2.9 *Voronoi sets and polyhedral decomposition.* Let $x_0, \dots, x_K \in \mathbf{R}^n$ be distinct. Consider the set of points that are closer (in Euclidean norm) to x_0 than the other x_i, *i.e.*,

$$V = \{x \in \mathbf{R}^n \mid \|x - x_0\|_2 \le \|x - x_i\|_2,\ i = 1, \dots, K\}.$$

V is called the *Voronoi region* around x_0 with respect to $x_1, \dots, x_K$.

(a) Show that V is a polyhedron. Express V in the form $V = \{x \mid Ax \preceq b\}$.

$$\|x - x_0\|_2 \le \|x - x_i\|_2 \Rightarrow (x - x_0)^T (x - x_0) \le (x - x_i)^T (x - x_i) \Rightarrow$$

$$\Rightarrow x^T x - 2x_0^T x + \|x_0\|_2^2 \le x^T x - 2x_i^T x + \|x_i\|_2^2 \rightsquigarrow 2(x_i - x_0)^T x \le \|x_i\|_2^2 - \|x_0\|_2^2 \Rightarrow$$

$$A = \begin{pmatrix} 2(x_1 - x_0)^T \\ 2(x_2 - x_0)^T \\ \vdots \\ 2(x_K - x_0)^T \end{pmatrix} \quad b = \begin{pmatrix} \|x_1\|_2^2 - \|x_0\|_2^2 \\ \|x_2\|_2^2 - \|x_0\|_2^2 \\ \vdots \\ \|x_K\|_2^2 - \|x_0\|_2^2 \end{pmatrix} \Rightarrow V = \{x \mid Ax \le b\}$$

(b) Conversely, given a polyhedron P with nonempty interior, show how to find $x_0, \dots, x_K$ so that the polyhedron is the Voronoi region of x_0 with respect to $x_1, \dots, x_K$.

$$V = \{x \mid Ax \le b\} \rightarrow \left\{ A = \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \right.$$

برای بدست آوردن x_iها با توجه به قسمت قبل داریم

$$\rightarrow b_i = \lambda_i \{\|x_i\|^2 - \|x_0\|^2\}$$

و

$$a_i = 2\lambda_i \{x_i - x_0\}$$

بدین صورت می‌توان برای یک x_0، x_iها را بدست آورد.

$$\frac{a_i}{2\lambda_i} + x_0 = x_i \rightsquigarrow \|x_i\|^2 = \|x_0\|^2 + \frac{\|a_i\|^2}{4\lambda_i^2} + \frac{x_0^T a_i}{\lambda_i} \rightsquigarrow$$

$$\rightsquigarrow b_i = \lambda_i \left\{ \frac{\|a_i\|^2}{4\lambda_i^2} + \frac{x_0^T a_i}{\lambda_i} \right\} \rightsquigarrow b_i = \frac{\|a_i\|^2}{4\lambda_i} + x_0^T a_i \Rightarrow \lambda_i = \frac{\|a_i\|^2}{4(b_i - x_0^T a_i)}$$

$$x_i = x_0 + \frac{a_i}{2\lambda_i} \rightsquigarrow x_i = x_0 + \frac{2(b_i - x_0^T a_i)}{\|a_i\|^2} a_i$$

(c) We can also consider the sets

$$V_k = \{x \in \mathbf{R}^n \mid \|x - x_k\|_2 \le \|x - x_i\|_2,\ i \ne k\}.$$

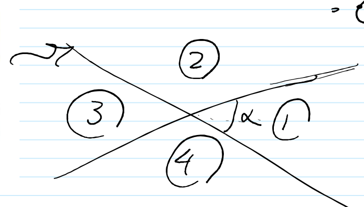
The set V_k consists of points in $\mathbf{R}^n$ for which the closest point in the set $\{x_0, \dots, x_K\}$ is x_k.

The sets $V_0, \dots, V_K$ give a polyhedral decomposition of $\mathbf{R}^n$. More precisely, the sets V_k are polyhedra with nonempty interior, $\bigcup_{k=0}^K V_k = \mathbf{R}^n$, and $\mathbf{int}\, V_i \cap \mathbf{int}\, V_j = \emptyset$ for $i \ne j$, *i.e.*, V_i and V_j intersect at most along a boundary.

