&+ \frac{4(-1)^{(n/2+1)}}{\pi(n-1)(n+1)}\cos(2000n\pi t) + \cdots
\end{aligned}
$$

in which *n* assumes even integer values. Use MATLAB to compute and plot the sum through $n = 60$ for $0 \leq t \leq 2\ \text{ms}$. Compare your plot with the waveform shown in **Figure P6.3**.

P6.4. The Fourier series for the **half-wave rectified cosine** shown in **Figure P6.4** is

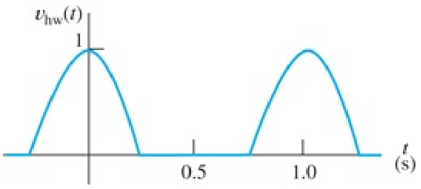


Figure P6.4

$$v_{hw}(t) = \frac{1}{\pi} + \frac{1}{2}\cos(2\pi t) + \frac{2}{\pi(1)(3)}\cos(4\pi t)$$
$$- \frac{2}{\pi(3)(5)}\cos(8\pi t) + \cdots$$
$$+ \frac{2(-1)^{(n/2+1)}}{\pi(n-1)(n+1)}\cos(2n\pi t) + \cdots$$

in which $n = 2, 4, 6,$ etc. Use MATLAB to compute and plot the sum through $n = 4$ for $-0.5 \le t \le 1.5$ s. Then plot the sum through $n = 50.$ Compare your plots with the waveform in **Figure P6.4**.

P6.5. Fourier analysis shows that the **sawtooth waveform** of **Figure P6.5** can be written as

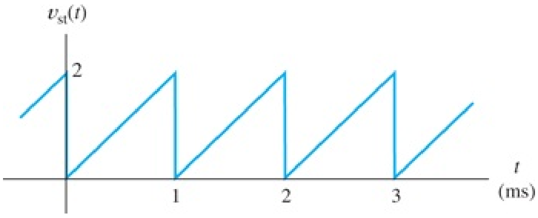


Figure P6.5

$$v_{st}(t) = 1 - \frac{2}{\pi}\sin(2000\pi t)$$
$$- \frac{2}{2\pi}\sin(4000\pi t) - \frac{2}{3\pi}\sin(6000\pi t)$$
$$- \cdots - \frac{2}{n\pi}\sin(2000n\pi t) - \cdots$$

Use MATLAB to compute and plot the sum through $n = 3$ for $0 \le t \le 2$ ms. Repeat for the sum through $n = 50.$

P6.6. What is the transfer function of a filter? Describe how the transfer function of a filter can be determined using laboratory methods.

P6.7. How does a filter process an input signal to produce the output signal in terms of sinusoidal components?

***P6.8.** The transfer function $H(f) = V_{out}/V_{in}$ of a filter is shown in **Figure P6.8**. The input signal is given by

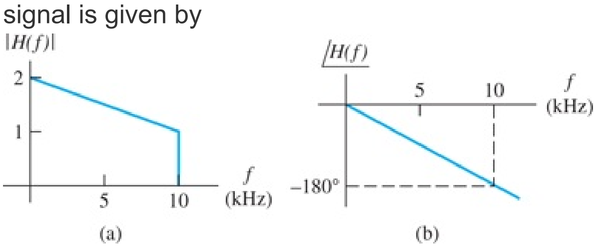


Figure P6.8

$$v_{in}(t) = 5 + 2\cos(5000\pi t + 30°)$$
$$+ 2\cos(15000\pi t)$$

Find an expression (as a function of time) for the steady-state output of the filter.

P6.9. Repeat **Problem P6.8** for the input voltage given by

$$v_{\text{in}}(t) = 4 + 5\cos(10^4\pi t - 30^\circ) + 2\sin(24000\pi t)$$

P6.10. Repeat **Problem P6.8** for the input voltage given by

$$v_{\text{in}}(t) = 6 + 2\cos(6000\pi t) - 4\cos(12000\pi t)$$

***P6.11.** The input to a certain filter is given by

$$v_{\text{in}}(t) = 2\cos(10^4\pi t - 25^\circ)$$

and the steady-state output is given by

$$v_{\text{out}}(t) = 2\cos(10^4\pi t + 20^\circ)$$

Determine the (complex) value of the transfer function of the filter for $f = 5000\ \text{Hz}$.

***P6.12.** The input and output voltages of a filter operating under sinusoidal steady-state conditions are observed on an oscilloscope. The peak amplitude of the input is 5 V and the output is 15 V. The period of both signals is 4 ms. The input reaches a positive peak at $t = 1\ \text{ms},$ and the output reaches its positive peak at $t = 1.5\ \text{ms}$. Determine the frequency and the corresponding value of the transfer function.

***P6.13.** The triangular waveform of **Problem P6.2** is the input for a filter with the transfer function shown in **Figure P6.13**. Assume that the phase of the transfer function is zero for all frequencies. Determine the steady-state output of the filter.

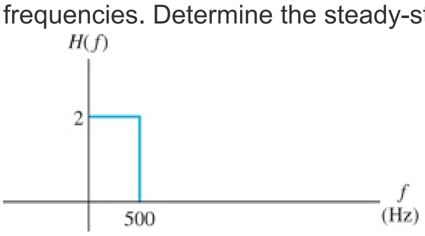


Figure P6.13

***P6.14.** Consider a circuit for which the output voltage is the running-time integral of the input voltage, as illustrated in **Figure P6.14**. If the input voltage is given by $v_{\text{in}}(t) = V_{\max}\cos(2\pi f t)$, find an expression for the output voltage as a function of time. Then, find an expression for the transfer function of the integrator. Plot the magnitude and phase of the transfer function versus frequency.

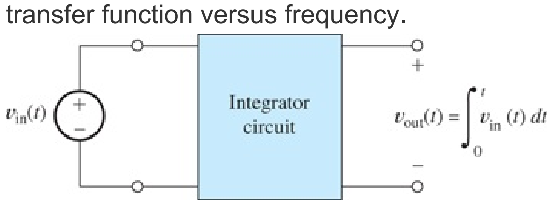


Figure P6.14

P6.15. The sawtooth waveform of **Problem P6.5** is applied as the input to a filter with the transfer function shown in **Figure P6.15**. Assume that the phase of the transfer function is zero for all frequencies. Determine the steady-state output of the filter.

