AMS 361 R01/R03

Week 10 : Cauchy-Euler

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Cauchy-Euler (Special case: Homogeneous, Second order)

Consider the differential equation

$$a_2 x^2 y'' + a_1 x y' + a_0 y = 0,$$

where a_2, a_1, a_0 are constants.

Steps

- Obtain the *characteristic equation* (or *auxiliary equation*)

$$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1 = \frac{-(a_1 - a_2) + \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2},$$

$$\lambda_2 = \frac{-(a_1 - a_2) - \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2}.$$

Cauchy-Euler (Special case: Homogeneous, Second order)

Steps

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is
$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}.$$
2. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution is
$$y(x) = C_1 x^{\lambda} + C_2 x^{\lambda} \ln x.$$
3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$, then the general solution is
$$y(x) = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x).$$

Cauchy-Euler (Special case: Homogeneous, Second order)

Example

Find the GS to the following DE

$$x^2y'' - 2xy' - 10y = 0.$$

$x^2 y'' - 2xy' - 10y = 0$

$a_2 = 1, \quad a_1 = -2, \quad a_0 = -10$

- Obtain the *characteristic equation* (or *auxiliary equation*)

$$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1 = \frac{-(a_1 - a_2) + \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2},$$

$$\lambda_2 = \frac{-(a_1 - a_2) - \sqrt{(a_1 - a_2)^2 - 4a_2 a_0}}{2a_2}.$$

$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0$

$1 \cdot \lambda^2 + (-2-1)\lambda + (-10) = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$(\lambda+2)(\lambda-5) = 0$

$\lambda_1 = -2, \quad \lambda_2 = 5$

1. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}.$$

G.S. $\quad y(x) = C_1 x^{-2} + C_2 x^5$

Cauchy-Euler (Special case: Homogeneous, Second order)

Example (Test 2 Problem 3, Fall 2019)

Find the GS of

$$x^2y'' - 3xy' + 4y = 0.$$

Remark

Besides this method (week 10 Cauchy-Euler), we also have another method (week 11 Variable coefficients) to solve this ODE.

$$x^2 y'' - 3x y' + 4y = 0$$

$$a_2 = 1, \quad a_1 = -3, \quad a_0 = 4$$

- Obtain the *characteristic equation* (or *auxiliary equation*)

$$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1 = \frac{-(a_1 - a_2) + \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2},$$

$$\lambda_2 = \frac{-(a_1 - a_2) - \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2}.$$

$$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0$$

$$\lambda^2 + (-3-1)\lambda + 4 = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda_1 = \lambda_2 = 2$$

2. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution is

$$y(x) = C_1 x^\lambda + C_2 x^\lambda \ln x.$$

$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2} \ln x$$

$$= C_1 x^2 + C_2 x^2 \ln x$$

Cauchy-Euler (Special case: Homogeneous, Second order)

Example

Solve

$$\begin{cases} 4x^2y'' + 17y = 0 \\ y(1) = -1 \\ y'(1) = -\frac{1}{2} \end{cases} .$$

$$4x^2y'' + 17y = 0$$

- Obtain the *characteristic equation* (or *auxiliary equation*)

$$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0,$$

and its solutions

$$\lambda_1 = \frac{-(a_1 - a_2) + \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2},$$

$$\lambda_2 = \frac{-(a_1 - a_2) - \sqrt{(a_1 - a_2)^2 - 4a_2a_0}}{2a_2}.$$

$$4\lambda^2 + (0-4)\lambda + 17 = 0$$

$$4\lambda^2 - 4\lambda + 17 = 0$$

$$\lambda = \frac{-(-\cancel{4}) \pm \sqrt{\cancel{4}^2 - \cancel{4\cdot4}\cdot17}}{2\cdot\cancel{4}} = \frac{1 \pm \sqrt{1-17}}{2} = \frac{1 \pm \sqrt{-16}}{2}$$

$$= \frac{1 \pm 4i}{2}$$

$$\lambda_1 = \frac{1}{2} - 2i, \qquad \lambda_2 = \frac{1}{2} + 2i$$

3. If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i \in \mathbb{C}$, then the general solution is

$$y(x) = C_1x^\alpha \cos(\beta \ln x) + C_2x^\alpha \sin(\beta \ln x).$$

$$\alpha = \frac{1}{2}, \qquad \beta = 2$$

$$y(x) = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha \sin(\beta \ln x)$$

