A new operator splitting algorithm for elastoviscoplastic flow problems

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Operator notations

Operators :

$$\nabla\vec{u} = \underbrace{\mathbf{D}\left(\vec{u}\right)}_{deformation\ rate} + \underbrace{\mathbf{W}\left(\vec{u}\right)}_{vorticity} = \left[\frac{\nabla\vec{u}+\nabla\vec{u}^T}{2}\right] + \left[\frac{\nabla\vec{u}-\nabla\vec{u}^T}{2}\right]$$

$$\tau = \underbrace{\tau_d}_{\substack{deviatoric\ stress \\ (shear)}} + \underbrace{\tau_s}_{\substack{hydrostatic\ stress \\ (pressure)}} = \left[\tau - \frac{1}{d}tr(\tau)\mathbf{I_d}\right] + \left[\frac{1}{d}tr(\tau)\mathbf{I_d}\right]$$

$\forall a \in [-1,1]$ $\quad \dfrac{\mathscr{D}_{\mathbf{a}}\tau}{\mathscr{D}\mathbf{t}} = \dfrac{\partial\tau}{\partial t} + \left(\vec{u}.\vec{\nabla}\right)\tau + \beta_{\mathbf{a}}\left(\tau, \nabla\vec{u}\right)$ the **Jaumann objective stress rate.**

$$\beta_a\left(\tau, \nabla\vec{u}\right) = \tau.\mathbf{W}\left(\vec{u}\right) - \mathbf{W}\left(\vec{u}\right).\tau - a\left[\mathbf{D}\left(\vec{u}\right).\tau + \tau.\mathbf{D}\left(\vec{u}\right)\right]$$

$\forall\tau \in \mathbb{R}^{d\times d}$ $\quad |\tau|^2 = \tau : \tau = tr\left(\tau^T\tau\right) = \sum_{1\le i,j\le d}\tau_{ij}^2$ $\quad$ the usual **matrix Euclidean norm**

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Model parameters

Dimension: $d \in \{2, 3\}$

Jaumann operator parameter: $a \in [-1, 1]$

Fluid properties:

$$\left[\begin{array}{ll} \rho & \textit{solution density} \\ \eta_s & \textit{solvent viscosity} \\ \eta_m & \textit{polymer viscosity} \\ \lambda & \textit{elastic relaxation time} \\ \tau_Y & \textit{yield stress} \end{array}\right]$$

Additional notations

Cauchy stress tensor: $$\tau_{tot} = \underbrace{\tau}_{\textit{elastic stress}} - \underbrace{p\mathbf{Id}}_{\textit{pressure}} + \underbrace{2\eta_s \mathbf{D}\left(\vec{u}\right)}_{\textit{viscous friction}}$$

Yield response:

$$\mathcal{K}(|\tau_d|) = \left[\frac{|\tau_d| - Bi}{n|\tau_d|}\right]_+^n = \max\left(0, 1 - \frac{B_i}{|\tau_d|}\right) \text{ with } n = 1 \text{ (Bingham).}$$

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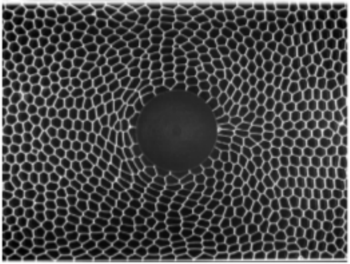
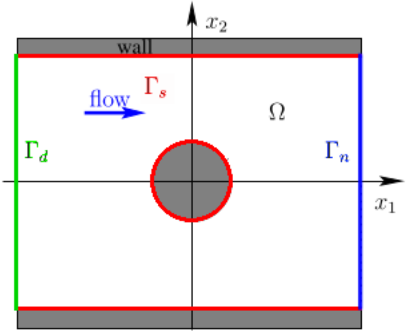
Domain

