AMS 361 R01/R03

Week 12 : Constant coefficients (E-Analysis)

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Constant coefficients (Homogeneous)

Remark

Uppercase letter means vector or matrix, and lowercase letter means value.

Consider the homogeneous linear system

$$\begin{cases} x_1' = ax_1 + bx_2 \; + 0 \\ x_2' = cx_1 + dx_2 \; + 0 \end{cases},$$

where a, b, c, d are constants (see lecture note for 3×3 matrix case).

Steps

- Rewrite the system as

$$X' = AX,$$

where

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Constant coefficients (Homogeneous)

Steps

- Obtain the *characteristic equation* of A

$$\det(A - \lambda I) = (\lambda - a)(\lambda - d) - bc = 0,$$

  and its solutions λ_1, λ_2 that are eigenvalues of A.

- Find the corresponding eigenvectors

$$V_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, \quad V_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix},$$

  by

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$(A - \lambda_2 I)V_2 = \begin{bmatrix} a - \lambda_2 & b \\ c & d - \lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Constant coefficients (Homogeneous)

Steps

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is
$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}.$$
2. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution is
$$X(t) = c_1 U_1 e^{\lambda t} + c_2 (U_1 t + U_2) e^{\lambda t},$$
where U_1, U_2 are *generalised eigenvectors* satisfying
$$(A - \lambda I)^2 U_2 = 0, \quad (A - \lambda I) U_2 = U_1.$$
3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$ with $V_1 = B_1 + B_2 i, V_2 = B_1 - B_2 i$, then the general solution is
$$X(t) = c_1 e^{\alpha t}(B_1 \cos(\beta t) - B_2 \sin(\beta t)) + c_2 e^{\alpha t}(B_2 \cos(\beta t) + B_1 \sin(\beta t)).$$

Constant coefficients (Homogeneous)

Example (Final Problem 3, Fall 2016)

Use LT and another method to find the PS:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \text{ with } X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Remark

LT method: week 14 Laplace transform;

Another method: week 12 E-Analysis.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Rewrite the system as

$$X' = AX,$$

where

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

$a = 1$ $\quad b = -2$

$c = 3$ $\quad d = -4$

- Obtain the *characteristic equation* of A

$$\det(A - \lambda I) = (\lambda - a)(\lambda - d) - bc = 0,$$

and its solutions λ_1, λ_2 that are eigenvalues of A.

$$\begin{aligned} \det(A - \lambda I) &= (\lambda - a)(\lambda - d) - bc \\ &= (\lambda - 1)(\lambda + 4) - (-2)\cdot 3 \\ &= \lambda^2 + 3\lambda - 4 + 6 \\ &= \lambda^2 + 3\lambda + 2 = 0 \\ &(\lambda + 1)(\lambda + 2) = 0 \end{aligned}$$

$\lambda_1 = -2, \quad \lambda_2 = -1$ eigenvalues of A

- Find the corresponding eigenvectors

$$V_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, \quad V_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix},$$

by

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$(A - \lambda_2 I)V_2 = \begin{bmatrix} a - \lambda_2 & b \\ c & d - \lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $\lambda_1 = -2$, $(A - \lambda_1 I) V_1 = 0$

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+2 & -2 \\ 3 & -4+2 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3V_{11} - 2V_{12} = 0 \\ 3V_{11} - 2V_{12} = 0 \end{cases} \Rightarrow 3V_{11} = 2V_{12}$$

Let $V_{11} = 2$. Then $V_{12} = 3$

$$V_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Can not let $V_{11} = V_{12} = 0$

For $\lambda_2 = -1$, $(A - \lambda_2 I) V_2 = 0$

$$\begin{bmatrix} a-\lambda_2 & b \\ c & d-\lambda_2 \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+1 & -2 \\ 3 & -4+1 \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2V_{21} - 2V_{22} = 0 \\ 3V_{21} - 3V_{22} = 0 \end{cases} \Rightarrow V_{21} = V_{22}$$

Let $V_{21} = 1$. Then $V_{22} = 1$

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Can not let $V_{21} = V_{22} = 0$

