AMS 315, Spring 2021, April 20, 2021, Midterm Examination 3

Penalties by question

A and B forms

1. Both forms:
   - -15 Incorrect degrees of freedom
   - -10 incorrect sum of squares or mean squares (-10 for first error, no points for consistent errors)
   - -25 no decision or inconsistent decision
   - -5 major computational error (more than 5% in magnitude) with correct formula
   - -2 minor computational error
2. A form: 10 points for linear contrast: -5 wrong linear contrast, correct formula

   10 points for linear sum of squares: -5 correct formula wrong answer

   10 points for Scheffe interval;

   -2 no entry of endpoints into text box.

   B form: 30 points for correct Tukey W

   -15 no 4.91

   -15 no square root (MSE/7)
3. Right or wrong: 10 points.

C and D forms

4. Both forms: 5 points for each contrast.

   -2 no entry into text box.
5. Both forms:
   - -15 Incorrect degrees of freedom
   - -10 incorrect sum of squares or mean squares (-10 for first error, no points for consistent errors)
   - -25 no decision or inconsistent decision
   - -5 major computational error (more than 5% in magnitude) with correct formula
   - -2 minor computational error
   - -2 no entry into text box.
6. Both forms:
   - -15 Incorrect degrees of freedom
   - -10 incorrect sum of squares or mean squares (-10 for first error, no points for consistent errors)
   - -25 no decision or inconsistent decision
   - -5 major computational error (more than 5% in magnitude) with correct formula
   - -2 minor computational error
   - -2 no entry into text box.

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E and F forms

7. Both forms
   - +15 correct M matrix
   - -20 incorrect variance of W_1
   - -20 incorrect variance of W_2
   - -20 incorrect covariance of W_1, W_2
   - -10 one correct covariance, second wrong
   - -10 forget σ^2
   - -2 no entry into text box.

G form

8. +30 correct formula for σ_Y
   +30 correct statement of delta method applied to *var(W)*.

Late Penalty:

Papers submitted by email with recorded time after 6:35 pm were assessed a 15% late penalty.

A form

Common Information for Questions 1, 2, and 3

A. A research team studied *Y*, the protein production of a laboratory animal, and sought to set the dose of the medicine so that $E(Y)$ is minimized. They used four doses of the medicine: 1, 2, 3, and 4 units respectively. They randomly assigned 12 animals to dosage 1, 12 to dosage 2, 12 to dosage 3, and 12 to dosage 4. They observed that the average values of *Y* at each dosage were $y_{1\bullet} = 525, y_{2\bullet} = 505,\ y_{3\bullet} = 380,\ and\ y_{4\bullet} = 338,$ where $y_{i.}$ was the average of the observations taken with dosage $i = 1,2,3,4$, respectively. They also observed that $s_1^2 = 28{,}634, s_2^2 = 34{,}537,\ s_3^2 = 36{,}491,$ and $s_4^2 = 22{,}434,$ where s_i^2 was the unbiased estimate of the variance for the observations taken with dosage $i = 1,2,3,4$ respectively.

1. Complete the analysis of variance table for these results; that is, be sure to specify the degrees of freedom, sums of squares, mean squares, F-test, and your conclusion. Test the null hypothesis that all treatment means are equal using significance levels 0.10, 0.05, and 0.01. Also enter your F statistic and conclusion in the text box underneath this question. This question is worth 45 points.
2. Find the estimated linear contrast, the sum of squares due to the linear contrast and the 99% Scheffe confidence interval for the linear contrast. The coefficients of the linear contrast are $-3, -1, 1, 3$. Enter the left and right values of the Scheffe confidence interval for the linear contrast in the text box underneath this question. This question is worth 30 points.
3. What is the optimal setting of dosage, and how do you document it? This question is worth 10 points.

End of Application of Common Information

WORK TABLE

DOSE	J_i	$y_{i\bullet}$	$y_{i.} - y_{\bullet\bullet}$	s_i^2
1	12	525	88	28,634
2	12	505	68	34,537
3	12	380	-57	36,491
4	12	338	-99	22,434
SUM	48	1748.	0 ✓	122,096.

$y_{\bullet\bullet} = \frac{1748}{4} = 437$

$SS_{TREAT} = 12[88^2 + 68^2 + (-57)^2 + (-99)^2] = 12(25{,}418)$

$= 305{,}016$ ON 3 DF.

