TEAM = PHOENIX 03/08.21 HW #2 ①

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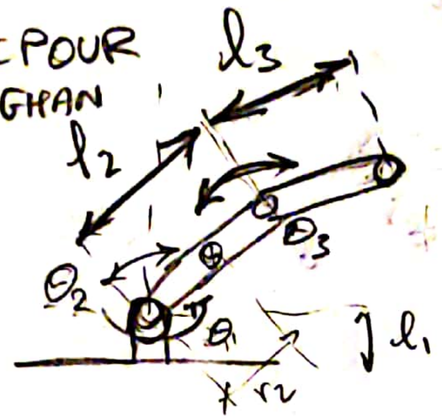


$\theta_1 \rightarrow$ ROTATION ALONG Z AXIS

$\theta_2 \rightarrow$ ROTATION ALONG Y AXIS

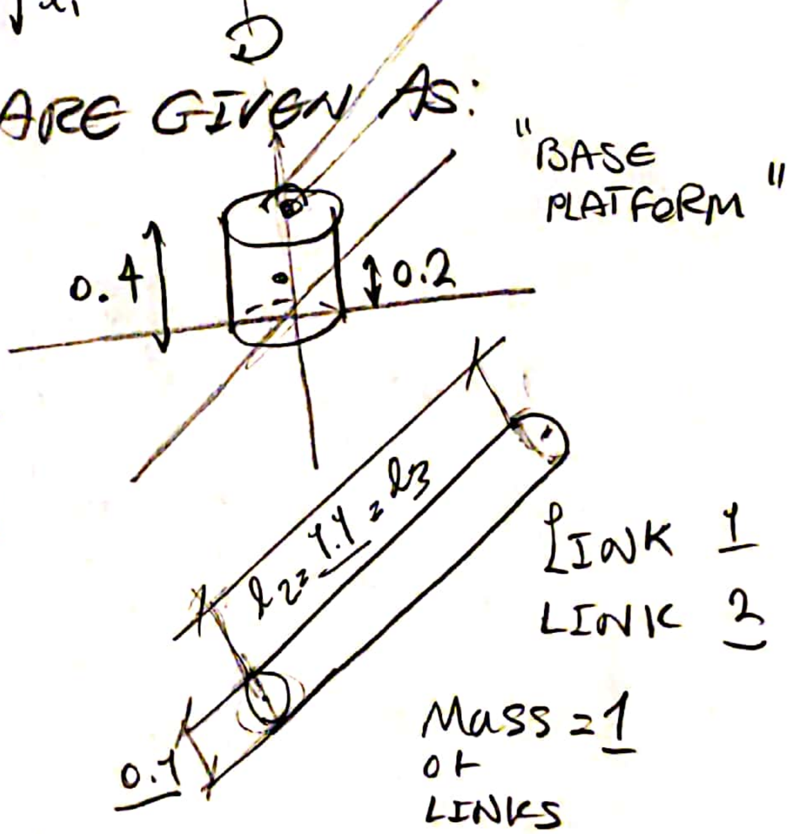
$\theta_3 \rightarrow$ ROTATION ALONG Y AXIS

THE ROBOT DIMENSIONS ARE GIVEN AS:

$l_1 = 0.4$ WIDTH $= 0.4$

$l_2 = 1.1$ WIDTH $= 0.1$

$l_3 = 1.1$ WIDTH $= 0.1$



DYNAMICAL MODEL:

BY DEFINING THE POTENTIAL V AND KINETIC K ENERGIES OF THE SYSTEM, THE LAGRANGIAN $\mathcal{L}$ IS CALCULATED AS THE DIFFERENCE BETWEEN THE KINETIC AND POTENTIAL ENERGIES.

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q$$

$Q \rightarrow$ GENERALIZED FORCE

THREE LINKS $\longrightarrow$ 3 DoF

3 GENERALIZED COORDINATES $(q_i) \rightarrow$ 3 EQUATIONS

KINETIC ENERGY:

TO CALCULATE KINETIC ENERGY OF ROBOT, WE SUM THE KINETIC ENERGY OF EACH LINK. HENCE, TOTAL KINETIC ENERGY IS CALCULATED AS FOLLOWS:

$$K(\theta, \dot{\theta}) = \sum_{i=1}^{n} K_i(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T D(\theta)\dot{\theta} \quad (1)$$

$D(\theta) \in R^{n \times n}$ IS THE MANIPULATOR INERTIA MATRIX, GIVEN AS:

$$D(\theta) = \sum_{i=1}^{n} J_i^T M_i J_i \quad (A)$$

, $J_i \rightarrow$ JACOBIAN MATRIX

$M_i \rightarrow$ GENERALIZED MASS MATRIX. FOR ith LINK.

