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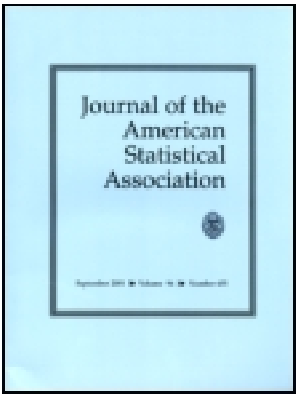

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A MULTIVARIATE EXPONENTIAL DISTRIBUTION

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A number of multivariate exponential distributions are known, but they have not been obtained by methods that shed light on their applicability. This paper presents some meaningful derivations of a multivariate exponential distribution that serves to indicate conditions under which the distribution is appropriate. Two of these derivations are based on "shock models," and one is based on the requirement that residual life is independent of age. It is significant that the derivations all lead to the same distribution.

For this distribution, the moment generating function is obtained, comparison is made with the case of independence, the distribution of the minimum is discussed, and various other properties are investigated. A multivariate Weibull distribution is obtained through a change of variables.

1. INTRODUCTION

EXPONENTIAL distributions play a central role in life testing, reliability and other fields of application. Though the assumption of independence can often be used to obtain joint distributions, sometimes such an assumption is questionable or clearly false. Thus, an understanding of multivariate exponential distributions (i.e., multivariate distributions with exponential marginals) is desirable.

A number of such distributions have been obtained by methods that do not shed much light on their applicability. The purpose of this paper is to present some meaningful derivations of a multivariate exponential distribution. These derivations serve to indicate conditions under which the distribution is appropriate.

In considering the general problem of constructing bivariate distributions H with given marginals F and G, Fréchet (1951) obtained the condition

$$\max\,[F(x) + G(y) - 1,\, 0] \le H(x, y) \le \min\,[F(x),\, G(y)]. \tag{1.1}$$

These upper and lower bounds are themselves bivariate distributions with the given marginals, and so constitute solutions to the problem. Recently Plackett (1965) constructed a one parameter family of bivariate distributions which includes these solutions as well as the solution $H(x, y) = F(x)G(y)$; he also surveyed previous work on the problem.

The family of solutions

$$H(x, y) = F(x)G(y)\{1 + \alpha[1 - F(x)][1 - G(y)]\}, \qquad |\alpha| \le 1, \tag{1.2}$$

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due to Morgenstern (1956) has been studied by Gumbel (1960) when F and G are exponential. Gumbel also studied the bivariate distribution

$$H(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\delta xy}, \qquad 0 \le \delta \le 1,$$

which has exponential marginals. However, we know of no model or other basis for determining how these distributions might arise in practice.

An interesting model based on the exponential distribution has been used by Freund (1961) for deriving a bivariate distribution. However, the distribution obtained does not have exponential marginals.

The models and characterization investigated in this paper lead to the multivariate distribution with exponential marginals, which in the bivariate case is given by

$$P\{X > s, Y > t\} = \exp[-\lambda_1 s - \lambda_2 t - \lambda_{12} \max(s, t)], \quad s, t > 0. \quad (1.3)$$

Each approach used to derive this distribution was chosen for its intuitive appeal, and it is significant that each leads to the same distribution.

For convenience we say that X and Y are BVE $(\lambda_1, \lambda_2, \lambda_{12})$ if (1.3) holds; we refer to the distribution of (1.3) as the bivariate exponential, BVE$(\lambda_1, \lambda_2, \lambda_{12})$.

We begin by considering the bivariate case (§2) and its properties (§3) before investigating the multivariate case (§4). Ramifications of the condition that residual life is independent of age are also explored (§6).

2. DERIVATION OF THE BIVARIATE EXPONENTIAL DISTRIBUTION

The first defining properties (§2.1, §2.2) are motivated by reliability considerations and are based on models in which a two-component system survives or dies according to the occurrences of "shocks" to each or both of the components. (Shock models in one dimension have been utilized by several authors; see, e.g., Epstein (1958), Esary (1957), Gaver (1963).)

The defining property of §2.3 is based on a bivariate extension of a central property of the exponential distribution that the distribution of residual life is independent of age, i.e., $P\{\text{survival to time } t+s \mid \text{survival to time } t\} = P\{\text{survival to time } s\}$.

2.1 *A "fatal shock" model*

Suppose that the components of a two-component system die after receiving a shock which is always fatal. Independent Poisson processes $Z_1(t; \lambda_1)$, $Z_2(t; \lambda_2)$, $Z_{12}(t; \lambda_{12})$ govern the occurrence of shocks. (By $Z(t; \lambda) \equiv \{Z(t), t \ge 0; \lambda\}$ we mean a Poisson process with parameter λ.) Events in the process $Z_1(t; \lambda_1)$ are shocks to component 1, events in the process $Z_2(t; \lambda_2)$ are shocks to component 2, and events in the process $Z_{12}(t; \lambda_{12})$ are shocks to both components. Thus if X and Y denote the life of the first and second components,

$$\begin{aligned}\bar{F}(s, t) &\equiv P\{X > s, Y > t\} \\ &= P\{Z_1(s; \lambda_1) = 0, Z_2(t; \lambda_2) = 0, Z_{12}(\max(s, t); \lambda_{12}) = 0\} \\ &= \exp[-\lambda_1 s - \lambda_2 t - \lambda_{12} \max(s, t)].\end{aligned}$$

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2.2 *Non-fatal shock models*

Again consider a two-component system and three independent Poisson processes $Z_1(t;\, \delta_1)$, $Z_2(t;\, \delta_2)$, $Z_{12}(t;\, \delta_{12})$ governing the occurrence of shocks, with the modification that shocks need not be fatal.

Describe the state of the system by the ordered pairs (0, 0), (0, 1), (1, 0), (1, 1), where a 1 in the first (second) place indicates that the first (second) component is operating and a 0 indicates that it is not.