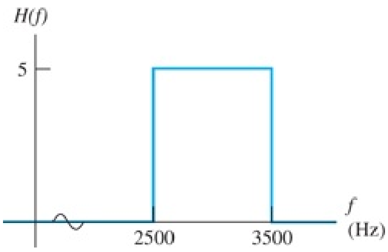


Figure P6.15

P6.16. Figure P6.16 shows the input and output voltages of a certain filter operating in steady state with a sinusoidal input. Determine the frequency and the corresponding value of the transfer function.

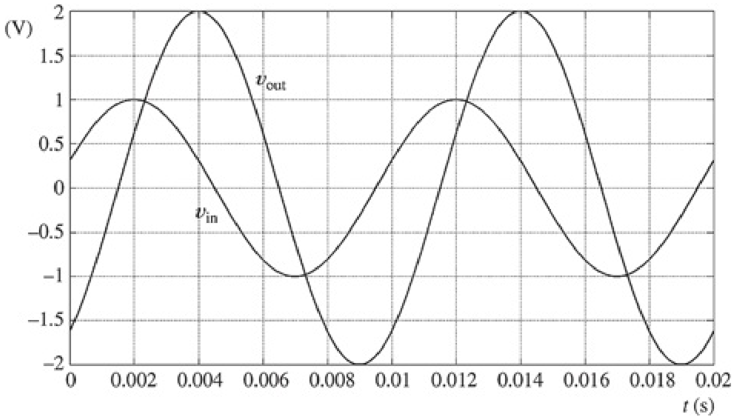


Figure P6.16

P6.17. List the frequencies in hertz for which the transfer function of a filter can be determined given that the input to the filter is

$$v_{in}(t) = 2 + 3\cos(1000\pi t) + 3\sin(2000\pi t) + \cos(3000\pi t)\,\mathrm{V}$$

and the output is

$$v_{out}(t) = 3 + 2\cos(1000\pi t + 30°) + 3\cos(3000\pi t)\,\mathrm{V}$$

Compute the transfer function for each of these frequencies.

P6.18. Consider a system for which the output voltage is $v_o(t) = v_{in}(t) + v_{in}\left(t - 10^{-3}\right)$. (In other words, the output equals the input plus the input delayed by 1 ms.) Given that the input voltage is $v_{in}(t) = V_{max}\cos(2\pi f t)$, find an expression for the output voltage as a function of time. Then, find an expression for the transfer function of the system. Use MATLAB to plot the magnitude of the transfer function versus frequency for the range from 0 to 2000 Hz. Comment on the result.

P6.19. Suppose we have a system for which the output voltage is

$$v_o(t) = 1000\int_{t-10^{-3}}^{t} v_{in}(t)\,dt$$

Given the input voltage $v_{in}(t) = V_{max}\cos(2\pi f t)$, find an expression for the output voltage as a function of time. Then, find an expression for the transfer function of the system. Use MATLAB to plot the magnitude of the transfer function versus frequency for the range from 0 to 2000 Hz. Comment on the result.

P6.20. Suppose we have a circuit for which the output voltage is the time derivative of the input voltage, as illustrated in **Figure P6.20**. For an input voltage given by $v_{\text{in}}(t) = V_{\max}\cos(2\pi f t)$, find an expression for the output voltage as a function of time. Then, find an expression for the transfer function of the differentiator. Plot the magnitude and phase of the transfer function versus frequency.

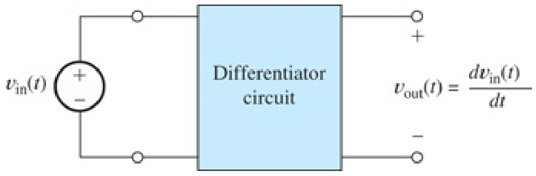


Figure P6.20

Section 6.2: First-Order Lowpass Filters

P6.21. Draw the circuit diagram of a first-order *RC* lowpass filter and give the expression for the half-power frequency in terms of the circuit components. Sketch the magnitude and phase of the transfer function versus frequency.
P6.22. Repeat **Problem P6.21** for a first-order *RL* filter.
***P6.23.** Consider a first-order *RC* lowpass filter. At what frequency (in terms of f_B) is the phase shift equal to $-1°? -10°? -89°?$
P6.24. In **Chapter 4**, we used the time constant to characterize first-order *RC* circuits. Find the relationship between the half-power frequency and the time constant.
***P6.25.** An input signal given by

$$v_{\text{in}}(t) = 5\cos(500\pi t) + 5\cos(1000\pi t) + 5\cos(2000\pi t)$$

is applied to the lowpass *RC* filter shown in **Figure P6.25**. Find an expression for the output signal.

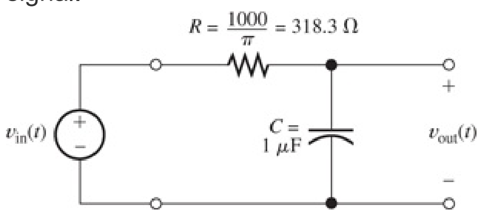


Figure P6.25

P6.26. The input signal of a first-order lowpass filter with the transfer function given by **Equation 6.9** on page 297 and a half-power frequency of 200 Hz is

$$v_{\text{in}}(t) = 3 + 2\sin(800\pi t + 30°) + 5\cos\left(20 \times 10^3 \pi t\right)$$

Find an expression for the output voltage.
P6.27. Suppose that we need a first-order *RC* lowpass filter with a half-power frequency of 1 kHz. Determine the value of the capacitance, given that the resistance is $5\ \text{k}\Omega$.
P6.28. The input signal to a filter contains components that range in frequency from 100 Hz to 50 kHz. We wish to reduce the amplitude of the 50-kHz component by a factor of 200 by passing the signal through a first-order lowpass filter. What half-power frequency is required for the filter? By what factor is a component at 2 kHz changed in amplitude in passing through this filter?
P6.29. Suppose we have a first-order lowpass filter that is operating in sinusoidal steady-state conditions at a frequency of 5 kHz. Using an oscilloscope, we observe that the positive-going zero

crossing of the output is delayed by $30\ \mu\mathrm{s}$ compared with that of the input. Determine the break frequency of the filter.