$MS_{TREAT} = \frac{SS_{TREAT}}{3} = 101{,}672.$

$MSE = \frac{122{,}096}{4} = 30{,}524$ ON 44 DF.

$SSE = 44(MSE) = 1{,}343{,}056$

ANOVA TABLE

SOURCE	DF	SS	MS	F
TREATMENT (DOSE)	3	305,016	101,672	$\boxed{3.33}$
(PURE) ERROR	44	1,343,056	30,524	
TOTAL	47	1,648,072		

A1A CONTINUED

α	$F(3,44)$	DECISION.
.10	2.213	REJECT
.05	2.816	REJECT
.01	4.261.	ACCEPT

REJECT H_0 AT $\alpha = .05$ AND $\alpha = .10$; ACCEPT H_0 ALL MEANS EQUAL AT $\alpha = .01$.

A2. $\hat{\lambda}_L = -3(525) - (505) + 380 + 3(338) = -686.$

$$SS_L = \frac{(\hat{\lambda}_L)^2}{20/12} = 282,357.6$$

99% SCHEFFE CI FOR λ_L:

$$-686 \pm \sqrt{3(4.261)}\sqrt{30,524\left(\frac{20}{12}\right)}$$

$$-686 \pm \sqrt{12.783}\quad\sqrt{50,873.33}$$

$$-686 \pm (3.575)(225.55)$$

$$-686 \pm 806.35$$

$$\boxed{-1492.35 \text{ TO } 120.35}$$

A3. AT $\alpha = .05$, THE RESULTS ARE SIGNIFICANT.

THE 95% SCHEFFE CI FOR λ_L IS.

$$-686 \pm \sqrt{3(2.816)}\sqrt{50,873.33}$$

$$-686 \pm (2.907)(225.55)$$

$$-686 \pm 655.57.$$

$\hat{\lambda}_L$ IS NEGATIVE AND SIGNIFICANT

TO MINIMIZE $E(Y)$, CHOOSE DOSE = 4

OR HIGHER IF POSSIBLE

B form

Common Information for Questions 1, 2, and 3

B. A research team studied Y, the protein production of a laboratory animal, and sought to set the dose of the medicine so that $E(Y)$ is maximized. They used four doses of the medicine: 1, 2, 3, and 4 units respectively. They randomly assigned 7 animals to dosage 1, 7 to dosage 2, 7 to dosage 3, and 7 to dosage 4. They observed that the average values of Y at each dosage were $y_{1\bullet} = 152, y_{2\bullet} = 164,\ y_{3\bullet} = 104,\ and\ y_{4\bullet} = 148$, where $y_{i.}$ was the average of the observations taken with dosage $i = 1,2,3,4$ respectively. They also observed that $s_1^2 = 3{,}210, s_2^2 = 4{,}180,\ s_3^2 = 2{,}575,$ and $s_4^2 = 2{,}895,$ where s_i^2 was the unbiased estimate of the variance for the observations taken with dosage $i = 1,2,3,4$ respectively.

1. Complete the analysis of variance table for these results; that is, be sure to specify the degrees of freedom, sums of squares, mean squares, F-test, and your conclusion. Test the null hypothesis that all treatment means are equal using significance levels 0.10, 0.05, and 0.01. Also enter your F statistic and conclusion in the text box underneath this question. This question is worth 45 points.
2. Apply Tukey's W procedure to obtain the 99% confidence interval for $E(Y_{2j}) - E(Y_{3j})$. Enter the left and right values of the Tukey W confidence interval for $E(Y_{2j}) - E(Y_{3j})$ in the text box underneath this question. This question is worth 30 points.
3. What is the optimal setting of dosage, and how do you document it? This question is worth 10 points.