Suppose that events in process $Z_1(t;\, \delta_1)$ are shocks to the first component which cause a transition from (1, 1) to (0, 1) with probability p_1, and from (1, 1) to (1, 1) with probability $1-p_1$. Similarly, events in process $Z_2(t;\, \delta_2)$ are transitions from (1, 1) to (1,0) or (1, 1) which occur with probability p_2 and $1-p_2$, respectively. Events in process $Z_{12}(t;\, \delta_{12})$ are shocks to both components which cause a transition from state (1, 1) to states (0, 0), (0, 1), (1, 0), (1, 1) with respective probabilities p_{00}, p_{01}, p_{10}, p_{11}. Furthermore, assume that each shock to a component represents an independent opportunity for failure.

Let X and Y denote the life length of the first and second components. Since $Z_1(t;\, \delta_1)$, $Z_2(t;\, \delta_2)$, $Z_{12}(t;\, \delta_{12})$ are independent and have independent increments, we have for $t \geq s \geq 0$,

$$
\begin{aligned}
&P\{X > s,\ Y > t\} \qquad (2.1)\\
&= \left\{ \sum_{k=0}^{\infty} e^{-\delta_1 s} \frac{(\delta_1 s)^k}{k!} (1 - p_1)^k \right\} \left\{ \sum_{l=0}^{\infty} e^{-\delta_2 t} \frac{(\delta_2 t)^l}{l!} (1 - p_2)^l \right\} \\
&\quad \cdot \left\{ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[e^{-\delta_{12} s} \frac{(\delta_{12} s)^m}{m!} p_{11}^m \right] \left[e^{-\delta_{12}(t-s)} \frac{(\delta_{12}(t - s))^n}{n!} (p_{11} + p_{01})^n \right] \right\} \\
&= \exp \left\{ -s[\delta_1 p_1 + \delta_{12} p_{01}] - t[\delta_2 p_2 + \delta_{12}(1 - p_{11} - p_{01})] \right\}.
\end{aligned}
$$

By symmetry, for $s \geq t \geq 0$,

$$
P\{X>s,\ Y>t\} = \exp \left\{ -s[\delta_1 p_1 + \delta_{12}(1 - p_{11} - p_{10})] - t(\delta_2 p_2 + \delta_{12} p_{10}) \right\}. \qquad (2.2)
$$

Consequently, by combining (2.1) and (2.2), it follows that

$$
P\{X > s,\ Y > t\} = \exp\left[-\lambda_1 s - \lambda_2 t - \lambda_{12} \max(s, t) \right], \qquad (2.3)
$$

where

$$
\lambda_1 = \delta_1 p_1 + \delta_{12} p_{01}, \quad \lambda_2 = \delta_2 p_2 + \delta_{12} p_{10}, \quad \lambda_{12} = \delta_{12} p_{00}.
$$

When $p_1 = p_2 = 1$, $p_{00} = 1$, we have the specialized fatal model. When $p_1 = p_2 = 0$, we effectively eliminate the first two processes; but the joint distribution obtained from the process Z_{12} is of the same form.

2.3 Residual life independent of age

The univariate exponential distribution is characterized by

$$
\bar{F}(s + t) = \bar{F}(s) \bar{F}(t), \qquad (2.4)
$$

for all $s \geq 0$, $t \geq 0$. Of course, this is equivalent to $P\{X > s+t \mid X > s\} = P\{X > t\}$, i.e., the probability of surviving to time $s+t$ given survival to time s is exactly the unconditional probability of survival to time t.

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Because this characterization is so fundamental in the univariate case, it is important to investigate its multivariate extensions. If $\bar{F}(s, t) = P\{X > s, Y > t\}$, one obvious extension of (2.4) is

$$\bar{F}(s_1 + t_1, s_2 + t_2) = \bar{F}(s_1, s_2)\bar{F}(t_1, t_2) \tag{2.5}$$

for all $s_1, s_2, t_1, t_2 > 0$. The solution of this equation is given by

Lemma 2.1. If (2.5) holds, then

$$\bar{F}(s, t) = \exp\{-(\theta_1 s + \theta_2 t)\}, \qquad 0 < \theta_1, \theta_2. \tag{2.6}$$

Proof. Setting $s_2 = t_2 = 0$ in (2.5) yields $\bar{F}_1(s_1 + t_1) \equiv \bar{F}(s_1 + t_1, 0) = \bar{F}_1(s_1)\bar{F}_1(t_1)$, so that $\bar{F}_1(s) = \exp(-\theta_1 s)$, for some $\theta_1 > 0$. Similarly, $\bar{F}_2(t) = \exp(-\theta_2 t)$, for some $\theta_2 > 0$. By choosing $s_2 = t_1 = 0$, we obtain $\bar{F}(s_1, t_2) = \bar{F}_1(s_1)\bar{F}_2(t_2)$, from which the result follows.||

We see from the lemma that the functional equation (2.5) yields a joint distribution which is a product of marginal distributions. Consequently, the assumption of (2.5) is too strong to yield an interesting multivariate exponential distribution, but it may be a convenient way to justify the assumption of independence.

Let us examine the functional equation (2.5) more critically. Consider again a two-component system and suppose both components have survived to time t. A physically meaningful extension of (2.4) which leads to (1.3) is obtained if the conditional probability of both components surviving an additional time $s = (s_1, s_2)$ is set equal to the unconditional probability of surviving to time (s_1, s_2) starting at the origin, i.e.,

$$P\{X > s_1 + t, Y > s_2 + t \mid X > t, Y > t\} = P\{X > s_1, Y > s_2\}, \tag{2.7}$$

or

$$\bar{F}(s_1 + t, s_2 + t) = \bar{F}(s_1, s_2)\bar{F}(t, t), \tag{2.8}$$

for all $s_1 \geq 0$, $s_2 \geq 0$, $t \geq 0$. This represents a weakening of (2.5).

