DATE 400109843 پردازش SUBJECT : سیگنال سیستم

سوال ①

برای ~~علیت~~ پایداری باید داشته باشیم که $\sum_{k=-\infty}^{+\infty} |h[k]| < B$ یا $\int_{-\infty}^{+\infty} h(x)\,dx < B$

برای علیت باید داشته باشیم که: $\forall t<0 : h[t]=0$ یا $\forall t<0 : h(t)=0$

الف) $h[n]=\left(\frac{1}{3}\right)^n u[n+2]$

$h[-1]=\left(\frac{1}{3}\right)^{-1} u[1]=3 \rightarrow$ علی نیست

$\sum_{k=-\infty}^{k=+\infty}\left(\frac{1}{3}\right)^k u[k+2]=\sum_{k=-2}^{\infty}\left(\frac{1}{3}\right)^k=\frac{\left(\frac{1}{3}\right)^{-2}}{1-\frac{1}{3}}=\frac{27}{2} \rightarrow$ پایدار است

ب) $\left(\frac{1}{4}\right)^n u[n-3]=h[n]$

این تابع به ازای $n<3$، 0 است در نتیجه $\forall n<0 : h[n]=0$

پس این تابع، تابعی علی است.

$\sum_{k=-\infty}^{+\infty}\left(\frac{1}{4}\right)^k u[k-3]=\sum_{k=3}^{\infty}\left(\frac{1}{4}\right)^k=\frac{\left(\frac{1}{4}\right)^3}{1-\frac{1}{4}}=\frac{1}{48} \rightarrow$ پایدار است.

ج) $h[n]=\frac{1}{n}u[n-1]$

$\rightarrow h[n]\begin{cases}\frac{1}{n} & n\geq 1\\ 0 & n<1\end{cases} \rightsquigarrow \forall n<0 : h[n]=0 \rightarrow$ علی است

$\sum_{k=-\infty}^{\infty}\frac{1}{n}u[n-1]=\sum_{n=1}^{\infty}\frac{1}{n} \rightarrow$ Converges to infinity

پایدار نیست

iman NOTEBOOK

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د) $h[n] = 10^{n} u[4-n]$

$n = -2 \rightsquigarrow h[n] = 10^{-2} u[6] = \frac{1}{100} \neq 0 \rightarrow$ علّی نیست

$\sum_{k=-\infty}^{+\infty} h[k] = \sum_{k=-\infty}^{+\infty} 10^{k} u[4-k] = \sum_{k=-\infty}^{4} 10^{k} = \frac{10^5}{10-1} = \frac{10^5}{9}$

پایدار است.

ه) ~~$h[n] =$~~ $h(t) = e^{-t} u(2-t)$

$h(-2) = e^{2} u(4) = e^{2} \neq 0 \rightarrow$ علّی نیست

$\int_{-\infty}^{+\infty} h(t) dt = \int_{-\infty}^{+\infty} e^{-t} u(2-t) dt = \int_{-\infty}^{2} e^{-t} dt = -e^{-t} \Big|_{-\infty}^{2} \rightarrow$ Converges to infinity

پایدار نیست

و) $h(t) = e^{-3|t|}$

$t = -2 \rightsquigarrow h(t) = e^{-6} \rightsquigarrow$ علّی نیست

$\int_{-\infty}^{+\infty} e^{-3|t|} dt = 2 \times \int_{0}^{+\infty} e^{-3t} dt = -\frac{2}{3} e^{-3t} \Big|_{0}^{+\infty} = \frac{2}{3} \rightarrow$ پایدار است

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سوال 2)

آ) $x_1[n] = u[n-1] - u[n]$, $x_2[n] = u[n]$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{+\infty} x_1[k]\, x_2[n-k] = \sum_{k=-\infty}^{+\infty} \left(u[k-1] - u[k]\right) u[n-k]$$

$$= \sum_{k=-\infty}^{n} \left(u[k-1] - u[k]\right) = \sum_{k=-\infty}^{n} -\delta[k-1] = -1$$

ب) $x_2[n] = u[n+2]$, $x_1[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{k=-\infty}^{+\infty} \left(\frac{1}{2}\right)^{k-2} u[k-2] \times u[n-k+2]$$

$$= \sum_{k=-\infty}^{n+2} \left(\frac{1}{2}\right)^{k-2} u[k-2] = \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$$

$$\Rightarrow x_1[n] * x_2[n] = 2 \times \left(1 - \left(\frac{1}{2}\right)^{n+1}\right)$$

