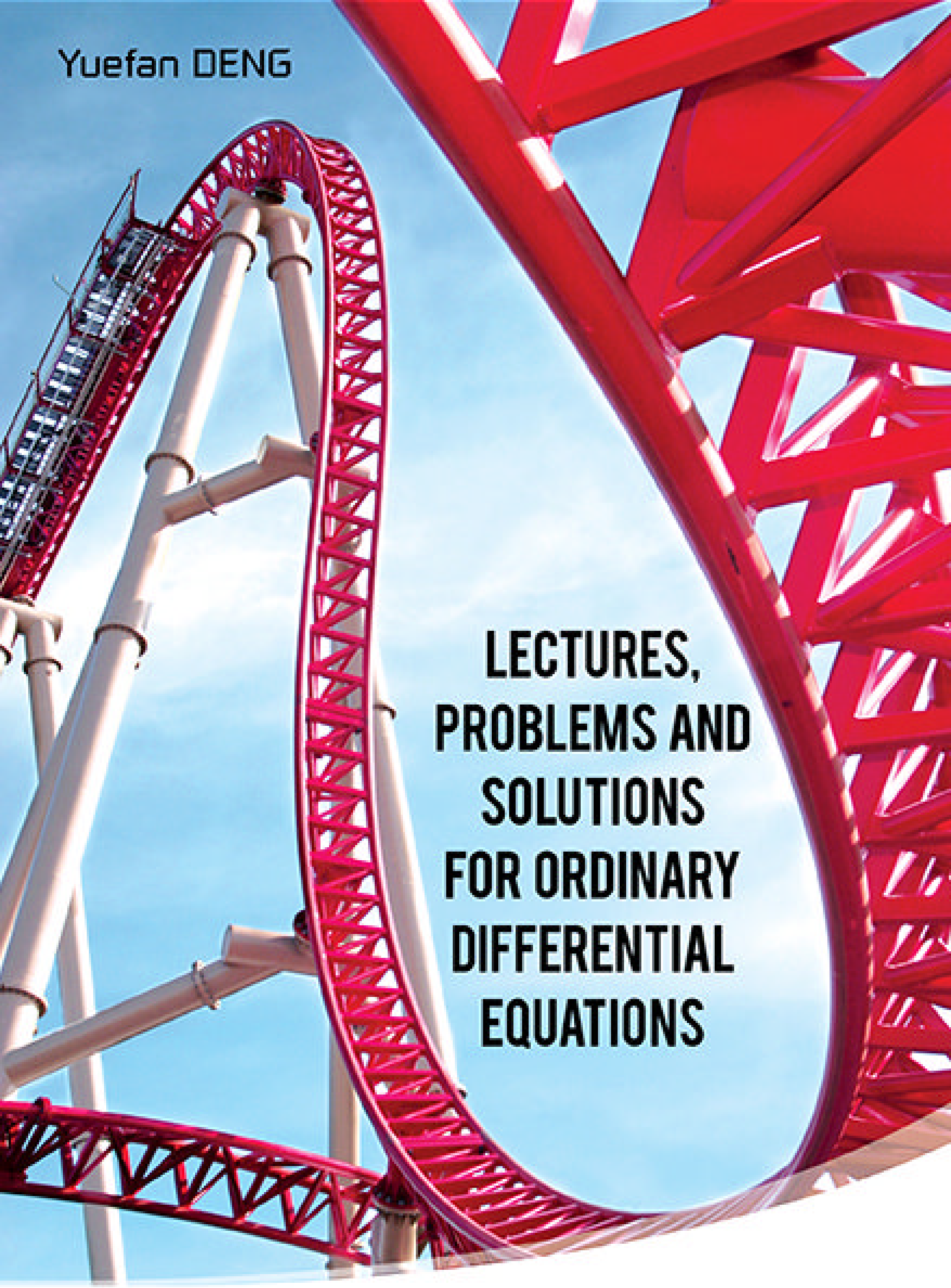
Yuefan DENG



LECTURES, PROBLEMS AND SOLUTIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Preface

This book, *Lectures, Problems and Solutions for Ordinary Differential Equations,* results from more than 20 revisions of lectures, exams, and homework assignments to approximately 5,000 students in the College of Engineering and Applied Sciences at Stony Brook University over the past 30 semesters. The book contains notes for twenty-five 80-minute lectures and approximately 1,000 problems with solutions. Creating another ODEs book is probably more unnecessary than reinventing the wheel. Yet, I constructed this manuscript *differently* by focusing on gathering and inventing examples from physical and life sciences, engineering, economics, and other real-world applications. These examples, partly adapted from well-known textbooks and partly freshly composed, focus on illustrating the applications of ordinary differential equations. The examples are much stronger stimuli to students for learning to prove the theorems while mastering the art of applying them.

While preparing this manuscript, I benefit immensely from many people including undergraduate students who took the class and graduate teaching assistants who helped teach it. Changnian Han has triple roles: one of the undergraduate students, one of the TAs, and one of the editors. I also thank the World Scientific Publishing and its fantastic senior editor Ms. Ji Zhang.

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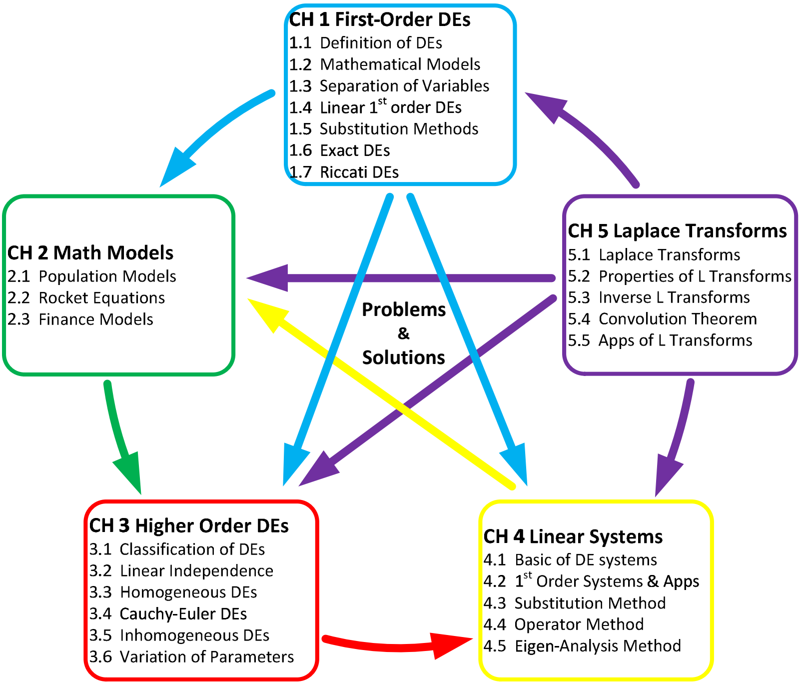
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First-Order
Differential Equations

1.1 Definition of Differential Equations

A differential equation (DE) is a mathematical equation that relates some functions of one or more variables with its derivatives. A DE is used to describe changing quantities and it plays a major role in quantitative studies in many disciplines such as all areas of engineering, physical sciences, life sciences, and economics.

Examples

Are they DEs or not?

$ax^2 + bx + c = 0$ No

$ax^2 + bx' + c = 0$	Yes	Here $x' = \frac{dx}{dt}$
$ax^2 + bx' + cy' = 0$	Yes	Here $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$
$y'' = x^3$	Yes	Here $y' = \frac{dy}{dt}$

To solve a DE is to express the solution of the unknown function (the dependent variable) in mathematical terms without the derivatives.

Example

$$ax' + b = 0$$

$$x' = -\frac{b}{a}$$ is not a solution

$$x = -\int \frac{b}{a} dt$$ is a solution

In general, there are two common ways in solving DEs, analytic and numerical. Most DEs, difficult to solve by analytical methods, must be "solved" by numerical methods although many DEs are too stiff to solve using numerical techniques. Solving DEs by numerical methods is a different subject requiring basic knowledge of computer programming and numerical analysis; this book focuses on introducing analytical methods for solving very small families of DEs that are truly solvable. Although the DEs are quite simple, the solution methods are not and the essential solution steps and terminologies involved are fully applicable to much more complicated DEs.

Classification of DEs

Classification of DEs is itself another subject in studying DEs. We will introduce classifications and terminologies for flowing the contents of the book but one may still need to lookup terms undefined here. First, we introduce the terms of dependent and independent variables or functions. A dependent variable represents the output or effect while the independent variable represents the input or the cause. Truly, a DE is an equation that relates these two variables. A DE may have more than one variable for each and the DE with one independent variable and one dependent variable is called ordinary differential equation or ODE. The ODE, and simply referred to as DE, is the object of our study. Continuing, you understand why this 1-to-1 relationship is called *ordinary*. A DE that has one dependent variable and $N \geq 2$ independent variables, 1-to-N relationship, is called partial differential equation or PDE. Studying PDEs, out of the scope of this book, requires solid understanding of partial derivatives and, more desirably, full knowledge of multiple variable calculus. By now, you certainly want to see two other types of DEs. With $N \geq 2$ dependent variables and 1 independent variable, you may compose an N-to-1 *system* of ODEs. Similarly, with $N \geq 2$ dependent variables and $M \geq 2$ independent variables, you may compose an N-to-M *system* of PDEs.

We have several other classifications to categorize DEs.

Order of DEs

The order of a DE is determined by the highest order derivative of the dependent variable (and sometimes we interchangeably call it unknown function). If you love to generalize things, the algebraic equations may be classified as the 0th order DEs as there are no derivatives for the unknowns in the algebraic equations.

Examples

Determine the order of the following DEs:

$x' + ax = 0$ $x' + ax^2 = 0$	1st-order DE
$x'' + bx = 0$ $x'' + bx^4 = 0$	2nd-order DE
$x^{(n)} + cx = 0$	n-th order DE

	Definition	**Examples**
ODEs vs. PDEs	ODEs contain only one independent variable. PDEs contain two or more independent variables.	ODE: $x'' + \omega^2 x = f(t)$ PDE: $u_{tt} = u_{xx} + u_{yy}$
Order of DEs	The highest order of the derivatives of the independent variables.	1st order: $y' = x + y$ 2nd order: $y'' = x + y^2$
Linear vs. Nonlinear	DEs containing nonlinear term(s) for the dependent variables are nonlinear regardless the nature of the independent variables	Linear: $y' = x^2 + y$ Nonlinear: $y'' = x + y^2$
Homogenous vs. Inhomogeneous	A 1st-order DE whose both sides are homogeneous functions of the same degree is a 1st-order homogeneous and otherwise it inhomogeneous. A linear DE whose terms contain only independent variable or its derivatives is homogeneous. (See text)	1st order Homogeneous: $(y')^2 = x^2 + xy$ 1st order Inhomogeneous: $y'' = x^2 + y^2$ Homogeneous: $y'' + y^2 \cos x = 0$ Inhomogeneous: $y'' + y = \cos x$

Linearity of DEs

DEs can also be classified as linear or nonlinear according to the linearity of the dependent variables regardless of the nature of the independent variables. A DE that contains only linear terms for the dependent variable or its derivative(s) is a linear DE. Otherwise, a DE that contains at least one nonlinear term for the dependent variable or its derivative(s) is a nonlinear DE.

Linear	**Nonlinear**
$y'' + y = 0$	$y'' + y^2 = 0$
$y^{(n)} + x + y = 0$	$y^{(n)} + xy + y^3 = 0$
$y^{(n)} + x^3 + y = 0$	$\left(y^{(n)}\right)^2 + x^3 + y = 0$

Solutions

A function that satisfies the DE is a solution to that DE. Seeking such function is the main content of the book while composing the DEs, which may excite engineering majors more, is the secondary objective of this book.

Solutions can also be classified into several categories, for example, general solutions, particular solutions and singular solutions. A general solution (G.S.), *i.e.*, complete solution, is the set of all solutions to the DE and it usually can be expressed in a function with arbitrary constant(s). A particular solution (P.S.) is a subset of the general solutions whose arbitrary constant(s) are determined by initial or boundary conditions or both. A singular solution is a solution that is singular. In a less convoluting term, a singular solution is one for which the DE (or the initial value problem or the Cauchy problem) fails to have a unique solution at some point on the solution. The set on which the solution is singular can be a single point or the entire real line.

Examples

Try to *guess* the solutions of the following DEs:

(1) $\frac{dx}{dt} + \omega^2 x = 0$

(2) $\frac{dx}{dt} + \omega^2 x = t$

(3) $\frac{d^2x}{dt^2} + \omega^2 x = 0$

(4) $\frac{d^2x}{dt^2} + \omega^2 x = \sin(\omega_1 t)$ where, in general, $\omega_1 \neq \omega$

(5) $\frac{d^2x}{dt^2} + \omega^2 x = \sin(\omega t)$

As briefly mentioned before, there are several methods to find the solution of DEs. The following two are common:

1. To obtain analytical (closed form) solutions. Only a small percentage of linear DEs and a few special nonlinear DEs are simple enough to allow findings of analytical solutions. In such studies, basic concepts and theorems concerning the properties of the DEs or their solutions are introduces.
2. To obtain numerical solutions. Most DEs in science, engineering, and finance are too complicated to allow findings of analytical solutions and numerical methods are the only approach to finding *approximate* numerical solutions. Unfortunately, most of these DEs are the most important for practical purposes. Indeed, every rose has its thorn. To solve DEs numerically, one has to acquire a different set of skills: numerical analysis and computer programming.

Problems

Problem 1.1.1 Verify by substitution that each given function is a solution of the given DE. Throughout these problems, primes denote derivatives with respect to (WRT) x.

$$y'' + y = 3\cos 2x\,, \qquad y_1 = \cos x - \cos 2x\,, \qquad y_2 = \sin x - \cos 2x$$

Problem 1.1.2 Verify by substitution that each given function is a solution of the given DE. Throughout these problems, primes denote derivatives WRT x.

$$x^2y'' - xy' + 2y = 0, \qquad y_1 = x\cos(\ln x), \qquad y_2 = x\sin(\ln x)$$

Problem 1.1.3 A function $y = g(x)$ is described by some geometric property of its graph. Write a DE of the form $dy/dx = f(x, y)$ having the function g as its solution (or as one of its solutions). The graph of g is normal to every curve of the form $y = x^2 + k$ (k is a constant) where they meet.

Problem 1.1.4 Determine by inspection at least one solution of DE $xy' + y = 3x^2$. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

Problem 1.1.5 Determine by inspection at least one solution of DE $y'' + y = 0$. That is, use your knowledge of derivatives to make an intelligent guess. Then test your hypothesis.

Problem 1.1.6 Verify by substitution that the given function is a solution of the given DE. Primes denote derivatives WRT x.

$$y' + 2xy^2 = 0, \qquad y = \frac{1}{1 + x^2}$$

Problem 1.1.7 Verify that $y(x)$ satisfies the given DE and then determine a value of the constant Cso that $y(x)$ satisfies the given initial condition (I.C.).

$$y' + 3x^2y = 0, \qquad y(x) = C\exp(-x^3), \qquad y(0) = 7$$

Problem 1.1.8 Verify that $y(x)$ satisfies the given DE and then determine a value of the constant C so that $y(x)$ satisfies the given I.C..

$$y' + y\tan x = \cos x, \qquad y(x) = (x + C)\cos x, \qquad y(\pi) = 0$$

Problem 1.1.9 Verify that $y(x)$ satisfies the given DE and then determine a value of the constant C so that $y(x)$ satisfies the given I.C..

$$y' = 3x^2(y^2 + 1), \qquad y(x) = \tan(x^3 + C), \qquad y(0) = 1$$

Problem 1.1.10 Verify that $y(x)$ satisfies the given DE and then determine a value of the constant C so that $y(x)$ satisfies the given I.C..

$$xy' + 3y = 2x^5, \qquad y(x) = \frac{1}{4}x^5 + \frac{C}{x^3}, \qquad y(2) = 1$$

Problem 1.1.11 Verify by substitution that the given functions are solutions of the given DE. Primes denote derivatives WRT x.

$$y'' = 9y, \qquad y_1 = \exp(3x), \qquad y_2 = \exp(-3x)$$

Problem 1.1.12 Verify by substitution that the given functions are solutions of the given DE. Primes denote derivatives WRT x.

$$\exp(y)\, y' = 1, \quad y(x) = \ln(x + C), \qquad y(0) = 0$$

Problem 1.1.13 Verify that $y(x)$ satisfies the given DE and then determine a value of the constant C so that $y(x)$ satisfies the given I.C.. In the equation and the solution n is a given constant parameter.

$$y' + nx^{n-1}y = 0; \quad y(x) = C\exp(-x^n), \qquad y(0) = 2014$$

1.2 Mathematical Models

Given below are a few examples that illustrate the process of translating scientific laws and principles into DEs. The independent variable in these examples is the time t while the other variables are dependent variables.

1.2.1 Newton's Law of Cooling

The time rate of change (the rate of change WRT time) of the temperature $T(t)$ of a body is proportional to the difference between $T(t)$ and the temperature A of the surrounding medium, which is assumed to be so big that the energy exchange between the medium and the body does not change the temperature of the medium. This is an approximation of true heat transfer in which the total energy is conserved.

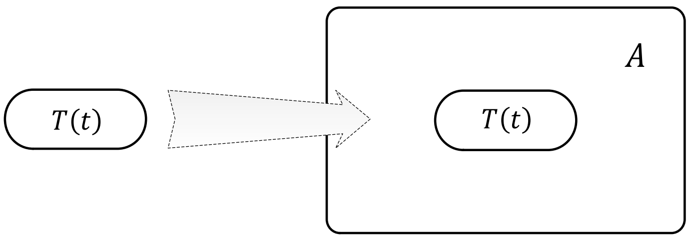

Figure 1.1 Place an object in a medium of temperature A.

Consider an object at a temperature $T(t)$ that is placed in a medium at a constant temperature A (Figure 1.1). We establish the DE that describes the temperature change of the object while it exchanges energy (heat) with the medium.

- A is the temperature of the medium.
- $T(t)$is the temperature of the object at time t; $T(t = 0) = T_0$ is the initial temperature.

- t is the time.
- $\frac{dT(t)}{dt} = k(A - T)$ is the rate of temperature variation, or the DE, and k is a positive constant that characterizes the heat conductivity of the medium with the body.

Solution

$$\frac{dT}{dt} = k(A - T)$$

$$\frac{dT}{T - A} = -kdt$$

Integrating both sides of the DE, we get

$$\int_{T_0}^{T} \frac{dT}{T - A} = -\int_0^t kdt$$

$$\ln(T - A) - \ln(T_0 - A) = -kt$$

$$\ln\left(\frac{T - A}{T_0 - A}\right) = -kt$$

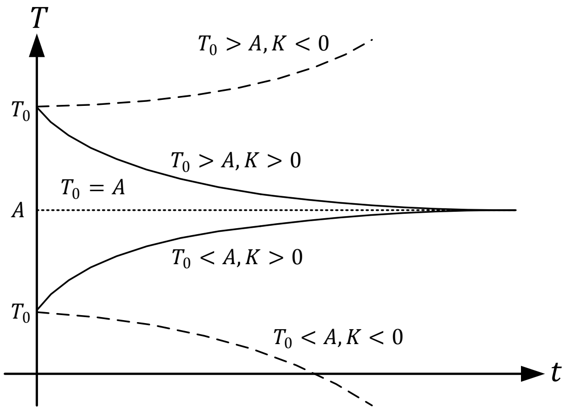

Figure 1.2 Plot of the object's temperature change T with time t.

$$\frac{T - A}{T_0 - A} = \exp(-kt)$$

$$T - A = (T_0 - A)\exp(-kt)$$

$$T(t) = A(t) + (T_0 - A)\exp(-kt)$$

Finally, we get the solution of the DE,

$$T(t) = T_0 \exp(-kt) + (1 - \exp(-kt))A$$

Thus, we have converted a scientific law into a DE and found its solution. Now, if given the values of A and k, we can predict the temperature of the object at any time.

Remarks

(1) If $A \approx T_0$ we can get $\frac{dT}{dt} \approx 0$ according to the original DE, meaning that $T(t)$ does not change with time much, approximately a constant.

(2) If $A > T_0$, we have $\frac{dT}{dt}$ >0.
The temperature of the object increases while heat is transferred from the medium to the object. This process will last till the object reaches the same temperature of the medium.

(3) If $A < T_0$, we have $\frac{dT}{dt} < 0$.
The temperature of the object decreases while heat is transferred to the medium from the object. This process will last till the object reaches the same temperature of the medium.

1.2.2 Newton's Law of Motion

Let the motion of a particle (a mathematical dot) of mass m along the straight line be described by its position function $x = f(t)$. Then, the velocity of the particle is

$$v(t) = \frac{dx}{dt} = x'(t) = f'(t)$$

and the acceleration is

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = v'(t) = x''(t) = f''(t)$$

If a force $F(t)$ acts on a particle and it is directed along its line of motion, then

$$F(t) = ma$$

Initial position: $x(t = 0) = x_0$
Initial speed: $v(t = 0) = v_0$

We can establish an equation of motion for the particle as

$$\begin{cases} m\dfrac{d^2x}{dt^2} = F(t) \\ v(t = 0) = x'(t = 0) = v_0 \\ x(t = 0) = x_0 \end{cases}$$

Solution

Let us suppose that we are given the acceleration of the particle and that it is a constant.

$$\frac{dv}{dt} = a$$

So to find v we integrate both sides of the above equation.

$$v = \int a dt + C_1 = at + C_1$$

We know that the initial velocity at time $t = 0$ is v_0. So after plugging this information into the above equation we get $C_1 = v_0$, and thus

$$v = \frac{dx}{dt} = at + v_0$$

Now, integrating the speed, we get the position of the particle as

$$\begin{aligned} x(t) &= \int v(t)dt + C_2 \\ &= \int (at + v_0)dt + C_2 \\ &= \frac{1}{2}at^2 + v_0 t + C_2 \end{aligned}$$

Applying the I.C. $x(t = 0) = x_0$, we get $C_2 = x_0$.

Finally, we get the position of the particle as a function of time (the independent variable) and other parameters such as the initial position, speed, and the acceleration

$$x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$

So, if given the acceleration (through the force), the initial position and speed, we can determine the position and speed of the particle at any time.

1.2.3 Torricelli's Law for Draining

The time rate of change of the volume V of water or other liquid in a draining sink is proportional to the square root of the depth y of water in the tank.

$V(t)$ is the volume of water in the sink at time t. Although a hemisphere is used in this example (Figure 1.3), the draining model is applicable to containers of any conceivable shapes.

$y(t)$ is the height of the water surface, from the draining point, at time t. The draining model introduced by Torricelli can be written as

$$\frac{d\mathrm{V}}{dt} = k\sqrt{y} \propto -\sqrt{y}$$

where k is the so-called draining constant which depends on

1) Liquid property such as viscosity. The k value will be different for the following liquids: tea, honey, water, or gasoline, etc.
2) Size of the hole.
3) Pressure differential between inside and outside of the container.

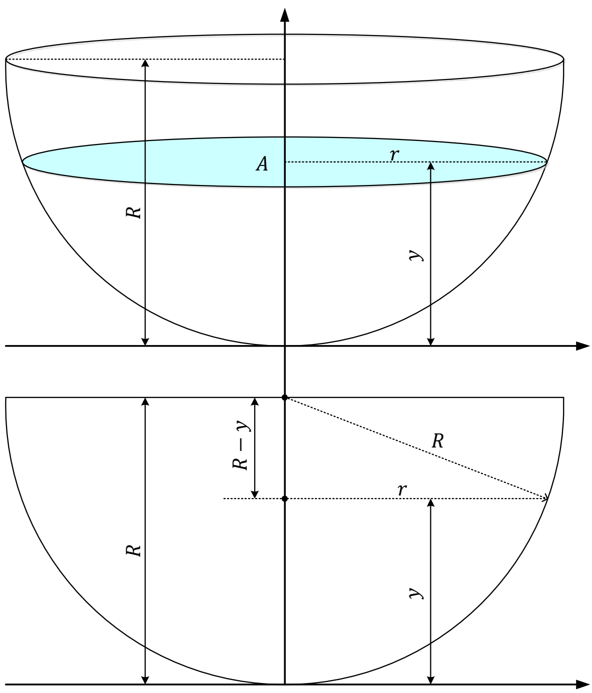

Figure 1.3 A hemisphere draining model of radius R.

Let $A(y)$ be the cross section area of the container at height y. We can write the volume of the container as a function of area and height.

$$dV = A(y)dy$$

where dV is the volume element of the container as a function of area and height variation from $y + dy$ to y. With the draining equation

$$\frac{dV}{dt} = -k\sqrt{y}$$

We can now write

$$\begin{cases} \frac{A(y)dy}{dt} = -k\sqrt{y} \\ y(t = 0) = y_0 \end{cases} \tag{1.1}$$

where y_0 is the initial height of liquid surface in the container. We have converted the original draining equation to involving independent variable t and the dependent variable y, the height.

Solution

Let us solve the above draining problem for a peculiar case: the container is a hemisphere whose inner radius is R. We can easily write the cross-section area at any height y as

$$\begin{aligned} A(y) &= \pi r^2 = \pi(R^2 - (R - y)^2) \\ &= \pi(R^2 - R^2 + 2Ry - y^2) \\ &= \pi y(2R - y) \end{aligned} \tag{1.2}$$

Plugging (1.2) into (1.1), we get

$$\frac{\pi y(2R - y)dy}{dt} = -k\sqrt{y}$$

$$\pi\sqrt{y}(2R - y)dy = -kdt$$

$$\int_R^y \pi\sqrt{y}(2R - y)dy = \int_0^t -kdt$$

$$\int_R^y \pi\left(2Ry^{\frac{1}{2}} - y^{\frac{3}{2}}\right)dy = -kt$$

$$\pi\left(2R\int_R^y y^{\frac{1}{2}}dy - \int_R^y y^{\frac{3}{2}}dy\right) = -kt$$

$$\pi\left(\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_R^y = -kt$$

$$\pi y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) - \pi R^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}R\right) = -kt$$

$$\pi y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) - \frac{14}{15}\pi R^{\frac{5}{2}} = -kt$$

Therefore, the resulting solution of the draining equation can be written as:

$$\pi y^{\frac{3}{2}}\left(\frac{4}{3}R - \frac{2}{5}y\right) = \frac{14}{15}\pi R^{\frac{5}{2}} - kt$$

This equation relates the height of the surface of the water in hemisphere container with time, in an implicit form. The initial state is that the hemisphere container is filled with water to the rim.

Remarks

(1) For $t = 0$, the only way the solution to make sense is $y = R$.
(2) The time for the tank to be empty is that $y = 0$. Plugging in $y = 0$ to the above equation, we can compute the time needed to empty the container with

$$y = 0 \Rightarrow kt = \frac{14}{15}\pi R^{\frac{5}{2}}$$

$$t_{\text{empty}} = \frac{14}{15}\frac{\pi}{k}R^{\frac{5}{2}}$$

a) From the above equation we have, $t_{\text{empty}} \propto R$. So the bigger the tank (the larger its radius), the longer the time needed to empty it: $R \uparrow \Rightarrow t_{\text{empty}} \uparrow$.
A special case, if the container is tiny, little time is needed to empty it: $R \to 0 \Rightarrow t_{\text{empty}} \to 0$.

b) From the above equation we also have, $t_{\text{empty}} \propto \frac{1}{k}$. Thus, $k \uparrow \Rightarrow t_{\text{empty}} \downarrow$.
A special case, if the draining constant is huge (a huge leaking hole, or a very slippery liquid, or a massive pressure differential), little time is needed to empty it: $k \to \infty \Rightarrow t_{\text{empty}} \to 0$.
The opposite is also true: if the draining constant is tiny (a tiny leaking hole, or a very stuffy liquid, or no pressure differential), it takes eternity to empty it: $k \to 0 \Rightarrow t_{\text{empty}} \to \infty$.

So far, everything appears to make sense!

1.2.4 Population Models

The time rate of change of a population $P(t)$ with constant birth and death rates is in many cases proportional to the size of the population.

$$P(t) = \text{Population at time } t$$

We can establish a simple initial value problem (IVP) to describe the population as a function of time.

$$\begin{cases} \dfrac{dP}{dt} \propto P \Rightarrow \dfrac{dP}{dt} = kP \\ P(t=0) = P_0 \end{cases}$$

Solution

$$\frac{dP}{dt} = kP$$
$$\frac{dP}{p} = kdt$$
$$\int_{P_0}^{P} \frac{dP}{P} = \int_0^t kdt$$
$$\ln P - \ln P_0 = kt$$
$$P(t) = P_0 \exp(kt)$$

For example, if $P(t = 0) = 1000$ and $P(t = 1) = 2000$, we can get

$$\begin{cases} P_0 = 1000 \\ k = \ln 2 \end{cases}$$

with which we can construct the model of exponential growth of populations as

$$P(t) = 1000 \exp(t \ln 2) = 1000 \cdot 2^t$$

In this case, the population doubles every time unit.

1.2.5 A Swimmer's Problem

The picture (Figure 1.4) shows a northward flowing river of width $w = 2a$. The lines $x = \pm a$ represent the banks of the river and the y-axis is the center. Suppose that the velocity v_R at which the water flows increases as one approaches the center of the river and is given in terms of distance x from the center by

$$v_R = v_0 \left(1 - \frac{x^2}{a^2}\right)$$

where $v_R(x)$ is the water speed at location x, v_0 is the water speed at the center of the river and v_s is the swimmer speed.

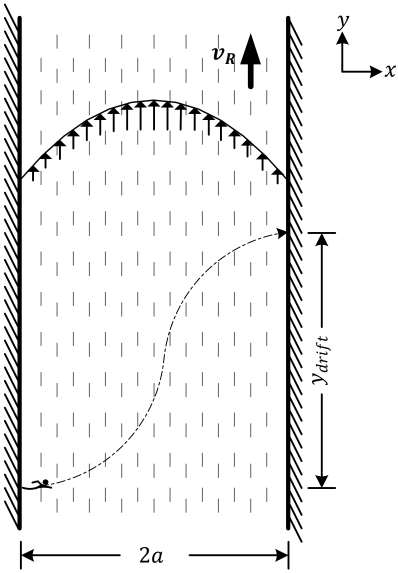

Figure 1.4 A swimmer model.

Remarks

(1) The water speed at the center of the river is maximum:

$$x = 0, v_R = v_0(1 - 0) = v_0$$

(2) The water speed at either bank of the river is zero:

At left bank where $x = -a$

$$v_R = v_0(1 - 1) = 0$$

At right bank where $x = a$

$$v_R = v_0(1 - 1) = 0$$

The two cases where the speed of water is zero at the riverbanks are called no-slip conditions, a well-known result in fluid mechanics.