Since (2.7) can also be written

$$P\{X > s_1 + t, Y > s_2 + t \mid X > s_1, Y > s_2\} = P\{X > t, Y > t, \tag{2.9}$$

we may also check the physical meaning of (2.9). In the univariate case if we suppose a functioning component is of age s, the probability that it will function to time t units from now is the same as if the component were new. With the same interpretation for a two-component system with functioning components of ages s_1 and s_2, equation (2.9) asserts that the probability that both components are functioning t time units from now is the same as if both components were new.

The solution of the functional equation (2.8) is given by

Lemma 2.2. If (2.8) holds, then

$$\bar{F}(x, y) = \begin{cases} e^{-\theta y}\bar{F}_1(x - y), & x \geq y, \\ e^{-\theta x}\bar{F}_2(y - x) & x \leq y, \end{cases} \tag{2.10}$$

where the marginal distributions $F(t, 0)$ and $F(0, t)$ are denoted by $F_1(t)$ and $F_2(t)$, respectively.

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Proof. Setting $(s_1, s_2)=(s, s)$ in (2.8) yields $\bar{F}(s+t, s+t)=\bar{F}(s, s)\bar{F}(t, t)$, which implies that $\bar{F}(s, s)=\exp(-\theta s)$, for some $\theta>0$. With $s_2=0$ in (2.8),

$$\bar{F}(s_1+t, t) = \bar{F}(s_1, 0)\bar{F}(t, t) = \bar{F}(s_1, 0)\exp(-\theta t),$$

from which the result follows.||

The requirement of exponential marginal distributions yields

$$\bar{F}(x, y) = \begin{cases}\exp[-\theta y-\delta_1(x-y)], & x \geq y,] \\ \exp[-\theta x-\delta_2(y-x)], & x \leq y,]\end{cases} \tag{2.11}$$

where $\theta \geq \delta_1, \delta_2$ in order that $\bar{F}$ be monotone. If, in addition, $\delta_1+\delta_2 \geq \theta$, then $\lambda_1=\theta-\delta_2$, $\lambda_2=\theta-\delta_1$, and $\lambda_{12}=\delta_1+\delta_2-\theta$ are all positive and the substitution $\delta_1=\lambda_1+\lambda_{12}$, $\delta_2=\lambda_2+\lambda_{12}$, $\theta=\lambda_1+\lambda_2+\lambda_{12}$ in (2.11) yields the BVE given by (1.3). We show later (§5) that the condition $\delta_1+\delta_2 \geq \theta$ is necessary for F given by (2.11) to be a distribution function.

3. PROPERTIES OF THE BIVARIATE EXPONENTIAL DISTRIBUTION

3.1 *The distribution function*

An interesting facet of the BVE is that it has both an absolutely continuous and a singular part. Though distributions in one dimension with this property are usually pathological and of no practical importance, they do arise naturally in higher dimensions.

In the case of the BVE, the presence of a singular part is a reflection of the fact that if X and Y are BVE, then $X=Y$ with positive probability, whereas the line $x=y$ has two-dimensional Lebesgue measure zero. If X and Y are lifetimes, the event $X=Y$ can occur when failure is caused by a shock simultaneously felt by both items, as indicated in §2.1 and §2.2. Simultaneous failure also occurs with the failure of an essential input, common to both items. Sometimes $X=Y$ because one component (say, a jet engine) explodes and the other component (an adjacent engine) is destroyed by the explosion.

Another example where $X=Y$ with positive probability is the case that X and Y are waiting times for the registration of an event by two adjacent geiger counters. Counters are sometimes placed in a specific orientation, say one above the other, so that a simultaneous event in each counter records particles with nearly perpendicular paths.

Theorem 3.1. *If* $\bar{F}(x, y)$ *is* BVE$(\lambda_1, \lambda_2, \lambda_{12})$ *and* $\lambda=\lambda_1+\lambda_2+\lambda_{12}$, *then*

$$\bar{F}(x, y) = \frac{\lambda_1+\lambda_2}{\lambda}\bar{F}_a(x, y) + \frac{\lambda_{12}}{\lambda}\bar{F}_s(x, y),$$

where

$$\bar{F}_s(x, y) = \exp[-\lambda \max(x, y)]$$

is a singular distribution, and

$$\bar{F}_a(x, y) = \frac{\lambda}{\lambda_1+\lambda_2}\exp[-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x, y)] - \frac{\lambda_{12}}{\lambda_1+\lambda_2}\exp[-\lambda\max(x, y)]$$

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is absolutely continuous.

Proof. To find the absolutely continuous part F_a of F from $\overline{F}(x, y) = \alpha\overline{F}_a(x, y) + (1-\alpha)\overline{F}_s(x, y)$, $0 \leq \alpha \leq 1$, we compute

$$\frac{\partial^2 \overline{F}(x, y)}{\partial x\, \partial y} = \alpha f_a(x, y) = \begin{cases} \lambda_2(\lambda_1 + \lambda_{12})\overline{F}(x, y), & x > y, \\ \lambda_1(\lambda_2 + \lambda_{12})\overline{F}(x, y), & x < y, \end{cases}$$

from which α may be obtained as the integral of $\alpha f_a(x, y)$. With α and F_a determined, F_s can be obtained by subtraction.

Alternatively, we can obtain the theorem via the following probabilistic argument. Let Z_1, Z_2, Z_{12} denote the respective waiting times to the first event in the processes $Z_1(t; \lambda_1)$, $Z_2(t; \lambda_2)$, $Z_{12}(t; \lambda_{12})$ defined in §2.1. Then $\overline{F}(x, y)$ can be decomposed as

$$\overline{F}(x, y) = P\{X > x, Y > y \mid A\}P\{A\} + P\{X > x, Y > y \mid A^c\}P\{A^c\},$$

where $A = \{Z_{12} > \min(Z_1, Z_2)\}$, and A^c is the complement of A. It is well known and easily checked that $P\{A^c\} = \lambda_{12}/\lambda$. Since $\min(Z_1, Z_2, Z_{12})$ is an exponential random variable with parameter λ,

$$P\{X > x, Y > y \mid A^c\} = P\{Z_{12} > \max(x,y) \mid Z_{12} < \min(Z_1, Z_2)\} = e^{-\lambda \max(x, y)}.$$

From this, $P\{A\} = (\lambda_1 + \lambda_2)/\lambda$ and $P\{X > x, Y > y \mid A\}$ is obtained by subtraction.