$$= C_1 x^{\frac{1}{2}} \cos(2\ln x) + C_2 x^{\frac{1}{2}} \sin(2\ln x)$$

$$y'(x) = C_1\left(\frac{1}{2}x^{-\frac{1}{2}}\cos(2\ln x) + x^{\frac{1}{2}}\left(-2\frac{1}{x}\sin(2\ln x)\right)\right)$$
$$+ C_2\left(\frac{1}{2}x^{-\frac{1}{2}}\sin(2\ln x) + x^{\frac{1}{2}}\left(2\frac{1}{x}\cos(2\ln x)\right)\right)$$
$$= \left(\frac{1}{2}C_1 + 2C_2\right)x^{-\frac{1}{2}}\cos(2\ln x)$$
$$+ \left(-2C_1 + \frac{1}{2}C_2\right)x^{-\frac{1}{2}}\sin(2\ln x)$$

$$\begin{cases} y(1) = -1 \\ y'(1) = -\frac{1}{2} \end{cases}$$

$\ln 1 = 0$

$$\begin{cases} C_1 1^{\frac{1}{2}}\cos(2\ln 1) + C_2 1^{\frac{1}{2}}\sin(2\ln 1)) \\ \left(\frac{1}{2}C_1 + 2C_2\right)1^{-\frac{1}{2}}\cos(2\ln 1) + \left(-2C_1 + \frac{1}{2}C_2\right)1^{-\frac{1}{2}}\sin(2\ln 1) \end{cases}$$

$$\begin{cases} C_1 = -1 \\ \frac{1}{2}C_1 + 2C_2 = -\frac{1}{2} \end{cases}$$

$$\begin{cases} C_1 = -1 \\ C_2 = 0 \end{cases}$$

$$y(x) = -x^{\frac{1}{2}}\cos(2\ln x)$$

Cauchy-Euler (General case)

Consider the differential equation

$$a_n(x-a)^n y^{(n)} + a_{n-1}(x-a)^{n-1} y^{(n-1)} + \cdots + a_1(x-a)y' + a_0 y = f(x),$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ and $a < x$ are constants.

Steps

1. Method 1: Backward to constant coefficients (week 10 Cauchy-Euler).
   - Let $t = \ln(x-a)$. Then $x = e^t + a$.
   - Solve (week 9 Constant coefficients)

$$\begin{aligned} &(a_n D(D-1)(D-2)\cdots(D-(n-1)) + \cdots \\ &+ a_3 D(D-1)(D-2) + a_2 D(D-1) + a_1 D + a_0)y = f(e^t + a). \end{aligned}$$

   - Finally plug $t = \ln(x-a)$ back in.
2. Method 2: Forward to variable coefficients (week 11 Variable coefficients).

Cauchy-Euler

Example (Final Problem 1, Fall 2016)

Find the GS of the following DE by any method of your choice:

$$x^2y'' + 5xy' + 4y = x^2 - x^{-2}.$$

Remark

Week 10 Cauchy-Euler or week 11 Order reduction.

$x^2 y'' + 5xy' + 4y = x^2 - x^{-2}$

$a_2 = 1, \quad a_1 = 5, \quad a_0 = 4, \quad a = 0, \quad n = 2, \quad f(x) = x^2 - x^{-2}$

Consider the differential equation

$$a_n(x-a)^n y^{(n)} + a_{n-1}(x-a)^{n-1} y^{(n-1)} + \cdots + a_1(x-a)y' + a_0 y = f(x),$$

where $a_0, a_1, \ldots, a_{n-1}, a_n$ and $a < x$ are constants.

- Let $t = \ln(x-a)$. Then $x = e^t + a$.