***P6.30.** Sketch the magnitude of the transfer function $H(f) = V_{\mathrm{out}}/V_{\mathrm{in}}$ to scale versus frequency for the circuit shown in **Figure P6.30**. What is the value of the half-power frequency? [*Hint:* Start by finding the Thévenin equivalent circuit seen by the capacitance.]

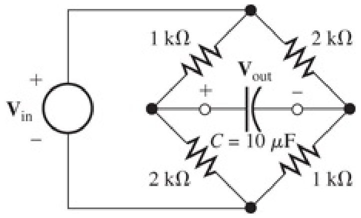


Figure P6.30

P6.31. In steady-state operation, a first-order *RC* lowpass filter has the input signal $v_{\mathrm{in}}(t) = 5\cos\left(20\times10^3\pi t\right)$ and the output signal $v_{\mathrm{out}}(t) = 0.2\cos\left(20\times10^3\pi t-\theta\right)$. Determine the break frequency of the filter and the value of θ.

P6.32. Consider the circuit shown in **Figure P6.32(a)**. This circuit consists of a source having an internal resistance of R_s, an *RC* lowpass filter, and a load resistance R_L.

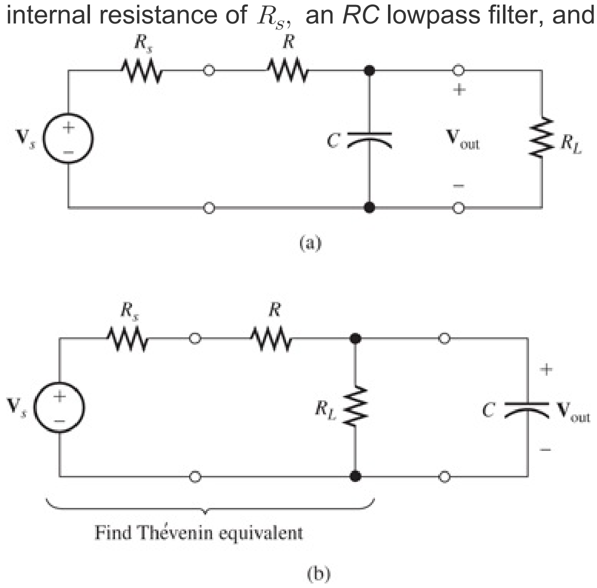


Figure P6.32

a. Show that the transfer function of this circuit is given by

$$H(f) = \frac{V_{\mathrm{out}}}{V_s} = \frac{R_L}{R_s+R+R_L}\times\frac{1}{1+j(f/f_B)}$$

in which the half-power frequency f_B is given by

$$f_B = \frac{1}{2\pi R_t C} \quad \text{where} \quad R_t = \frac{R_L(R_s+R)}{R_L+R_s+R}$$

Notice that R_t is the parallel combination of R_L and (R_s+R). [*Hint:* One way to make this problem easier is to rearrange the circuit as shown in **Figure P6.32(b)** and then to find the Thévenin equivalent for the source and resistances.]

b. Given that $C = 0.2\ \mu\mathrm{F}$, $R_s = 2\ \mathrm{k}\Omega$, $R = 47\ \mathrm{k}\Omega$, and $R_L = 1\ \mathrm{k}\Omega$, sketch (or use MATLAB to plot) the magnitude of *H*(*f*) to scale versus f/f_B from 0 to 3.

P6.33.

a. Derive an expression for the transfer function $H(f) = V_{out}/V_{in}$ for the circuit shown in **Figure P6.33**. Find an expression for the half-power frequency.

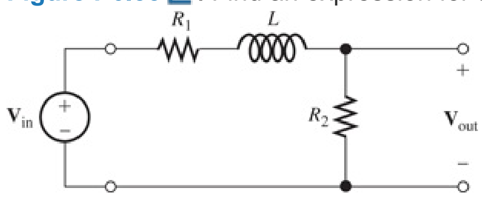


Figure P6.33

b. Given $R_1 = 50\ \Omega,\ R_2 = 50\ \Omega,$ and $L = 15\ \mu\text{H},$ sketch (or use MATLAB to plot) the magnitude of the transfer function versus frequency.

P6.34. We apply a 5-V-rms 20-kHz sinusoid to the input of a first-order *RC* lowpass filter, and the output voltage in steady state is 0.5 V rms. Predict the steady-state rms output voltage after the frequency of the input signal is raised to 150 kHz and the amplitude remains constant.

P6.35. Perhaps surprisingly, we can apply the transfer-function concept to mechanical systems. Suppose we have a mass *m* moving through a liquid with an applied force *f* and velocity *v*. The motion of the mass is described by the first-order differential equation

$$f = m\frac{dv}{dt} + kv$$

in which *k* is the coefficient of viscous friction. Find an expression for the transfer function

$$H(f) = \frac{\mathbf{V}}{\mathbf{F}}$$

Also, find the half-power frequency (defined as the frequency at which the transfer function magnitude is $1/\sqrt{2}$ times its dc value) in terms of *k* and *m*. [*Hint:* To determine the transfer function, assume a steady-state sinusoidal velocity $v = V_m \cos(2\pi f t)$, solve for the force, and take the ratio of their phasors.]

Section 6.3: Decibels, the Cascade Connection, and Logarithmic Frequency Scales

P6.36. What is a logarithmic frequency scale? A linear frequency scale?

P6.37. What is a notch filter? What is one application?

P6.38. What is the main advantage of converting transfer function magnitudes to decibels before plotting?

P6.39. What is the passband of a filter?

***P6.40.**

a. Given $|H(f)|_{dB} = -10\ \text{dB},$ find $|H(f)|$.
b. Repeat for $|H(f)|_{dB} = 10\ \text{dB}$.

***P6.41.**

a. What frequency is halfway between 100 and 3000 Hz on a logarithmic frequency scale?
b. On a linear frequency scale?

