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Lectures for CH2 Math Models

Lecture plan for this Chapter:

Lecture	Date	Contents	Sec.
Week04.1	02/14	More Exact DE example	1.7
Week04.2	02/16	Basic concepts of mathematical models	2.1
Week05.1	02/21	**Test 1** (only material in CH1)	
Week05.2	02/23	Basic concepts of mathematical models **Math Model 1**: Newton's Law of Cooling **Math Model 2**: Torricelli's Law for Draining	2.1 2.2
Week06.1	02/28	**Math Model 3**: The Population Model	2.3
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-------- Start of Lecture Week04.2 (02/16/2023) --------

CH2 Math Models

Several (six) Basic Mathematical Models

There are many applications of DEs, that we call Mathematical Models. Let me introduce a few and then perform in-depth analysis including three steps:

(1) **Construction of DEs** (requiring some understanding of engineering/physics/economics… domain knowledge.)
(2) **Solution of DEs** (requiring knowledge of solving DE's, the key purpose of this course.)
(3) **Interpretations of Solution(s)** (making sense of what you have done in the previous 2 steps.)

Math Model 1: Newton's Law of Cooling (introduced by Newton in 1701): Placing an object of temperature T_0 into a large medium of fixed temperature A.

(See textbook, p. 68)

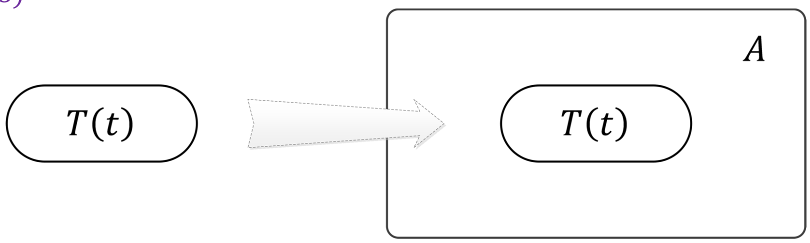


Figure 1. Illustration of the Newton's Law of Cooling

The ODE to express this problem:

$$\begin{cases} \dfrac{dT(t)}{dt} = k(A - T) \\ T(t = 0) = T_0 \end{cases}$$

Solution:

$$\int_{T_0}^{T} \frac{dT}{T - A} = \int_0^t (-k)dt$$

With some efforts,

$$\ln(T - A) - \ln(T_0 - A) = -kt$$
$$\ln\left(\frac{T - A}{T_0 - A}\right) = -kt$$
$$\frac{T - A}{T_0 - A} = \mathrm{e}^{-kt}$$
$$T - A = (T_0 - A)\mathrm{e}^{-kt}$$

The final PS for the heat DE:

$$T(t) = A + (T_0 - A)\mathrm{e}^{-kt}$$

Let's sketch a few special cases:

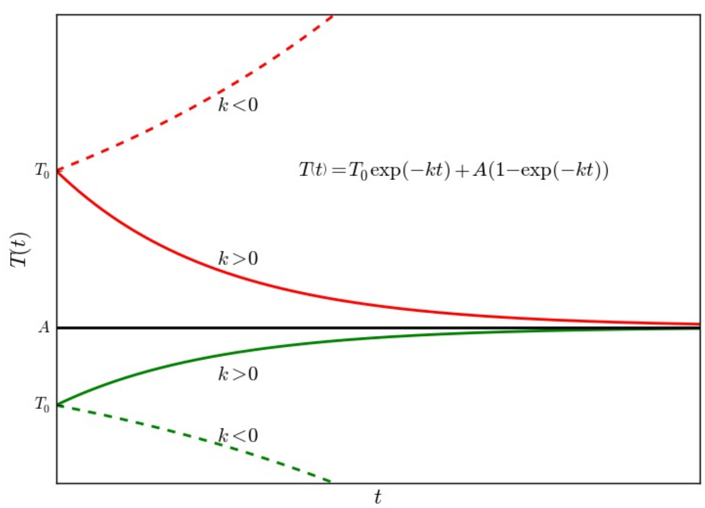


Figure 2. The multiple scenarios of the PS of heat DE.

-------- Start of Lecture Week05.1 (02/21/2023) --------
Test1

-------- Start of Lecture Week05.2 (02/23/2023) --------

Math Model 2: Torricelli's Law for Draining (introduced by Torricelli in 1643)

(See textbook, p. 72)

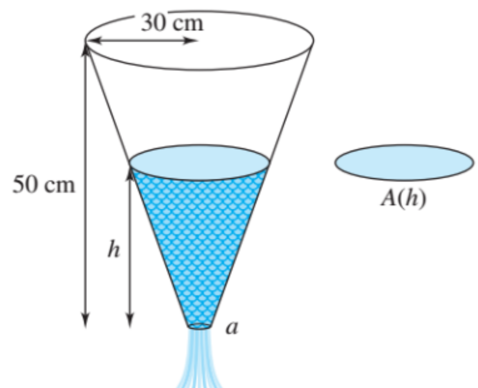


Figure 3. Illustration of Torricelli's Law for Draining

The original Torricelli's Law for Draining

$$\begin{cases} \dfrac{dV}{dt} = -k\sqrt{y} \\ y(t=0) = y_0 \end{cases}$$

where V is the volume of liquid in the container. We have three variables V, y, t in one IVP that must be resolved, of course, by leveraging on more conditions.

In an ODE, we express

$$\begin{cases} \dfrac{A(y)dy}{dt} = -k\sqrt{y} \\ y(t=0) = y_0 \end{cases}$$

where $A(y)$ is the cross-section area of liquid surface at height y.

Now, we choose to work on a special case: draining from a spherical container (Quite common?!)

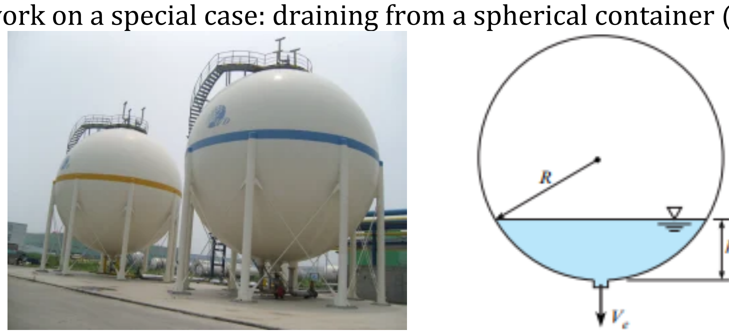


Figure 4. Draining from a spherical container.