(3) The water speed profile formula can be derived from more basic laws of physics. The derivation is out of the scope and we use it without proof.

A Swimmer's Problem

Suppose that the swimmer starts at $(-a, 0)$ on the west bank and swims at the east direction, as shown in Figure 1.4, the swimmer's velocity vector relative to the ground had horizontal component v_S and vertical component v_R. Hence the swimmer's instantaneous directional angle α is given by

$$\frac{dy}{dx} = y' = \tan\alpha = \frac{v_R}{v_S} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)$$

Thus, we have an IVP that describes the trajectory of the swimmer.

$$\begin{cases} y' = \dfrac{v_0}{v_S}\left(1 - \dfrac{x^2}{a^2}\right) \\ y(x = -a) = 0 \end{cases}$$

Solution

Solving this problem, we have

$$dy = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$\int_0^y dy = \int_{-a}^x \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$y = \frac{v_0}{v_S}\left(x - \frac{x^3}{3a^2}\right)\Bigg|_{-a}^{x}$$

Finally, we obtain the equation for describing the trajectory of the swimmer as

$$y = \frac{v_0}{v_S}\left(x - \frac{x^3}{3a^2} + \frac{2a}{3}\right)$$

Remarks

We can see at the east bank of the river $x = a$, the swimmer would have drifted by

$$\begin{aligned} y_{\text{drift}}(x = a) &= \frac{v_0}{v_S}\left(a - \frac{a^3}{3a^2} + \frac{2a}{3}\right) \\ &= \frac{v_0}{v_S}\frac{4a}{3} \end{aligned}$$

(1) If $a \uparrow \Rightarrow y_{\text{drift}} \uparrow$, meaning that the wider the river, the longer the drift.

(2) If $V_0 \uparrow \Rightarrow y_{\text{drift}} \uparrow$, meaning that the faster the river flow, the longer the drift.

(2a) $v_0 \to 0 \implies y \to 0$, a special case: if the river water does not move (e.g., a big lake), you would not drift.

(2b) $v_0 \to \infty \implies y \to \infty$, a special case: if the river flows very fast (e.g., Niagara River), you would drift very far.

(3) If $V_S \uparrow \Rightarrow y_{\text{drift}} \downarrow$, meaning that the faster the swimmer, the shorter the drift.

(3a) $v_S \to \infty \implies y \to 0$

(3b) $v_s \to 0 \implies y \to \infty$

1.2.6 Slope Fields & Solution Curves

Suppose we have a DE

$$\begin{cases} y' = f(x, y) \\ y(a) = b \end{cases}$$

(1) Does the solution exist?
(2) How many solutions exist? Is it unique?
(3) How to find it when it exists?

Here we define some concepts that can help us study the properties of the given DE

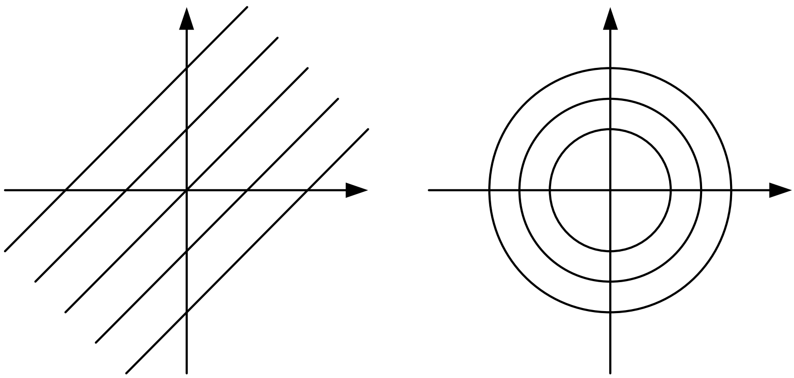

Figure 1.5 Examples of slope field and isocline.

Slope Curves: A line in x, y plane whose slope $\tan\alpha$ is $f(x, y)$.

Slope Field: A collection of the slope curves. (Figure 1.5)

Isocline: A series of lines with the same slope, such as a family of curves $f(x, y) = c$. (Figure 1.5)

Example 1

Find the isoclines

$$y' = x^2 + y^2 = C$$

Solution

The Isocline equation

$$y' = f(x, y) = x^2 + y^2 = C$$

It is a circle

Example 2

Find the isoclines

$$y' = \sin(x - y)$$

Solution

$$x - y = \arcsin c$$
$$y = x - \arcsin c$$

Problems

Problem 1.2.1 A car traveling at 60 mph (88ft/s) skids 176ft after its brakes are suddenly applied. Under the assumption that the braking system provides constant deceleration, what is that deceleration? For how long does the skid continue?

Problem 1.2.2 Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x(0) = x_0$ and initial velocity $v(0) = v_0$.

$$\begin{cases} a(t) = 50\sin(5t) \\ v_0 = -10 \\ x_0 = 8 \end{cases}$$

Problem 1.2.3 Find the position function $x(t)$ of a moving particle with the given acceleration $a(t)$, initial position $x(0) = x_0$ and initial velocity $v(0) = v_0$.

$$\begin{cases} a(t) = \dfrac{1}{(t+1)^n} \\ v_0 = 0 \\ x_0 = 0 \end{cases}, \text{where } n \geq 3$$

Problem 1.2.4 We drain a fully filled spherical container of inner radius *R* by drilling a hole at the bottom. We assume the draining process follows Torricelli's law and the draining constant *k*.

(1) Compute the time needed to empty the container.

(2) If we double the radius of the container which is also fully filled and keep all other conditions unchanged, compute the draining time again.

Problem 1.2.5 A complete spherical container of radius R has two small holes. One at the top for air to come in, and one at the bottom for liquid to drain out (think of it as a hollowed bowling ball). At $t = 0$, the container is filled with liquid. Besides the radius R, you are also given the draining constant k. The draining process follows Torricelli's law. Now please do the following,
(1) Compute the time T_1 needed to drain the upper half of the container (i.e., the water level dropping from the top hole to the equator line of the ball).
(2) Compute the time T_2 needed to drain the lower half of the container (i.e., the water level dropping from the equator line of the ball to the bottom hole).
(3) Which relationship is correct $T_1 > T_2$, $T_1 < T_2$, or $T_1 = T_2$?

Problem 1.2.6 A coffee cup from the "Moonbucks" coffee shop is like a cylinder with decreasing horizontal cross-section area. The radius of the top circle is R, and that for the bottom is 10% smaller. The cup height is H. Now, a nutty kid drills a pinhole at the half height point of the cup, allowing coffee to leak under the Draining Equation with a draining constant k. Compute the time for coffee to leak to the pinhole. Compute the time again if we turn the cup upside down with full cup to start.

Problem 1.2.7 A spherical container of inner radius R is originally filled up with liquid. We drill one hole at the bottom, and another hole at the top (for air in-take), to allow liquid to drain from the bottom hole. The draining process follows the Torricelli's law, and the draining constant depends on the liquid properties; the hole size, and the inside-outside pressure differential, etc. While draining the upper half, we make the draining constant as k. What draining constant must we set (e.g., by adjusting the bottom hole size) so that the lower half will take the same amount of time to drain as the upper half did?

Problem 1.2.8 A spherical container of inner radius R is filled up with liquid. We drill one hole at the bottom (and another at the top to allow air to come in). The draining process follows the Torricelli's law with the usual draining constant k. We start draining the container, when it is full, until the leftover volume reaches a "magic" value at which the draining time for the top portion is equal to that for the bottom (left-over) portion. Please calculate the leftover volume.

Problem 1.2.9 During the 2014 Stony Brook University Roth Quad Regatta boat race, students propelled their boats from one end to the other of a long pond (not a river) of length L. Since it is not a river, water movement and wind have little effect on the boats. Suppose your boat's mass is M and each of your boaters' mass is m (all boaters are equal in mass). The propelling force from each boater during the entire race is an equal constant f_0. Water resistance is approximated to be proportional to the speed of the boat with a constant μ_0 regardless of the weight of the boat plus boaters. Your boat is always still at the start, and the boaters are the only power source.

(1) Establish the DE, for n boats for relating the boat's speed with time and the given parameters.

(2) Solve the DE established above to find the speed as a function of time.

(3) Which leads to the shortest travel time with all other conditions fixed? One boater or as many as possible?

1.3 Separation of Variables

All 1st-order IVP can be written as

$$\begin{cases} y' = F(x, y) \\ y(x = a) = b \end{cases}$$

If $F(x, y)$ can be written as

$$y' = \frac{g(x)}{f(y)}$$

Then the 1st-order IVP is said to be separable.

Solution

$$\frac{dy}{dx} = \frac{g(x)}{f(y)}$$
$$\int f(y)dy = \int g(x)dx$$
$$F(y) = G(x) + C$$

where C is a constant determined by the I.C.: $y(a) = b$.

Example 1A

Solve

$$\begin{cases} y' = -6xy \\ y(0) = 7 \end{cases}$$

Solution

$$\frac{dy}{dx} = -6xy \xRightarrow{\text{Separable DE}} \int \frac{dy}{y} = \int -6x dx$$
$$\ln y = -3x^2 + \ln C$$
$$y = C\exp(-3x^2)$$

which is the G.S. of the DE. Plugging in I.C. $y(0) = 7$ into $y = C\exp(-3x^2)$, we will get

$$y(0) = C\ \exp(0) = 7$$

This gives

$$C = 7$$

Therefore, the P.S. is

$$y(x) = 7\exp(-3x^2)$$

Example 1B

Is $y = 0$ a solution for equation $y' = -6xy$?

Solution

$$\text{Left Hand Side (LHS)} = y' = 0$$
$$\text{Right Hand Side (RHS)} = -6xy = -6x \cdot 0 = 0$$
$$\text{LHS} = \text{RHS}$$

Therefore

$$y = 0$$

is a P.S. for $y' = -6xy$.

Example 2

Solve IVP

$$\begin{cases} y' = \dfrac{4-2x}{3y^2-5} \\ y(1) = 3 \end{cases}$$

Solution

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$$

$$\int (3y^2-5)dy = \int (4-2x)dx$$

Integrate with both side, we have

$$y^3 - 5y = 4x - x^2 + C$$

as the G.S.. Plugging the I.C. $y(1) = 3$ gives

$$3^3 - 15 = 4 - 1 + C$$
$$C = 9$$

Therefore, the P.S. for the IVP is

$$y^3 - 5y = 4x - x^2 + 9$$

Example 3

Solve

$$y^2 + x^2y' = 0$$

Solution

$$y^2 + x^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2}$$
$$\frac{1}{y^2}dy = -\frac{1}{x^2}dx$$
$$\int \frac{1}{y^2}dy = -\int \frac{1}{x^2}dx$$
$$-\frac{1}{y} = \frac{1}{x} + C$$

Therefore,

$$\frac{1}{x} + \frac{1}{y} = C$$

is the G.S. of the given DE.

Singular Solutions

Suppose we have a DE

$$y' = H(y)G(x)$$

To solve this DE, we have to divide both sides by $H(y)$.

$$\frac{dy}{H(y)} = G(x)dx$$

That gives

$$\int \frac{dy}{H(y)} = \int G(x)dx$$

Such division is mathematically allowed if and only if $H(y) \neq 0$. Therefore

$$H(y) = 0$$

is a solution for the equation because it indeed satisfies the original DE. This solution is peculiar so it is called singular solution.

Problems

Problem 1.3.1 Find G.S. of the DE.

$$3x(y-2)dx + (x^2+1)dy = 0$$

Problem 1.3.2 Find G.S. (implicit if necessary, explicit if convenient) of the DE in

$$y' = (64xy)^{\frac{1}{3}}$$

Prime denotes derivatives WRT x.

Problem 1.3.3 Find G.S. (implicit if necessary, explicit if convenient) of the DE in

$$y' = 1 + x + y + xy$$

Prime denotes derivatives WRT x. (Suggestion: Factor the right-hand side.)

Problem 1.3.4 Find a function satisfying the given DE and the prescribed I.C..

$$\begin{cases} \dfrac{dy}{dx} = x\exp(-x) \\ y(0) = 1 \end{cases}$$

Problem 1.3.5 Find explicit P.S. of the IVP in

$$\begin{cases} y' = -2\cos 2x \\ y(0) = 2014 \end{cases}$$

Problem 1.3.6 Find explicit P.S. of the IVP in

$$\begin{cases} 2\sqrt{x}y' = \cos^2 y \\ y(4) = \dfrac{\pi}{4} \end{cases}$$

Problem 1.3.7 Find the explicit P.S. of the IVP in

$$\begin{cases} y' = 2xy + 3x^2 y\exp(x^3) \\ y(0) = 5 \end{cases}$$

Problem 1.3.8 Find the G.S. (implicit if necessary, explicit if convenient) of the DE in the following problem. Primes denote derivatives WRT x.

$$y^3 y' = (y^4 + 1)\cos x$$

Problem 1.3.9 Find the G.S. of following DE.

$$4xy^2 + y' = 5x^4y^2$$

Problem 1.3.10 Find explicit P.S. of the IVP in

$$\begin{cases} y' = 2xy^2 + 3x^2y^2 \\ y(1) = -1 \end{cases}$$

Problem 1.3.11 Find the G.S. (implicit if necessary, explicit if convenient) of the DE in the following problem. Primes denote derivatives WRT x.

$$y' = 2x\sec y$$

Problem 1.3.12 Find the explicit P.S. of the IVP in

$$\begin{cases} \tan x\, y' = y \\ y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \end{cases}$$

Problem 1.3.13 Find the G.S. (implicit if necessary, explicit if convenient) of the DE in the following problem. Primes denote derivatives WRT x.

$$(x^2 + 1)\tan(y)\, y' = x$$

Problem 1.3.14 Find the G.S. (implicit if necessary, explicit if convenient) of the DE in the following problem.

$$(x + 3)^3 y' = (y - 2)^2$$

Problem 1.3.15 Solve the following DE.

$$\begin{cases} xy' - x^2y = 3xy \\ y(1) = 1 \end{cases}$$

Problem 1.3.16 Solve the following DE.

$$\begin{cases} (1 + x)y' + y = \cos(x) \\ y(0) = 1 \end{cases}$$

Problem 1.3.17 Solve the following DE

$$\frac{dy}{dt} = \beta y(\alpha - \ln y)$$

where, obviously, α and β are parameters independent of y and t. If $y(t = 0) = y_0$ and $y(t \to \infty) = y_\infty$, can you express α and/or β in terms of y_0 and y_∞? If so, do it.

1.4 Linear First-Order DEs

All linear 1st-order DEs can be written in the following form:

$$A(x)y' + B(x)y = C(x) \tag{1.3}$$

If $A(x) = 0$, then it is not a DE. Now suppose $A(x) \neq 0$.

If $B(x) = 0$,

$$A(x)y' = C(x)$$

It is a separable DE.

If $C(x) = 0$,

$$A(x)y' + B(x)y = 0$$

Dividing both side of above equation by y gives

$$\frac{y'}{y} = -\frac{B(x)}{A(x)}$$

It is a separable DE.

For the general form of linear 1st-order DE, we divide both sides of (1.3) by $A(x)$,

$$y' + \frac{B(x)}{A(x)}y = \frac{C(x)}{A(x)}$$

Let

$$\frac{B(x)}{A(x)} = P(x) \quad \text{and} \quad \frac{C(x)}{A(x)} = Q(x)$$

We have

$$y' + P(x)y = Q(x) \tag{1.4}$$

Let us now derive the method to solve the general linear 1st-order DE in the form of (1.4). The idea to get rid of $P(x)y$ term is to compose a whole derivative as a function of $\rho(x)y$:

$$\rho(x)(y' + P(x)y) = Q(x)\rho(x)$$

$$\rho(x)y' + \rho(x)P(x)y = Q(x)\rho(x) \tag{1.5}$$

If we can make the LHS of (1.5)

$$\rho(x)y' + \rho(x)P(x)y = \frac{d}{dx}(\rho(x)y) \tag{1.6}$$

then (1.5) can be easily solved as a function of $\rho(x)y$. We will prove later that the above relation we introduced is proper. Eq-(1.6) means we have

$$\rho(x)y' + \rho(x)P(x)y = \rho(x)y' + \rho'(x)y$$

Cancelling $\rho(x)y'$ on both sides of the above equation, we have

$$\rho(x)P(x)y = \rho'(x)y$$

$$\rho(x)P(x) = \rho'(x)$$

This is a separable DE with unknown function $\rho(x)$. To solve this DE, we have

$$\frac{d\rho(x)}{\rho(x)} = P(x)dx$$

$$\int \frac{d\rho}{\rho} = \int P(x)dx$$

$$\ln \rho = \int P(x)dx$$

Therefore, we have

$$\rho(x) = \exp\left(\int P(x)dx\right) \tag{1.7}$$

This is called the integrating factor of DE (1.4). Now we know that $\rho(x)$ from Eq-(1.7) satisfies the property of (1.6). Plugging (1.6) back to (1.5), we have

$$\frac{d}{dx}(\rho(x)y) = Q(x)\rho(x) \tag{1.8}$$

which is easy to solve

$$\rho(x)y = \int Q(x)\rho(x)dx$$

Thus, the solution to the original DE (1.4) is

$$y(x) = \frac{1}{\rho(x)}\left(\int Q(x)\rho(x)dx\right)$$

Plugging back (1.7) generates the solution

$$y(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right) \tag{1.9}$$

Now we show the proof of how (1.6) satisfies. Since we have the integrating factor

$$\rho(x) = \exp\left(\int P(x)dx\right)$$

We can prove

$$\exp\left(\int P(x)dx\right)y' + \exp\left(\int P(x)dx\right)P(x)y = \frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right)$$

Here we have

$$\begin{aligned}
\text{RHS} &= \frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right) \\
&= y'\exp\left(\int P(x)dx\right) + y\left(\exp\left(\int P(x)dx\right)\right)' \\
&= y'\exp\left(\int P(x)dx\right) + y\exp\left(\int P(x)dx\right)\left(\int P(x)dx\right)' \\
&= y'\exp\left(\int P(x)dx\right) + y\exp\left(\int P(x)dx\right)P(x) \\
&= \text{LHS}
\end{aligned}$$

Hence the proof.

We now outline the steps on how to solve linear DE without memorizing the formula.

STEP 1: Convert DE to the form $y' + P(x)y = Q(x)$
STEP 2: Identify $P(x)$
STEP 3: Compute integrating factor

$$\rho(x) = \exp\left(\int P(x)dx\right)$$

STEP 4: Multiply both sides of the converted DE with the integrating factor

$$\exp\left(\int P(x)dx\right)y' + \exp\left(\int P(x)dx\right)P(x)y = Q(x)\exp\left(\int P(x)dx\right)$$

STEP 5: Write the LHS as a whole derivative

$$\exp\left(\int P(x)dx\right)y' + \exp\left(\int P(x)dx\right)P(x)y = \frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right)$$

Therefore

$$\frac{d}{dx}\left(\exp\left(\int P(x)dx\right)y\right) = Q(x)\exp\left(\int P(x)dx\right)$$

$$\exp\left(\int P(x)dx\right)y = \int Q(x)\exp\left(\int P(x)dx\right)dx$$

Finally, the solution of the DE is

$$y(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)$$

Example 1
Solve

$$(x^2+1)y' + 3xy = 6x$$

Solution
STEP 1: Converting the DE

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

STEP 2: Identify

$$P(x) = \frac{3x}{x^2+1}$$

STEP 3: Computing the integrating factor.

$$\begin{aligned}\rho(x) &= \exp\left(\int \frac{3x}{x^2+1}dx\right)\\ &= \exp\left(\frac{3}{2}\int \frac{d(x^2+1)}{x^2+1}\right)\\ &= \exp\left(\frac{3}{2}\ln(x^2+1)\right)\\ &= (x^2+1)^{\frac{3}{2}}\end{aligned}$$

STEP 4: Multiplying the DE by the integrating factor.

$$y'(x^2+1)^{\frac{3}{2}} + \frac{3x}{x^2+1}y(x^2+1)^{\frac{3}{2}} = \frac{6x}{x^2+1}(x^2+1)^{\frac{3}{2}}$$

$$y'(x^2+1)^{\frac{3}{2}} + 3xy(x^2+1)^{\frac{1}{2}} = 6x(x^2+1)^{\frac{1}{2}}$$

STEP 5: Completing the solution.

$$\frac{d}{dx}\left(y(x^2+1)^{\frac{3}{2}}\right) = 6x(x^2+1)^{\frac{1}{2}}$$

$$y(x^2+1)^{\frac{3}{2}} = \int 6x(x^2+1)^{\frac{1}{2}}dx$$

$$y = \frac{1}{(x^2+1)^{\frac{3}{2}}}\int 3(x^2+1)^{\frac{1}{2}}d(x^2+1)$$

$$y = (x^2+1)^{-\frac{3}{2}}\left(2(x^2+1)^{\frac{3}{2}} + C\right)$$

$$y = 2 + C(x^2+1)^{-\frac{3}{2}}$$

This is the solution to the original DE.

Example 2

Solve

$$\begin{cases} x^2y' + xy = \sin x \\ y(1) = y_0 \end{cases}$$

Solution

Method I: Step-by-step solution with the integrating factor.

$$y' + \frac{y}{x} = \frac{\sin x}{x^2}$$

The integrating factor

$$\rho(x) = \exp\left(\int \frac{1}{x}dx\right) = \exp(\ln x) = x$$

Multiplying the integrating factor yields

$$(xy)' = \frac{\sin x}{x^2} \cdot x$$
$$xy = \int \frac{\sin x}{x} dx$$
$$y = \frac{1}{x} \int \frac{\sin x}{x} dx + \frac{C}{x}$$
$$y(x) = \frac{1}{x} \int_{x_0}^{x} \frac{\sin t}{t} dt + \frac{C}{x}$$

This is the G.S. to the original DE.
Using the I.C. $y(1) = y_0$

$$y(1) = y_0 = \frac{1}{1} \int_{x_0}^{1} \frac{\sin t}{t} dt + \frac{C}{1}$$
$$C = y_0 - \int_{x_0}^{1} \frac{\sin t}{t} dt$$

Now plugging the C back into $y(x)$, we have

$$y(x) = \frac{1}{x} \int_{x_0}^{x} \frac{\sin t}{t} dt + \frac{y_0}{x} - \frac{1}{x} \int_{x_0}^{1} \frac{\sin t}{t} dt$$
$$= \frac{y_0}{x} + \frac{1}{x} \int_{1}^{x} \frac{\sin t}{t} dt$$

Method II: Direct use the formula (1.9)

$$y = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)$$
$$P(x) = \frac{1}{x}, \qquad Q(x) = \frac{\sin x}{x^2}$$
$$y(x) = \exp\left(\int \frac{1}{x} dx\right)\left(\int \frac{\sin x}{x^2} \exp\left(\int \frac{1}{x} dx\right) dx + C\right)$$
$$= \frac{1}{x} \int_{x_0}^{x} \frac{\sin t}{t} dt + \frac{C}{x}$$

Using the I.C. $y(1) = y_0$, we have

$$C = y_0 - \int_{x_0}^{1} \frac{\sin t}{t} dt$$

Therefore,

$$y(x) = \frac{y_0}{x} + \frac{1}{x} \int_{1}^{x} \frac{\sin t}{t} dt$$

Problems

Problem 1.4.1 Find G.S. and the corresponding P.S. of the following DE.

$$\begin{cases} (x^2+1)y' + 3x^3y = 6x\exp\left(-\dfrac{3}{2}x^2\right) \\ y(0) = 1 \end{cases}$$

Problem 1.4.2 (a) Show that

$$y_c(x) = C\exp\left(-\int P(x)dx\right)$$

is a G.S. of $y' + P(x)y = 0$.
(b) Show that

$$y_p(x) = \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)$$

is a P.S. of $y' + P(x)y = Q(x)$.
(c) Suppose that $y_c(x)$ is any G.S. of $y' + P(x)y = 0$ and that $y_p(x)$ is any P.S. of $y' + P(x)y = Q(x)$. Show that $y(x) = y_c(x) + y_p(x)$ is a G.S. of $y' + P(x)y = Q(x)$.

Problem 1.4.3 Solve the following IVP.

$$\begin{cases} xy' + 2y = 7x^2 \\ y(2) = 5 \end{cases}$$

Problem 1.4.4 Find the G.S. of the following DE.

$$y' + y\cot x = \cos x$$

Problem 1.4.5 Find the G.S. of the following DE by two different methods.

$$y' = 3(y+7)x^2$$

Problem 1.4.6 Find the G.S. of the following DE.

$$xy' = 2y + x^3\cos x$$

Problem 1.4.7 Find the G.S. of the following DE.

$$(2x+1)y' + y = (2x+1)^{\frac{3}{2}}$$

Problem 1.4.8 Find the G.S. of the following DE by two different methods.

$$y' = \frac{2xy + 2x}{x^2 + 1}$$

Problem 1.4.9 Find the G.S. of the following DE.

$$(1 + 2xy)y' = 1 + y^2$$

(Hint: regard x as depending variable and y as independent)

Problem 1.4.10 Find the G.S. of the following DE.

$$2xy' + y = 10\sqrt{x}$$

Problem 1.4.11 Find the G.S. of the following DE.

$$(x + y\exp(y))y' = 1$$

(Hint: regarding x as depending variable and y as independent)

Problem 1.4.12 Find the G.S. of the following DE.

$$2y + (x + 1)y' = 3(x + 1)$$

Problem 1.4.13 Solve the following IVP.

$$\begin{cases} (1 + x)y' + y = \sin x \\ \quad y(x = 0) = 1 \end{cases}$$

Problem 1.4.14 Find the G.S. of the following second-order DE.

$$x^2y'' + 3xy' = 4x^4$$

(Hint: introduce substitution $v = y'$)

1.5 Substitution Methods

Substitution methods are those of introducing one or more new variables to represent one or more variables that will convert the original DE to a separable or other more easily solvable DE.

For example, one can change one variable x to u.

$$F_1(x,y) = 0 \xRightarrow{f(x)\to u} F_2(u,y)$$

One may also consider changing both variables.

$$G_1(x,y) = 0 \xRightarrow[]{\substack{g(x,y)\to u \\ h(x,y)\to v}} G_2(u,v)$$

Let us see experience power of the substitution methods by solving several families of DEs.

1.5.1 Polynomial Substitution

Solve

$$y' = F(ax + by + c)$$

where a, b, and c are constants.

STEP 1: Introducing a new variable v.

$$v = ax + by + c$$

STEP 2: Transform the DE into a DE of v.

$$v' = a + by'$$

$$y' = \frac{1}{b}v' - \frac{a}{b}$$

After the substitution, the original equation now becomes

$$y' = F(v)$$

Plugging back the equation for y', we have

$$\frac{1}{b}v' - \frac{a}{b} = F(v)$$

$$\frac{1}{b}v' = F(v) + \frac{a}{b}$$

which is now separable.

STEP 3: Solve the above DE.

$$\frac{1}{b}\frac{dv}{dx} = F(v) + \frac{a}{b}$$

$$\frac{\frac{1}{b}dv}{F(v) + \frac{a}{b}} = dx$$

$$\frac{dv}{bF(v) + a} = dx$$

$$x = \int \frac{dv}{bF(v) + a}$$

Example

Solve

$$y' = (x + y + 3)^2$$

Solution

Let

$$v = x + y + 3$$
$$v' = 1 + y'$$
$$y' = v' - 1$$

After the substitution, the original equation becomes

$$v' - 1 = v^2$$

$$\frac{dv}{v^2+1} = dx$$
$$\int \frac{dv}{v^2+1} = \int dx$$
$$\tan^{-1} v = x + C$$
$$v = \tan(x + C)$$

Substituting the value of v back into the above equation, we now obtain

$$x + y + 3 = \tan(x + C)$$

That is

$$y = \tan(x + C) - x - 3$$

1.5.2 Homogeneous DEs

For the 1st-order homogeneous DEs

$$y' = F\left(\frac{y}{x}\right)$$

we make a different substitution,

$$v = \frac{y}{x}$$

Let

$$v = \frac{y}{x}$$

$$y = xv$$

$$y' = xv' + v$$

Substituting this in the original DE we have

$$xv' + v = F(v)$$

$$v' = \frac{F(v) - v}{x}$$

which is a separable DE. Solving this DE, we have

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

$$\int \frac{dv}{F(v) - v} + C = \ln x$$

$$x = \exp\left(\int \frac{dv}{F(v) - v} + C\right)$$

$$x = C \exp\left(\int \frac{dv}{F(v) - v}\right)$$

Example
Solve

$$2xyy' = 4x^2 + 3y^2$$

Solution
Diving the original DE by $2xy$

$$y' = \frac{2x}{y} + \frac{3y}{2x}$$

This DE is homogeneous.
Let

$$v = \frac{y}{x}$$

$$y = vx$$

$$y' = v + xv'$$

Substituting this back to the DE, we have

$$v + xv' = \frac{2}{v} + \frac{3}{2}v$$

$$v' = \frac{1}{x}\left(\frac{2}{v} + \frac{v}{2}\right)$$

$$\frac{2vdv}{4 + v^2} = \frac{dx}{x}$$

$$\int \frac{d(v^2 + 4)}{4 + v^2} = \int \frac{dx}{x}$$

$$\ln(4 + v^2) = \ln x + C$$

$$4 + v^2 = Cx$$

Plugging v back, we have

$$4 + \left(\frac{y}{x}\right)^2 = Cx$$

$$y^2 + 4x^2 = Cx^3$$

1.5.3 Bernoulli DEs

Given the DE

$$y' + P(x)y = Q(x)y^n \tag{1.10}$$

We have a few cases depending on the value of n.

CASE 1 ($n < 0$):

(1.10) is a general nonlinear 1st-order DE.

CASE 2 ($n = 0$):

(1.10) is the linear 1st-order DE $y' + P(x)y = Q(x)$ whose solution method was introduced before.

CASE 3 ($n = \frac{1}{2}$):

(1.10) is nonlinear 1st-order DE.

CASE 4 ($n = 1$):

(1.10) is linear 1st-order DE $y' + \big(P(x) - Q(x)\big)y = 0$ which is actually separable.

CASE 5 ($n = 2$):

(1.10) is nonlinear 1st-order DE.

CASE 6 ($n > 2$):

(1.10) is nonlinear 1st-order DE.

To sum up, for $n \neq 0, 1$, we know that (1.10) is a nonlinear 1st-order DE and we have not learned any methods to solve the DE for such cases. Now, let us use a substitution to solve the Bernoulli DEs for $n \neq 0, 1$.

