AMS 361 R01/R03

Week 9 : Constant coefficients (Inhomogeneous (Undetermined coefficients, Variational principle))

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Spring 2023

Constant coefficients (Inhomogeneous)

Consider the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x),$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants and $f(x) = \sum \prod_{j=1,2,\ldots,k} f_j(x)$.

Steps

- Find the general solution $y_c = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$ for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

- Find a particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients (UC or MUC), Variational principle (VP or VOP))

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

- General solution is $y = y_c + y_p$.

Undetermined coefficients

Steps

- For each $j = 1, 2, \ldots, k$, guess y_{p_j} based on the following table.

Term in $f_j(x)$	Choice for y_{p_j}
e^{rx}	Ce^{rx}
x^m	$A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0$
$\cos(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$

- If $f_j(x)$ happens to be a single (or double, or triple, ..., or n-fold) root of the corresponding homogeneous equation, i.e. $f_j = y_k$, then multiply your choice of y_{p_j} by x (or x^2, or x^3, ..., or x^n).
- Let $y_p = \sum \prod_{j=1,2,\ldots,k} y_{p_j}$.
- Compute $y_p', y_p'', \ldots, y_p^{(n)}$.
- Substitute $y_p, y_p', y_p'', \ldots, y_p^{(n)}$ above into the original ODE to find the coefficients $A, B, C, A_0, A_1, \ldots, A_m$.

Undetermined coefficients

Example (Final Problem 2, Spring 2018)

Use LT method and another method to find the PS of (Convolutions, in any, must be evaluated):

$$\begin{cases} x'' + x = \sin(\omega t) \\ x(0) = x'(0) = 0 \end{cases}$$

where $\omega \neq 0$ is a constant; three cases $\omega = 1, \omega = -1$, and $\omega =$ anything else must be considered.

Remark

LT method: week 14 Laplace transform;
Another method: week 9 Undetermined coefficients.

$$x'' + x = \sin(wt)$$

- Find general solution y_c for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

$$x_c'' + x_c = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda_1 = -i, \quad \lambda_2 = i \qquad (\lambda = \alpha \pm \beta i \Rightarrow \alpha = 0, \beta = 1)$$

$$x_1 = e^{\alpha t} \cos \beta t = \cos t$$

$$x_2 = e^{\alpha t} \sin \beta t = \sin t$$

$$x = C_1 x_1 + C_2 x_2$$

$$= C_1 \cos t + C_2 \sin t$$

- Find particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients, Variational principle)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

$$x_p'' + x_p = \sin wt$$

$$f(t) = \sin wt$$

$$f(t) = \begin{cases} \sin t, & w = 1 \\ -\sin t, & w = -1 \\ \sin wt, & w \notin \{-1, 1\} \end{cases} \Rightarrow x_p = \begin{cases} A t \cos t + B t \sin t \\ A t \cos t + B t \sin t \\ A \cos wt + B \sin wt \end{cases}$$

① Case 1: $w = 1$

$$f(t) = \sin t,$$

$$x_p = A t \cos t + B t \sin t$$

$$x_p' = (A+Bt)\cos t + (-At+B)\sin t$$

$$x_p'' = (-At+2B)\cos t + (-2A-Bt)\sin t$$

$$x_p'' + x_p = \sin t$$

$$(-\cancel{At}+2B)\cos t + (-2A-\cancel{Bt})\sin t + \cancel{At}\cos t + \cancel{Bt}\sin t$$

$$\left.\begin{aligned} &= 2B\cos t - 2A\sin t \\ &= \sin t \end{aligned}\right\} \Rightarrow \begin{cases} 2B = 0 \\ -2A = 1 \end{cases}$$

$$\begin{cases} A = -\frac{1}{2} \\ B = 0 \end{cases}$$

$$x_p = -\frac{1}{2}t\cos t$$

$$x = x_c + x_p$$

$$\boxed{x(t) = C_1\cos t + C_2\sin t - \frac{1}{2}t\cos t}$$ G.S.

$$x' = -C_1\sin t + C_2\cos t + \frac{1}{2}t\sin t - \frac{1}{2}\cos t$$

$$x(0) = x'(0) = 0$$

$$\begin{cases} x(0) = C_1\cos 0 + C_2\sin 0 - \frac{1}{2}0\cos 0 = C_1 = 0 \\ x'(0) = -C_1\sin 0 + C_2\cos 0 + \frac{1}{2}0\sin 0 - \frac{1}{2}\cos 0 = C_2 - \frac{1}{2} = 0 \end{cases}$$

$$\begin{cases} C_1 = 0 \\ C_2 = \frac{1}{2} \end{cases}$$

$$\boxed{x(t) = \frac{1}{2}\sin t - \frac{1}{2}t\cos t}$$ P.S.