The following is the geometrical way of finding the A(y), the cross-section area at height y:

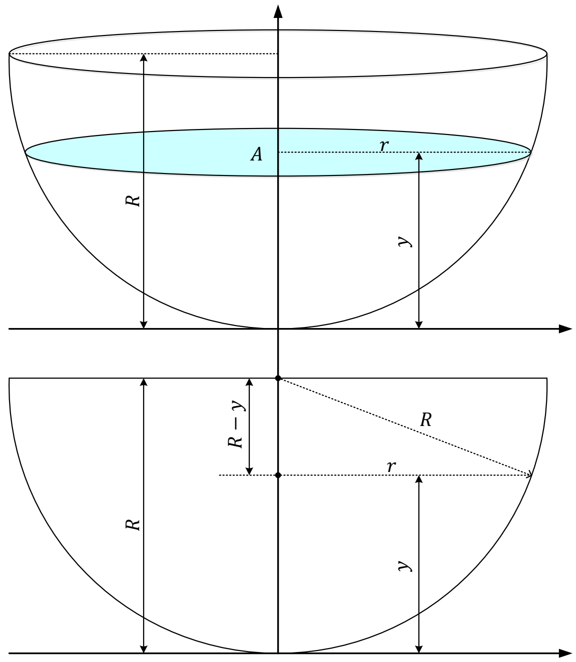


Figure 5. Setup for draining from a spherical container and geometrical description of finding $A(y)$.

Solution:

(Of course, we can have much smarter way, if you are skillful in the basic analytical geometry, for solving this problem):

1. Let's figure out

$$\begin{aligned} A(y) &= \pi r^2 = \pi(R^2 - (R-y)^2) \\ &= \pi(R^2 - R^2 + 2Ry - y^2) \\ &= \pi y(2R - y) \end{aligned}$$

2. Solve the IVP

$$\begin{cases} \dfrac{\pi y(2R-y)dy}{dt} = -k\sqrt{y} \\ y(t=0) = R \end{cases}$$

Solution of the IVP:

$$\pi\sqrt{y}(2R-y)dy = -kdt$$
$$\pi\int_R^y \sqrt{y}(2R-y)dy = \int_0^t (-k)dt$$
$$\pi\int_R^y \left(2Ry^{\frac{1}{2}} - y^{\frac{3}{2}}\right)dy = -kt$$

$$\pi\left(2R\int_R^y y^{\frac{1}{2}}dy - \int_R^y y^{\frac{3}{2}}dy\right) = -kt$$
$$\pi\left(\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_R^y = -kt$$
$$\pi y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) - \pi R^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}R\right) = -kt$$

The PS, an implicit function between "y" and "t":

$$\pi y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) - \frac{14}{15}\pi R^{\frac{5}{2}} = -kt$$

Let's sketch this PS:

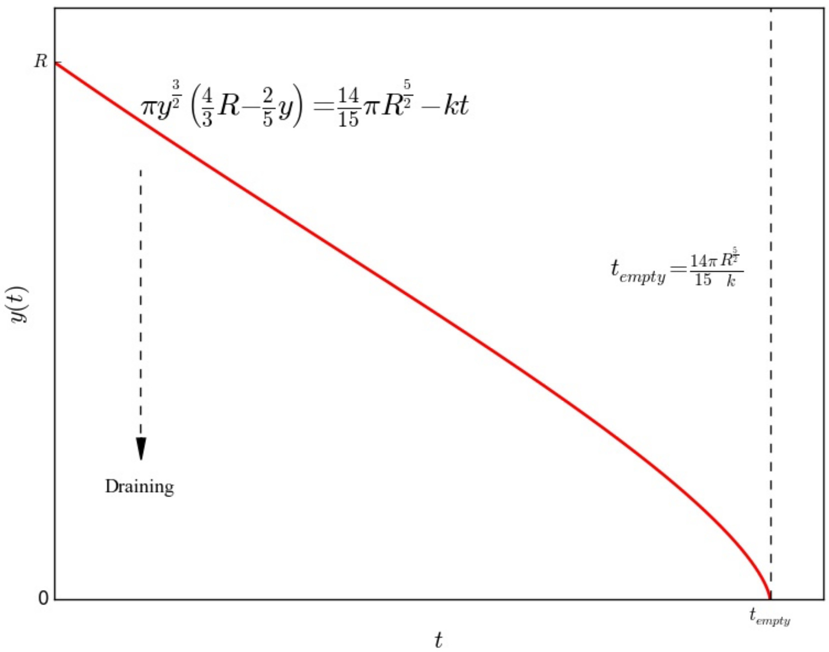


Figure 6. PS for draining from a spherical container.

Remarks:
(1) Let's check at $t = 0$,

$$\pi y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) - \frac{14}{15}\pi R^{\frac{5}{2}} = -k\cancel{t}0$$

Or

$$\cancel{\pi} y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) - \frac{14}{15}\cancel{\pi} R^{\frac{5}{2}} = 0$$

Or (multiplying 15/2 on both sides):

$$y^{\frac{3}{2}}(10R - 3y) - 7R^{\frac{5}{2}} = 0$$
$$10Ry^{\frac{3}{2}} - 3y^{\frac{5}{2}} - 7R^{\frac{5}{2}} = 0$$
$$7R\left(y^{\frac{3}{2}} - R^{\frac{3}{2}}\right) + 3(R - y)y^{\frac{3}{2}} = 0$$

The only root for this equation is

$$y = R$$

which is the IC.