End of Application of Common Information

WORK TABLE

DOSE	J_i	$y_{i\bullet}$	$y_{i\bullet} - y_{\bullet\bullet}$	s_i^2
1	7	152	10	3,210
2	7	164	22	4,180
3	7	104	-38	2,575
4.	7	148	6	2,895
SUM	28	568	0 ✓	12,860

$$y_{\bullet\bullet} = \frac{568}{4} = 142$$

$$\sum (y_{i\bullet} - y_{\bullet\bullet})^2 = [10^2 + 22^2 + (-38)^2 + 6^2] = 2064.$$

$$SS_{TREAT} = 7\sum (y_{i\bullet} - y_{\bullet\bullet})^2 = 14{,}448.$$

$$MS_{TREAT} = \frac{SS_{TREAT}}{3} = 4816 \text{ ON } 3 \text{ DF.}$$

$$MSE = \frac{12{,}860}{4} = 3{,}215 \text{ ON } 24 \text{ DF}$$

$$SSE = 24(MSE) = 77{,}160. \text{ ON } 24 \text{ DF}$$

$$SS_{TOT} = SS_{TREAT} + SSE = 91{,}608.$$

ANOVA TABLE.

SOURCE	DF	SS	MS	F.
DOSE (TREATMENT)	3	14,448	4,816	$\boxed{1.50}$
(PURE) ERROR	24	77,160	3,215	
TOTAL	27	91,608		

B1 CONTINUED

α	$F(3, 24)$	DECISION.
.10	2.327	ACCEPT
.05	3.009.	ACCEPT
.01	4.718.	ACCEPT

ACCEPT H_0 ALL TREATMENT MEANS EQUAL AT $\alpha = .10$ (AND $\alpha = .05$ AND $\alpha = .01$).

B2 TUKEY'S 99% CI FOR $E(Y_{2j}) - E(Y_{3j})$ HAS CENTER

$y_{2\cdot} - y_{3\cdot} = 164 - 108 = 56.$

$w = q(.01, 4, 24)\sqrt{\frac{MSE}{7}} = 4.91\sqrt{\frac{3{,}215}{7}} = 4.91\sqrt{459.3}$

$= 105.2.$

TUKEY'S 99% CI FOR $E(Y_{2j}) - E(Y_{3j})$ IS.

$60 \pm 105.2 = \boxed{-45.2 \text{ TO } 165.2}$

B3. THERE IS NO OPTIMUM BECAUSE H_0 ALL TREATMENT MEANS WERE EQUAL WAS ACCEPTED.

C. A research team studied how *Y*, the protein production of a laboratory animal, could be minimized by choice of dosage. They used four doses: 1, 2, 3, and 4 units. They randomly assigned 22 animals to 1 unit, 22 to 2 units, 22 to 3 units, and 22 to 4 units. The average values of *Y* at each dosage were $y_{1\bullet} = 1435$, $y_{2\bullet} = 961$, $y_{3\bullet} = 834$, and $y_{4\bullet} = 1042$, where $y_{i.}$ was the average of the observations taken with dosage $i = 1,2,3,4$. The within dosage variances were $s_1^2 = 295{,}636, s_2^2 = 189424$, $s_3^2 = 382{,}178$, and $s_4^2 = 304{,}972$. They found that $y_{\bullet\bullet} = 1068$ and that the average s_i^2 was 293,052.5. The total sum of squares was 29,050,950. The coefficients of the linear contrast were $-3, -1, 1, 3;$, and $\hat{\lambda}_{Lin} = -1{,}306$. The coefficients of the quadratic contrast were $1, -1, -1, 1$; and $\hat{\lambda}_{Quad} = 682$. The coefficients of the cubic contrast were $-1, 3, -3, 1$; and $\hat{\lambda}_{Cubic} = -12$.

4. What are the values of the sum of squares due to the linear contrast, the sum of squares dues to the quadratic contrast, and the sum of squares due to the cubic contrast? Report the sum of squares due to the linear contrast in the text box beneath this question. (15 points)
5. Find the analysis of variance table for the linear regression of *Y* on dosage, using the sum of squares due to the linear contrast as the sum of squares for the regression of *Y* on dosage. Test the null hypothesis that there is no linear association at the 0.10, 0.05, and 0.01 levels of significance. Enter the test statistic for this hypothesis in the text box beneath this question. (40 points)
6. Test the null hypothesis that the linear model is adequate at the 0.10, 0.05, and 0.01 levels of significance. Report the analysis of variance table including the sum of squares due to lack of fit. Enter the value of the test statistic in the text box beneath this question. (50 points).