Eigenvectors $V_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is

$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}.$$

$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$$

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$$

G. S.

$$\begin{cases} x(t) = 2c_1 e^{-2t} + c_2 e^{-t} \\ y(t) = 3c_1 e^{-2t} + c_2 e^{-t} \end{cases}$$

$$X(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{cases} x(0) = 1 \\ y(0) = 2 \end{cases}$$

$$\begin{cases} x(0) = 2c_1 e^0 + c_2 e^0 = 2c_1 + c_2 = 1 \\ y(0) = 3c_1 e^0 + c_2 e^0 = 3c_1 + c_2 = 2 \end{cases}$$

$$\begin{cases} c_1 = 1 \\ c_2 = -1 \end{cases}$$

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{-t}}$$

P. S.

Constant coefficients (Homogeneous)

Example (Final Problem 2, Spring 2022)

Solve the following DE using three different methods: (1) the Eigen-Analysis method, (2) the Substitution method, and (3) the Operator method.

$$X'(t) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} X.$$

Remark

(1) Week 12 E-Analysis;
(2)(3) Week 13 Separation of variables.

$$X'(t) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} X$$

- Rewrite the system as

$$X' = AX,$$

where

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix}$$

- Obtain the *characteristic equation* of A

$$\det(A - \lambda I) = (\lambda - a)(\lambda - d) - bc = 0,$$

and its solutions λ_1, λ_2 that are eigenvalues of A.

$$\begin{aligned} \det(A - \lambda I) &= (\lambda - 2)(\lambda - 4) - (-1)\cdot 1 \\ &= \lambda^2 - 6\lambda + 8 + 1 \\ &= \lambda^2 - 6\lambda + 9 \\ &= (\lambda - 3)^2 = 0 \end{aligned}$$

$\lambda_1 = \lambda_2 = 3$ eigenvalues of A

2. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution is

$$X(t) = c_1 U_1 e^{\lambda t} + c_2 (U_1 t + U_2) e^{\lambda t},$$

where U_1, U_2 are *generalised eigenvectors* satisfying

$$(A - \lambda I)^2 U_2 = 0, \quad (A - \lambda I) U_2 = U_1.$$

$$(A - \lambda I)^2 U_2 = 0$$

$$\begin{bmatrix} 2-3 & -1 \\ 1 & 4-3 \end{bmatrix}^2 \begin{bmatrix} U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2-3 & -1 \\ 1 & 4-3 \end{bmatrix}\begin{bmatrix} 2-3 & -1 \\ 1 & 4-3 \end{bmatrix}\right)\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (-1)(-1)+(-1)(1) & (-1)(-1)+(-1)(1) \\ (1)(-1)+(1)(1) & (1)(-1)+(1)(1) \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 0=0$$

Let $u_{21}=0$ and $u_{22}=1$

Cannot let $u_{21}=u_{22}=0$

$$U_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Common mistake

$c_2 U_2 e^{\lambda_2 t}$ X

$c_2 U_1 t e^{\lambda_2 t}$ X

$$U_1 = (A-\lambda I)\, U_2$$

$$\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X(t) = c_1 U_1 e^{\lambda_1 t} + c_2 (U_1 t + U_2) e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} + c_2\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{3t}$$

$$\boxed{X(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -t \\ t+1 \end{bmatrix} e^{3t}}$$

G.S.

Constant coefficients (Homogeneous)

Example

Solve the following Homo system

$$\begin{cases} x' = 4x - 3y \\ y' = 3x + 4y \\ x(0) = 2 \\ y(0) = 3 \end{cases} .$$

$$\begin{cases} x' = 4x - 3y \\ y' = 3x + 4y \\ x(0) = 2 \\ y(0) = 3 \end{cases}$$

- Rewrite the system as

$$X' = AX,$$

where

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$X' = AX$$

$$X' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

- Obtain the *characteristic equation* of A

$$\det(A - \lambda I) = (\lambda - a)(\lambda - d) - bc = 0,$$

and its solutions λ_1, λ_2 that are eigenvalues of A.