(2) The time to empty the spherical container is

$$y = 0$$

in the above PS, or

$$\cancel{y}0^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}\cancel{y}0\right) - \frac{14}{15}\pi R^{\frac{5}{2}} = -kt$$

Thus,

$$-\frac{14}{15}\pi R^{\frac{5}{2}} = -kt_{\text{Empty}}$$

Or

$$t_{\text{Empty}} = \frac{14}{15}\frac{\pi R^{\frac{5}{2}}}{k}$$

I made three plots with $k = 10, 20, 30$

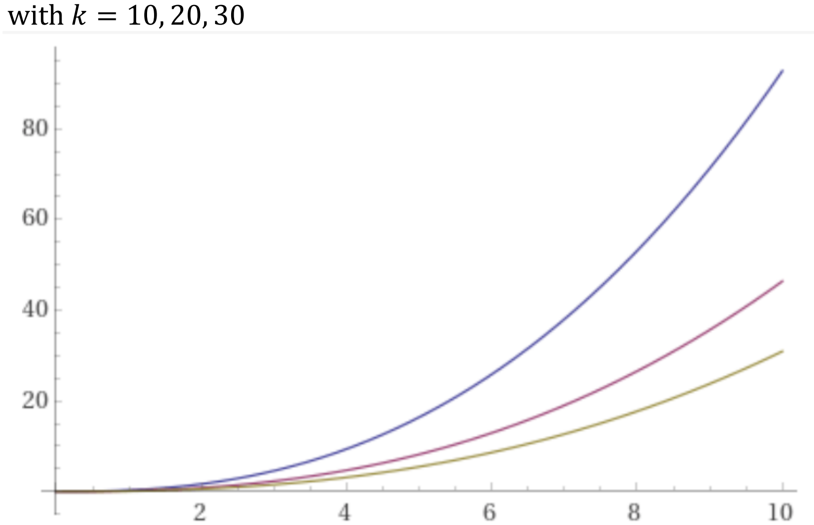


Figure 7. Time vs. radius for k=10 (blue), 20 (red) and 30 (yellow) for emptying sphere.

In fact, if we use the volume of a half spherical container

$$V_{0.5} = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)$$

We get

$$t_{\text{Empty}} = \left(\frac{14}{15}\frac{\pi R^{\frac{5}{2}}}{k}\right)\left(\frac{V_{0.5}}{\frac{2}{3}\pi R^3}\right)$$

$$= \frac{7}{5}\frac{V_{0.5}}{k\sqrt{R}}$$

This form does not look too interesting or inspiring (as we can't come up any useful stories), but it looks neater.

Alternatively, this spherical container for draining can be solve by "analytical" geometry if that's your cup of tea ☺

For a **sphere** of radius R that can be expressed as

$$x^2 + y^2 + z^2 = R^2$$

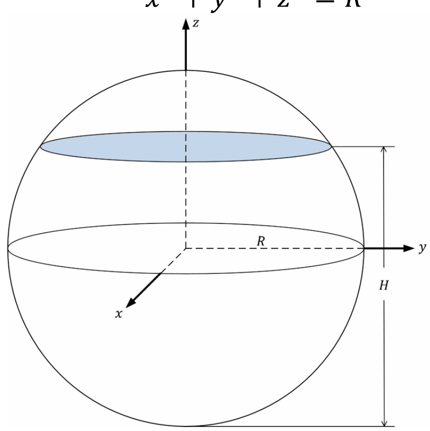


Figure 8. Figure 2.5 The Cartesian coordinates for a spherical container.

Assuming the draining hole is placed at the south pole

$$(x, y, z)_{\text{South Pole}} = (0, 0, -R)$$

while an air-intake hole is placed the north pole

$$(x, y, z)_{\text{North Pole}} = (0, 0, R)$$

although they can be placed anywhere if the draining problem makes sense. Given the initial liquid surface height H measured from the draining hole, the height at any instance is $z + R$. At this height, the liquid surface is a disc whose radius is $r = \sqrt{R^2 - z^2}$. Thus, applying the Torricelli's law for draining, we get

$$\begin{cases} \dfrac{\pi(R^2 - z^2)dz}{dt} = -k\sqrt{z + R} \\ z(t = 0) = H - R \end{cases}$$

One can conveniently solve this IVP to obtain the solution. For example, we consider draining from $z(t=0)=H-R$ to $z(t=T)=h-R$, *i.e.*,

$$\int_{H-R}^{h-R} \frac{(R^2-z^2)}{\sqrt{z+R}} dz = -\frac{k}{\pi} T$$

To evaluate the integral, we introduce a substitution $u=z+R$ and the LHS becomes

$$\begin{aligned} LHS &= \int_H^h \frac{(R^2-(u-R)^2)}{\sqrt{u}} du \\ &= \int_H^h \left(2Ru^{1/2}-u^{3/2}\right) du \\ &= \frac{2}{15} u^{3/2}(10R-3u)\Big|_H^h \\ &= \frac{2}{15}\left(h^{3/2}(10R-3h)-H^{3/2}(10R-3H)\right) \end{aligned}$$

Thus, the time is

$$T = \frac{\pi}{k}\frac{2}{15}\left(H^{3/2}(10R-3H)-h^{3/2}(10R-3h)\right)$$

Remarks

(1) To empty a half-filled spherical tank, *i.e.*, $H = R$ and $h = 0$, we get the consistent time as before,

$$T_{0.5\mathrm{L}} = \frac{\pi}{k}\frac{14}{15}R^{\frac{5}{2}}$$

(2) To empty a fully filled spherical tank, *i.e.*, $H = 2R$ and $h = 0$, we get

$$\begin{aligned} T_{1.0} &= \frac{\pi}{k}\frac{16}{15}\sqrt{2}R^{\frac{5}{2}} \\ &= \frac{8\sqrt{2}}{7}\left(\frac{\pi}{k}\frac{14}{15}R^{\frac{5}{2}}\right) \\ &= \frac{8\sqrt{2}}{7}T_{0.5\mathrm{L}} \\ &\approx (1.61624)T_{0.5\mathrm{L}} \end{aligned}$$

(3) To empty the upper half of a fully filled spherical tank from the same draining hole, *i.e.*, $H = 2R$ and $h = R$, we use

$$\begin{aligned} T_{0.5\mathrm{U}} &= T_{1.0} - T_{0.5\mathrm{L}} \\ &= \frac{\pi}{k}\frac{16}{15}\sqrt{2}R^{\frac{5}{2}} - \frac{\pi}{k}\frac{14}{15}R^{\frac{5}{2}} \\ &= \left(\frac{8\sqrt{2}}{7} - 1\right)T_{0.5\mathrm{L}} = 0.61624T_{0.5\mathrm{L}} \end{aligned}$$

Thus,

$$\frac{T_{0.5\mathrm{U}}}{T_{0.5\mathrm{L}}} = 0.61624$$

i.e., the upper half drains significantly faster than the lower half of a spherical tank.