Let $t = \ln x$

Then $x = e^t$

- Solve (week 9 Constant coefficients)

$$(a_n D(D-1)(D-2)\cdots(D-(n-1)) + \cdots$$
$$+ a_3 D(D-1)(D-2) + a_2 D(D-1) + a_1 D + a_0)y = f(e^t + a).$$

$\begin{pmatrix} n=2 \\ a=0 \end{pmatrix}$

$$(a_2 D(D-1) + a_1 D + a_0)\, y = f(e^t)$$
$$(D(D-1) + 5D + 4)\, y = (e^t)^2 - (e^t)^{-2}$$
$$(D^2 + 4D + 4)\, y = e^{2t} - e^{-2t}$$
$$D^2 y + 4Dy + 4y = e^{2t} - e^{-2t}$$
$$y'' + 4y' + 4y = e^{2t} - e^{-2t}$$

$$y_c'' + 4y_c' + 4y_c = 0$$
$$\lambda^2 + 4\lambda + 4 = 0$$
$$(\lambda + 2)^2 = 0$$
$$\lambda_1 = \lambda_2 = -2$$

double root

$$y_1 = e^{\lambda_1 t} = e^{-2t}$$

$$y_2 = te^{\lambda_2 t} = te^{-2t}$$

$$y_c = C_1 y_1 + C_2 y_2$$
$$= C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y_p'' + 4y_p' + 4y_p = e^{2t} - e^{-2t}$$
$$f(t) = e^{2t} - e^{-2t}$$
$$= f_1 + f_2$$
$$f_1 = e^{2t} \qquad \lambda = -2$$
$$f_2 = -e^{-2t} = -y_1$$
$$te^{-2t} = -y_2$$

$$y_{p_1} = Ae^{2t}$$
$$y_{p_2} = Bt^2 e^{-2t}$$

- If $f_j(x)$ happens to be a single (or double, or triple, ..., or n-fold) root of the corresponding homogeneous equation, i.e. $f_j = y_k$, then multiply your choice of y_{p_j} by x (or x^2, or x^3, ..., or x^n).

$$y_p = y_{p_1} + y_{p_2}$$
$$= Ae^{2t} + Bt^2 e^{-2t}$$
$$y_p' = 2Ae^{2t} + (2Bte^{-2t} - 2Bt^2 e^{-2t})$$
$$y_p'' = 4Ae^{2t} + (2Be^{-2t} - 4Bte^{-2t}) - (4Bte^{-2t} - 4Bt^2 e^{-2t})$$
$$= 4Ae^{2t} + 2Be^{-2t} - 8Bte^{-2t} + 4Bt^2 e^{-2t}$$

$$y_p'' + 4y_p' + 4y_p = e^{2t} - e^{-2t}$$

$$e^{2t} - e^{-2t}$$

$$= (4Ae^{2t} + 2Be^{-2t} - 8Bte^{-2t} + 4Bt^2e^{-2t})$$
$$+ 4(2Ae^{2t} + 2Bte^{-2t} - 2Bt^2e^{-2t})$$
$$+ 4(Ae^{2t} + Bt^2e^{-2t})$$
$$= 4Ae^{2t} + 2Be^{-2t} - \cancel{8Bte^{-2t}} + \cancel{4Bt^2e^{-2t}}$$
$$+ 8Ae^{2t} + \cancel{8Bte^{-2t}} - \cancel{8Bt^2e^{-2t}}$$
$$+ 4Ae^{2t} + \cancel{4Bt^2e^{-2t}}$$
$$= 16Ae^{2t} + 2Be^{-2t}$$
$$= e^{2t} - e^{-2t}$$
$$\begin{cases} 16A = 1 \\ 2B = -1 \end{cases}$$
$$\begin{cases} A = \frac{1}{16} \\ B = -\frac{1}{2} \end{cases}$$
$$y_p = \frac{1}{16}e^{2t} - \frac{1}{2}t^2e^{-2t}$$
$$y = y_c + y_p$$
$$= c_1e^{-2t} + c_2te^{-2t} + \frac{1}{16}e^{2t} - \frac{1}{2}t^2e^{-2t}$$

- Finally plug $t = \ln(x - a)$ back in.

$$t = \ln x$$
$$y = c_1e^{-2\ln x} + c_2(\ln x)e^{-2\ln x} + \frac{1}{16}e^{2\ln x} - \frac{1}{2}(\ln x)^2e^{-2\ln x}$$
$$= c_1e^{\ln x^{-2}} + c_2(\ln x)e^{\ln x^{-2}} + \frac{1}{16}e^{\ln x^2} - \frac{1}{2}(\ln x)^2e^{\ln x^{-2}}$$
$$\boxed{y(x) = c_1x^{-2} + c_2x^{-2}\ln x + \frac{1}{16}x^2 - \frac{1}{2}x^{-2}(\ln x)^2}$$

G.S.