P6.42. Find the decibel equivalent for $|H(f)| = 0.5.$ Repeat for $|H(f)| = 2,\ |H(f)| = 1/\sqrt{2} \cong 0.7071,$ and $|H(f)| = \sqrt{2}$.

P6.43. Find the frequency that is

a. one octave higher than 800 Hz;
b. two octaves lower;
c. two decades lower;
d. one decade higher.

P6.44. Explain what we mean when we say that two filters are cascaded.

P6.45. We have a list of successive frequencies 2, f_1, f_2, f_3, 50 Hz. Determine the values of f_1, f_2, and f_3 so that the frequencies are evenly spaced on:

a. a linear frequency scale, and
b. a logarithmic frequency scale.

***P6.46.** Two first-order lowpass filters are in cascade as shown in **Figure P6.46**. The transfer functions are

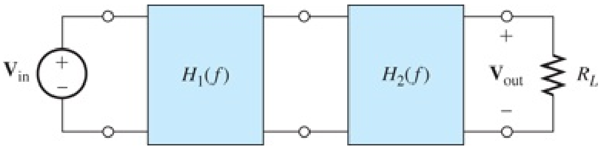


Figure P6.46

$$H_1(f) = H_2(f) = \frac{1}{1 + j(f/f_B)}$$

a. Write an expression for the overall transfer function.
b. Find an expression for the half-power frequency for the overall transfer function in terms of f_B.

[Comment: This filter cannot be implemented by cascading two simple *RC* lowpass filters like the one shown in **Figure 6.7** on page **296** because the transfer function of the first circuit is changed when the second is connected. Instead, a buffer amplifier, such as the voltage follower discussed in **Section 14.3**, must be inserted between the *RC* filters.]

P6.47. How many decades are between $f_1 = 20$ Hz and $f_2 = 4.5$ kHz?

a. How many octaves?

P6.48. We have two filters with transfer functions $H_1(f)$ and $H_2(f)$ cascaded in the order 1–2. Give the expression for the overall transfer function of the cascade. Repeat if the transfer function magnitudes are expressed in decibels denoted as $|H_1(f)|_{\text{dB}}$ and $|H_2(f)|_{\text{dB}}$. What caution concerning $H_1(f)$ must be considered?

P6.49. Two filters are in cascade. At a given frequency f_1, the transfer function values are $|H_1(f_1)|_{\text{dB}} = -30$ and $|H_2(f_1)|_{\text{dB}} = +10$. Find the magnitude of the overall transfer function in decibels at $f = f_1$.

Section 6.4: Bode Plots

P6.50. What is a Bode plot?

P6.51. What is the slope of the high-frequency asymptote for the Bode magnitude plot for a first-order lowpass filter? The low-frequency asymptote? At what frequency do the asymptotes meet?

***P6.52.** A transfer function is given by

$$H(f) = \frac{100}{1 + j(f/1000)}$$

Sketch the asymptotic magnitude and phase Bode plots to scale.

P6.53. Suppose that three filters, having identical first-order lowpass transfer functions, are cascaded, what will be the rate at which the overall transfer function magnitude declines above the break frequency? Explain.

P6.54. Solve for the transfer function $H(f) = V_{out}/V_{in}$ and sketch the asymptotic Bode magnitude and phase plots to scale for the circuit shown in **Figure P6.54**.

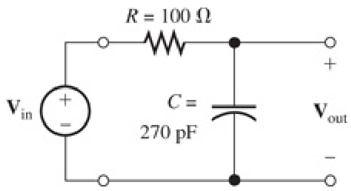


Figure P6.54

P6.55. A transfer function is given by

$$H(f) = \frac{10}{1 - j(f/500)}$$

Sketch the asymptotic magnitude and phase Bode plots to scale. What is the value of the half-power frequency?

P6.56. Consider a circuit for which

$$v_{out}(t) = v_{in}(t) - 200\pi \int_0^t v_{out}(t)\, dt$$

a. Assume that $v_{out}(t) = A\cos(2\pi f t)$, and find an expression for $v_{in}(t)$.
b. Use the results of part (a) to find an expression for the transfer function $H(f) = \mathbf{V}_{out}/\mathbf{V}_{in}$ for the system.
c. Draw the asymptotic Bode plot for the transfer function magnitude.

P6.57. Solve for the transfer function $H(f) = \mathbf{V}_{out}/\mathbf{V}_{in}$ and draw the asymptotic Bode magnitude and phase plots for the circuit shown in **Figure P6.57**.

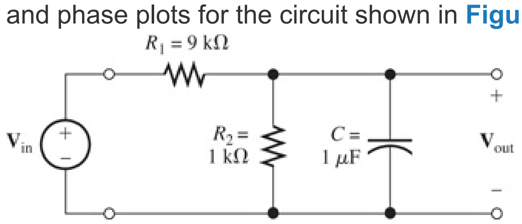


Figure P6.57

P6.58. Sketch the asymptotic magnitude and phase Bode plots to scale for the transfer function

$$H(f) = \frac{1 - j(f/100)}{1 + j(f/100)}$$

P6.59. Solve for the transfer function $H(f) = \mathbf{V}_{out}/\mathbf{V}_{in}$ and draw the Bode magnitude and phase plots for the circuit shown in **Figure P6.59**.

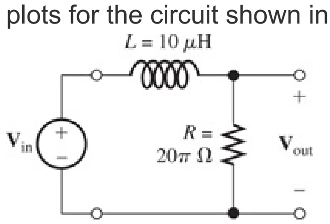


Figure P6.59

***P6.60.** In solving **Problem P6.14**, we find that the transfer function of an integrator circuit is given by $H(f) = 1/(j2\pi f)$. Sketch the Bode magnitude and phase plots to scale. What is the slope of the magnitude plot?

P6.61. In solving **Problem P6.20**, we find that the transfer function of a differentiator circuit is given by $H(f) = j2\pi f$. Sketch the Bode magnitude and phase plots to scale. What is the slope of the magnitude plot?

Section 6.5: First-Order Highpass Filters

P6.62. Draw the circuit diagram of a first-order *RC* highpass filter and give the expression for the half-power frequency in terms of the circuit components.
P6.63. What is the slope of the high-frequency asymptote for the Bode magnitude plot for a first-order highpass filter? The low-frequency asymptote? At what frequency do the asymptotes meet?
***P6.64.** Consider the circuit shown in **Figure P6.64**. Sketch the asymptotic Bode magnitude and phase plots to scale for the transfer function $H(f) = V_{out}/V_{in}$.