Experimental configuration :

$$\Omega \subset \mathbb{R}^d \qquad \partial\Omega = \Gamma_d \cup \Gamma_n \cup \Gamma_s \qquad \underbrace{\partial\Omega_- = \Gamma_- = \left\{ x \in \partial\Omega \mid \vec{u}\,.\,\vec{n} < 0 \right\}}_{upstream\ \ boundary}$$





Model

Find $\mathcal{U} = (\tau, \overrightarrow{u}, p)^T :]0, T[\times \Omega \to \mathbb{R}^{d\times d} \times \mathbb{R}^d \times \mathbb{R}$ such that :

$$
\left\{
\begin{array}{lllll}
(1) & W_e \left[\dfrac{\mathscr{D}_{\mathbf{a}}\tau}{\mathscr{D}\mathbf{t}}\right] + & \tau - 2\alpha \mathbf{D}(\overrightarrow{u}) & = & 0_{\mathbb{R}^{d\times d}} \\
(2) & R_e \left[\dfrac{\partial \overrightarrow{u}}{\partial t} + \left(\overrightarrow{u}.\overrightarrow{\nabla}\right)\overrightarrow{u}\right] & \underbrace{-\overrightarrow{div}\left(\tau - p\mathbf{I}_d + 2\eta_s \mathbf{D}(\overrightarrow{u})\right)}_{-\overrightarrow{div}(\tau_{tot}) = -\overrightarrow{div}(\tau) + \overrightarrow{\nabla} p - (1-\alpha)\Delta \overrightarrow{u}} & = & 0_{\mathbb{R}^d} \\
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(5) \quad \vec{F} = \tau \vec{n} = 0 \text{ in }]0, T[\times \Gamma_n \qquad \vec{F_t} = \tau \vec{n} - \tau_{nn} \vec{n} = 0 \text{ in }]0, T[\times \Gamma_s \\
(6) \quad \tau = \tau_- \text{ in }]0, T[\times \Gamma_-
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Rewrite the model as:

Find $\mathcal{U} = (\tau, \vec{u}, p)^T$: $]0, T[\times \Omega \to \mathbb{R}^{d\times d} \times \mathbb{R}^d \times \mathbb{R}$ such that :

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Θ-Scheme

Three-steps θ-scheme time-approximation of the previous equation :

Let $\Delta t > 0$ and $\theta \in]0, 1/2[$ and $\mathcal{U}_n$ be given.

To compute $\mathcal{U}_{\mathbf{n+1}}$ compute successively the following subproblems :

$$\mathcal{P}_1\left(\mathcal{U}^{n+\theta}, \mathcal{U}^n\right) \Leftrightarrow \mathcal{M}\frac{\mathcal{U}^{n+\theta} - \mathcal{U}^n}{\theta\Delta t} + \mathcal{A}_1\left(\mathcal{U}^{n+\theta}\right) + \mathcal{A}_2\left(\mathcal{U}^n\right) = 0$$

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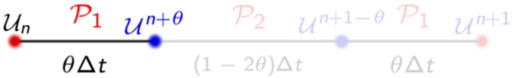

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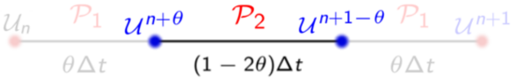
$$\mathcal{P}_2\left(\mathcal{U}^{n+1-\theta}, \mathcal{U}^{n+\theta}\right) \Leftrightarrow \mathcal{M}\frac{\mathcal{U}^{n+1-\theta} - \mathcal{U}^{n+\theta}}{(1-2\theta)\Delta t} + \mathcal{A}_1\left(\mathcal{U}^{n+\theta}\right) + \mathcal{A}_2\left(\mathcal{U}^{n+1-\theta}\right) = 0$$

$$\mathcal{P}_1\left(\mathcal{U}^{n+1}, \mathcal{U}^{n+1-\theta}\right) \Leftrightarrow \mathcal{M}\frac{\mathcal{U}^{n+1} - \mathcal{U}^{n+1-\theta}}{\theta\Delta t} + \mathcal{A}_1\left(\mathcal{U}^{n+1}\right) + \mathcal{A}_2\left(\mathcal{U}^{n+1-\theta}\right) = 0$$



The third step is similar to the first one.

