5-10- Optimal Experiment design:

a) D-Optimal design

$$\text{minimize} \quad \log\det X^{-1}$$
$$\text{s.t.} \quad X = \sum x_i v_i v_i^T$$
$$x \succeq 0, \quad \mathbf{1}^T x = 1$$

$$L(x, Z, \lambda, \alpha) = \log\det X^{-1} + tr\left(Z\left(X - \sum x_i v_i v_i^T\right)\right) - \lambda^T x + \alpha(\mathbf{1}^T x - 1)$$

$$= \log\det X^{-1} + tr(ZX) - \sum x_i v_i^T Z v_i - \lambda^T x + \alpha \mathbf{1}^T x - \alpha =$$

$$= \log\det X^{-1} + tr(ZX) + \sum x_i\{-v_i^T Z v_i - \lambda_i + \alpha\} - \alpha$$

L is affine on $x_i \Rightarrow$

$$g(Z, \lambda) = \begin{cases} \log\det Z + n - \alpha & \alpha = \lambda_i + v_i^T Z v_i \\ -\infty & \text{otherwise} \end{cases}$$

$$\frac{\partial L(x, Z, \lambda, \alpha)}{\partial X} = 0 \rightsquigarrow X^{-1} = Z$$

dual Problem: $\text{maximize} \quad \log\det Z + n - \alpha$

$\text{s.t.} \quad \alpha \geq v_i^T Z v_i \quad i \in (1, p)$

Change of variables: $Y = Z/\alpha$

dual Problem: $\text{maximize} \quad \log\det Y + n + n\log\alpha - \alpha$

$\text{s.t.} \quad v_i^T Y v_i \leq 1 \quad i \in (1, p)$

We Can easily maximize $n \log \alpha - \alpha$ over α:

$$\frac{\partial}{\partial \alpha}(n \log \alpha - \alpha) = 0 \rightsquigarrow n/\alpha = 1 \rightsquigarrow n = \alpha$$

The Dual Problem:

$$\begin{aligned} &\text{maximize} \quad \log\det Y + n \log n \\ &\text{s.t.} \quad v_i^T Y v_i \leq 1 \end{aligned}$$

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6) A-Optimal Design:

$$\begin{aligned} &\text{minimize} \quad tr(X^{-1}) \\ &\text{subject to} \quad X = \sum_{i=1}^{p} x_i v_i v_i^T \\ &\qquad\qquad x \succeq 0,\ \mathbf{1}^T x = 1 \end{aligned}$$

$$L(X, Z, \lambda, \alpha) = tr(X^{-1}) + tr\left(Z\left(X - \sum x_i v_i v_i^T\right)\right) - \lambda^T x + \alpha(\mathbf{1}^T x - 1)$$

$$= tr(X^{-1}) + tr(ZX) + \sum x_i \{-v_i^T Z v_i - \lambda_i + \alpha\} - \alpha$$

$$\frac{\partial L}{\partial X} = -X^{-2T} + Z^T = 0 \rightsquigarrow X = Z^{-1/2}$$

$$g(Z, \lambda, \alpha) = \begin{cases} -\alpha + 2tr(Z^{+1/2}) & Z \succeq 0,\ v_i^T Z v_i + \lambda_i = \alpha \\ -\infty & \text{otherwise} \end{cases}$$

The Dual Problem: maximize $-\alpha + 2tr(Z^{1/2})$

$$\begin{aligned} \text{s.t.} \quad & Z \succeq 0 \\ & v_i^T Z v_i \leq \alpha \quad \forall i \in (1, p) \end{aligned}$$

With Substituting $Y = Z/\alpha$

$$\text{maximize} \quad -\alpha + 2\sqrt{\alpha}\, tr(Y^{1/2})$$
$$\text{s.t.} \quad Y \succcurlyeq 0$$
$$v_i^T Y v_i \leqslant 1 \quad \forall i \in (1, p)$$

we can easily maximize obj. over α:

$$\frac{\partial}{\partial \alpha}(obj) = -1 + \frac{1}{\sqrt{\alpha}} tr(Y^{1/2}) \Rightarrow \alpha = (tr(Y^{1/2}))^2$$

The Dual Problem:

$$\text{maximize} \quad (tr(Y^{1/2}))^2$$
$$\text{s.t.} \quad Y \succcurlyeq 0$$
$$v_i^T Y v_i \leqslant 1 \quad \forall i \in (1, p)$$