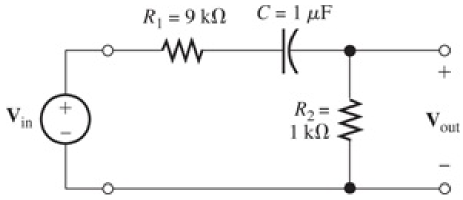


Figure P6.64

***P6.65.** Consider the first-order highpass filter shown in **Figure P6.65**. The input signal is given by

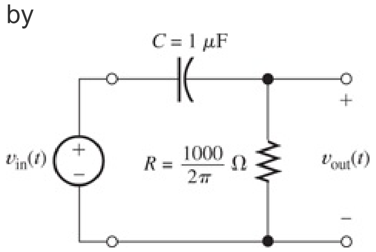


Figure P6.65

$$v_{in}(t) = 5 + 5\cos(2000\pi t)$$

Find an expression for the output $v_{out}(t)$ in steady-state conditions.
P6.66. Repeat **Problem P6.65** for the input signal given by

$$v_{in}(t) = 10\cos(400\pi t) + 20\cos(4000\pi t)$$

P6.67. Suppose we need a first-order highpass filter (such as **Figure 6.19** on page **309**) to attenuate a 60-Hz input component by 60 dB. What value is required for the break frequency of the filter? By how many dB is the 600-Hz component attenuated by this filter? If $R = 5\ k\Omega$, what is the value of *C*?
P6.68. Consider the circuit shown in **Figure P6.68**. Sketch the Bode magnitude and phase plots to scale for the transfer function $H(f) = V_{out}/V_{in}$.

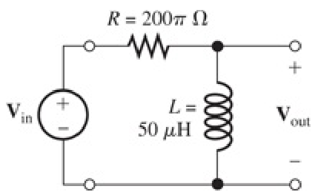


Figure P6.68

P6.69. Consider the circuit shown in **Figure P6.69**. Sketch the Bode magnitude and phase plots to scale for the transfer function $H(f) = V_{out}/V_{in}$.

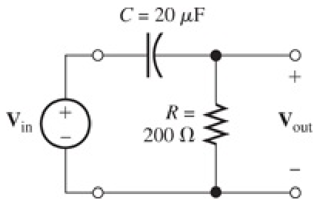


Figure P6.69

Section 6.6: Series Resonance

P6.70. What can you say about the impedance of a series *RLC* circuit at the resonant frequency? How are the resonant frequency and the quality factor defined?

P6.71. What is a *bandpass filter*? How is its bandwidth defined?

***P6.72.** Consider the series resonant circuit shown in **Figure P6.72**, with $L = 20\ \mu\text{H},\ R = 14.14\ \Omega,$ and $C = 1000\ \text{pF}$. Compute the resonant frequency, the bandwidth, and the half-power frequencies. Assuming that the frequency of the source is the same as the resonant frequency, find the phasor voltages across the elements and sketch a phasor diagram.

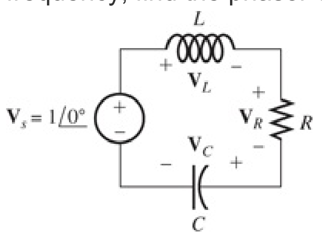


Figure P6.72

P6.73. Work **Problem P6.72** for $L = 80\ \mu\text{H},\ R = 14.14\ \Omega,$ and $C = 1000\ \text{pF}$.

P6.74. Suppose we have a series resonant circuit for which $B = 30\ \text{kHz},\ f_0 = 300\ \text{kHz},$ and $R = 40\ \Omega$. Determine the values of *L* and *C*.

***P6.75.** At the resonant frequency $f_0 = 1\ \text{MHz},$ a series resonant circuit with $R = 50\ \Omega$ has $|V_R| = 2\ \text{V}$ and $|V_L| = 20\ \text{V}$. Determine the values of *L* and *C*. What is the value of $|V_C|$?

P6.76. Suppose we have a series resonant circuit for which $f_0 = 12\ \text{MHz}$ and $B = 600\ \text{kHz}$. Furthermore, the minimum value of the impedance magnitude is $20\ \Omega$. Determine the values of *R, L*, and *C*.

P6.77. Derive an expression for the resonant frequency of the circuit shown in **Figure P6.77**. (Recall that we have defined the resonant frequency to be the frequency for which the impedance is purely resistive.)

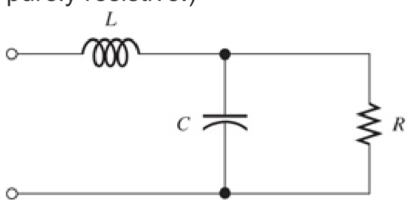


Figure P6.77

Section 6.7: Parallel Resonance

P6.78. What can you say about the impedance of a parallel *RLC* circuit at the resonant frequency? How is the resonant frequency defined? Compare the definition of quality factor for the parallel resonant circuit with that for the series resonant circuit.
***P6.79.** A parallel resonant circuit has $R = 5\ \mathrm{k\Omega}, L = 50\ \mu\mathrm{H},$ and $C = 200\ \mathrm{pF}$. Determine the resonant frequency, quality factor, and bandwidth.
P6.80. A parallel resonant circuit has $f_0 = 20\ \mathrm{MHz}$ and $B = 200\ \mathrm{kHz}$. The maximum value of $|Z_p|$ is $5\ \mathrm{k\Omega}$. Determine the values of *R, L*, and *C*.
P6.81. Consider the parallel resonant circuit shown in **Figure 6.29** on page 319. Determine the *L* and *C* values, given $R = 1\ \mathrm{k\Omega},\ f_0 = 10\ \mathrm{MHz},$ and $B = 500\ \mathrm{kHz}$. If $\boldsymbol{I} = 10^{-3}\angle 0°,$ draw a phasor diagram showing the currents through each of the elements in the circuit at resonance.
P6.82. A parallel resonant circuit has $f_0 = 100\ \mathrm{MHz},\ B = 5\ \mathrm{MHz},$ and $R = 2\ \mathrm{k\Omega}$. Determine the values of *L* and *C*.