Θ-Scheme

Three-steps θ-scheme time-approximation of the previous equation :

Let $\Delta t > 0$ and $\theta \in]0, 1/2[$ and $\mathcal{U}_n$ be given.

To compute $\mathcal{U}_{\mathbf{n+1}}$ compute successively the following subproblems :

$$\mathcal{P}_1\left(\mathcal{U}^{n+\theta}, \mathcal{U}^n\right) \Leftrightarrow \mathcal{M}\frac{\mathcal{U}^{n+\theta} - \mathcal{U}^n}{\theta \Delta t} + \mathcal{A}_1\left(\mathcal{U}^{n+\theta}\right) + \mathcal{A}_2\left(\mathcal{U}^n\right) = 0$$

$$\mathcal{P}_2\left(\mathcal{U}^{n+1-\theta}, \mathcal{U}^{n+\theta}\right) \Leftrightarrow \mathcal{M}\frac{\mathcal{U}^{n+1-\theta} - \mathcal{U}^{n+\theta}}{(1-2\theta)\Delta t} + \mathcal{A}_1\left(\mathcal{U}^{n+\theta}\right) + \mathcal{A}_2\left(\mathcal{U}^{n+1-\theta}\right) = 0$$

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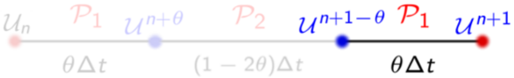

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Let $\Delta t > 0$ and $\theta \in]0, 1/2[$ and $\mathcal{U}_n$ be given.

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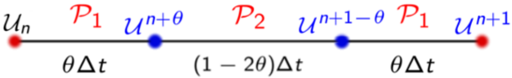
$$
\begin{array}{lclclclcl}
\mathcal{P}_1\left(\mathcal{U}^{n+\theta}, \mathcal{U}^n\right) & \Leftrightarrow & \mathcal{M}\dfrac{\mathcal{U}^{n+\theta} - \mathcal{U}^n}{\theta \Delta t} & + & \mathcal{A}_1\left(\mathcal{U}^{n+\theta}\right) & + & \mathcal{A}_2\left(\mathcal{U}^n\right) & = & 0 \\
\mathcal{P}_2\left(\mathcal{U}^{n+1-\theta}, \mathcal{U}^{n+\theta}\right) & \Leftrightarrow & \mathcal{M}\dfrac{\mathcal{U}^{n+1-\theta} - \mathcal{U}^{n+\theta}}{(1-2\theta)\Delta t} & + & \mathcal{A}_1\left(\mathcal{U}^{n+\theta}\right) & + & \mathcal{A}_2\left(\mathcal{U}^{n+1-\theta}\right) & = & 0 \\
\mathcal{P}_1\left(\mathcal{U}^{n+1}, \mathcal{U}^{n+1-\theta}\right) & \Leftrightarrow & \mathcal{M}\dfrac{\mathcal{U}^{n+1} - \mathcal{U}^{n+1-\theta}}{\theta \Delta t} & + & \mathcal{A}_1\left(\mathcal{U}^{n+1}\right) & + & \mathcal{A}_2\left(\mathcal{U}^{n+1-\theta}\right) & = & 0
\end{array}
$$



The third step is similar to the first one.