② Case 2: $w = -1$

$f(t) = -\sin t$,

$$y_p = At\cos t + Bt\sin t$$

$$x_p' = (A+Bt)\cos t + (-At+B)\sin t$$

$$x_p'' = (-At+2B)\cos t + (-2A-Bt)\sin t$$

$$x_p'' + x_p = -\sin t$$

$$(-\cancel{At}+2B)\cos t + (-2A-\cancel{Bt})\sin t + \cancel{At}\cos t + \cancel{Bt}\sin t$$

$$\left.\begin{aligned} &= 2B\cos t - 2A\sin t \\ &= -\sin t \end{aligned}\right\} \Rightarrow \begin{cases} 2B = 0 \\ -2A = -1 \end{cases}$$

$$\begin{cases} A = \frac{1}{2} \\ B = 0 \end{cases}$$

$$x_p = \frac{1}{2}t\cos t$$

$$x = x_c + x_p$$

$$\boxed{x(t) = C_1\cos t + C_2\sin t + \frac{1}{2}t\cos t}$$ G.S.

$$x' = -C_1\sin t + C_2\cos t - \frac{1}{2}t\sin t + \frac{1}{2}\cos t$$

$$x(0) = x'(0) = 0$$

$$\begin{cases} x(0) = C_1\cos 0 + C_2\sin 0 + \frac{1}{2}0\cos 0 = C_1 = 0 \\ x'(0) = -C_1\sin 0 + C_2\cos 0 - \frac{1}{2}0\sin 0 + \frac{1}{2}\cos 0 = C_2 + \frac{1}{2} = 0 \end{cases}$$

$$\begin{cases} C_1 = 0 \\ C_2 = -\frac{1}{2} \end{cases}$$

$$\boxed{x(t) = -\frac{1}{2}\sin t + \frac{1}{2}t\cos t}$$ P.S.

③ Case 3: $\omega \notin \{-1, 1\}$,

$$f(t) = \sin wt$$

$$x_p = A\cos wt + B\sin wt$$

$$x_p' = -Aw\sin wt + Bw\cos wt$$

$$x_p'' = -Aw^2\cos wt - Bw^2\sin wt$$

$$x_p'' + x_p = \sin wt$$

$$-Aw^2\cos wt - Bw^2\sin wt + A\cos wt + B\sin wt$$

$$= A(1-w^2)\cos wt + B(1-w^2)\sin wt$$

$$= \sin wt$$

$$\begin{cases} A(1-w^2) = 0 \\ B(1-w^2) = 1 \end{cases} \qquad (w \neq -1, 1) \Rightarrow (w^2 \neq 1) \Rightarrow (1-w^2 \neq 0)$$

$$\begin{cases} A = 0 \\ B = \dfrac{1}{1-w^2} \end{cases}$$

$$x_p = \frac{1}{1-w^2}\sin wt$$

$$x = x_c + x_p$$

$$\boxed{x(t) = C_1\cos t + C_2\sin t + \frac{1}{1-w^2}\sin wt} \qquad \text{G.S.}$$

$$x' = -C_1\sin t + C_2\cos t + \frac{w}{1-w^2}\cos wt$$

$$x(0) = x'(0) = 0$$

$$\begin{cases} x(0) = C_1\cos 0 + C_2\sin 0 + \dfrac{1}{1-w^2}\sin 0 = C_1 = 0 \\ x'(0) = -C_1\sin 0 + C_2\cos 0 + \dfrac{w}{1-w^2}\cos 0 = C_2 + \dfrac{w}{1-w^2} = 0 \end{cases}$$

$$\begin{cases} C_1 = 0 \\ C_2 = \dfrac{w}{w^2-1} \end{cases}$$

$$\boxed{x(t) = \frac{w}{w^2-1} \sin t - \frac{1}{w^2-1} \sin wt}$$

P.S.

In conclusion,

$$\boxed{x(t) = \begin{cases} w\left(\frac{1}{2} \sin t - \frac{1}{2} t \cos t\right) & \text{if } w = -1, 1 \\ \frac{1}{w^2-1}\left(w \sin t - \sin wt\right) & \text{if } w \neq -1, 1 \end{cases}}$$

P.S.

Undetermined coefficients

Example (Test 3 Problem 2, Spring 2020)

Find, by any method, the GS of

$$y'' + \beta^2 y = 1 + e^{\beta x} + \cos(\beta x) + \sin(\beta x)$$

where integer $\beta > 0$.

Undetermined coefficients

Example (Final Problem 4, Fall 2020)

Use LT and one non-LT method to find the PS of

$$\begin{cases} y''' + y'' + y' + y = \sin x + \cos x \\ y''(0) = y'(0) = y(0) = 0 \end{cases}$$

Your solution may not contain convolution ($\otimes$). Thus, convolutions, if any, must be evaluated.

Remark

LT method: week 14 Laplace transform;
Another method: week 9 Undetermined coefficients.

Undetermined coefficients

Example (Test 3 Problem 1, Spring 2022)

Find the G.S. of the DE and the P.S. of the following IVP:

$$\begin{cases} x'' - 3x' - 10x = e^{2t} + e^{-5t} \\ x(0) = x'(0) = 0 \end{cases}.$$

Primes denote derivatives WRT t.

$$x'' - 3x' - 10x = e^{2t} + e^{-5t}$$

- Find general solution y_c for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

$$x_c'' - 3x_c' - 10x_c = 0$$

- Obtain the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

and its solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda + 2)(\lambda - 5) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = -2$$

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

$$y_1 = e^{\lambda_1 t} = e^{5t}$$

$$y_2 = e^{\lambda_2 t} = e^{-2t}$$

$$y_c = c_1 y_1 + c_2 y_2 = c_1 e^{5t} + c_2 e^{-2t}$$

- Find particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients, Variational principle)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

$$x_p'' - 3x_p' - 10x_p = e^{2t} + e^{-5t}$$

$$f(t) = \underbrace{e^{2t}}_{f_1} + \underbrace{e^{-5t}}_{f_2}$$

$$f_1(t) = e^{2t}$$

$$f_2(t) = e^{-5t}$$

$f = f_1 + f_2$

- For each $j = 1, 2, \ldots, k$, guess y_{p_j} based on the following table.