Section 6.8: Ideal and Second-Order Filters

P6.83. Name four types of ideal filters and sketch their transfer functions.
***P6.84.** An ideal bandpass filter has cutoff frequencies of 9 and 11 kHz and a gain magnitude of two in the passband. Sketch the transfer-function magnitude to scale versus frequency. Repeat for an ideal band-reject filter.
P6.85. An ideal lowpass filter has a cutoff frequency of 10 kHz and a gain magnitude of two in the passband. Sketch the transfer-function magnitude to scale versus frequency. Repeat for an ideal highpass filter.
P6.86. Each AM radio signal has components ranging from 10 kHz below its carrier frequency to 10 kHz above its carrier frequency. Various radio stations in a given geographical region are assigned different carrier frequencies so that the frequency ranges of the signals do not overlap. Suppose that a certain AM radio transmitter has a carrier frequency of 980 kHz. What type of filter should be used if we want the filter to pass the components from this transmitter and reject the components of all other transmitters? What are the best values for the cutoff frequencies?
P6.87. In an electrocardiograph, the heart signals contain components with frequencies ranging from dc to 100 Hz. During exercise on a treadmill, the signal obtained from the electrodes also contains noise generated by muscle contractions. Most of the noise components have frequencies exceeding 100 Hz. What type of filter should be used to reduce the noise? What cutoff frequency is appropriate?
***P6.88.** Draw the circuit diagram of a second-order highpass filter. Suppose that $R = 1\ \mathrm{k\Omega},\ Q_s = 1,$ and $f_0 = 100\ \mathrm{kHz}$. Determine the values of *L* and *C*.
P6.89. Draw the circuit diagram of a second-order highpass filter. Given that $R = 50\ \Omega,\ Q_s = 0.5,$ and $f_0 = 30\ \mathrm{MHz},$ determine the values of *L* and *C*.
P6.90. Suppose that sinewave interference has been inadvertently added to an audio signal that has frequency components ranging from 20 Hz to 15 kHz. The frequency of the interference slowly varies in the range 950 to 1050 Hz. A filter that attenuates the interference by at least 20 dB and passes most of the audio components is desired. What type of filter is needed? Sketch the magnitude Bode plot of a suitable filter, labeling its specifications.

Section 6.9: Transfer Functions and Bode Plots with MATLAB

P6.91. Consider the filter shown in **Figure P6.91**.

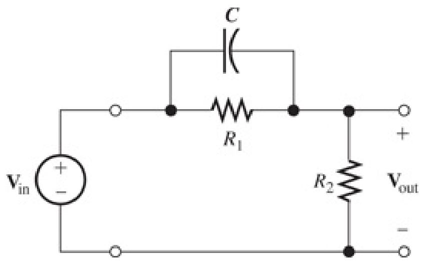


Figure P6.91

a. Derive an expression for the transfer function $H(f) = \boldsymbol{V}_{\text{out}} / \boldsymbol{V}_{\text{in}}$.
b. Use MATLAB to obtain a Bode plot of the transfer-function magnitude for $R_1 = 9\ \text{k}\Omega,\ R_2 = 1\ \text{k}\Omega,$ and $C = 0.01\ \mu\text{F}$. Allow frequency to range from 10 Hz to 1 MHz.
c. At very low frequencies, the capacitance becomes an open circuit. In this case, determine an expression for the transfer function and evaluate for the circuit parameters of part (b). Does the result agree with the value plotted in part (b)?
d. At very high frequencies, the capacitance becomes a short circuit. In this case, determine an expression for the transfer function and evaluate for the circuit parameters of part (b). Does the result agree with the value plotted in part (b)?

P6.92. Repeat **Problem P6.91** for the circuit of **Figure P6.92**.

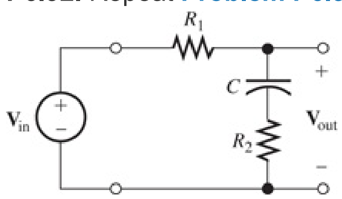


Figure P6.92

P6.93. Suppose that we need a filter with the Bode plot shown in **Figure P6.93(a)**. We decide to cascade a highpass circuit and a lowpass circuit as shown in **Figure P6.93(b)**. So that the second (i.e., right-hand) circuit looks like an approximate open circuit across the output of the first (i.e., left-hand) circuit, we choose $R_2 = 100R_1$.

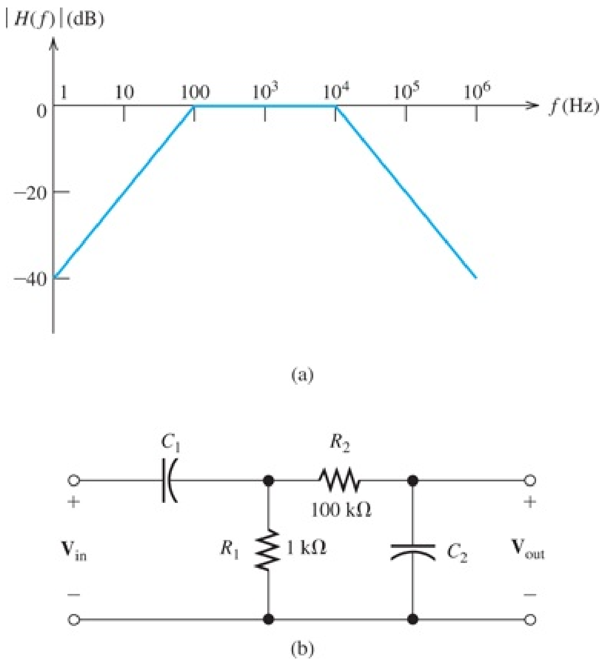


Figure P6.93

a. Which of the components form the lowpass filter? Which form the highpass filter?
b. Compute the capacitances needed to achieve the desired break frequencies, making the approximation that the left-hand circuit has an open-circuit load.
c. Write expressions that can be used to compute the exact transfer function $H(f) = V_{out}/V_{in}$ and use MATLAB to produce a Bode magnitude plot for *f* ranging from 1 Hz to 1 MHz. The result should be a close approximation to the desired plot shown in **Figure P6.93(a)**.