Suppose that $P_1, \dots, P_m$ are polyhedra with nonempty interior such that $\bigcup_{i=1}^m P_i = \mathbf{R}^n$, $\mathbf{int}\, P_i \cap \mathbf{int}\, P_j = \emptyset$ for $i \ne j$. Can this polyhedral decomposition of $\mathbf{R}^n$ be described as the Voronoi regions generated by an appropriate set of points?

خیر لزوماً همچین اتفاقی نمی‌افتد

مثال نقض:



در نظر بگیرید که $0 < \alpha < \pi/2$ باشد. در این صورت x_0 هر جای درون فضای 1 باشد.

طبق فضای voronoi داریم که x_0 درون فضای 2 دقیقاً ... به خط جدا کننده این دو فضا

در نظر بگیرید که $0 \leq \alpha \leq \frac{\pi}{2}$ باشه. در این صورت x_p هر جای درون فضای (1) باشه.
طبق فضای varanoi داریم که x_p درون فضای 2 دقیقاً باید نسبت به خط جداکننده این دو فضا
به صورت آینه باشه.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \le a^T x \le \beta\}$.

$\hookrightarrow \alpha \le a^T x \to$ half Space (1)
$a^T x \le \beta \to$ half Space (II)
$\Big\} \Rightarrow$ Intersection of two half Space $\to$ Convex set

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \dots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

$\{x \in \mathbb{R}^n \mid \alpha_i \le x_i \le \beta_i,\ i = 1, \dots, n\} \to \{x \in \mathbb{R}^n \mid \alpha \le a^T x \le \beta\} \to$ Convex (در قسمت a اثبات کردیم)

$\alpha = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{Bmatrix},\ \beta = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{Bmatrix},\ a = \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}$

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\}$.

$\{x \in \mathbb{R}^n \mid a_1^T x \le b_1,\ a_2^T x \le b_2\} = \{x \in \mathbb{R}^n \mid a_1^T x \le b_1\} \cap \{x \in \mathbb{R}^n \mid a_2^T x \le b_2\}$

$\hookrightarrow$ Intersection of two $\underbrace{\text{half-space}}_{\text{Convex Set}} \to$ Convex Set

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

$\{x \mid \|x - x_0\|_2 \le \|x - z\|_2,\ z \in S\} \rightsquigarrow \|x - x_0\|_2 \le \|x - z\|_2 \to (x - x_0)^T (x - x_0) \le (x - z)^T (x - z)$

$\Rightarrow \|x\|_2^2 - 2x_0^T x + \|x_0\|_2^2 \le \|x\|_2^2 - 2z^T x + \|z\|_2^2 \to 2(z - x_0)^T x \le \|z\|_2^2 - \|x_0\|_2^2$

$\hookrightarrow a^T x \le b \to$ Convex Set

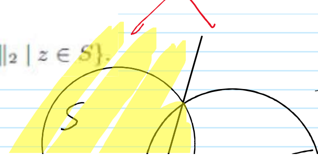
Desired Set $= \bigcap_S \{x \mid \|x - x_0\|_2 \le \|x - z\|_2,\ z \in S\} \to$ Intersection of Convex Sets
$\hookrightarrow$ Convex Set

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

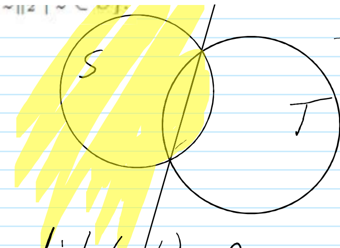
$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

if $S \cap T \ne \emptyset \to$ 

$\to$ this set isn't Convex

$\mathrm{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$.

if $S \cap T \neq \emptyset \rightarrow$



$\rightarrow$ this set isn't Convex

$x \in S \cap T \rightarrow \mathrm{dist}(x, S) = \mathrm{dist}(x, T) = 0 \rightarrow x \in \text{Set}$

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

$x + S_2 \subseteq S_1 \rightarrow \forall y \in S_2: \ x + y \in S_1$

$\{x \mid x + S_2 \subseteq S_1\} \rightarrow \bigcap_{\forall y \in S_2} \{x \mid x + y \in S_1\} \rightarrow \bigcap_{\forall y \in S_2} \{S_1 - y\} \rightsquigarrow$ Intersection of Convex Sets