5-40- E-optimal experiment design:

$$\text{minimize} \quad 1/t$$
$$\text{s.t.} \quad \sum x_i v_i v_i^T \succeq tI$$
$$x \succeq 0 \quad , \quad 1^T x = 1$$

$$L(x,Z,t,\lambda,\alpha) = 1/t - tr\left(Z\left(\sum x_i v_i v_i^T - tI\right)\right) - \lambda^T x + \alpha(1^T x - 1)$$

$$= \frac{1}{t} + t\,tr(Z) + \sum x_i(-v_i^T Z v_i - \lambda_i + \alpha) - \alpha$$

$$\inf(L) = \begin{cases} 2\sqrt{tr(Z)} - \alpha & Z \succeq 0,\ v_i^T Z v_i + \lambda_i = \alpha \\ -\infty & \text{otherwize} \end{cases}$$

$$\frac{\partial L}{\partial t} = 0 \rightsquigarrow -\frac{1}{t^2} + tr(Z) = 0 \Rightarrow t = \sqrt{tr(Z)}$$

$$\rightarrow Y := \frac{Z}{\alpha}$$

Dual Problem: $\max \ 2\sqrt{\alpha}\sqrt{tr(Y)} - \alpha$

$$\text{s.t.} \quad v_i^T Y v_i \preceq 1 \quad \forall i \in (1,p)$$
$$Y \succeq 0$$

$$\frac{\partial}{\partial \alpha}(obj) = 0 \rightsquigarrow \frac{1}{\sqrt{\alpha}}\sqrt{tr(Y)} = 1 \Rightarrow \alpha = tr(Y)$$

Dual Problem: maximize $tr(Y)$

$$\text{s.t.} \quad Y \succeq 0$$
$$v_i^T Y v_i \preceq 1 \quad \forall i \in (1,p)$$

5-12- Analytic Centering

$$\text{minimize} \quad -\sum_{i=1}^{m} \log y_i$$
$$\text{s.t.} \quad y = b - Ax$$

$$L(x, y, \lambda) = -\sum \log y_i + \lambda^T (y - b + Ax) =$$
$$= -\sum \log y_i + \lambda^T (y - b) + \lambda^T A x \rightarrow A^T \lambda = 0$$

$$\frac{\partial L}{\partial y_i} = 0 \rightarrow -\frac{1}{y_i} + \lambda_i = 0 \Rightarrow y_i = 1/\lambda_i \Rightarrow$$

$$\Rightarrow g(\lambda) = \begin{cases} \sum \log \lambda_i + m - \lambda^T b & A^T \lambda = 0, \ \lambda > 0 \\ -\infty & \text{otherwize} \end{cases}$$

The Dual Problem:

$$\text{maximize} \quad \sum_{i=1}^{m} \log \lambda_i + m - \lambda^T b$$
$$\text{s.t.} \quad A^T \lambda = 0$$

5_14_ A penalty method for equality Constraints

$$\phi(x) = f(x) + \alpha \|Ax - b\|_2^2$$

$\hat{x}$ minimizes $\phi(x)$ $\longrightarrow$ $\nabla f_0(\hat{x}) + 2\alpha A^T(A\hat{x} - b) = 0$

The ~~dual function~~ of Problem:
Langrangian

$$L(x, \lambda) = f_0(x) + \lambda^T(Ax - b)$$

$\nabla L = \nabla f_0(x) + \cancel{\lambda} A^T \lambda$ $\Rightarrow$ $\hat{x}$ is also a minimizer

$\Rightarrow$ $\lambda = 2\alpha(A\hat{x} - b)$ $\Rightarrow$ for $L(x, \lambda)$

with $\lambda = 2\alpha(A\hat{x} - b)$

λ is dual feasible with ~~$g(\lambda) = \inf$~~

$$g(\lambda) = \inf_x \left(f_0(x) + \lambda^T(Ax - b)\right) = f_0(\hat{x}) + 2\alpha \|A\hat{x} - b\|_2^2$$