Solution steps

STEP 1: Select a proper substitution (why this sub?)

$$v = y^{1-n}$$

STEP 2: Find the relation of $v' = g(y')$

$$v' = (1-n)y^{-n}y'$$

STEP 3: Substituting v into the DE by inserting the above formula

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

Since $n \neq 1$, we have

$$\frac{1}{1-n}\left((1-n)y^{-n}y'\right) + P(x)y^{1-n} = Q(x)$$

Plugging v and v' into the DE yields

$$\frac{v'}{1-n} + P(x)v = Q(x)$$

That is

$$v' + (n-1)P(x)v = (n-1)Q(x) \tag{1.11}$$

STEP 4: Solve this DE in terms of v. Since it becomes a linear DE, we can solve this DE by any of the methods that were introduced earlier.
STEP 5: After solving (1.11), we find v.
STEP 6: Back-substitute variable v with y to express the solution in the original variables.

Example
Solve

$$xy' + 6y = 3xy^{\frac{4}{3}}$$

Solution
It is a Bernoulli DE. Let us follow the step-by-step procedure.

STEP 1:

$$v = y^{1-n} = y^{-\frac{1}{3}}$$
$$y = v^{-3}$$

STEP 2:

$$v' = -\frac{1}{3}y^{-\frac{4}{3}}y'$$
$$\begin{aligned} y' &= -3v'(v^{-3})^{\frac{4}{3}} \\ &= -3v'v^{-4} \end{aligned}$$

STEP 3:

$$x(-3v'v^{-4}) + 6v^{-3} = 3x(v^{-3})^{\frac{4}{3}}$$
$$-3xv'v^{-4} + 6v^{-3} = 3xv^{-4}$$
$$v'x - 2v = -x$$
$$v' - \frac{2v}{x} = -1$$

STEP 4: Now we solve the above DE as a linear 1st-order DE

$$\rho(x) = \exp\left(\int P(x)dx\right)$$
$$P(x) = -\frac{2}{x}$$
$$\int P(x)dx = -\int \frac{2}{x}dx = -2\ln x$$
$$\rho(x) = \exp(-2\ln x) = x^{-2}$$

Multiplying the integrating factor to both side of the DE

$$\frac{1}{x^2}v' - \frac{2v}{x^3} = -\frac{1}{x^2}$$
$$\left(\frac{1}{x^2}v\right)' = -\frac{1}{x^2}$$
$$\frac{1}{x^2}v = -\int \frac{1}{x^2}dx$$
$$\frac{v}{x^2} = \frac{1}{x} + C$$
$$v = x + Cx^2$$

STEP 5: Plug y back

$$y^{-\frac{1}{3}} = x + Cx^2$$
$$y = \frac{1}{(x + Cx^2)^3}$$
$$y = \frac{1}{x^3(1 + Cx)^3}$$

Problems

Problem 1.5.1 Find G.S. of the following DE.

$$x^2y' = xy + y^2$$

Problem 1.5.2 Find G.S. of the following DE.

$$xyy' = y^2 + x\sqrt{4y^2 + x^2}$$

Problem 1.5.3 Show that the substitution $v = \ln y$ transforms the DE $y' + P(x)y = Q(x)(y \ln y)$ into the linear DEy $v' + P(x) = Q(x)v(x)$

Problem 1.5.4 Find the G.S. of the following DE. Prime denotes derivatives WRT x.

$$5y^4y' = x^2y' + 2xy$$

Problem 1.5.5 Find the G.S. of the following DE.

$$xy' - 4x^2y + 2y \ln y = 0$$

Problem 1.5.6 Find the G.S. of the following DE.

$$tx' - (m+1)t^m x + 2x \ln x = 0$$

Problem 1.5.7 Find the G.S. of the following DE.

$$xy' = 6y + 12x^4y^{\frac{2}{3}}$$

Problem 1.5.8 Find the G.S. of the following 2nd-order DE.

$$yy'' = 3(y')^2$$

Problem 1.5.9 Find the G.S. of the following DE by two different methods.

$$y' = xy^3 - xy$$

Problem 1.5.10 Show that the solution curves of the following DE.

$$y' = -\frac{y(2x^3 - y^3)}{x(2y^3 - x^3)}$$

are of the form $x^3 + y^3 = 3Cxy$.
(Hint: Use substitution $u = y/x$)

Problem 1.5.11 Find the G.S. of the following DE.

$$x^2y' = xy + x^2 \exp\left(\frac{y}{x}\right)$$

Problem 1.5.12 Find the G.S. of the following DE.

$$(x + y)y' = 1$$

Problem 1.5.13 Find the G.S. of the following DE.

$$(2x \sin y \cos y)y' = 4x^2 + \sin^2 y$$

Problem 1.5.14 Find the G.S. of the following second-order DE.

$$y'' = (x + y')^2$$

(Hint: Introduce substitution $v = y'$)

Problem 1.5.15 Find the G.S. of the following DE.

$$(x + \exp(y))y' = x \exp(-y) - 1$$

Problem 1.5.16 Find the G.S. of the following DE.

$$(3xy)^2 + x^{\frac{3}{2}}y' = y^2$$

Problem 1.5.17 Find the G.S. of the following DE.

$$y^2(xy' + 1)(1 + x^4)^{\frac{1}{2}} = x$$

Problem 1.5.18 Find the G.S. of the following DE.

$$x \exp(y)\, y' = 2(\exp(y) + x^3 \exp(2x))$$

Problem 1.5.19 Use your favorite method to find the G.S. of the following 1st-order nonlinear DE.

$$\frac{dy}{dx} - 4xy \ln y + 2\frac{y}{x}(\ln y)^n = 0$$

where $n \geq 2$ is a constant integer.

Problem 1.5.20 Find the G.S. of the following DE.

$$yy'' + (y')^2 = yy'$$

Problem 1.5.21 Find the G.S. of the following DE.

$$6xy^3 + 2y^4 + (9x^2y^2 + 8xy^3)y' = 0$$

Problem 1.5.22 Given a 1st-order nonlinear DE

$$y' + 7yx^{-1} - 3y^2 = 3x^{-2}$$

(1) Prove that after substitution, $y(x) = x^{-1} + u(x)$, one can transform this DE to a Bernoulli equation.
(2) Solve the resulting Bernoulli equation.

Problem 1.5.23 Find the G.S. of the following DE.

$$x^2y'' + 3xy' = 4$$

(Hint: introduce substitution $v = y'$)

Problem 1.5.24 Find the G.S. of the following DE.

$$tx' - 1000t^{1000}x + 2x^2 = 0$$

Problem 1.5.25 Find the G.S. of the following DE.

$$(x^2 + xy)dx - (xy + y^2)dy = 0$$

Problem 1.5.26 Find the G.S. of the following DE.

$$2x^2yy' + 2xy^2 + 1 = 0$$

Problem 1.5.27 Find the G.S. of the following DE by two different methods.

$$(x^2 + 1)y' - 2xy - 2x = 0$$

Problem 1.5.28 Solve the following IVP.

$$\begin{cases} y' = \dfrac{y}{x} - b\left(1 + \left(\dfrac{y}{x}\right)^2\right)^{1/2} \\ y(x = a) = 0 \end{cases}$$

where a and b are two constants and x in $[0, a]$. Please also do the following,
(1) Sketch the solution for $b < 1$.
(2) Sketch the solution for $b = 1$.

(3) Sketch the solution for $b > 1$.
(4) In which of the above cases, $y(x = 0) = 0$ is possible.

Problem 1.5.29 Find the G.S. of the following DE.

$$x' = 3x^2 - \frac{8}{t}x + \frac{4}{t^2}$$

(Hint: introduce substitution $x = \frac{1}{t} + u$*)*

Problem 1.5.30 Find the G.S. of the following DE.

$$y' = (K(x) + y + \beta)(y - \beta)$$

When the function $K(x)$ is given as $K(x) = x^{2011}$ and $\beta = 1$, find the specific form of the solution.

Problem 1.5.31 Find the G.S. of the following DE.

$$tx' - x = \beta(x'x + t)$$

Problem 1.5.32 Find the G.S. of the following DE.

$$y' = by^2 + cx^n$$

where $b, c \neq 0$ are constants and $n = 0, -2$. You need to consider both values of n.

1.6 The Exact DEs

Consider DE

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

It is

- 1st-order
- Nonlinear
- Not homogeneous
- Not Separable
- Not Bernoulli

New methods must be discovered in order to solve DEs of this kind.

The G.S. of all DEs can be written as

$$F(x, y) = C$$

With this claim we can produce

$$d\big(F(x, y)\big) = dC = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

Therefore we can see

$$\left(\frac{\partial F}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y}\right) dy = 0$$

Let

$$M(x, y) = \frac{\partial F}{\partial x} \text{ and } N(x, y) = \frac{\partial F}{\partial y}$$

This gives

$$M(x, y)dx + N(x, y)dy = 0 \tag{1.12}$$

All 1st-order DEs can be written in the above manner.

The condition for the DEs to be exact is

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

That is, if $N_x = M_y$, then (1.12) is exact. In fact, a theorem can be proven that the necessary and sufficient condition for (1.12) to be exact is $N_x = M_y$.

Example 1

Is DE

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

exact?

Solution

We have

$$M(x,y) = 6xy - y^3 \text{ and } N(x,y) = 4y + 3x^2 - 3xy^2$$

Here

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(4y + 3x^2 - 3xy^2) = 6x - 3y^2$$

Since

$$M_y = N_x = 6x - 3y^2$$

The DE is exact.

Example 2

Is DE

$$ydx + 3xdy = 0$$

exact?

Solution

From DE, we get

$$M(x,y) = y \text{ and } N(x,y) = 3x$$

Here

$$M_y = \frac{\partial}{\partial y}(y) = 1$$

$$N_x = \frac{\partial}{\partial x}(3x) = 3$$

That means

$$M_y \neq N_x$$

The DE is not exact.

Example 3

Is DE

$$y^3 dx + 3xy^2 dy = 0$$

exact?

Solution

We get

$$M(x,y) = y^3 \text{ and } N(x,y) = 3xy^2$$

Here

$$M_y = \frac{\partial}{\partial y}(y^3) = 3y^2$$

$$N_x = \frac{\partial}{\partial x}(3xy^2) = 3y^2$$

That means

$$N_x = M_y = 3y^2$$

The DE is exact.

Since the above DE is exact, let us try to find its solution.

There is a new set of methods for solving exact DEs. As mentioned in the beginning of this section the solution of any DE can be written as

$$F(x,y) = C \tag{1.13}$$

And from above example, we have

$$M(x,y) = \frac{\partial F}{\partial x} = y^3 \tag{1.14}$$

$$N(x,y) = \frac{\partial F}{\partial y} = 3xy^2 \tag{1.15}$$

So from (1.14) we have

$$\begin{aligned} F(x,y) &= \int y^3 dx \\ &= y^3 x + C \end{aligned}$$

Here we have C as a constant WRT x. Consider the constant be a function of y: $C = g(y)$. Now we have

$$F(x,y) = y^3 x + g(y) \tag{1.16}$$

Plugging (1.16) back into (1.15), we get

$$\frac{\partial}{\partial y}\big(y^3 x + g(y)\big) = 3xy^2$$

$$3xy^2 + g'(y) = 3xy^2$$

$$g'(y) = 0$$

$$g(y) = C_1$$

So we find that $g(y)$ is actually a constant.

Plugging $g(y) = C_1$ into (1.16), we get

$$F(x,y) = y^3 x + C_1$$

Therefore, from (1.13) we have

$$y^3 x + C_1 = C_2$$

Thus, the solution is

$$y^3 x = C$$

Steps for solving exact DEs

STEP 1: Write the DE into form

$$M(x,y)dx + N(x,y)dy = 0 \tag{1.17}$$

and get the formula

$$M(x,y) = \frac{\partial F}{\partial x} \tag{1.18}$$

$$N(x,y) = \frac{\partial F}{\partial y} \tag{1.19}$$

STEP 2: Find

$$F(x,y) = \int M(x,y)dx + g(y)$$

STEP 3: Plug $F(x,y)$ into (1.19) to obtain $g(y)$.
STEP 4: The solution is

$$\int M(x,y)dx + g(y) = C$$

Remarks

In the above steps, we start from (1.18), integrate it and then plug it back to (1.19) to get the solution. Similarly, we can start from (1.19), integrate it and plug back to (1.18) to get the *same* solution, different by a constant.

Now let us clarify a few points:

(1) Not all DEs are exact. Some non-exact DEs can be converted to exact DEs and some cannot. So, the question is that under what condition(s), a non-exact DE can be converted to an exact DE.
(2) If a non-exact DE can be converted to an exact DE, is the conversion unique? In other words, is there more than one method to convert a non-exact DE to exact?

Now let us try to convert a non-exact DE into an exact DE. Suppose we have DE

$$M(x,y)dx + N(x,y)dy = 0$$

If we have integrating factor $I(x)$ such that

$$I(x)M(x,y)dx + I(x)N(x,y)dy = 0$$

be exact, we should have

$$\frac{\partial}{\partial y}\big(I(x)M(x,y)\big) = \frac{\partial}{\partial x}\big(I(x)N(x,y)\big)$$

That means

$$I(x)\frac{\partial}{\partial y}M(x,y) = N(x,y)\frac{\partial}{\partial x}I(x) + I(x)\frac{\partial}{\partial x}N(x,y)$$

$$I \cdot M_y = N \cdot I' + I \cdot N_x$$

$$I\big(M_y - N_x\big) = I' \cdot N$$

$$\frac{I'}{I} = \frac{M_y - N_x}{N}$$

$$I(x) = \exp\left(\int \frac{M_y - N_x}{N}dx\right)$$

Here the integrand

$$\frac{M_y - N_x}{N}$$

must be a function with a single variable x.

Exact DE Theorem

For DE

$$M(x,y)dx + N(x,y)dy = 0$$

1) If

$$\frac{M_y - N_x}{N} = f(x)$$

is a function of one variable x, we have the integrating factor

$$\rho(x) = \exp\left(\int f(x)dx\right)$$

2) If

$$\frac{M_y - N_x}{M} = g(y)$$

is a function of one variable y, we have the integrating factor

$$\rho(y) = \exp\left(-\int g(y)dy\right)$$

Remarks

(1) For one DE, $\rho(x)$ and $\rho(y)$ may both exist. In this case, use the most convenient form.
(2) It is possible none of the $\rho(x)$ and $\rho(y)$ exist.
(3) You must have noticed that $\rho(x)$ has no negative sign in the exponential term while $\rho(y)$ has a negative sign in the exponential term. So given below is the proof of why we need the negative sign for $\rho(y)$ but is not required for $\rho(x)$.

Proof of the negative sign in $\rho(y)$ in

$$\rho(y) = \exp\left(-\int g(y)dy\right)$$

$M(x,y)dx + N(x,y)dy = 0$ is the given DE. We will choose our integrating factor as $\rho(y)$

$$\rho(y)M(x,y)dx + \rho(y)N(x,y)dy = 0$$

Let

$$\bar{M}(x,y) = \rho(y)M(x,y)$$

$$\bar{N}(x,y) = \rho(y)N(x,y)$$

Therefore, we have the new DE as $\bar{M}(x,y)dx + \bar{N}(x,y)dy = 0$

Since we want the above new DE to be exact, we should have

$$\bar{M}_y = \bar{N}_x$$

That is

$$\frac{\partial}{\partial y}\big(\rho(y)M(x,y)\big) = \frac{\partial}{\partial x}\big(\rho(y)N(x,y)\big)$$

$$M(x,y)\frac{\partial}{\partial y}\rho(y) + \rho(y)\frac{\partial}{\partial y}M(x,y) = \rho(y)\frac{\partial}{\partial x}N(x,y)$$

$$M \cdot \rho' + \rho \cdot M_y = \rho \cdot N_x$$

$$\frac{\rho'}{\rho} = \frac{N_x - M_y}{M}$$

That is

$$\frac{\rho'}{\rho} = -g(y)$$

where

$$g(y) = \frac{M_y - N_x}{M}$$

as noted above.

Solving this DE, we have

$$\rho(y) = \exp\left(-\int g(y)dy\right)$$

We sum up above discussion in the following theorems:

Theorem 1

For a given 1st-order DE $M(x,y)dx + N(x,y)dy = 0$, if

$$\frac{M_y - N_x}{N} = f(x)$$

is a function of pure x, then the DE can be converted to an exact DE by multiplying the original DE with

$$\rho(x) = \exp\left(\int f(x)dx\right)$$

Theorem 2

For a given 1st-order DE $M(x,y)dx + N(x,y)dy = 0$, if

$$\frac{M_y - N_x}{M} = g(y)$$

is a function of pure y, then the DE can be converted to a exact DE by multiplying the original DE with

$$\rho(y) = \exp\left(-\int g(y)dy\right)$$

Next, let us work on one example to understand the above theorem and make several remarks.

Example

Determine whether the DE

$$ydx + 3xdy = 0$$

is exact or not? If not convert it into an exact DE and solve it.

Solution

From the given DE, we get

$$M(x,y) = y \text{ and } N(x,y) = 3x$$

Here, we have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y) = 1$$
$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(3x) = 3$$

Since $M_y \neq N_x$, we know that the given DE is not exact. For solving this DE, there are two methods of doing this.

Method I:

$$\frac{M_y - N_x}{N} = \frac{1-3}{3x} = -\frac{2}{3x} = f(x)$$

It is clear that $f(x)$ is a function of pure x

$$\begin{aligned}\rho(x) &= \exp\left(\int f(x)dx\right) \\ &= \exp\left(-\int \frac{2}{3x}dx\right) \\ &= \exp\left(-\frac{2}{3}\ln x\right) \\ &= x^{-\frac{2}{3}}\end{aligned}$$

Multiplying $\rho(x)$ on both sides of the original DE, we have

$$x^{-\frac{2}{3}}ydx + 3x \cdot x^{-\frac{2}{3}}dy = 0$$

This is now our new DE. Here we have

$$\frac{\partial F}{\partial x} = \bar{M}(x,y) = x^{-\frac{2}{3}}y \tag{1.20}$$

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = 3x^{\frac{1}{3}} \tag{1.21}$$

Thus, we have

$$\bar{M}_y = \frac{\partial}{\partial y}\left(x^{-\frac{2}{3}}y\right) = x^{-\frac{2}{3}}$$

$$\bar{N}_x = \frac{\partial}{\partial x}\left(3x^{\frac{1}{3}}\right) = x^{-\frac{2}{3}}$$

Since $\bar{M}_y = \bar{N}_x$ we know the new DE is now exact.

So we can now use either (1.20) or (1.21) to find $F(x,y)$. Since the function in (1.21) is easier to integrate, we choose (1.21) to find $F(x,y)$. We have

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = 3x^{\frac{1}{3}}$$

$$\begin{aligned}F(x,y) &= \int 3x^{\frac{1}{3}}dy + g(x) \\ &= 3x^{\frac{1}{3}}y\end{aligned} \tag{1.22}$$

Substituting (1.22) back into (1.20), we now have

$$\frac{\partial}{\partial x}\left(3x^{\frac{1}{3}}y + g(x)\right) = x^{-\frac{2}{3}}y + g'(x) = x^{-\frac{2}{3}}y$$

This solves $g'(x) = 0$. That means $g(x) = C_1$. Substituting this back to (1.22), we now have

$$F(x,y) = 3x^{\frac{1}{3}}y + C_1$$

That gives the solution of the DE

$$F(x,y) = C_2$$

That is

$$3x^{\frac{1}{3}}y = C$$

Method II:

We now discuss the second method of converting a non-exact DE to an exact DE.

$$\frac{M_y - N_x}{M} = \frac{1-3}{y} = -\frac{2}{y} = g(y)$$

The integrating factor

$$\begin{aligned}\rho(y) &= \exp\left(-\int g(y)dy\right)\\ &= \exp\left(\int \frac{2}{y}dy\right)\\ &= y^2\end{aligned}$$

Multiplying $\rho(y)$ on both sides of the original DE, we have

$$y^3dx + 3xy^2dy = 0$$

as our new DE. Here we have

$$\frac{\partial F}{\partial x} = \bar{M}(x,y) = y^3 \tag{1.23}$$

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = 3xy^2 \tag{1.24}$$

and

$$\bar{M}_y = \frac{\partial}{\partial y}(y^3) = 3y^2$$

$$\bar{N}_x = \frac{\partial}{\partial x}(3xy^2) = 3y^2$$

Since $\bar{M}_y = \bar{N}_x$, the new DE formed is now exact.

We can now use either (1.23) or (1.24) to find $F(x,y)$. Since the function in (1.23) is easier to integrate, we choose (1.23) to find $F(x,y)$. We now have

$$\begin{aligned}F(x,y) &= \int y^3dx + g(y)\\ &= xy^3 + g(y) \qquad (1.25)\end{aligned}$$

Substituting (1.25) back into (1.24), we now have

$$\frac{\partial}{\partial y}\big(xy^3 + g(y)\big) = 3xy^2 + g'(y) = 3xy^2$$

It gives

$$g'(y) = 0$$

which means

$$g(y) = C_1$$

Substituting this back into (1.25), we now have

$$F(x, y) = xy^2 + C_1$$

Thus, the solution of the DE is

$$xy^3 + C_1 = C_2$$

That is

$$xy^3 = C$$

This is the same answer as we found in Method I.

SUMMARY

We have 4 ways of solving a non-exact DE, which is summarized as follows.

Steps for solving a non-Exact DE by converting it into an Exact DE

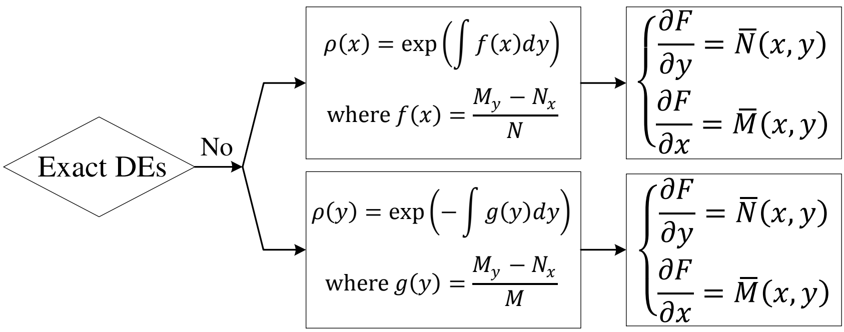

Figure 1.6 Steps of converting a non-Exact DE to an Exact DE.

STEP 1: Determine $M(x, y)$ and $N(x, y)$.
STEP 2: Check if $M_y = N_x$? If YES then go to STEP 6:.
STEP 3: Check if

$$\frac{M_y - N_x}{N} = f(x)$$

is a function of pure x or if

$$\frac{M_y - N_x}{M} = g(y)$$

is a function of pure y.

STEP 4: Find the easiest integrating factor to solve

$$\rho(x) = \exp\left(\int f(x)dx\right) \text{ or } \rho(y) = \exp\left(-\int g(y)dy\right)$$

STEP 5: Compute the new $\bar{M}$ and $\bar{N}$ for either $\rho(x)$ or $\rho(y)$

$$\bar{M}(x,y) = \rho\, M(x,y)$$

$$\bar{N}(x,y) = \rho\, N(x,y)$$

STEP 6: Construct the partial DEs

$$\begin{cases} \dfrac{\partial F}{\partial x} = \bar{M}(x,y) \\ \dfrac{\partial F}{\partial y} = \bar{N}(x,y) \end{cases}$$

STEP 7: Use either of the above DE to find $F(x,y)$

$$F(x,y) = \int \bar{M}(x,y)dx + g(y)$$

or equivalently,

$$F(x,y) = \int \bar{N}(x,y)dy + g(x)$$

STEP 8: Substituting the DE we get from STEP 7: back to the DE of STEP 6:, we have either

$$\frac{\partial}{\partial y}\left(\int \bar{M}(x,y)dx + g(y)\right) = \bar{N}(x,y)$$

or

$$\frac{\partial}{\partial x}\left(\int \bar{N}(x,y)dy + g(x)\right) = \bar{M}(x,y)$$

STEP 9: Solving the above DE produces the final solution to the DE $F(x,y)$ =C.

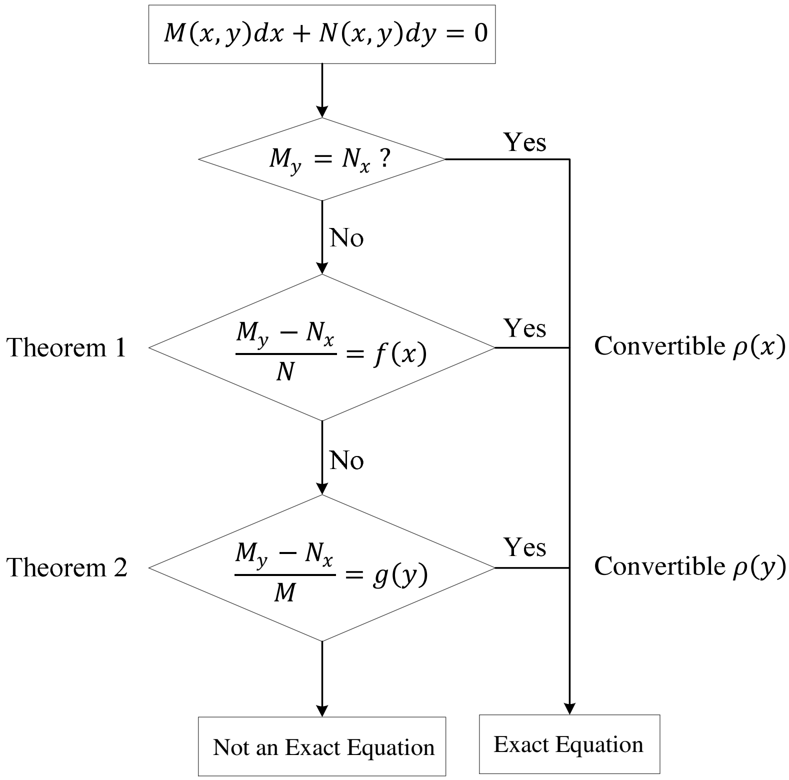

Figure 1.7 Steps of solving a non-Exact DE.

Example 1

Solve exact DE

$$M(x,y)dx + N(x,y)dy = 0$$

Solution

We can compose two simple partial DEs with F as the dependent variable and x and y as the independent variables

$$\frac{\partial F}{\partial x} = M(x,y) \tag{1.26}$$

$$\frac{\partial F}{\partial y} = N(x,y) \tag{1.27}$$

Solving (1.26), we get

$$F(x,y) = \int M(x,y)dx + g(y)$$

Plugging the above into (1.27), we get

$$\frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + g'(y) = N(x,y)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)dx\right)$$

Solving this, we have

$$g(y) = \int N(x,y)dy - \int \left(\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)\right)dy$$

Thus, we get the solution of the DE as

$$\int M(x,y)dx + \int N(x,y)dy - \int \left(\frac{\partial}{\partial y}\left(\int M(x,y)dx\right)\right)dy = C$$

Example 2

Is DE exact? Solve it.

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

Solution

From the given DE, we have

$$\frac{\partial F}{\partial x} = M(x,y) = 6xy - y^3$$

$$\frac{\partial F}{\partial y} = N(x,y) = 4y + 3x^2 - 3xy^2$$

Now we evaluate

$$M_y = \frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2$$

$$N_x = \frac{\partial}{\partial x}(4y + 3x^2 - 3xy^2) = 6x - 3y^2$$

Since $N_x = M_y$, we know that the given DE is exact. Now we can find $F(x,y)$

$$\frac{\partial F}{\partial x} = M(x,y) = 6xy - y^3$$
$$F(x,y) = \int (6xy - y^3)dx + g(y)$$
$$= 3x^2y - y^3x + g(y)$$

Plugging it back to
$$\frac{\partial F}{\partial y} = N(x,y)$$

We have
$$\frac{\partial}{\partial y}\left(3x^2y - y^3x + g(y)\right) = 4y + 3x^2 - 3xy^2$$
$$3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$
$$g'(y) = 4y$$
$$g(y) = 2y^2$$

Therefore, the solution of the DE is
$$3x^2y - xy^3 + 2y^2 = C$$

Let us do one more example to show that not all DEs can be solved in this way.

Example 3

Solve the DE
$$(3y^2 + 5x^2y)dx + (3xy + 2x^3)dy = 0$$

Solution

From the given DE, we have
$$M(x,y) = 3y^2 + 5x^2y$$
$$N(x,y) = 3xy + 2x^3$$
$$M_y = \frac{\partial}{\partial y}(3y^2 + 5x^2y) = 6y + 5x^2$$
$$N_x = \frac{\partial}{\partial x}(3xy + 2x^3) = 3y + 6x^2$$

Since $M_y \neq N_x$, the given DE is not exact. Now let us see if we can convert it into an exact DE.
$$\frac{M_y - N_x}{N} = \frac{3y - x^2}{3xy + 2x^3}$$

is not a function of pure x, and
$$\frac{M_y - N_x}{M} = \frac{3y - x^2}{3y^2 + 5x^2y}$$

is not a function of pure y. Therefore, the given DE cannot be converted into an exact DE. Therefore it is not possible to solve this DE.

Finally, we show the relation between the exact DE method and the linear DE we previously studied.

Converting a Linear DE into an Exact DE

Now the question arises whether linear DEs are exact. And if they are not, is it possible to convert them into such a form? So now let us try to answer these queries. The most general form of a linear DE is

$$y' + P(x)y = Q(x) \tag{1.28}$$

That is

$$\frac{dy}{dx} + P(x)y - Q(x) = 0$$

$$\big(P(x)y - Q(x)\big)dx + dy = 0 \tag{1.29}$$

Now we have

$$M(x,y) = P(x)y - Q(x)$$

$$N(x,y) = 1$$

$$M_y = P(x)$$

$$N_x = 0$$

Since in general $M_y \neq N_x$, (1.28) is not an exact DE.

As we have shown that the given general form of linear DE is not exact, we have to now try to convert it into an exact DE.