Term in $f_j(x)$	Choice for y_{p_j}
e^{rx}	Ce^{rx}

$f_1(t) = e^{2t} \longrightarrow x_{p_1}(t) = Ae^{2t}$

$f_2(t) = e^{-5t} \longrightarrow x_{p_2}(t) = Be^{-5t}$

- If $f_j(x)$ happens to be a single (or double, or triple, ..., or n-fold) root of the corresponding homogeneous equation, then multiply your choice of y_{p_j} by x (or x^2, or x^3, ..., or x^n).

Not happen. Skip this step.

- Let $y_p = \sum \prod_{j=1,2,\ldots,k} y_{p_j}$.

$$\begin{aligned} x_p &= x_{p_1} + x_{p_2} \\ &= Ae^{2t} + Be^{-5t} \end{aligned}$$

- Compute $y_p', y_p'', \ldots, y_p^{(n)}$.

$$x_p = Ae^{2t} + Be^{-5t}$$

$$x_p' = 2Ae^{2t} - 5Be^{-5t}$$

$$x_p'' = 4Ae^{2t} + 25Be^{-5t}$$

- Substitute $y_p, y_p', y_p'', \ldots, y_p^{(n)}$ above into the original ODE to find the coefficients $A, B, C, A_0, A_1, \ldots, A_m$.

$$x_p'' - 3x_p' - 10x_p = e^{2t} + e^{-5t}$$

$$(4Ae^{2t} + 25Be^{-5t}) - 3(2Ae^{2t} - 5Be^{-5t}) - 10(Ae^{2t} + Be^{-5t}) = e^{2t} + e^{-5t}$$

$$\begin{matrix} e^{2t}: \\ e^{-5t}: \end{matrix} \quad \begin{cases} 4A - 6A - 10A = 1 \\ 25B + 15B - 10B = 1 \end{cases}$$

$$\begin{cases} A = -\frac{1}{12} \\ B = \frac{1}{30} \end{cases}$$

$$x_p = Ae^{2t} + Be^{-5t}$$
$$= -\frac{1}{12}e^{2t} + \frac{1}{30}e^{-5t}$$

- General solution is $y = y_c + y_p$.

$$x = x_c + x_p$$

$$\boxed{x(t) = c_1 e^{5t} + c_2 e^{-2t} - \frac{1}{12}e^{2t} + \frac{1}{30}e^{-5t}}$$ G.S.

IC: $x(0) = x'(0) = 0$

$$x(t) = c_1 e^{5t} + c_2 e^{-2t} - \frac{1}{12}e^{2t} + \frac{1}{30}e^{-5t}$$
$$x'(t) = 5c_1 e^{5t} - 2c_2 e^{-2t} - \frac{1}{6}e^{2t} - \frac{1}{6}e^{-5t}$$

$$\begin{cases} x(0) = c_1 e^0 + c_2 e^0 - \frac{1}{12}e^0 + \frac{1}{30}e^0 = 0 \\ x'(0) = 5c_1 e^0 - 2c_2 e^0 - \frac{1}{6}e^0 - \frac{1}{6}e^0 = 0 \end{cases}$$

$$\begin{cases} c_1 + c_2 - \frac{1}{20} = 0 \\ 5c_1 - 2c_2 - \frac{1}{3} = 0 \end{cases}$$

$$\begin{cases} c_1 + c_2 = \frac{1}{20} \\ 5c_1 - 2c_2 = \frac{1}{\ } \end{cases}$$

$$\begin{cases} c_1 = \frac{13}{210} \\ c_2 = -\frac{1}{84} \end{cases}$$

$$\boxed{x = \frac{13}{210}e^{5x} - \frac{1}{84}e^{-2x} - \frac{1}{12}e^{2x} + \frac{1}{30}e^{-5x}}$$ P.S.

Undetermined coefficients

Example (Test 3 Problem 3, Spring 2022)

Find the solution of the IVP

$$\begin{cases} y''''(x) + y''(x) = \cos x \\ y'''(0) = y''(0) = y'(0) = y(0) = 0 \end{cases} .$$

$$y'''' + y'' = \cos x$$

$$y_c'''' + y_c'' = 0$$

$$\lambda^4 + \lambda^2 = 0$$

$$\lambda^2(\lambda^2+1) = 0 \qquad \lambda_{3,4} = 0 \pm i = \alpha \pm \beta i$$

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = i, \quad \lambda_4 = -i \qquad (\alpha = 0, \ \beta = 1)$$

$$y_1 = e^{\lambda_1 x} = 1 \qquad y_2 = x e^{\lambda_2 x} = x$$

$$y_3 = e^{\alpha x}\cos\beta x = \cos x \ , \qquad y_4 = e^{\alpha x}\sin\beta x = \sin x$$

$$y_c = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4$$

$$= C_1 + C_2 x + C_3 \sin x + C_4 \cos x$$

$$y_p'''' + y_p'' = \cos x$$

$$f(x) = \cos x \longrightarrow y_p = x(A\cos x + B\sin x)$$

$$y_p = Ax\cos x + Bx\sin x$$

$$y_p' = \cdots \ \cdot\cdot$$

$$y_p'' = (2B - Ax)\cos x - (2A + Bx)\sin x$$

$$y_p''' = \cdots\cdots$$

$$y_p'''' = (Ax - 4B)\cos x + (4A + Bx)\sin x$$

$$y_p'''' + y_p'' = \cos x$$

$$(Ax - 4B)\cos x + (4A + Bx)\sin x + (2B - Ax)\cos x - (2A + Bx)\sin x$$

$$= \cos x$$

$$\begin{cases} \cancel{Ax} - 4B + 2B - \cancel{Ax} = 1 \\ 4A + \cancel{Bx} - (2A + \cancel{Bx}) = 0 \end{cases}$$