First subproblem

Solving the first subproblem $\mathcal{P}_1$:

Let $\mathcal{U}^k = \left[\tau^k, \overrightarrow{u}^k, p^k\right]^T$ be given. $\mathcal{P}_1$ is a **non linear Stokes problem** :

$$\mathcal{P}_1\left(\mathcal{U}^{k+1}, \mathcal{U}^k\right) \quad \Leftrightarrow \quad \mathcal{M}\frac{\mathcal{U}^{k+1}}{\theta\Delta t} + \mathcal{A}_1\left(\mathcal{U}^{k+1}\right) = \mathcal{M}\frac{\mathcal{U}^k}{\theta\Delta t} - \mathcal{A}_2\left(\mathcal{U}^k\right) = \mathcal{V}^k$$

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$$\Leftrightarrow \begin{cases} W_e\dfrac{\tau^{k+1}}{\theta\Delta t} + \mathcal{K}\left(\left|\tau_d^{k+1}\right|\right)\tau^{k+1} - 2\alpha\mathbf{D}\left(\overrightarrow{u}^{\mathbf{k+1}}\right) & = & \mathcal{V}_1^k \\ Re\dfrac{\overrightarrow{u}^{k+1}}{\theta\Delta t} - \overrightarrow{div}\left(\tau^{k+1}\right) + \overrightarrow{\nabla}p^{k+1} - (1-\alpha)\Delta\overrightarrow{u}^{k+1} & = & \mathcal{V}_2^k \\ div\left(\overrightarrow{u}^{k+1}\right) & = & 0 \end{cases}$$

Linearizing the first subproblem

Fixed point algorithm :

Linearize problem $\mathcal{P}_1$ with $\mathcal{K}\left(|\tau_d^{k+1}|\right)\tau^{k+1} \simeq \underbrace{\mathcal{K}\left(|\tau_d^{k}|\right)}_{explicit}\tau^{k+1}$

$\Rightarrow \mathcal{P}_1^{lin}$ becomes a **"simple Stokes subproblem"**

Step s = 0: Let $\mathcal{U}_0^k = \mathcal{U}^k$ *i.e.* $\left(\tau_0^k, \vec{u}_0^k, p_0^k\right) = \left(\tau^k, \vec{u}^k, p^k\right)$

Step s > 0: Solve $\mathcal{P}_1^{lin}\left(\mathcal{U}_{s+1}^k, \mathcal{U}_s^k\right)$ with boundary conditions (4-5) to compute $\mathcal{U}_{s+1}^k$
.

Finally : Set $\mathcal{U}^{k+1} = \mathcal{U}_{s_\infty}^k$ *i.e.* $\left(\tau^{k+1}, \vec{u}^{k+1}, p^{k+1}\right) = \left(\tau_{s_\infty}^k, \vec{u}_{s_\infty}^k, p_{s_\infty}^k\right)$

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Linearizing the first subproblem

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Second subproblem

Solving the second subproblem $\mathcal{P}_2$

Let $\mathcal{U}^k = \left[\tau^k, \vec{u}^{\,k}, p^k\right]^T$ and $\theta' = 1 - 2\theta$ be given. $\mathcal{P}_2$ is a **stress transport problem** :

$$\mathcal{P}_2\left(\mathcal{U}^{k+1}, \mathcal{U}^k\right) \quad \Leftrightarrow \quad \mathcal{M}\frac{\mathcal{U}^{k+1}}{\theta'\Delta t} + \mathcal{A}_2\left(\mathcal{U}^{k+1}\right) = \mathcal{M}\frac{\mathcal{U}^k}{\theta'\Delta t} - \mathcal{A}_1\left(\mathcal{U}^k\right) = \mathcal{W}^k$$

Solving the second subproblem $\mathcal{P}_2$

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$$\Leftrightarrow \begin{cases} \dfrac{\tau^{k+1}}{\theta'\Delta t} + \left(\vec{u}^{\,k+1}.\vec{\nabla}\right)\tau^{k+1} + \beta_{\mathbf{a}}\left(\tau^{\mathbf{k+1}}, \nabla\vec{u}^{\,\mathbf{k+1}}\right) & = \quad \mathcal{W}_1^k \\ Re\dfrac{\vec{u}^{\,k+1}}{\theta'\Delta t} = Re\dfrac{\vec{u}^{\,k}}{\theta'\Delta t} + \overrightarrow{div}\left(\tau^k\right) - \vec{\nabla}p^k + (1-\alpha)\Delta\vec{u}^{\,k} & = \quad \mathcal{W}_2^k \end{cases}$$

