AMS 361 R01/R03

Week 11 : Variable coefficients (Homogeneous, Inhomogeneous (Order reduction, Variational principle))

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Variable coefficients (Homogeneous)

Consider the differential equation

$$A(x)y'' + B(x)y' + C(x)y = 0,$$

with a given solution $y_1(x)$.

Steps

- Solve (week 2 Separable)

$$Av'' + (2A(\ln y_1)' + B)v' = 0.$$

- Obtain

$$v(x) = C_1 \int \frac{e^{-\int \frac{B}{A} dx}}{y_1^2} dx + C_2.$$

- The general solution is

$$y(x) = v(x)y_1(x).$$

Variable coefficients (Homogeneous)

Example

Find the GS to the following DE

$$x^2y'' - x(x-1)y' + (x-1)y = 0,$$

with one given solution $y_1(x) = x$.

$$x^2 y'' - x(x-1) y' + (x-1) y = 0$$

$$A = x^2 \qquad B = -x(x-1) \qquad C = x-1$$

$$y_1(x) = x$$

Check:
$$y_1 = x$$
$$y_1' = 1$$
$$y_1'' = 0$$

$$\begin{aligned}
LHS &= x^2 y_1'' - x(x-1) y_1' + (x-1) y_1 \\
&= x^2 \cdot 0 - x(x-1) \cdot 1 + (x-1)x \\
&= 0 - \cancel{x(x-1)} + \cancel{x(x-1)} \\
&= 0 \\
&= RHS
\end{aligned}$$

Yes, y_1 is a solution.

- Solve (week 2 Separable)
$$Av'' + (2A(\ln y_1)' + B)v' = 0.$$

$$Av'' + (2A(\ln y_1)' + B)v' = 0$$
$$x^2 v'' + (2x^2 (\ln x)' + (-x(x-1)))v' = 0$$

- Obtain
$$v(x) = C_1 \int \frac{e^{-\int \frac{B}{A} dx}}{y_1^2} dx + C_2.$$

$$\begin{aligned}
V(x) &= C_1 \int \frac{e^{-\int \frac{B}{A} dx}}{y_1^2} dx + C_2 \\
&= C_1 \int \frac{e^{-\int \frac{-x(x-1)}{x^2} dx}}{x^2} dx + C_2
\end{aligned}$$

$$= C_1 \int \frac{e^{\int \frac{x-1}{x} dx}}{x^2} dx + C_2$$

$$= C_1 \int \frac{e^{\int (1-\frac{1}{x}) dx}}{x^2} dx + C_2$$

$$= C_1 \int \frac{e^{x - \ln x}}{x^2} dx + C_2$$

$$= C_1 \int \frac{e^{x} \cdot e^{\ln x^{-1}}}{x^2} dx + C_2$$

$$= C_1 \int \frac{e^{x}}{x^3} dx + C_2$$

- The general solution is

$$y(x) = v(x) y_1(x).$$

$$y = v\, y_1$$

$$= \left(C_1 \int x^{-3} e^{x} dx + C_2\right) x$$

$$\boxed{y(x) = C_1 x \int x^{-3} e^{x} dx + C_2 x}$$

G.S.

$$= C_1\, \delta(x) + C_2 x$$

will learn it in Laplace Transform

Variable coefficients (Homogeneous)

Example (Test 2 Problem 3, Fall 2019)

Find the GS of

$$x^2y'' - 3xy' + 4y = 0.$$

Remark

Besides this method (week 11 Variable coefficients), we also have another method (week 10 Cauchy-Euler) to solve this ODE.

Guess $y_1 = x^2$

$$x^2 y'' - 3x y' + 4y = 0$$

$$A = x^2 \qquad B = -3x \qquad C = 4$$

Guess $y_1 = x^2$

Check: $\text{LHS} = x^2 (x^2)'' - 3x (x^2)' + 4(x^2)$

$$= x^2(2) - 3x(2x) + 4x^2$$

$$= 2x^2 - 6x^2 + 4x^2$$

$$= 0$$

$$= \text{RHS}$$

- Solve (week 2 Separable)