$$\begin{cases} -2B = 1 \\ 2A = 0 \end{cases} \Rightarrow \begin{cases} B = -\frac{1}{2} \\ A = 0 \end{cases}$$

$$y_p = Ax\cos x + Bx\sin x = -\frac{1}{2}x\sin x.$$

$$\boxed{y = y_c + y_p = C_1 + C_2 x + C_3 \sin x + C_4 \cos x - \frac{1}{2}x\sin x}$$

G.S.

$y' = \cdots$

$y'' = \cdots$

$y''' = \cdots$

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 0 \\ y'''(0) = 0 \end{cases}$$

$$\begin{cases} C_1 = 1 \\ C_2 = 0 \\ C_3 = 0 \\ C_4 = -1 \end{cases}$$

$$\boxed{y = 1 - \cos x - \frac{1}{2}x\sin x.}$$

P.S.

Undetermined coefficients

Example (Test 3 Problem 5, Spring 2022)

Find the G.S. of the DE:

$$y''' + y'' - y' - y = 1 + \cos x + \cos 2x + e^x.$$

$$y''' + y'' - y' - y = 1 + \cos x + \cos 2x + e^x$$

- Find general solution y_c for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

$$y''' + y'' - y' - y = 0$$

$$\lambda^3 + \lambda^2 - \lambda - 1 = 0$$

$$\lambda^2(\lambda+1) - (\lambda+1) = 0$$

$$(\lambda+1)(\lambda^2 - 1) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-1) = 0$$

$$\lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 1.$$

$$y_1 = e^{\lambda_1 x} = e^{-x}, \quad y_2 = x e^{\lambda_2 x} = x e^{-x}, \quad y_3 = e^{\lambda_3 x} = e^x.$$

$$y_c = C_1 y_1 + C_2 y_2 + C_3 y_3$$

$$= C_1 e^{-x} + C_2 x e^{-x} + C_3 e^x$$

- Find particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients, Variational principle)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

$$f(x) = \underset{f_1}{1} + \underset{f_2}{\cos x} + \underset{f_3}{\cos 2x} + \underset{f_4}{e^x} = f_1 + f_2 + f_3 + f_4$$

$f_1(x) = 1$ | $y_{p_1}(x) = A$

$f_2(x) = \cos x$ | $y_{p_2}(x) = B \sin x + C \cos x$

$f_3(x) = \cos 2x$ | $y_{p_3}(x) = D \sin 2x + E \cos 2x$

$f_4(x) = e^x$ | $y_{p_4}(x) = F x e^x$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4}$$

$$y_p = A + B\sin x + C\cos x + D\sin 2x + E\cos 2x + Fxe^x$$
$$y_p' = B\cos x - C\sin x + 2D\cos 2x - 2E\sin 2x + Fe^x + Fxe^x$$
$$y_p'' = -B\sin x - C\cos x - 4D\sin 2x - 4E\cos 2x + 2Fe^x + Fxe^x$$
$$y_p''' = -B\cos x + C\sin x - 8D\cos 2x + 8E\sin 2x + 3Fe^x + Fxe^x$$

$$y_p''' + y_p'' - y_p' - y_p = 1 + \cos x + \cos 2x + e^x$$

$$-B\cos x + C\sin x - 8D\cos 2x + 8E\sin 2x + 3Fe^x + Fxe^x$$
$$+(-B\sin x - C\cos x - 4D\sin 2x - 4E\cos 2x + 2Fe^x + Fxe^x)$$
$$-(B\cos x - C\sin x + 2D\cos 2x - 2E\sin 2x + Fe^x + Fxe^x)$$
$$-(A + B\sin x + C\cos x + D\sin 2x + E\cos 2x + Fxe^x)$$
$$= 1 + \cos x + \cos 2x + e^x$$

$$\begin{cases} -A = 1 \\ -B - C - B - C = 1 \\ -8D - 4E - 2D - E = 1 \\ 3F + 2F - F = 1 \\ C - B + C - B = 0 \\ 8E - 4D + 2E - D = 0 \\ F + F - F - F = 0 \end{cases} \Rightarrow \begin{cases} A = -1 \\ 2B + 2C = -1 \\ 10D + 5E = -1 \\ F = \frac{1}{4} \\ B = C \\ 2E = D \\ 0 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A = -1 \\ B = -\frac{1}{4} \\ C = -\frac{1}{4} \\ D = -\frac{2}{25} \\ E = -\frac{1}{25} \\ F = \frac{1}{4} \end{cases}$$

$$y_p(x) = -1 - \frac{1}{4}\sin x - \frac{1}{4}\cos x - \frac{2}{25}\sin 2x - \frac{1}{25}\cos 2x + \frac{1}{4}xe^{x}$$

- General solution is $y = y_c + y_p$.

$$y = y_c + y_p$$

$$\boxed{\begin{aligned} y(x) = {} & c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x} \\ & -1 - \frac{1}{4}\sin x - \frac{1}{4}\cos x - \frac{2}{25}\sin 2x - \frac{1}{25}\cos 2x + \frac{1}{4}xe^{x} \end{aligned}}$$

G.S.