Example (Final Problem 1, Spring 2018)

Use any method of your choice to find the GS of

$$x^2y'' + xy' + 9y = \cos(3\ln x) + 3\sin(\ln x).$$

Cauchy-Euler

Example (Test 3 Problem 1, Spring 2019)

Find and verify the GS of

$$x^2y'' + xy' - 9y = x^3 + x^{-3}.$$

Cauchy-Euler

Example (Final Problem 3, Spring 2022)

Find the GS of the following DE:

$$(x+1)^2y'' + (x+1)y' + y = \cos(\ln(x+1)),$$

where $x+1>0$.

$$(x+1)^2 y'' + (x+1)y' + y = \cos(\ln(x+1))$$

$a_2 = 1$ $\quad a_1 = 1$ $\quad a_0 = 1$

$a = -1$ $\quad n = 2$ $\quad f(x) = \cos(\ln(x+1))$

- Let $t = \ln(x-a)$. Then $x = e^t + a$.

Let $t = \ln(x+1)$

Then $x = e^t - 1$

$f = \cos(\ln(x+1)) = \cos t$

- Solve (week 9 Constant coefficients)

$$(a_n D(D-1)(D-2)\cdots(D-(n-1)) + \cdots$$
$$+a_3 D(D-1)(D-2) + a_2 D(D-1) + a_1 D + a_0)y = f(e^t + a).$$

$$(a_2 D(D-1) + a_1 D + a_0)y = f$$

$$(D(D-1) + D + 1)\ y = \cos t$$

$$(D^2 - D + D + 1)\ y = \cos t$$

$$(D^2 + 1)\ y = \cos t$$

$$D^2 y + y = \cos t$$

$$y'' + y = \cos t$$

$y_c'' + y_c = 0$

$\lambda^2 + 1 = 0$

$\lambda_1 = -i, \quad \lambda_2 = i$

$y_1 = \cos t, \quad y_2 = \sin t$

$y_c = C_1 y_1 + C_2 y_2 = C_1 \cos t + C_2 \sin t$

$$y_p'' + y_p = \cos t$$

$$f(t) = \cos t = y_1$$

$$y_p = t(A\cos t + B\sin t)$$

- If $f_j(x)$ happens to be a single (or double, or triple, ..., or n-fold) root of the corresponding homogeneous equation, i.e. $f_j = y_k$, then multiply your choice of y_{p_j} by x (or x^2, or x^3, ..., or x^n).

$$y_p = At\cos t + Bt\sin t$$

$$y_p' = (A\cos t + B\sin t) + t(-A\sin t + B\cos t)$$

$$y_p'' = -A\sin t + B\cos t + (-A\sin t + B\cos t) + t(-A\cos t - B\sin t)$$

$$= (-2A\sin t + 2B\cos t) + t(-A\cos t - B\sin t)$$

$$y_p'' + y_p = \cos t$$

$$(-2A\sin t + 2B\cos t) + \cancel{t(-A\cos t - B\sin t)}$$

$$+ \cancel{t(A\cos t + B\sin t)} = \cos t$$

$$-2A\sin t + 2B\cos t = \cos t$$

$$\begin{cases} -2A = 0 \\ 2B = 1 \end{cases}$$

$$\begin{cases} A = 0 \\ B = \frac{1}{2} \end{cases}$$

$$y_p = \frac{1}{2}t\sin t$$

$$y(t) = y_c + y_p = C_1\cos t + C_2\sin t + \frac{1}{2}t\sin t$$

- Finally plug $t = \ln(x-a)$ back in.

$$t = \ln(x+1)$$

G.S.

$$t = \ln(x+1)$$

$$\boxed{y(x) = C_1\cos(\ln(x+1)) + C_2\sin(\ln(x+1)) + \frac{1}{2}\ln(x+1)\sin(\ln(x+1))}$$

Cauchy-Euler

Example (Final Problem 1, Fall 2022)

Use any method to find the GS:

$$y''' - \frac{2}{(x-a)^2}y' = \frac{1}{(x-a)^3}$$

where $x > a$, a constant.