Of course, the problem can get fancy with the container shapes or where to plant the drain hole(s):

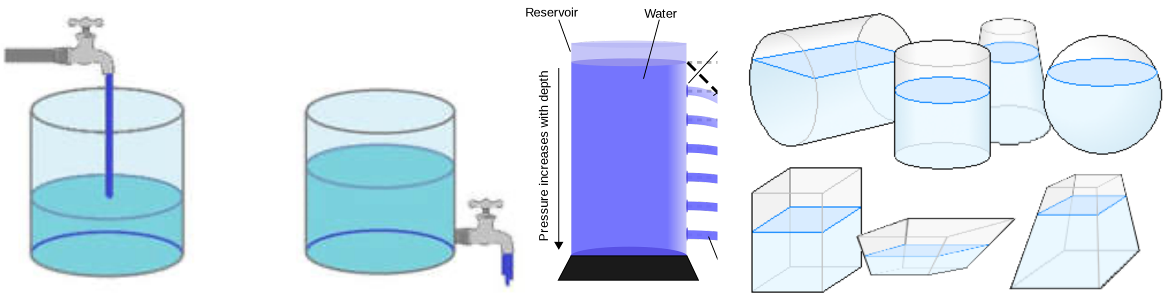


Figure 9. Variations of the draining problem: position of draining holes(s) and container shapes.

-------- Start of Lecture Week06.1 (02/28/2023) --------

Math Model 3A: The Population Model

(See textbook, p. 79)

The time rate of change of a population with only births (and no death, e.g., cancer cells)

$$\begin{cases} \dfrac{dP}{dt} = kP \\ P(t=0) = P_0 \end{cases}$$

We can easily solve this IVP and find its PS as

$$P(t) = P_0 \mathrm{e}^{kt}$$

I make a plot to show what "exponential" population growth looks like:

$$P_0 = 100$$
$$k = 0.911$$

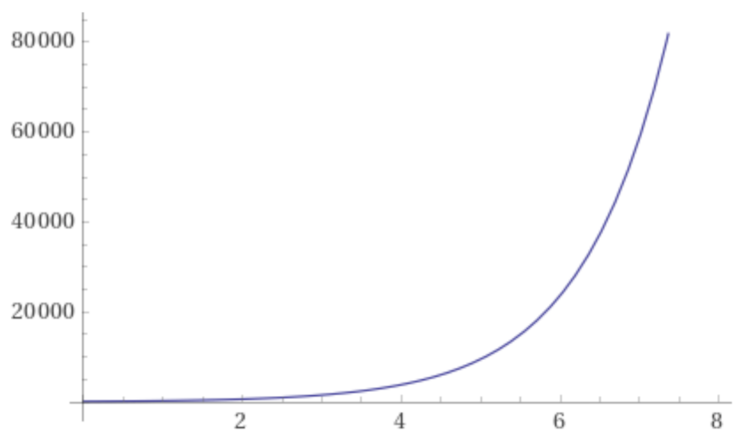


Figure 10. "exponential" population growth.

However, in general Population Model is much more complex!

Math Model 3B: The Population Model B: The time rate of change of a population with both births and deaths.

$$\begin{cases} \dfrac{dP}{dt} = \underbrace{(kM)P}_{\text{Birth rate}} - \underbrace{kP^2}_{\text{Death rate}} = kP(M-P) \\ P(t=0) = P_0 \end{cases}$$

Note: Constants k and M at this moment are just "meaningless" constants. We will soon assign them meanings in terms of population analysis.

Solution:
This IVP can be converted to

$$\begin{cases} \dfrac{dP}{P(M-P)} = kdt \\ P(t=0) = P_0 \end{cases}$$

Thus,

$$\int_{P_0}^{P} \frac{dP}{P(M-P)} = \int_0^t kdt$$

$$\int_{P_0}^{P} \left(\frac{1}{P} - \frac{1}{P-M}\right) dP = kMt$$

$$(\ln P - \ln(P-M))|_{P_0}^{P} = kMt$$

$$\ln\left(\frac{P}{P_0}\right) - \ln\left(\frac{P-M}{P_0-M}\right) = kMt$$

$$\frac{P}{P_0}\frac{M-P_0}{M-P} = \mathrm{e}^{kMt}$$

After much labor (save space to work out), we get the following PS

This is the well-known solution of the Population Model expressed as an explicit function of the independent variable "t":

$$P(t) = \frac{M}{1 + \left(\frac{M}{P_0} - 1\right) e^{-kMt}}$$

In this formula, you would be able to compute the population "P" at a given time "t".

We have many interesting observations:

We first check the IC: At $t = 0$, $P(t = 0) = P_0$.

Now, let me make a table (9 cases) to examine the population counts, especially asymptotically: $\lim_{t\to\infty} P(t)$:

	$k > 0$	$k = 0$	$k < 0$
$P_0 > M$	$\lim_{t\to\infty} P(t) = M$	$P(t) = P_0$	$P(t_c) = \infty$ Doomsday where $t_c = \frac{1}{kM} \ln\left(1 - \frac{M}{P_0}\right)$
$P_0 = M$	$P(t) = M$	$P(t) = M$	$P(t) = M$
$P_0 < M$	$\lim_{t\to\infty} P(t) = M$	$P(t) = P_0$	$\lim_{t\to\infty} P(t) = 0$ Extinction

This very little DE can have very fancy outcomes (summarized in a figure):

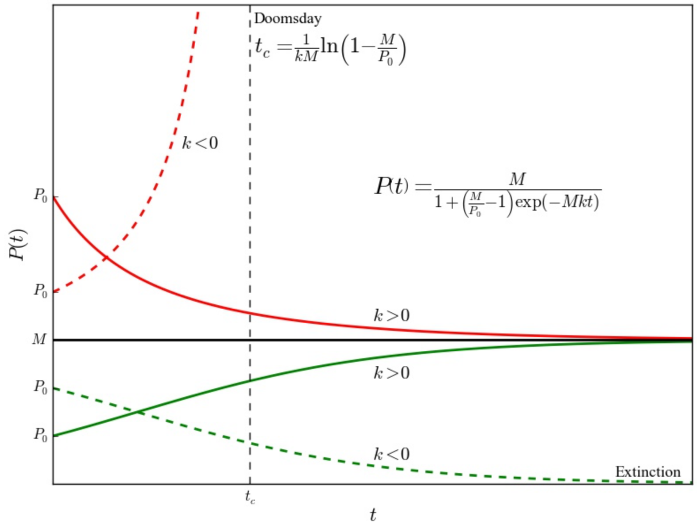


Figure 11. Special cases of the PS for the Population model

Alternatively, we can express the solution of the Population Model differently, i.e., treat time "t" as the dependent variable and population "P" as the independent variable.

