AMS 315, Spring 2023

Study Guide
Chapter Six, Inferences Comparing Two Population Central Values

This chapter presents the two independent sample t-test, which is the most important single procedure in statistics. It is the basis of evaluating a clinical trial.

The framework is that a random sample of size n_1 is taken from the random variable Y_1 which has mean $E(Y_1) = \mu_1$ and $\text{var}(Y_1) = \sigma_1^2$. An independent random sample of size n_2 is taken from the random variable Y_2 which has mean $E(Y_2) = \mu_2$ and $\text{var}(Y_2) = \sigma_2^2$. The null hypothesis is $H_0 : E(Y_1 - Y_2) = 0$. In the paired variance (equal variance) t-test, $\sigma_1^2 = \sigma_2^2 = \sigma^2$. In a clinical trial, the standard Food and Drug Administration alternative hypothesis is the two-sided alternative $H_1 : E(Y_1 - Y_2) \neq 0$.

6.1. Introduction and abstract of research study

Theorem 6.1 is the basis from probability theory for the analysis in this chapter. I suggest that you also master the more general result (using the specifications in Theorem 6.1) that $\text{var}(Y_1 - Y_2) = \text{var}(Y_1) + \text{var}(Y_2) - 2\,\text{cov}(Y_1, Y_2)$. The box describing the properties of the sampling distribution for the difference between two means contains fundamental material. Review the case study.

6.2. Inferences about $\mu_1 - \mu_2$: independent samples

This section first specifies the pooled variance t-test; that is, the procedure to be used when $\text{var}(Y_1) = \text{var}(Y_2)$. Under Fisher's statement of the null hypothesis that the only difference between the two distributions is an arbitrary label, the assumption of equal variance is reasonable. The box defining the confidence interval for $\mu_1 - \mu_2$ specified the pooled variance estimate of the common variance and is fundamental material which will be tested repeatedly. The pooled variance estimate is a consolidation of two independent variance estimates and must have a value between the two estimates. Also, note the box defining the statistical test. This section also specified the approximate t-test for independent samples with unequal variance. In practice, the unequal variance t-test and unequal variance confidence interval is my choice of the best procedure. I will not ask you to calculate the unequal variance t-test procedures or to calculate the approximate degrees of freedom in an examination. These values are calculated in most statistical packages.

6.3. A nonparametric alternative: the Wilcoxon rank sum test

Statisticians, especially those working in psychology, typically recommend the Wilcoxon test because it does not assume normality. I will not ask homework or examination questions on this procedure. R.A. Fisher and E.J.G. Pitman recommend a permutation test approach to dealing with the issue of sampling from a non-normal distribution. Pitman showed that the moments of the equal variance two independent sample t-test under a permutation distribution are equal to the moments of this test assuming normality. Consequently, statisticians say that the equal variance independent sample t-test has robustness of level of significance to the assumption of normality. The crucial issue in applying a t-test is the independence of the observations rather than the normality. The independence is guaranteed when there is random assignment of participants to treatment group in a clinical trial.

6.4. Inferences about $\mu_1 - \mu_2$: paired data

The sampling model appropriate for a paired t-test is that there is a random sample of *n* elements. The data from each element *i* is a pair of random variables (Y_{1i}, Y_{2i}). The null hypothesis is $H_0 : E(Y_1 - Y_2) = 0$ as before. The paired t-test is a one-sample t-test using the data $D_i = Y_{1i} - Y_{2i}$. The procedure is used for correlated data such as the difference between post-test and pre-test measures to assess the effectiveness of a training program. There are homework questions about this procedure and will be examination questions about it.

6.5. A nonparametric alternative: the Wilcoxon signed-rank test

This is an effective procedure. I will not ask examination or homework questions about it.

6.6. Choosing sample sizes for inferences about $\mu_1 - \mu_2$

The fundamental design equation from the previous chapter can be used here. First, a good design principle is to use the same sample size for each group. The reason is that the robustness of the t-test to non-normal null distributions is better when the sample sizes are equal. Additionally, one would want to use a larger sample size in the group with the larger variance. An incorrect choice can lead to a design increased type II error rate.

Under the null hypothesis with equal sample size *n* for each group, the test statistic $\bar{Y}_1 - \bar{Y}_2$ is $N(0, \frac{2\sigma^2}{n})$. When $E(\bar{Y}_1 - \bar{Y}_2) = E_1$ and the variances are the same, then test statistic $\bar{Y}_1 - \bar{Y}_2$ is $N(E_1, \frac{2\sigma^2}{n})$. The structure of the null and alternative distributions is the same as the last chapter. That is, you can use the formula $\sqrt{n} \ge \frac{|z_\alpha|\sigma_0 + |z_\beta|\sigma_1}{|E_0 - E_1|}$ when $\alpha \le \frac{1}{2}$ and $\beta \le \frac{1}{2}$ with the specification that $E_0 = 0$, E_1 is the value specified, $\sigma_0 = \sigma_1 = \sigma\sqrt{2}$.

6.7 Research study: effects of oil spill on plant growth

Read the case study.

6.8. Summary and key formulas

The summary is complete. Again, I will not test on the Wilcoxon procedures. You should be aware that they exist and that they are accepted procedures. The validity of the null distribution of the t-test under a permutation distribution is my justification for not focusing on these procedures.

