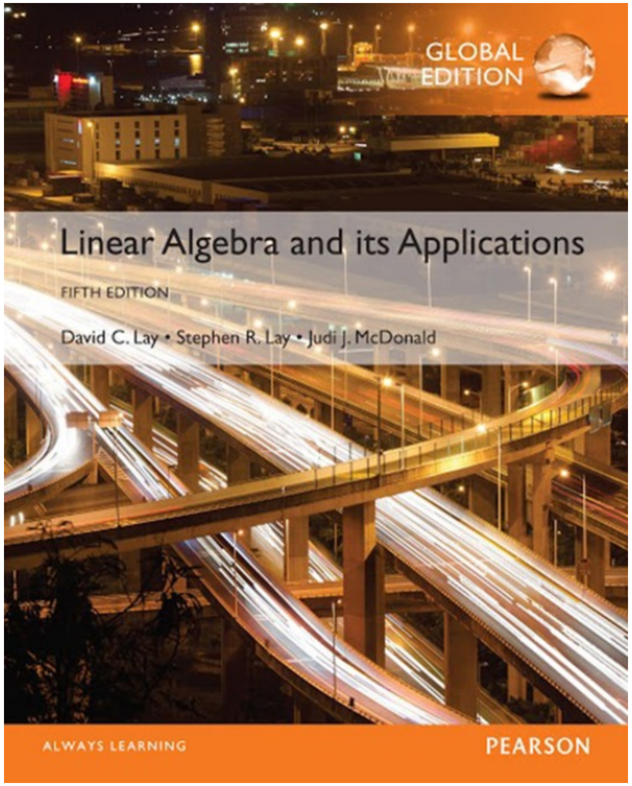
5 Eigenvalues and Eigenvectors

5.5

COMPLEX EIGENVALUES



COMPLEX EIGENVALUES

- The matrix eigenvalue-eigenvector theory already developed for $\mathbb{R}^n$ applies equally well to $\mathbb{C}^n$.

- So a complex scalar λ satisfied $\det(A\text{-}\lambda I) = 0$ if and only if there is a nonzero vector x in $\mathbb{C}^n$ such that $A\mathrm{x} = \lambda\mathrm{x}$.

- We call λ a **(complex) eigenvalue** and x a **(complex) eigenvector** corresponding to λ.

COMPLEX EIGENVALUES

- **Example 1** If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ on $\mathbb{R}^2$ rotates the plane counterclockwise through a quarter-turn.
- The action of A is periodic, since after four quarter-turns, a vector is back where it started.
- Obviously, no nonzero vector is mapped into a multiple of itself, so A has no eigenvectors in $\mathbb{R}^2$ and hence no real eigenvalues.
- In fact, the characteristic equation of A is

$$\lambda^2 + 1 = 0$$

COMPLEX EIGENVALUES

- The only roots are complex: $\lambda = i$ and $\lambda = -i$. However, if we permit A to act on $\mathbb{C}^2$, then

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- Thus i and $-i$ are eigenvalues, with $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ and $\begin{bmatrix} 1 \\ i \end{bmatrix}$ as corresponding eigenvectors.

REAL AND IMAGINARY PARTS OF VECTORS

- The complex conjugate of a complex vector x in $\mathbb{C}^n$ is the vector $\bar{x}$ in $\mathbb{C}^n$ whose entries are the complex conjugates of the entries in x.

- The **real** and **imaginary parts** of a complex vector x are the vectors Re x and Im x in $\mathbb{R}^n$ formed from the real and imaginary parts of the entries of x.

REAL AND IMAGINARY PARTS OF VECTORS

- **Example 4** If $x = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, then

$\text{Re } \mathrm{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\text{Im } \mathrm{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, and $\bar{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix}$

EIGENVALUES AND EIGENVECTORS OF A REAL MATRIX THAT ACTS ON

- **Theorem 9**: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector v in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \text{ where } P = [\operatorname{Re} \mathbf{v} \quad \operatorname{Im} \mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$