$$\Rightarrow f_0(x) \geqslant f_0(\hat{x}) + 2\alpha \|A\hat{x} - b\|_2^2$$

5-26- minimize $x_1^2 + x_2^2$

$$\text{s.t.} \quad (x_1-1)^2 + (x_2-1)^2 \leq 1$$

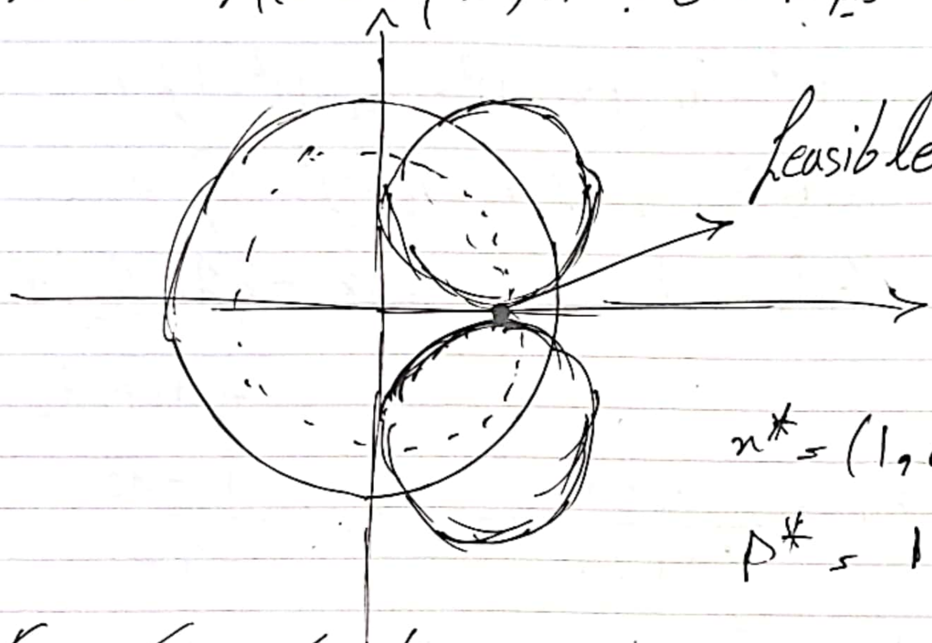
$$(x_1-1)^2 + (x_2+1)^2 \leq 1$$

a)

$x_1^2 + x_2^2 \longrightarrow$ دایره‌های به مرکز $(0,0)$

$(x_1-1)^2 + (x_2-1)^2 \leq 1 \longrightarrow$ دایره به شعاع 1 به مرکز $(1,1)$

$(x_1-1)^2 + (x_2+1)^2 \leq 1 \longrightarrow$ دایره به شعاع 1 به مرکز $(1,-1)$



$x^* = (1,0)$

$p^* = 1$

b) ~~The KKT Conditions are:~~

$$L: x_1^2 + x_2^2 + \lambda_1 \{(x_1-1)^2 + (x_2-1)^2 - 1\} +$$
$$+ \lambda_2 \{(x_1-1)^2 + (x_2+1)^2 - 1\}$$

The KKT Conditions are:

$(x_1-1)^2 + (x_2-1)^2 \leq 1, \quad (x_1-1)^2 + (x_2+1)^2 \leq 1$

$\lambda_1 \geq 0, \quad \lambda_2 \geq 0,$

$$2x_1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) = 0$$

$$2x_2 + 2\lambda_2(x_2-1) + 2\lambda_2(x_2+1) = 0$$

$$\lambda_1\{(x_1-1)^2+(x_2-1)^2-1\} = \lambda_2\{(x_1-1)^2+(x_2+1)^2 \cancel{+1}\} = 0$$

at $x^* = (1,0)$ The KKT Conditions are:

$1 \leq 1$, $1 \leq 1$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\boxed{2=0}$, $0=0$

which has no solution

c) $$L(x,\lambda) = x_1^2+x_2^2+\lambda_1\{(x_1-1)^2+(x_2-1)^2-1\} + \lambda_2\{(x_1-1)^2+(x_2+1)^2-1\} =$$

$$= (1+\lambda_1+\lambda_2)(x_1^2+x_2^2) - 2(\lambda_1+\lambda_2)x_1 - 2(\lambda_1-\lambda_2)x_2 + \lambda_1+\lambda_2$$

$$\frac{\partial L}{\partial x_1} = 0 \rightsquigarrow x_1 = \frac{\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2}, \quad x_2 = \frac{\lambda_1-\lambda_2}{1+\lambda_1+\lambda_2}$$

$$g(\lambda_1,\lambda_2) = \begin{cases} -\dfrac{(\lambda_1+\lambda_2)^2+(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} + \lambda_1+\lambda_2 & 1+\lambda_1+\lambda_2 \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

The Dual Problem:

$$\text{Maximize} \quad \frac{\lambda_1+\lambda_2-(\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2}$$

$$\text{S.t.} \quad \lambda_1 \geq 0, \ \lambda_2 \geq 0$$

ماکسیمم این تابع زمانی است که $\lambda_1 = \lambda_2$ باشد،

$$\text{Maximize} \quad \frac{2\lambda_1}{1+2\lambda} \qquad \lambda \to \infty$$

$$\hookrightarrow d^* = p^* = 1$$

اما هیچگاه نمی‌توان به مقدار λ رسید.

در این سوال Strong Duality نداریم، زیرا به مقدار optimum نمی‌توان رسید.