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\}$. You can assume $a \neq b$ and $0 \leq \theta \leq 1$.

$$\{x \mid \|x - a\|_2 \leq \theta \|x - b\|_2\} = \{x \mid \|x - a\|_2^2 \leq \theta^2 \|x - b\|_2^2\} = \{x \mid (x-a)^T (x-a) \leq \theta^2 (x-b)^T (x-b)\}$$

$$= \{x \mid (1 - \theta^2) x^T x - 2(a - \theta^2 b)^T x + a^T a - \theta^2 b^T b \leq 0\}$$

if $\theta = 1 \rightsquigarrow \text{Set} = \{x \mid -2(a - b)^T x + a^T a - b^T b \leq 0\} \rightarrow$ half-Space $\rightarrow$ Convex

if $\theta \neq 1 \rightarrow \text{Set} = \left\{x \mid x^T x - \frac{2(a - \theta^2 b)^T}{1 - \theta^2} x + \frac{a^T a - \theta^2 b^T b}{1 - \theta^2} \leq 0\right\} =$

$$= \left\{x \mid \left\| x - \frac{a - \theta^2 b}{1 - \theta^2} \right\|_2^2 \leq \frac{\theta^2 b^T b - a^T a}{1 - \theta^2} - \left\| \frac{a - \theta^2 b}{1 - \theta^2} \right\|_2^2 \right\}$$

$\hookrightarrow$ a ball $\rightarrow$ Convex

2.19 *Linear-fractional functions and convex sets.* Let $f : \mathbf{R}^m \to \mathbf{R}^n$ be the linear-fractional function

$$f(x) = (Ax + b)/(c^T x + d), \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}.$$

In this problem we study the inverse image of a convex set C under f, *i.e.*,

$$f^{-1}(C) = \{x \in \mathbf{dom}\, f \mid f(x) \in C\}.$$

For each of the following sets $C \subseteq \mathbf{R}^n$, give a simple description of $f^{-1}(C)$.

(a) The halfspace $C = \{y \mid g^T y \leq h\}$ (with $g \neq 0$).

$$f^{-1}(C) = \{x \in \mathrm{dom} f \mid g^T f(x) \leq h\} = \{x \mid g^T \frac{(Ax+b)}{c^T x + d} \leq h,\ c^T x + d > 0\} =$$

$$= \{x \mid g^T (Ax+b) \leq h c^T x + hd,\ c^T x + d > 0\} = \{x \mid (g^T A - h c^T) x \leq hd - g^T b,\ c^T x + d > 0\}$$

$$= \{x \mid (A^T g - hc)^T x \leq hd - g^T b,\ c^T x + d > 0\} \rightarrow$$ Intersection of a halfspace with dom f

(b) The polyhedron $C = \{y \mid Gy \preceq h\}$.

$$f^{-1}(C) = \{x \in \mathrm{dom} f \mid G f(x) \preceq h\} = \{x \mid G \frac{(Ax+b)}{c^T x + d} \preceq h,\ c^T x + d > 0\} =$$

$$= \{x \mid GAx + Gb \preceq h c^T x + hd,\ c^T x + d > 0\} = \{x \mid (GA - hc^T) x \preceq hd - Gb,\ c^T x + d > 0\}$$

$\hookrightarrow$ Intersection of a polyhedron with dom f

(c) The ellipsoid $\{y \mid y^T P^{-1} y \leq 1\}$ (where $P \in \mathbf{S}^n_{++}$).

$$f^{-1}(C) = \{x \in \mathrm{dom} f \mid f(x)^T P^{-1} f(x) \leq 1\} = \{x \in \mathrm{dom} f \mid \frac{(Ax+b)^T}{(c^T x + d)} P^{-1} \frac{(Ax+b)}{c^T x + d} \leq 1\} =$$

$$= \{x \in \mathrm{dom} f \mid (Ax+b)^T P^{-1} (Ax+b) \leq (c^T x + d)^2 = (c^T x + d)^T (c^T x + d)\} =$$

$$= \{x \in \mathrm{dom} f \mid x^T A^T P^{-1} A x + b^T P^{-1} b + 2 b^T P^{-1} A x \leq x^T c c^T x + d^T d + 2d c^T x\} =$$