Undetermined coefficients

Example (Final Problem 4, Spring 2022)

Find the GS of the DE:

$$y''' + y'' + y' + y = 1 + e^{x} + e^{-x} + e^{2x} + e^{-2x}.$$

$$y''' + y'' + y' + y = 1 + e^{x} + e^{-x} + e^{2x} + e^{-2x}$$

- Find general solution y_c for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

$$y_c''' + y_c'' + y_c' + y_c = 0$$

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

$$\lambda^2(\lambda+1) + (\lambda+1) = 0$$

$$(\lambda+1)(\lambda^2+1) = 0$$

$$(\lambda+1)(\lambda+i)(\lambda-i) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -i, \quad \lambda_3 = i \qquad (\alpha = 0, \quad \beta = 1)$$

$$y_1 = e^{\lambda_1 x} = e^{-x},$$

$$y_2 = e^{\alpha x} \cos \beta x = \cos x, \qquad y_3 = e^{\alpha x} \sin \beta x = \sin x$$

$$y_c = C_1 y_1 + C_2 y_2 + C_3 y_3$$

$$= C_1 e^{-x} + C_2 \cos x + C_3 \sin x$$

- Find particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients, Variational principle)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

$$y_p''' + y_p'' + y_p' + y_p = 1 + e^{x} + e^{-x} + e^{2x} + e^{-2x}$$

$$f(x) = 1 + e^{x} + e^{-x} + e^{2x} + e^{-2x}$$

$$= f_1 + f_2 + f_3 + f_4 + f_5$$

- For each $j = 1, 2, \ldots, k$, guess y_{p_j} based on the following table.

Term in $f_j(x)$	Choice for y_{p_j}
e^{rx}	Ce^{rx}

$f_1 = 1$ $\qquad y_{p_1} = A$

$f_2 = e^x$ $\qquad y_{p_2} = Be^x$

$f_3 = e^{-x} = y_1$ $\qquad y_{p_3} = Cxe^x$

$f_4 = e^{2x}$ $\qquad y_{p_4} = Fe^{2x}$

$f_5 = e^{-2x}$ $\qquad y_{p_5} = Ge^{-2x}$

- If $f_j(x)$ happens to be a single (or double, or triple, ..., or n-fold) root of the corresponding homogeneous equation, then multiply your choice of y_{p_j} by x (or x^2, or x^3, ..., or x^n).

- Let $y_p = \sum \prod_{j=1,2,...,k} y_{p_j}$.

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4} + y_{p_5}$$
$$= A + Be^x + Cxe^{-x} + Fe^{2x} + Ge^{-2x}$$

- Compute $y'_p, y''_p, \ldots, y_p^{(n)}$.

$$y_p' = Be^x + Ce^{-x} - Cxe^{-x} + 2Fe^{2x} - 2Ge^{-2x}$$
$$y_p'' = Be^x + 2Ce^{-x} + Cxe^{-x} + 4Fe^{2x} - 4Ge^{-2x}$$
$$y_p''' = Be^x - Cxe^{-x} + 3Ce^{-x} + 8Fe^{2x} - 8Ge^{-2x}$$

- Substitute $y_p, y'_p, y''_p, \ldots, y_p^{(n)}$ above into the original ODE to find the coefficients $A, B, C, A_0, A_1, \ldots, A_m$.

$$\begin{aligned}
y_p''' + y_p'' + y_p' + y_p &= Be^x - Cxe^{-x} + 3Ce^{-x} + 8Fe^{2x} - 8Ge^{-2x} \\
&\quad + Be^x + 2Ce^{-x} + Cxe^{-x} + 4Fe^{2x} - 4Ge^{-2x} \\
&\quad + Be^x + Ce^{-x} - Cxe^{-x} + 2Fe^{2x} - 2Ge^{-2x} \\
&\quad + A + Be^x + Cxe^{-x} + Fe^{2x} + Ge^{-2x} \\
&= A + 4Be^x + 2Ce^{-x} + 15Fe^{2x} - 5Ge^{-2x} \\
&= 1 + e^x + e^{-x} + e^{2x} + e^{-2x}
\end{aligned}$$

$$\begin{cases} A = 1 \\ B = \frac{1}{4} \\ C = \frac{1}{2} \\ F = \frac{1}{15} \\ G = -\frac{1}{5} \end{cases}$$

$$y_p = 1 + \frac{1}{4}e^{x} + \frac{1}{2}xe^{-x} + \frac{1}{15}e^{2x} - \frac{1}{5}e^{-2x}$$

- General solution is $y = y_c + y_p$.

$$y = y_c + y_p$$

$$\boxed{\begin{aligned} y = {} & C_1 e^{-x} + C_2 \cos x + C_3 \sin x + \\ & 1 + \frac{1}{4}e^{x} + \frac{1}{2}xe^{-x} + \frac{1}{15}e^{2x} - \frac{1}{5}e^{-2x} \end{aligned}}$$ G.S.