C-Eq: $\det(A - \lambda I) = (\lambda - 4)(\lambda - 4) - (-3)3 = (\lambda-4)^2 + 9 = 0$

$$(\lambda - 4)^2 = -9$$

$$\lambda - 4 = \pm 3i$$

$$\lambda_1 = 4 + 3i, \quad \lambda_2 = 4 - 3i \qquad (\alpha = 4, \ \beta = 3)$$

- Find the corresponding eigenvectors

$$V_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, \quad V_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix},$$

by

$$(A - \lambda_1 I)V_1 = \begin{bmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$(A - \lambda_2 I)V_2 = \begin{bmatrix} a - \lambda_2 & b \\ c & d - \lambda_2 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For $\lambda_1 = 4+3i$,

$$(A-\lambda_1 I)V_1 = \begin{bmatrix} 4-(4+3i) & -3 \\ 3 & 4-(4+3i) \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -3iV_{11} - 3V_{12} = 0$$

$$V_{12} = -iV_{11}$$

$\divideontimes (i^2=-1)$

Let $V_{12}=1$. Then $V_{11} = \frac{V_{12}}{-i} = \frac{1}{-i} = \frac{-1}{i} = \frac{i^2}{i} = i$.

$$V_1 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i \quad \left(B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

For $\lambda_2 = 4-3i$,

$$(A-\lambda_2 I)V_2 = \begin{bmatrix} 4-(4-3i) & -3 \\ 3 & 4-(4-3i) \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 3i & -3 \\ 3 & 3i \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 3iV_{21} - 3V_{22} = 0$$

$$V_{22} = iV_{21}$$

Let $V_{22}=1$. Then $V_{21} = \frac{1}{i} = \frac{-1}{-i} = \frac{i^2}{-i} = -i$.

$$V_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$ with $V_1 = B_1 + B_2 i$, $V_2 = B_1 - B_2 i$, then the general solution is

$$X(t) = c_1 e^{\alpha t}(B_1 \cos(\beta t) - B_2 \sin(\beta t)) + c_2 e^{\alpha t}(B_2 \cos(\beta t) + B_1 \sin(\beta t)).$$

$$X(t) = c_1 e^{\alpha t}\left(B_1 \cos(\beta t) - B_2 \sin(\beta t)\right) + c_2 e^{\alpha t}\left(B_2 \cos(\beta t) + B_1 \sin(\beta t)\right)$$

$$= c_1 e^{4t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(3t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(3t)\right) + c_2 e^{4t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(3t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(3t)\right)$$

$$\boxed{X(t) = c_1 e^{4t}\begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}}$$ G.S.

$$= e^{4t}\left(\begin{bmatrix} c_2 \\ c_1 \end{bmatrix}\cos(3t) + \begin{bmatrix} -c_1 \\ c_2 \end{bmatrix}\sin(3t)\right)$$

IVP: $X(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}$

$$\begin{cases} c_1 = 3 \\ c_2 = 2 \end{cases}$$

$$X(t) = e^{4t}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\cos(3t) + \begin{bmatrix} -3 \\ 2 \end{bmatrix}\sin(3t)\right)$$

$$\boxed{X(t) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}\cos 3t + \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{4t}\sin 3t}$$ P.S.

Constant coefficients (Inhomogeneous)

Consider the inhomogeneous linear system

$$\begin{cases} x_1' = ax_1 + bx_2 + g_1(t) \\ x_2' = cx_1 + dx_2 + g_2(t) \end{cases},$$

where a, b, c, d are constants (see lecture note for 3×3 matrix case).

Steps

- Rewrite the system as

$$X' = AX + G(t),$$

where

$$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}.$$

Constant coefficients (Inhomogeneous)

Steps

- Find general solution X_c for the associated homogeneous system (week 12 Homogeneous)

$$X' = AX,$$

where

$$X_c = c_1 X_1 + c_2 X_2.$$

- Find particular solution X_p for the inhomogeneous system (week 12 Undetermined coefficients)

$$X' = AX + G(t),$$

where

$$G(t) = \sum \prod_{j=1,2,\dots,k} G_j(x).$$

- General solution is $X = X_c + X_p$.

Undetermined coefficients

Steps

- For each $j = 1, \ldots, k$, guess X_{p_j} based on the following table.

