# AMS 361 R01/R03

## Week 8 : Constant coefficients (Homogeneous (Second order, Higher order))

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# Constant coefficients (Homogeneous (second order))

Consider the differential equation

$$ay'' + by' + cy = 0,$$

where $a, b, c$ are constants.

## Steps

- Obtain the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

and its solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

# Constant coefficients (Homogeneous (second order))

**Steps**

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

2. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}.$$

3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$, then the general solution is

$$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x).$$

# Constant coefficients (Homogeneous (second order))

**Example**

Find the PS to the following IVP:

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 6 \end{cases}.$$

$$y'' + 3y' + 2y = 0$$

- Obtain the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

and its solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda_1 = -2, \qquad \lambda_2 = -1$$

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}.$$

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{-x}} \qquad G.S.$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 6 \end{cases}$$

$$y(x) = C_1 e^{-2x} + C_2 e^{-x}$$

$$y'(x) = -2C_1 e^{-2x} - C_2 e^{-x}$$

$$\begin{cases} C_1 e^{-2\cdot 0} + C_2 e^{-0} = 1 \\ -2C_1 e^{-2\cdot 0} - C_2 e^{-0} = 6 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 1 & (1) \\ -2C_1 - C_2 = 6 & (2) \end{cases}$$

$$(1)+(2): \quad C_1 + \cancel{C_2} - 2C_1 - \cancel{C_2} = 1 + 6 = 7 \Rightarrow C_1 = -7$$

$$\begin{cases} C_1 = -7 \\ C_2 = 8 \end{cases}$$

$$\boxed{y(x) = -7e^{-2x} + 8e^{-x}} \qquad P.S.$$

# Constant coefficients (Homogeneous (second order))

**Example**

Find the GS to the following DE:

$$9y'' - 12y' + 4y = 0.$$

$9y'' - 12y' + 4y = 0$

- Obtain the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

and its solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$9\lambda^2 - 12\lambda + 4 = 0$

$(3\lambda - 2)^2 = 0$

$\lambda_1 = \lambda_2 = \frac{2}{3} = \lambda$

2. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}.$$

$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$

$\boxed{y(x) = C_1 e^{\frac{2}{3}x} + C_2 x e^{\frac{2}{3}x}}$ G.S.

# Constant coefficients (Homogeneous (second order))

**Example (Final Problem 1, Spring 2022)**

Find the GS of the DE and the solution of the IVP:

$$\begin{cases} y'' - 4y' + 5y = 0 \\ y(0) = 1, y'(0) = 2 \end{cases}.$$

$y'' - 4y' + 5y = 0$

- Obtain the *characteristic equation*

$$a\lambda^2 + b\lambda + c = 0,$$

and its solutions

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$\lambda^2 - 4\lambda + 5 = 0$

$\lambda^2 - 4\lambda + 4 + 1 = 0$

$(\lambda - 2)^2 + 1 = 0$

$(\lambda - 2)^2 = -1$

$\lambda - 2 = \pm i$

$\lambda_1 = 2 + i, \quad \lambda_2 = 2 - i \qquad (\alpha = 2, \ \beta = 1)$

3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$, then the general solution is

$$y(x) = C_1 e^{\alpha x} \cos(\beta x) + C_2 e^{\alpha x} \sin(\beta x).$$

$y(x) = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$

$\boxed{y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x}$ G.S.

$\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases}$

$y(x) = C_1 e^{2x} \cos x + C_2 e^{2x} \sin x$

$y'(x) = C_1 (2e^{2x} \cos x + e^{2x}(-\sin x)) + C_2 (2e^{2x} \sin x + e^{2x} \cos x)$

$= (2C_1 + C_2) e^{2x} \cos x + (C_1 + 2C_2) e^{2x} \sin x$

$\begin{cases} C_1 e^{2\cdot 0} \cos 0 + C_2 e^{2\cdot 0} \sin 0 = 1 \\ (2C_1 + C_2) e^{2\cdot 0} \cos 0 + (C_1 + 2C_2) e^{2\cdot 0} \sin 0 = 2 \end{cases}$

$$\Rightarrow \begin{cases} C_1 = 1 \\ 2C_1 + C_2 = 2 \end{cases}$$

$$\Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 0 \end{cases}$$

$$y(x) = 1 \cdot e^{2x} \cos x + 0 \cdot e^{2x} \sin x$$

$$\boxed{y(x) = e^{2x} \cos x}$$

P.S.