Compare the following two formulas.

$$\frac{M_y - N_x}{N} = P(x)$$

$$\frac{M_y - N_x}{M} = \frac{P(x)}{P(x)y - Q(x)}$$

As we can see, working with first one is much easier than working with second one. Hence we have our integrating factor.

$$\rho(x) = \exp\left(\int P(x)dx\right)$$

Multiplying $\rho(x)$ on both sides of (1.29), we have

$$\exp\left(\int P(x)dx\right)\big(P(x)y - Q(x)\big)dx + \exp\left(\int P(x)dx\right)dy = 0$$

which is now the new DE. Thus, we have

$$\frac{\partial F}{\partial x} = \bar{M}(x,y) = \exp\left(\int P(x)dx\right)\big(P(x)y - Q(x)\big) \quad (1.30)$$

$$\frac{\partial F}{\partial y} = \bar{N}(x,y) = \exp\left(\int P(x)dx\right) \quad (1.31)$$

$$\begin{aligned}
\bar{M}_y &= \frac{\partial}{\partial y}\left(\exp\left(\int P(x)dx\right)\big(P(x)y - Q(x)\big)\right) \\
&= \exp\left(\int P(x)dx\right)\frac{\partial}{\partial y}\big(P(x)y - Q(x)\big) \\
&= P(x)\exp\left(\int P(x)dx\right)
\end{aligned}$$

$$\begin{aligned}
\bar{N}_x &= \frac{\partial}{\partial x}\exp\left(\int P(x)dx\right) \\
&= P(x)\exp\left(\int P(x)dx\right)
\end{aligned}$$

Since $\bar{M}_y = \bar{N}_x$, the new DE is exact.

Now we can use either (1.30) or (1.31) to find $F(x,y)$. Since the function in (1.31) is easier to integrate, we use this to find $F(x,y)$

$$F(x,y)\& = \int \exp\left(\int P(x)dx\right)dy$$

$$= y \exp\left(\int P(x)dx\right) + g(x)$$

Substituting this back to (1.30), we have

$$\frac{\partial}{\partial x}\left(y \exp\left(\int P(x)dx\right) + g(x)\right) = \exp\left(\int P(x)dx\right)\left(P(x)y - Q(x)\right)$$

$$y\,P(x) \exp\left(\int P(x)dx\right) + g'(x)$$

$$= y\,P(x) \exp\left(\int P(x)dx\right) - Q(x) \exp\left(\int P(x)dx\right)$$

$$g'(x) = -Q(x) \exp\left(\int P(x)dx\right)$$

$$g(x) = -\int Q(x) \exp\left(\int P(x)dx\right) dx$$

Substituting this back to $F(x, y)$, we now have

$$F(x, y) = y \exp\left(\int P(x)dx\right) - \int Q(x) \exp\left(\int P(x)dx\right) dx$$

Therefore, the solution to the DE is

$$y \exp\left(\int P(x)dx\right) - \int Q(x) \exp\left(\int P(x)dx\right) dx = C$$

That is

$$y = \exp\left(-\int P(x)dx\right)\left(\int Q(x) \exp\left(\int P(x)dx\right) dx + C\right)$$

Problems

Problem 1.6.1 Solving the following DE using the exact DE method.

$$(1 + \ln(xy))dx + \left(\frac{x}{y}\right) dy = 0$$

Problem 1.6.2 Given a 1st-order DE

$$A(s,y)dx + B(x,y)dy = 0$$

which is not an exact equation in general, but the function $A(x,y)$ and $B(x,y)$ satisfy the following relationship

$$\left(\frac{\partial A(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial x}\right) / B(x,y) = P(x)$$

where $P(x)$ is a function of one variable x. Prove that, after multiplying the original DE by an integrating factor $\rho(x) = \exp(\int P(x)\,dx)$, one can transform the original non-exact DE to an exact DE.

Problem 1.6.3 Given 1st-order linear DE

$$\alpha(x)\frac{dy}{dx} + \beta(x)y + \gamma(x) = 0$$

where $\alpha(x) \neq 0$.

(1) Using exact equation method to solve the equation and express the solution in terms of $\alpha(x), \beta(x),$ and $\gamma(x)$.

(2) Using 1st-order linear equation method to solve the equation and express the solution in terms of $\alpha(x), \beta(x),$ and $\gamma(x)$.

Problem 1.6.4 The 1st-order linear DE can be expressed alternatively as

$$(P(x)y - Q(x))dx + dy = 0$$

where functions$P(x) \neq 0$ and $Q(x) \neq 0$ are given. Please

(1) Check if this DE is Exact.

(2) If not, convert it to an Exact DE.

(3) Solve the DE using the Exact Equation method and your solution may be expressed in terms of the functions $P(x)$ and $Q(x)$.

(4) If $P(x) = \frac{1}{x}$ and $Q(x) = \frac{\cos x}{x}$, get the specific solution.

Problem 1.6.5 Solving the following DE using the exact DE method.

$$(2x - y^2)dx + xydy = 0$$

Problem 1.6.6 Solving the following DE by two different methods.

$$y' = -\frac{3x^2 + 2y^2}{4xy}$$

Problem 1.6.7 Solving the following DE by two different methods.

$$y' = \frac{x + 3y}{y - 3x}$$

Problem 1.6.8 Solving the following DE using the exact DE method.

$$ydx + (2x + y^4)dy = 0$$

Problem 1.6.9 Solving the following DE using the exact DE method.

$$y'(2xy + 1) = 2x - y^2$$

Problem 1.6.10 Solving the following DE using the exact DE method.

$$y^3 + xy^2y' - y' = 0$$

Problem 1.6.11 Check whether the following equation is exact

$$(\cos x + \ln y)dx + \left(\frac{x}{y} + \exp(y)\right)dy = 0$$

If it is an exact equation according to your verification above, solve the equation using the exact equation solution technique. If it is not, then use some other technique to solve the equation.

Problem 1.6.12 Solving the following DE by two different methods.

$$\exp(y) + y\cos x + (x\exp(y) + \sin x)y' = 0$$

1.7 Riccati DEs

DEs that can be expressed as follows are called Riccati DEs:

$$y' = A(x) + B(x)y + C(x)y^2$$

If $A(x) = 0$, it reduces to the Bernoulli's DE.
If $C(x) = 0$, it reduces to the 1st-order linear DE.

By some method (including guessing), one may obtain one P.S. $y_1(x)$. With this P.S., one may propose the following G.S.:

$$y(x) = y_1(x) + \frac{1}{Z(x)}$$

Thus,

$$y' = y_1' - \frac{Z'}{Z^2}$$

$$y^2 = y_1^2 + \frac{1}{Z^2} + \frac{2y_1}{Z}$$

Plugging in the above trial G.S. and other terms to the original Riccati DE, we have

$$y_1' - \frac{Z'}{Z^2} = A + B\left(y_1 + \frac{1}{Z}\right) + C\left(y_1^2 + \frac{1}{Z^2} + \frac{2y_1}{Z}\right)$$

which can be written as follows, after re-organizing the terms:

$$y_1' - A - By_1 - Cy_1^2 = \frac{Z'}{Z^2} + B\left(\frac{1}{Z}\right) + C\left(\frac{1}{Z^2} + \frac{2y_1}{Z}\right)$$

Because y_1 is the P.S. of the original Riccati DE, the LHS above must be zero:

$$y_1' - A - By_1 - Cy_1^2 = 0$$

Thus,

$$\frac{Z'}{Z^2} + B\left(\frac{1}{Z}\right) + C\left(\frac{1}{Z^2} + \frac{2y_1}{Z}\right) = 0$$

Or, multiplying the above equation by Z^2, we get

$$Z' + (B + 2y_1)Z + C = 0$$

which is a simple 1st-order linear DE that can be easily solved.

Example

Find the G.S. of the following DE

$$y' + y^2 = \frac{2}{x^2}$$

Solution

This is a simple Riccati DE and, by inspection, we may assume a P.S. in the following form $y_1 = \frac{A}{x}$ where A is a constant to be determined. Since it is a P.S., it must satisfy the original DE, i.e.,

$$-\frac{A}{x^2} + \frac{A^2}{x^2} = \frac{2}{x^2}$$

We can easily find two roots $A = -1$ and $A = 2$ to satisfy the above equation. Thus, we found two P.S.'s: $y_1 = -\frac{1}{x}$ or $y_1 = \frac{2}{x}$.

Next, we may select any one of the P.S.'s to compose the G.S. as (We selected P.S.: $y_1 = -{}^{1}/_{x}$):

$$y = -\frac{1}{x} + \frac{1}{Z}$$

Thus,

$$y' = \frac{1}{x^2} - \frac{Z'}{Z^2}$$

$$y^2 = \frac{1}{x^2} - \frac{2}{xZ} + \frac{1}{Z^2}$$

Plugging in the above two formulas to the original DE:

$$\left(\frac{1}{x^2} - \frac{Z'}{Z^2}\right) + \left(\frac{1}{x^2} - \frac{2}{xZ} + \frac{1}{Z^2}\right) = \frac{2}{x^2}$$

We get

$$Z' + \frac{2}{x}Z = 1$$

whose solution is

$$Z = \frac{x}{3} + \frac{C}{x^2}$$

Finally, the G.S. for the DE is

$$y(x) = -\frac{1}{x} + \frac{3x^2}{x^3 + C_1}$$

Problems

Problem 1.7.1 The DE

$$y' = A(x)y^2 + B(x)y + C(x)$$

is called Riccati DE. Suppose that one P.S. y_1 of this DE is known. Show that the substitution $y = y_1 + 1/v$ can transform the Riccati DE into the linear DE $v' + (B + 2Ay_1)v = -A$.

Problem 1.7.2 Solve the DE

$$y' + 2xy = 1 + x^2 + y^2$$

given that $y_1 = x$ is a solution.

Problem 1.7.3 Solve the DE

$$y' = 1 + \frac{1}{4}(x - y)^2$$

given that $y_1 = x$ is a solution.

Problem 1.7.4 Solve the DE

$$y' - 13(x^2 + y^2) + 26x = 1$$

given that $y_1 = x$ is a solution.

Problem 1.7.5 Solve the DE

$$\frac{dy}{dx} = (f(x) + y + a)(y - a)$$

given that $y_1 = a$ is a solution.

Problem 1.7.6 Solve the DE

$$y' = -(a^2 + 4ax^3) + 4x^3y + y^2$$

given that $y_1 = a$ is a solution.

Problem 1.7.7 Solve the DE
$$y' = \frac{2\cos^2 x - \sin^2 x + y^2}{2\cos x}$$
given that $y_1 = \sin x$ is a solution.

Problem 1.7.8 Find the G.S. of the following DE
$$y' = y^2 + \alpha(x)(y - x^2) + 2x - x^4$$
and express it (of course) in terms of the given function $\alpha(x)$. We know that this equation has one solution $y = x^2$.

Problem 1.7.9 Solve the DE
$$x^3 y' + x^2 y - y^2 = 2x^4$$
given that $y_1 = x^2$ is a solution.

Chapter 2
Mathematical Models

2.1 Population Model

2.1.1 General Population Equation

It is customary to track the growth or decline of a population in terms of its birth rate and death rate functions, which are defined as follows:

- $\beta(t)$ is the number of births per unit population per unit time at time t
- $\delta(t)$ is the number of deaths per unit population per unit time at time t

Then the numbers of births and deaths that occur during the time interval $(t, t+\Delta t)$ is approximately given by births: $\beta(t)P(t)\Delta t$ and deaths: $\delta(t)P(t)\Delta t$. The change ΔP in the population during the time interval $(t, t+\Delta t)$ of the length Δt is

$$\begin{aligned}\Delta P &= \text{births} - \text{deaths}\\ &= \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t\\ &= \big(\beta(t) - \delta(t)\big)P(t)\Delta t\end{aligned}$$

Thus,

$$\frac{\Delta P}{\Delta t} = \big(\beta(t) - \delta(t)\big)P(t)$$

Taking the limit $\Delta t \to 0$, we get the DE

$$\frac{dP}{dt} = (\beta - \delta)P \tag{2.1}$$

which is the General Population Equation where $\beta = \beta(t)$ and $\delta = \delta(t)$ where β and δ can be either a constant or functions of t or they may, indirectly, depend on the unknown function $P(t)$.

Example 1

Given the following information, find the population after 10 years.

$$P(0) = P_0 = 100$$
$$\beta = 0.0005P$$
$$\delta = 0$$

Solution

Substituting the given information in the above formula we have

$$\frac{dP}{dt} = (\beta - \delta)P$$
$$\frac{dP}{dt} = 0.0005P^2$$
$$\frac{dP}{P^2} = 0.0005dt$$
$$\int \frac{dP}{P^2} = \int 0.0005dt$$

$$-\frac{1}{P} = 0.0005t + C$$

Substitution of $P(0) = 100$ in the above DE given

$$C = -\frac{1}{100}$$

Therefore

$$P(t) = \frac{2000}{20 - t}$$

and

$$P(10) = 200$$

Thus, the population after 10 years is 200.

2.1.2 The Logistic Equation

Let us consider the case of a fruit-fly population in a container. It is often observed that the birth rate decreases as the population itself increases due to limited space and food supply. Suppose that the birth rate β is a linear decreasing function of the population size P so that $\beta = \beta_0 - \beta_1 P$ where β_0 and β_1 are positive constants. If the death rate $\delta = \delta_0$ remains constant, then the general population equation takes the form

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0)P$$

Let $a = \beta_0 - \delta_0$ and $b = \beta_1$, then

$$\frac{dP}{dt} = aP - bP^2 \tag{2.2}$$

Eq-(2.2) is called the logistic equation for population model. We can rewrite the logistic equation as

$$\frac{dP}{dt} = kP(M - P) \tag{2.3}$$

where $k = b$ and $M = a/b$ are constants. M is sometimes called the carrying capacity of the environment, that is to say, the maximum population that the environment can support on a long-term basis. This DE can be solved using the separation of variables as follows.

$$\begin{cases} \dfrac{dP}{P(M-P)} = kdt \\ P(t=0) = P_0 \end{cases}$$

$$\int_{P_0}^{P} \frac{dP}{P(M-P)} = \int_0^t kdt$$

$$\frac{1}{M}\int_{P_0}^{P} \frac{M}{P(M-P)} dP = kt$$

$$\int_{P_0}^{P} \left(\frac{1}{M-P} + \frac{1}{P}\right) dP = Mkt$$

$$(\ln P - \ln(M-P))_{P_0}^{P} = Mkt$$

$$\left(\ln\left(\frac{P}{P_0}\right) + \ln\left(\frac{M-P_0}{M-P}\right)\right) = Mkt$$

$$\frac{P}{P_0}\frac{M-P_0}{M-P} = \exp(Mkt)$$

This gives us a solution as an implicit function of P. We can write it explicitly.

$$P = (M-P)\frac{P_0 \exp(Mkt)}{M-P_0}$$

$$P\left(1 + \frac{P_0 \exp(Mkt)}{M-P_0}\right) = M\frac{P_0 \exp(Mkt)}{M-P_0}$$

$$P = \frac{MP_0 \exp(Mkt)}{M - P_0 + P_0 \exp(Mkt)}$$

$$P = \frac{MP_0}{P_0 + (M-P_0)\exp(-Mkt)} \tag{2.4}$$

This is the final formula for the logistic equation.

Remarks

(1) If $t = 0$, $P(t = 0) = P_0$

(2) The limit cases: $\lim_{t \to \infty} P(t) = \begin{cases} P_0, & k = 0 \\ M, & k > 0 \\ 0, & k < 0 \end{cases}$

(3) Limited Environment Situation

A certain environment can support at most M individuals. It is then reasonable to expect the growth rate $\beta - \delta$ (the combined birth and death rates) to be proportional to $M - P$ because we may think of $M - P$ as the potential for further expansion. Then

$$\beta - \delta = k(M - P)$$

so that

$$\frac{dP}{dt} = (\beta - \delta)P = kP(M - P)$$

An example for the limited environment situation is the fruit-fly population.

(4) Competition Situation

If the birth rate β is constant but the death rate δ is proportional to P, so that $\delta = \alpha P$, then

$$\frac{dP}{dt} = (\beta - \alpha P)P = kP(M - P)$$

Example

Suppose that at time $t = 0$, 10,000 people in a city with population $M =$ 100,000 are attacked by a virus. After a week the number $P(t)$ of those attacked with the virus has increased to $P(1) =$ 20,000. Assuming that $P(t)$ satisfies a logistic equation, when will 80% of the city's population have been attacked by that particular virus?

Solution

In order to eliminate the unnecessary manipulation of large numbers, we can consider the population unit of our problem as thousand. Now, substituting $\mathrm{P}_0 = 10$ and $\mathrm{M} = 100$ into the logistic equation we get

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)\exp(-Mkt)}$$

$$P(t) = \frac{1000}{10 + 90\exp(-100kt)}$$

Then substituting $P(t = 1) = 20$ into the equation

$$20 = \frac{1000}{1 + 90\exp(-100k)}$$

We solve for

$$\exp(-100k) = \frac{4}{9}$$

$$k = -\frac{1}{100}\ln\frac{4}{9}$$

With $P(t) = 80$, we have

$$80 = \frac{1000}{10 + 90\exp(-100kt)}$$

Which we solve for

$$\exp(-100kt) = \frac{1}{36}$$

Thus,

$$t = \frac{\ln 36}{100k} = \frac{\ln 36}{\ln 9/4} \approx 4.42$$

It follows that 80% of the population has been attacked by the virus after 4.42 weeks.

2.1.3 Doomsday vs. Extinction

Consider a population $P(t)$ of unsophisticated animals in which females rely solely on chance of encountering males for reproductive purposes. The rate of which encounters occur is proportional to the product of the number $P/2$ of males and $P/2$ of females, hence at a rate proportional to P^2. We therefore assume the births occur at the rate of kP^2(per unit time and k is constant). The birth rate (births per unit time per population) is then given by $\beta = kP$. If the death rate δ is constant then the general population equation gives the following

$$\frac{dP}{dt} = kP^2 - \delta P$$

Let $M = \delta/k > 0$, we have

$$\frac{dP}{dt} = kP(P - M) \tag{2.5}$$

Note

The RHS of the above DE is the NEGATIVE of the RHS of the logistic equation. The constant M is now called the threshold population. The behavior of the future population depends critically on whether the initial population P_0 is less than or greater than M.

Let us now solve this DE that is very similar to the logistic equation.

$$\frac{dP}{P(P-M)} = kdt$$

$$\int \frac{dP}{P(P-M)} = \int kdt$$

$$\frac{1}{M}\int \frac{M}{P(P-M)} dP = kt$$

$$\int \left(\frac{1}{P-M} - \frac{1}{P}\right) dP = Mkt$$

Plugging the I.C. $P(t = 0) = P_0$ to the left integration, we have

$$(\ln(P-M) - \ln P)|_{P_0}^{P} = Mkt$$

$$\frac{P_0}{P}\frac{P-M}{P_0 - M} = \exp(Mkt)$$

CASE 1: $P_0 > M$

Let

$$C_1 = \frac{P_0}{P_0 - M}$$

We know that $C_1 > 1$. To solve an explicit DE for P, we have

$$P = (P - M)C_1 \exp(-Mkt)$$

$$P(C_1 \exp(-Mkt) - 1) = MC_1 \exp(-Mkt)$$

$$P(t) = \frac{MC_1 \exp(-Mkt)}{C_1 \exp(-Mkt) - 1}$$

Note that the denominator in the above DE approaches zero as t approaches

$$T = \frac{\ln C_1}{kM} = \frac{1}{kM} \ln \frac{P_0}{P_0 - M} > 0$$

That is, $\lim_{t \to T} P(t) = \infty \to$ Doomsday.

CASE 2: $0 < P_0 < M$

Let

$$C_2 = \frac{P_0}{M - P_0}$$

We know that $C_2 > 0$. A similar solution for P will give

$$P = (M - P)C_2 \exp(-Mkt)$$

$$P(1 + C_2 \exp(-Mkt)) = MC_2 \exp(-Mkt)$$

$$P(t) = \frac{MC_2 \exp(-Mkt)}{C_2 \exp(-Mkt) + 1}$$

Since $C_2 > 0$, it follows that $\lim_{t \to \infty} P(t) = 0 \to$ Extinction.

Problems

Problem 2.1.1 Suppose that when a certain lake is stocked with fish, the birth and death rates β and δ are both inversely proportional to $\sqrt{P}$.
(1) Show that

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$$

where k is constant.
(2) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

Problem 2.1.2 Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$ and $P(t = 0) = P_0$.
(1) Show that

$$P(t) = \frac{P_0}{1 - kP_0 t}$$

where k is a constant.
(2) Prove that $P(t) \to \infty$ as $t \to 1/kP_0$. This is doomsday.
(3) Suppose that $P_0 = 6$ and there are 9 rabbits after ten months. When does doomsday occur?
(4) If $\beta < \delta$, what happens to the rabbit population in the long run?

Problem 2.1.3 Consider a population $P(t)$ satisfying the logistic equation

$$\frac{dP}{dt} = aP - bP^2$$

where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If initial population is $P(0) = P_0$ and B_0 births per month and D_0 deaths per month are occurring at time $t = 0$, show that the limiting population is

$$M = \frac{B_0 P_0}{D_0}$$

Problem 2.1.4 For the modified logistic population equation $P' = kP(\ln M - \ln P)$, $P(t = 0) = P_0$ where P is the population count as a function of time t; k, P_0 and M are positive constants.
(1) Solve the DE.

(2) Sketch the solution for $P_0 > M$ and show the limit when $t \to \infty$.
(3) Sketch the solution for $P_0 < M$ and show the limit when $t \to \infty$.

Problem 2.1.5 A tumor may be regarded as a population of multiplying cells. It is found empirically that birth rate of the cells in a tumor decreases exponentially with time, so that

$$\beta(t) = \beta_0 \exp(-\alpha t)$$

where α and β_0 are positive constants. Hence

$$\begin{cases} \dfrac{dP}{dt} = \beta_0 \exp(-\alpha t)\, P \\ \quad P(0) = P_0 \end{cases}$$

Solve this IVP to see that

$$P(t) = P_0 \exp\left(\frac{\beta_0}{\alpha}(1 - \exp(-\alpha t))\right)$$

Observe that $P(t)$ approaches the finite limiting population

$$P_0 \exp\left(\frac{\beta_0}{\alpha}\right)$$

as $t \to \infty$.

Problem 2.1.6 Consider two population functions $P_1(t)$ and $P_2(t)$, both of which satisfy the logistic equation with the limiting population M, but with different values k_1 and k_2 of the constant k in DE

$$\frac{dP}{dt} = kP(M - P)$$

Assume that $k_1 < k_2$. Which population approaches M the most rapidly? You can reason geometrically by examining slope fields (especially if appropriate software is available), symbolically by analyzing the solution given in DE

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)\exp(-Mkt)}$$

or numerically by substituting successive values of t.

Problem 2.1.7 Suppose that the fish population $P(t)$ in a lake is attacked by a disease at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (deaths per week per fish) is thereafter proportional to $1/\sqrt{P}$. If there were initially 900 fish in the lake and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?

Problem 2.1.8 Consider a rabbit population $P(t)$ satisfying the logistic equation $P' = aP - bP^2$, where $B = aP$ is the time rate at which births occur and $D = bP^2$ is the rate at which deaths occur. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M? Please sketch $P(t)$ with given parameters.

Problem 2.1.9 Consider a prolific animal whose birth and death rates, β and α, are each proportional to its population with $\beta > \alpha$ and $P_0 = P(t = 0)$.
(1) Compute the population as a function of time and parameters and I.C. given.
(2) Find the time for doomsday.
(3) Suppose that $P_0 = 2011$ and that there are 4027 animals after 12 time units (days or months or years), when is the doomsday?
(4) If $\beta < \alpha$, compute the population limit when time approaches infinity.

Problem 2.1.10 Fish population $P(t)$ in a lake is attacked by a disease (such as human beings who eats them) at time $t = 0$, with the result that the fish cease to reproduce (so that the birth rate is $\beta = 0$) and the death rate δ (death per week per fish) is thereafter proportional to $P^{-1/2}$. If there were initially 4000 fish in the lake and 2011 were left after 11 weeks, how long did it take all the fish in the lake to die? Can you change the "2011" to a different number such that the fish count never changes with time? To what number if so?

2.2 Acceleration-Velocity Model

2.2.1 Velocity and Acceleration Models

We know that

$$v = \frac{dy}{dt} \qquad\qquad y = \int v dt$$
$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} \qquad v = \int a dt \qquad y = \int \int a\, dt$$

where

$$\left.\begin{array}{l} a(t) = \text{acceleration} \\ v(t) = \text{velocity} \\ y(t) = \text{position} \end{array}\right\} \text{of the body}$$

Newton's Second Law of Motion, $F = ma$, implies that to gain an acceleration a for a mass m, one must exert force F.

Example

Suppose that a ball is thrown upwards from the ground ($y_0 = 0$) with initial velocity $v_0 = 49$ m/s. Neglecting air resistance, calculate the maximum height it can reach and the time required for it come back to its original position.

Solution

The body if thrown upwards reaches the maximum height when the velocity is zero that is $y_{\max} = y(v(t) = 0)$. The time required for the body to return back to its original is calculated when its position function is zero that is when $y = 0$. From the Newton's Law,

$$m\frac{dv}{dt} = F_G$$

where $F_G = -mg$ is the (downward-directed) force of gravity, where the gravitational acceleration is $g \approx 9.8\, m/s^2$. Using the above equation and plugging the I.C., we have

$$\frac{dv}{dt} = -9.8$$
$$v(t) = -9.8t + 49$$

Hence, the ball's height function $y(t)$ is given by

$$\begin{aligned} y(t) &= \int v dt \\ &= \int (-9.8t + 49)dt \\ &= -4.9t^2 + 49t + y_0 \end{aligned}$$

Since $y_0 = 0$, we have

$$y(t) = -4.9t^2 + 49t$$

So the ball reaches its maximum height when $v = 0$

$$-9.8t + 49 = 0$$
$$t = 5 \text{ sec}$$

Hence its maximum height is

$$y_{\max} = y(5) = 122.5\ m$$

The time required for the ball to return is

$$y(t) = 0$$
$$-4.9t^2 + 49t = 0$$
$$t = 10 \text{ sec}$$

2.2.2 Air Resistance Model

Now let us add the consideration of the air resistance in problems. In order to account for air resistance, the force F_R exerted by air, on the moving mass m must be added. So now we have

$$m\frac{dv}{dt} = F_G + F_R$$

It is found that $F_R = kv^p$ where $1 \le p \le 2$ and the value of k depends on the size and shape of the body as well as the density and viscosity of the air. Generally speaking $p = 1$ is for relatively low speeds and $p = 2$ for high speeds and we consider an intermediate value $1 < p < 2$ for intermediate speeds.

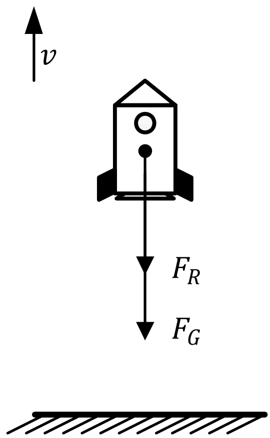

Figure 2.1 A rocket model with the consideration of air resistance.

The vertical motion of a body with mass m near the surface of the earth is subject to two kinds of force: a downward gravitational force F_G and a force F_R (we have taken $p = 1$ to simplify situations) of the air resistance that is proportional to velocity and directed in the opposite direction of motion of the body. We then have, $F_G = -mg$ and $F_R = -kv$ where k is a positive constant and v is the velocity of the body. Note that the negative sign for F_R makes the force positive (an upward force) when the body is falling down (v is negative) and negative (a downward force) if the body is thrown upwards (v is positive). So the total force on the body is now

$$F = F_R + F_G = -kv - mg$$

Therefore

$$m\frac{dv}{dt} = -kv - mg$$

Let

$$\rho = \frac{k}{m} > 0$$

Then the DE becomes

$$\frac{dv}{dt} = -\rho v - g$$

This DE is a separable DE that has a solution

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)\exp(-\rho t) - \frac{g}{\rho} \tag{2.6}$$

where v_0 is the initial velocity of the body $v(0) = v_0$. We now have

$$v_r = \lim_{t\to\infty} v(t) = -\frac{g}{\rho}$$

Thus the speed of a falling body with air resistance does not increase indefinitely. In fact it approaches a finite limiting speed or terminal speed.

$$|v_r| = \frac{g}{\rho} = \frac{mg}{k}$$

Thus the velocity of the body in terms of its terminal speed under the influence of air resistance is written as

$$v(t) = (v_0 - v_r)\exp(-\rho t) + v_r \tag{2.7}$$

Integration of the above DE will give us the position of the body in terms of its terminal velocity under the influence of air resistance as

$$y = \int v dt$$

$$\begin{aligned} y(t) &= \int \big((v_0 - v_r)\exp(-\rho t) + v_r\big)dt \\ &= -\frac{1}{\rho}(v_0 - v_r)\exp(-\rho t) + v_r t + C \end{aligned}$$

The initial height of the body at time $t = 0$ is $y(0) = y_0$. Substituting this is the above equation we have

$$C = y_0 + \frac{v_0 - v_r}{\rho}$$

Thus we get the position function of the body as

$$y(t) = y_0 + v_r t + \frac{1}{\rho}(v_0 - v_r)(1 - \exp(-\rho t)) \tag{2.8}$$

Example

We will again consider the ball that is thrown from the ground level with initial velocity$v_0 = 49\, m/s$. But now we will take the air resistance $\rho =$ 0.04 into account and find out the maximum height and the total time that the ball spent in the air.

Solution

We first find out the terminal velocity

$$v_r = -\frac{g}{\rho} = -\frac{9.8}{0.004} = -245$$

Substituting the values of $y_0 = 0$, $v_0 = 49$ and $v_r = 245$ in the equation of velocity and position we get

$$\begin{aligned} v(t) &= (v_0 - v_r)\exp(-\rho t) + v_r \\ &= \big(49 - (-245)\big)\exp(-0.04t) + (-245) \\ &= 294\exp\left(-\frac{t}{25}\right) - 245 \end{aligned}$$

$$\begin{aligned} y(t) &= y_0 + v_r t + \frac{1}{\rho}(v_0 - v_r)(1 - \exp(-\rho t)) \\ &= 0 + (-245)t + \frac{1}{0.04}(49 + 245)(1 - \exp(-0.04t)) \\ &= 7350 - 245t - 7350\exp\left(-\frac{t}{25}\right) \end{aligned}$$

The ball reaches maximum height when its velocity is zero. That is

$$\begin{aligned} v(t) &= 0 \\ 294\exp\left(-\frac{t}{25}\right) - 245 &= 0 \\ t_m &= -25\ln\frac{245}{294} \\ &\approx 4.558 \text{ sec} \end{aligned}$$

Here we notice that the time required for the ball to reach maximum height is $t_m = 4.558$ sec with air resistance as compared to 5 sec without air resistance. So for the maximum height of the ball substitute the value of t_m into the position function. So we get

$$\begin{aligned} y_{\max} &= y(t_m) \\ &= 7350 - 245t_m - 7350\exp\left(-\frac{t_m}{25}\right) \\ &\approx 108.3\ m \end{aligned}$$

Again here we have the maximum height as 108.3 m with air resistance as opposed to 122.5 m without air-resistance. Similarly, the time required for the ball to reach back in its original position is 9.41sec as compared to 10 sec from previous example.