Term in G_j	Choice for X_{p_j}
e^{rt}	Ee^{rt}
t^m	$E_0 + E_1 t + \cdots + E_m t^m$
$\cos(\omega t)$	$E_c \cos(\omega t) + E_s \sin(\omega t)$
$\sin(\omega t)$	$E_c \cos(\omega t) + E_s \sin(\omega t)$

- If one of terms in G_j is $e^{\lambda t}$, which happens to be X_1 or X_2, then

$$X_{p_j} = U_1 t e^{\lambda t} + U_2 e^{\lambda t}$$

- Let $X_p = \sum \prod_{j=1,2,\ldots,k} X_{p_j}$.
- Compute X_p'.
- Solve $X_p' = AX_p + G$ to find the coefficients $E, U_1, U_2, E_0, E_1, \ldots E_m, E_c, E_s$.

Constant coefficients (Inhomogeneous)

Example (Final Problem 4, Fall 2016)

Use LT and another method to find the PS (Convolutions, if any, need not to be evaluated):

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \text{ with } \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Remark

LT method: week 14 Laplace transform;
Another method: week 12 E-Analysis.

Constant coefficients (Inhomogeneous)

Example (Final Problem 4, Spring 2018)

Use LT and another method to find the PS of

$$\begin{cases} X'(t) = \begin{bmatrix} 4 & 4 \\ -9 & -8 \end{bmatrix} X(t) + \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix} \\ X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}.$$

Remark

LT method: week 14 Laplace transform;
Another method: week 12 E-Analysis.

Constant coefficients (Inhomogeneous)

Example (Test 3 Problem 3, Spring 2019)

Find the GS by any method and verify the GS

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}.$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}$$

- Find general solution X_c for the associated homogeneous system (week 12 Homogeneous)

$$X' = AX,$$

where

$$X_c = c_1X_1 + c_2X_2.$$

$X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $G = \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix}$

$X' = AX$

C-Eq: $\det(A - \lambda I) = (\lambda-2)(\lambda-1) - 6 = 0$

$$(\lambda^2 - 3\lambda + 2) - 6 = 0$$

$$(\lambda^2 - 3\lambda - 4) = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$\lambda_1 = 4$, $\lambda_2 = -1$

For $\lambda_1 = 4$,

$(A - \lambda_1 I)V_1 = 0$

$$\begin{bmatrix} 2-4 & 3 \\ 2 & 1-4 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -2V_{11} + 3V_{12} = 0 \\ 2V_{11} - 3V_{12} = 0 \end{cases} \Rightarrow 2V_{11} = 3V_{12}$$

Let $V_{12} = 2$, Then $V_{11} = 3$.

$$V_1 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

For $\lambda_2 = -1$

$(A - \lambda_2 I)V_2 = 0$

$$\begin{bmatrix} 2+1 & 3 \\ 2 & 1+1 \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 3V_{21} + 3V_{22} = 0 \\ 2V_{21} + 2V_{22} = 0 \end{cases} \Rightarrow V_{21} = -V_{22}$$

Let $V_{22} = 1$, Then $V_{21} = -1$

$$V_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

G.S. $X_c = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$

$$= c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

$$= c_1 X_1 + c_2 X_2$$

$$X_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

- Find particular solution X_p for the inhomogeneous system (week 12 Undetermined coefficients)

$$X' = AX + G(t),$$

where

$$G(t) = \sum \prod_{j=1,2,\dots,k} G_j(x).$$

$X' = AX + G$

$$G = \begin{bmatrix} e^{2t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} = G_1 + G_2$$

- For each $j = 1, \dots, k$, guess X_{p_j} based on the following table.

Term in G_j	Choice for X_{p_j}
e^{rt}	Ee^{rt}
t^m	$E_0 + E_1 t + \cdots + E_m t^m$
$\cos(\omega t)$	$E_c \cos(\omega t) + E_s \sin(\omega t)$
$\sin(\omega t)$	$E_c \cos(\omega t) + E_s \sin(\omega t)$

$$G_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} \qquad X_{p_1} = E_1 e^{2t}$$

$$G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t} \qquad X_{p_2} = E_2 e^{-2t}$$

- If one of terms in G_j is $e^{\lambda t}$, which happens to be X_1 or X_2, then

$$X_{p_j} = U_1 t e^{\lambda t} + U_2 e^{\lambda t}$$

Not happen. Skip!