Example Past Examination Problems

1. A research team took a random sample of 6 observations from a normally distributed random variable *Y* and observed that $\bar{y}_6 = 388.6$ and $s_Y^2 = 152.2$, where $\bar{y}_6$ is the average of the six observations sampled from *Y* and s_Y^2 is the unbiased estimate of $\text{var}(Y)$. A second research team took a random sample of 5 observations from a normally distributed random variable *X* and observed that $\bar{x}_5 = 379.6$ and $s_X^2 = 139.5$, where $\bar{x}_5$ is the average of the five observations sampled from *X* and s_X^2 is the unbiased estimate of $\text{var}(X)$. Test the null hypothesis $H_0 : E(X) = E(Y)$ against the alternative $H_1 : E(X) \neq E(Y)$ at the 0.10, 0.05, and 0.01 levels of significance using the pooled variance t-test. Answer: $s_P^2 = 146.56$, $t_9 = -1.23$, accept at 0.10, 0.05, and 0.01 levels of significance.

2. A research team took a random sample of 3 observations from a normally distributed random variable *Y* and observed that $\bar{y}_3 = 254.1$ and $s_Y^2 = 214.2$, where $\bar{y}_3$ is the average of the three observations sampled from *Y* and s_Y^2 is the unbiased estimate of $\text{var}(Y)$. A second research team took a random sample of 4 observations from a normally distributed random variable *X* and observed that $\bar{x}_4 = 359.1$ and $s_X^2 = 176.9$, where $\bar{x}_4$ is the average of the four observations sampled from *X* and s_X^2 is the unbiased estimate of $\text{var}(X)$. Calculate the 99% confidence interval for $E(X) - E(Y)$ using the pooled variance estimator. Answer: $s_P^2 = 191.82$; the 99% confidence interval is from 62.3 to 147.7.

3. In a clinical trial, 50 patients suffering from an illness will be randomly assigned to one of two groups so that 25 receive an experimental treatment and 25 receive the best available treatment. The random variable *X* is the response of a patient to the experimental medicine, and the random variable *B* is the response of a patient to the best currently available treatment. The random variables *X* and *B* are normally distributed with $\sigma_X = \sigma_B = 500$ under both the null and alternative

distributions. The null hypothesis to be tested is that $E(X)-E(B)=0$ against the alternative that $E(X)-E(B)>0$ at the 0.01 level of significance. What is the probability of a Type II error for the test of the null hypothesis when $E(X)-E(B)=500$?

Answer: $\beta = 0.113$

4. In a clinical trial, *2J* patients suffering from an illness will be randomly assigned to one of two groups so that *J* will receive an experimental treatment and *J* will receive the best available treatment. The random variable *X* is the response of a patient to the experimental medicine, and the random variable *B* is the response of a patient to the best currently available treatment. The random variables *X* and *B* are normally distributed and have $\sigma_X = \sigma_B = 500$ under both the null and alternative distributions. The null hypothesis to be tested is that $E(X)-E(B)=0$ against the alternative that $E(X)-E(B)>0$ at the 0.005 level of significance. What is the number *J* in each group that would have to be taken so that the probability of a Type II error for the test of the null hypothesis specified in the common section is 0.01 when $E(X)-E(B)=250$?

   Answer: There should be at least 193 in each group. That is, to detect a difference equal to 1/2 of the standard deviation, researchers need about 200 in each group.

5. A research time wished to estimate the reduction of the density of contaminant in a liquid due to filtering the liquid. They filtered four samples, called A, B, C, and D. Find the 99% confidence interval for the expected reduction in the density of contaminant using the data in the table below:

Sample	Density of contaminant before filtering	Density of contaminant after filtering
A	132	87
B	205	163
C	81	35
D	423	357

   Answer: $s_D^2 = 120.25$; the 99% confidence interval for the expected reduction in density is from 17.7 to 81.8.

6. In a clinical trial, *2J* patients suffering from an illness will be randomly assigned to one of two groups so that *J* will receive an experimental treatment and *J* will receive the best available treatment. The random variable *X* is the response of a patient to the experimental medicine, and the random variable *B* is the response of a patient to the best currently available treatment. The random variables *X* and *B* are normally distributed. The null hypothesis to be tested is that $E(X)-E(B)=0$ against the alternative that $E(X)-E(B)>0$ at the $\alpha, \alpha \le 0.5$, level of significance. When the null hypothesis is true, $\text{var}(X)=\text{var}(B)=\sigma_0^2$. When the alternative hypothesis is true, $\text{var}(B)=\sigma_0^2$, but $\text{var}(X)=\sigma_1^2 > \sigma_0^2$. What is the number *J* in each group that would have to be

taken so that the probability of a Type II error for the test of the null hypothesis specified in the common section is $\beta, \beta \le 0.5,$ when $E(X) - E(B) = \Delta > 0$?

Answer: $\sqrt{J} \ge \frac{|z_\alpha| \sqrt{\sigma_0^2 + \sigma_0^2} + |z_\beta| \sqrt{\sigma_0^2 + \sigma_1^2}}{\Delta}$. The scenario can be realistic. Ratio scale random variables with greater mean typically have greater variance; for example, the Poisson and the chi-square distributions. This greater variance requires a somewhat greater sample size in study design.

End of Study Guide for Chapter 6