AMS 361 R01/R03

Week 13 : Variable coefficients (Separation of variables)

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Variable coefficients

Consider the linear system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}.$$

(see lecture note for 3×3 matrix case).

Steps

- Rewrite the system as

$$\begin{cases} x' = a(t)x + b(t)y + f(t) \\ y' = c(t)x + d(t)y + g(t) \end{cases}.$$

- From the first equation, we have

$$y = \frac{1}{b(t)}x' - \frac{a(t)}{b(t)}x - \frac{f(t)}{b(t)}.$$

Variable coefficients

Steps

- Solve (week 8,9,10,11 Second order)

$$\left(\frac{1}{b(t)}x'-\frac{a(t)}{b(t)}x-\frac{f(t)}{b(t)}\right)' = c(t)x+d(t)\left(\frac{1}{b(t)}x'-\frac{a(t)}{b(t)}x-\frac{f(t)}{b(t)}\right)+g(t).$$

  to obtain

$$x(t) = c_1x_1 + c_2x_2 + x_p.$$

- Then

$$y(t) = \frac{1}{b(t)}x'(t) - \frac{a(t)}{b(t)}x(t) - \frac{f(t)}{b(t)} = c_1y_1 + c_2y_2 + y_p.$$

- The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix}.$$

Separation of variables

Example (Final Problem 3, Fall 2020)

Use LT and one other method to find the PS of

$$\begin{cases} x' = y + z + 1 \\ y' = z + x + 1 \\ z' = x + y + 1 \end{cases},$$

with ICs $x(0) = y(0) = z(0) = 0$. Convolutions, if any, must be evaluated.

Remark

LT method: week 14 Laplace transform;
One other method: week 12 E-Analysis or week 13 Separation of variables.

Separation of variables

Example (Final Problem 2, Spring 2022)

Solve the following DE using three different methods: (1) the Eigen-Analysis method, (2) the Substitution method, and (3) the Operator method.

$$X'(t) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} X.$$

Remark

(1) Week 12 E-Analysis;
(2)(3) Week 13 Separation of variables.

$$X'(t) = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} X$$

- Rewrite the system as

$$\begin{cases} x' = a(t)x + b(t)y + f(t) \\ y' = c(t)x + d(t)y + g(t) \end{cases}.$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x' = 2x - y & ① \\ y' = x + 4y & ② \end{cases}$$

- From the first equation, we have

$$y = \frac{1}{b(t)}x' - \frac{a(t)}{b(t)}x - \frac{f(t)}{b(t)}.$$

①: $y = -x' + 2x$

- Solve (week 8,9,10,11 Second order)

$$\left(\frac{1}{b(t)}x' - \frac{a(t)}{b(t)}x - \frac{f(t)}{b(t)}\right)' = c(t)x + d(t)\left(\frac{1}{b(t)}x' - \frac{a(t)}{b(t)}x - \frac{f(t)}{b(t)}\right) + g(t).$$

to obtain

$$x(t) = c_1x_1 + c_2x_2 + x_p.$$

②: $(-x' + 2x)' = x + 4(-x' + 2x)$

$-x'' + 2x' = x - 4x' + 8x$

$x'' - 6x' + 9x = 0$ (second order C.C.)

$\lambda^2 - 6\lambda + 9 = 0$

$(\lambda - 3)^2 = 0$

$$\lambda_1 = \lambda_2 = 3$$

$$x_1 = e^{\lambda_1 t} = e^{3t}, \qquad x_2 = t e^{\lambda_2 t} = t e^{3t}$$

$$x = c_1 x_1 + c_2 x_2$$
$$= c_1 e^{3t} + c_2 t e^{3t}$$

- Then

$$y(t) = \frac{1}{b(t)}x'(t) - \frac{a(t)}{b(t)}x(t) - \frac{f(t)}{b(t)} = c_1 y_1 + c_2 y_2 + y_p.$$

$$y = -x' + 2x$$
$$= -\left(c_1 e^{3t} + c_2 t e^{3t}\right)' + 2\left(c_1 e^{3t} + c_2 t e^{3t}\right)$$
$$= -\left(3c_1 e^{3t} + \left(c_2 e^{3t} + 3c_2 t e^{3t}\right)\right) + \left(2c_1 e^{3t} + 2c_2 t e^{3t}\right)$$
$$= (-c_1 - c_2)e^{3t} + (-c_2) t e^{3t}$$
$$= c_1 (-1) e^{3t} + c_2(-1-t) e^{3t}$$

- The general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_p \\ y_p \end{bmatrix}.$$

$$\begin{cases} x = c_1 e^{3t} + c_2 t e^{3t} \\ y = c_1 (-1) e^{3t} + c_2 (-1-t) e^{3t} \end{cases}$$

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} t \\ -1-t \end{bmatrix} e^{3t}}$$

G.S.

Variable coefficients

Example (Final Problem 3, Fall 2022)

Use any method to find the GS:

$$\begin{cases} tx' = 2x - y + (1 - t^2) \\ ty' = 3x - 2y + (2t) \end{cases} .$$

Separation of variables

Example (Final Problem 4, Fall 2022)

Use LT method and a non-LT method to find PS:

$$\begin{cases} -x' + y + z = e^{2t} \\ x - y' + z = e^{2t} \\ x + y - z' = e^{2t} \\ x(0) = y(0) = z(0) = 0 \end{cases}.$$

Remark

LT method: week 14 Laplace transform;
A non-LT method: week 12 E-Analysis or week 13 Separation of variables.