# Constant coefficients (Homogeneous)

Consider the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0,$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ are constants.

**Steps**

- Obtain the *characteristic equation*

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0,$$

  and its solutions

$$\lambda_1, \lambda_2 \ldots, \lambda_n.$$

- The general solution is

$$y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n.$$

# Constant coefficients (Homogeneous)

## Steps

1. For $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_k \in \mathbb{R}$,

$$y_1 = e^{\lambda_1 x},\ y_2 = e^{\lambda_2 x},\ \ldots,\ y_k = e^{\lambda_k x}.$$

2. For $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = \lambda_{k+r} = \lambda \in \mathbb{R}$,

$$y_{k+1} = e^{\lambda x},\ y_{k+2} = xe^{\lambda x},\ y_{k+3} = x^2 e^{\lambda x},\ \ldots,\ y_{k+r} = x^{r-1} e^{\lambda x}.$$

3. For $\lambda_{m+1} = \lambda_{m+3} = \lambda_{m+5} = \cdots = \lambda_{m+2l-1} = \alpha + \beta i$, $\lambda_{m+2} = \lambda_{m+4} = \lambda_{m+6} = \cdots = \lambda_{m+2l} = \alpha - \beta i \in \mathbb{C}$,

$$y_{m+1} = e^{\alpha x}\cos(\beta x),\ y_{m+3} = xe^{\alpha x}\cos(\beta x),\ y_{m+5} = x^2 e^{\alpha x}\cos(\beta x),$$
$$\cdots,\ y_{m+2l-1} = x^{l-1} e^{\alpha x}\cos(\beta x),$$
$$y_{m+2} = e^{\alpha x}\sin(\beta x),\ y_{m+4} = xe^{\alpha x}\sin(\beta x),\ y_{m+6} = x^2 e^{\alpha x}\sin(\beta x),$$
$$\cdots,\ y_{m+2l} = x^{l-1} e^{\alpha x}\sin(\beta x).$$

# Constant coefficients (Homogeneous)

**Example**

Find the GS to the following DE

$$(D-1)^3(D-2)^2(D-3)(D^2+9)y = 0,$$

where $D = \frac{d}{dx}$.

$(D-1)^3(D-2)^2(D-3)(D^2+9)y=0$ , $D=\frac{d}{dx}$

- Obtain the *characteristic equation*

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1, \lambda_2 \ldots, \lambda_n.$$

Replace every "D" by "$\lambda$"

$(\lambda-1)^3(\lambda-2)^2(\lambda-3)(\lambda^2+9)=0$

$\lambda_1=\lambda_2=\lambda_3=1$,

$\lambda_4=\lambda_5=2$,

$\lambda_6=3$

$\lambda_7=3i$, $\lambda_8=-3i$

2. For $\lambda_{k+1}=\lambda_{k+2}=\lambda_{k+3}=\cdots=\lambda_{k+r}=\lambda\in\mathbb{R}$,

$$y_{k+1}=e^{\lambda x},\ y_{k+2}=xe^{\lambda x},\ y_{k+3}=x^2e^{\lambda x},\ \ldots,\ y_{k+r}=x^{r-1}e^{\lambda x}.$$

$\lambda_1=\lambda_2=\lambda_3=1$

$y_1=e^{\lambda_1 x}=e^x$

$y_2=xe^{\lambda_2 x}=xe^x$

$y_3=x^2e^{\lambda_3 x}=x^2e^x$

2. For $\lambda_{k+1}=\lambda_{k+2}=\lambda_{k+3}=\cdots=\lambda_{k+r}=\lambda\in\mathbb{R}$,

$$y_{k+1}=e^{\lambda x},\ y_{k+2}=xe^{\lambda x},\ y_{k+3}=x^2e^{\lambda x},\ \ldots,\ y_{k+r}=x^{r-1}e^{\lambda x}.$$