- Let $X_p = \sum \prod_{j=1,2,\ldots,k} X_{p_j}$.

$$X_p = X_{p_1} + X_{p_2} = E_1 e^{2t} + E_2 e^{-2t} = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} e^{2t} + \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} e^{-2t}$$

- Compute X_p'.

$$X_p' = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} (2e^{2t}) + \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} (-2e^{-2t}) = \begin{bmatrix} 2e_{11} \\ 2e_{12} \end{bmatrix} e^{2t} + \begin{bmatrix} -2e_{21} \\ -2e_{22} \end{bmatrix} e^{-2t}$$

- Solve $X_p' = AX_p + G$ to find the coefficients $E, U_1, U_2, E_0, E_1, \ldots E_m, E_c, E_s$.

$$X_p' = AX_p + G$$

$$\begin{bmatrix} 2e_{11} \\ 2e_{12} \end{bmatrix} e^{2t} + \begin{bmatrix} -2e_{21} \\ -2e_{22} \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} e^{2t} + \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} e^{-2t} \right) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}$$

① For coefficients of e^{2t}: $\begin{bmatrix} 2e_{11} \\ 2e_{12} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{cases} 2e_{11} = 2e_{11} + 3e_{12} + 1 \\ 2e_{12} = 2e_{11} + e_{12} + 0 \end{cases} \Rightarrow \begin{cases} 0 = 3e_{12} + 1 \\ 0 = 2e_{11} - e_{12} \end{cases} \Rightarrow \begin{cases} e_{11} = -\frac{1}{6} \\ e_{12} = -\frac{1}{3} \end{cases}$$

$$E_1 = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix}$$

② For coefficients of e^{-2t}: $\begin{bmatrix} -2e_{21} \\ -2e_{22} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\begin{cases} -2e_{21} = 2e_{21} + 3e_{22} + 0 \\ -2e_{22} = 2e_{21} + e_{22} + 1 \end{cases} \Rightarrow \begin{cases} 0 = 4e_{21} + 3e_{22} \\ 0 = 2e_{21} + 3e_{22} + 1 \end{cases} \Rightarrow \begin{cases} e_{21} = \frac{1}{2} \\ e_{22} = -\frac{2}{3} \end{cases}$$

$$E_2 = \begin{bmatrix} e_{21} \\ e_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{2}{3} \end{bmatrix}$$

$$X_p = E_1 e^{2t} + E_2 e^{-2t}$$

$$= \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} e^{2t} + \begin{bmatrix} \frac{1}{2} \\ -\frac{2}{3} \end{bmatrix} e^{-2t}$$

- General solution is $X = X_c + X_p$.

G.S. $X(t) = X_c + X_p$

$$\boxed{X(t) = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \end{bmatrix} e^{2t} + \begin{bmatrix} \frac{1}{2} \\ -\frac{2}{3} \end{bmatrix} e^{-2t}}$$

$\Updownarrow$ equivalent

G.S.

We can rewrite the above solution as

$$\boxed{\begin{cases} x(t) = 3c_1 e^{4t} - c_2 e^{-t} - \frac{1}{6} e^{2t} + \frac{1}{2} e^{-2t} \\ y(t) = 2c_1 e^{4t} + c_2 e^{-t} - \frac{1}{3} e^{2t} - \frac{2}{3} e^{-2t} \end{cases}}$$

Constant coefficients (Inhomogeneous)

Example (Test 3 Problem 3, Spring 2020)

Find, by any method, the GS of

$$\begin{cases} x' = x + 4y + e^{5t} \\ y' = 2x + 3y + e^{-2t} \end{cases}.$$

Constant coefficients (Inhomogeneous)

Example (Final Problem 3, Fall 2020)

Use LT and one other method to find the PS of

$$\begin{cases} x' = y + z + 1 \\ y' = z + x + 1 \\ z' = x + y + 1 \end{cases},$$

with ICs $x(0) = y(0) = z(0) = 0$. Convolutions, if any, must be evaluated.

