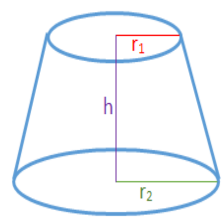
# AMS 361: Applied Calculus IV by Prof. Y. Deng
# Test 2

10/27/2020 8:15 pm-9:40 pm EDT Anywhere with Internet

**Email:** sbu.ams.361@gmail.com
**Subject:** Lastname-SBUID-Test2
**Filename:** Lastname-SBUID-Test2

(1) Open book, open notes, open everything including **allowed** functions of electronics.
**(2) No collaboration is allowed. Every student** must turn on **video** to allow sufficient viewing area including the student **during the test. No need to show ID for video recording but you must photocopy your ID to the test paper.**
(3) Do any two of the three problems on any 8.5x11 paper or a tablet as long as you can email **one** PDF to the above gmail with the required Subject.
(4) If all three problems are attempted, the best two (and only two) will be credited. Each problem is worth 7.5 points for a total of 15 points (max).
(5) **No** points for solutions without appropriate intermediate steps. Partial credits are given only for steps that are relevant to the solutions.
(6) **No** late papers are accepted for **any** reason(s). **Same time (EDT)** for all regardless of locations.
(7) Five extra minutes are given (to the usual class ending time: 9:35pm) to prepare for submission, i.e., your paper must leave your device before 9:40 pm for the posted gmail address and any papers late by even **one second** will not be graded.
(8) Test papers will be examined for abnormalities and ~5% students may be Zoom-interviewed.

**T2-1 (7.5 Points):** The radii of the two end-discs of a container (as shown) are $r_1$ and $r_2$ and its height $h$. I drill two identical holes at the centers of two end-discs to enable a draining constant $k$. Please do the following:




(1) Compute the time $t_1$ needed to empty the fully-filled container when it is placed as shown. (3.0 points)
(2) Compute the time $t_2$ needed to empty the fully-filled (same liquid) container after you turn it upside down. (2.5 points)
(3) Use your results above to show special cases (2.0 points)
   a) If $r_1 = r_2$, compute $t_1/t_2$
   b) If $r_1 = 0$, compute $t_1/t_2$

**Solution:**

(1)

We know

$$\begin{cases} y(t=0) = y_0 \\ A(y)\dfrac{dy}{dt} = -k\sqrt{y} \end{cases}$$

And make R the radius of the water level and y the height of the water level

First for draining the container as shown,

$$A(y) = \pi x^2$$
$$\frac{x - r_2}{y} = \frac{r_1 - r_2}{h}$$

$$x = \frac{r_1 - r_2}{h} y + r_2$$
$$\pi \left( \frac{r_1 - r_2}{h} y + r_2 \right)^2 \frac{dy}{dt} = -k\sqrt{y}$$
$$\pi \frac{\left( \frac{r_1 - r_2}{h} y + r_2 \right)^2}{\sqrt{y}} dy = -kdt$$
$$t_1 = \frac{\pi}{k} \int_0^h \frac{\left( \frac{r_1 - r_2}{h} y + r_2 \right)^2}{\sqrt{y}} dy$$
$$= \frac{\pi}{15k} (6r_1^2 + 8r_1 r_2 + 16r_2^2)\sqrt{h}$$

(2) Then flipping the container,

$$A(y) = \pi x^2$$
$$x = \frac{r_2 - r_1}{h} y + r_1$$
$$t_2 = \frac{\pi}{k} \int_0^h \frac{1}{\sqrt{y}} \left( \frac{r_2 - r_1}{h} y + r_1 \right)^2 dy$$
$$= \frac{\pi}{15k} (6r_2^2 + 8r_1 r_2 + 16r_1^2)\sqrt{h}$$

(3) Then making $r_1 = r_2$,

$$t_1 = \frac{2\pi\sqrt{h}}{k} r_1^2 = t_2$$

So,

$$\frac{t_1}{t_2} = 1$$

Making $r_1 = 0$,

$$t_1 = \frac{\pi\sqrt{h}}{15k} 16r_2^2$$
$$t_2 = \frac{\pi\sqrt{h}}{15k} 6r_2^2$$

So,

$$\frac{t_1}{t_2} = \frac{8}{3}$$

**T2-2 (7.5 Points):** In a room of constant temperature $A = 20°C$, a container with cooling constant $k = 0.1$ is poured 1 gallon of boiling water at $T_B = 100°C$ at time $t = 0$. Waiting till $t = 10$, I add 5 gallons of icy water $T_{Ice} = 0°C$ to the container, rapidly (ignoring pouring time). Compute the water temperature at $t = 15$.
***Hints****: The temperature of mixing* $m_1$ *gallons of water at* $T_1$ *with* $m_2$ *gallons at* $T_2$ *is* $(m_1 T_1 + m_2 T_2)/(m_1 + m_2)$

**Solution:**

Newton's law of cooling DE:

$$dT(t) = k(A - T)$$
$$T(t = 0) = T_0$$

PS of Newton's law of cooling:

$$T(t) = A + (T_0 - A)e^{-kt}$$

For $T_B = 100$°C, $k = 0.1$ and $A = 20$°C, $t = 10$ we have

$$\begin{aligned} T(10) &= 20 + (100 - 20)e^{-(0.1)(10)} \\ &= 20 + 80e^{-1} \end{aligned}$$

After adding 5 gallons of icy water

$$T_{MIX}(10) = \frac{m_1 T_1 + m_2 T_2}{m_1 + m_2} = \frac{(1)(20 + 80e^{-1}) + (5 * 0)}{1 + 5} \simeq 8.238$$

Now, for $T_{MIX}(15), k = 0.1, A = 20$°C, $t = 15 - 10 = 5$

$$T_{MIX}(15) = 20 + (8.238 - 20)e^{-(0.1)(5)} = 12.866$$

**T2-3 (7.5 Points):** Use any method to find the GS of

$$x^2y'' + xy' + y = \cos(\ln x) + \sin(\ln x) \qquad \forall x > 0$$

**Solution:**

Using substitution $t = \ln x$

$$xy' = \dot{y}$$
$$x^2y'' = \ddot{y} - \dot{y}$$
$$\ddot{y} + y = \cos t + \sin t$$

The C-Eq is

$$r^2 + 1 = 0$$
$$r_{1,2} = \pm i$$

The solution of the H.D.E is

$$y_c(t) = C_1 \cos t + C_2 \sin t$$

For the InHomo part

$$y_P(t) = t\,(\mathrm{A}\cos t + \mathrm{B}\sin t)$$
$$y_P'(t) = A\cos t + B\sin t + t(-A\sin t + B\cos t)$$
$$y_P''(t) = 2(-A\sin t + B\cos t) - t(A\cos t + B\sin t)$$

Plugging back into the original DE

$$[2(-A\sin t + B\cos t) - t(A\cos t + B\sin t)] + t(A\cos t + B\sin t)$$
$$= 2(-A\sin t + B\cos t) = \sin t + \cos t$$

Thus,

$$A = -\frac{1}{2}, \qquad B = \frac{1}{2}$$

The GS is

$$y = y_C + y_P = C_1\cos t + C_2\sin t + \frac{t}{2}(-\cos t + \sin t)$$

Back sub,

$$y(x) = C_1\cos(\ln x) + C_2\sin(\ln x) + \frac{1}{2}\ln x\,(-\cos(\ln x) + \sin(\ln x))$$