$$Av'' + (2A(\ln y_1)' + B)v' = 0.$$

$$A v'' + (2A(\ln y_1)' + B)v' = 0$$

$$x^2 v'' + (2x^2 (\ln x^2)' - 3x) v' = 0$$

- Obtain

$$v(x) = C_1 \int \frac{e^{-\int \frac{B}{A} dx}}{y_1^2} dx + C_2.$$

$$v(x) = C_1 \int \frac{e^{-\int \frac{B}{A} dx}}{y_1^2} dx + C_2$$

$$= C_1 \int \frac{e^{-\int \frac{-3x}{x^2} dx}}{(x^2)^2} dx + C_2$$

$$= C_1 \int \frac{e^{3\int \frac{1}{x} dx}}{x^4} dx + C_2$$

$$= C_1 \int \frac{e^{3 \ln x}}{x^4} dx + C_2$$

$$= C_1 \int \frac{e^{\ln x^3}}{x^4}\, dx + C_2$$

$$= C_1 \int \frac{x^3}{x^4}\, dx + C_2$$

$$= C_1 \int \frac{1}{x}\, dx + C_2$$

$$= C_1 \ln x + C_2$$

- The general solution is

$$y(x) = v(x)y_1(x).$$

$$y(x) = v(x)\, y_1(x)$$

$$= (C_1 \ln x + C_2)\, x^2$$

$$\boxed{y(x) = C_2 x^2 + C_1 x^2 \ln x}$$

G.S.

Variable coefficients (Inhomogeneous)

Consider the differential equation

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Steps

1. Method 1: Solve it directly (week 11 Order reduction).
2. Method 2:
   - Find the general solution $y_c = C_1 y_1 + C_2 y_2$ for the associated homogeneous equation (week 8,10,11 Homogeneous)

     $$A(x)y'' + B(x)y' + C(x)y = 0.$$

   - Find a particular solution y_p for the inhomogeneous equation (week 11 Variational principle (VP or VOP))

     $$A(x)y'' + B(x)y' + C(x)y = F(x).$$

   - General solution is $y = y_c + y_p$.

Order reduction

Steps

- Get a solution $y_1(x)$ for the associated homogeneous equation (given in problem or week 8,10 Homogeneous or guess)

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

- Obtain $v'(x)$ by solving (week 3 Linear)

$$y_1(Av'' + (2A(\ln y_1)' + B)v') = F.$$

- Then

$$v(x) = \int v'(x)dx + C_2.$$

- The general solution is

$$y(x) = v(x)y_1(x).$$

Order reduction

Example (Final Problem 1, Fall 2016)

Find the GS of the following DE by any method of your choice:

$$x^2y'' + 5xy' + 4y = x^2 - x^{-2}.$$

Remark

Week 10 Cauchy-Euler or week 11 Order reduction.

Order reduction

Example (Final Problem 2, Fall 2016)

Use LT method and another method to find the PS (Convolutions, if any, must be evaluated):

$$\begin{cases} x'' + x = \tan(t) \\ x(0) = x'(0) = 0 \end{cases} .$$

Remark

LT method: week 14 Laplace transform;
Another method: week 9 Variational principle or week 11 Order reduction.

$$x'' + x = \tan t$$

$A=1 \quad B=0 \quad C=1 \qquad F = \tan t$

- Get a solution $y_1(x)$ for the associated homogeneous equation (given in problem or week 8,10 Homogeneous or guess)

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$x_1 = \cos t, \qquad x_2 = \sin t$$

- Obtain $v'(x)$ by solving (week 3 Linear)

$$y_1(Av'' + (2A(\ln y_1)' + B)v') = F.$$

$$y_1(Av'' + (2A(\ln y_1)' + B)v') = F$$

$$\cos t\,(1 \cdot v'' + (2 \cdot 1\,(\ln \cos t)' + 0)\,v') = \tan t$$

$$\cos t\,\left(v'' + \left(2\,\frac{1}{\cos t}\,(-\sin t)\right)v'\right) = \frac{\sin t}{\cos t}$$

$$v'' - 2\tan t\, v' = \tan t \sec t \qquad (\text{linear})$$

$$P = -2\tan t \qquad Q = \frac{\tan t}{\cos t}$$

$$\rho(t) = e^{\int P(t)\,dt} = e^{\int -2\tan t\,dt} = e^{-2\ln|\sec t|}$$

- $\int \tan x\,dx = \ln|\sec x| + C$

$$= e^{\ln(\sec t)^{-2}} = \frac{1}{\sec^2 t} = \cos^2 t$$

$$v'(t) = \frac{1}{\rho}\left(\int Q\rho\,dt + C_1\right)$$

$$= \frac{1}{\cos^2 t}\left(\int \frac{\tan t}{\cos t}\cos^2 t\,dt + C_1\right)$$

$$= \frac{1}{\cos^2 t}\left(\int \tan t \cos t \, dt + C_1\right)$$
$$= \frac{1}{\cos^2 t}\left(\int \sin t \, dt + C_1\right)$$
$$= \frac{1}{\cos^2 t}\left(-\cos t + C_1\right)$$
$$= -\frac{1}{\cos t} + C_1 \frac{1}{\cos^2 t}$$