$$= \{x \in \mathrm{dom} f \mid x^T (A^T P^{-1} A - cc^T) x + 2(b^T P^{-1} A - dc^T) x \leq d^2 - b^T P^{-1} b\} =$$

$$= \{x \mid x^T (A^T P^{-1} A - cc^T) x + 2(A^T P^{-1} b - dc)^T x \leq d^2 - b^T P^{-1} b,\ c^T x + d > 0\}$$

if $A^T P^{-1} A - cc^T \succ 0 \longrightarrow$ ellipsoid intersecting with dom f

(d) The solution set of a linear matrix inequality, $C = \{y \mid y_1 A_1 + \cdots + y_n A_n \preceq B\}$, where $A_1, \ldots, A_n, B \in \mathbf{S}^p$.

$$f^{-1}(C) = \{x \in \mathrm{dom} f \mid \sum f_i(x) A_i \preceq B\} = \{x \in \mathrm{dom} f \mid \frac{\sum (a_i^T x + b_i)}{c^T x + d} A_i \preceq B\} =$$

$$= \{ x \in \mathrm{dom} f \mid \sum (a_i^T x + b_i) A_i \preceq B (c^T x + d) \} =$$

$$= \{ x \in \mathrm{dom} f \mid \sum ((a_i^T x) A_i) - B c^T x \preceq dB - \sum b_i A_i \}$$

Intersection of a linear matrix inequality with dom f

2.20. Strictly positive solution of LE

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2.20 *Strictly positive solution of linear equations.* Suppose $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, with $b \in \mathcal{R}(A)$. Show that there exists an x satisfying

$$x \succ 0, \qquad Ax = b$$

if and only if there exists no λ with

$$A^T\lambda \succeq 0, \qquad A^T\lambda \neq 0, \qquad b^T\lambda \leq 0.$$

Hint. First prove the following fact from linear algebra: $c^Tx = d$ for all x satisfying $Ax = b$ if and only if there is a vector λ such that $c = A^T\lambda$, $d = b^T\lambda$.

اول Hint را ثابت می‌کنیم

$\hookrightarrow c = A^T\lambda$ و $d = b^T\lambda$ چون $Ax = b \rightsquigarrow \lambda^T A x = \lambda^T b \Rightarrow c^T x = d$

حال به عکس این عبارت را اثبات می‌کنیم:

فرض کنیم $x_0 \in R(A)$ در نتیجه $Ax_0 = b$

حال x را به صورت $x = Fy + x_0$ در نظر بگیریم که $R(F) = N(A)$

$\forall x: c^Tx = d \Rightarrow \forall y: c^TFy + c^Tx_0 = d$

که این اتفاق تنها در حالتی ممکن است که $F^Tc = 0$ باشد.

$\hookrightarrow c \in N(F^T) = R(A^T) \rightsquigarrow \exists \lambda: A^T\lambda = c \Rightarrow$

$\Rightarrow c^Tx_0 = d = \lambda^TAx_0 = b^T\lambda$

حال به سراغ مسئله اصلی می‌رویم:

دو فضای $D = \{x \mid Ax = b\}$ و $C = R^n_{++}$ را در نظر بگیریم

اگر این دو فضا اشتراک با هم نداشته باشند، می‌توانیم hyperplane ای در نظر بگیریم که

$c^Tx \geq d : \forall x \in C$ و $c^Tx \leq d : \forall x \in D$, $c \neq 0$

$\hookrightarrow c \geq 0$ و $d \leq 0$

با توجه به اینکه D یک affine set است، مقدار c^Tx روی آن ثابت است،

$c^Tx = d$

حال با توجه به اینکه $D = \{x \mid Ax = b\}$ و hint وجود دارد λ به صورتی که

$c = A^T\lambda$, $d = b^T\lambda$

$c \geq 0 \to A^T\lambda \geq 0$, $d \leq 0 \to b^T\lambda \leq 0$ و $c \neq 0 \to A^T\lambda \neq 0$

2.24. Supporting hyperplanes

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2.24 *Supporting hyperplanes.*

(a) Express the closed convex set $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