It is easily verified that $P\{X > x, Y > y \mid A^c\}$ is a singular distribution since its mixed second partial derivative is zero where $x \neq y$, and that $P\{X > x, Y > y \mid A\}$ is absolutely continuous since its mixed second partial derivative is a density.||

3.2 *Moment generating function*

Since we are considering positive random variables, the Laplace transform (moment generating function) exists and is natural to compute in place of the characteristic function.

This transform is given in

Lemma 3.2. The moment generating function for the BVE is given by

$$\psi(s, t) = \int_0^\infty \int_0^\infty e^{-sx-ty} dF(x, y) = \frac{(\lambda + s + t)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + st\lambda_{12}}{(\lambda + s + t)(\lambda_1 + \lambda_{12} + s)(\lambda_2 + \lambda_{12} + t)},$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.

Proof. Treating the absolutely continuous and singular parts of F separately (see Theorem 3.1), we obtain

$$\begin{aligned} \psi(s, t) = &\iint_{x>y} e^{-sx-ty}\lambda_2(\lambda_1 + \lambda_{12})\overline{F}(x, y)\,dx\,dy \\ &+ \iint_{x<y} e^{-sx-ty}\lambda_1(\lambda_2 + \lambda_{12})\overline{F}(x, y)\,dx\,dy \\ &+ \int_0^\infty e^{-sx-ty}\lambda_{12}\overline{F}_s(x, x)\,dx. \end{aligned}$$

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The result follows upon the evaluation of these integrals.||

Remark. The decomposition of the integral into parts as in the above lemma can be avoided by exploiting a result of Young (1917) on integration by parts in two or more dimensions. If $G(0, y) \equiv 0 \equiv G(x, 0)$, and G is of bounded variation on finite intervals, then

$$\int_0^\infty \int_0^\infty G(x, y)dF(x, y) = \int_0^\infty \int_0^\infty \bar{F}(x, y)dG(x, y). \tag{3.1}$$

This change is of particular use when $G(x, y)$ is absolutely continuous and $\bar{F}$ is easy to compute. To satisfy the conditions for G, we replace $\exp\{-(sx+ty)\}$ of the Laplace transform by $G(x, y) = (1-e^{-sx})(1-e^{-ty})$, obtaining

$$\phi(s, t) = \int_0^\infty \int_0^\infty (1 - e^{-sx})(1 - e^{-ty})dF(x, y) = \int_0^\infty \int_0^\infty \bar{F}(x, y)ste^{-(sx+ty)}dx\, dy$$

$$= \frac{st(\lambda + \lambda_{12} + s + t)}{(\lambda + s + t)(\lambda_1 + \lambda_{12} + s)(\lambda_2 + \lambda_{12} + t)}.$$

The Laplace transform $\psi(s, t)$ is obtained from the relation

$$\psi(s, t) = \phi(s, t) - \phi(\infty, t) - \phi(s, \infty) + 1.$$

To obtain the moments of the BVE we compute

$$EX = \frac{1}{\lambda_1 + \lambda_{12}}, \qquad \operatorname{Var} X = \frac{1}{(\lambda_1 + \lambda_{12})^2},$$

$$EY = \frac{1}{\lambda_2 + \lambda_{12}} \qquad \operatorname{Var} Y = \frac{1}{(\lambda_2 + \lambda_{12})^2},$$

from the marginal distributions and

$$EXY = \left.\frac{\partial^2 \psi}{\partial s \partial t}\right|_{s=t=0} = \left.\frac{\partial^2 \Phi}{\partial s \partial t}\right|_{s=t=0} = \frac{1}{\lambda}\left(\frac{1}{\lambda_1 + \lambda_{12}} + \frac{1}{\lambda_2 + \lambda_{12}}\right).$$

Hence the covariance is given by

$$\operatorname{Cov}(X, Y) = \frac{\lambda_{12}}{\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})},$$

and the correlation is $\rho(X, Y) = \lambda_{12}/\lambda$. Note that $0 \le \rho(X, Y) \le 1$.

Higher moments are not difficult to compute directly from the equality $\int x^i y^j dF(x, y) = \int ijx^{i-1}y^{j-1}\bar{F}(x, y)dxdy\,(i, j > 0)$ which follows from (3.1). If i and j are positive integers, we obtain with $\gamma_i = \lambda_i + \lambda_{12}$, $i = 1, 2$, that

$$EX^iY^j = j\Gamma(i + 1)\sum_{k=0}^{i-1} \frac{\Gamma(j + k)}{\Gamma(k + 1)\gamma_1^{i-k}\lambda^{j+k}} + i\Gamma(j + 1)\sum_{k=0}^{j-1} \frac{\Gamma(i + k)}{\Gamma(k + 1)\gamma_2^{j-k}\lambda^{i+k}}.$$

3.3 *Distributions obtained by a change of variables*

If X is an exponential random variable, then aX is exponential for all $a > 0$. However, if (X, Y) is BVE, then (aX, bY) is BVE only if $a = b > 0$. The distribution of (aX, bY) for $a, b>0$ is easily seen to be of the form

$$\overline{F}(x, y) = \exp^{[-\lambda_1 x-\lambda_2 y-\max(\lambda_3 x,\ \lambda_4 y)]}.$$

This distribution has exponential marginals and includes the BVE as the special case $\lambda_3=\lambda_4=\lambda_{12}$. It also includes the upper bound of (1.1) when F_1 and F_2 are exponential ($\lambda_1=\lambda_2=0$).