5-30- minimize $\operatorname{tr} X - \log\det X$

KKT Conditions:

$$X \succ 0 \ , \ Xs = y \ , \ I - X^{-1} + \tfrac{1}{2}(\lambda s^T + s\lambda^T) = 0 \quad \text{II}$$

$$\left.\begin{aligned} &Xs = y \Rightarrow s = X^{-1}y \\ &\text{II} \Rightarrow X^{-1} = I + \tfrac{1}{2}(\lambda s^T + s\lambda^T) \end{aligned}\right\} \Rightarrow s = y + \tfrac{1}{2}(\lambda + (\lambda^T y)s)$$

$$\Rightarrow y^T s = y^T y + \tfrac{1}{2}(\lambda^T y + (\lambda^T y)(y^T s)) \Rightarrow$$

$$\Rightarrow 1 - y^T y = \lambda^T y \Rightarrow$$

$$\Rightarrow s = y + \tfrac{1}{2}\lambda + s/2 - \frac{y^T y}{2} s \Rightarrow$$

$$\Rightarrow \lambda = s(1 + y^T y) - 2y \Rightarrow$$

$$\Rightarrow X^{-1} = I + \tfrac{1}{2}\left\{2ss^T(1 + y^T y) - 2ys^T - 2sy^T\right\} =$$

$$= I + ss^T(1 + y^T y) - ys^T - sy^T$$

$$X^{-1}X^* = \left(I + ss^T(1+y^Ty) - ys^T - sy^T\right)\left(I + yy^T - \frac{ss^T}{s^Ts}\right) =$$

$$= I + yy^T - \frac{ss^T}{s^Ts} + ss^T(1+y^Ty) + \cancel{1 + y^Ty - ss^T(1+y^Ty)}$$

$$\cancel{-sy^T - sy^T(y^Ty) - \frac{ss^T}{s^Ts}}$$

$$\cancel{-ys^T - yy^T + ys^T}$$

$$= I + yy^T - \frac{ss^T}{s^Ts} + ss^T(1+y^Ty) + sy^T(1+y^Ty) - ss^T(1+y^Ty)$$

$$- ys^T - yy^T + ys^T - sy^T - sy^Tyy^T + \frac{ss^T}{s^Ts} = I$$

در نتیجه X^* جواب optimal است.

تنها باید اثبات کنیم که $X^* \succ 0$

$$X^* = I + \frac{yy^T}{\cancel{\quad}} - \frac{ss^T}{s^Ts} = \cancel{\left(I + \frac{ys^T}{\|s\|}\right.}$$

$$= \left(I + \frac{ys^T}{\|s\|_2} - \frac{ss^T}{s^Ts}\right)\left(I + \frac{ys^T}{\|s\|_2} - \frac{ss^T}{s^Ts}\right)^T$$

5-39. SDP Relaxation of two-way partitioning problem

a) $tr(Wxx^T) = tr(x^TWx) = x^TWx$

$(xx^T)_{ii} = x_i^2 = 1 \Longleftrightarrow X_{ii} = 1$

دو معادله دقیقاً با هم برابر هستند.

b) این مسئله دوم، فضای جستجوی بیشتری نسبت به مسئله a دارد

زیرا در مسئله a) شرط $Rank(X) = 1$ را نیز داریم

در نتیجه مسئله a یک زیرمسئله از این قسمت است.

c) Minimize $1^T v$

$\rightarrow 5.144$

s.t. $W + diag(v) \succeq 0$

Dual:

$$L((v), X) = 1^T v - tr(X(W + diag(v))) =$$

$$= 1^T v - tr(XW) - \sum X_{ii} v_i =$$

$$= -tr(XW) + \sum v_i (1 - X_{ii})$$

Dual Problem, Maximize $-tr(XW)$ = Minimize $tr(XW)$

s.t. $X \succeq 0$

$X_{ii} = 1$

سوال 7_ شرایط KKT برای مسئله QCQP غیرمحدب.

KKT Conditions:

$$\cancel{L(y \geq 0 \wedge \beta^T y = 0)}$$

$$L = \alpha^T y - 2|\gamma|^T \sqrt{y} - \lambda \sum \beta_i y_i =$$

$$= \sum \alpha_i y_i - 2 \sum |\gamma_i| y_i^{1/2} - \lambda \sum \beta_i y_i$$

$$\frac{\partial L}{\partial y_i} = 0 \Rightarrow \alpha_i - \frac{|\gamma_i|}{\sqrt{y_i}} - \lambda \beta_i = 0 \Rightarrow$$

$$\Rightarrow \sqrt{y_i} = \frac{|\gamma_i|}{\alpha_i - \lambda \beta_i} \Rightarrow y_i = \frac{\gamma_i^2}{(\alpha_i - \lambda \beta_i)^2}$$

$$\beta^T y = 0 \Rightarrow \sum \frac{\beta_i \gamma_i^2}{(\alpha_i - \lambda \beta_i)^2} = 0 \Rightarrow$$

$$\Rightarrow \sum \beta_i \gamma_i^2 \prod_{i \neq j} (\alpha_j - \lambda \beta_j)^2 = 0$$