特殊方程作业 11

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问题 1 采用拉普拉斯变换法求解下列定解问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1,\ t > 0 \\ u|_{x=0} = 0,\ u|_{x=1} = 0, & t \geqslant 0 \\ u|_{t=0} = 0,\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = \sin 2\pi x, & 0 \leqslant x \leqslant 1 \end{cases}$$

问题 2 采用积分变换法求解下列定解问题

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1,\ t > 0 \\ u|_{x=0} = 100,\ u|_{x=1} = 100, & t \geqslant 0 \\ u|_{t=0} = 3\sin(5\pi x) + 100, & 0 \leqslant x \leqslant 1 \end{cases}$$

问题 3 采用适当方法求解下列定解问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 1, & x > 0,\ y > 0 \\ u|_{x=0} = y, & x \geqslant 0 \\ u|_{y=0} = x, & y \geqslant 0 \end{cases}$$

问题 #1 **Grade:**

根据初始条件，确定对 $u(x,t)$ 做关于 t 的拉普拉斯变换，定义

$$U(x,s) \triangleq \int_0^\infty u(x,t)e^{-st}\mathrm{d}t$$

由拉普拉斯变换的微分性质

$$\mathscr{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x,s) + u(x,0) = sU(x,s)$$

Faculty Comments

问题 #1 Grade:

Faculty Comments

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2U(x,s) + u'(x,0) = s^2U(x,s) + \sin 2\pi x$$

另一方面

$$\mathscr{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2}$$

代入偏微分方程

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} - \frac{s^2}{4}U(x,s) = \frac{1}{4}\sin 2\pi x$$

对应的齐次方程为

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} - \frac{s^2}{4}U(x,s) = 0$$

解之得齐次方程通解

$$\widetilde{U}(x,s) = C_1 e^{-\frac{s}{2}x} + C_2 e^{\frac{s}{2}x}$$

观察得到非齐次方程的一个特解

$$U^*(x,s) = -\frac{\sin 2\pi x}{s^2 + 16\pi^2}$$

从而可以得出非齐次方程的通解

$$U(x,s) = C_1 e^{-\frac{s}{2}x} + C_2 e^{\frac{s}{2}x} - \frac{\sin 2\pi x}{s^2 + 16\pi^2}$$

代入边界条件 $U(0,s) = 0$，$U(1,s) = 0$，得到齐次方程组

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{-\frac{s}{2}} + C_2 e^{\frac{s}{2}} = 0 \end{cases}$$

由于

$$\begin{vmatrix} 1 & 1 \\ e^{-\frac{s}{2}} & 2e^{\frac{s}{2}} \end{vmatrix} \neq 0$$

说明该齐次方程组只有零解，即 $C_1 = 0$，$C_2 = 0$ 因此可得

$$U(x,s) = -\frac{\sin 2\pi x}{s^2 + 16\pi^2}$$

相应的可以写出拉普拉斯逆变换

问题 #1 **Grade:**

Faculty Comments

$$\mathscr{L}^{-1}\left\{U(x,s)\right\} = -\frac{1}{4\pi}\sin 2\pi x \sin 4\pi t$$

问题 #2 **Grade:**

Faculty Comments

设 $u(x,t) = w(x,t) + 100$，将定解问题转换成

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial^2 w}{\partial x^2}, & 0 < x < 1,\ t > 0 \\ w(0,t) = w(1,t) = 0, & t \geqslant 0 \\ w(s,0) = 4\sin(5\pi x) \end{cases}$$

定义

$$W(x,s) \triangleq \mathscr{L}\{w(x,t)\} = \int_0^{+\infty} w(x,t)e^{-st}\mathrm{d}t$$

$$\mathscr{L}\left\{\frac{\partial w}{\partial t}\right\} = sW(x,s) + w(x,0) = sW(x,s) + 3\sin(5\pi x)$$

$$\mathscr{L}\left\{\frac{\partial^2 w}{\partial x^2}\right\} = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

代入偏微分方程，得到常微分方程

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - sW(x,s) = 3\sin(5\pi x)$$

分别解得齐次方程通解

$$\widetilde{W}(x,s) = C_1 e^{-\sqrt{s}x} + C_2 e^{\sqrt{s}x}$$

以及非齐次方程特解

$$W^*(x,s) = \frac{3}{s + 25\pi^2}\sin(5\pi x)$$

得到非齐次方程通解

$$W(x,s) = C_1 e^{-\sqrt{s}x} + C_2 e^{\sqrt{s}x} + \frac{3}{s + 25\pi^2}\sin(5\pi x)$$