$x_1 x_2 \geq 1 \rightsquigarrow x_1 \geq 1/x_2 \rightarrow$ این Convex Set را می‌توانیم با اشتراک نیم‌صفحه‌هایی که

از شیب تمام نقاط روی تابع $x_1 x_2 = 1$ ایجاد می‌شوند

پیدا کنیم. نقطه‌ای مانند $x = (t, 1/t)$ در نظر بگیرید. $\rightarrow t > 0$

$$\frac{x_2 - 1/t}{x_1 - t} = -1/t^2 \Rightarrow x_2 - 1/t = -\frac{x_1}{t^2} + 1/t \Rightarrow x_2 + \frac{x_1}{t^2} = 2/t$$

$$\text{Set: } \bigcap_{t>0} \left\{ x \;\middle|\; x_2 + \frac{x_1}{t^2} \geq \frac{2}{t} \right\}$$

(b) Let $C = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbf{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

$$C = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\} \longrightarrow \text{Boundary } C: \{x \in \mathbb{R}^n \mid \|x\|_\infty = 1\}$$

$$\text{hyperplane} \rightarrow \forall x \in C: a^T x \geq a^T \hat{x}$$

$x^{(i)}$ را به این صورت تعریف می‌کنیم که عنصر iام برابر با 1 است و بقیه برابر با صفر هستند.

$$a^T \hat{x} \leq a^T x^{(i)} \quad \text{for } 1 \leq i \leq n$$

$$\hookrightarrow \begin{cases} a_i \geq 0 & \text{if } \hat{x}_i = 1 \\ a_i \leq 0 & \text{if } \hat{x}_i = -1 \\ a_i = 0 & \text{if } -1 \leq \hat{x}_i \leq 1 \end{cases}$$

2.33. Monotone Nonnegative cone.

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2.33 *The monotone nonnegative cone.* We define the *monotone nonnegative cone* as

$$K_{m+} = \{x \in \mathbf{R}^n \mid x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order.

(a) Show that K_{m+} is a proper cone.

if $x \in K_{m+} \Rightarrow -x \in K_{m+}$ if $x = 0$

$$x \in K_{m+} \Rightarrow x_1 \ge x_2 \ge \cdots \ge x_n \ge 0 \xrightarrow{\theta \ge 0} \theta x_1 \ge \theta x_2 \ge \cdots \ge \theta x_n \Rightarrow \theta x \in K_{m+}$$

Proof of Convexity:

$$\left.\begin{array}{l} x^{(1)} \in K_{m+} \Rightarrow x_1^{(1)} \ge x_2^{(1)} \ge \cdots \ge x_n^{(1)} \ge 0 \\ x^{(2)} \in K_{m+} \Rightarrow x_1^{(2)} \ge x_2^{(2)} \ge \cdots \ge x_n^{(2)} \ge 0 \\ 0 \le \alpha \le 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \alpha x_1^{(1)} \ge \alpha x_2^{(1)} \ge \cdots \ge \alpha x_n^{(1)} \ge 0 \\ (1-\alpha) x_1^{(2)} \ge (1-\alpha) x_2^{(2)} \ge \cdots \ge (1-\alpha) x_n^{(2)} \ge 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \alpha x_1^{(1)} + (1-\alpha) x_1^{(2)} \ge \alpha x_2^{(1)} + (1-\alpha) x_2^{(2)} \ge \cdots \ge \alpha x_n^{(1)} + (1-\alpha) x_n^{(2)} \ge 0 \Rightarrow$$

$$\Rightarrow \alpha x^{(1)} + (1-\alpha) x^{(2)} \in K_{m+} \Rightarrow$$

$\Rightarrow K_{m+}$ is a proper Cone

(b) Find the dual cone K^*_{m+}. *Hint.* Use the identity

$$\sum_{i=1}^n x_i y_i = (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n).$$

$$K^*_{m+} : \{ y \mid x^T y \ge 0 \text{ for all } x \in K_{m+} \}$$

K_{m+} طبق تعریف

$$x^T y = \sum_{i=1}^n x_i y_i = \sum_{i=1}^n \alpha_i \sum_{j=1}^i y_j \rightarrow \alpha_i = \begin{cases} x_i - x_{i+1} & i < n \\ x_i & i = n \end{cases} \rightarrow \alpha_i \ge 0$$

$$x = (t, 0, 0, \ldots, 0) \in K_{m+} \rightarrow \alpha_i = \delta_{i,1} \rightarrow y_1 \ge 0$$

$$x = (t, t, 0, \ldots, 0) \in K_{m+} \longrightarrow \alpha_i = \delta_{i,2} \rightarrow y_1 + y_2 \ge 0$$

