AMS 361: Applied Calculus IV

Homework 4 Solution

Assignment Date: When available in Brightspace
Due Date: See Brightspace
Submission to: Brightspace (1 PDF)
Grades: See individual problems

Problem 4.1

Newton's Law of Cooling

$$\Rightarrow \frac{dT}{dt} \propto T - T_0$$

$$\Rightarrow \frac{dT}{dt} = -k(T - 90)$$

$$\Rightarrow \int \frac{dt}{T - 90} = \int -kdt$$

$$\Rightarrow \int_{T_0}^{T_{15}} \frac{d}{T - A} = -\int kdt$$

$$\Rightarrow k = -\frac{1}{t} \ln\left(\frac{T_{15} - A}{T_0 - A}\right)$$

$$\Rightarrow k = -\frac{1}{15} \ln\left(\frac{180 - 90}{250 - 90}\right)$$

$$\Rightarrow k = \frac{1}{15} \ln\left(\frac{16}{9}\right)$$

$$T_0 = T_{15} = 180F,\ T_x = 125F$$

$$\Rightarrow T_x = 125F$$

$$\Rightarrow t = -\frac{1}{k} \ln\left(\frac{T_x - A}{T_{15} - A}\right)$$

$$\Rightarrow t = -\frac{1}{\frac{1}{15}\ln\left(\frac{16}{9}\right)} \ln\left(\frac{125 - 90}{180 - 90}\right)$$

$$\Rightarrow\ t = -\frac{15}{\ln\left(\frac{16}{9}\right)}\cdot\ln\left(\frac{35}{90}\right)$$

$$\Rightarrow\ t \approx 24.62\ \text{min}$$

Problem 4.2

Torricelli; law of cooling

$$\begin{cases}\dfrac{A(y)dy}{dt} = -k\sqrt{y}\\ y(t=0) = 40\end{cases}$$

Radius at height h,

$$r(y) = \frac{r_1 - r_2}{h}y + r_2$$

Area of cross-section

$$A(y) = \pi y^2\left(\frac{r_1 - r_2}{n}\right)^2 + \pi r_2^2 + 2\pi y r_2^2\left(\frac{r_1 - r_2}{h}\right)$$

Apply toricelis law

$$\frac{A(y)dy}{dt} = -k\sqrt{y}$$

$$\Rightarrow\quad \int_h^0 \frac{A(y)dy}{\sqrt{y}} = -\int_0^t kdt$$

$$\Rightarrow\quad \int_h^0 \frac{A(y)dy}{\sqrt{y}} = -kt|_0^t$$

Evaluating LHS

$$\int_h^0 dy\left[\pi r_2^2\frac{1}{\sqrt{y}} + 2\pi\sqrt{y}r_2\left(\frac{r_1 - r_2}{h}\right) + \pi\left(\frac{r_1 - r_2}{h}\right)^2 y^{3/2}\right)^2$$

$$= 2\pi\sqrt{y}\frac{4}{3}\pi r_2\left(\frac{r_1 - r_2}{h}\right)y^{3/2} + \frac{2}{5}\pi\left(\frac{r_1 - r_2}{h}\right)^2 y^{5/2}$$

$$= -2\pi\sqrt{h}\left[r_2{}^2 + \frac{2}{3}r_2(r_1 - r_2) + \frac{1}{5}(r_1 - r_2)^2\right]$$

$$\Rightarrow -\frac{2\pi\sqrt{n}}{15}(8r_2^2 + 4r_1r_2 + 3r_1{}^2)$$

Evaluating RHs

$$= kt]_0^T = -kT_1$$

$$\Rightarrow T_1 \quad = \frac{2\pi\sqrt{h}}{15k}(8r_2^2 + 4r_1r_2 + 3r_1^2)$$

Turned upside down

Evaluate

$$\int_h^0 dy \left[\pi y^{3/2}\left(\frac{r_2 - r_1}{h}\right)^2 + 2\pi r_1\left(\frac{r_2 - r_1}{h}\right)y^{1/2} + \pi r_1^2 \frac{1}{\sqrt{y}}\right]$$

$$T_2 = \frac{2\pi\sqrt{h}}{15k}(8r_1^2 + 4r_1r_2 + 3r_2^2)$$

if r_1 = 0, then T_1 and T_2 is

$$T_1 = \frac{2\pi\sqrt{h}}{15k} \cdot 8r_2^2$$

$$T_2 = \frac{2\pi\sqrt{h}}{15k} \cdot 3r_2^2$$

Problem 4.3

DE's

$$\frac{dP}{dt} = -\alpha P(M - P)$$

$$\frac{dP}{dt} = \alpha P(P - M)$$

Plug in values

$$\frac{dP}{dt} = 0.001P(P - 100)$$

$$P(0) = 1000$$

Taking the limit

$$\lim_{t\to\infty} \quad P(t) = \lim_{t\to\infty} \quad \frac{100}{1 - 0.9e^{-0.1t}} = 100$$

With the given conditions, T_P:

$$T_p = \frac{\ln \quad C_1}{\alpha M}, \; C_1 = \frac{P_0}{P_0 - M}$$

$$T_p = \frac{\ln\left(\frac{10}{9}\right)}{0.1} \approx 1.0536$$

$$\lim_{t \to T_p} P(t) = \infty$$