代入边界条件

$$w(0,t) = w(1,t) = 0 \xrightarrow{\mathscr{L}} W(0,s) = W(1,s) = 0$$

问题 #2

Grade:

Faculty Comments

得到齐次方程组

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{-\sqrt{s}} + C_2 e^{\sqrt{s}} = 0 \end{cases}$$

由于

$$\begin{vmatrix} 1 & 1 \\ e^{-\sqrt{s}} & e^{\sqrt{s}} \end{vmatrix} \neq 0$$

所以仅存在零解

$$W(x,s) = \frac{3}{s + 25\pi^2} \sin(5\pi x)$$

做拉普拉斯逆变换

$$w(x,t) = e^{-(5\pi)^2} \sin(5\pi x)$$

问题 #3

Grade:

Faculty Comments

由于

$$\frac{\partial^2 u}{\partial x \partial y} = 1$$

可以设

$$u(x,y) = F(x) + G(y) + xy$$

代入边界条件

$$u(0,y) = F(0) + G(y) = y$$

$$u(x,0) = F(x) + G(0) = x$$

整理得

$$F(x) = x - G(0),\ F(0) = -G(0)$$

$$G(x) = y - F(0),\ G(0) = -F(0)$$

相应的

$$F(x) = x + F(0)$$

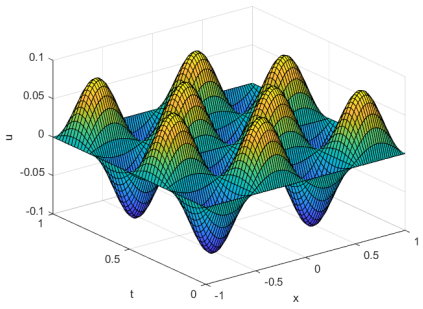
$$G(y) = y + G(0)$$

代回原式

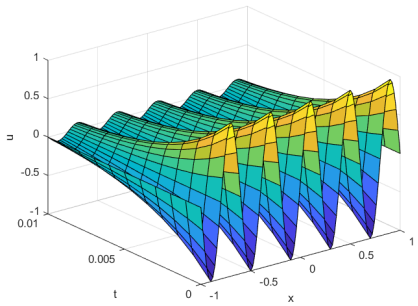
$$u(x,y) = xy + x + y + F(0) + G(0)$$

问题 #3	Grade:
由于 $F(0)+G(0)=0$，因此 $$u(x,y)=xy+x+y$$	*Faculty Comments*

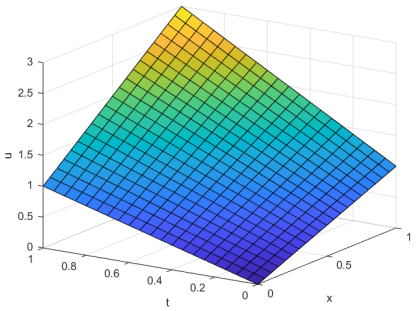
三个问题的结果图示以及 MATLAB 计算程序附在后面。



(a)



(b)



(c)

```m
% 问题1结果图示
clear;

x = -1:0.02:1;
t = 0:0.02:1;
[X,T] = meshgrid(x,t);

k = 1/(4*pi);
uxt = -sin(4*pi*T).*sin(2*pi*X)*k;

% 绘制图像
figure;
surf(X,T,uxt);
xlabel('x');
ylabel('t');
zlabel('u');
```

(a)

```m
% 问题2结果图示
clear;

x = -1:0.02:1;
t = 0:0.001:0.01;
[X,T] = meshgrid(x,t);

k = 5*pi;
uxt = exp(-k*k*T).*sin(k*X);

% 绘制图像
figure;
surf(X,T,uxt);
xlabel('x');
ylabel('t');
zlabel('u');
```

(b)

```m
% 问题3结果图示
clear;

x = 0:0.05:1;
y = 0:0.05:1;
[X,Y] = meshgrid(x,y);

uxy = X.*Y+X+Y;

% 绘制图像
figure;
surf(X,Y,uxy);
xlabel('x');
ylabel('t');
zlabel('u');
```

(c)