به همین صورت می‌توان اثبات کرد $\sum_{i=1}^k y_i \ge 0$ for $1 \le k \le n$

در نتیجه فضای dual cone K^*_{m+} را می‌توان به صورت زیر ساخت

$$K^*_{m+} : \{ y \mid \sum_{i=1}^k y_i \ge 0 \text{ for all } 1 \le k \le n \}$$

2.5. Dual of intersection of cones

Friday, November 10, 2023 7:35 PM

2.5 *Dual of intersection of cones.* Let C and D be closed convex cones in $\mathbf{R}^n$. In this problem we will show that

$$(C \cap D)^* = C^* + D^*$$

when $C^* + D^*$ is closed. Here, $+$ denotes set addition: $C^* + D^*$ is the set $\{u + v \mid u \in C^*,\ v \in D^*\}$. In other words, the dual of the intersection of two closed convex cones is the sum of the dual cones. (A sufficient condition for of $C^* + D^*$ to be closed is that $C \cap \mathbf{int}\, D \neq \emptyset$. The general statement is that $(C \cap D)^* = \mathbf{cl}(C^* + D^*)$, and that the closure is unnecessary if $C \cap \mathbf{int}\, D \neq \emptyset$, but we won't ask you to show this.)

(a) Show that $C \cap D$ and $C^* + D^*$ are convex cones.

$$y \in C \cap D \Rightarrow \left\{ \begin{array}{l} y \in C \xrightarrow[\text{Cone}]{\theta \geq 0} \theta y \in C \\ y \in D \xrightarrow[\text{Cone}]{\theta \geq 0} \theta y \in D \end{array} \right\} \Rightarrow \theta y \in C \cap D$$

$$y_1, y_2 \in C \cap D \Rightarrow \left\{ \begin{array}{l} y_1, y_2 \in C \xrightarrow[\text{Convex}]{0 \leq \alpha \leq 1} \alpha y_1 + (1-\alpha) y_2 \in C \\ y_1, y_2 \in D \xrightarrow[\text{Convex}]{0 \leq \alpha \leq 1} \alpha y_1 + (1-\alpha) y_2 \in D \end{array} \right\} \Rightarrow \alpha y_1 + (1-\alpha) y_2 \in C \cap D$$

$C \cap D$ is a Convex Cone

C^*, D^* are Convex Cones.

$$C^* + D^* = \{x + y \mid x \in C^*, y \in D^*\}$$

$$\left. \begin{array}{l} x \in C^* \\ y \in D^* \end{array} \right\} \rightarrow x + y \in C^* + D^* \qquad \left. \begin{array}{l} x \in C^* \rightarrow \theta x \in C^* \\ y \in D^* \rightarrow \theta y \in D^* \end{array} \right\} \Rightarrow \theta(x+y) \in C^* + D^*$$

$$\left. \begin{array}{l} x_1 \in C^* \\ y_1 \in D^* \end{array} \right\} \Rightarrow x_1 + y_1 \in C^* + D^* \qquad \left. \begin{array}{l} x_2 \in C^* \\ y_2 \in D^* \end{array} \right\} \Rightarrow x_2 + y_2 \in C^* + D^*$$

$$\left. \begin{array}{l} x_1, x_2 \in C^* \Longrightarrow \alpha x_1 + (1-\alpha) x_2 \in C^* \\ y_1, y_2 \in D^* \Longrightarrow \alpha y_1 + (1-\alpha) y_2 \in C^* \end{array} \right\} \Rightarrow \alpha(x_1 + y_1) + (1-\alpha)(x_2 + y_2) \in C^* + D^*$$

$\Rightarrow C^* + D^*$ is a Convex Cone

(b) Show that $(C \cap D)^* \supseteq C^* + D^*$.