Other changes of variables in the BVE may be of interest, in particular, the distribution of $(X^{1/\beta}, Y^{1/\gamma})$ is a bivariate Weibull distribution, namely,

$$\overline{F}(x, y) = \exp^{[-\lambda^1 x^\beta-\lambda^2 y^\gamma-\lambda^{12} \max(x^\beta, y^\gamma)]}.$$

3.4 *Representation in terms of independent random variables.*

Theorem 3.2. *(X, Y) is BVE if and only if there exist independent exponential random variables U, V and W such that $X=\min(U, W)$, $Y=\min(V, W)$.*

This theorem is an immediate consequence of the fatal shock model discussed in §2.1. It can be an aid in reducing questions concerning dependent exponential random variables to questions concerning independent exponential random variables. An illustration of its usefulness is given in §3.5.

3.5 *Minima of exponential random variables*

An important property of independent exponential random variables X and Y is that $\min(X, Y)$ is exponential. If X and Y are dependent, but (X, Y) is BVE, then

$$P\{\min(X, Y) > x\} = P\{X > x, Y > x\} = e^{-\lambda x},$$

where $\lambda=\lambda_1+\lambda_2+\lambda_{12}$, so that the minimum of X and Y is again exponential. (This fact is also an immediate consequence of Theorem 3.2.)

On the other hand, the minimum of dependent exponential random variables need not be exponential, e.g., if X and Y have one of the bivariate distributions studied by Gumbel (1960),

$$\overline{F}(x, y) = e^{-x-y-\delta xy},$$

or

$$\overline{F}(x, y) = e^{-x-y}[1 + \alpha(1 - e^{-x})(1 - e^{-y})],$$

then $\min(X, Y)$ is exponential *only* in the case of independence (δ or $\alpha=0$). Similarly, if F is the lower bound of (1.1), then $\min(X, Y)$ is not exponential even when F_1 and F_2 are exponential.

3.6 *Comparison of the bivariate exponential with the case of independence*

It is common practice in reliability theory to assume that the components of a system have independent life lengths. It is of interest to see the effect of this assumption when in fact the lives have a BVE distribution.

Let us suppose the marginal distributions are known to be given by

$$\overline{F}_1(x) = e^{-(\lambda_1+\lambda_{12})x}, \quad \overline{F}_2(x) = e^{-(\lambda_2+\lambda_{12})x};$$

suppose that we operate under the assumption that the joint distribution

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$F(x, y)$ is $F_1(x)F_2(y)$, when in fact, $\bar{F}(x, y)$ is given by (1.3). Clearly, the difference

$$\bar{F}(x, y) - \bar{F}_1(x)\bar{F}_2(y) = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)}(1 - e^{-\lambda_{12} \min(x,y)})$$

is positive for all x and y, so the probability that both items survive is actually greater than the assumption of independence would lead us to believe. However, it is easily verified that for any bivariate distribution F with marginals F_1 and F_2,

$$\bar{F}(x, y) - \bar{F}_1(x)\bar{F}_2(y) = F(x, y) - F_1(x)F_2(y),$$

so the probability that both items fail is also greater than that computed under the assumption of independence. This means that in the case of a series system (which functions only when both items function), system survival probability is greatest in the case of dependence. On the other hand, in the case of a parallel system (which fails only when both items fail), system survival probability is greatest in the case of independence.

To determine the greatest discrepancy, $\max_{x,y} [\bar{F}(x, y) - \bar{F}_1(x)\bar{F}_2(y)]$, note that if $x < y$,

$$\bar{F}(x, y) - \bar{F}_1(x)\bar{F}_2(y) = e^{-\lambda_1 x - (\lambda_2 + \lambda_{12})y}(1 - e^{-\lambda_{12} x})$$

is decreasing in y, so that

$$\bar{F}(x, y) - \bar{F}_1(x)\bar{F}_2(y) \le e^{-\lambda \min(x,y)}(1 - e^{-\lambda_{12} \min(x,y)}) = e^{-\lambda t}(1 - e^{-\lambda_{12} t}),$$

where $t = \min (x, y)$. The maximum of the right-hand side occurs at $t = \lambda_{12}^{-1} \log (1 + \lambda_{12}/\lambda)$ and is equal to

$$\max_{x,y} [\bar{F}(x, y) - \bar{F}_1(x)\bar{F}_2(y)] = \delta^\delta/(1 + \delta)^{1+\delta},$$

where $\delta = \lambda/\lambda_{12}$. In terms of the correlation $\rho(x, y) = \lambda_{12}/\lambda$, the maximum discrepancy is $[\rho^\rho/(1+\rho)^{(1+\rho)}]^{1/\rho}$.

4. THE MULTIVARIATE EXPONENTIAL DISTRIBUTION

4.1 *Derivations*

To fix ideas, we consider first an extension of the fatal shock model to a three-component system. Let the independent Poisson processes $Z_1(t; \lambda_1)$, $Z_2(t; \lambda_2)$, $Z_3(t; \lambda_3)$ govern the occurrence of shocks to components 1, 2, 3, respectively; $Z_{12}(t; \lambda_{12})$, $Z_{13}(t; \lambda_{13})$, $Z_{23}(t; \lambda_{23})$ govern the occurrence of shocks to the component pairs 1 and 2, 1 and 3, 2 and 3, respectively; and $Z_{123}(t; \lambda_{123})$ governs the occurrence of simultaneous shocks to components 1, 2, 3. If X_1, X_2, X_3 denote the life length of the first, second, and third components, then

$$\begin{aligned}\bar{F}(x_1, x_2, x_3) &= P\{X_1 > x_1, X_2 > x_2, X_3 > x_3\} \\ &= P\{Z_1(x_1) = 0, Z_2(x_2) = 0, Z_3(x_3) = 0, Z_{12}(\max (x_1, x_2)) = 0, \\ &\qquad Z_{13}(\max (x_1, x_3)) = 0, Z_{23}(\max (x_2, x_3)) = 0, Z_{123}(\max (x_1, x_2, x_3)) = 0\} \qquad (4.1) \\ &= \exp [-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_{12} \max (x_1, x_2) - \lambda_{13} \max (x_1, x_3) \\ &\qquad - \lambda_{23} \max (x_2, x_3) - \lambda_{123} \max (x_1, x_2, x_3)].\end{aligned}$$