Using,

$$\ln\left(\frac{P}{P_0}\right) - \ln\left(\frac{P-M}{P_0-M}\right) = kMt$$

we get,

$$t = \frac{1}{kM}\ln\left(\frac{1-\frac{M}{P_0}}{1-\frac{M}{P}}\right)$$

Let's examine one special, and likely most interesting, case: $P_0 > M$ and $k < 0$.
Taking a limit,

$$\begin{aligned} t_c &= \lim_{P\to\infty}\left[\frac{1}{kM}\ln\left(\frac{1-\frac{M}{P_0}}{1-\frac{M}{P}}\right)\right] \\ &= \frac{1}{kM}\ln\left(\frac{1-\frac{M}{P_0}}{1-0}\right) \\ &= \frac{1}{kM}\ln\left(1-\frac{M}{P_0}\right) \end{aligned}$$

we obtain the formula for t_c which is the time when the doomsday will arrive.

$$t_c = \frac{1}{kM}\ln\left(1-\frac{M}{P_0}\right)$$

We examine the trends,

(1) $\frac{M}{P_0} \ll 1$ or $M \ll P_0$, i.e., a patient has huge number of disease cells or tiny containing capacity (low disease tolerance) or both,

$$t_c = -\frac{1}{kM}\left(\frac{M}{P_0} + \frac{1}{2}\left(\frac{M}{P_0}\right)^2 + \frac{1}{3}\left(\frac{M}{P_0}\right)^3 + \cdots\right) \approx -\frac{1}{kM}\left(\frac{M}{P_0}\right) = -\frac{1}{kP_0}$$

The patient's doomsday will arrive at,

$$t_c \propto \frac{1}{P_0}$$

(2) $\frac{M}{P_0} \to 1$ or $M \approx P_0$, i.e., a patient's diagnosed disease cell count is near the containing capacity,

$$\begin{aligned} t_c &= \frac{1}{kM}\ln\left(1-\frac{M}{P_0}\right) \\ &\to -\frac{1}{kM}\ln(1-1) \\ &\to \infty \end{aligned}$$

The patient's doomsday will never arrive (by this disease.)

Let's examine in greater details how t_c depends on k, M, P_0:

(1) For given M, P_0, we found a trivial case,

$$t_c \propto \frac{1}{k}$$

(2) For given M, k, we plot how t_c varies with P_0:

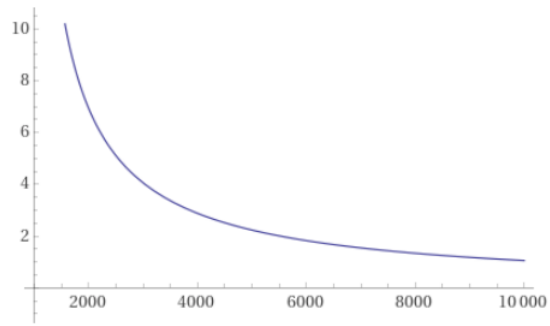


Figure 12. Doomsday time t_c vs. the initial population P_0.

In this plot, $k = -0.0001$, $M = 1000$. The plot is reasonable that the larger the initial population P_0, the shorter the doomsday time t_c. In other words, the more tumor cells a patient is initially diagnosed (compared with M, the patient "containing" capacity), the shorter time the patient has before the doomsday. Vice versa.

Analytically,

(3) For given P_0, k, we plot how t_c varies with M:

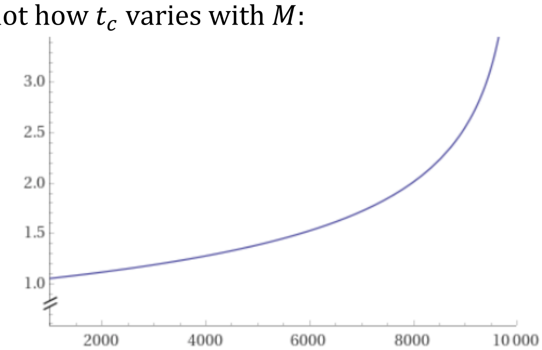


Figure 13. Doomsday time t_c vs. the containing capacity M.

In this plot, $k = -0.0001$, $P_0 = 10{,}000$. The plot is reasonable that the larger the containing capacity M, the longer the doomsday time t_c. In other words, the more containing capacity (or more tolerant to disease) for a given initial tumor count P_0, the longer time the patient has before the doomsday. Vice versa.

-------- Start of Lecture Week06.2 (03/02/2023) --------

Math Model 4: Newton's Laws of Motion (Introduced by Newton in 1687 at the age of 44).

(See textbook, p. 92)

Basically, we apply Newton's law of motion.
The motion velocity

$$v(t) = \frac{dx}{dt} = x'(t)$$

the acceleration

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v'(t) = x''(t)$$

Given an extremal force

$$F(t, x(t), x'(t))$$

we have, according to Newton,

$$F\big(t, x(t), x'(t)\big) = mx''(t)$$

where, in general, $m = m(t, x(t), x'(t))$.

In general, this is a 2nd order nonlinear DE.