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POTENTIAL ENERGY CALCULATION :

LETS INTRODUCE $h_i \rightarrow$ THE HEIGHT OF THE MASS CENTER OF THE ith LINK

THE POTENTIAL ENERGY IS GIVEN BY :

$$V_i(\theta) = m_i g h_i(\theta) \quad (2)$$

(1) AND (2) $\Longrightarrow$ LAGRANGIAN BECOMES :

$$\mathcal{L}(\theta,\dot{\theta}) = \sum_{i=1}^{n} \left(K_i(\theta,\dot{\theta}) - V_i(\theta)\right) = \frac{1}{2}\dot{\theta}^T D(\theta)\dot{\theta} - V(\theta)$$

REWRITING THE EULER-LAGRANGIAN DYNAMIC MODEL IN COMPACT FORM :

$$D(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + g(\theta) = \tau$$

WHERE,

$\tau \rightarrow$ VECTOR OF ACTUATOR TORQUES.

$g(\theta) \rightarrow$ VECTOR OF GRAVITY FORCES

$C(\theta,\dot{\theta}) \longrightarrow$ CORIOLIS TERM. B

$$C(\theta,\dot{\theta}) = \Gamma_{ijk}\dot{\theta}_k = \frac{1}{2}\left(\frac{\partial D_{ij}}{\partial \theta_k} + \frac{\partial D_{ik}}{\partial \theta_j} - \frac{\partial D_{kj}}{\partial \theta_i}\right)\dot{\theta}_k.$$

D IS SQUARE MATRIX 3x3.

C, g, τ ARE COLUMN VECTOR OF THE SIZE 4x1.

TO FIND THE INERTIA MATRIX D USING EQ. A WE NEED TO DEFINE THE JACOBIAN OF EVERY LINK. BY ATTACHING COORDINATE FRAME AT THE MASS CENTER OF EACH LINK, THE GENERALIZED INERTIA MATRIX M_i CAN BE WRITTEN IN A DIAGONAL FORM.

$$M_i = \begin{bmatrix} m_i & 0 & 0 & 0 & 0 & 0 \\ 0 & m_i & 0 & 0 & 0 & 0 \\ 0 & 0 & m_i & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xi} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{yi} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{zi} \end{bmatrix}$$

SO, BY EXPANDING EQ. A WE GET.

$$D(\theta) = J_1^T M_1 J_1 + J_2^T M_2 J_2 + J_3^T M_3 J_3 + J_4^T M_4 J_4$$

ALSO, THE JACOBIAN MATRIX J CORRESPONDING TO EACH LINK IS GIVEN BY:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad J_2 = \begin{bmatrix} 0 & 0 & 0 \\ r_2 c_2 & 0 & 0 \\ 0 & r_2 & 0 \\ s_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 0 & l_2 s_3 & 0 \\ r_3 c_{23} + l_2 c_2 & 0 & 0 \\ 0 & r_3 + l_2 c_3 & 0 \\ s_{23} & 0 & 0 \\ 0 & -1 & -1 \\ c_{23} & 0 & 0 \end{bmatrix}$$

$r_i \rightarrow$ STANDS FOR DISTANCE OF THE ith LINK TO ITS MASS CENTER

$s_i \rightarrow \sin\theta_i$

$c_i \rightarrow \cos\theta_i$

$s_{ij}, c_{ij} \rightarrow \sin(\theta_i + \theta_j), \cos(\theta_i + \theta_j)$

HAVING THE INERTIA MATRIX $\underline{D}$, THE CORIOLIS AND CENTRIFUGAL FORCES ARE COMPUTED FROM EQ. A.