- Then
$$v(x) = \int v'(x)dx + C_2.$$

$$v(t) = \int v'(t)\, dt$$
$$= \int \left(-\frac{1}{\cos t} + C_1 \frac{1}{\cos^2 t}\right) dt + C_2$$
$$= -\int \frac{1}{\cos t}\, dt + C_1 \int \frac{1}{\cos^2 t}\, dt + C_2$$

- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
- $\int \sec^2 x \, dx = \tan x + C$

$$= -\ln|\sec t + \tan t| + C_1 \tan t + C_2$$

- The general solution is
$$y(x) = v(x)y_1(x).$$

$$x(t) = v(t)\, x_1(t)$$
$$= \left(-\ln|\sec t + \tan t| + C_1 \tan t + C_2\right)\cos t$$
$$\boxed{x(t) = C_2 \cos t + C_1 \sin t - \cos t \ln|\sec t + \tan t|}$$ G.S.

Example (Test 3 Problem 3, Fall 2022)

Find a GS of

$$xy'' - (2x+1)y' + (x+1)y = x^2e^x.$$

Hint: The homo portion of the DE may have a solution e^x.

$$xy'' - (2x+1)y' + (x+1)y = x^2e^x$$

$A = x$ $\quad B = -(2x+1)$ $\quad C = x+1$ $\quad F = x^2e^x$

- Get a solution $y_1(x)$ for the associated homogeneous equation (given in problem or week 8,10 Homogeneous or guess)

$$A(x)y'' + B(x)y' + C(x)y = 0.$$

$y_1 = e^x$

Check: $LHS = x(e^x)'' - (2x+1)(e^x)' + (x+1)(e^x)$

$= xe^x - (2x+1)e^x + (x+1)e^x$

$= (x - 2x - 1 + x + 1)e^x$

$= 0$

$= RHS$

- Obtain $v'(x)$ by solving (week 3 Linear)

$$y_1(Av'' + (2A(\ln y_1)' + B)v') = F.$$

$y_1(Av'' + (2A(\ln y_1)' + B)v') = F$

$e^x(xv'' + (2x(\ln e^x)' - (2x+1))v') = x^2e^x$

$\cancel{e^x}(xv'' + (2x(x)' - 2x - 1)v') = x^2\cancel{e^x}$

$xv'' + (2x - 2x - 1)v' = x^2$

$xv'' - v' = x^2$ $\quad$ (Linear)

$v'' - \frac{1}{x}v' = x$

$P = -\frac{1}{x}$ $\quad Q = x$

$\rho = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$

$v' = \frac{1}{\rho}\left(\int Q\rho\, dx + C_1\right)$

$$= x\left(\int x \frac{1}{x}\, dx + C_1\right)$$
$$= x\left(\int dx + c_1\right)$$
$$= x\,(x + C_1)$$
$$= x^2 + C_1 x$$

- Then
$$v(x) = \int v'(x)dx + C_2.$$

$$v = \int v'\, dx + C_2$$
$$= \int (x^2 + C_1 x)\, dx + C_2$$
$$= \int x^2 dx + C_1 \int x\, dx + C_2$$
$$= \frac{1}{3} x^3 + \frac{1}{2} C_1 x^2 + C_2$$

- The general solution is
$$y(x) = v(x)y_1(x).$$

$$y = v\, y_1$$
$$= \left(\frac{1}{3} x^3 + \frac{1}{2} C_1 x^2 + C_2\right) e^x$$
$$= \frac{1}{2} C_1 x^2 e^x + C_2 e^x + \frac{1}{3} x^3 e^x$$
$$\boxed{y(x) = C_1 x^2 e^x + C_2 e^x + \frac{1}{3} x^3 e^x}$$

G.S.

Variational principle (Variable coefficients)

Steps

- The *Wronskian* of two solutions y_1, y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

- Compute

$$u_1(x) = \int \frac{-y_2 \frac{f(x)}{A(x)}}{W(y_1, y_2)} dx, \quad u_2(x) = \int \frac{y_1 \frac{f(x)}{A(x)}}{W(y_1, y_2)} dx.$$

  In the indefinite integrals above, it is not necessary to write an arbitrary constant C.

- The particular solution is

$$y_p = u_1(x) y_1 + u_2(x) y_2.$$

Variational principle

Example (Test 3 Problem 1, Spring 2020)

Find, by the method of VOP, the GS of

$$x^2y'' + xy' - \alpha^2 y = x^\alpha + x^{-\alpha}$$

where integer $\alpha > 0$