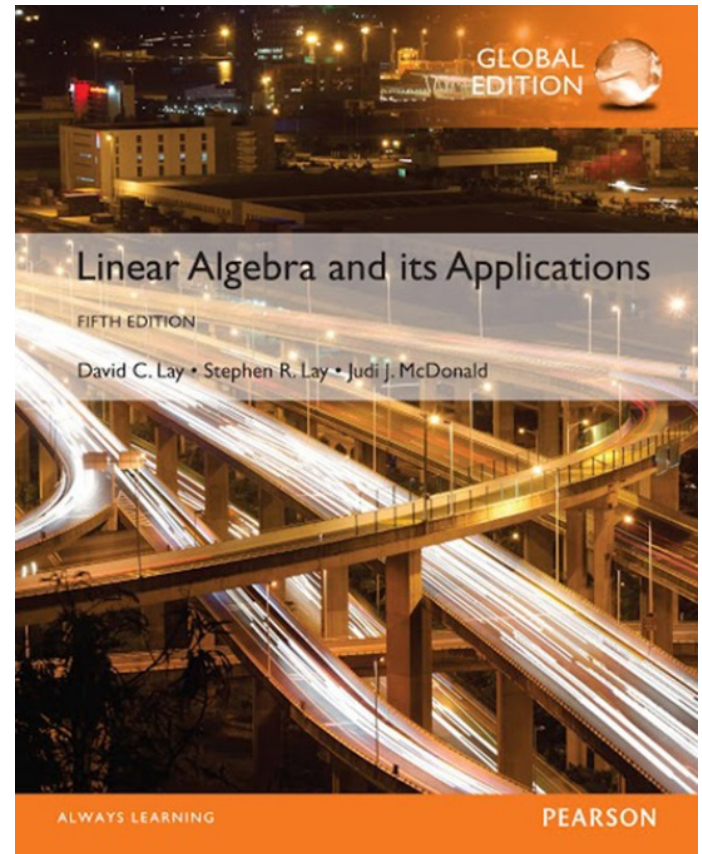


# 4

## Vector Spaces

### 4.3

## LINEARLY INDEPENDENT SETS; BASES



# LINEAR INDEPENDENT SETS; BASES

- An indexed set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $V$  is said to be **linearly independent** if the vector equation

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p = \mathbf{0} \quad (1)$$

has *only* the trivial solution,  $c_1 = 0, \dots, c_p = 0$ .

- The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is said to be **linearly dependent** if (1) has a nontrivial solution, *i.e.*, if there are some weights,  $c_1, \dots, c_p$ , *not all zero*, such that (1) holds.
- In such a case, (1) is called a **linear dependence relation** among  $\mathbf{v}_1, \dots, \mathbf{v}_p$ .

# LINEAR INDEPENDENT SETS; BASES

- **Theorem 4:** An indexed set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  of two or more vectors, with  $\mathbf{v}_1 \neq \mathbf{0}$ , is linearly dependent if and only if some  $\mathbf{v}_j$  (with  $j > 1$ ) is a linear combination of the preceding vectors,  $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$ .
- **Definition:** Let  $H$  be a subspace of a vector space  $V$ . An indexed set of vectors  $B = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$  in  $V$  is a basis for  $H$  if
  - (i)  $B$  is a linearly independent set, and
  - (ii) The subspace spanned by  $B$  coincides with  $H$ ; that is,  $H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$

# LINEAR INDEPENDENT SETS; BASES

- The definition of a basis applies to the case when  $H = V$ , because any vector space is a subspace of itself.
- Thus a basis of  $V$  is a linearly independent set that spans  $V$ .
- When  $H \neq V$ , condition (ii) includes the requirement that each of the vectors  $\mathbf{b}_1, \dots, \mathbf{b}_p$  must belong to  $H$ , because  $\text{Span } \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$  contains  $\mathbf{b}_1, \dots, \mathbf{b}_p$ .

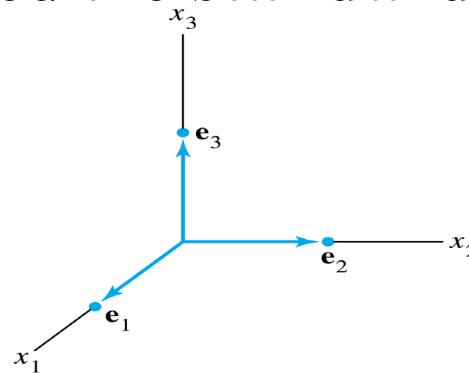
# STANDARD BASIS

- Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be the columns of the  $n \times n$  matrix,  $I_n$ .

- That is,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

- The set  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  is called the **standard basis** for  $\mathbb{R}^n$ .  
See the following figure.



The standard basis for  $\mathbb{R}^3$ .

# THE SPANNING SET THEOREM

- **Theorem 5:** Let  $S = \{v_1, \dots, v_p\}$  be a set in  $V$ , and let  $H = \text{Span}\{v_1, \dots, v_p\}$ .
  - a. If one of the vectors in  $S$ —say,  $v_k$ —is a linear combination of the remaining vectors in  $S$ , then the set formed from  $S$  by removing  $v_k$  still spans  $H$ .
  - b. If  $H \neq \{0\}$ , some subset of  $S$  is a basis for  $H$ .
- **Proof:**
  - a. By rearranging the list of vectors in  $S$ , if necessary, we may suppose that  $v_p$  is a linear combination of  $v_1, \dots, v_{p-1}$ —say,

# THE SPANNING SET THEOREM

$$\mathbf{v}_p = a_1 \mathbf{v}_1 + \dots + a_{p-1} \mathbf{v}_{p-1} \quad (3)$$

- Given any  $\mathbf{x}$  in  $H$ , we may write

$$\mathbf{x} = c_1 \mathbf{v}_1 + \dots + c_{p-1} \mathbf{v}_{p-1} + c_p \mathbf{v}_p \quad (4)$$

for suitable scalars  $c_1, \dots, c_p$ .