P6.94. Suppose that we need a filter with the Bode plot shown in **Figure P6.93(a)**. We decide to cascade a highpass circuit and a lowpass circuit, as shown in **Figure P6.94**. So that the second (i.e., right-hand) circuit looks like an approximate open circuit across the output of the first (i.e., left-hand) circuit, we choose $C_2 = C_1/100$.

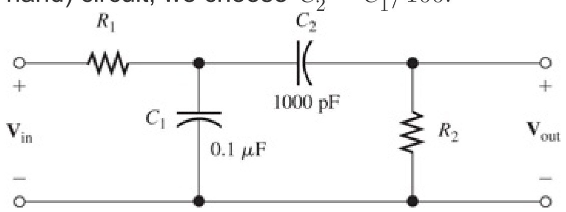


Figure P6.94

a. Which of the components form the lowpass filter? Which form the highpass filter?
b. Compute the resistances needed to achieve the desired break frequencies, making the approximation that the left-hand circuit has an open-circuit load.
c. Write expressions that can be used to compute the exact transfer function $H(f) = V_{out}/V_{in}$ and use MATLAB to produce a Bode magnitude plot for *f* ranging from 1 Hz to 1 MHz. The result should be a close approximation to the desired plot shown in **Figure P6.93(a)**.

P6.95. Other combinations of *R, L,* and *C* have behaviors similar to that of the series resonant circuit. For example, consider the circuit shown in **Figure P6.95**.

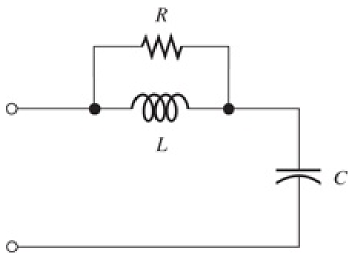


Figure P6.95

a. Derive an expression for the resonant frequency of this circuit. (We have defined the resonant frequency to be the frequency for which the impedance is purely resistive.)
b. Compute the resonant frequency, given $L = 1\text{ mH},\ R = 1000\ \Omega,$ and $C = 0.25\ \mu\text{F}$.
c. Use MATLAB to obtain a plot of the impedance magnitude of this circuit for *f* ranging from 95 to 105 percent of the resonant frequency. Compare the result with that of a series *RLC* circuit.

P6.96. Consider the circuit of **Figure P6.77** with $R = 1\text{ k}\Omega,\ L = 1\text{ mH},$ and $C = 0.25\ \mu\text{F}$.

a. Using MATLAB, obtain a plot of the impedance magnitude of this circuit for *f* ranging from 9 to 11 kHz.
b. From the plot, determine the minimum impedance, the frequency at which the impedance is minimum, and the bandwidth (i.e., the band of frequencies for which the impedance is less than $\sqrt{2}$ times the minimum value).
c. Determine the component values for a series *RLC* circuit having the same parameters as those found in part (b).
d. Plot the impedance magnitude of the series circuit on the same axes as the plot for part (a).

P6.97. Other combinations of *R, L*, and *C* have behaviors similiar to that of the parallel circuit. For example, consider the circuit shown in **Figure P6.97**.

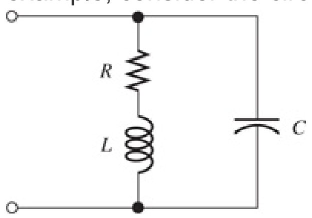


Figure P6.97

a. Derive an expression for the resonant frequency of this circuit. (We have defined the resonant frequency to be the frequency for which the impedance is purely resistive. However, in this case you may find the algebra easier if you work with admittances.)
b. Compute the resonant frequency, given $L = 1\text{ mH},\ R = 1\ \Omega,$ and $C = 0.25\ \mu\text{F}$.
c. Use MATLAB to obtain a plot of the impedance magnitude of this circuit for *f* ranging from 95 to 105 percent of the resonant frequency. Compare the result with that of a parallel *RLC* circuit.

P6.98. Consider the filter shown in **Figure P6.98**.

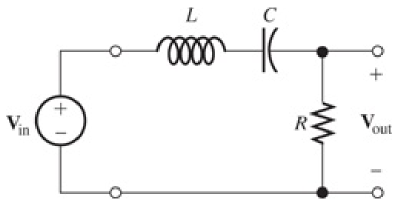


Figure P6.98

a. Derive an expression for the transfer function $H(f) = V_{out}/V_{in}$.
b. Use MATLAB to obtain a Bode plot of the transfer function magnitude for $R = 10\ \Omega,\ L = 10\ \text{mH},$ and $C = 0.02533\ \mu\text{F}$. Allow frequency to range from 1 kHz to 100 kHz.
c. At very low frequencies, the capacitance becomes an open circuit and the inductance becomes a short circuit. In this case, determine an expression for the transfer function and evaluate for the circuit parameters of part (b). Does the result agree with the value plotted in part (b)?
d. At very high frequencies, the capacitance becomes a short circuit and the inductance becomes an open circuit. In this case, determine an expression for the transfer function and evaluate for the circuit parameters of part (b). Does the result agree with the value plotted in part (b)?

P6.99. Repeat **Problem P6.98** for the circuit of **Figure P6.99**.

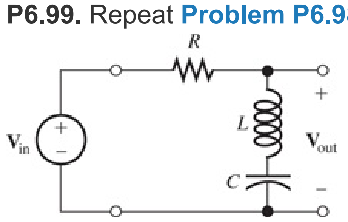


Figure P6.99

Section 6.10: Digital Signal Processing

P6.100. Develop a digital filter that mimics the action of the *RL* filter shown in **Figure P6.100**. Determine expressions for the coefficients in terms of the time constant and sampling interval *T*. [*Hint:* If your circuit equation contains an integral, differentiate with respect to time to obtain a pure differential equation.]