For our purpose, we make many assumptions to simplify the IVP:

$$\begin{cases} \dfrac{d^2x}{dt^2} = a\big(t, x(t), x'(t)\big) \\ v(t=0) = x'(t=0) = v_0 \\ x(t=0) = x_0 \end{cases}$$

where

$$a\big(t, x(t), x'(t)\big) = \frac{F\big(t, x(t), x'(t)\big)}{m\big(t, x(t), x'(t)\big)}$$

The following is how a few solutions of this problem:

Air Resistance Model (constant mass and no engine):

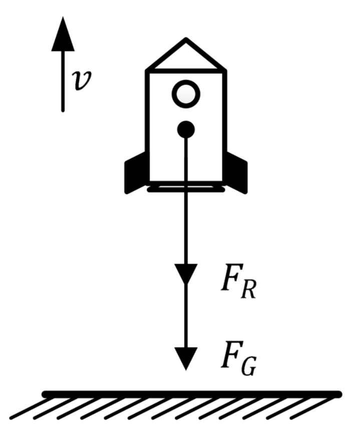


Figure 14. A simple "rocket" model.

$$F = F_R + F_G = -kv - mg$$

Thus, the IVP becomes

$$\begin{cases} m\dfrac{dv}{dt} = -kv - mg \\ v(t=0) = y'(t=0) = v_0 \\ y(t=0) = y_0 \end{cases}$$

Now, let's solve this

$$\frac{dv}{dt} = -\rho v - g$$

where we use (for convenience):

$$\rho = \frac{k}{m} > 0$$

Thus,

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)\mathrm{e}^{-\rho t} - \frac{g}{\rho}$$

This little PS has a lot of interesting stories to tell.

Remarks:

1. Checking out the IC: $v(t=0) = \left(v_0 + \frac{g}{\rho}\right)\mathrm{e}^{-\rho \cancel{t}0} - \frac{g}{\rho} = v_0$
2. Checking the long-time velocity

$$v_T = \lim_{t\to\infty} v(t) = -\frac{g}{\rho} = -\frac{mg}{k}$$

Leading to the terminal speed:

$$|v_T| = \frac{g}{\rho} = \frac{mg}{k}$$

3. The height one can reach:

First, let's rewrite the velocity for convenience (You don't have to)

$$\begin{aligned} v(t) &= \left(v_0 + \frac{g}{\rho}\right)\mathrm{e}^{-\rho t} - \frac{g}{\rho} \\ &= (v_0 - v_T)\mathrm{e}^{-\rho t} + v_T \end{aligned}$$

Second, compute the height vs. t:

$$\begin{aligned} y(t) &= \int_0^t v(\tau)d\tau \\ &= \int_0^t \left((v_0 - v_T)\mathrm{e}^{-\rho\tau} + v_T\right)d\tau \\ &= \frac{1}{\rho}(v_0 - v_T)(1 - \mathrm{e}^{-\rho t}) + v_T t + y_0 \end{aligned}$$

Now, let's analyze these results a little more:

1. The time to reach the max height (going up):

$$v(T_{UP}) = \left(v_0 + \frac{g}{\rho}\right)\mathrm{e}^{-\rho T_{UP}} - \frac{g}{\rho} = 0$$

Solving this leads to

$$\begin{aligned} T_{UP} &= \frac{1}{\rho}\ln\left(1+\frac{\rho v_0}{g}\right) \\ &= \frac{m}{k}\ln\left(1+\frac{kv_0}{mg}\right) \end{aligned}$$

One can also compute T_{DN} here but let me postpone.

2. We can find the max height

In fact, you can use the following relationship,

$$y(t) \Leftrightarrow v(t) \Leftrightarrow t$$

With this and the chain rule, we have

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$$

Essentially, it's like a sub that leads to

$$\begin{cases} m\dfrac{dv}{dy}v = -kv - mg \\ v(y=0) = v_0 \end{cases}$$

Thus,

$$\begin{aligned} dy &= -\frac{mv}{kv+mg}dv \\ &= -\left(\frac{m}{k}\right)\frac{v}{v+\frac{mg}{k}}dv \\ &= -\left(\frac{m}{k}\right)\left(1+\frac{v_T}{v-v_T}\right)dv \end{aligned}$$

To slove this, we get

$$\begin{aligned} y &= -\left(\frac{m}{k}\right)\int_{v_0}^{v}\left(1+\frac{v_T}{v-v_T}\right)dv \\ &= \frac{v_T}{g}\left((v-v_0)+v_T\ln\frac{v-v_T}{v_0-v_T}\right) \end{aligned}$$

At the max height, $v = 0$

$$\begin{aligned} H &= \frac{v_T}{g}\left((0-v_0)+v_T\ln\frac{0-v_T}{v_0-v_T}\right) \\ &= -\frac{v_T}{g}\left(v_0+v_T\ln\left(1-\frac{v_0}{v_T}\right)\right) \\ &= \frac{m}{k}\left(v_0-\frac{mg}{k}\ln\left(1+\frac{kv_0}{mg}\right)\right) \end{aligned}$$

3. We can find the time to return the launching point

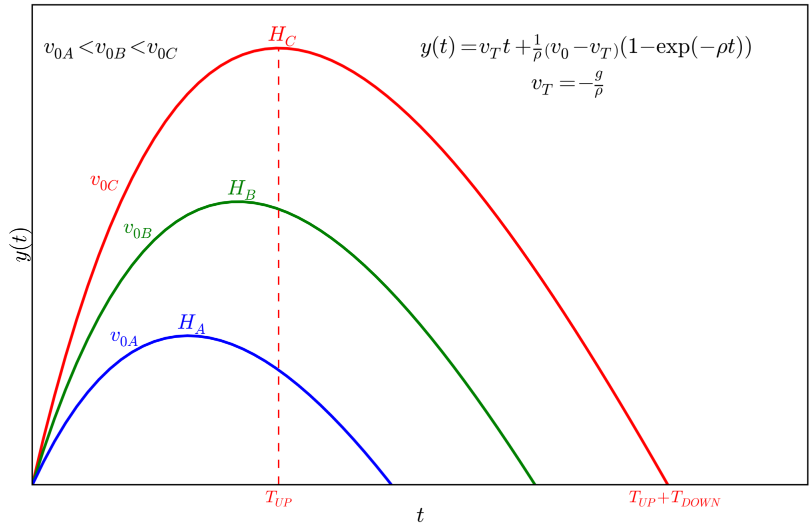


Figure 15. Rocket heights vs. time.

Math Model 5: Windy Day Plane Landing

(See textbook, p. 123)

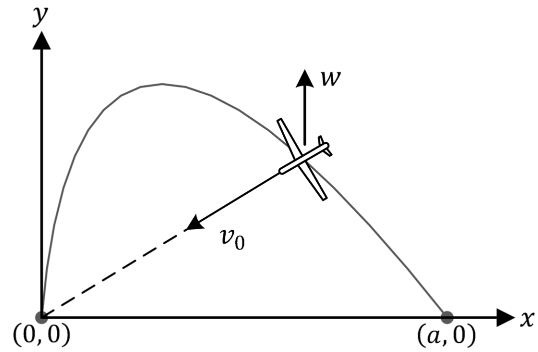


Figure 16. Illustration of plane landing in winds.