We can also formulate DEs for the velocity and position of the ball without using their terminal velocity. This follows directly from (2.6)

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)\exp\left(-\frac{k}{m}t\right) \tag{2.9}$$

We can substitute ρ back to k/m.

Remarks

(1) The time required for the ball to reach maximum height is when its velocity is zero. At that point in time the ball reverses its motion to start falling down. So the time required for the ball to reverse its direction is t_{rev}.

$$-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)\exp\left(-\frac{k}{m}t_{rev}\right) = 0$$

$$\exp\left(-\frac{k}{m}t_{rev}\right) = \frac{kv_0}{mg} + 1$$

$$t_{\text{rev}} = \frac{m}{k}\ln\left(\frac{kv_0}{mg} + 1\right)$$

(1a) When the resistance is huge, it takes no time to reverse

$$\lim_{k\to\infty} t_{\text{rev}} = 0$$

(1b) When the resistance is tiny, we have to evaluate t_{rev} with care.

$$
\begin{aligned}
t_{\text{rev}} &= \lim_{k\to\infty} \frac{m}{k} \ln\left(\frac{kv_0}{mg} + 1\right) \\
&= \lim_{k\to\infty} \frac{m}{k} \frac{kv_0}{mg} \\
&= \frac{v_0}{g}
\end{aligned}
$$

which is just the value for no air resistance case.

(2) If $t \to \infty$,

$$v(t) \Rightarrow -\frac{mg}{k} = v_{\text{terminal}}$$

We see that:

v_{terminal} has no dependence on v_0

$$v_{\text{terminal}} \propto m$$

$$v_{\text{terminal}} \propto g$$

$$v_{\text{terminal}} \propto \frac{1}{k} = \begin{cases} \infty, & k \to 0 \\ 0, & k \to \infty \end{cases}$$

2.2.3 Gravitational Acceleration

According to Newton's Law of Gravitation, the gravitational force of attraction between two point masses M and m located at a distance r apart is given by

$$F = \frac{GMm}{r^2}$$

where G is gravitational constant given as $G = 6.67 \times 10^{-11}\, N \cdot (m/kg)^2$ in MKS units.

Also the initial velocity necessary for a projectile to escape from earth altogether is called the escape velocity and is found out to be

$$v_0 = \sqrt{\frac{2G}{R}}$$

where R is the radius of the earth.

Problems

Problem 2.2.1 Suppose that a body moves horizontally through a medium whose resistance is proportional to its velocity v, so that $v' = -kv$.
(1) Show that its velocity and position at time t are given by

$$v(t) = v_0 \exp(-kt) \quad \text{and} \quad x(t) = x_0 + \left(\frac{v}{k}\right)(1 - \exp(-kt))$$

(2) Conclude that the body travels only a finite distance, and find that distance.

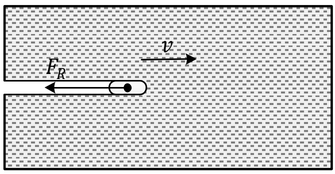

Figure 2.2 The motion model for Problem 2.2.1 and Problem 2.2.2.

Problem 2.2.2 Consider a body that moves horizontally through a medium whose resistance is of the $v' = -kv^{\frac{3}{2}}$. Show that

$$v(t) = \frac{4v_0}{\left(kt\sqrt{v_0} + 2\right)^2} \quad \text{and} \quad x(t) = x_0 + \frac{2}{k}\sqrt{v_0}\left(1 - \frac{2}{kt\sqrt{v_0} + 2}\right)$$

Conclude that under a $3/2$-power resistance a body coasts only a finite distance before coming to a stop.

Problem 2.2.3 In Jules Verne's original problem, the projectile launched from the surface of the earth is attracted by both the earth and the moon, so its distance $r(t)$ form the center of the earth satisfies the IVP

$$\begin{cases} \dfrac{d^2r}{dt^2} = -\dfrac{GM_e}{r^2} + \dfrac{GM_m}{(S-r)^2} \\ r(0) = R \\ r'(0) = v_0 \end{cases}$$

where $M_e$and M_m denote the masses of the earth and the moon, respectively; R is the radius of the earth and $S = 384{,}400\ km$ is the distance between the centers of the earth and the moon. To reach the moon, the projectile must only just pass the point between the moon and earth where its net acceleration vanishes. Thereafter it is "under the control" of the moon, and falls from there to the lunar surface. Find the minimal launch velocity v_0 that suffices for the projectile to make it "From the Earth to the Moon."

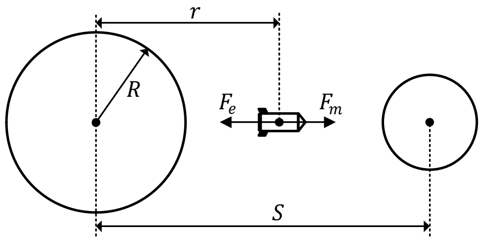

Figure 2.3 The projectile model for Problem 2.2.3.

Problem 2.2.4 One pushes straight up a toy rocket of fixed mass m at an initial velocity of v_0 from the surface of earth. Assume earth's gravitational constant to be g), and air frictional force on the rocket $-kv$ where k is the frictional constant and v is the velocity of the rock. (Figure 2.1) Ignore the rocket "launcher" height. Please
(1) Compute the max height h_{max} the rocket can reach.
(2) Compute the time T_{up} for the rocket to reach max height.

Problem 2.2.5 Suppose that a motorboat is moving at 40 ft/s when its motor suddenly quits, and that 10s later the boat has slowed to 20 ft/s.
(1) Assume that the resistance it encounters while coasting is proportional to its velocity. How far will the boat coast in all?
(2) Assuming that the resistance is proportional to the square of the velocity. How far does the motorboat coast in the first minute after its motor quits?

Figure 2.4 The motorboat model for Problem 2.2.5.

Problem 2.2.6 The resistance of an object (mass m) moving in Earth's gravitational field (with gravitational constant g) is proportional to its speed $v(t)$.

(1) Write down the equation of motion by Newton's second law, in terms of speed of the object, with assumption that the object was initially thrown upward at speed $v_{0,}$ and air frictional constant k.

(2) Naturally, the object will move upward in the sky until at one point when it reverses its direction to move downward to the earth. Find the time when the direction of the object has just reversed. Does this time (at which reversing occurs) depend on the initial velocity? How?

(3) Eventually, the object will move at a constant speed, the Terminal Speed. Find this terminal speed. Does this terminal speed depend on the initial velocity? How?

Problem 2.2.7 On January 15, 2009, the US Airways Flight 1549 "landed" safely in the Hudson River a couple of minutes after departing from LGA. The air (straight line) distance between LGA and the landing spot on the river is M. We assume (I) the flight's Captain Sullenberger flies the plane at a constant speed v_0 relative to the wind, after realizing the bird problem immediately after taking off; (II) the wind blows at a constant speed w and at a direction perpendicular to the line linking the landing spot and LGA; (III) the plane maintains its heading directly toward the landing spot. Compute the critical wind speed w_c above which the plane would have been blown away. Compute the lengths of the flight trajectories for $w < w_c$ and $w \neq 0$ and $w = 0$.

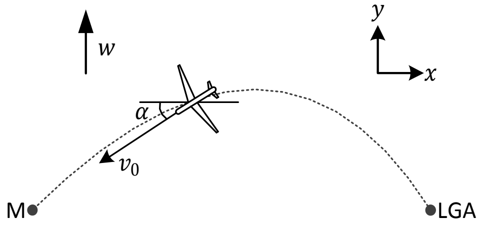

Figure 2.5 The aircraft model for Problem 2.2.7.

Problem 2.2.8 On a straight stretch of road of length L there were two stop signs at the two ends. Assume you follow traffic laws and stop at these signs. Your poor car can accelerate at a constant a_1 and decelerate at another constant a_2. Compute the shortest time to travel from one stop sign to another if there is no speed limit in between. Do it again if the speed limit between the signs is v_m.

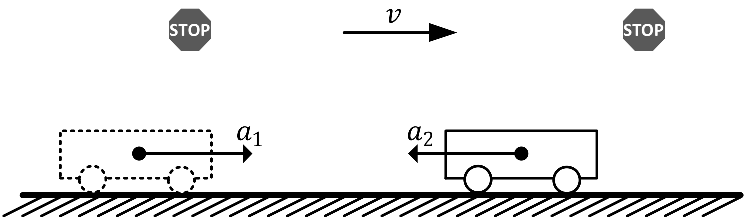

Figure 2.6 The moving car model for Problem 2.2.8.

Problem 2.2.9 A man with a parachute jumps out of a hovering helicopter at height H. During the fall, the man's drag coefficient is k (with closed parachute) and nk (with open parachute) and air resistance is taken as proportional to velocity. The total weight of the man and his parachute is m. Take the initial velocity when he jumped to be zero. Gravitational constant is g. Find the best time for the man to open his parachute after he leaves the helicopter for the quickest fall and yet "soft" landing at touchdown speed $\leq v_0$.

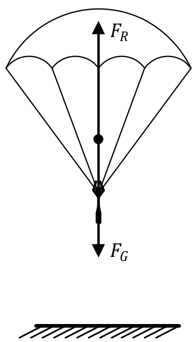

Figure 2.7 The jumper model for Problem 2.2.9 and Problem 2.2.14.

Problem 2.2.10 If a projectile leaves Earth's surface at velocity v_0, please derivate the formula for the later velocity as a function of the distance r from the projectile to the center of the earth. The earth's radius R and gravitational constant G are given. Also, naturally, the velocity has to be a real number. Compute the condition at which the velocity will become imaginary.

Problem 2.2.11 Suppose that a projectile is fired straight upward from the surface of the earth with initial velocity v_0 and assume its height $y(t)$ above the surface satisfies the IVP

$$\begin{cases} y'' = -\dfrac{GM}{(y+R)^2} - \beta \exp(-y) \\ y(0) = 0 \\ y'(0) = v_0 \end{cases}$$

Compute the maximum altitude the projectile reaches and the time (from launch) for the projectile to reach the maximum altitude. In the above, G, M, v_0 and β are all given constants.

Problem 2.2.12 A guy (of body mass m) jumps out of a horizontally flying plane of speed v_0 at height H. The horizontal component of the jumper's initial velocity is equal to that of the plane's and the jumper's initial vertical speed can be regarded as 0. The air resistance is $-\alpha\vec{v}$ with the parachute open where α is a given constant and $\vec{v}$ is the jumper's velocity. Please compute the total distance, from when the guy was thrown off the plane to the landing spot, if the parachute is opened at the middle height point. For clarification, you compute the length of the trajectory the poor guy draws while in sky (before hitting ground). You may express your solution implicitly.

(Hint: For a curve described by $F(x, y) = 0$, one can calculate its length with $x \in [a, b]$ by $L = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$)

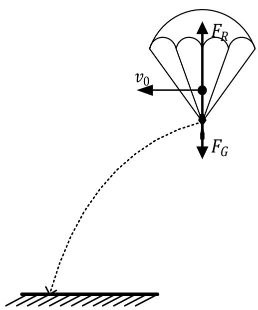

Figure 2.8 The jumper model for Problem 2.2.12 and Problem 2.2.13.

Problem 2.2.13 A guy (of body mass m) jumps out of a horizontally flying plane of speed v_0 at height H. The horizontal component of the jumper's initial velocity is equal to that of the plane's and the jumper's initial vertical speed can be regarded as 0. The air resistance is $\alpha|\vec{v}|$ with the parachute open where

α is a given constant and $|\vec{v}| = \left(v_x^2 + v_y^2\right)^{1/2}$ is the magnitude of jumper's velocity. Please compute the times taken the jumper to reach land if the parachute is opened at heights $\frac{1}{2}H$ and $\frac{1}{4}H$.

Problem 2.2.14 A man with a parachute jumps out of a "frozen" helicopter at height H. During the fall, the man's drag coefficient is k (with closed parachute) and nk (with open parachute) and air resistance is taken as proportional to velocity. The total weight of the man and his parachute is m. Take the initial velocity when he jumped to be zero. Gravitational constant is g. The man opens the parachute t_0 time after he jumps off the helicopter. Please find the total falling time, and the speed he hits the ground. Can you adjust the t_0 to enable the quickest fall, and lightest hit on the ground?

Problem 2.2.15 A fighter jet taking off from one aircraft carrier C_1 wishes to land on another carrier C_2, currently located precisely on the northeast of C_1. The distance between the two carriers is L. The carrier C_2 moves east at a constant speed of v_2 just after the jet took off. Wind blows from south to north at a constant speed of v_W and its impact to the carriers is negligible while the impact to the jet must be taken into account fully. Obviously, given the scale of flight distance (usually a thousand miles), the carriers and jet can be treated as mathematical points. Compute the shortest trajectory of the jet to land as it wishes.

Problem 2.2.16 A bullet of mass m is fired off the barrel of a gun at an initial speed v_0 to a medium whose resistance follows $\alpha v + \beta v^2$ where α and β are constants and v is the bullet's instantaneous speed. Compute the maximal distance that the bullet can travel in this medium. If we double the initial speed and keep other conditions unchanged, compute the maximal distance again. You may ignore gravity.

Problem 2.2.17 Consider shooting a bullet of mass m to three connected (and fixed) media each of equal length L. The resistance in Medium-1 is kv, in Medium-2 is kv^2, and Medium-3 is kv. Compute the bullet's speed to enter Medium-1 such that the bullet will pass through Medium-1 and then Medium-2 and then Medium-3, and magically stop precisely at the edge of Medium-3. You may neglect gravity.

Problem 2.2.18 Consider shooting a bullet of mass m to a first medium whose resistance is proportional to the speed, to a second medium whose resistance is proportional to $v^{\frac{3}{2}}$, and a third medium whose resistance is proportional to v^2. Compute in all three cases the farthest the bullet can travel with a given initial speed of v_0. The medium in all three cases is big (you may imagine the water of the ocean).

Problem 2.2.19 Consider shooting, at an initial speed v_0, a bullet of mass m to a medium whose resistance is $-\alpha(\beta + v^2)$ where v is the instatenous speed, α is the given resistance constant and β is another given constant. You may assume the medium is infinite (for example, the ocean) and you may also neglect the gravity for this problem. Please compute

(1) the farthest distances the bullet can travel in the medium;

(2) the time the bullet is in motion.

Problem 2.2.20 We set a 2-dimensional Cartesian coordinate system for a north running river, so that the *y*-axis lies on the river's west bank (pointing from south to north), and the *x*-axis across the river, pointing from west to east. A ferryboat wants to cross the river from a point $(a, 0)$ on the east bank to the west bank, and it hopes to dock at $(0,0)$. The water speed, depending on the location in the river, is measured by a function where $W(x) = \omega_0(a - x)/a^2$ where both ω_0 and a are positive constants. Now, we further assume the ferryboat keeps facing the docking point $(0, 0)$ at all times during the trip, and it moves at a constant speed v_0 relative to the water. Please do the following:

(1) Establish the equation for the boat's trajectory.

(2) Solve the equation.

(3) Sketch the boat's trajectory if the river flows much slower than the boat.

Problem 2.2.21 An egg is dropped vertically to a fully-filled swimming pool of depth H at an initial speed 0 at the surface. Three forces act on the egg: gravity, water floating force, and friction with water. Assume water density 1, egg density $\rho > 1$, egg mass m. Floating force is given as $\frac{m}{\rho} \times 1 \times g$ where g is the usual gradational constant while the friction force is μv where μ is a friction-related constant and v is egg's instantaneous speed. Compute the egg impact speed at the pool bottom. You may express the solution in implicit form.

2.3 An Example in Finance

Consider a situation when one owes a certain amount of money to the bank.

We consider the following terms:

- $z(t)$ is the amount owed to the bank at time t.
- r is the interest rate which is a constant
- dt is the time period from the time interval $[t, t + dt]$
- w is the payment rate which is also a constant
- dz is the change of debts to bank during time dt

So now let us calculate the decrement of loan due to payments made to the bank.

$$\text{The decrement of loan due to payments made} = (\text{total amount with interest}) - (\text{payments})$$

$$dz = z(t)\, rdt - wdt$$

where

- dz is the decrement of loan due to payments made
- $z(t)\, rdt$ is the increase of loan due to interest
- wdt is the payment rate×time=payments made

Therefore

$$dz = (zr - w)dt$$

$$z' = zr - w \tag{2.10}$$

is the loan equation.

To solve this DE, we simply use the separation of variable method.

$$\frac{z'}{r} = z - \frac{w}{r}$$

$$\frac{dz}{z - \frac{w}{r}} = rdt$$

$$\int_{z_0}^{z} \frac{dz}{z - \frac{w}{r}} = \int_{t=0}^{t} rdt$$

$$\ln\left(z - \frac{w}{r}\right)\Big|_{z_0}^{z} = rt$$

$$\ln\left(\frac{z - \frac{w}{r}}{z_0 - \frac{w}{r}}\right) = rt$$

$$z - \frac{w}{r} = \left(z_0 - \frac{w}{r}\right)\exp(rt)$$

Thus the solution to the financial model problem is

$$z(t) = \frac{w}{r} - \left(\frac{w}{r} - z_0\right)\exp(rt) \tag{2.11}$$

Remarks

(1) At $t = 0, z(t = 0) = z_0$

(2) If $r = 0, z \neq z_0$ (Since w/r will become indeterminate)

(3) If the rate by which the payments are made is greater than the interest rate

$$w > z_0 r$$

then we have

$$\frac{w}{r} - z_0 > 0$$

As the payments are higher than the interest rate, the loan will decrease.

If he increases his rate of payments (i.e., he starts paying biweekly/weekly/daily instead of monthly) his loan will decrease faster as shown in the figure.

(4) If the rate by which the payments are made is equal to the interest rate then

$$\frac{w}{r} - z_0 = 0$$

The time required to complete the entire payment amount would be when $z = 0$

$$\frac{w}{r} - \left(\frac{w}{r} - z_0\right)\exp(rt) = 0$$

$$\frac{w}{r} = \left(\frac{w}{r} - z_0\right)\exp(rt)$$

$$\exp(rt) = \frac{\frac{w}{r}}{\frac{w}{r} - z_0}$$

$$rt = -\ln\left(1 - \frac{rz_0}{w}\right)$$

$$t = -\frac{1}{r}\ln\left(1 - \frac{rz_0}{w}\right)$$

Since time t cannot be negative, we have

$$\ln\left(1 - \frac{rz_0}{w}\right) < 0$$

$$0 < \left(1 - \frac{rz_0}{w}\right) < 1$$

$$0 < \frac{rz_0}{w} < 1$$

Hence the time required to complete the loan is

$$t = -\frac{1}{r}\ln\left(1 - \frac{rz_0}{w}\right)$$

(5) If the rate by which the payments are made is smaller to the interest rate, then

$$\left(\frac{w}{r} - z_0\right) < 0$$

the payment will become indefinite and the loan will never be paid off.

Problems

Problem 2.3.1 A person borrowed Z_0 amount of funds from a bank which charges a fixed (constant) interest rate r. The periodical payment the borrower makes to the bank is $Q_0 t$, where Q_0 is a constant, and t is the time.
(1) Establish the DE governing the time-varying loan balance $Z(t)$ with other parameters.
(2) Solve the equation to express $Z(t)$ explicitly as a function of t.

Problem 2.3.2 Mr. and Mrs. Young borrowed Z amount of money from a bank at a daily interest rate r. The Youngs make the same "micro-payment" of D amount to the bank each day.
(1) Set up the DE for amount x owed to the bank at the end of each day after the loan closing.
(2) Solve the above DE for x as a function of t, Z, and r.
(3) Obviously, if D is too small (paid too little) to the bank, the Youngs will never pay off the loan. Find the critical D at which the loan will not change.
(4) The Youngs wish to pay off the loan in N days. What do they have to pay each day? Write it as a function of Z, N, and r.
(5) The Youngs wish to pay off the loan in $N/2$ days. Find the daily payment D as a function of Z, N, and r.

Problem 2.3.3 Mr. A borrowed Z_0 from a bank at a fixed interest rate r. Mr. A pays the bank fixed amount W periodically so that he can pay of the loan at time T. Please do the following.
(1) Derive a formula for T in terms of Z_0, W, and r. In other words, find the exact form for function $T = T(Z_0, W, r)$.
(2) For given values of Z_0 and W, compute

(a) $T_1 = T\left(Z_0, W, r \to 0\right) =?$

(Hint: $\lim_{x \to 0} ln(1 + x) = x$*)*

(b) $T_2 = T\left(Z_0, W, r = \frac{W}{2Z_0}\right) =?$

(3) For given values of Z_0 and r, how much does Mr. A have to add to his current payment W to pay off the loan at $T/2$ instead of T?

Problem 2.3.4 Mr. Wyze and Mr. Fulesch each borrows z_0 from a bank at the same time. Wyze got it at a fixed interest rate r and will pay it off at time T with a fixed periodic payment W_0. Fulesch's rate changes with time according to $r_F = \frac{1}{5}r(1+t)$ so Fulesch's rate is only $1/5$ of Wyze's at the start. For example, if Wyze's rate is 6.0%, Fulesch's starting rate is only 1.2%. If Fulesch also pays W_0 periodically, his debt drops faster than Wyze's initially before it drops slower and then grows. Compute the time when both loaners own the bank the same amount, again. The first time when they own the same is when they just got the loans.

Problem 2.3.5 Given that a loan of Z_0 with a fixed interest rate r would be paid off at time T with a fixed periodic payment W_0. Now, one pays a new periodic payment $(1+\alpha)W_0$, compute the new time to pay off the loan. Compute total interests paid to the bank in both cases.

Problem 2.3.6 One borrows Z_0 from a bank at a fixed rate r. Now, we assume the time to pay off the loan with fixed periodic payment W is T_1. Please do the following.

(1) If the periodic payment is doubled while all other terms remain unchanged, derive a formula for the new payoff time.

(2) If the interested rate is doubled while all other terms remain unchanged, derive a formula for the new periodic payment.

Problem 2.3.7 One borrows Z_0 dollars from a bank at a fixed rate r. Now, we assume the time to pay off the loan with fixed periodic payment W is T. Please do the following.

(1) If the periodic payment is nW while all other terms remain unchanged, derive a formula for the new payoff time.

(2) If the interested rate is br while all other terms remain unchanged, derive a formula for the new periodic payment.

Problem 2.3.8 One borrows Z_0 from a bank at a fixed interest rate r. We assume the time to pay off the loan with a fixed periodical payment W_0 is T_0. Now, if the payment is changed to αW_0 while all other terms remain unchanged, derive a formula for the new payoff time T_1 in terms of the given parameters.

(1) If $\alpha > 1$, which of the following is true?

(a) $T_1 > T_0$

(b) $T_1 = T_0$

(c) $T_1 < T_0$

(2) If $0 < \alpha < 1$, which of the following is true?

(a) $T_1 > T_0$

(b) $T_1 = T_0$

(c) $T_1 < T_0$

(3) If $\alpha < 0$, compute T_1.

Chapter 3
Linear DEs of Higher Order

3.1 Classification of DEs

	Linear DE	Nonlinear DE
Constant Coefficients	$y'' + y = 0$	$y'' + y^2 = 0$
Variable Coefficients	$y'' + P(x)y = 0$	$y'' + P(x)y^2 = 0$

We will study two types of DEs, namely,

(1) The general form of a variable coefficient, linear, 2nd-order DE.

$$y'' + P(x)y' + Q(x)y = R(x)$$

If $R(x) = 0$, the DE is "reduced" to a homogeneous DE.

(2) The general form of a constant coefficient, linear, 2nd-order DE.

$$y'' + ay' + by = c$$

where a, b and c are constants and if $c = 0$, the DE is "reduced" to a homogeneous DE.

Theorem 1

For homogeneous DE

$$y'' + P(x)y' + Q(x)y = 0 \tag{3.1}$$

of a inhomogeneous DE $y'' + P(x)y' + Q(x)y = R(x)$, if $y_1(x)$ is a solution and $y_2(x)$ is another solution, then the linear combination of y_1 and y_2 is also a solution. Further, if $y_1(x)$ and $y_2(x)$ are linearly independent (L.I.) then their linear combination $y(x) = C_1y_1(x) + C_2y_2(\text{x})$ is the G.S..

Corollaries of Theorem 1

If $y_1, y_2, \dots, y_n$ are n solutions for DE

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

then the linear combination of $y_1, y_2, \dots, y_n$ is also a solution to the DE. If $y_1, y_2, \dots, y_n$ are n L.I. solutions of the DE, then their linear combination is the G.S. to the DE.

Proof of Theorem 1

Let

$$y_3 = C_1y_1 + C_2y_2$$

be the linear combination of y_1 and y_2 where C_1 and C_2 are constants. Since y_1 and y_2 are the solutions to the DE (3.1), they satisfy

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \tag{3.2}$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0 \tag{3.3}$$

$C_1 \times$ (3.2) gives

$$C_1 y_1'' + C_1 P(x) y_1' + C_1 Q(x) y_1 = 0$$

Since

$$C_1 y_1'' = (C_1 y_1)'' \text{ and } C_1 y_1' = (C_1 y_1)'$$

we have

$$(C_1 y_1)'' + P(x)(C_1 y_1)' + Q(x)(C_1 y_1) = 0 \tag{3.4}$$

Similarly, $C_2 \times$ (3.3) gives

$$(C_2 y_2)'' + P(x)(C_2 y_2)' + Q(x)(C_2 y_2) = 0 \tag{3.5}$$

Adding the two equations above generates

$$((C_1 y_1)'' + (C_2 y_2)'') + P(x)((C_1 y_1)' + (C_2 y_2)') + Q(x)\left(C_1 y_1 + C_{2y_2}\right) = 0$$

$$\begin{aligned}\left(C_1 y_1 + C_{2y_2}\right)'' &+ P(x)(C_1 y_1 + C_2 y_2)' \\ &+ Q(x)(C_1 y_1 + C_2 y_2) = 0\end{aligned} \tag{3.6}$$

That is

$$y_3'' + P(x) y_3' + Q(x) y_3 = 0$$

Therefore $y_3 = C_1 y_1 + C_2 y_2$ is a solution to the DE.

Example 1

Check if y_1 and y_2 are solutions of the DE

$$y'' + \omega^2 y = 0$$

where ω is a constant and find y_3. Given $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$

Solution

Check y_1

$$y_1' = -\omega \sin \omega x \quad \text{and} \quad y_1'' = -\omega^2 \cos \omega x$$

$$y_1'' + \omega^2 y_1 = 0$$

Thus y_1 is a solution of the given DE.

Check y_2

$$y_2' = \omega \cos \omega x \quad \text{and} \quad y_2'' = -\omega^2 \sin \omega x$$

$$y_2'' + \omega^2 y_2 = 0$$

Thus y_2 is a solution of the given DE.

As per Theorem 1, we choose

$$y_3 = C_1 y_1 + C_2 y_2 = C_1 \cos \omega x + C_2 \sin \omega x$$

Check y_3

$$y_3' = -\omega C_1 \sin \omega x + \omega C_2 \cos \omega x \quad \text{and} \quad y_3'' = -\omega^2 C_1 \cos \omega x - \omega^3 C_2 \sin \omega x$$

$$y_3'' + \omega^2 y_3 = 0$$

Therefore y_3 is also a solution of the DE

$$y'' + \omega^2 y = 0$$

Example 2

Given $y_1 = \sin \omega x$, $y_2 = 5 \sin \omega x$. Both are solutions of the DE

$$y'' + \omega^2 y = 0$$

Find the third solution.

Solution

According to theorem 1

$$\begin{aligned} y_3 &= C_1 y_1 + C_2 y_2 \\ &= C_1 \sin \omega x + 5C_2 \sin \omega x \\ &= (C_1 + 5C_2) \sin \omega x \\ &= C_3 \sin \omega x \end{aligned}$$

where $C_3 = C_1 + 5C_2$

Problems

Problem 3.1.1 Show that $y = 1/x$ is a solution of $y' + y^2 = 0$, but that if $c \neq 0$ and $c \neq 1$, then $y = c/x$ is not a solution.

Problem 3.1.2 Show that $y = x^3$ is a solution of $yy'' = 6x^4$, but that if $c \neq 1$, then $y = cx^3$ is not a solution.

Problem 3.1.3 Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of $yy'' + (y')^2 = 0$, but that their sum $y = y_1 + y_2$ is not a solution.

Problem 3.1.4 Let y_p be a P.S. of the inhomogeneous DE $y'' + py' + qy = f(x)$ and let y_c be a solution of its associated homogeneous DE $y'' + py' + qy = 0$. Show that $y = y_c + y_p$ is a solution of the given inhomogeneous DE. With $y_p = 1$ and $y_c = C_1 \cos x + C_2 \sin x$ in the above notation, find a solution of $y'' + y = 1$ satisfying the I.C. $y(0) = y'(0) = -1$.

3.2 Linear Independence

Definition

For functions y_1 and y_2, if y_1 is not a constant multiple of y_2, then y_1 is L.I. of y_2. That is, if one can find two constants $c_1$and c_2 that are not zero simultaneously to make $c_1y_1 + c_2y_2 = 0$, then y_1 and y_2 are called linear dependent (L.D.). O.W., they are L.I..

For the two-function case, if $\frac{y_1}{y_2}$ ≠constant, then they are L.I..

Example

Determine if y_1, y_2 are L.I.

y_1	y_2	$\frac{y_1}{y_2}$	L.I.
$\sin x$	$\cos x$	$\tan x$	Y
$\sin 2x$	$\sin x$	$2\cos x$	Y
$2\sin x$	$\sin x$	2	N
$2\exp(x)$	$\exp(x)$	2	N
$\exp(2x)$	$\exp(x)$	$\exp(x)$	Y
$\exp(ax)$	$\exp(bx)$	$\exp\big((a-b)x\big)$	Y $(a \neq b)$

Now Let us define a new operator

$$D = \frac{d}{dx}$$

Example

$$Dy = \left(\frac{d}{dx}\right)y = \frac{dy}{dx} = y'$$

$$D^2y = \left(\frac{d}{dx}\right)^2 y = \frac{d^2y}{dx^2} = y''$$

and so on. So for the n^{th} order we have

$$D^ny = \left(\frac{d}{dx}\right)^n y = \frac{d^ny}{dx^n} = y^{(n)}$$

So now, our n^{th} order DE

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{3.7}$$

can be written as

$$D^n y + P_1(x)D^{n-1}y + \cdots + P_n(x)y = 0$$

That is

$$\left(D^n + P_1(x)D^{n-1} + \cdots + P_n(x)\right)y = 0 \tag{3.8}$$

Definition

A new linear differential operator L as

$$L = D^n + P_1(x)D^{n-1} + \cdots + P_n(x)$$

so that (3.8) now can be written as $Ly = 0$.