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It is clear that similar arguments yield the n-dimensional exponential distribution given by

$$\bar{F}(x_1, x_2, \cdots, x_n) = \exp\Big[-\sum_1^n \lambda_i x_i - \sum_{i<j} \lambda_{ij} \max(x_i, x_j) \tag{4.2}$$
$$- \sum_{i<j<k} \lambda_{ijk} \max(x_i, x_j, x_k)$$
$$- \cdots - \lambda_{12\cdots n} \max(x_1, x_2, \cdots, x_n)\Big].$$

To obtain a more compact notation for this distribution, let S denote the set of vectors $(s_1, \cdots, s_n)$ where each $s_j = 0$ or 1 but $(s_1, \cdots, s_n) \neq (0, \cdots, 0)$. For any vector $s \in S$, $\max(x_i s_i)$ is the maximum of the x_i's for which $s_i = 1$. Thus,

$$\bar{F}(x_1, \cdots, x_n) = \exp\Big[-\sum_{s \in S} \lambda_s \max(x_i s_i)\Big]. \tag{4.3}$$

For example, for $n=3$ the correspondence with (4.1) is $\lambda_{100} = \lambda_1$, $\lambda_{010} = \lambda_2$, $\lambda_{001} = \lambda_3$, $\lambda_{110} = \lambda_{12}$, $\lambda_{101} = \lambda_{13}$, $\lambda_{011} = \lambda_{13}$, $\lambda_{111} = \lambda_{123}$.

We call the distribution given by (4.2) or (4.3) the *multivariate exponential distribution* (abbreviated MVE). Note that the $(n-1)$-dimensional marginals (hence k-dimensional marginals, $k = 1, 2, \cdots, n-1$) are MVE. In particular, the two-dimensional marginals are BVE, so the one-dimensional marginals are exponential.

In the bivariate case (§2.2) the fatal and non-fatal shock models both yield the BVE. Indeed, if we assume in the multivariate case that shocks need not be fatal but instead cause transitions with varying probabilities, then by a tedious but direct calculation we again obtain the MVE.

Consider now the requirement that the residual life is independent of age, i.e.,

$$P\{X_1 > s_1 + t, \cdots, X_n > s_n + t \mid X_1 > t, \cdots, X_n > t\}$$
$$= P\{X_1 > s_1, \cdots, X_n > s_n\}$$

or

$$\bar{F}(s_1 + t, \cdots, s_n + t) = \bar{F}(s_1, \cdots, s_n)\bar{F}(t, \cdots, t). \tag{4.4}$$

If, in addition, the $(n-1)$-dimensional marginals are MVE, then the solution of (4.4) is the n-dimensional MVE. To see this, note that $\bar{F}(s, \cdots, s) = e^{-\theta s}$ follows from (4.4) with $s_1 = \cdots = s_n = s$. The choice $s_n = 0$ in (4.4) yields

$$\bar{F}(s_1 + t, \cdots, s_{n-1} + t, t) = \bar{F}(s_1, \cdots, s_{n-1}, 0)\bar{F}(t, \cdots, t) \tag{4.5}$$
$$= e^{-\theta t}\bar{F}_{n-1}(s_1, \cdots, s_{n-1}),$$

where $\bar{F}_{n-1}$ is an $(n-1)$-dimensional marginal. Using the assumption that $\bar{F}_{n-1}$ is MVE, (4.5) yields (4.2) or (4.3) on the domain $x_n \leq x_i$, $i = 1, 2, \cdots, n-1$.

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4.2 *Properties of the MVE*

The MVE of dimension n is not absolutely continuous (except, of course, for $n=1$ or in the special case of independence). As in the bivariate case, this is because a singular part is present: at least one of the hyperplanes $x_i=x_j(i\neq j)$, $x_i=x_j=x_k$ (i, j, k distinct), etc., has positive probability. For example, by referring to the fatal shock model (§4.1), we see that $X_i=X_j\neq X_k$, for all $k\neq i, j$ when the first event in a process governing shocks to components i or j occurs in the process $Z_{ij}(t)$. Because it is quite cumbersome and seems to be of little importance, we do not further discuss the decomposition of the MVE into parts absolutely continuous with respect to Lebesgue measures on various hyperplanes.

In the bivariate case the evaluation of the moment generating function was carried out both by a direct computation and by integration by parts. For the multivariate case, the direct computation becomes incredibly complicated. However, there is considerable simplification if in place of the Laplace transform we compute

$$\phi(s_1, \cdots, s_n) = \int \prod_1^n (1 - e^{-s_i x_i}) dF(x_1, \cdots, x_n) \tag{4.6}$$

using integration by parts. To do this, it is convenient to replace the parameters λ_s by new parameters g_s, $s\in S$, defined by

$$g_s = \sum_{rs\neq 0} \lambda_r;$$

i.e., g_s is the sum of all λ_r such that some coordinate is 1 in both r and s. For example, with $n=3$, g_{101} is the sum over all λ_s where s_1 or s_3 equals 1:

$$g_{101} = \lambda_{111} + \lambda_{110} + \lambda_{101} + \lambda_{011} + \lambda_{100} + \lambda_{001}.$$

Theorem 4.1. *If ϕ is defined by (4.6), then*

$$\phi(s_1, \cdots, s_n) = \left(\prod_1^n s_i\right) \sum{}' (g_{10\cdots 0} + s_1)^{-1}(g_{110\cdots 0} + s_1 + s_2)^{-1} \cdots \cdot (g_{1\cdots 1} + \sum s_j)^{-1}, \tag{4.7}$$

where Σ' is the summation over all permutations of the indices.

