AMS 361: Applied Calculus IV by Prof. Y. Deng

Short Test 2: 10/17/2019 7p-8:20p Frey 100

Test 2

(1) Closed Book with 1-page (double-sided 8.5x11) self-prepared.
(2) Do any two of the three problems.
(3) If all three are attempted, the best two (and only two) will be credited.
(4) Each problem is worth 7.5 points for a total of 15 points (max).
(5) No points for solutions without appropriate intermediate steps.
(6) Partial credits are given only for steps that are relevant to the solutions.
(7) No name, no grade and no request will be answered.
(8) No SBU ID card, no test.

SB ID		
Name		
Problems	Score	Remarks
T2-1		
T2-2		
T2-3		
Total Score		

T2-1 (7.5 Points): In a hot summer day of constant temperature $A_1 = 100F$, my car overheated to $T_0 = 250F$. I pulled it over and waited for 20 minutes to drop its temperature to $T_{20} = 200F$. I found, and moved my car to, a cool garage of temperature $A_2 = 70F$ (ignore the moving time and temperature increase due to move). The car can function properly only at (or below) $T_x = 100F$. How long should I wait to reach this T_x?

Solution:
Newton's law of cool for constant ambient temperature A,

$$\begin{cases} \dfrac{dT(t)}{dt} = k(A - T) \\ T(t = 0) = T_0 \end{cases}$$

This give us the PS:

$$T(t) = A + (T_0 - A)e^{-kt}$$

and

$$t = \frac{1}{k} \ln \frac{T_0 - A}{T - A}$$

Then, for $T_0 = 250F, T_{20} = 200F$ and $A_1 = 100F, t = 20$, we have

$$k = \frac{1}{20} \ln \frac{250 - 100}{200 - 100}$$

$$= \frac{1}{20}\ln\left(\frac{3}{2}\right)$$

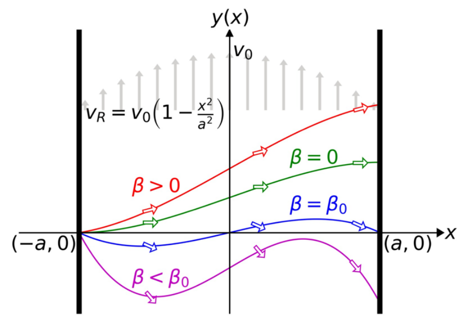
Then use $k = \frac{1}{20}\ln\frac{3}{2}$, and $A_2 = 70F$, notice now the T_0 become 200F

$$t = \frac{1}{\frac{1}{20}\ln\left(\frac{3}{2}\right)}\ln\frac{200-70}{100-70}$$

$$= \frac{20}{\ln\left(\frac{3}{2}\right)}\ln\left(\frac{13}{3}\right)$$

T2-2 (7.5 Points): A swimmer crosses a river, whose water speed and other settings are shown in the figure, at a constant speed v_s and constant angle β measured from the x-axis. Find (1) the swimmer's trajectory (6.0 Points) for any β and (2) the special β_0 at which the swimmer will arrive precisely at $(a, 0)$ as shown by the blue trajectory.

This problem is a minor adjustment of the swimmer's problem in textbook (p.131) and my Lecture #12.



First, we generalize this with angle β

$$\begin{cases} \dfrac{dx}{dt} = v_s\cos\beta \\ \dfrac{dy}{dt} = v_0\left(1 - \dfrac{x^2}{a^2}\right) + v_s\sin\beta \end{cases}$$

Thus, we use an IVP to describes the trajectory of the swimmer

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{v_0}{v_s}\left(1 - \dfrac{x^2}{a^2}\right)\dfrac{1}{\cos\beta} + \dfrac{\sin\beta}{\cos\beta} \\ y(x = -a) = 0 \end{cases}$$

Solving this IVP leads to

$$dy = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)\frac{1}{\cos\beta} + \frac{\sin\beta}{\cos\beta}dx$$

$$\int_0^y dy = \frac{v_0}{v_s}\int_{-a}^{x}\left(1 - \frac{x^2}{a^2}\right)\frac{1}{\cos\beta} + \frac{\sin\beta}{\cos\beta}dx$$

Then, the trajectory of the swimmer is

$$y = \frac{v_0}{v_s}\left(x - \frac{x^3}{3a^2} + \frac{2a}{3}\right)\frac{1}{\cos\beta} + (x + a)\frac{\sin\beta}{\cos\beta}$$

The total drift at $(a, 0)$ for any angle

$$y_{\text{drift}}(x = a) = \frac{v_0}{v_s}\left(\frac{4a}{3}\right)\frac{1}{\cos\beta} + 2a\frac{\sin\beta}{\cos\beta}$$

To make the drift vanish, we must choose a special angle β_0 to make

$$\frac{v_0}{v_s}\left(\frac{4a}{3}\right)\frac{1}{\cos\beta_0} + 2a\frac{\sin\beta_0}{\cos\beta_0} = 0$$

$$\sin\beta_0 = -\left(\frac{2}{3}\right)\frac{v_0}{v_s}$$

Interestingly, the special angle β_0 does not depend on the river width.

T2-3 (7.5 Points): Find the GS of

$$x^2y'' - 3xy' + 4y = 0$$

Method 1: S-method to replace IV
Let

$$x = e^t \text{ and } t = \ln x$$

Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$
$$\frac{d^2y}{dx^2} = e^{-2t}(\frac{d^2y}{dt^2} - \frac{dy}{dt})$$

So, the DE becomes

$$\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 3\frac{dy}{dt} + 4y = 0$$
$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$$
$$r^2 - 4r + 4 = 0$$
$$r_{1,2} = 2, 2$$

Then,

$$y(t) = A_1e^{2t} + A_2te^{2t}$$

Back substitute,

$$y(x) = C_1x^2 + C_2\ln|x|x^2$$

Method 2: TS Method
Let $y = x^\lambda$, then,

$$y' = \lambda x^{\lambda-1}$$
$$y'' = \lambda(\lambda - 1)x^{\lambda-2}$$

Therefore, we have

$$\lambda(\lambda - 1)x^\lambda - 3\lambda x^\lambda + 4x^\lambda = 0$$
$$\lambda^2 - 4\lambda + 4 = 0$$
$$\lambda_{1,2} = 2, 2$$

Then the GS is

$$y = C_1x^2 + C_2\ln|x|x^2$$