$$z \in C^* + D^* \rightarrow z = x + y \left\{ \begin{array}{l} x \in C^* \rightarrow x^T u \geq 0 : \forall u \in C \\ y \in D^* \rightarrow y^T u \geq 0 : \forall u \in D \end{array} \right\} \Rightarrow (x+y)^T u \geq 0 \Rightarrow$$

$\Rightarrow z^T u \geq 0$

$\hookrightarrow \forall u \in C \cap D$

$$C^* + D^* \subseteq (C \cap D)^* \Longleftarrow z \in (C \cap D)^*$$

(c) Now let's show $(C \cap D)^* \subseteq C^* + D^*$ when $C^* + D^*$ is closed. You can do this by first showing

$$(C \cap D)^* \subseteq C^* + D^* \iff C \cap D \supseteq (C^* + D^*)^*.$$

You can use the following result:

If K is a closed convex cone, then $K^{**} = K$.

(c) Now let's show $(C\cap D)^* \subseteq C^* + D^*$ when $C^* + D^*$ is closed. You can do this by first showing

$$(C\cap D)^* \subseteq C^* + D^* \iff C\cap D \supseteq (C^* + D^*)^*.$$

You can use the following result:

If K is a closed convex cone, then $K^{**} = K$.

Next, show that $C\cap D \supseteq (C^* + D^*)^*$ and conclude $(C\cap D)^* = C^* + D^*$.

ابتدا اثبات می‌کنیم $X \subseteq Y \iff X^* \supseteq Y^*$

$a \in X$, $b \in Y^*$ $\quad X \subseteq Y \Rightarrow a \in Y \Rightarrow \forall_{a\in X}\ a^T b \geq 0 \Rightarrow b \in X^* \Rightarrow X^* \supseteq Y^*$

برای برعکس $(X^* \supseteq Y^* \Rightarrow X \subseteq Y)$ نیز می‌توانیم مشابه همین کار را انجام دهیم

$$(C\cap D)^* \subseteq C^* + D^* \iff C\cap D \supseteq (C^* + D^*)^*$$

$x \in (C^* + D^*)^* \longrightarrow x^T y \geq 0 \rightsquigarrow y \in C^* + D^*$

$y = u + v : u \in C^*, v \in D^*$

$$\left.\begin{array}{l} v = 0 \rightarrow x^T u \geq 0 \rightsquigarrow x \in C^{**} = C \\ u = 0 \rightarrow x^T v \geq 0 \rightsquigarrow x \in D^{**} = D \end{array}\right\} \Rightarrow x \in C\cap D \Rightarrow$$

$\Rightarrow (C^* + D^*)^* \subseteq C\cap D \Rightarrow (C\cap D)^* \subseteq C^* + D^*$

از قسمت قبل داریم که $(C\cap D)^* \supseteq C^* + D^*$ در نتیجه $(C\cap D)^* = C^* + D^*$

(d) Show that the dual of the polyhedral cone $V = \{x \mid Ax \succeq 0\}$ can be expressed as

$$V^* = \{A^T v \mid v \succeq 0\}.$$

$V = \{x \mid Ax \geq 0\} = \{x \mid a_i^T x \geq 0$ for تمامی $i\}$

با توجه به قسمت قبل : $V^* = \bigcap_i \{x \mid a_i^T x \geq 0\}^*$

Dual of a half space $S = \{x \mid x^T a \geq 0\} \rightsquigarrow S^* = \{y \mid x^T y \geq 0 : \forall x \in S\}$

$x^T a \geq 0 \rightsquigarrow \lambda x^T a \geq 0$ for any $\lambda \geq 0 \rightsquigarrow S^* = \{\lambda a \mid \lambda \geq 0\}$

$V^* = \bigcap_i \{\lambda_i a_i \mid \lambda_i \geq 0\} = \{A^T \lambda \mid \lambda \geq 0\}$

2.6 *Polar of a set.* The *polar* of $C \subseteq \mathbf{R}^n$ is defined as the set

$$C^\circ = \{y \in \mathbf{R}^n \mid y^T x \le 1 \text{ for all } x \in C\}.$$

(a) Show that C° is convex (even if C is not).

$$\left.\begin{array}{l} y_1 \in C^o \to \forall x \in C: y_1^T x \le 1 \to \forall x \in C: \alpha y_1^T x \le \alpha \\ y_2 \in C^o \to \forall x \in C: y_2^T x \le 2 \to \forall x \in C: (1-\alpha) y_2^T x \le 1-\alpha \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\alpha y_1 + (1-\alpha) y_2)^T x \le 1 : \forall x \in C \Rightarrow \alpha y_1 + (1-\alpha) y_2 \in C^o \Rightarrow$$

$$\Rightarrow C^o \text{ is Convex}$$

(b) What is the polar of a cone?