End of application of common information

C4. $$SS_{LIN} = \frac{(\hat{\lambda}_{LIN})^2}{20/22} = \frac{(-1306)^2}{20/22} = \boxed{1{,}876{,}199.6.}$$

$$SS_{QUAD} = \frac{(\hat{\lambda}_{QUAD})^2}{4/22} = \frac{(682)^2}{4/22} = 2{,}558{,}182$$

$$SS_{CUBIC} = \frac{(\hat{\lambda}_{CUBIC})^2}{20/22} = \frac{(-12)^2}{20/22} = 158.4.$$

C5. ANOVA TABLE LINEAR REGRESSION

SOURCE	DF	SS	MS	F.
LINEAR	1	1,876,199.6	1,876,199.6	$\boxed{5.94.}$
ERROR	86	27,174,750.4	315,985.5	
TOTAL	87	29,050,950		

α	F(1, 86).	DECISION
.10	2.765	REJECT
.05	3.952.	REJECT
.01	6.939.	ACCEPT

REJECT H_0 NO LINEAR ASSOCIATION AT $\alpha = .10$ AND $\alpha = .05$. ACCEPT H_0: NO LINEAR ASSOCIATION AT $\alpha = .01$.

C6.

ANOVA TABLE
LACK OF FIT OF LINEAR MODEL.

SOURCE	DF	SS	MS
LINEAR	1	1,876,199.6	
LACK OF FIT	2	2,558,340.4	1,279,170.2
PURE ERROR	84	24,616,410	293,052.5
TOTAL	87	29,050,950	

$$F_{LOF} = \frac{1,279,170.2}{293,052.5} = \boxed{4.36.}$$

α	F(2,84)	DECISION
.10	2.367	REJECT
.05	3.105	REJECT
.01	4.867	ACCEPT

REJECT H_0 LINEAR MODEL ADEQUATE AT $\alpha = .05$

AND $\alpha = .10$; ACCEPT IT AT $\alpha = .01$.

CHOOSE DOSAGE 3 TO MINIMIZE E(Y)

D. A research team studied how Y, the protein production of a laboratory animal, could be minimized by choice of dosage. They used four doses: 1, 2, 3, and 4 units. They randomly assigned 16 animals to 1 unit, 16 to 2 units, 16 to 3 units, and 16 to 4 units. The average values of Y at each dosage were $y_{1\bullet} = 76,\ y_{2\bullet} = 144,\ y_{3\bullet} = 303,$ and $y_{4\bullet} = 317$, where $y_{i.}$ was the average of the observations taken with dosage $i = 1,2,3,4$. The within dosage variances were $s_1^2 = 67{,}720, s_2^2 = 87{,}436,\ s_3^2 = 57{,}685,$ and $s_4^2 = 55{,}643$. They found that $y_{\bullet\bullet} = 210$ and that the average s_i^2 was 67,121. The total sum of squares was 4,705,820. The coefficients of the linear contrast were $-3,-1,1,3;$, and $\hat{\lambda}_{Lin} = 882$. The coefficients of the quadratic contrast were $1,-1,-1,1;$ and $\hat{\lambda}_{Quad} = -54$. The coefficients of the cubic contrast were $-1,3,-3,1;$ and $\hat{\lambda}_{Cubic} = -236$.

4. What are the values of the sum of squares due to the linear contrast, the sum of squares dues to the quadratic contrast, and the sum of squares due to the cubic contrast? Report the sum of squares due to the linear contrast in the text box beneath this question. (15 points)
5. Find the analysis of variance table for the linear regression of Y on dosage, using the sum of squares due to the linear contrast as the sum of squares for the regression of Y on dosage. Test the null hypothesis that there is no linear association at the 0.10, 0.05, and 0.01 levels of significance. Enter the test statistic for this hypothesis in the text box beneath this question. (40 points)
6. Test the null hypothesis that the linear model is adequate at the 0.10, 0.05, and 0.01 levels of significance. Report the analysis of variance table including the sum of squares due to lack of fit. Enter the value of the test statistic in the text box beneath this question. (50 points).

End of application of common information

D4. $SS_{LIN} = \dfrac{(\hat{\lambda}_{LIN})^2}{20/16} = \boxed{622{,}339.2}$

$SS_{QUAD} = \dfrac{(\hat{\lambda}_{QUAD})^2}{4/16} = 11{,}664.0$

$SS_{CUBIC} = \dfrac{(\hat{\lambda}_{CUBIC})^2}{20/16} = 44556.8$

D5.