Remark. We illustrate the notation of (4.7) for the case $n=3$:

$$\begin{aligned}\phi(s_1, s_2, s_3) \frac{g_{111} + s_1 + s_2 + s_3}{s_1 s_2 s_3} &= (g_{110} + s_1 + s_2)^{-1}[(g_{100} + s_1)^{-1} + (g_{010} + s_2)^{-1}] \\ &+ (g_{101} + s_1 + s_3)^{-1}[(g_{100} + s_1)^{-1} + (g_{001} + s_3)^{-1}] \\ &+ (g_{011} + s_2 + s_3)^{-1}[(g_{010} + s_2)^{-1} + (g_{001} + s_3)^{-1}].\end{aligned}$$

Proof. The extension of (3.1),

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$$\int_0^\infty \cdots \int_0^\infty G(x_1, \cdots, x_n) dF(x_1, \cdots, x_n)$$
$$= \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1, \cdots, x_n) dG(x_1, \cdots, x_n),$$

holds whenever G is of bounded variation on finite intervals and G is identically zero if any argument of G is zero. Using this integration by parts formula, we obtain

$$\phi(s_1, \cdots, s_n) \tag{4.8}$$
$$= \left(\prod_1^n s_i\right) \int_0^\infty \cdots \int_0^\infty \bar{F}(x_1, \cdots, x_n) \exp\left(-\sum_1^n s_i x_i\right) \prod_1^n dx_i.$$

The evaluation of this integral is straightforward, though somewhat troublesome because it must be evaluated as a sum over all $n!$ regions $x_{i_1} > \cdots > x_{i_n}$.

If $x_1 > \max(x_2, \cdots, x_n)$, then

$$\bar{F}(x_1, \cdots, x_n) = \bar{F}(0, x_2, x_3, \cdots, x_n) \exp^{[-g_{100\cdots 0} x_1]}$$

so that

$$\left(\prod s_i\right) \int_{x_1 > \max(x_2, \cdots, x_n)} \bar{F}(x_1, \cdots, x_n) \exp\left[-\sum_1^n s_i x_i\right] \prod_1^n dx_i$$
$$= \frac{\prod s_i}{g_{10\cdots 0} + s_1} \int e^{-(g_{10\cdots 0} + s_1)\max(x_2, \cdots, x_n)} \bar{F}_{n-1}(x_2, \cdots, x_n) \cdot \exp\left(-\sum_1^n s_i x_i\right) \prod_2^n dx_i.$$

This integral is of dimension $n-1$ but otherwise is of the same form as (4.8). Assuming the ordering $x_1 > x_2 > \cdots > x_n$, we obtain by iteration that

$$\left(\prod_1^n s_i\right) \int_{x > \cdots > x_n > 0} \bar{F}(x_1, \cdots, x_n) \exp\left(-\sum_1^n s_i x_i\right) \prod_1^n dx_i$$
$$= (g_{10\cdots 0} + s_1)^{-1} (g_{110\cdots 0} + s_1 + s_2)^{-1} \cdots (g_{11\cdots 1} + s_1 + s_2 + \cdots + s_n)^{-1}$$
$$\cdot \prod_1^n s_i. \|$$

A final but very important property of the MVE that we mention is its representation in terms of independent exponentials. As in the bivariate case (Theorem 3.2), we obtain from the fatal shock model that if $X_1, \cdots, X_n$ are MVE, there exist independent exponential random variables Z_s, $s \in S$ such that $X_i = \min_{s_i = 1} Z_s$.

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5. FURTHER RESULTS FOR THE FUNCTIONAL EQUATION $\overline{F}(s_1+t,\ s_2+t) = \overline{F}(s_1,\ s_2)\overline{F}(t,\ t)$

In Section 2.3 we introduced and motivated the functional equation (2.8) and found the general solution to be

$$\overline{F}(x,\ y) = \begin{cases} e^{-\theta y}\overline{F}_1(x-y), & x \geq y, \\ e^{-\theta x}\overline{F}_2(y-x), & x \leq y. \end{cases} \tag{5.1}$$

When F_1, F_2 are exponential distributions with parameters δ_1, δ_2 satisfying $\delta_1, \delta_2 \leq \theta \leq \delta_1+\delta_2$, $\overline{F}(x,\ y)$ is the BVE. However, $F(x,\ y)$ specified by (5.1) is a distribution only for certain marginals F_1 and F_2.

In order that $F(x,\ y)$ be a distribution function, it is necessary, for any two points $(x_1,\ y_1)$ and $(x_2,\ y_2)$, that

$$\overline{F}(x_1,\ y_1) + \overline{F}(x_2,\ y_2) - \overline{F}(x_1,\ y_2) - \overline{F}(x_2,\ y_1) \geq 0. \tag{5.2}$$

(5.2) is equivalent to conditions on $\overline{F}_1$ and $\overline{F}_2$ which depend upon $(x_1,\ y_1)$ and $(x_2,\ y_2)$; e.g., if $x_1 \leq x_2 \leq y_1 \leq y_2$, (5.2) becomes

$$[\overline{F}_2(y_1-x_1) - \overline{F}_2(y_2-x_1)]\exp^{[\theta(x_2-x_1)]} \geq \overline{F}_2(y_1-x_2) - \overline{F}_2(y_2-x_2).$$

Such conditions are not easily verified; we obtain some alternative conditions when the marginal distributions have densities f_1 and f_2 satisfying certain regularity conditions.

Theorem 5.1. *Let $F_j(x)$ be a distribution function with absolutely continuous density $f_j(x)$ for which $\lim_{z\to\infty} f_j(z)=0$, $j=1, 2$. In order that $F(x,\ y)$ given by* (5.1) *be a bivariate distribution, it is necessary and sufficient that*

(i) $\theta \leq f_1(0) + f_2(0) \leq 2\theta$,

(ii) $\dfrac{d \log f_j(z)}{dz} \geq -\theta$, *for all* $z \geq 0, j = 1, 2$.