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If we previously have solved $\mathcal{P}_1\left(\mathcal{U}^{\mathbf{k}},\mathcal{U}^{\mathbf{k-1}}\right)$ **:** $\vec{u}^{\,k+1} = \underbrace{\dfrac{1-\theta}{\theta}\vec{u}^{\,k} - \dfrac{1-2\theta}{\theta}\vec{u}^{\,k-1}}_{explicit}$

Solving the second subproblem $\mathcal{P}_2$

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If we previously have solved $\mathcal{P}_1\left(\mathcal{U}^{\mathbf{k}}, \mathcal{U}^{\mathbf{k-1}}\right)$ **:** $\vec{u}^{k+1} = \underbrace{\frac{1-\theta}{\theta}\vec{u}^k - \frac{1-2\theta}{\theta}\vec{u}^{k-1}}_{\textit{explicit}}$

$\Rightarrow$ $\mathcal{P}_2$ becomes a **linear Friedrich's first-order system**.

Solving the second subproblem $\mathcal{P}_2$

Let $\mathcal{U}^k = \left[\tau^k, \vec{u}^k, p^k\right]^T$ and $\theta' = 1 - 2\theta$ be given. $\mathcal{P}_2$ is a **stress transport problem** :

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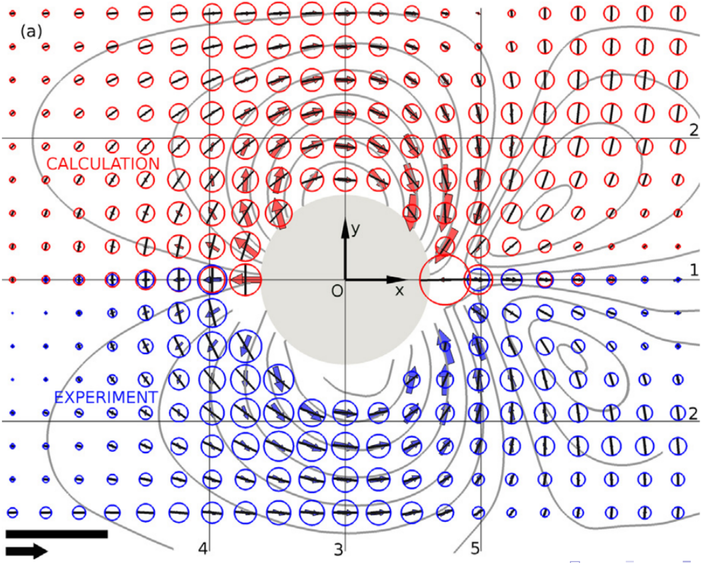
If we previously have solved $\mathcal{P}_1\left(\mathcal{U}^{\mathbf{k}}, \mathcal{U}^{\mathbf{k-1}}\right)$ **:** $\vec{u}^{k+1} = \underbrace{\dfrac{1-\theta}{\theta}\vec{u}^k - \dfrac{1-2\theta}{\theta}\vec{u}^{k-1}}_{\textit{explicit}}$

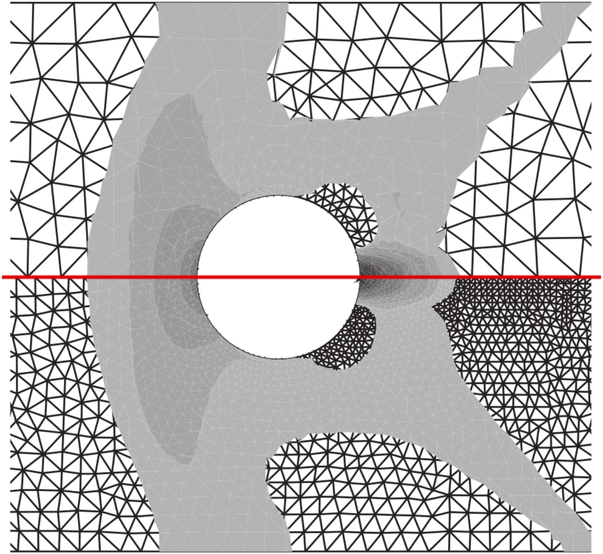
$\Rightarrow \mathcal{P}_2$ becomes a **linear Friedrich's first-order system**.