ج) $x_2[n] = 3^{n} u[-n-1]$, $x_1[n] = \left(\frac{1}{3}\right)^{n} u[n]$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{+\infty} x_2[k] \cancel{x_1[k-n]} = x_1[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} 3^{k} u[-k-1] \times \left(\frac{1}{3}\right)^{n-k} \times u[n-k] = \cancel{\sum_{k=-\infty}^{n} 3^{2k-k}}$$

$$= \sum_{k=-\infty}^{n} 3^{2k-n} u[-k-1] = \sum_{k=-\infty}^{-1} 3^{2k-n} = 3^{-n} \times \sum_{k=1}^{k=\infty} 9^{-k} =$$

$$= 3^{-n} \times \frac{1/9}{1-1/9} = \frac{1}{8} \times 3^{-n}$$

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2) $x_2(t) = \sin t \; , \; x_1(t) = e^{-t}$

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{+\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau = \int_{\tau=-\infty}^{+\infty} e^{-(t-\tau)} \times \sin \tau \, d\tau =$$

$$= \int_{\tau=-\infty}^{+\infty} e^{-(t-\tau)} \times \frac{e^{i\tau} - e^{-i\tau}}{2i}\, d\tau = e^{-t} \int_{\tau=-\infty}^{+\infty} \frac{e^{(i+1)\tau} - e^{(1-i)\tau}}{2i}\, d\tau$$

$$= \frac{e^{-t}}{2i} \times \left\{ \frac{1}{i+1} e^{(1+i)\tau} - \frac{1}{1-i} e^{(1-i)\tau} \right\} \Bigg|_{\tau=-\infty}^{+\infty}$$

3) $x_2(t) = u(t-2) - u(t+2) \; , \; x_1(t) = e^{-|t|}$

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau =$$

$$= \int_{\tau=-\infty}^{+\infty} u(t-\tau-2)\, e^{-|\tau|}\, d\tau - \int_{\tau=-\infty}^{+\infty} u(t-\tau+2)\, e^{-|\tau|}\, d\tau =$$

$$= -\int_{\tau=+\infty}^{t-2} e^{-|\tau|}\, d\tau + \int_{\tau=+\infty}^{t+2} e^{-|\tau|}\, d\tau = \int_{\tau=t-2}^{t+2} e^{-|\tau|}\, d\tau = A$$

$$A \begin{cases} t>2 : \displaystyle\int_{t-2}^{t+2} e^{-\tau}\, d\tau = e^{-(t-2)} - e^{-(t+2)} \\ t<-2 : \displaystyle\int_{t-2}^{t+2} e^{\tau}\, d\tau = e^{t+2} - e^{(t-2)} \\ -2<t<2 : \displaystyle\int_{t-2}^{0} e^{\tau}\, d\tau + \int_{0}^{t+2} e^{-\tau}\, d\tau = 2 - e^{t-2} - e^{-(t+2)} \end{cases}$$

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سؤال 3)

الف) $f(t) * g(t) = \int_{\tau=-\infty}^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$

$$\left.\begin{aligned} & t-\tau = \tau' \Rightarrow d\tau' = -d\tau \\ & \quad \hookrightarrow \tau = t-\tau' \\ & \lim_{\tau \to -\infty} \tau' = +\infty \ , \ \lim_{\tau \to +\infty} \tau' = -\infty \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow f(t) * g(t) = \int_{\tau'=+\infty}^{-\infty} -f(t-\tau')\, g(\tau')\, d\tau' = \int_{\tau'=-\infty}^{+\infty} g(\tau')\, f(t-\tau')\, d\tau' = g(t) * f(t) \Rightarrow$$

$$\Rightarrow f(t) * g(t) = g(t) * f(t)$$

ب) $(f(t) * g(t)) * h(t) = \int_{\tau=-\infty}^{+\infty} (f(\tau) * g(\tau))\, h(t-\tau)\, d\tau =$

$$= \int_{\tau=-\infty}^{+\infty} \int_{\tau'=-\infty}^{+\infty} (f(\tau')\, g(\tau-\tau')\, d\tau') \overbrace{h(t-\tau)}^{h(t-\tau)} d\tau = \int_{\tau=-\infty}^{+\infty} \int_{\tau'=-\infty}^{+\infty} f(\tau')\, g(\tau-\tau')\, h(t-\tau)\, d\tau'\, d\tau$$

$$= \int_{\tau'=-\infty}^{+\infty} f(\tau') \left\{ \int_{\tau=-\infty}^{+\infty} g(\tau-\tau')\, h(t-\tau)\, d\tau \right\} d\tau' =$$