- Substituting the expression for  $\mathbf{v}_p$  from (3) into (4), it is easy to see that  $\mathbf{x}$  is a linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$ .
- Thus  $\{\mathbf{v}_1, \dots, \mathbf{v}_{p-1}\}$  spans  $H$ , because  $\mathbf{x}$  was an arbitrary element of  $H$ .

# THE SPANNING SET THEOREM

- b. If the original spanning set  $S$  is linearly independent, then it is already a basis for  $H$ .
  - Otherwise, one of the vectors in  $S$  depends on the others and can be deleted, by part (a).
  - So long as there are two or more vectors in the spanning set, we can repeat this process until the spanning set is linearly independent and hence is a basis for  $H$ .
  - If the spanning set is eventually reduced to one vector, that vector will be nonzero (and hence linearly independent) because  $H \neq \{0\}$ .



# THE SPANNING SET THEOREM

- **Example 7:** Let  $v_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ ,  $v_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$

and  $H = \text{Span}\{v_1, v_2, v_3\}$ .

Note that  $v_3 = 5v_1 + 3v_2$ , and show that

$\text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2\}$ . Then find a basis for the subspace  $H$ .

- **Solution:** Every vector in  $\text{Span}\{v_1, v_2\}$  belongs to  $H$  because
$$c_1 v_1 + c_2 v_2 = c_1 v_1 + c_2 v_2 + 0 v_3$$

# THE SPANNING SET THEOREM

- Now let  $\mathbf{x}$  be any vector in  $H$ —say,

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3.$$

- Since  $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$ , we may substitute

$$\begin{aligned}\mathbf{x} &= c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 (5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3) \mathbf{v}_1 + (c_2 + 3c_3) \mathbf{v}_2\end{aligned}$$

- Thus  $\mathbf{x}$  is in  $\text{Span} \{\mathbf{v}_1, \mathbf{v}_2\}$ , so every vector in  $H$  already belongs to  $\text{Span} \{\mathbf{v}_1, \mathbf{v}_2\}$ .
- We conclude that  $H$  and  $\text{Span} \{\mathbf{v}_1, \mathbf{v}_2\}$  are actually the set of vectors.
- It follows that  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is a basis of  $H$  since  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly independent.

# BASIS FOR COL $B$

- **Example 8:** Find a basis for  $\text{Col } B$ , where

$$B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_5] = \begin{bmatrix} \overset{\text{pivot}}{1} & 4 & \overset{\text{pivot}}{0} & 2 & \overset{\text{pivot}}{0} \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Reduced echelon form}$$

- **Solution:** Each nonpivot column of  $B$  is a linear combination of the pivot columns.
- In fact,  $\mathbf{b}_2 = 4\mathbf{b}_1$  and  $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$ .
- By the Spanning Set Theorem, we may discard  $\mathbf{b}_2$  and  $\mathbf{b}_4$ , and  $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$  will still span  $\text{Col } B$ .

## BASIS FOR COL $B$

- Let

$$S = \{b_1, b_3, b_5\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- Since  $b_1 \neq 0$  and no vector in  $S$  is a linear combination of the vectors that precede it,  $S$  is linearly independent. (Theorem 4).
- Thus  $S$  is a basis for  $\text{Col } B$ .

# BASES FOR NUL $A$ AND COL $A$

- **Theorem 6:** The pivot columns of a matrix  $A$  form a basis for Col  $A$ .
- **Proof:** Let  $B$  be the reduced echelon form of  $A$ .
- The set of pivot columns of  $B$  is linearly independent, for no vector in the set is a linear combination of the vectors that precede it.
- Since  $A$  is row equivalent to  $B$ , the pivot columns of  $A$  are linearly independent as well, because any linear dependence relation among the columns of  $A$  corresponds to a linear dependence relation among the columns of  $B$ .

## BASES FOR $\text{NUL } A$ AND $\text{COL } A$

- For this reason, every nonpivot column of  $A$  is a linear combination of the pivot columns of  $A$ .
- Thus the nonpivot columns of  $A$  may be discarded from the spanning set for  $\text{Col } A$ , by the Spanning Set Theorem.
- This leaves the pivot columns of  $A$  as a basis for  $\text{Col } A$ .
- **Warning:** The pivot columns of a matrix  $A$  are evident when  $A$  has been reduced only to echelon form.
- But, be careful to use the pivot columns of  $A$  itself for the basis of  $\text{Col } A$ .

# BASES FOR NUL $A$ AND COL $A$

- Row operations can change the column space of a matrix.
- The columns of an echelon form  $B$  of  $A$  are often not in the column space of  $A$ .
- **Two Views of a Basis**
- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.
- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span  $V$ .

# TWO VIEWS OF A BASIS

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- Thus a basis is a spanning set that is as small as possible.
- A basis is also a linearly independent set that is as large as possible.
- If  $S$  is a basis for  $V$ , and if  $S$  is enlarged by one vector—say,  $\mathbf{w}$ —from  $V$ , then the new set cannot be linearly independent, because  $S$  spans  $V$ , and  $\mathbf{w}$  is therefore a linear combination of the elements in  $S$ .