We can see the trajectory DEs for the plane:

$$\begin{cases} \dfrac{dx}{dt} = -v_0 \cos\alpha = -\, v_0 \dfrac{x}{\sqrt{x^2+y^2}} \\ \dfrac{dy}{dt} = -v_0 \sin\alpha + w = -\, v_0 \dfrac{y}{\sqrt{x^2+y^2}} + w \end{cases}$$

We can simplify by eliminating "t" also assume

$$k = \frac{w}{v_0}$$

Thus,

$$\frac{dy}{dx} = \frac{y}{x} - k\sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Since this is a Homo DE (nonlinear!), we make sub

$$u = \frac{y}{x}$$

Thus,

$$y = ux$$
$$y' = u'x + u$$

The original DE now becomes

$$\underbrace{\frac{dy}{dx}}_{u'x+u} = \underbrace{\frac{y}{x}}_{u} - k\sqrt{1 + \underbrace{\left(\frac{y}{x}\right)^2}_{u^2}}$$

Thus,

$$u'x + u = u - k\sqrt{1+u^2}$$

Or

$$\frac{du}{\sqrt{1+u^2}} = -k\frac{dx}{x}$$

Now, integrating both sides, we get

$$\ln\left(u + \sqrt{1+u^2}\right) = -k \ln x + C$$

Remember this formula?

$$\int \frac{du}{\sqrt{1+u^2}} = \ln\left(u+\sqrt{1+u^2}\right) + C_1$$

Check Textbook p.124: You just need to make a sub: $u = \frac{e^z - e^{-z}}{2}$. In the interests of time, I'll skip deriving this formula.

Applying the IC $u(x=a) = \frac{y(a)}{a} = \frac{0}{a} = 0$, we get $c = -k\ln a$ and

$$\ln\left(u+\sqrt{1+u^2}\right) = -k\ln\frac{x}{a}$$

$$u+\sqrt{1+u^2} = \left(\frac{x}{a}\right)^{-k}$$

Thus,

$$u(x) = \frac{1}{2}\left(\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right)$$

Now, back sub

$$u = \frac{y}{x}$$

Thus,

$$\begin{aligned} y(x) &= xu \\ &= x\underbrace{\frac{1}{2}\left(\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right)}_{u} \\ &= \left(\frac{x}{a}\right)\frac{a}{2}\underbrace{\left(\left(\frac{x}{a}\right)^{-k} - \left(\frac{x}{a}\right)^{k}\right)}_{u} \\ &= \frac{a}{2}\left(\left(\frac{x}{a}\right)^{1-k} - \left(\frac{x}{a}\right)^{1+k}\right) \end{aligned}$$

Discuss the special cases:

(1) If $k = \frac{w}{v_0} = 1$, meaning the plane flies at the same speed as the wind, the plane's vertical coordinate is $y(x \to 0) = \frac{a}{2}$ for $x \to 0$, *i.e.,* the plane will reach at $\left(0, \frac{a}{2}\right)$ and miss the airport.

(2) If $k = \frac{w}{v_0} > 1$, meaning the plane flies slower than the wind, the plane's vertical coordinate is $y(x \to 0) \to \infty$ for $x \to 0$, *i.e.,* the plane will get lost.

(3) If $k = \frac{w}{v_0} < 1$, meaning the plane flies faster than the wind, the plane's vertical coordinate is $y(x \to 0) \to 0$ and the plane lands properly.

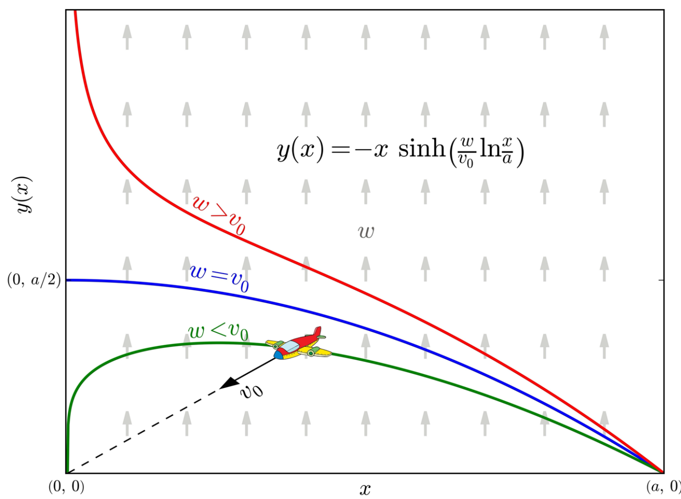


Figure 17. Plane trajectories in 3 cases.

-------- Start of Lecture Week07.1 (03/07/2023) --------

Math Model 6: Swimmer's Problem

(See textbook, p. 131)

The picture shows a river of width $w = 2a$ that flows north. The lines $x = 0, \pm a$ represent the river's center line and banks. The river water speed v_R increases as one approaches the river center, and the speed can be proven to be

$$v_R(x) = v_0\left(1 - \frac{x^2}{a^2}\right)$$

where x is the distance from the river center line, $v_R(x)$ is the water speed at location x and v_0 is the water speed at the center of the river.

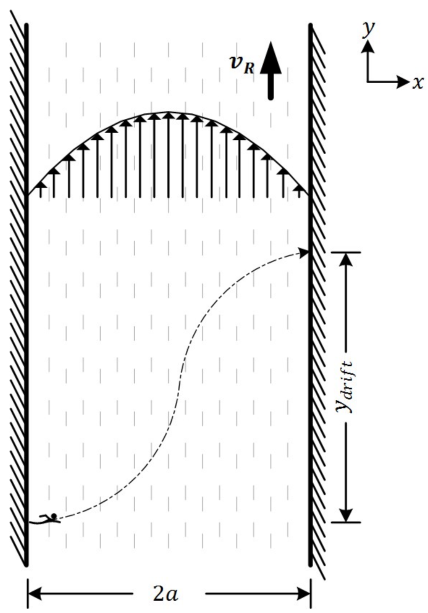


Figure 18. Set up of the swimmer problem. River width = 2a.

