13 Variance-Covariance Matrix of MY

The correlation matrix of the random variables $(Y_1, Y_2, Y_3, Y_4)^T$ is $\begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$, $0 < \rho < 1$,

and each random variable has variance σ^2. Let

$$W_1 = Y_1 + Y_2 - Y_3 - Y_4, \text{ and}$$
$$W_2 = -3Y_1 - Y_2 + Y_3 + 3Y_4.$$

Find the variance covariance matrix of $(W_1, W_2)^T$.

FIRST FIND M SUCH THAT $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = MY$

$$\begin{array}{c|cccc} & Y_1 & Y_2 & Y_3 & Y_4 \\ \hline W_1 & 1 & 1 & -1 & -1 \\ W_2 & -3 & -1 & 1 & 3 \end{array}.$$

THAT IS, $M = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -3 & -1 & 1 & 3 \end{bmatrix}$.

$VCV(W) = M(\sigma^2 C) M^T$ WHERE $\sigma^2 C = \sigma^2 \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix}$.

SECOND STEP:

$$\sigma^2 MC = \sigma^2 \begin{bmatrix} 1 & 1 & -1 & -1 \\ -3 & -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} 1-\rho & 1-\rho & -1+\rho & -1+\rho \\ -3+3\rho & -1+\rho & 1-\rho & 3-3\rho \end{bmatrix}$$

THIRD STEP

$$\sigma^2 MCM^T = \sigma^2 \begin{bmatrix} 1-\rho & 1-\rho & -1+\rho & -1+\rho \\ -3+3\rho & -1+\rho & 1-\rho & 3-3\rho \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ -1 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= \boxed{\sigma^2 \begin{bmatrix} 4-4\rho & -8+8\rho \\ -8+8\rho & 20-20\rho \end{bmatrix}}$$

$$= \sigma^2\, 4(1-\rho) \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$$

INTERPRETATION

$VAR(w_1) = 4(1-\rho)\sigma^2$

$VAR(w_2) = 20(1-\rho)\sigma^2$

$COV(w_1, w_2) = -8(1-\rho)\sigma^2$

$$CORR(w_1, w_2) = \frac{COV(w_1, w_2)}{\sqrt{VAR(w_1)\,VAR(w_2)}}$$

$$= \frac{-8(1-\rho)\sigma^2}{\sqrt{4(1-\rho)\sigma^2(20(1-\rho)\sigma^2)}}$$

$$= \frac{-8}{\sqrt{80}} = -0.894.$$