$\lambda_4=\lambda_5=2$,

$y_4=e^{\lambda_4 x}=e^{2x}$

$y_5=xe^{\lambda_5 x}=xe^{2x}$

1. For $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_k \in \mathbb{R}$,

$$y_1 = e^{\lambda_1 x}, \; y_2 = e^{\lambda_2 x}, \; \ldots, \; y_k = e^{\lambda_k x}.$$

$\lambda_6 = 3$

$y_6 = e^{\lambda_6 x} = e^{3x}$

3. For $\lambda_{m+1} = \lambda_{m+3} = \lambda_{m+5} = \cdots = \lambda_{m+2l-1} = \alpha + \beta i$,
$\lambda_{m+2} = \lambda_{m+4} = \lambda_{m+6} = \cdots = \lambda_{m+2l} = \alpha - \beta i \in \mathbb{C}$,

$$y_{m+1} = e^{\alpha x}\cos(\beta x), \; y_{m+3} = xe^{\alpha x}\cos(\beta x), \; y_{m+5} = x^2 e^{\alpha x}\cos(\beta x),$$
$$\cdots, \; y_{m+2l-1} = x^{l-1}e^{\alpha x}\cos(\beta x),$$
$$y_{m+2} = e^{\alpha x}\sin(\beta x), \; y_{m+4} = xe^{\alpha x}\sin(\beta x), \; y_{m+6} = x^2 e^{\alpha x}\sin(\beta x),$$
$$\cdots, \; y_{m+2l} = x^{l-1}e^{\alpha x}\sin(\beta x).$$

$\lambda_7 = 3i = 0 + 3i = \alpha + \beta i$, $\quad \lambda_8 = -3i = 0 - 3i = \alpha - \beta i$

$(\alpha = 0, \; \beta = 3)$

$y_7 = e^{\alpha x}\cos\beta x = e^{0\cdot x}\cos 3x = \cos 3x$

$y_8 = e^{\alpha x}\sin\beta x = e^{0\cdot x}\sin 3x = \sin 3x$

- The general solution is

$$y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n.$$

$y(x) = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4 + C_5 y_5 + C_6 y_6 + C_7 y_7 + C_8 y_8$

$= C_1 e^x + C_2 x e^x + C_3 x^2 e^x + C_4 e^{2x} + C_5 x e^{2x}$

$+ C_6 e^{3x} + C_7 \cos 3x + C_8 \sin 3x$

# Constant coefficients (Homogeneous)

**Example (Test 3 Problem 4, Spring 2022)**

Find the G.S. of the DE:

$$y^{(3)} + 3y'' + 4y' + 12y = 0.$$

$$y''' + 3y'' + 4y' + 12y = 0$$

- Obtain the *characteristic equation*

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1, \lambda_2 \ldots, \lambda_n.$$

$$\lambda^3 + 3\lambda^2 + 4\lambda + 12 = 0$$

$$\lambda^2(\lambda+3) + 4(\lambda+3) = 0$$

$$(\lambda+3)(\lambda^2+4) = 0$$

$$(\lambda+3)(\lambda^2-(2i)^2) = 0$$

$$(\lambda+3)(\lambda-2i)(\lambda+2i) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i$$