Example 1

Find the linear differential operator L in $y'' + y = 0$

Solution

$$\frac{d^2y}{dx^2} + y = 0$$
$$\left(\frac{d^2}{dx^2} + 1\right)y = 0$$
$$L = D^2 + 1$$

Example 2

Find the linear differential operator L in $y'' - 5y' + 6y = 0$

Solution

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$
$$\left(\frac{d^2}{dx^2} - 5\frac{d}{dx} + 6\right)y = 0$$
$$L = D^2 - 5D + 6$$

Properties of the Linear Differential Operator

(a) $(L_1 + L_2)y = L_1 y + L_2 y$
(b) $C(Ly) = L(Cy)$

From the above two properties, we can conclude that the linear differential operator L satisfies the linearity property.

Theorem 2

For an n^{th} order linear DE $Ly = 0$, if $y_1, y_2 \dots y_n$ are n L.I. solutions to the DE, then the linear combination

$$y_{\text{combo}} = \sum_{i=1}^{n} C_i y_i$$

is the G.S. to the DE.

Proof

Since $\forall\ 1 \le i \le n$, y_i is a solution to $Ly = 0$, then

$$Ly_i = 0 \tag{3.9}$$

Multiplying both sides of (3.9) by C_i, we have

$$C_i L y_i = 0$$

By the linearity of L, that gives

$$L(C_i y_i) = 0 \ \ \forall\ 1 \le i \le n$$

Summing over i, we have

$$\sum_{i=1}^{n} L(C_i y_i) = 0$$

That is

$$L\left(\sum_{i=1}^{n} C_i y_i\right) = 0 \tag{3.10}$$

By the definition, we know this is

$$Ly_{\text{combo}} = 0$$

L.I. for More than Two Functions

For more than two functions, the definition of linear independence becomes a lot more complicated. The definition at the beginning of the section is still valid as the definition for pair wise linear independence. However, a set of functions that are mutually pair-wise L.I. does not necessarily indicate they are L.I. as a whole.

Example

Given $y_1 = \sin x$, $y_2 = \cos x$ and $y_3 = \sin x + \cos x$, prove that they are mutually pair wise linear independent.

Solution

It is easy to verify that

$$\frac{y_1}{y_2} = \tan x$$
$$\frac{y_3}{y_1} = 1 + \cot x$$
$$\frac{y_3}{y_2} = \tan x + 1$$

These prove that they are mutually linear independent.

Remarks

Despite the fact they are mutually pair wise L.I., intuitively we think they are not L.I. as a whole since we can easily find that $y_3 = y_1 + y_2$. From the definition we will introduce below, they are indeed L.D..

For a formal definition of the linear independence for more than two functions, we have to expand the definition for two functions from a new direction. For any two L.I. functions y_1 and y_2, from definition, we know that we cannot find a constant c such that

$$\frac{y_1}{y_2} = c \tag{3.11}$$

for all x where y_1 and y_2 are defined. Now let us write this definition in a balanced fashion. Consider the equation

$$C_1y_1 + C_2y_2 = 0 \tag{3.12}$$

If we can find a non-zero vector $[C_1, C_2]^T$ that satisfies (3.12), we know we can find a constant C that satisfies (3.11). Note that the zero in (3.12) means the zero function that is 0 everywhere y_1 and y_2 are defined. Thus, we know that the linear independence of y_1 and y_2 means that the only solution for (3.12) is $[C_1, C_2]^T = [0,0]^T$. Now we can expand this definition to n functions.

Definition

For n functions $y_1, y_2, \dots, y_n$ said to be L.I. if

$$C_1y_1 + C_2y_2 + \cdots + C_ny_n = 0 \tag{3.13}$$

has only one solution which is $[C_1, C_2, \dots, C_n]^T = [0,0, \dots ,0]^T$. The zero in the RHS of (3.13) is the zero function defined where $y_1, \dots, y_n$ are defined.

It is very difficult to verify directly from the above original definition of L.I. if a set of functions are L.I.. In practice, several other approaches can be adopted. One of them is that of the Wronskian method.

Let us now introduce the Wronskian of n functions.

Definition

The Wronskian of n functions $y_1, y_2 \dots y_n$ is the following determinant

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_{n\times n} \tag{3.14}$$

Theorem 3

For n functions $y_1, y_2, \ldots, y_n$, if the Wronskian $W(y_1, \ldots, y_n) \neq 0$, then they are L.I.. Conversely, if $y_1, y_2, \ldots, y_n$ are L.D., then the Wronskian $W(y_1, \ldots, y_n) = 0$.

Example 1

Check if $y_1 = \exp(r_1 x)$and $y_2 = \exp(r_2 x)$ $(r_1 \neq r_2)$ are L.I..

Solution

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \exp(r_1 x) & \exp(r_2 x) \\ r_1 \exp(r_1 x) & r_2 \exp(r_2 x) \end{vmatrix} \\ &= \exp(r_1 x) \cdot r_2 \exp(r_2 x) - \exp(r_2 x) \cdot r_1 \exp(r_1 x) \\ &= (r_2 - r_1) \exp\big((r_2 - r_1)x\big) \end{aligned}$$

Since $r_1 \neq r_2$, we know that $W(y_1, y_2) \neq 0$. Thus, $y_1$and y_2 are L.I..

Example 2

Check if $y_1 = \exp(rx)$ and $y_2 = x\exp(rx)$ are L.I..

Solution

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \exp(rx) & x\exp(rx) \\ r\exp(rx) & \exp(rx) + rx\exp(rx) \end{vmatrix} \\ &= (1 + rx)\exp(2rx) - rx\exp(2rx) \\ &= \exp(2rx) \neq 0 \end{aligned}$$

Thus, y_1 and y_2 are L.I..

Example 3

Check if $y_1 = \sin\omega x$ and $y_2 = \cos\omega x$ are L.I., where $\omega \neq 0$ is a constant.

Solution

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \sin\omega x & \cos\omega x \\ \omega\cos\omega x & -\omega\sin\omega x \end{vmatrix} \\ &= -\omega\sin^2\omega x - \omega\cos^2\omega x \end{aligned}$$

$$= -\omega(\sin^2 \omega x + \cos^2 \omega x)$$
$$= -\omega \neq 0$$

Thus, y_1 and y_2 are L.I..

Remarks

Please note that if Wronskian is non-zero, the functions are L.I.. But, if the Wronskian is zero, it does not necessarily imply that the functions are L.D.. For example, for $y_1 = x^2$ and $y_2 = x|x|$, though they are L.D. on either $[0, +\infty)$ or $(-\infty, 0]$, we cannot find a set of non-zero $[C_1, C_2]^T$ such that (3.12) holds for the entire real axis. This means y_1 and y_2 are L.I. when considered as a function on $(-\infty, +\infty)$, but the Wronskian gives for $x \in [0, +\infty)$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$$

and for $x \in (-\infty, 0)$, we have

$$W(y_1, y_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$$

These two mean $W(y_1, y_2) = 0\ \forall x$. This example indicates the Wronskian is zero does not give much information about the linear independence of the functions.

Problems

Problem 3.2.1 Define two operators with given constants $\alpha_1, \alpha_2, \beta_1, \beta_2$.

$$L_1 \equiv D^2 + \alpha_1 D + \beta_1$$
$$L_2 \equiv D^2 + \alpha_2 D + \beta_2$$

Applying these two operators to a function $x(t)$, we may get $L_1 L_2 x(t)$ and $L_2 L_1 x(t)$ check if $L_1 L_2 x(t) = L_2 L_1 x(t)$. What if we define the two operators as follows, check it again.

$$L_1 x(t) \equiv Dx(t) + tx(t)$$
$$L_2 x(t) \equiv tDx(t) + x(t)$$

Problem 3.2.2 Check the linear independence of the following three functions:

$$\exp(x)\,,\ \cosh x\,,\ \sinh x$$

Problem 3.2.3 Check the linear independence of the following three functions:

$$0,\ \sin x\,, \exp(x)$$

Problem 3.2.4 Check the linear independence of the following three functions:

$$2x,\ \ 3x^2,\ \ 5x - 8x^2$$

Problem 3.2.5 Determine if $f(x)$ and $g(x)$ are L.I..

$$f(x) = \pi \qquad g(x) = \cos^2 x + \sin^2 x$$

Problem 3.2.6 Let y_1 and y_2 be two solutions of $A(x)y'' + B(x)y' + C(x)y = 0$ on an open closed interval I where, A, B and C are continuous and $A(x)$ is never zero.

(1) Let $W = W(y_1, y_2)$, Show that

$$A(x)\frac{dW}{dx} = y_1(Ay_2'') - y_2(Ay_1'')$$

Then substitute for Ay_1'' and Ay_2'' from the original DE to show that

$$A(x)\frac{dW}{dx} = -B(x)W(x)$$

(2) Solve this 1st-order DE to deduce Abel's formula

$$W(x) = K\exp\left(-\int \frac{B(x)}{A(x)}dx\right)$$

Problem 3.2.7 Prove directly that the functions $f_0(x) = 1,\ f_1(x) = x, f_2(x) = x^2, \dots, f_n(x) = x^n$ are L.I..

Problem 3.2.8 Show that the Wronskian of the following n^{th} order inhomogeneous DE

$$x^{(n)} + P_1(t)x^{(n-1)} + \cdots + P_n(t)x = f(t)$$

is a function of $P_1(t)$ only, and find the expression of the Wronskian in terms of this $P_1(t)$.

Problem 3.2.9 Suppose that the three numbers r_1, r_2 and r_3 are distinct. Show that the three functions $\exp(r_1 x)$, $\exp(r_2 x)$ and $\exp(r_3 x)$ are L.I. by showing that their Wronskian

$$W = \exp\big((r_1 + r_2 + r_3)x\big)\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} \neq 0, \forall x$$

Problem 3.2.10 Compute the Vandermonde determinant:

$$V = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-2} & r_2^{n-2} & \cdots & r_n^{n-1} \end{vmatrix}$$

Problem 3.2.11 Given that the Vandermonde determinant:

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-2} & r_2^{n-2} & \cdots & r_n^{n-1} \end{vmatrix}$$

is non-zero if the numbers $r_1, r_2, \dots, r_n$ are distinct. Prove that the functions $f_i(x) = \exp(r_i x)$, $1 \leq i \leq n$ are L.I..

3.3 Constant Coefficient Homogeneous DEs

Consider a 2nd-order DE with constant coefficients $a \neq 0$, b and c

$$ay'' + by' + cy = 0 \tag{3.15}$$

Let us try out a solution. Without proper preparation, we can assume the solution to DE (3.15) is $\exp(rx)$. So, we can try a trial solution

$$y_t(x) = \exp(rx)$$

$$y_t' = r\exp(rx), \qquad y_t'' = r^2\exp(rx)$$

Plugging these to (3.15), we have

$$ar^2\exp(rx) + br\exp(rx) + c\exp(rx) = 0$$

That is

$$(ar^2 + br + c)\exp(rx) = 0$$

an algebraic equation. Since $\exp(rx) > 0$, to enable the above equation, we must have

$$ar^2 + br + c = 0 \tag{3.16}$$

This is called the characteristic equation (C-Eq) of the original DE (3.15). Solving this C-Eq, we can get two solutions.

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

So through "back substitution", we get the solutions of the DE.

$$y_1 = \exp(r_1 x) \text{ and } y_2 = \exp(r_2 x)$$

Since $a \neq 0$, there are three possible cases for r_1 and r_2 depending on the value of $b^2 - 4ac$. They are:

- Case (1) $b^2 - 4ac > 0$
- Case (2) $b^2 - 4ac = 0$
- Case (3) $b^2 - 4ac < 0$

CASE 1: $(b^2 - 4ac > 0)$

The C-Eq (3.16) has two distinct real roots $r_1 \neq r_2$. The two trial solutions are

$$y_1 = \exp(r_1 x) \text{ and } y_2 = \exp(r_2 x)$$

From last section, we know that $\exp(r_1 x)$ and $\exp(r_2 x)$ are L.I. and thus the G.S. for the DE is

$$y = C_1 \exp(r_1 x) + C_2 \exp(r_2 x) \tag{3.17}$$

CHECK

Since r_1 and r_2 are the solutions of the algebraic equation (3.16), we have

$$\begin{cases} ar_1^2 + br_1 + c = 0 \\ ar_2^2 + br_2 + c = 0 \end{cases} \tag{3.18}$$

Plugging the two trial solutions $\exp(r_1 x)$ and $\exp(r_2 x)$ into the DE (3.16), we have

$$\begin{aligned} &a(\exp(r_1 x))'' + b(\exp(r_1 x))' + c\exp(r_1 x) \\ &= ar_1^2 \exp(r_1 x) + br_1 \exp(r_1 x) + c\exp(r_1 x) \\ &= (ar_1^2 + br_1 + c)\exp(r_1 x) \end{aligned}$$

From (3.18), we know that the above formula gives 0. Hence $\exp(r_1 x)$ is a solution of the DE and similarly, we can prove it for $\exp(r_2 x)$.

Example

Solve the DE.

$$y'' - 5y' + 6y = 0$$

Solution

STEP 1: Introduce the trial solution $y = \exp(rx)$

STEP 2: Plugging the trial solution into the DE, we have

$$y' = r\exp(rx) \quad \text{and} \quad y'' = r^2\exp(rx)$$
$$r^2\exp(rx) - 5r\exp(rx) + 6\exp(rx) = 0$$
$$(r^2 - 5r + 6)\exp(rx) = 0$$

We get the C-Eq

$$r^2 - 5r + 6 = 0$$
$$(r-2)(r-3) = 0$$
$$r_1 = 2 \ \text{ and } \ r_2 = 3$$

STEP 3: Because $r_1 \neq r_2$ and both are real, we conclude it is case 1

STEP 4: Compose the G.S.

$$y = C_1\exp(2x) + C_2\exp(3x)$$

STEP 5: Check

$$y = C_1\exp(2x) + C_2\exp(3x)$$
$$y' = 2C_1\exp(2x) + 3C_2\exp(3x)$$
$$y'' = 4C_1\exp(2x) + 9C_2\exp(3x)$$

$$\begin{aligned} & y'' - 5y' + 6y \\ &= 4C_1\exp(2x) + 9C_2\exp(3x) - 5(2C_1\exp(2x) + 3C_2\exp(3x)) \\ &\qquad + 6(C_1\exp(2x) + C_2\exp(3x)) \\ &= C_1(4 - 10 + 6)\exp(2x) + C_2(9 - 15 + 6)\exp(3x) \\ &= 0 \end{aligned}$$

Thus $y = C_1\exp(2x) + C_2\exp(3x)$ is the G.S..

CASE 2: $(b^2 - 4ac = 0)$

The C-Eq (3.16) has two equal real roots

$$r_1 = r_2 = r = -\frac{b}{2a}$$

It is clear that $\exp(r_1x) = \exp(r_2x)$ and they are not L.I. In order to get the G.S., let us now try a new trial solution.

$$y_2 = x\exp(rx)$$

We have

$$y_2' = rx\exp(rx) + \exp(rx)$$
$$y_2'' = 2r\exp(rx) + r^2x\exp(rx)$$

Substituting these back to the given DE we will have

$$\begin{aligned} &a(r^2 x \exp(rx) + 2r\exp(rx)) + \\ &\quad b(rx\exp(rx) + \exp(rx)) + cx\exp(rx) \\ &= (ar^2 + br + c)x\exp(rx) + (2ar + b)\exp(rx) \end{aligned}$$

From discussion of case 1, we know that the first term is 0. Noticing that $r = -b/2a$, it is clear the second term of the above formula is also 0. This means that y_2 is a solution to the original DE. Hence, we now have

$$y_1 = \exp(rx)$$

$$y_2 = x\exp(rx)$$

And the G.S. is

$$y = C_1\exp(rx) + C_2 x\exp(rx) \tag{3.19}$$

Sometimes, we also write it as form

$$y = (C_1 + C_2 x)\exp(rx) \tag{3.20}$$

Example 1

Solve the following DE

$$y'' - 2y' + y = 0$$

Solution

Introduce the trial solution $y = \exp(rx)$

Plugging the trial solution into the DE, we have

$$y' = r\exp(rx) \text{ and } y'' = r^2\exp(rx)$$
$$r^2\exp(rx) - 2r\exp(rx) + \exp(rx) = 0$$
$$(r^2 - 2r + 1)\exp(rx) = 0$$

Hence we get the C-Eq

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0$$
$$r_1 = r_2 = 1$$
$$y_1 = \exp(x)$$

and we have

$$y_2 = x\exp(x)$$

So we have the G.S.

$$y = C_1\exp(x) + C_2 x\exp(x)$$

Example 2

Suppose the C-Eq of the given DE is

$$(r - r_1)^3 = 0$$

Find the G.S. of the DE.

Solution

$(r - r_1)^3 = 0$ means that the C-Eq has three identical roots, thus, we have the following

$$y_1 = \exp(r_1 x) \quad y_2 = x\exp(r_1 x) \quad y_3 = x^2\exp(r_1 x)$$

and the G.S. is

$$y = (C_1 + C_2 x + C_3 x^2)\exp(r_1 x)$$

CASE 3: $(b^2 - 4ac < 0)$

The C-Eq (3.16) has two complex roots

$$r_1 = \alpha + i\beta$$

$$r_2 = \alpha - i\beta$$

where

$$\alpha = -\frac{b}{2a} \text{ and } \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

are both real number. Now we can form our trail solution

$$y_1 = \exp\big((\alpha + i\beta)x\big) \text{ and } y_2 = \exp\big((\alpha - i\beta)x\big)$$

It is easy to check that for $\beta \neq 0$, y_1 and $y_2$are L.I. (You do not need to bother with Wronskian since there are only two functions here).

$$\begin{aligned}\frac{y_1}{y_2} &= \frac{\exp\big((\alpha + i\beta)x\big)}{\exp\big((\alpha - i\beta)x\big)} \\ &= \exp(2i\beta x) \neq \text{constant}\end{aligned}$$

So now the G.S. is

$$y = C_1 \exp\big((\alpha + i\beta)x\big) + C_2 \exp\big((\alpha - i\beta)x\big) \tag{3.21}$$

Since the DE is raised in real domain, we also want to write the solutions with the real coefficients. The trick here is to manipulate the constants in (3.21). We can rewrite (3.21) as

$$y = \exp(\alpha x)\,(C_1 \exp(i\beta x) + C_2 \exp(-i\beta x))$$

From the most well-known Euler's formula

$$\exp(\pm i\beta x) = \cos\beta x \pm i \sin\beta x$$

we have

$$\begin{aligned} y &= \exp(\alpha x)\,\big(C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)\big) \\ &= \exp(\alpha x)\,\big((C_1 + C_2)\cos\beta x + (iC_1 - iC_2)\sin\beta x\big) \end{aligned}$$

Let

$$c_1 = C_1 + C_2 \quad \text{and} \quad c_2 = iC_1 - iC_2$$

We have the G.S. in the real form

$$y = \exp(\alpha x)\,(c_1 \cos\beta x + c_2 \sin\beta x) \tag{3.22}$$

Example 1

Solve the IVP

$$\begin{cases} y'' - 4y' + 5y = 0 \\ y(0) = 1, \;\; y'(0) = 5 \end{cases}$$

Solution

After seeing many examples, now we are able to find the C-Eq directly from the DE.

$$r^2 - 4r + 5 = 0$$
$$(r-2)^2 + 1 = 0$$

This C-Eq has two roots $2 + i$ and $2 - i$. Hence we have the G.S.

$$y(x) = \exp(2x)\,(c_1 \cos x + c_2 \sin x)$$

To find the P.S., we plug the I.C. into it. First, we have

$$y(0) = c_1 = 1$$

For y', we have

$$y'(x) = 2\exp(2x)\,(c_1 \cos x + c_2 \sin x) + \exp(2x)\,(-c_1 \sin x + c_2 \cos x)$$

and

$$y'(0) = 2c_1 + c_2 = 5$$

Plugging $c_1 = 1$ into above equation, we can get

$$c_2 = 3$$

So the solution to the IVP is

$$y(x) = \exp(2x)\,(\cos x + 3\sin x)$$

Example 2

Find the G.S. of the following DE.

$$(D^2 + 6D + 13)^2 y = 0$$

Where D is the differential operator

$$D = \frac{d}{dx}$$

Solution

The C-Eq of the given DE is

$$(r^2 + 6r + 13)^2 = 0$$

That is

$$((r+3)^3 + 4)^2 = 0$$

and gives

$$x_1 = x_2 = -3 + 2i \text{ and } x_3 = x_4 = -3 - 2i$$

Hence the G.S. of the given DE is

$$y(x) = \exp(-3x)\,(c_1 \cos 2x + d_1 \sin 2x) + x\exp(-3x)\,(c_2 \cos 2x + d_2 \sin 2x)$$

SUMMARY

For homogeneous 2nd-order constant coefficient DEs

$$ay'' + b'y + cy = 0$$

We have trial solution

$$y(x) = \exp(rx)$$

and the C-Eq

$$ar^2 + br + c = 0$$

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$\Delta = \boldsymbol{b^2} \boldsymbol{-4ac}$	$\boldsymbol{r_1, r_2}$	**L.I. Solution**	**G.S.**
$\Delta > 0$	$r_1 \neq r_2$ Real roots	$\exp(r_1 x)$, $\exp(r_1 x)$	$C_1 \exp(r_1 x) + C_2 \exp(r_2 x)$
$\Delta = 0$	$r_1 = r_2 = r$ Real root	$\exp(rx)$, $x \exp(rx)$	$(C_1 + C_2 x) \exp(rx)$
$\Delta < 0$	$r_1 = \alpha + i\beta$ $r_2 = \alpha - i\beta$ Complex roots	$\exp\big((\alpha + i\beta)x\big)$ $\exp\big((\alpha - i\beta)x\big)$	$C_1 \exp\big((\alpha + i\beta)x\big)$ $+C_2 \exp\big((\alpha - i\beta)x\big)$ $= \exp(\alpha x)\,(C_1 \cos\beta x + C_2 \sin\beta x)$

Theorem 4

If a homogeneous constant coefficient DE has n repeated and identical roots $r_1 = r_2 = \cdots = r_n = r$, then there are n L.I. solutions

$$\exp(rx)\,, x\exp(rx)\,, \ldots, x^{n-1}\exp(rx) \tag{3.23}$$

The G.S. for such DE is

$$y = C_1 \exp(rx) + C_2 x \exp(rx) + \cdots + C_n x^{n-1} \exp(rx) \tag{3.24}$$

Or equivalently,

$$y = (C_1 + C_2 x + \cdots + C_n x^{n-1}) \exp(rx) \tag{3.25}$$

Proof

Suppose we have DE in the form of

$$(D - r_1)^n y = 0 \tag{3.26}$$

where D is the differential operator

$$D \equiv \frac{d}{dx}$$

We prove that the functions in (3.23) are the solutions to DE (3.26) by mathematical induction.

First, we have that for $y_1 = \exp(r_1 x)$

$$\begin{aligned}(D - r_1)y_1 &= Dy_1 - r_1 y_1 \\ &= \frac{d}{dx}\exp(r_1 x) - r_1 \exp(r_1 x) \\ &= r_1 \exp(r_1 x) - r_1 \exp(r_1 x) \\ &= 0\end{aligned}$$

This means y_1 is a solution of $(D - r_1)y = 0$, and hence a solution of $(D - r_1)^n y = 0$.

Now supposed for $k < n$, $y_k = x^{k-1}\exp(r_1 x)$ is a solution of

$$(D - r_1)^k y = 0$$

and hence is a solution of (3.26). For $k + 1$, we prove

$$y_{k+1} = x^k \exp(r_1 x)$$

is also a solution.

$$\begin{aligned}&(D - r_1)^{k+1} y_{k+1} \\ &\quad = (D - r_1)^k (Dy_{k+1} - r_1 y_{k+1}) \\ &\quad = (D - r_1)^k \big(x^k r_1 \exp(r_1 x) + kx^{k-1}\exp(r_1 x) - r_1 x^k \exp(r_1 x)\big) \\ &\quad = (D - r_1)^k kx^{k-1}\exp(r_1 x) \\ &\quad = k(D - r_1)^k y_k \\ &\quad = 0\end{aligned}$$

This means y_{k+1} is a solution of

$$(D - r_1)^{k+1} y = 0$$

To prove that the functions in (3.23) are L.I., we can use the conclusion of Problem 3.3.17 and directly apply the definition of L.I. on them. The detail of this part of proof is left as a homework problem for you.

Example 1

Find the G.S. of the following DE.

$$y''' + y'' + y' + y = 0$$

Solution

The DE can be written in terms of differential operator as

$$(D^3 + D^2 + D + 1)y = 0$$

Thus, the C-Eq is

$$r^3 + r^2 + r + 1 = 0$$
$$r^2(r+1) + (r+1) = 0$$
$$(r^2+1)(r+1) = 0$$
$$r_1 = i, r_2 = -i, r_3 = -1$$

Therefore,

$$\exp(-x), \exp(ix) \text{ and } \exp(-ix)$$

are solutions to the original DE and they are L.I. Thus, the G.S. is

$$y(x) = C_1 \exp(-x) + C_2 \cos x + C_3 \sin x$$

Example 2

Find the G.S. of the following DE.

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

Solution

The DE can be written in terms of differential operators

$$(9D^5 - 6D^4 + D^3)y = 0$$

So the C-Eq is

$$9r^5 - 6r^4 + r^3 = 0$$
$$r^3(9r^2 - 6r + 1) = 0$$
$$r^3(3r-1)^2 = 0$$
$$r_1 = r_2 = r_3 = 0, r_4 = r_5 = \frac{1}{3}$$
$$y(x) = C_1 + C_2 x + C_3 x^2 + (C_4 + C_5 x)\exp\left(\frac{1}{3}x\right)$$

Example 3

Find the G.S. for the following DE.

$$(D^2 - 5D + 6)(D - 1)^2(D^2 + 1)y = 0$$

Where

$$D = \frac{d}{dx}$$

is the differential operator.

Solution

The C-Eq is

$$(r^2 - 5r + 6)(r - 1)^2(r^2 + 1) = 0$$
$$(r - 2)(r - 3)(r - 1)^2(r^2 + 1) = 0$$
$$r_1 = 2, r_2 = 3, r_3 = r_4 = 1, r_5 = i, r_6 = -i$$

Hence the G.S. is

$$y(x) = C_1 \exp(2x) + C_2 \exp(3x) + (C_3 + C_4 x) \exp(x) + C_5 \cos x + C_6 \sin x$$

Problems

Problem 3.3.1 Use the quadratic formula to solve the following equations (Note in each case that the roots are not complex).

(1) $x^2 + ix + 2 = 0$

(2) $x^2 - 2ix + 3 = 0$

Problem 3.3.2 (a) Use Euler's formula to show that every complex number can be written in the form $r\exp(i\theta)$, where $r \geq 0$ and $-\pi < \theta \leq \pi$.

(b) Express the numbers 4, -2, $3i$, $1 + i$ and $-1 + i\sqrt{3}$ in the form of $r\exp(i\theta)$.

(c) The two square roots of $r\exp(i\theta)$ are $\pm\sqrt{r}\exp\left(\frac{i\theta}{2}\right)$. Find the square roots of the numbers $2 - 2i\sqrt{3}$ and $-2 + 2i\sqrt{3}$.

Problem 3.3.3 Compute the Wronskian of the three L.I. solutions (x_1, x_2, x_3) of the following 3rd order DE.

$$t^3 x''' + 6t^2 x'' + 7tx' + x = 0$$

Problem 3.3.4 Find the highest point on the solution curve in

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 6 \end{cases}$$

Problem 3.3.5 Find the third quadrant point of intersection of the solution curves with different I.C..

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y_1(0) = 3, y_1'(0) = 1 \\ y_2(0) = 0, y_2'(0) = 1 \end{cases}$$

Problem 3.3.6 Solve the IVP.

$$\begin{cases} 3y''' + 2y'' = 0 \\ y(0) = -1 \\ y'(0) = 0 \\ y''(0) = 1 \end{cases}$$

Problem 3.3.7 Find a linear homogeneous constant-coefficient DE with the given G.S..

$$y(x) = (A + Bx + Cx^2)\cos 2x + (D + Ex + Fx^2)\sin 2x$$

Problem 3.3.8 Solve the IVP

$$\begin{cases} y^{(3)} = y \\ y(0) = 1 \\ y'(0) = y''(0) = 0 \end{cases}$$

Problem 3.3.9 Solve the IVP

$$\begin{cases} y^{(4)} = y''' + y'' + y' + 2y \\ y(0) = y'(0) = y''(0) = 0 \\ y'''(0) = 30 \end{cases}$$

Problem 3.3.10 The DE

$$y'' + (\sin x)y = 0$$

has the discontinuous coefficient function

$$\sin x = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Show that this DE nevertheless has two L.I. solutions $y_1(x)$ and $y_2(x)$ defined for all x such that

- Each satisfies the DE at each point $x \neq 0$
- Each has a continuous derivative at $x = 0$
- $y_1(0) = y_2'(0) = 1$ and $y_0(0) = y_2'(0) = 0$

Problem 3.3.11 Find the G.S. of the following DE.

$$D^3(D-2)(D+3)(D^2+1)y = 0$$

where $D = d/dx$

Problem 3.3.12 Solve the following IVP.

$$\begin{cases} y'' - 2y' + 2y = 0 \\ y(0) = 0, y'(0) = 5 \end{cases}$$

Problem 3.3.13 Solve the following IVP.

$$\begin{cases} y''' + 9y' = 0 \\ y(0) = 3, y'(0) = -1, y''(0) = 2 \end{cases}$$

Problem 3.3.14 Find the G.S. of the following DE.

$$(D-1)^3(D-2)^2(D-3)(D^2+9)y(x) = 0$$

where $D = d/dx$.