Two speed components:

$$\begin{cases} \dfrac{dx}{dt} = v_S \\ \dfrac{dy}{dt} = v_0\left(1 - \dfrac{x^2}{a^2}\right) \end{cases}$$

with some IC:

$$y(x = -a) = 0$$

Neglecting "t", we compose the IVP:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{v_0}{v_S}\left(1 - \dfrac{x^2}{a^2}\right) \\ y(x = -a) = 0 \end{cases}$$

Solution:

This is a first order, separable, and **LINEAR** DE (I stated "NL" by mistake on 10/6/2022 lecture)

$$\frac{dy}{dx} = \tan\alpha = \frac{v_R}{v_S} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)$$

This is a very simple problem:

$$dy = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)dx$$

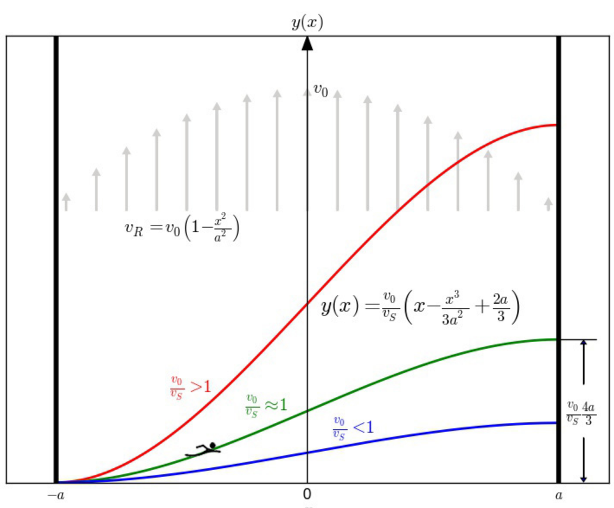
$$\int_0^y dy = \int_{-a}^{x} \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$y = \frac{v_0}{v_S}\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{x}$$

whose PS is

$$y = \frac{v_0}{v_S}\left(x - \frac{x^3}{3a^2} + \frac{2a}{3}\right)$$

which can be plotted as



Now, let's examine how much the swimmer has drift: At the east bank of the river $x = a$, the swimmer's drift is

$$y_D = y(x = a) = \frac{v_0}{v_S}\left(a - \frac{a^3}{3a^2} + \frac{2a}{3}\right) = \frac{v_0}{v_S}\frac{4a}{3}$$

We can make a ton of remarks of this result.

		$a \to 0$ Narrow River	$a \to \infty$ Wide River
$\frac{v_0}{v_S} \to 0$	$v_0 \ll v_S$ A Lake	$y_D \to 0$	$y_D \to \infty$
$\frac{v_0}{v_S} \to 1$	$v_0 \approx v_S$	$y_D \to 0$	$y_D \to \infty$
$\frac{v_0}{v_S} \to \infty$	$v_0 \gg v_S$ Niagara Falls	$y_D \to \infty$	$y_D \to ?$

By the way, typical river flow speeds:

Mississippi River (US): 1.2 mph

The Nile River: 2.1 m/s = 4.7 mph

The Amazon River: <7 km/h = 4.35 mph

The Yangtze River (China): 0.73~1.24 m/s = 1.6~2.8 mph

For a reference,

Typical adult human walking speed: 3-4 mph.

Fast human (Usain Bolt) running speed: 27.5 mph.

Math Model 7: River Ferry-Boat Docking

Will skip this model.

Math Model 8: Compound Interest

(See textbook, p. 141)

Step 1: **Construction of DE:**

Your loan has a fixed interest rate r. You pay bank or get more loan from the account at a rate "k". The loan increment from time t to t+dt

$$\begin{aligned} dZ &= \text{Interest charge} - \text{payment} \\ &= Z * r * dt - w * dt \end{aligned}$$

Thus, you will have the following DE

$$\frac{dZ}{dt} = rZ - w$$

where $w > 0$ means you made payment and $w < 0$ you borrow more (usually banks do allow that.)

Step 2: **Solution of DE:**

This is a very simple first order linear DE and solvable by both methods we have learned (PQ equation method and separation of variables).

$$\frac{Z'}{r} = Z - \frac{w}{r}$$

$$\frac{dZ}{Z - \frac{w}{r}} = rdt$$

$$\int_{z_0}^{z} \frac{dZ}{Z - \frac{w}{r}} = \int_{t=0}^{t} rdt$$

$$\ln\left(Z - \frac{w}{r}\right)\Big|_{z_0}^{z} = rt$$

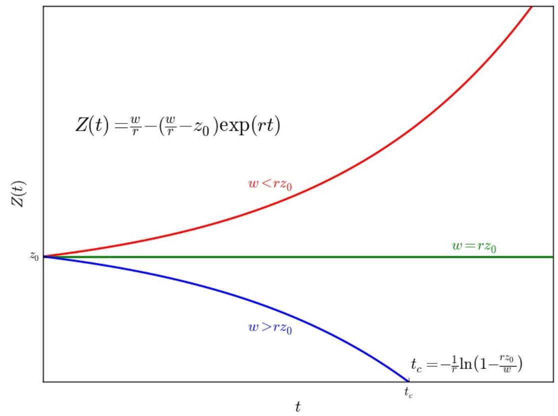
$$\ln\left(\frac{Z - \frac{w}{r}}{z_0 - \frac{w}{r}}\right) = rt$$

$$Z - \frac{w}{r} = \left(z_0 - \frac{w}{r}\right) \mathrm{e}^{rt}$$

If $Z(t = 0) = Z_0$, then PS:

$$Z(t) = \frac{w}{r} - \left(\frac{w}{r} - z_0\right) \mathrm{e}^{rt}$$

Step 3**: Analysis of Solution(s)**



(1) If $\frac{w}{r} - z_0 > 0$ (you pay more than the interest), your loan $Z(t)$ decreases with time.

(2) If $\frac{w}{r} - z_0 < 0$ (you pay less than the interest), your loan $Z(t)$ increase with time, until you bankrupt.

(3) If $\frac{w}{r} - z_0 = 0$ (you precisely pay the interest), your loan $Z(t)$ will stay steady!

*** The End of CH2 ***