Proof. $F(x,\ y)$ will be a distribution function if and only if both the absolutely continuous part $F_a(x,\ y)$ and the singular part $F_s(x,\ y)$ are distribution functions and $F(x,\ y)$ is a convex mixture of $F_a(x,\ y)$ and $F_s(x,\ y)$, i.e.,

$$\overline{F}(x,\ y) = \alpha\overline{F}_a(x,\ y) + (1-\alpha)\overline{F}_s(x,\ y), \quad 0 \leq \alpha \leq 1. \tag{5.3}$$

To determine the conditions on $\overline{F}_a(x,\ y)$ and $\overline{F}_s(x,\ y)$ we compute $\partial^2\overline{F}/\partial x\,\partial y = \alpha f_a(x,\ y)$ and obtain α from $F_a(\infty,\ \infty)=1$. The singular part $F_s(x,\ y)$ is concentrated on $x=y$, so that we can obtain it from $(1-\alpha)\overline{F}_s(x,\ x)=\overline{F}(x,\ x) - \alpha\overline{F}_a(x,\ x)$. Carrying out these steps, we have

$$\frac{\partial^2\overline{F}(x,\ y)}{\partial x\ \partial y} \equiv \alpha f_a(x,\ y) = \begin{cases} e^{-\theta y}[f_1'(x-y) + \theta f_1(x-y)], & x \geq y, \\ e^{-\theta x}[f_2'(y-x) + \theta f_2(y-x)], & x \leq y, \end{cases} \tag{5.4}$$

$$\int_{x\geq y} \alpha f_a(x,\ y)dx\,dy = 1 - \frac{1}{\theta}f_1(0),$$

$$\int_{x\leq y} \alpha f_a(x,\ y)dx\,dy = 1 - \frac{1}{\theta}f_2(0),$$

so that

$$\alpha \equiv \int_0^\infty \int_0^\infty \alpha f_a(x, y)dx\, dy = 2 - \frac{1}{\theta}\,[f_1(0) + f_2(0)]. \tag{5.5}$$

Thus, the absolutely continuous part $F_a(x, y)$ of $F(x, y)$ has density $f_a(x, y)$ given by (5.4) and (5.5). We compute

$$\begin{aligned}\overline{F}_a(x, x) &= \alpha^{-1}\int_x^\infty du \int_u^\infty dv\, e^{-\theta u}[f'(v-u) + \theta f(v-u)] \\ &+ \alpha^{-1}\int_x^\infty dv \int_v^\infty du\, e^{-\theta u}[f'(u-v) + \theta f(u-v)] \\ &= e^{-\theta x}.\end{aligned}$$

But $\overline{F}(x, x) = \exp(-\theta x)$, so that

$$\overline{F}_s(x, x) = [\overline{F}(x, x) - \alpha \overline{F}_a(x, x)]/(1-\alpha) = \exp(-\theta x).$$

Since $F_s(x, y)$ is concentrated on $x = y$, we conclude that

$$\overline{F}_s(x, y) = \exp[-\theta \max(x, y)]. \tag{5.6}$$

Thus, F is a valid distribution function if:

(i) F is a convex mixture of F_a and F_s, i.e., $0 \le \alpha \le 1$. From (5.5) this condition is

$$\theta \le f_1(0) + f_2(0) \le 2\theta.$$

(ii) F_a is a valid distribution function, i.e., $f_a(x, y) \ge 0$. From (5.4) this is $f_i'(z) + \theta f_i'(z) \ge 0$, $i = 1, 2$, or equivalently,

$$d \log f_i(z)/dz \ge -\theta. \|$$

We now check the conditions of the theorem for the important Weibull and gamma distributions. The respective density functions

$$w(z) \equiv w(z; \beta, \delta) = \beta\delta z^{\beta-1}\exp(-\delta z^\beta), \qquad z > 0, \beta > 0, \delta > 0,$$

$$g(z) \equiv g(z; \beta, \delta) = \frac{1}{\Gamma(\beta)}\,\delta^\beta z^{\beta-1}\exp(-\delta z), \qquad z > 0, \beta > 0, \delta > 0,$$

satisfy the regularity conditions. It is easily checked that

$$\lim_{z\to\infty}\frac{d \log w(z)}{dz} = \lim_{z\to\infty}\frac{d \log g(z)}{dz} = -\infty \quad \text{if} \quad \alpha > 1,$$

and

$$\lim_{z\to 0}\frac{d \log w(z)}{dz} = \lim_{z\to 0}\frac{d \log g(z)}{dz} = -\infty \quad \text{if} \quad \alpha < 1.$$

Because of condition (ii), F_1 or F_2 can be Weibull or gamma distributions only in the special case that they are exponential.

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In the exponential case, condition (*i*) becomes $\theta \le \delta_1 + \delta_2 \le 2\theta$ where δ_1 and δ_2 are the parameters of f_1 and f_2. Condition (*ii*) requires $\delta_1 \le \theta$, $\delta_2 \le \theta$. Thus,

$$\overline{F}(x, y) = \begin{cases} e^{-\theta y}\overline{F}_1(x - y), & x \ge y, \\ e^{-\theta x}\overline{F}_2(y - x), & x \le y, \end{cases}$$

is a bivariate distribution with exponential marginals f_1 and f_2 if and only if $\delta_1 \le \theta$, $\delta_2 \le \theta$, $\delta_1 + \delta_2 \ge \theta$.

Remark. Suppose (f_1, f_2) satisfies conditions (*i*) and (*ii*) so that $\overline{F}(x, y)$ defined by (5.1) is a valid bivariate distribution, and similarly suppose (g_1, g_2) satisfies (*i*) and (*ii*). Then the mixture $(\gamma f_1 + (1-\gamma)g_1, \gamma f_2 + (1-\gamma)g_2)$ satisfies the conditions and yields another solution to the functional equation (2.8). In particular, marginal distributions which are mixtures of certain exponential distributions yield solutions.

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