Problem 3.3.15 Find the G.S. of the following DE.

$$y'' - y' - 15y = 0$$

Problem 3.3.16 Find the G.S. of the following DE.

$$9y'' - 12y' + 4y = 0$$

Problem 3.3.17 For a homogeneous equation with repeated real roots, please do the following:

(1) For homogeneous equation $(D - r_1)^{k_1} y(x) = 0$ where $D = \frac{d}{dx}$ is the usual derivative operator, r_1 is a real constant and k_1 is a positive integer. Find the G.S. for the equation.

(2) Compose the G.S. of the following homogeneous equation

$$(D - r_1)^{k_1}(D - r_2)^{k_2} \cdots (D - r_n)^{k_n} = 0$$

where $r_1, r_2, \dots, r_n$ are known distinct real constants while $k_1, k_2, \dots, k_n$ are known positive integers. You may use existing theorems without deriving them, but indicate which one(s) you use.

Problem 3.3.18 Find the G.S. of the following DE:

$$\left(x\frac{d}{dx} - \alpha\right)^n y(x) = x$$

where $x > 0, \alpha =$ constant and $n =$ positive integer.

3.4 Cauchy-Euler DEs

Linear DEs of higher orders with variable coefficients are usually not solvable analytically even though they are linear. But one class of DEs of special coefficients can be solved and this class is called Cauchy-Euler DEs. The *N*-th order Cauchy-Euler DEs can be expressed as

$$\sum_{n=0}^{N} a_n x^n y^{(n)} = 0$$

A special case is the 2nd-order Cauchy-Euler DEs:

$$\sum_{n=0}^{2} a_n x^n y^{(n)} = a_0 y + a_1 x y' + a_2 x^2 y'' = 0$$

We may use this special case to demonstrate the solution methods and, in fact, higher order can be solved in the similar fashion. The real *geist* of the solution methods is to transform the coefficients of the DEs from variable to constant.

<u>Method I:</u> Assuming trial solution

$$y(x) = x^{\lambda}$$

Thus,

$$y'(x) = \lambda x^{\lambda-1}$$

$$y''(x) = \lambda(\lambda-1)x^{\lambda-2}$$

Then the original DE becomes

$$a_0 x^{\lambda} + a_1 \lambda x^{\lambda} + a_2 \lambda(\lambda-1)x^{\lambda} = 0$$

Or

$$\big(a_0 + a_1\lambda + a_2\lambda(\lambda-1)\big)x^{\lambda} = 0$$

Since $x^{\lambda} \neq 0$, we must have

$$a_0 + a_1\lambda + a_2\lambda(\lambda - 1) = 0$$

the C-Eq and we can adopt all ideas in solving constant coefficient DEs.

Method II: Substituting to replace independent variable

$$x = \exp(t)$$

With this, we can get

$$y' = \frac{dy}{dx} = \frac{dt}{dt}\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt}$$

$$xy' = \frac{dy}{dt} \equiv \dot{y}$$

Similarly,

$$y'' = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\frac{dy}{dt} = -\frac{1}{x^2}\dot{y} + \frac{1}{x}\frac{1}{x}\frac{d}{dt}\frac{dy}{dt}$$
$$= -\frac{1}{x^2}\dot{y} + \frac{1}{x^2}\ddot{y}$$

Thus,

$$x^2y'' = -\dot{y} + \ddot{y}$$

This formula can be further generalized.

With the above, we transform our original DE to

$$a_0y + a_1\dot{y} + a_2(-\dot{y} + \ddot{y}) = 0$$

Or

$$a_0y + (a_1 - a_2)\dot{y} + a_2\ddot{y} = 0$$

a constant coefficient DE and all methods discussed before can be adopted.

Example:

Find the G.S. of the following DE

$$x^2y'' + xy' - y = 0$$

Solution

Method I: Assuming trial solution

$$y(x) = x^\lambda$$

Thus,

$$y'(x) = \lambda x^{\lambda-1}$$
$$y''(x) = \lambda(\lambda-1)x^{\lambda-2}$$

Then the original DE becomes

$$\lambda(\lambda-1) + \lambda - 1 = 0$$

whose roots are

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

The G.S. is

$$y(x) = c_1x^1 + c_2x^{-1}$$

Method II: Substituting to replace independent variable

$$x = \exp(t)$$

We generate a new DE:

$$y'' - y = 0$$

whose G.S. is

$$y(t) = c_1 \exp(t) + c_2 \exp(-t)$$

Back substitute leads to

$$y(x) = c_1x^1 + c_2x^{-1}$$

Problems

Problem 3.4.1 Find the G.S. of the following DE.

$$(ax^2D + bxD + c)y = 0$$

where $D = d/dx$, and a, b and c are constants.

Problem 3.4.2 Find the G.S. of the following DE.

$$x^2y'' + xy' - 9y = 0$$

Problem 3.4.3 Find the G.S. of the following DE.

$$x^2y'' + bxy' + cy = 0$$

Problem 3.4.4 Solve the following IVP.

$$\begin{cases} x^2y'' - 2xy' + 2y = 0 \\ y(1) = 3, \quad y'(1) = 1 \end{cases}$$

Problem 3.4.5 Find the G.S. of the following DE.

$$(x+2)^2y'' - (x+2)y' + y = 0$$

Problem 3.4.6 Find the G.S. of the following DE.

$$x^2y'' - 2xy' - 10y = 0$$

Problem 3.4.7 Find the G.S. of the following DE.

$$x^3y''' + x^2y'' - xy' + y = 0$$

where $x > 0$.

3.5 Inhomogeneous Higher Order DEs

Theorem 1

If y_c is a G.S. to $Ly = 0$ and y_p is a P.S. to $Ly = f(x)$, then $y = y_c + y_p$ is the G.S. to the inhomogeneous DE $Ly = f(x)$.

Proof

Since y_c is a G.S. to $Ly = 0$, we have

$$Ly_c = 0$$

and y_p is a P.S. to $Ly = f(x)$ gives

$$Ly_p = f(x)$$

Then

$$\begin{aligned} L(y_c + y_p) &= Ly_c + Ly_p \\ &= 0 + f(x) \\ &= f(x) \end{aligned}$$

Therefore $y_c + y_p$ is a G.S. to $Ly = f(x)$.

Now let us discuss the way to find the P.S..

Suppose that RHS $f(x) = a\cos kx + b\sin kx$, then it is reasonable to expect a P.S. of the same form:

$$y_p = A\cos kx + B\sin kx$$

which is a linear combination with undetermined coefficients A and B. The reason is that any derivative of such a linear combination of $\cos kx$ and $\sin kx$ has the same form. Therefore, we may substitute this form of y_p in the given DE to determine the coefficients A and B by equating coefficients of $\cos kx$ and $\sin kx$ on both sides. The technique of finding a P.S. for a inhomogeneous DE is rather tricky and tedious; one must remember several rules.

Let us consider a few examples.

Example 1
Find the P.S. of
$$y'' + 3y' + 4y = 3x + 2$$

Solution
Since $f(x) = 3x + 2$, let us guess that
$$y_p = Ax + B$$
Therefore
$$y_p' = A \text{ and } y_p'' = 0$$
Substituting these in the given DE, we have
$$y_p'' + 3y_p' + 4y_p = 3x + 2$$
$$0 + 3A + 4(Ax + B) = 3x + 2$$
$$A = \frac{3}{4} \text{ and } B = -\frac{1}{16}$$
Therefore, we have a P.S.
$$y_p = \frac{3}{4}x - \frac{1}{16}$$

Example 2
Find the P.S. of
$$y'' - 4y = 2\exp(3x)$$

Solution
Since any derivative of $\exp(3x)$ is a constant multiple of $\exp(3x)$, let us try
$$y_p = A\exp(3x)$$
Therefore we now have
$$y_p'' = 9A\exp(3x)$$
Substituting it in the given DE, we have
$$9A\exp(3x) - 4A\exp(3x) = 2\exp(3x)$$
$$A = \frac{2}{5}$$
Therefore we find a P.S. of the given DE.
$$y_p = \frac{2}{5}\exp(3x)$$

Example 3

Find the P.S. of

$$3y'' + y' - 2y = 2\cos x$$

Solution

Though our first guess might be $y_p = A\cos x$, the presence of y' on the LHS shows us that we will have to include a term of $\sin x$ as well. So let our trial solution be

$$y_p = A\cos x + B\sin x$$

Therefore

$$y_p' = -A\sin x + B\cos x \quad \text{and} \quad y_p'' = -A\cos x - B\sin x$$

Substituting these into the given DE, we have

$$3(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x)$$
$$= 2\cos x$$

Equating the terms of $\cos x$ and $\sin x$, we have

$$\begin{cases} -5A + B = 2 \\ -A - 5B = 0 \end{cases}$$

Solving the above two simultaneous equations we have

$$A = -\frac{5}{13}$$
$$B = \frac{1}{13}$$

Hence the P.S. is

$$y_p = -\frac{5}{13}\cos x + \frac{1}{13}\sin x$$

Next, let us introduce several rules for popular cases.

Rule 1

If $f(x), f'(x)$ and $f''(x)$ do not satisfy $Ly = 0$ (i.e., $f(x)$ and its derivatives are not solution of $Ly = 0$), then the trial P.S. y_p can be linear combination of all terms in $f(x)$.

Meaning of the Rule

Suppose that no term appearing in either $f(x)$ or any of its derivatives satisfies the associated homogeneous DE $Ly = 0$. Then take as a trial solution for y_p a linear combination of all such L.I. terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the inhomogeneous DE $Ly = f(x)$.

Necessary Conditions

We check the supposition made in Rule 1 by first using the C-Eq to find the complementary function y_c, and then write a list of all terms appearing in $f(x)$ and its successive derivatives. If none of the terms in this list duplicates a term in y_c, then we proceed with Rule 1.

Example

Try to find the G.S. of the following DE using undetermined coefficient method

$$y'' - 3y' - 4y = 15\exp(4x)$$

Solution

First let us find the G.S. to the homogeneous DE.

$$y'' - 3y' - 4y = 0$$

The C-Eq is

$$r^2 - 3r - 4 = 0$$

$$r_1 = 4, \qquad r_2 = -1$$

Therefore

$$y_c = C_1\exp(4x) + C_2\exp(-x)$$

Now let us try $y_p = A\exp(4x)$ as the P.S. to the inhomogeneous DE. Note that here we have $f(x) = 15\exp(4x)$

$$y_p' = 4A\exp(4x) \text{ and } y_p'' = 16A\exp(4x)$$

Plugging this back to the DE, we have

$$\begin{aligned} \text{LHS} &= y_p'' - 3y_p' - 4y \\ &= 16A\exp(4x) - 3(4A\exp(4x)) - 4(A\exp(4x)) \\ &= (16 - 12 - 4)A\exp(4x) \\ &= 0 \end{aligned}$$

While

$$\text{RHS} = 15\exp(4x)$$

Hence LHS $\neq$ RHS $\forall A$.

No matter what value of A we choose we will never be able to find the solution. So after reading the Rule mentioned above and the fact that $\exp(4x)$ term in y_c ($y_c = C_1 \exp(4x) + C_2 \exp(-x)$) also appears in $f(x)$ ($15 \exp(4x)$), we cannot use the term $A \exp(4x)$ as the trial P.S., as the P.S. should be L.I. with all terms present in $f(x)$ (that is the P.S. should not be a constant multiple of the terms present in $f(x)$). So now the question arises of how to solve the given DE. Consider Rule 1A.

Rule 1A

If $f(x)$ is a solution to $Ly = 0$, then one way of composing a trial solution is to multiply the solution by x.

Example (continued)

Now we introduce a new trial solution to the DE $y'' - 3y' - 4y = 15 \exp(4x)$:

$$y_p = Ax \exp(4x)$$

Hence

$$y_p' = A \exp(4x) + 4Ax \exp(4x) \quad \text{and} \quad y_p'' = 8A \exp(4x) + 16Ax \exp(4x)$$

Plugging these back to the DE, we have

$$\begin{aligned}
\text{LHS} &= y_p'' - 3y_p' - 4y_p \\
&= 8A \exp(4x) + 16Ax \exp(4x) - 3(A \exp(4x) + 4Ax \exp(4x)) \\
&\qquad - 4Ax \exp(4x) \\
&= 5A \exp(4x) \\
&= \text{RHS} = 15 \exp(4x)
\end{aligned}$$

This gives

$$A = 3$$

Therefore we have a P.S.

$$y_p = 3x \exp(4x)$$

and the G.S. is

$$y = C_1 \exp(4x) + C_2 \exp(-x) + 3x \exp(4x)$$

Example 2

Find the P.S. of

$$y'' + 4y = 3x^3$$

Solution

We can find the solution to the homogeneous part of the above DE as

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

The function $f(x) = 3x^3$ and its derivatives are constant multiples of the L.I. functions x^3, x^2, x and 1. Because none of these appears in y_c, we try

$$y_p = Ax^3 + Bx^2 + Cx + D$$

Therefore

$$y_p' = 3Ax^2 + 2Bx + C \text{ and } y_p'' = 6Ax + 2B$$

Substituting the above in the DE, we have

$$(6Ax + 2B) + 4(Ax^3 + Bx^2 + Cx + D) = 3x^3$$

Equating the coefficients of the like terms, we have

$$\begin{cases} 4A = 3 \\ 4B = 0 \\ 6A + 4C = 0 \\ 2B + 4D = 0 \end{cases}$$

These give

$$A = \frac{3}{4}, \qquad B = 0, \qquad C = -\frac{9}{8}, \qquad D = 0$$

Therefore, we have the P.S.

$$y_p = \frac{3}{4}x^3 - \frac{9}{8}x$$

Example 3

Find the general form of a P.S. for

$$y''' + 9y' = x \sin x + x^2 \exp(2x)$$

Solution

The C-Eq is

$$r^3 + 9r = 0$$

$$r_1 = 0, r_2 = 3i, r_3 = -3i$$

So the G.S. to the homogeneous DE is

$$y_c = C_1 + C_2 \cos 3x + C_3 \sin 3x$$

The derivatives of the inhomogeneous RHS involves the terms

$$\cos x \,, \sin x \,, x \cos x \,, x \sin x$$

and

$$\exp(2x) \,, x \exp(2x) \,, x^2 \exp(2x)$$

Since there is no duplication with the terms of the G.S., the trial solution takes the form

$$y_p = A\cos x + B\sin x + Cx\cos x + Dx\sin x + E\exp(2x) + Fx\exp(2x) + Gx^2\exp(2x)$$

Upon substituting y_p in the DE and equating the coefficient terms we get seven equations determining the seven coefficients A, B, C, D, E, F and G

Rule 2 (The case of duplication)

If $f(x)$ contains terms such as $P_m(x)\exp(rx)\cos kx$ or $P_m(x)\exp(rx)\sin kx$, then we can select trial solution

$$y_p(x) = x^s\big((A_0 + A_{1x} + \cdots + A_m x^m)\exp(rx)\cos kx + (B_0 + B_1 x + \cdots + B_m x^m)\exp(rx)\sin kx\big)$$

where P_m is a polynomial of order m and s is the smallest non-negative integer which does not allow duplication of the above solution as the solution of the homogeneous DE. Then determine the coefficients in y_p by substituting y_p into the inhomogeneous DE.

$\boldsymbol{f(x)}$	$\boldsymbol{y_p}$
$P_m(x) = b_0 + b_1 x + \cdots + b_m x^m$	$x^s(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$\exp(rx)$	$\exp(rx)$
$\exp(rx)\,(a\cos kx + b\sin kx)$	$\exp(rx)\,(A\cos kx + B\sin kx)$
$P_m(x)\exp(rx)$	$\exp(rx)\,(A_0 + A_1 x + \cdots + A_m x^m)$
$P_m(x)(a\cos kx + b\sin kx)$	$(A\cos kx + B\sin kx)(A_0 + A_1 x + \cdots + A_m x^m)$

The above table lists the functions that can be chosen as the trial solutions for a given RHS.

Example 1

Find the particular and G.S. for

$$y''' + y'' = 3\exp(x) + 4x^2$$

Solution

We have the C-Eq

$$r^3 + r^2 = 0$$
$$r_1 = r_2 = 0, r_3 = -1$$

The complementary solution is

$$y_c(x) = C_1 + C_2 x + C_3 \exp(-x)$$

We now try to form our P.S. as

$$Ae^x + B + Cx + Dx^2$$

$f(x)$	y_p
$3\exp(x)$	$\exp(x)$
$4x^2$	$x^s(B + Cx + Dx^2)$

Since we have C_1 and $C_2 x$ in complementary solution, we will have duplication in $B + Cx + Dx^2$ terms in y_c. Therefore $s \neq 0$. If we chose $s = 1$, $Bx + Cx^2 + Dx^3$ will still duplicate the x term. Now let $s = 2$, $Bx^2 + Cx^3 + Dx^4$ will not duplicate any terms in y_c. Therefore, we have $s = 2$. The part $A\exp(x)$ corresponding to $3\exp(x)$ does not duplicate any part of the complementary function, but the part $B + Cx + Dx^2$ must be multiplied by x^2 to eliminate duplication. Hence we have

$$y_p = A\exp(x) + Bx^2 + Cx^3 + Dx^4$$
$$y_p' = A\exp(x) + 2Bx + 3Cx^2 + 4Dx^3$$
$$y_p'' = A\exp(x) + 2B + 6Cx + 12Dx^2$$
$$y_p''' = A\exp(x) + 6C + 24Dx$$

Substituting these back to the given DE gives

$$A\exp(x) + 6C + 24Dx + A\exp(x) + 2B + 6Cx + 12Dx^2 = 3\exp(x) + 4x^2$$
$$2A\exp(x) + (2B + 6C) + (6C + 24D)x + 12Dx^2 = 3\exp(x) + 4x^2$$

Equating the like terms, we have

$$\begin{cases} 2A = 3 \\ 2B + 6C = 0 \\ 6C + 24D = 0 \\ 12D = 4 \end{cases}$$

Solving the above simultaneous equations gives us

$$A = \frac{3}{2}, \qquad B = 4, \qquad C = -\frac{4}{3}, \quad D = \frac{1}{3}$$

A P.S. is

$$y_p = \frac{3}{2}\exp(x) + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4$$

So the G.S. is

$$y = C_1 + C_2 x + C_3 \exp(-x) + \frac{3}{2}\exp(x) + 4x^2 - \frac{4}{3}x^3 + \frac{1}{3}x^4$$

Example 2

Determine the appropriate form for a complementary and P.S. if given the roots of the C-Eq and $f(x)$. The roots are $r = 2, 2, 2$ and $-2 \pm 3i$ and $f(x) = x^2 \exp(2x) + x \sin 3x$.

Solution

Since we are already given the roots of the C-Eq, let us compose the complementary function. So the complementary function is

$$y_c(x) = (C_1 + C_2 x + C_3 x^2)\exp(2x) + \exp(-2x)\,(C_4 \cos 3x + C_5 \sin 3x)$$

As a first step toward forming the P.S., we examine the sum

$$(A + Bx + Cx^2)\exp(2x) + \exp(-2x)\big((D + Ex)\cos 3x + (F + Gx)\sin 3x\big)$$

To eliminate duplication with terms of $y(x)$, the first part corresponding to $x^2 \exp(2x)$ must be multiplied by x^2, and the second part corresponding to $x \sin 3x$ must be multiplied by x. Hence we would take

$$\begin{aligned} y_p(x) &= (Ax^3 + Bx^4 + Cx^5)\exp(2x) \\ &\quad + \exp(-2x)\big((Dx + Ex^2)\cos 3x + (Fx + Gx^2)\sin 3x\big) \end{aligned}$$

Example 3

$$y''' + 9y' = x \sin x + x^2 \exp(2x)$$

Solution

First, let us find the G.S.. We have the C-Eq

$$r^3 + 9r = 0$$

$$r_1 = 0, \qquad r_2 = 3i, \qquad r_3 = -3i$$

The G.S. is

$$y_c = C_1 + C_2 \cos 3x + C_3 \sin 3x$$

Now let us find the P.S. from observing the RHS. We compose our trial.

$$\begin{aligned} y_p &= A_1 \sin x + A_2 \cos x + A_3 x \sin x + A_4 x \cos x + A_5 \exp(2x) \\ &\quad + A_6 x \exp(2x) + A_7 x^2 \exp(2x) \end{aligned}$$

Problems

Problem 3.5.1 Find the G.S of the following DE.

$$y'' - 2y' - 8y = \exp(4x)$$

Problem 3.5.2 Find the G.S. of the following DE.

$$y'''' - y = 1$$

Problem 3.5.3 Find the G.S. of the following DE.

$$y'''' - y = 4\exp(x)$$

Problem 3.5.4 Find the G.S. of the following DE.

$$y^{(4)} - y''' - y'' - y' - 2y = 18x^5$$

Problem 3.5.5 Find a P.S. of the following DE.

$$y''' + y'' + y' + y = x^3 + x^2$$

Problem 3.5.6 Find a P.S. of the following DE.

$$y''' + y'' + y' + y = 1 + \exp(x) + \exp(2x) + \exp(3x)$$

Problem 3.5.7 Find the G.S. of the following DE.

$$y^{(4)} + \omega^2 y'' = (x^2 + 1)\sin(\omega x)$$

Problem 3.5.8 Find the G.S. of the following DE with the following two conditions: $\omega \neq \omega_0$ and $\omega = \omega_0$.

$$y'' + \omega^2 y = \exp(i\omega_0 x)$$

Problem 3.5.9 Find the G.S. of the following DE.

$$y'' + 9y = \cos 3x$$

Problem 3.5.10 Find the G.S. of the following DE.

$$y'' + 2y' + y = 5(\sin x + \cos x)$$

Problem 3.5.11 Find a P.S. of the following DE.

$$y''' + y'' + y' + y = 1 + \cos x + \sin 2x + \exp(-x)$$

Problem 3.5.12 Find the G.S. of the following DE.

$$x^2y'' - 4xy' + 6y = x^3$$

Problem 3.5.13 Find the G.S. of the following DE.

$$x^6y'' + 2x^5y' - 12x^4y = 1$$

Problem 3.5.14 Find the G.S. of the following DE.

$$x^2y'' + 2xy' - 6y = 72x^5$$

Problem 3.5.15 Find the G.S. of the following DE.

$$4x^2y'' - 4xy' + 3y = 8x^{\frac{4}{3}}$$

Problem 3.5.16 Find the G.S. of the following DE.

$$x^2y'' + xy' + y = \ln x$$

Problem 3.5.17 Find the G.S. of the following DE.

$$x^3y''' + x^2y'' + xy' - y = 1 + x + x^2$$

Problem 3.5.18 Find the G.S. of the following DE.

$$x^2y'' + xy' - 4y = x^2 + x^{-2}$$

Problem 3.5.19 Given the three L.I. solutions of the homogeneous portion of $y^{(3)} + \alpha y^{(2)} + \beta y^{(1)} + \gamma y = f(x)$ as $y_1(x)$, $y_2(x)$, $y_3(x)$, find the G.S. for the inhomogeneous equation in terms of α, β, γ, and the given functions.

Problem 3.5.20 Please use any method of your choice to find a P.S. $x_p(t)$ for the following DE

$$\frac{d^2x}{dt^2} + \omega^2x(t) = f(t)$$

where ω is a constant and $f(t)$ is a given function for the following three situations. You may express the P.S. $x_p(t)$ in terms of the given $f(t)$ in some integral form if you do not have enough information to integrate it out.

(1) Function $f(t)$ is a general form;
(2) Function $f(t)$ is a specific function $f(t) = \exp(i\omega t)$;
(3) Function $f(t)$ is a specific function $f(t) = \exp(\omega t)$.

3.6 Variation of Parameters

Consider an example $y'' + y = \tan x$. Since $f(x) = \tan x$ has infinitely many L.I. derivatives $\sec^2 x$, $2\sec^2 x \tan x$, $4\sec^2 x \tan^2 x + 2\sec^4 x$, …, etc. Therefore, we do not have available a finite linear combination to use as a trial solution and we do not have corresponding rules to follow for solving DEs of this type. We have to introduce a new method, i.e., variation of parameters.

For $Ly = f(x)$, the homogeneous DE $Ly = 0$ has n solutions y_1, y_2, ..., y_n.

(1) The G.S. of homogeneous DE is $y_c = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n$

(2) Let us propose to change the C_1, C_2 ... C_n in the complementary function to variables $u_1(x), u_2(x)$ ... $u_n(x)$ for composing a trial P.S. for the inhomogeneous DE

$$\begin{aligned} y_p(x) &= u_1(x)y_1 + u_2(x)y_2 + \cdots + u_n(x)y_n \\ &= \sum_{i=1}^{n} u_i(x) y_i(x) \\ &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= u^T Y \end{aligned}$$

Example

Consider a 2nd-order inhomogeneous DE

$$y'' + P(x)y' + Q(x)y = f(x) \tag{3.27}$$

Compose a formula for the P.S. for the above DE.

Solution

1) Assume the solutions to the homogeneous DE are y_1 and y_2. This gives us

$$\begin{cases} y_1'' + P(x)y_1' + Q(x)y_1 = 0 \\ y_2'' + P(x)y_2' + Q(X)y_2 = 0 \end{cases} \tag{3.28}$$

2) Compose a trial P.S. y_p as

$$y_p = u_1(x)y_1 + u_2(x)y_2 \tag{3.29}$$

Therefore we have

$$\begin{aligned} y_p' &= (u_1y_1 + u_2y_2)' \\ &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ &= (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2') \end{aligned}$$

In the above process, we introduced two "free" parameters $u_1(x)$ and $u_2(x)$. With such, we can impose two constraints. Let us force the following condition (although one may impose many other conditions).

$$u_1'y_1 + u_2'y_2 = 0 \tag{3.30}$$

Therefore, we have

$$y_p' = u_1y_1' + u_2y_2' \tag{3.31}$$

$$\begin{aligned} y_p'' &= (u_1y_1' + u_2y_2')' \\ &= u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \end{aligned}$$

$$y_p'' = (u_1y_1'' + u_2y_2'') + (u_1'y_1' + u_2'y_2') \tag{3.32}$$

From (3.28) we have

$$y_1'' = -P(x)y_1' - Q(x)y_1$$

$$y_2'' = -P(x)y_2' - Q(x)y_2$$

Substitute these into $u_1y_1'' + u_2y_2''$

$$
\begin{aligned}
u_1 y_1'' + u_2 y_2''& \\
&= u_1(-P(x)y_1' - Q(x)y_1) \\
&+ u_2(-P(x)y_2' - Q(x)y_2) \\
&= -P(u_1y_1' + u_2y_2') - Q(u_1y_1 + u_2y_2) \\
&= -Py_p' - Qy_p
\end{aligned}
\tag{3.33}
$$

Finally, plugging back (3.33) into (3.32)

$$y_p'' = -Py_p' - Qy_p + u_1'y_1' + u_2'y_2'$$

$$y_p'' + Py_p' + Qy_p = u_1'y_1' + u_2'y_2' \tag{3.34}$$

But according to (3.27), we have

$$y_p'' + Py_p' + Qy_p = f(x)$$

So we have

$$u_1'y_1' + u_2'y_2' = f(x) \tag{3.35}$$

So from (3.30) and (3.35)

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

In theory, one may easily GENERALIZE the above formulation for higher-order DEs.

$$y^{(n)} + P_1 y^{(n-1)} + \cdots + P_n y = f(x)$$

$$y_c = c_1y_1 + \cdots + c_ny_n$$

$$y_p = u_1y_1 + \cdots + u_ny_n$$

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f(x) \end{bmatrix}$$

Next, let us derive a formula for u_1' and u_2' and, ultimately, for y_p. From (3.30), we have

$$u_1' = -\frac{y_2}{y_1}u_2{}' \tag{3.36}$$

and plugging (3.36) into (3.35) gives

$$\begin{aligned} f(x) &= -y_1'\left(\frac{y_2}{y_1}u_2'\right) + y_2'u_2' \\ &= \frac{u_2'}{y_1}\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \frac{u_2'}{y_1}W(y_1, y_2) \end{aligned}$$

where $W(y_1, y_2)$ is the Wronskian. Thus we have

$$u_2' = \frac{y_1 f(x)}{W(y_1, y_2)}$$

$$u_2(x) = \int \frac{y_1(t)f(t)}{W\big(y_1(t), y_2(t)\big)}dt \tag{3.37}$$

Plugging u_2' back to (3.36), we have

$$u_1' = -\frac{y_2 f(x)}{W(y_1, y_2)}$$

and

$$u_1(x) = -\int \frac{y_2(t)f(t)}{W\big(y_1(t), y_2(t)\big)}dt \tag{3.38}$$

Plugging (3.37) and (3.38) back into (3.29), we find the trial solution

$$\begin{aligned} y_p(x) &= u_1 y_1 + u_2 y_2 \\ &= -y_1(x)\int \frac{y_2(t)f(t)}{W\big(y_1(t), y_2(t)\big)}dt + y_2(x)\int \frac{y_1(t)f(t)}{W\big(y_1(t), y_2(t)\big)}dt \end{aligned}$$

$$= \int \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W\big(y_1(t), y_2(t)\big)} f(t)dt$$

$$y_p(x) = \int K(x,t)f(t)dt$$

where

$$K(x,t) = \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W\big(y_1(t), y_2(t)\big)}$$

is called the *kernel* mathematically. The kernel is also called the *Green's function* in physics.

SUMMARY

Consider solving a 2nd-order DE with constant coefficients

$$y'' + ay' + by = f(x)$$

STEP 1: Find the C-Eq for the homogeneous portion

$$Ly = 0$$

That is

$$r^2 + ar + b = 0$$

STEP 2: Find the two solutions y_1 and y_2 and compose the complementary solutions y_c of the homogeneous DE.

$$y_c = c_1y_1 + c_2y_2$$

STEP 3: Compose the trial P.S. of the inhomogeneous DE $Ly = f(x)$

$$y_{p(x)} = u_1(x)y_1 + u_2(x)y_2$$

where $u_1(x) \neq u_2(x)$ and are function of x.