Remark

LT method: week 14 Laplace transform;
One other method: week 12 E-Analysis or week 13 Separation of variables.

Constant coefficients (Inhomogeneous)

Example (Final Problem 6, Spring 2022)

Use the Eigen-Analysis method to solve the DE:

$$X'(t) = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} X(t) + \begin{bmatrix} e^{-2t} \\ e^{3t} \end{bmatrix}.$$

$$X'(t) = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} X(t) + \begin{bmatrix} e^{-2t} \\ e^{3t} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} e^{-2t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}$$

$$\det(A - \lambda I) = (\lambda - 2)(\lambda - (-1)) - 4 \cdot 1 = 0$$

$$(\lambda - 2)(\lambda + 1) - 4 = 0$$

$$\lambda^2 - \lambda - 2 - 4 = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$\lambda_1 = 3, \quad \lambda_2 = -2.$

For $\lambda_1 = 3$,

$$(A - \lambda_1 I) V_1 = \begin{bmatrix} 2-3 & 4 \\ 1 & -1-3 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} -V_{11} + 4V_{12} = 0 \\ V_{11} - 4V_{12} = 0 \end{cases} \Rightarrow V_{11} = 4V_{12}$$

Let $V_{12} = 1$. Then $V_{11} = 4 \cdot 1 = 4$

$$V_1 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$,

$$(A - \lambda_2 I) V_1 = \begin{bmatrix} 2-(-2) & 4 \\ 1 & -1-(-2) \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 4V_{21} + 4V_{22} = 0 \\ V_{21} + V_{22} = 0 \end{cases} \Rightarrow V_{21} = -V_{22}$$

Let $V_{22} = 1$. Then $V_{21} = -1$

$$V_2 = \begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$X_c(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

$$G = \begin{bmatrix} e^{-2t} \\ e^{3t} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}$$

$$= G_1 + G_2$$

$$G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} \rightarrow X_{p_1} = Ae^{3t} + Bte^{3t} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^{3t}$$

$$G_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t} \rightarrow X_{p_2} = Ce^{-2t} + Dte^{-2t} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} te^{-2t}$$

$$X_p = X_{p_1} + X_{p_2}$$

$$= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^{3t} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} te^{-2t}$$

$$X_p' = 3\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{3t} + 3\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^{3t}$$

$$-2\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^{-2t} - 2\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} te^{-2t}$$

$$= \begin{bmatrix} 3a_1 + b_1 \\ 3a_2 + b_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 3b_1 \\ 3b_2 \end{bmatrix} te^{3t}$$

$$+ \begin{bmatrix} -2c_1 + d_1 \\ -2c_2 + d_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} -2d_1 \\ -2d_2 \end{bmatrix} te^{-2t}$$

$$X_p'(t) = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} X_p(t) + G(t)$$

$$\begin{bmatrix} 3a_1 + b_1 \\ 3a_2 + b_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 3b_1 \\ 3b_2 \end{bmatrix} te^{3t} + \begin{bmatrix} -2c_1 + d_1 \\ -2c_2 + d_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} -2d_1 \\ -2d_2 \end{bmatrix} te^{-2t}$$

$$= \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^{3t} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{-2t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} te^{-2t} \right)$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-2t}$$

① e^{3t}: $$\begin{bmatrix} 3a_1 + b_1 \\ 3a_2 + b_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3a_1 + b_1 \\ 3a_2 + b_2 \end{bmatrix} = \begin{bmatrix} 2a_1 + 4a_2 \\ a_1 - a_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} 3a_1 + b_1 = 2a_1 + 4a_2 \\ 3a_2 + b_2 = a_1 - a_2 + 1 \end{cases}$$

$$\begin{cases} b_1 = -a_1 + 4a_2 \\ b_2 = a_1 - 4a_2 + 1 \end{cases}$$

② te^{3t}: $$\begin{bmatrix} 3b_1 \\ 3b_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} 3b_1 \\ 3b_2 \end{bmatrix} = \begin{bmatrix} 2b_1 + 4b_2 \\ b_1 - b_2 \end{bmatrix}$$