Cone: $C^o = \{y \in \mathbb{R}^n \mid y^T x \le 1 : \forall x \in C\}$

$$\left.\begin{array}{l} x_1 \in C \Rightarrow y^T x_1 \le 1 \\ \forall \theta > 0: \theta x_1 \in C \Rightarrow \theta y^T x_1 \le 1 \quad \forall \theta > 0 \end{array}\right\} \to$$

اگر $y^T x \le 0$ نباشه، می‌توان θ را مقداری پیدا کرد که $\theta y^T x > 1$ باشه که Convex بودن C^o را خراب می‌کند

$$\Rightarrow C^o = \{y \in \mathbb{R}^n \mid y^T x \le 0 : \forall x \in C\}$$

Dual of Cone: $C^* = \{y \in \mathbb{R}^n \mid y^T x \ge 0 : \forall x \in C\} \Rightarrow C^o = -C^*$

(c) What is the polar of the unit ball for a norm $\|\cdot\|$?

$$C = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$$

$$C^o = \{y \in \mathbb{R}^n \mid y^T x \le 1 : \forall x \in C\} = \{y \in \mathbb{R}^n \mid y^T x \le 1 \text{ و } \|x\| \le 1\}$$

unit ball of dual norm

(d) What is the polar of the set $C = \{x \mid \mathbf{1}^T x = 1, \ x \succeq 0\}$?

$$C = \{x \mid \sum x_i = 1, x \ge 0\}$$

$C^o = \{y \mid y^T x \le 1 : \forall x \in C\}$ پس $\sum y_i x_i \le 1$

$\to u^{(i)} \in C$

بردار $u^{(i)}$ را به این صورت تعریف می‌کنیم که تنها درایه iام آن برابر با یک است و بقیه صفر.

بردار u را به این صورت تعریف می‌کنیم که همه درایه‌های آن برابر با یک است و بقیه صفر.

$$y \in C^\circ \Rightarrow y^T u^{(i)} \leq 1 \Rightarrow \sum y_i u_i^{(i)} \leq 1 \Rightarrow y_i u_i^{(i)} \leq 1 \Rightarrow y_i \leq 1 \quad for \ 1 \leq i \leq n$$

$$C^\circ = \{ y \mid \|y\|_\infty \leq 1 \}$$

(e) Show that if C is closed and convex, with $0 \in C$, then $(C^\circ)^\circ = C$.

$$\left.\begin{array}{l} x \in C \\ y \in C^\circ \end{array}\right\} \Rightarrow y^T x \leq 1 \Rightarrow x^T y \leq 1 \Rightarrow x \in (C^\circ)^\circ \Rightarrow C \subseteq (C^\circ)^\circ$$

حال می‌خواهیم اثبات کنیم $(C^\circ)^\circ \subseteq C$

$$x \in (C^\circ)^\circ / C \Rightarrow x \in (C^\circ)^\circ, \ x \notin C$$

می‌توان یک hyperplane پیدا کرد که C و $\{x\}$ را جدا کند.

$$\left.\begin{array}{l} a^T x \geq b, \quad a^T y \leq b : \forall y \in C \\ 0 \in C \end{array}\right\} \Rightarrow b \geq 0$$

$$a^T x \geq b \xrightarrow{\text{Scaling } a} \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) x \geq 1 \text{ پس } \tilde{a}^T x \geq 1$$

$$\tilde{a}^T y \leq 1 : \forall y \in C \text{ پس } \tilde{a} \in C^\circ \Rightarrow a \in C^\circ \Big\} \Rightarrow x \notin C$$

در نتیجه داریم که $(C^\circ)^\circ \subseteq C$

$$(C^\circ)^\circ \subseteq C, \quad C \subseteq (C^\circ)^\circ \Rightarrow C = (C^\circ)^\circ$$