It admits a solution if $\Delta t < \dfrac{1}{\theta'} - 2\,|a|\,||\mathbf{D}(\vec{u}^{\mathbf{k}})||_\infty$

Calculation vs Experiment

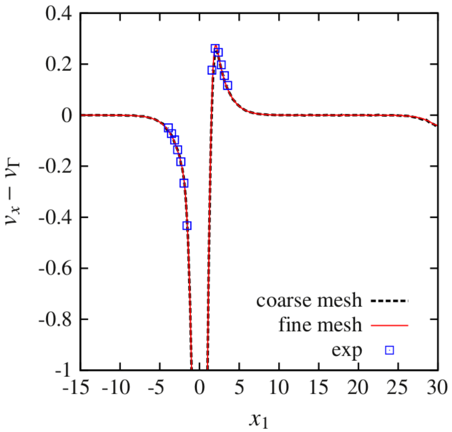


Yielded regions



Overshoot

"Negative wake"



References

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- Pierre Saramito.
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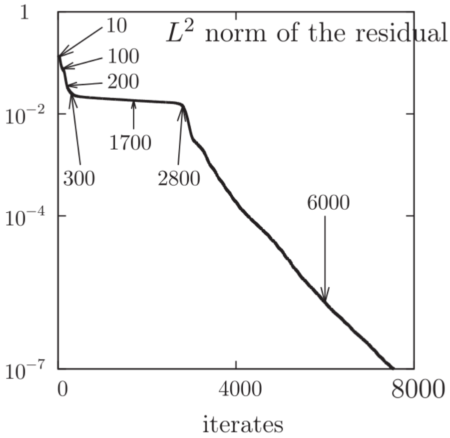
  ESAIM: Proceedings, 2:275–281, 1997.

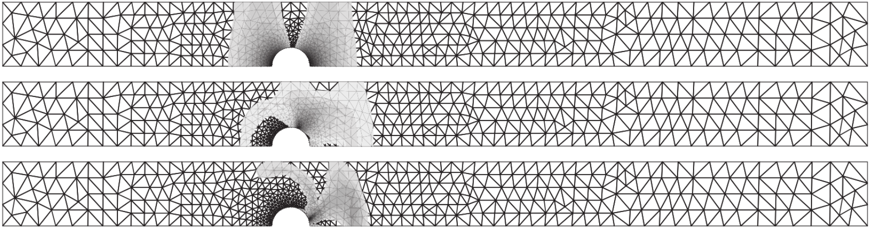
Any questions ?

Convergence of the scheme



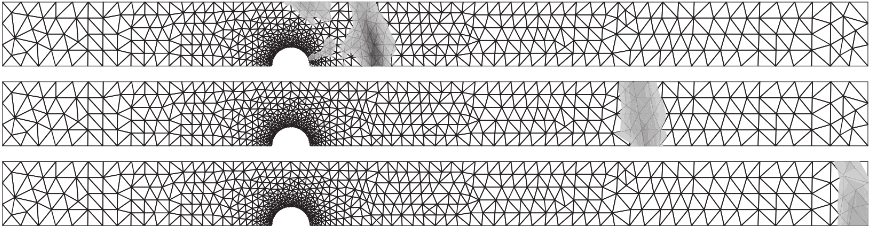
Convergence of the scheme

(b) iterates 10, 100, 200



Convergence of the scheme

(c) iterates 300, 1700, 2800



Convergence of the scheme

(d) iterate 6000