$$\begin{cases} 3b_1 = 2b_1 + 4b_2 \\ 3b_2 = b_1 - b_2 \end{cases} \Rightarrow b_1 = 4b_2$$

$$\begin{cases} b_1 = -a_1 + 4a_2 \\ b_2 = a_1 - 4a_2 + 1 \\ b_1 = 4b_2 \end{cases}$$

$$-a_1 + 4a_2 = 4(a_1 - 4a_2 + 1)$$

$$-a_1 + 4a_2 = -4(-a_1 + 4a_2) + 4$$

$$5(-a_1 + 4a_2) = 4$$

$$-a_1 + 4a_2 = \frac{4}{5} \rightarrow \text{Let } a_2 = 0. \text{ Then } a_1 = -\frac{4}{5}$$

$$b_1 = \frac{4}{5}$$

$$b_2 = \frac{1}{4} b_1 = \frac{1}{5}$$

$$\begin{cases} a_1 = -\frac{4}{5} \\ a_2 = 0 \\ b_1 = \frac{4}{5} \\ b_2 = \frac{1}{5} \end{cases} \Rightarrow \begin{cases} A = \begin{bmatrix} -\frac{4}{5} \\ 0 \end{bmatrix} \\ B = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} \end{cases}$$

③ e^{-2t}: $\begin{bmatrix} -2c_1 + d_1 \\ -2c_2 + d_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{cases} -2c_1 + d_1 = 2c_1 + 4c_2 + 1 \\ -2c_1 + d_2 = c_1 - c_2 \end{cases}$$

$$\begin{cases} d_1 = 4(c_1 + c_2) + 1 \\ d_2 = c_1 + c_2 \end{cases}$$

te^{-2t}: $\begin{bmatrix} -2d_1 \\ -2d_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

$$\begin{cases} -2d_1 = 2d_1 + 4d_2 \\ -2d_2 = d_1 - d_2 \end{cases} \Rightarrow d_1 + d_2 = 0$$

$$\begin{cases} d_1 = 4(c_1+c_2)+1 \\ d_2 = c_1 + c_2 \\ d_1 + d_2 = 0 \end{cases}$$

$$4(c_1+c_2)+1 + (c_1+c_2) = 0$$

$$5(c_1+c_2) = -1$$

$$c_1 + c_2 = -\frac{1}{5} \longrightarrow \text{Let } c_2 = 0. \text{ Then } c_1 = -\frac{1}{5}$$

$$d_2 = -\frac{1}{5}$$

$$d_1 = -d_2 = \frac{1}{5}$$

$$\begin{cases} c_1 = -\frac{1}{5} \\ c_2 = 0 \\ d_1 = \frac{1}{5} \\ d_2 = -\frac{1}{5} \end{cases} \Rightarrow \begin{cases} C = \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} \\ D = \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{5} \end{bmatrix} \end{cases}$$

$$X_p = Ae^{3t} + Bte^{3t} + Ce^{-2t} + Dte^{-2t}$$

$$= \begin{bmatrix} -\frac{4}{5} \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} te^{3t} + \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{5} \end{bmatrix} te^{-2t}$$

$$X(t) = X_c + X_p$$

$$\boxed{\begin{aligned} X(t) = \; & c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} \\ & + \begin{bmatrix} -\frac{4}{5} \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} te^{3t} + \begin{bmatrix} -\frac{1}{5} \\ 0 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{1}{5} \\ -\frac{1}{5} \end{bmatrix} te^{-2t} \end{aligned}}$$

G.S.

Constant coefficients (Inhomogeneous)

Example (Final Problem 4, Fall 2022)

Use LT method and a non-LT method to find PS:

$$\begin{cases} -x' + y + z = e^{2t} \\ x - y' + z = e^{2t} \\ x + y - z' = e^{2t} \\ x(0) = y(0) = z(0) = 0 \end{cases}.$$

Remark

LT method: week 14 Laplace transform;
A non-LT method: week 12 E-Analysis or week 13 Separation of variables.