Undetermined coefficients

Example (Test 3 Problem 1, Fall 2022)

Find the GS of

$$y'' + y = \cos x + \sin x.$$

$y'' + y = \cos x + \sin x$

- Find general solution y_c for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

$y_c'' + y_c = 0 \longrightarrow a = 1, \; b = 0, \; c = 1$

- Obtain the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

and its solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$1 \cdot \lambda^2 + 0 \cdot \lambda + 1 = 0$

$\lambda^2 + 1 = 0$

$\lambda_1 = -i, \quad \lambda_2 = i \qquad (\lambda = \alpha \pm \beta i \Rightarrow \alpha = 0, \; \beta = 1)$

3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$, then the general solution is

$$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x).$$

$y_1 = e^{\alpha x} \cos \beta x = e^0 \cos x = \cos x$

$y_2 = e^{\alpha x} \sin \beta x = e^0 \sin x = \sin x$

$y_c = C_1 y_1 + C_2 y_2$

$\quad = C_1 \cos x + C_2 \sin x$

- Find particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients, Variational principle)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

$y_p'' + y_p = \cos x + \sin x$

- For each $j = 1, 2, \ldots, k$, guess y_{p_j} based on the following table.

Term in $f_j(x)$	Choice for y_{p_j}
e^{rx}	Ce^{rx}
x^m	$A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0$
$\cos(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$

$$f(x) = \cos x + \sin x$$
$$= f_1(x) + f_2(x)$$

$f_1(x) = \cos x = y_1$ $\qquad$ $y_{p_1} = x(A_1 \cos x + B_1 \sin x)$

$f_2(x) = \sin x = y_2$ $\qquad$ $y_{p_2} = x(A_2 \cos x + B_2 \sin x)$

- If $f_j(x)$ happens to be a single (or double, or triple, $\ldots$, or n-fold) root of the corresponding homogeneous equation, then multiply your choice of y_{p_j} by x (or x^2, or x^3, $\ldots$, or x^n).

y_1, y_2

- Let $y_p = \sum \prod_{j=1,2,\ldots,k} y_{p_j}$.

$$y_p = y_{p_1} + y_{p_2}$$
$$= x(A_1 \cos x + B_1 \sin x) + x(A_2 \cos x + B_2 \sin x)$$
$$= (A_1 + A_2)\, x \cos x + (B_1 + B_2)\, x \sin x$$
$$= A\, x \cos x + B\, x \sin x \qquad (A = A_1 + A_2,\quad B = B_1 + B_2)$$

- Compute $y_p', y_p'', \ldots, y_p^{(n)}$.

$$y_p = A\, x \cos x + B\, x \sin x$$
$$y_p' = A(\cos x - x \sin x) + B(\sin x + x \cos x)$$
$$y_p'' = -A(2 \sin x + x \cos x) + B(2 \cos x - x \sin x)$$

- Substitute $y_p, y_p', y_p'', \ldots, y_p^{(n)}$ above into the original ODE to find the coefficients $A, B, C, A_0, A_1, \ldots, A_m$.

$$y_p'' + y_p = \cos x + \sin x$$

$$\cos x + \sin x$$

$$= -A(2\sin x + \cancel{x\cos x}) + B(2\cos x - \cancel{x\sin x}) + \cancel{Ax\cos x} + \cancel{Bx\sin x}$$

$$= 2B\cos x - 2A\sin x$$

$$\begin{cases} 2B = 1 \\ -2A = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{2} \\ B = \frac{1}{2} \end{cases}$$

$$y_p = A\,x\cos x + B\,x\sin x$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}x\sin x$$

- General solution is $y = y_c + y_p$.

$$y = y_c + y_p$$

$$\boxed{y = C_1\cos x + C_2\sin x - \frac{1}{2}x\cos x + \frac{1}{2}x\sin x}$$ G.S.

Undetermined coefficients

Example (Test 3 Problem 2, Fall 2022)

Find the GS of

$$y''' - y'' + y' - y = e^x.$$

$$y''' - y'' + y' - y = e^x$$

- Find general solution y_c for the associated homogeneous equation (week 8 Homogeneous)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

$$y_c''' - y_c'' + y_c' - y_c = 0$$

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

$$\lambda^2(\lambda-1) + (\lambda-1) = 0$$

$$(\lambda^2+1)(\lambda-1) = 0$$

$$(\lambda-1)(\lambda+i)(\lambda-i) = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = -i, \quad \lambda_3 = i \qquad (\lambda = \alpha \pm \beta i \Rightarrow \alpha = 0, \ \beta = 1)$$

$$y_1 = e^{\lambda_1 x} = e^x$$

$$y_2 = e^{\alpha x} \cos \beta x = \cos x$$

$$y_3 = e^{\alpha x} \sin \beta x = \sin x$$

$$y_c = C_1 y_1 + C_2 y_2 + C_3 y_3$$

$$= C_1 e^x + C_2 \cos x + C_3 \sin x$$

- Find particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients, Variational principle)

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

$$y_p''' - y_p'' + y_p' - y_p = e^x$$

$$f(x) = e^x$$

- For each $j = 1, 2, \ldots, k$, guess y_{p_j} based on the following table.

Term in $f_j(x)$	Choice for y_{p_j}
e^{rx}	Ce^{rx}
x^m	$A_m x^m + A_{m-1}x^{m-1} + \cdots + A_1 x + A_0$
$\cos(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$
$\sin(\omega x)$	$A\cos(\omega x) + B\sin(\omega x)$

$f(x) = e^x = y_1$ $\qquad$ $y_p = x\,Ce^x = Cxe^x$

- If $f_j(x)$ happens to be a single (or double, or triple, ..., or n-fold) root of the corresponding homogeneous equation, then multiply your choice of y_{p_j} by x (or x^2, or x^3, ..., or x^n).