FOR THE POTENTIAL PART, COMPONENTS OF GRAVITY FORCES ON THE ROBOTIC MANIPULATOR CAN BE DERIVED FROM:

$$g(\theta) = \frac{\partial V}{\partial \theta_i}$$

$$V(\theta) = g(m_1 h_1(\theta)) + m_2 h_2(\theta) + m_3 h_3(\theta) + m_4 h_4(\theta))$$

h_i IS THE HEIGHT OF THE MASS CENTER OF THE ith LINK GIVEN BY:

$$\underline{h_1(\theta) = r_1}, \quad \underline{h_2(\theta) = l_1 + r_2 \sin\theta_2}.$$

$$h_2(\theta) = l_1 + l_2 \sin\theta_2 + r_3 \sin(\theta_2 + \theta_3)$$

$$J_2^T M_2 J_2 =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ r_2 c_2 & 0 & 0 \\ 0 & r_2 & 0 \\ \delta_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & r_2 c_2 & 0 & \delta_2 & 0 & c_2 \\ 0 & 0 & r_2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5×2 $\quad 2 \times 4 \quad 3 \times 5$

$$\underbrace{\begin{bmatrix} 0 & r_2 c_2 & 0 & \delta_2 & 0 & c_2 \\ 0 & 0 & r_2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{3\times 6} \underbrace{\begin{bmatrix} m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{x_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{y_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{z_2} \end{bmatrix}}_{6\times 6}$$

$$0 + m_2 r_2 c_2 + \delta_2 I_{x_2} + c_2 I_{z_2}$$

$$\begin{bmatrix} 0 & m_2 r_2 c_2 & 0 & \delta_2 I_{x_2} & 0 & c_2 I_{z_2} \\ 0 & 0 & r_2 m_2 & 0 & -I_{y_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & m_2 r_2 c_2 & 0 & \delta_2 I_{x_2} & 0 & c_2 I_{z_2} \\ 0 & 0 & r_2 m_2 & 0 & -I_{y_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{3\times 6} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ r_2 c_2 & 0 & 0 \\ 0 & r_2 & 0 \\ \delta_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}^T}_{6\times 3}$$

$$\begin{bmatrix} m_2 r_2^2 c_2^2 + s_2^2 I_{x_2} + c_2^2 I_{z_2} & 0 & 0 \\ 0 & r_2^2 m_2 + I_{y_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = J_2^T M_2 J_2$$

$J_1^T M_1 J_1 =$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{3\times 6} \underbrace{\begin{bmatrix} M_1 \end{bmatrix}}_{6\times 6} =$$

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & I_{z_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{3\times 6} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{6\times 3} = \begin{bmatrix} I_{z_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$J_3^T M_3 J_3 =$

$$J_3^T = \begin{bmatrix} 0 & r_3 c_{23} + l_2 c_2 & 0 & s_{23} & 0 & c_{23} \\ l_2 s_3 & 0 & r_3 + l_2 c_3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$J_3^T M_3 = \begin{bmatrix} 0 & r_3 c_{23} + l_2 c_2 & 0 & s_{23} & 0 & c_{23} \\ l_2 s_3 & 0 & r_3 + l_2 c_3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} m_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & m_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{x_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{y_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{z_3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (r_3 c_{23} + l_2 c_2) m_3 & 0 & s_{23} I_{x_3} & 0 & c_{23} I_{z_3} \\ m_3 l_2 s_3 & 0 & (r_3 + l_2 c_3) m_3 & 0 & -I_{y_3} & 0 \\ 0 & 0 & 0 & 0 & -I_{y_3} & 0 \end{bmatrix}$$

$J_3^T M_3 J_3 =$

$$\begin{bmatrix} 0 & (r_3 c_{23} + l_2 c_2) m_3 & 0 & s_{23} I_{x_3} & 0 & c_{23} I_{z_3} \\ m_3 l_2 s_3 & 0 & (r_3 + l_2 c_3) m_3 & 0 & -I_{y_3} & 0 \\ 0 & 0 & 0 & 0 & -I_{y_3} & 0 \end{bmatrix}_{3\times 6} \begin{bmatrix} 0 & l_2 s_3 & 0 \\ r_3 c_{23} + l_2 c_2 & 0 & 0 \\ 0 & r_3 + r_2 c_3 & 0 \\ S_{23} & 0 & 0 \\ 0 & -1 & -1 \\ C_{23} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (r_3 c_{23} + l_2 c_2)^2 m_3 + S_{23}^2 I_{x_3} + c_{23}^2 I_{z_3} & 0 & 0 \\ 0 & m_3 (l_2 s_3)^2 + (r_3 + l_2 c_3)^2 m_3 + I_{y_3} & 0 \\ 0 & + I_{y_3} & 0 \end{bmatrix}$$