1. For $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_k \in \mathbb{R}$,

$$y_1 = e^{\lambda_1 x}, \ y_2 = e^{\lambda_2 x}, \ \ldots, \ y_k = e^{\lambda_k x}.$$

$$\lambda_1 = -3 \Rightarrow y_1 = e^{\lambda_1 x} = e^{-3x}$$

3. For $\lambda_{m+1} = \lambda_{m+3} = \lambda_{m+5} = \cdots = \lambda_{m+2l-1} = \alpha + \beta i$, $\lambda_{m+2} = \lambda_{m+4} = \lambda_{m+6} = \cdots = \lambda_{m+2l} = \alpha - \beta i \in \mathbb{C}$,

$$y_{m+1} = e^{\alpha x}\cos(\beta x), \ y_{m+3} = xe^{\alpha x}\cos(\beta x), \ y_{m+5} = x^2e^{\alpha x}\cos(\beta x),$$
$$\cdots, \ y_{m+2l-1} = x^{l-1}e^{\alpha x}\cos(\beta x),$$
$$y_{m+2} = e^{\alpha x}\sin(\beta x), \ y_{m+4} = xe^{\alpha x}\sin(\beta x), \ y_{m+6} = x^2e^{\alpha x}\sin(\beta x),$$
$$\cdots, \ y_{m+2l} = x^{l-1}e^{\alpha x}\sin(\beta x).$$

$$\lambda_2 = 2i = 0 + 2i, \quad \lambda_3 = -2i = 0 - 2i \quad (\alpha = 0, \quad \beta = 2)$$

$$y_2 = e^{\alpha x}\cos\beta x = e^{0x}\cos 2x = \cos 2x$$

$$y_3 = e^{\alpha x}\sin\beta x = e^{0\cdot x}\sin 2x = \sin 2x$$

- The general solution is

$$y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n.$$

$$y(x) = C_1 y_1 + C_2 y_2 + C_3 y_3$$

$$\boxed{y(x) = C_1 e^{-3x} + C_2 \cos 2x + C_3 \sin 2x}$$

G.S.

# Constant coefficients (Homogeneous)

**Example (Final Problem 5, Spring 2022)**

Find the GS of the DE and the solution of the IVP:

$$\begin{cases} y''' - 3y'' + 7y' - 5y = 0 \\ y(0) = 1, y'(0) = y''(0) = 0 \end{cases}.$$

$$y''' - 3y'' + 7y' - 5y = 0$$

- Obtain the *characteristic equation*

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1, \lambda_2 \ldots, \lambda_n.$$

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

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Method 1:

$$\lambda^3 - \lambda^2 - 2\lambda^2 + 2\lambda + 5\lambda - 5 = 0$$

$$\lambda^2(\lambda-1) - 2\lambda(\lambda-1) + 5(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2 - 2\lambda + 5) = 0$$

$$(\lambda-1)\left((\lambda-1)^2 + 4\right) = 0$$

$$(\lambda-1)\left((\lambda-1)^2 - (2i)^2\right) = 0$$

$$(\lambda-1)(\lambda-1+2i)(\lambda-1-2i) = 0$$

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Method 2:

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = 0$$

Guess $\lambda = 0$ ? ✗ $LHS = 0^3 - 3\cdot 0^2 + 7\cdot 0 - 5 = -5 \neq 0 = RHS$

Guess $\lambda = -1$ ✗ $LHS = (-1)^3 - 3(-1)^2 + 7(-1) - 5$

$= -1 - 3 - 7 - 5 = -16 \neq 0 = RHS$

Guess $\lambda = 1$ ✓ $LHS = 1^3 - 3\cdot 1^2 + 7\cdot 1 - 5$

$= 1 - 3 + 7 - 5 = 0 = RHS$

Now I know $\lambda = 1$ is a solution

So $\lambda^3 - 3\lambda^2 + 7\lambda - 5 = (\lambda - 1)(\qquad) = 0$

$$
\begin{array}{r|l}
 & \lambda^2 - 2\lambda + 5 \\
\hline
\lambda - 1 & \lambda^3 - 3\lambda^2 + 7\lambda - 5 \\
 & \lambda^3 - \lambda^2 \\
\hline
 & 0 - 2\lambda^2 + 7\lambda - 5 \\
 & \quad -2\lambda^2 + 2\lambda \\
\hline
 & \qquad 0 + 5\lambda - 5 \\
 & \qquad\quad 5\lambda - 5 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$\lambda^3 - 3\lambda^2 + 7\lambda - 5 = (\lambda - 1)(\lambda^2 - 2\lambda + 5) = 0$$