- Compute $y_p', y_p'', \ldots, y_p^{(n)}$.

$$y_p = Cxe^x$$
$$y_p' = C(x+1)e^x$$
$$y_p'' = C(x+2)e^x$$
$$y_p''' = C(x+3)e^x$$

- Substitute $y_p, y_p', y_p'', \ldots, y_p^{(n)}$ above into the original ODE to find the coefficients $A, B, C, A_0, A_1, \ldots, A_m$.

$$y_p''' - y_p'' + y_p' - y_p = e^x$$
$$C(\cancel{x}+3)e^x - C(\cancel{x}+2)e^x + C(\cancel{x}+1)e^x - C\cancel{x}e^x = e^x$$
$$(3C - 2C + C)e^x = e^x$$
$$2Ce^x = e^x$$
$$2C = 1$$
$$C = \tfrac{1}{2}$$

$$y_p = \tfrac{1}{2}xe^x$$

- General solution is $y = y_c + y_p$.

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{x} + c_2 \cos x + c_3 \sin x + \frac{1}{2} x e^{x}}$$

G.S.

Undetermined coefficients

Example (Final Problem 2, Fall 2022)

Use LT and a non-LT method to find PS (No need to evaluate convolutions):

$$\begin{cases} y''' + y'' + y' + y = e^{-x} + \cos x + \sin x \\ y(0) = y'(0) = y''(0) = 0 \end{cases}.$$

Remark

LT method: week 14 Laplace transform;
Non-LT method: week 9 Undetermined coefficients.

Steps

- The *Wronskian* of two solutions y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

- Compute

$a_2 y'' + a_1 y' + a_0 = f(x)$

$$u_1(x) = \int \frac{-y_2 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx, \quad u_2(x) = \int \frac{y_1 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx.$$

In the indefinite integrals above, it is not necessary to write an arbitrary constant C.

- The particular solution is

$$y_p = u_1(x) y_1 + u_2(x) y_2.$$

Variational principle

Example (Final Problem 2, Fall 2016)

Use LT method and another method to find the PS (Convolutions, if any, must be evaluated):

$$\begin{cases} x'' + x = \tan(t) \\ x(0) = x'(0) = 0 \end{cases} .$$

Remark

LT method: week 14 Laplace transform;
Another method: week 9 Variational principle or week 11 Order reduction.

$x'' + x = \tan(t)$

- Find the general solution $y_c = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ for the associated homogeneous equation (week 8 Homogeneous)

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

$x_c'' + x_c = 0$

$\lambda^2 + 1 = 0$

$\lambda_1 = -i, \quad \lambda_2 = i \qquad (\lambda = \alpha \pm \beta i \Rightarrow \alpha = 0, \ \beta = 1)$

$x_1 = e^{\alpha t} \cos \beta t = \cos t$

$x_2 = e^{\alpha t} \sin \beta t = \sin t$

$x_c = C_1 x_1 + C_2 x_2 = C_1 \cos t + C_2 \sin t$

- Find a particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients (UC or MUC), Variational principle (VP or VOP))

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x).$$

$x_p'' + x_p = \tan t \longrightarrow a_2 = 1, \ a_1 = 0, \ a_0 = 1$

$f(t) = \tan t \qquad a_2 x_p'' + a_1 x_p' + a_0 x_p = f(t)$

- The *Wronskian* of two solutions y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

$x_1 = \cos t \qquad x_2 = \sin t$

$x_1' = -\sin t \qquad x_2' = \cos t$

$$W(x_1, x_2) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = x_1x_2' - x_2x_1'$$

$$= (\cos t)(\cos t) - (\sin t)(-\sin t)$$
$$= \cos^2 t + \sin^2 t$$
$$= 1$$

- Compute

$$u_1(x) = \int \frac{-y_2 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx, \quad u_2(x) = \int \frac{y_1 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx.$$

In the indefinite integrals above, it is not necessary to write an arbitrary constant C.

$$u_1(t) = \int \frac{-x_2 \frac{f(t)}{a_2}}{W(x_1, x_2)} dt$$
$$= \int \frac{(-\sin t)\frac{\tan t}{1}}{1} dt$$
$$= -\int (\sin t)(\tan t)\, dt$$
$$= \int \frac{-\sin^2 t}{\cos t} dt \qquad (\sin^2 t + \cos^2 t = 1)$$
$$= \int \frac{\cos^2 t - 1}{\cos t} dt \qquad \left(\frac{1}{\cos t} = \sec t\right)$$
$$= \int \left(\cos t - \frac{1}{\cos t}\right) dt$$
$$= \int \cos t\, dt - \int \frac{1}{\cos t} dt$$

- $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

$$= \sin t - \ln|\sec t + \tan t|$$

$$u_2(t) = \int \frac{x_1 \frac{f(t)}{a_2}}{W(x_1, x_2)} dt$$

$$= \int \frac{(\cos t)\frac{\tan t}{1}}{1}\, dt$$

$$= \int (\cos t)(\tan t)\, dt \qquad \left(\tan t = \frac{\sin t}{\cos t}\right)$$

$$= \int \sin t \, dt$$

$$= -\cos t$$

- The particular solution is

$$y_p = u_1(x)y_1 + u_2(x)y_2.$$

$$x_p = u_1 x_1 + u_2 x_2$$

$$= (\sin t - \ln|\sec t + \tan t|)\cos t + (-\cos t)\sin t$$

$$= \sin t \cos t - \cos t \ln(\sec t + \tan t) - \sin t \cos t$$

$$= -\cos t \ln(\sec t + \tan t)$$

- General solution is $y = y_c + y_p$.

$$x = x_c + x_p$$

$$\boxed{x(t) = C_1 \cos t + C_2 \sin t - \cos t \ln(\sec t + \tan t)} \qquad 6.5.$$

Lecture notes page 58, Example 15.
page 63, Example 15A.