$= J_3^T M_3 J_3$, $D(\theta) = J_1^T M_1 J_1 + J_2^T M_2 J_2 + J_3^T M_3 J_3 + J_4^T M_4 J_4$

$$\begin{bmatrix} I z_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} m_2 r_2^2 c_2^2 + S_2^2 I x_2 + C_2^2 I z_2 & 0 & 0 \\ 0 & r_2^2 m_2 + I y_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} (r_3 c_{23} + l_2 c_2)^2 m_3 + S_{23}^2 I_{x_3} + C_{23}^2 I_{z_3} & 0 & 0 \\ 0 & m_3 (l_2 s_3)^2 + (r_3 + l_2 c_3)^2 m_3 + I_{y_3} & 0 \\ 0 & I_{y_3} & 0 \end{bmatrix}$$

$$\Rightarrow \underline{D_{11}} = I_{z_1} + m_2 r_2^2 C_2^2 + S_2^2 I_{x_2} + C_2^2 I_{z_2} + (r_3 C_{23} + l_2 C_3)^2 m_3 + S_{23}^2 I_{x_3} + C_{23}^2 I_{z_3}$$

$$\underline{D_{12}} = \underline{D_{13}} = \underline{D_{21}} = \underline{D_{23}} = \underline{D_{31}} = \underline{D_{33}} = 0$$

$$\underline{D_{22}} = r_2^2 m_2 + I_{y_2} + (r_3 + l_2 C_3)^2 m_3 + I_{y_3} + m_3 (l_2 S_3)^2$$

$$D_{32} = I_{y_3}$$

$$\Rightarrow D_{11} = I_{z_1} + m_2 r_2^2 C_2^2 + I_{x_2} S_2^2 + I_{z_2} C_2^2 + m_3 (r_3 C_{23} + l_2 C_3)^2 + I_{x_3} S_{23}^2 + I_{z_3} C_{23}^2$$

$$\Rightarrow D_{22} = m_2 r_2^2 + I_{y_2} + m_3 (r_3 + l_2 C_3)^2 + I_{y_3} + m_3 (l_2 S_3)^2$$

$$D_{32} = I_{y_3}$$

$$\underline{\text{NOW}} \quad C(\theta, \dot{\theta}) = \frac{1}{2} \left(\frac{\partial D_{ij}}{\partial \theta_k} + \frac{\partial D_{ik}}{\partial \theta_j} - \frac{\partial D_{kj}}{\partial \theta_i} \right) \dot{\theta}_k$$

FOR THE EFFECT OF GRAVITY FORCES ON THE ROBOTIC MANIPULATOR

$$g(\theta) = \frac{\partial v}{\partial \theta_i}$$

$$V(\theta) = g(m_1 h_1(\theta) + m_2 h_2(\theta) + m_3 h_3(\theta))$$

WHERE h_i THE HEIGHT OF THE MASS CENTER of ith LINK:

$h_1(\theta) = 0$, $h_2(\theta) = l_1 + r_2 \sin\theta_2$

$h_3(\theta) = l_1 + l_2 \sin\theta_2 + r_3 \sin(\theta_2 + \theta_3)$

$$\hat{g}(\theta) = \begin{bmatrix} \frac{\partial v}{\partial \theta_1} \\ \frac{\partial v}{\partial \theta_2} \\ \frac{\partial v}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} 0 \\ g(m_2 \dot{\theta}_2 r_2 \cos\theta_2 + m_3 \dot{\theta}_2 l_2 \cos\theta_2 + m_3 \dot{\theta}_2 r_3 \cos(\theta_2+\theta_3)) \\ g m_3 \dot{\theta}_3 r_3 \cos(\theta_2 + \theta_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ g(m_2 \dot{\theta}_2 r_2 \cos\theta_2 + m_3 \dot{\theta}_2 l_2 \cos\theta_2 + m_3 \dot{\theta}_2 r_3 \cos(\theta_2+\theta_3)) \\ g m_3 \dot{\theta}_3 r_3 \cos(\theta_2 + \theta_3) \end{bmatrix}$$

GRAVITY TERM