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$$(\lambda - 1)(\lambda - 1 + 2i)(\lambda - 1 - 2i) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 1 - 2i, \quad \lambda_3 = 1 + 2i \qquad (\alpha = 1, \quad \beta = 2)$$

1. For $\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_k \in \mathbb{R}$,

$$y_1 = e^{\lambda_1 x}, \; y_2 = e^{\lambda_2 x}, \; \ldots, \; y_k = e^{\lambda_k x}.$$

$$\lambda_1 = 1$$

$$y_1 = e^{\lambda_1 x} = e^{x}$$

3. For $\lambda_{m+1} = \lambda_{m+3} = \lambda_{m+5} = \cdots = \lambda_{m+2l-1} = \alpha + \beta i$,
$\lambda_{m+2} = \lambda_{m+4} = \lambda_{m+6} = \cdots = \lambda_{m+2l} = \alpha - \beta i \in \mathbb{C}$,

$$y_{m+1} = e^{\alpha x}\cos(\beta x), \; y_{m+3} = xe^{\alpha x}\cos(\beta x), \; y_{m+5} = x^2 e^{\alpha x}\cos(\beta x),$$
$$\cdots, \; y_{m+2l-1} = x^{l-1}e^{\alpha x}\cos(\beta x),$$
$$y_{m+2} = e^{\alpha x}\sin(\beta x), \; y_{m+4} = xe^{\alpha x}\sin(\beta x), \; y_{m+6} = x^2 e^{\alpha x}\sin(\beta x),$$
$$\cdots, \; y_{m+2l} = x^{l-1}e^{\alpha x}\sin(\beta x).$$

$$\lambda_2 = 1 - 2i, \quad \lambda_3 = 1 + 2i \qquad (\alpha = 1, \quad \beta = 2)$$

$$y_2 = e^{\alpha x}\cos\beta x = e^{x}\cos 2x$$

$$y_3 = e^{\alpha x}\sin\beta x = e^{x}\sin 2x$$

- The general solution is

$$y(x) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n.$$

$$y(x) = C_1 y_1 + C_2 y_2 + C_3 y_3$$

$$\boxed{y(x) = C_1 e^x + C_2 e^x \cos 2x + C_3 e^x \sin 2x} \quad \text{G.S.}$$

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \\ y''(0) = 0 \end{cases}$$

$$y(x) = C_1 e^x + C_2 e^x \cos 2x + C_3 e^x \sin 2x$$

$$y'(x) = C_1 e^x + (C_2 + 2C_3) e^x \cos 2x + (C_3 - 2C_2) e^x \sin 2x$$

$$\begin{aligned} y''(x) &= C_1 e^x + (C_2 + 2C_3) e^x \cos 2x + (C_3 - 2C_2) e^x \sin 2x \\ &\quad - (2C_2 + 4C_3) e^x \sin 2x + (2C_3 - 4C_2) e^x \cos 2x \\ &= C_1 e^x + (-3C_2 + 4C_3) e^x \cos 2x - (3C_2 + 4C_3) e^x \sin 2x \end{aligned}$$

$$\begin{cases} y(0) = C_1 + C_2 = 1 \\ y'(0) = C_1 + C_2 + 2C_3 = 0 \\ y''(0) = C_1 - 3C_2 + 4C_3 = 0 \end{cases}$$

$$\begin{cases} C_1 = \frac{5}{4} \\ C_2 = -\frac{1}{4} \\ C_3 = -\frac{1}{2} \end{cases}$$

$$y(x) = C_1 e^x + C_2 e^x \cos 2x + C_3 e^x \sin 2x$$

$$\boxed{y(x) = \frac{5}{4} e^x - \frac{1}{4} e^x \cos 2x - \frac{1}{2} e^x \sin 2x} \quad \text{P.S.}$$