$u = \tau - \tau' \rightsquigarrow du = d\tau$

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$$= \int_{\tau'=-\infty}^{+\infty} h(\tau') \left\{ \int_{u=-\infty}^{+\infty} g(u)\, h(t-u-\tau')\, du \right\} d\tau' =$$

$$= \int_{\tau'=-\infty}^{+\infty} h(\tau') \left\{ \int_{u=-\infty}^{+\infty} g(u)\, h((t-\tau')-u)\, du \right\} d\tau' =$$

$$= \int_{\tau'=-\infty}^{+\infty} h(\tau') \left(g(t-\tau') * h(t-\tau') \right) d\tau' =$$

$$= h(t) * \left(g(t) * h(t) \right)$$

سوال ۴) پاسخ ضربه مطلقاً جمع‌پذیر نیست $\rightarrow \sum_{-\infty}^{+\infty} |h[k]| = +\infty$

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{+\infty} h[n-k]\, x[k] \rightsquigarrow$$

$$\rightsquigarrow y[0] = \sum_{k=-\infty}^{+\infty} x[k]\, h[-k]$$

سیگنال $x[k] = \mathrm{Sgn}(h[-k])$ را در نظر بگیرید. این سیگنال کراندار است.

$$x[k] = \mathrm{Sgn}(h[-k]) \rightsquigarrow y[0] = \sum_{k=-\infty}^{+\infty} \mathrm{Sgn}(h[-k])\, h[-k] \Rightarrow$$

$$\Rightarrow y[0] = \sum_{k=-\infty}^{+\infty} |h[-k]| \rightarrow$$ طبق تعریف بی‌نهایت است

بنابراین فرض اثبات شد.

DATE. $x(t)=c_1e^{-t}+c_2e^{-2t}$ است SUBJECT:

سوال 5) جواب کلی برای تمامی این معادلات به صورت $\cancel{x(t)=4e^{-t}}$

آ) $\ddot{x}_{(t)}+3\dot{x}_{(t)}+2x(t)=0$, $x(0)=2$ و $x'(0)=0$

$$x(t)=e^{\omega t} \Rightarrow \omega^2+3\omega+2=0 \Rightarrow \omega=\begin{cases}-1\\-2\end{cases}$$

$$x(t)=c_1e^{-t}+c_2e^{-2t}$$

$$\left.\begin{aligned} x(0)&=c_1+c_2=2\\ x'(0)&=-c_1-2c_2=0\end{aligned}\right\} \Rightarrow c_2=-2,\ c_1=4 \Rightarrow x(t)=4e^{-t}-2e^{-2t}$$

ب) $\ddot{x}_{(t)}+3\frac{dx}{dt}+2x(t)=0$, $x(0)=0$ و $x'(0)=2$

$$\left.\begin{aligned} x(0)&=c_1+c_2=0\\ x'(0)&=-c_1-2c_2=2\end{aligned}\right\} \Rightarrow c_2=-2 \Rightarrow c_1=2 \Rightarrow x(t)=2e^{-t}-2e^{-2t}$$

ج) $x(0)=1$, $x'(0)=-1$ و $\ddot{x}_{(t)}+3\dot{x}_{(t)}+2x_{(t)}=0$

$$\left.\begin{aligned} x(0)&=c_1+c_2=1\\ x'(0)&=-c_1-2c_2=-1\end{aligned}\right\} \Rightarrow c_2=-2 \Rightarrow c_1=3 \Rightarrow x(t)=3e^{-t}-2e^{-2t}$$

د) $\ddot{x}_{(t)}+3\dot{x}_{(t)}+2x_{(t)}=0$, $x(0)=0$ و $x'(0)=0$

$$\left.\begin{aligned} x(0)&=c_1+c_2=0\\ x'(0)&=-c_1-2c_2=0\end{aligned}\right\} \Rightarrow c_1=c_2=0 \Rightarrow x(t)=0$$

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$\forall n < n_0 \,,\ x[n] = 0 \Rightarrow y[n] = 0 \quad n < n_0$ (آ

↳ Initial Rest

$y[n_0] = x[n_0] \rightsquigarrow y[n_0+1] = x[n_0+1] + \frac{x[n_0]}{2} \rightsquigarrow$