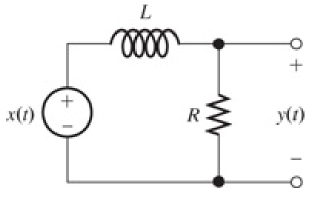


Figure P6.100

a. Given $R = 10\ \Omega$ and $L = 200\ \mathrm{mH},$ sketch the step response of the circuit to scale.
b. Use MATLAB to determine and plot the step response of the digital filter for several time constants. Use the time constant of part (b) and $f_s = 500\ \mathrm{Hz}.$ Compare the results of parts (b) and (c).

P6.101. Repeat **Problem P6.100** for the filter shown in **Figure P6.101**.

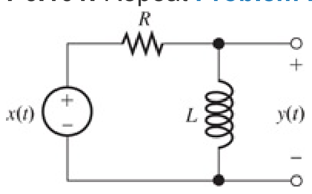


Figure P6.101

***P6.102.** Consider the second-order bandpass filter shown in **Figure P6.102**.

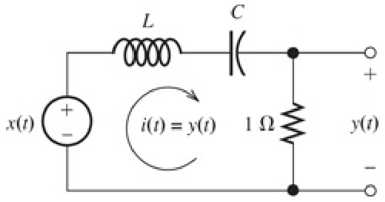


Figure P6.102

a. Derive expressions for *L* and *C* in terms of the resonant frequency ω_0 and quality factor Q_s.
b. Write the KVL equation for the circuit and use it to develop a digital filter that mimics the action of the *RLC* filter. Use the results of part (a) to write the coefficients in terms of the resonant frequency $\omega_0,$ circuit quality factor $Q_s,$ and sampling interval *T*. [*Hint:* The circuit equation contains an integral, so differentiate with respect to time to obtain a pure differential equation.]

Practice Test

Here is a practice test you can use to check your comprehension of the most important concepts in this chapter. Answers can be found in Appendix D and complete solutions are included in the Student Solutions files. See Appendix E for more information about the Student Solutions.

T6.1. What is the basic concept of Fourier theory as it relates to real-world signals? How does the transfer function of a filter relate to this concept?

T6.2. An input signal given by

$$\begin{aligned} v_{\text{in}}(t) &= 3 + 4\cos(1000\pi t) \\ &\quad + 5\cos(2000\pi t - 30°) \end{aligned}$$

is applied to the *RL* filter shown in **Figure T6.2**. Find the expression for the output signal $v_{\text{out}}(t)$.

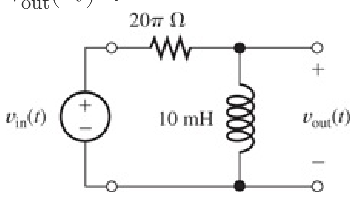


Figure T6.2

T6.3. Consider the Bode magnitude plot for the transfer function of a certain filter given by

$$H(f) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = 50\frac{j(f/200)}{1 + j(f/200)}$$

a. What is the slope of the low-frequency asymptote?
b. What is the slope of the high-frequency asymptote?
c. What are the coordinates of the point at which the asymptotes meet?
d. What type of filter is this?
e. What is the value of the break frequency?

T6.4. A series resonant circuit has $R = 5\ \Omega, L = 20\ \text{mH},$ and $C = 1\ \mu\text{F}$. Determine the values of:

a. the resonant frequency in Hz.
b. Q.
c. bandwidth in Hz.
d. the impedance of the circuit at the resonant frequency.
e. the impedance of the circuit at dc.
f. **f.** the impedance of the circuit as the frequency approaches infinity.

T6.5. Repeat **question T6.4** for a parallel resonant circuit with $R = 10\ \text{k}\Omega,\ L = 1\ \text{mH},$ and $C = 1000\ \text{pF}$.

T6.6. We have a signal consisting of voice conversations and music with frequency components from about 30 Hz to 8 kHz plus a loud sinusoidal tone of 800 Hz. Specify the type of ideal filter and cutoff frequencies if

a. we want nearly all of the voice and music components to pass through the filter with the 800 Hz tone eliminated, so we can monitor the conversations better.
b. we want to eliminate nearly all of the voice and music components and pass the 800 Hz tone through the filter so we can monitor slow variations in its amplitude, which can give information about movement of the persons speaking.

T6.7. Consider the transfer function $V_{\text{out}}/V_{\text{in}}$ for each of the circuits shown in **Figure T6.7**. Classify each circuit as a first-order lowpass filter, second-order bandpass filter, etc. Justify your answers.

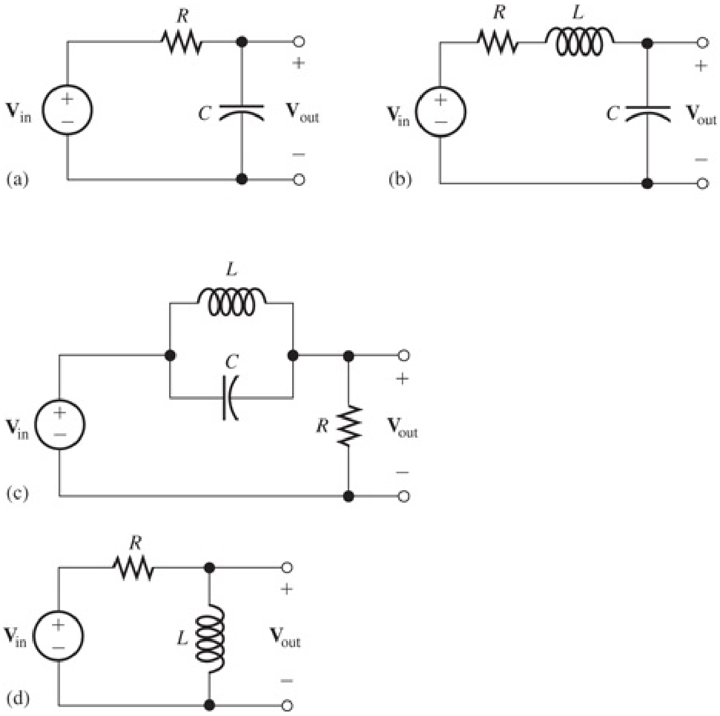


Figure T6.7

T6.8. Give a list of MATLAB commands to produce the magnitude Bode plot for the transfer function of **question T6.3** for frequency ranging from 10 Hz to 10 kHz.