ANOVA TABLE
LINEAR REGRESSION

SOURCE	DF	SS	MS	F
LINEAR	1	622,339.2	622,339.2.	$\boxed{9.45.}$
ERROR	62	4,083,480.8	65,862.6	
TOTAL	63	4,705,820		

α	F(1, 62).	DECISION
.10	2.788	REJECT
.05	3.996.	REJECT
.01	7.062	REJECT

REJECT H_0 NO LINEAR ASSOCIATION AT $\alpha = .01$ (AND $\alpha = .05$ AND $\alpha = .10$).

D6

ANOVA TABLE

LACK OF LINEAR FIT

SOURCE	DF.	SS.	MS	
REGRESSION	1	622,339.2		
LACK OF FIT	2	56,220.8	28110.4	$\boxed{0.42}$
PURE ERROR	60	4,027,260	67,121	
TOTAL	63	4,705,820		

α	F(2,60).	
.10	2.393.	ACCEPT
.05	3.150	ACCEPT
.01	4.977.	ACCEPT

ACCEPT H_0 LINEAR MODEL ADEQUATE AT $\alpha = .10$

(AND $\alpha = .05$ AND $\alpha = .01$).

CHOOSE DOSAGE AS SMALL AS POSSIBLE

E form

7. The correlation matrix of the random variables $(Y_1, Y_2, Y_3, Y_4)^T$ is $\begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$,

   $0 < \rho < 1$, and each random variable has variance σ^2. Let
   $W_1 = Y_1 + Y_2 + Y_3 + Y_4$, and let
   $W_2 = Y_1 + 2Y_2 + 3Y_3 + 4Y_4$.
   Find the variance covariance matrix of $(W_1, W_2)^T$. Please enter $cov(W_1, W_2)$ in the text box beneath this question. This problem is worth 50 points.

7E $\quad M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

$$\mathrm{vcv}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \sigma^2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 1+3\rho & 1+3\rho & 1+3\rho & 1+3\rho \\ 1+9\rho & 2+8\rho & 3+7\rho & 4+6\rho \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 4+12\rho & 10+30\rho \\ 10+30\rho & 30+70\rho \end{bmatrix}$$

$$\boxed{\mathrm{Cov}(W_1, W_2) = (10+30\rho)\,\sigma^2.}$$

F form

7. The correlation matrix of the random variables $(Y_1, Y_2, Y_3, Y_4)^T$ is $\begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$,

   $0 < \rho < 1$, and each random variable has variance σ^2. Let
   $W_1 = Y_1 + Y_2 + Y_3 + Y_4$, and let
   $W_2 = Y_1 + 3Y_2 + 3Y_3 + Y_4$. Find the variance covariance matrix of $(W_1, W_2)^T$.
   Please enter $cov(W_1, W_2)$ in the text box beneath this question. This problem is worth 50 points.

7F

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

$$\text{vcv}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 1+3\rho & 1+3\rho & 1+3\rho & 1+3\rho \\ 1+7\rho & 3+5\rho & 3+5\rho & 1+7\rho \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 4+12\rho & 8+24\rho \\ 8+24\rho & 20+44\rho \end{bmatrix}$$

$$\boxed{Cov(W_1, W_2) = \sigma^2(8+24\rho).}$$

GGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGG

8. The random variable $Y > 0$ has mean μ_Y and standard deviation σ_Y such that
$$\ln(\sigma_Y) = \beta_0 + \beta_1 \ln(\mu_Y).$$
Let f be a differentiable function with continuous first derivative. For $W = f(Y)$, find the approximate value of $var(W)$. Prove your result. This problem is worth 60 points.

End of Examination

$$\ln(\sigma_Y) = \beta_0 + \beta_1 \ln(\mu_Y)$$
$$= \beta_0 + \ln(\mu_Y^{\beta_1})$$
$$\sigma_Y = e[\ln(\sigma_Y)] = e[\beta_0 + \ln(\mu_Y^{\beta_1})]$$
$$= e^{\beta_0} \mu_Y^{\beta_1}$$
$$VAR(W) \cong [f'(E(Y))]^2 \; VAR\,Y.$$
$$VAR(W) \cong [f'(\mu_Y)]^2 (e^{\beta_0} \mu_Y^{\beta_1})^2$$
$$VAR(W) \cong e^{2\beta_0} \mu_Y^{2\beta_1} [f'(\mu_Y)]^2.$$