$\rightsquigarrow y[n_0+2] = x[n_0+2] + \frac{x[n_0+1]}{2} + \frac{x[n_0]}{2^2} \Rightarrow$

$\Rightarrow y[n_0+k] = \sum_{i=0}^{k} \frac{x[n_0+i]}{2^{k-i}}$

$x_2 = ax_1 + bx_0 \longrightarrow y_2 = ay_1 + by_0$ ← اثبات خطی بودن

$y_1[n_0+k] = \sum_{i=0}^{k} \frac{x_1[n_0+i]}{2^{k-i}}$

$y_0[n_0+k] = \sum_{i=0}^{k} \frac{x_0[n_0+i]}{2^{k-i}}$

$y_2[n_0+k] = \sum_{i=0}^{k} \frac{x_2[n_0+i]}{2^{k-i}} = \sum_{i=0}^{k} \frac{ax_1[n_0+i] + bx_0[n_0+i]}{2^{k-i}} =$

$= a\sum_{i=0}^{k} \frac{x_1[n_0+i]}{2^{k-i}} + b\sum_{i=0}^{k} \frac{x_0[n_0+i]}{2^{k-i}} \Rightarrow$

$\Rightarrow y_2 = ay_1 + by_0$ ✓ خطی

$$y[n] = \sum_{i=0}^{k} \frac{x[n_0 + \dot{k}]}{2^{k-i}} \qquad \forall n < n_0 : x[n] = 0$$

$$y_2[n] = x_1[n-m] \rightsquigarrow y_2[n] = \sum_{i=0}^{k} \frac{x_1[n_0 + \dot{k} - m]}{2^{k-i}} =$$

$$\Rightarrow y_2[n] = \sum_{i=m}^{k} \frac{x_1[n_0 + i - m]}{2^{k-i}}$$

$$y_1[n-m] = \sum_{i=0}^{k-m} \frac{x_1[n_0 + \dot{i}]}{2^{k-m-i}} \Rightarrow$$

$$\Rightarrow y_2[n] = y_1[n-m] \Rightarrow \text{Time-Independent}$$

ب) $y[0] = \frac{1}{2} y[-1] + x[0] \Rightarrow y[-1] = -2x[0]$

تعریف علیت : $t < t_0 : x_1[t] = x_2[t] \longrightarrow y_1[t] = y_2[t]$

$x_1[t] = 0 \rightsquigarrow y_1[t] = 0$

$x_2[n] = \delta[n] \Rightarrow n_0 = 0$

$y_2[0] = \frac{1}{2} y_2[-1] + x_2[0] \Rightarrow y_2[-1] = -2$ (with $x_2[0] = 1$)

$$\left.\begin{array}{l} x_1[-1] = 0 \Rightarrow y_1[-1] = 0 \\ x_2[-1] = 0 \Rightarrow y_2[-1] = -2 \end{array}\right\} \Rightarrow y_1[-1] \neq y_2[-1]$$

علی نیست

سوال ۴ (ج

آ)

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

$$y_h[n] - \frac{1}{2} y_h[n-1] = 0 \Rightarrow \frac{1}{2} = \frac{y_h[n]}{y_h[n-1]} = \frac{(1/2)^n}{(1/2)^{n-1}} = \frac{1}{2} \checkmark$$

ب)

$$y_p[n] - \frac{1}{2} y_p[n-1] = \left(\frac{1}{3}\right)^n u(n) \Rightarrow$$

$$\Rightarrow B \times \left(\frac{1}{3}\right)^n - \frac{1}{2} \times B \times \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n \Rightarrow$$

$$\Rightarrow B\left(1 - \frac{3}{2}\right) = 1 \Rightarrow B = -2$$

ج)

$$\forall n < n_0 \quad x[n] = 0 \rightarrow y[n] = 0$$

با توجه به فرم $x[n]$، $n_0 = 0$ است.

$$\cancel{y[0] - \frac{1}{2} y[n-1] =} \quad y[n] = A\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n u[n]$$

$$y[0] - \frac{1}{2} y[-1] = x[0]$$

طبق فرض سکون اولیه $y[-1] = 0 \Rightarrow$ $y[0] = x[0]$

$$y[0] = A \times \left(\frac{1}{2}\right)^0 + B\left(\frac{1}{2}\right)^0 = \left(\frac{1}{3}\right)^0 \Rightarrow A + B = 1 \Rightarrow A = 3$$

$$\Rightarrow y[n] = \left\{3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right\} u[n]$$