Variational principle

Example (Test 3 Problem 1, Spring 2019)

Find the GS by MUC and method of VOP:

$$y'' - 9y = e^{3x}.$$

Remark

MUC: week 9 Undetermined coefficients;
VOP: week 9 Variational principle.

Variational principle

Example (Test 3 Problem 2, Spring 2022)

Use the Variation of Parameters method to find the GS of the DE:

$$y'' + 4y = \sin 2x.$$

$y'' + 4y = \sin 2x$

- Find the general solution $y_c = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ for the associated homogeneous equation (week 8 Homogeneous)

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

$y_c'' + 4y_c = 0$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4$

$\lambda_1 = -2i, \quad \lambda_2 = 2i \qquad (\lambda = \alpha \pm \beta i \Rightarrow \alpha = 0, \ \beta = 2)$

$y_1 = e^{\alpha x} \cos \beta x = \cos 2x$

$y_2 = e^{\alpha x} \sin \beta x = \sin 2x$

$y_c = C_1 y_1 + C_2 y_2 = C_1 \cos 2x + C_2 \sin 2x$

- Find a particular solution y_p for the inhomogeneous equation (week 9 Undetermined coefficients (UC or MUC), Variational principle (VP or VOP))

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x).$$

$f(x) = \sin 2x$

- The *Wronskian* of two solutions y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.$$

$y_1 = \cos 2x \qquad\qquad y_2 = \sin 2x$

$y_1' = -2\sin 2x \qquad\qquad y_2' = 2\cos 2x$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$

$$= (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$$
$$= 2\cos^2 2x + 2\sin^2 2x$$
$$= 2\underbrace{(\cos^2 2x + \sin^2 2x)}_{=1}$$
$$= 2$$

- Compute

$$u_1(x) = \int \frac{-y_2 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx, \quad u_2(x) = \int \frac{y_1 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx.$$

In the indefinite integrals above, it is not necessary to write an arbitrary constant C.

$$u_1(x) = \int \frac{-y_2 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx$$
$$= \int \frac{-(\sin 2x)\frac{\sin 2x}{1}}{2} dx$$
$$= -\frac{1}{2}\int \sin^2 2x \, dx \qquad \left(\cos 2x = 1 - 2\sin^2 x\right)$$
$$\Downarrow$$
$$= -\frac{1}{2}\int \frac{1-\cos 4x}{2} dx \qquad \leftarrow \cos 4x = 1 - 2\sin^2 2x$$
$$= \frac{1}{4}\int \cos 4x \, dx - \frac{1}{4}\int dx$$
$$= \frac{1}{16}\int \cos 4x \, d4x - \frac{1}{4}\int dx$$
$$= \frac{1}{16}\sin 4x - \frac{1}{4}x$$

$$u_2(x) = \int \frac{y_1 \frac{f(x)}{a_2}}{W(y_1, y_2)} dx$$
$$= \int \frac{(\cos 2x)\frac{\sin 2x}{1}}{2} dx$$
$$= \frac{1}{2}\int \sin 2x \cos 2x \, dx \qquad \left(\sin 2x = 2\sin x \cos x\right)$$
$$\Downarrow$$

$$= \frac{1}{2}\int \frac{1}{2}\sin 4x \, dx$$
$$= \frac{1}{4}\int \sin 4x \, dx$$
$$= \frac{1}{16}\int \sin 4x \, d4x$$
$$= -\frac{1}{16}\cos 4x$$

$\sin 4x = 2\sin 2x \cos 2x$

- The particular solution is

$$y_p = u_1(x)y_1 + u_2(x)y_2.$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \left(\frac{1}{16}\sin 4x - \frac{1}{4}x\right)\cos 2x - \frac{1}{16}\cos 4x \sin 2x$$
$$= \frac{1}{16}\sin 4x \cos 2x - \frac{1}{4}x\cos 2x - \frac{1}{16}\sin 2x \cos 4x$$
$$= \frac{1}{16}(2\sin 2x \cos 2x)\cos 2x - \frac{1}{4}x\cos 2x - \frac{1}{16}\sin 2x(1 - 2\sin^2 2x)$$
$$= \frac{1}{8}\sin 2x \cos^2 2x - \frac{1}{4}x\cos 2x - \frac{1}{16}\sin 2x + \frac{1}{8}\sin^3 2x$$
$$= \frac{1}{8}\sin 2x\underbrace{(\sin^2 2x + \cos^2 2x)}_{=1} - \frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x$$
$$= \frac{1}{8}\sin 2x - \frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x$$
$$= \frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x$$

- General solution is $y = y_c + y_p$.

$$y = y_c + y_p$$
$$= C_1\cos 2x + C_2\sin 2x + \frac{1}{16}\sin 2x - \frac{1}{4}x\cos 2x$$
$$= C_1\cos 2x + \left(C_2 + \frac{1}{16}\right)\sin 2x - \frac{1}{4}x\cos 2x$$
$$\boxed{y(x) = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}x\cos 2x}$$

G.S.