STEP 4: Solve the equation to find u_1 and u_2

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = f(x)$$

OR: Solve the matrix below to find u_1 and u_2

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

OR: We can find the values of u_1 and u_2 from the formulas given below

$$u_1(x) = -\int \frac{y_2(t)f(t)}{W\big(y_1(t), y_2(t)\big)} dt$$

$$u_2(x) = \int \frac{y_1(t)f(t)}{W\big(y_1(t), y_2(t)\big)} dt$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

STEP 5: Find y_p by substituting the values of u_1 and u_2 in the equation below

$$y_p = u_1 y_1 + u_2 y_2$$

OR: Find y_p by using the equation below

$$y_p(x) = \int \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W\big(y_1(t), y_2(t)\big)} f(t)dt$$

STEP 6: Now find the G.S.

$$y(x) = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

Example 1

Solve

$$y'' - 3y' - 4y = 15\exp(4x)$$

by variation of parameters

Solution

STEP 1: The C-Eq is

$$r^2 - 3r - 4 = 0$$

$$r_1 = 4, \text{ and } r_2 = -1$$

The solution to the homogeneous DE is

$$y_1 = \exp(4x) \quad \text{and} \quad y_2 = \exp(-x)$$

STEP 2: The complementary solution is

$$y_c = c_1 \exp(4x) + c_2 \exp(-x)$$

STEP 3: Composing the trial P.S.

$$y_p = u_1 \exp(4x) + u_2 \exp(-x)$$

STEP 4: Find u_1 and u_2

$$\exp(4x)\, u_1' + \exp(-x)\, u_2' = 0 \tag{3.39}$$

$$4\exp(4x)\, u_1' - \exp(-x)\, u_2' = 15\exp(4x) \tag{3.40}$$

$$5\exp(4x)\, u_1' = 15\exp(4x) \qquad \text{(3.39)+(3.40)}$$

$$u_1' = 3$$

$$u_1(x) = \int 3dx = 3x$$

$$-5u_2' \exp(-x) = 15\exp(4x) \qquad \text{(3.40)-4(3.39)}$$

$$u_2' = -3\exp(5x)$$

$$u_2(x) = \int -3\exp(5x)\, dx = -\frac{3}{5}\exp(5x)$$

STEP 5: We now find the P.S.

$$y_p(x) = 3x\exp(4x) - \frac{3}{5}\exp(4x)$$

or, we can find y_p directly using the formula given below

$$y_p(x) = \int \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W\big(y_1(t), y_2(t)\big)} f(t)dt$$

$$y_1 = \exp(4x) \quad \text{and} \quad y_2 = \exp(-x)$$

$$W[t] = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \exp(4x) & \exp(-t) \\ \exp(4t) & -\exp(-t) \end{vmatrix} = -5\exp(3t)$$

$$
\begin{aligned}
y_p(x) &= \int \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W[t]} f(t)dt \\
&= \int \big(y_2(x)y_1(t) - y_1(x)y_2(t)\big)\frac{f(x)}{W(t)}dt \\
&= \int (\exp(-x)\exp(4t) - \exp(4x)\exp(-t))\frac{15\exp(4t)}{-5\exp(3t)}dt \\
&= -3\int (\exp(-x)\exp(5t) - \exp(4x))dt \\
&= -3\exp(-x)\left(\frac{1}{5}\exp(5x)\right) + 3x\exp(4x) \\
&= -\frac{3}{5}\exp(4x) + 3x\exp(4x)
\end{aligned}
$$

which is the same as the y_p found in the previous page.

STEP 6: We now find the G.S. to the problem

$$
\begin{aligned}
y &= y_c + y_p \\
&= C_1\exp(4x) + C_2\exp(-x) + 3x\exp(4x) - \frac{3}{5}\exp(4x) \\
&= C_3\exp(4x) + C_2\exp(-x) + 3x\exp(4x)
\end{aligned}
$$

where

$$C_3 = C_1 - \frac{3}{5}$$

is the final solution to this problem which is same as the solution we solved using the undetermined coefficients method.

Example 2

Solve the DE

$$y'' + y = \tan x$$

Solution

Method I

Here we have $f(x) = \tan x$. The C-Eq to the homogeneous DE is

$$r^2 + 1 = 0$$
$$r_{1,2} = \pm i$$
$$y_c = C_1\cos x + C_2\sin x$$
$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x$$

Hence

$$y_1' = -\sin x \quad \text{and} \quad y_2' = \cos x$$

Plugging these into

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = f(x)$$

Solving for u_1 and u_2 we have

$$\begin{aligned} u_1' &= -\sin x \tan x \\ &= \cos x - \sec x \\ u_2' &= \cos x \tan x \\ &= \sin x \end{aligned}$$

Therefore

$$\begin{aligned} u_1 &= \int (\cos x - \sec x) dx \\ &= \sin x - \ln|\sec x + \tan x| \\ u_2 &= \int \sin x \, dx \\ &= -\cos x \end{aligned}$$

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (\sin x - \ln|\sec x + \tan x|)\cos x - \cos x \sin x \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned}$$

<u>Method II</u>

$$\begin{aligned} y_1(x) &= \cos x \\ y_2(x) &= \sin x \end{aligned}$$

$$\begin{aligned} K(x,t) &= \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W[t]} \\ &= \frac{\sin x \cos t - \cos x \sin t}{1} \\ &= \sin(x - t) \\ y_p &= \int K(x,t) f(t) dt \\ &= \int \sin(x - t) \tan t \, dt \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned}$$

Finally, we have

$$\begin{aligned} y &= y_c + y_p \\ &= (C_1 - \ln|\sec x + \tan x|)\cos x + C_2 \sin x \end{aligned}$$

Problems

Problem 3.6.1 For the following 2nd-order, linear, constant coefficient, inhomogeneous DE

$$y'' - (r_1 + r_2)y' + r_1 r_2 y = f(x)$$

where r_1 and r_2 are two different real constants while $f(x)$ is a real function, please find

(1) The two L.I. solutions, $y_1(x)$ and $y_2(x)$, for the homogeneous portion of DE in terms of r_1 and r_2.

(2) The trial functions $u_1(x)$ and $u_2(x)$ that form the P.S. $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ by method of variation of parameters.

(3) The G.S. for the original DE.

Problem 3.6.2 For a given variable coefficient homogeneous DE

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

one solution is given as $y_1(x) \neq 0$. Find the G.S. of this DE by finding another linearly independent solution $y_2(x) = Z(x)y_1(x)$. Please note that $y_1(x)$, $P(x)$ and $Q(x)$ are given functions while $Z(x)$ is not.

Problem 3.6.3 Find the P.S. for

$$y'' + y = \cot x$$

Problem 3.6.4 Find the P.S. for

$$y''' + y'' + y' + y = f(x)$$

where $f(x)$ is a well-defined function.

Problem 3.6.5 Find a P.S. for

$$y'' + 9y = \sin x \tan x$$

Problem 3.6.6 Find a P.S. for

$$y'' + \omega^2 y = \sin \omega x \tan \omega x$$

Problem 3.6.7 Find a P.S. (with constants a, b) for

$$y'' + ay = \tan bx$$

Problem 3.6.8 For inhomogeneous DE

$$y'' + y = 2\sin x$$

please find

(1) The two L.I. solutions for the homogeneous portion of DE.

(2) The trial functions $u_1(x)$ and $u_2(x)$ that form the P.S. $y_p(x)$ by method of variation of parameters.

(3) The G.S. for the original DE.

Problem 3.6.9 You can verify by substitution that $y_c = C_1 x + C_2 x^{-1}$ is a complementary function for the inhomogeneous second-order DE.

$$x^2 y'' + xy' - y = 72x^5$$

But before applying the method of variation of parameters, you must first divide this DE by its leading coefficient x^2 to rewrite it in the standard form.

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 72x^3$$

Thus $f(x) = 72x^3$ in the DE

$$L(y) = y'' + P(x)y' + Q(x)y = f(x)$$

Now processed to solve the DE in

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{cases}$$

and thereby derive the P.S.

$$y_p = 3x^5$$

Problem 3.6.10 Find the G.S. of the following DE by two methods,

$$x'' - 3x' - 4x = 15\exp(4t) + 5\exp(-t)$$

(1) Trail solution method.

(2) Variation of parameters.

Problem 3.6.11 Find a P.S. of

$$x''' + P_1(t)x'' + P_2(t)x' + P_3(t)x = f(t),$$

whose complementary solution is given as

$$x_c(t) = c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t).$$

Your solution should be expressed in terms of the given functions.

Chapter 4
Systems of Linear DEs

4.1 Basics of DE systems

Given below is a general 1st-order DE.

$$f(t, x, x') = 0 \tag{4.1}$$

Here t is the independent variable and x is the dependent variable.

The general form of a system of two 1st-order DEs with two dependent variables x_1 and x_2 is

$$\begin{cases} f(t, x_1, x_2, x_1', x_2') = 0 \\ g(t, x_1, x_2, x_1', x_2') = 0 \end{cases} \tag{4.2}$$

where

- t = independent variable
- x_1 = function of t =1st dependent variable,
- x_2 = function of t =2nd dependent variable.

Example 1

$$\begin{cases} x_1 + 2x_2 + 3x_1' + 4x_2' = f(t) \\ x_1 + x_2 + x_1' + x_2' = g(t) \end{cases}$$

is a Coupled ODEs.

Example 2

The following two DEs

$$\begin{cases} x_1 + x_2' + x_1'' = f(t) \\ x_2 + x_1' + x_2'' = g(t) \end{cases}$$

are coupled because x_1 and x_2 are mixed in the second DE although there is no x_2 in the first DE.

Example 3

While DEs

$$\begin{cases} x_1 + 3x_1' + 5x_1'' = f(t) \\ x_2 + 2x_2' + 4x_2'' = g(t) \end{cases}$$

are two independent (uncoupled) DEs because x_1 and x_2 are in no way related by the above two DEs. Solving the two DEs, independently, should yield solutions to the two DEs.

Generalization

A generalization of (4.2) to n^{th} order, m DEs and m dependent variables can be written as

$$f_1(t; x_1, x_1', x_1'', \dots, x_1^{(n)};\ x_2, x_2', x_2'', \dots, x_2^{(n)}; \dots$$

$$\dots; x_m, x_m', x_m'', \dots, x_m^{(n)}) = 0$$

$$f_2(t; x_1, x_1', x_1'', \dots, x_1^{(n)};\ x_2, x_2', x_2'', \dots, x_2^{(n)}; \dots$$

$$\dots; x_m, x_m', x_m'', \dots, x_m^{(n)}) = 0$$

$$f_n(t; x_1, x_1', x_1'', \dots, x_1^{(n)};\ x_2, x_2', x_2'', \dots, x_2^{(n)}; \dots$$
$$\dots; x_m, x_m', x_m'', \dots, x_m^{(n)}) = 0$$
(4.3)

4.2 First-Order Systems and Applications

Example 1

Consider a system of a particle of mass m and a spring with constant k that moves under the influence of a force field F where x is the distance by which it is stretched from the natural length of the spring.

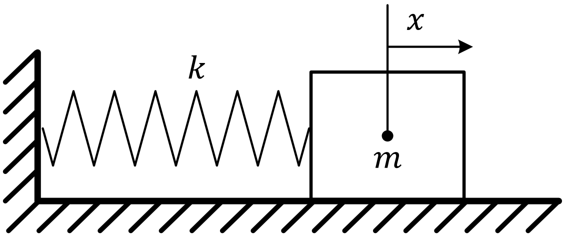

Figure 4.1 A block-spring system with mass m and spring constant k.

Solution

According to Newton's 2nd law,

$$F = ma = mx''$$

We know the force on the block by the spring is $F = -kx$, and then we have

$$mx'' = -kx$$

That is

$$x'' = -\frac{k}{m}x$$

Define $\omega^2 = \frac{k}{m}$ as constant, we have

$$x'' = -\omega^2 x$$

or

$$x'' + \omega^2 x = 0$$

So the C-Eq is

$$r^2 + \omega^2 = 0$$

Thus

$$r_{1,2} = \pm i\omega$$

and we have G.S.

$$x(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

By the I.C.

$$\begin{cases} x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

We can get

$$\begin{cases} C_1 = x_0 \\ C_2 = \dfrac{v_0}{\omega} \end{cases}$$

Finally, we can get the solution

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

Example 1A (Modified)

In this example we are studying the motion of the block under the influence of a force $f(t)$ which is applied to the block.

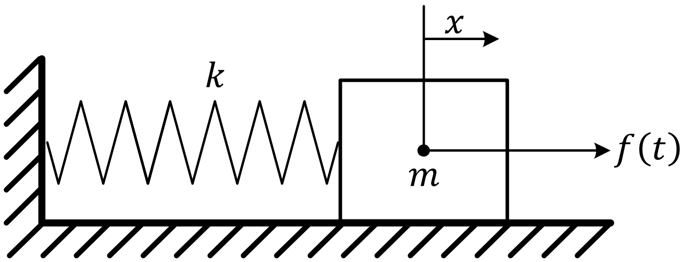

Figure 4.2 A block-spring system with an external force $f(t)$.

The equation of motion should be

$$mx'' = -kx + f(t)$$
$$x'' = -\frac{k}{m}x + \frac{f(t)}{m}$$

Let

$$\omega^2 = \frac{k}{m}$$

and

$$A(t) = \frac{f(t)}{m}$$

We have

$$x'' + \omega^2 x = A$$

We can now solve the above DE using variation of parameters learned previously

$$x(t) = x_c(t) + x_p(t)$$

Obviously one can complicate the system by tangling it with more spring or more blocks or both. Let us consider next a setup with two vibrating blocks.

Example 2

We move block 1 by x_1. So spring 1 is extended by x_1.
We move block 2 by x_2 .So spring 2 is extended by $(x_2 - x_1)$.

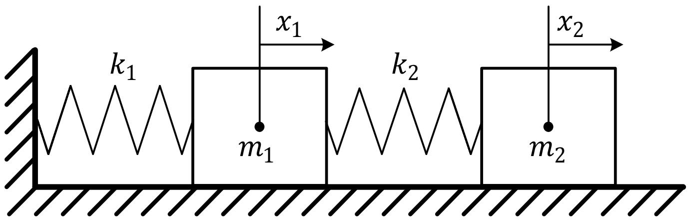

Figure 4.3 A system of two blocks connected by two springs.

Solution

Force on block 2 is

$$-k_2(x_2 - x_1)$$

Force on block 1 is

$$-k_1x_1 + k_2(x_2 - x_1)$$

The DEs for the system are

$$\begin{cases} m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) \\ m_2x_2'' = -k_2(x_2 - x_1) \end{cases}$$

From the first equation, we have

$$x_1'' = -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1)$$

And the second equation gives

$$x_2'' = -\frac{k_2}{m_2}(x_2 - x_1)$$

Let $\omega_1^2 = \frac{k_1}{m_1}$, $\omega_2^2 = \frac{k_2}{m_2}$ and $\omega_{12}^2 = \frac{k_2}{m_1}$ (coupled frequency), we can simplify the Des of the system as following

$$\begin{cases} x_1'' = -\omega_1^2x_1 + \omega_{12}^2(x_2 - x_1) \\ x_2'' = -\omega_2^2(x_2 - x_1) \end{cases}$$

Example 2A (A Variation of Example 2)

We study the motion of the two blocks under the influence of a force $f(t)$.

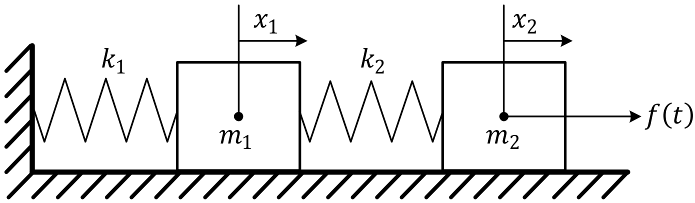

Figure 4.4 A system of two blocks connected by two springs and an external force on block-2.

Solution

The force on m_2 is $-k_2(x_2 - x_1) + f(t)$ while the force on m_1 is still

$$-k_1x_1 + k_2(x_2 - x_1)$$

The DEs for the system now become:

$$\begin{cases} m_1x_2'' = -k_2(x_2 - x_1) + f(t) \\ m_2x_1'' = -k_1x_1 + k_2(x_2 - x_1) \end{cases}$$

This gives

$$x_1'' = -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1)$$

$$x_2'' = -\frac{k_2}{m_2}(x_2 - x_1) + \frac{f(t)}{m_2}$$

$$= -\frac{k_2}{m_2}(x_2 - x_1) + B(t)$$

where $B(t) = \frac{f(t)}{m_2}$

Let $\omega_1^2 = \frac{k_1}{m_1}$, $\omega_2^2 = \frac{k_2}{m_2}$ and $\omega_{12}^2 = \frac{k_2}{m_1}$ (coupled frequency), we can have

$$\begin{cases} x_1'' = -\omega_1^2x_1 + \omega_{12}^2(x_2 - x_1) \\ x_2'' = -\omega_2^2(x_2 - x_1) + B(t) \end{cases}$$

Example 3: Kirchhoff circuits

According to Kirchhoff's voltage law, the left-hand loop of the network has

$$E_0 = L\frac{dI_1}{dt} + (I_1 - I_2)R_2 \tag{4.4}$$

The right-hand loop of the network has

$$\frac{1}{C}Q + I_2R_1 - (I_1 - I_2)R_2 = 0 \tag{4.5}$$

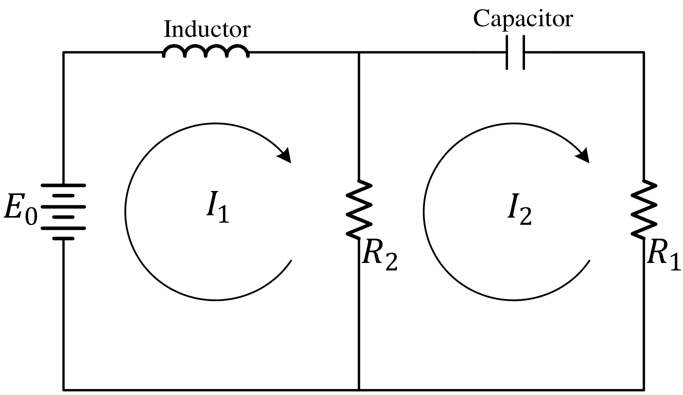

Figure 4.5 An example of Kirchhoff circuit.

Circuit Element	Voltage Drop
Inductor	$L\frac{dl}{dt}$
Resistor	RI
Capacitor	$\frac{1}{C}Q$

Table: Voltage drops across common circuit element.

Since $\frac{dQ}{dt} = I_2$, differentiation of each side of (4.4) and substitution of $\frac{dQ}{dt}$ yields

$$\frac{1}{C}\frac{dQ}{dt} + (R_1 + R_2)\frac{dI_2}{dt} - R_2\frac{dI_1}{dt} = 0$$

This gives

$$\frac{dI_2}{dt} = \frac{R_2}{R_1 + R_2}\frac{dI_1}{dt} - \frac{1}{C(R_1 + R_2)}I_2 \tag{4.6}$$

And from (4.5) we have

$$\frac{dI_1}{dt} = -\frac{R_2}{L}I_1 + \frac{R_2}{L}I_2 + \frac{E_0}{L} \tag{4.7}$$

Substitution of $\frac{dI_1}{dt}$ in (4.6), we have

$$\frac{dI_2}{dt} = \frac{R_2}{R_1 + R_2}\left(-\frac{R_2}{L}I_1 + \frac{R_2}{L}I_2 + \frac{E_0}{L}\right) - \frac{1}{C(R_1 + R_2)}I_2$$

$$\frac{dI_2}{dt} = -\frac{R_2^2}{L(R_1 + R_2)} I_1 + \frac{1}{R_1 + R_2}\left(\frac{R_2^2}{L} - \frac{1}{C}\right) I_2 + \frac{R_2}{R_! + R_2}\frac{E_0}{L} \tag{4.8}$$

From (4.7) and (4.8), we have

$$\frac{d}{dt}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_2}{L} & \frac{R_2}{L} \\ -\frac{R_2^2}{L(R_1 + R_2)} & \frac{1}{R_1 + R_2}\left(\frac{R_2^2}{L} - \frac{1}{C}\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} \frac{E_0}{L} \\ \frac{R_2}{R_1 + R_2}\frac{E_0}{L} \end{bmatrix}$$

Since, L, C, E_0, $R_1$and R_2 are all constants, we can write the above DEs as the following

$$\frac{dI}{dt} = AI + E$$

where $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$, A, and E are matrices with constant elements.

Coupled DE Can be Solved Using the Following Methods

(1) Substitution method

(2) Operator method

We will discuss each method in detail in the following sections.

Problems

Problem 4.2.1 For the following setting with typical assumptions that the springs are mass-less, and the surface the two massive blocks are placed is frictionless, please do the following.

(1) Construct the DE that governs the motion of the two blocks.

(2) Find the G.S. for the blocks if we break the middle spring.

(3) Find the G.S. for the blocks if $k_1 = k_2 = k_3$ for all cases, you may assume arbitrary I.C..

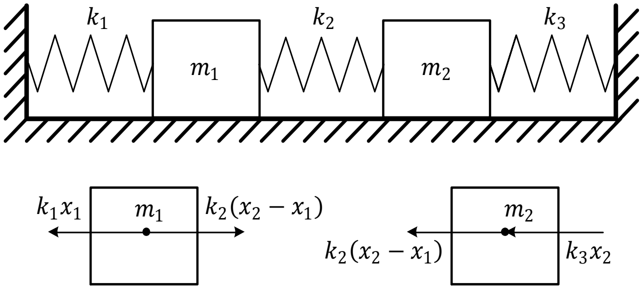

Figure 4.6 The block-spring system for Problem 4.2.1.

Problem 4.2.2 Two particles each of mass m move in the plane with co-ordinates $(x(t), y(t))$ under the influence of a force that is directed toward the origin and has an inverse-square central force field of

$$\frac{k}{x^2+y^2}.$$

Show that

$$\begin{cases} mx'' = -\dfrac{kx}{r^3} \\ my'' = -\dfrac{ky}{r^3} \end{cases}$$

where $r = \sqrt{x^2+y^2}$.

Problem 4.2.3 Suppose that a projectile of mass m moves in a vertical plane in the atmosphere near the surface of the earth under the influence of two forces: a downward gravitational force of magnitude mg, and a resistive force F_R that is directed opposite to the velocity vector v and has magnitude kv^2(where $v = |v|$ is the speed of the projectile). Show that the DEs of motion of the projectile are

$$\begin{cases} mx'' = -kvx' \\ my'' = -kvy' - mg \end{cases}$$

Problem 4.2.4 Two massless springs with spring constants k_1 and k_2 are hanging from a ceiling. The top end of the spring-1 is fixed at the ceiling, while the other end is connected to a block-1. The top end of the spring-2 is connected to block-1, while the other end is connected to block-2. The mass for block-1 is m_1, and that for block-2 is m_2. Initially, we hold the blocks so that the springs will stay at their nature lengths. At time $t = 0$, we will release both blocks to let them vibrate under the influence of the spring and gravity with gravitational constant g. Please find the motion of the blocks.

Problem 4.2.5 Three massless springs with spring constants k_1 , k_2 and k_3 are hanging on a ceiling. The top end of the spring-1 is fixed at the ceiling while the other end is connected to a block-1. The end of the spring-2 is connected to block-1 while the other end is connected to block-2. The top end of the spring-3 is connected to block-2 and the other end is connected to block-3. The mass for blocks are m_1, m_2 and m_3.
Initially, we hold the blocks so that the springs will stay at their nature lengths. At time *t*=0, we will release the blocks to let them vibrate under the influence of the spring and gravity with gravitational constant g. Please find the motion of the blocks.

Problem 4.2.6 One end of a massless spring of constant *k* is attached to a ceiling and the other end to a small ball of mass m. Normally, the spring stretches down a little, due to gravity, but stays perfectly vertical. Now, someone moves the ball up a little and the spring recoils to its natural length of L_0. If the ball is moved a little sideway from the natural state, without stretching or squeezing the spring, the spring stays on a line about $\theta_0$degrees from the vertical line. After being released, the ball starts some magic motion. Compute its motion. *(Hint: Since the move is so small that $\theta \approx \sin\theta$.)*

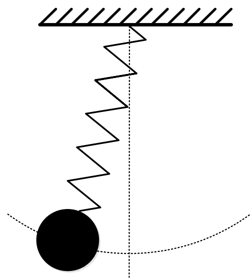

Figure 4.7 The pendulum system for Problem 4.2.6.

4.3 Substitution Method

Example 1

Solve the following two coupled DEs with I.C., i.e., system of DEs.

$$\begin{cases} x_1' = x_1 - 8x_2 \\ x_2' = -x_1 + 3x_2 \\ x_1(0) = x_2(0) = 2 \end{cases}$$

Solution

STEP 1: Solve for one of the two dependent variables x_1 or x_2 from the given two DEs.

The second DE can be written as

$$x_1 = 3x_2 - x_2' \tag{4.9}$$

Differentiating it gets

$$x_1' = 3x_2' - x_2''$$

STEP 2: Plugging x_1 and x_1' into the first DE, we get

$$(3x_2 - x_2')' = (3x_2 - x_2') - 8x_2$$

Therefore

$$3x_2' - x_2'' = 3x_2 - x_2' - 8x_2$$

Thus, we get a 2nd-order DE for x_2

$$x_2'' - 4x_2' - 5x_2 = 0 \tag{4.10}$$

STEP 3: Solving the resulting single-variable (decoupled) DE by C-Eq method, we have

$$r^2 - 4r - 5 = (r - 5)(r + 1) = 0$$

Thus, the solution for x_2 is

$$x_2 = C_1 \exp(5t) + C_2 \exp(-t)$$

STEP 4: Back substitute to solve for the other dependent variables, from (4.9) we get,

$$\begin{aligned} x_1 &= 3x_2 - x_2' \\ &= 3(C_1 \exp(5t) + C_2 \exp(-t)) - (C_1 \exp(5t) + C_2 \exp(-t))' \\ &= 3C_1 \exp(5t) + 3C_2 \exp(-t) - (5C_1 \exp(5t) - C_2 \exp(-t)) \\ &= -2C_1 \exp(5t) + 4C_2 \exp(-t) \end{aligned}$$

Thus, we have the G.S. for the system of DEs.

$$\begin{cases} x_1 = -2C_1 \exp(5t) + 4C_2 \exp(-t) \\ x_2 = C_1 \exp(5t) + C_2 \exp(-t) \end{cases}$$

STEP 5: Solve for the I.C.

$$\begin{cases} x_1(0) = 2 = -2C_1 + 4C_2 \\ x_2(0) = 2 = C_1 + C_2 \end{cases}$$

This gives

$$\begin{cases} C_1 = 1 \\ C_2 = 1 \end{cases}$$

The final solution is

$$\begin{cases} x_1 = -2\exp(5t) + 4\exp(-t) \\ x_2 = \exp(5t) + \exp(-t) \end{cases}$$

Solution Steps

STEP 1: Express one dependent variable by the other by solving the two DEs, i.e., decoupling variables.
STEP 2: Substitute the above variable to one DE to obtain a single-dependent variable DE.
STEP 3: Solve the resulting single-dependent variable DE from the above STEP 2:.
STEP 4: Back substitute to solve for other dependent variables.

Example 2

Solve the following two DEs.

$$\begin{cases} x' = 4x - 3y \\ y' = 6x - 7y \\ x(0) = 2 \\ y(0) = -1 \end{cases}$$

Solution

STEP 1: Solve the first DE for y in terms of x and x', or solve the second DE for x in terms of y and y'. (There is no written rule on which variable is better to eliminate first. This depends highly on experience and mathematical perception. Fortunately, many times, it would not matter which order to take.)

From the first DE $x' = 4x - 3y$, we get

$$y = \frac{4}{3}x - \frac{1}{3}x'$$

and

$$y' = \frac{4}{3}x' - \frac{1}{3}x''$$

STEP 2: Plugging y and y' into the second DE, we get

$$\frac{4}{3}x' - \frac{1}{3}x'' = 6x - 7\left(\frac{4}{3}x - \frac{1}{3}x'\right)$$

or

$$x'' + 3x' - 10x = 0$$

STEP 3: Solving the resulting single-dependent-variable DE by C-Eq method, we have

$$r^2 + 3r - 10 = 0$$

Solve this C-Eq gives

$$r_1 = -5 \text{ and } r_2 = 2$$

We can get the solution for x

$$x(t) = C_1 \exp(-5t) + C_2 \exp(2t)$$

STEP 4: Back-substitute to solve for $y(t)$.

$$\begin{aligned} y &= \frac{4}{3}x - \frac{1}{3}x' \\ &= \frac{4}{3}(C_1 \exp(-5t) + C_2 \exp(2t)) - \frac{1}{3}(C_1 \exp(-5t) + C_2 \exp(2t))' \\ &= \frac{4}{3}(C_1 \exp(-5t) + C_2 \exp(2t)) - \frac{1}{3}(-5C_1 \exp(-5t) + 2C_2 \exp(2t)) \\ &= 3C_1 \exp(-5t) + \frac{2}{3}C_2 \exp(2t) \end{aligned}$$

Therefore, we have the G.S. to the original system

$$\begin{cases} x = C_1 \exp(-5t) + C_2 \exp(2t) \\ y = 3C_1 \exp(-5t) + \frac{2}{3}C_2 \exp(2t) \end{cases}$$

STEP 5: Solve for the I.C.

$$\begin{cases} x(0) = 2 = C_1 + C_2 \\ y(0) = -1 = 3C_1 + \frac{2}{3}C_2 \end{cases}$$

This solves

$$\begin{cases} C_1 = -1 \\ C_2 = 3 \end{cases}$$

Thus, the final solution is

$$\begin{cases} x = -\exp(-5t) + 3\exp(2t) \\ y = -3\exp(-5t) + 2\exp(2t) \end{cases}$$

Example 3

Solve the following two DEs.

$$\begin{cases} x' + 2y' = 3 \\ 2x' - y' = 1 \end{cases}$$

Solution

We can easily find $x' = y' = 1$, thus we get (very strange solutions)

$$\begin{cases} x = t + C_1 \\ y = t + C_2 \end{cases}$$

Example 4

Solve the following two DEs.

$$\begin{cases} x + 2y' = 3 \\ 2x' - y = 1 \end{cases}$$

Solution

STEP 1: From the first DE, we get $x = 3 - 2y'$ and that gives

$$x' = (3 - 2y')' = -2y''$$

STEP 2: Substituting the above to the second DE, we get

$$4y'' + y = -1$$

This can solve

$$\begin{cases} y = C_1 \cos\frac{t}{2} + C_2 \sin\frac{t}{2} - 1 \\ x = -C_2 \cos\frac{t}{2} + C_1 \sin\frac{t}{2} + 3 \end{cases}$$

Problems

Problem 4.3.1 Solve the following system of DEs.

$$\begin{cases} x'' - y'' + x = 2\exp(-t) \\ x'' + y'' - x = 0 \end{cases}$$

Problem 4.3.2 Solve the following system of DEs.

$$\begin{cases} x' = -y \\ y' = 10x - 7y \\ x(0) = 2 \\ y(0) = -7 \end{cases}$$

Problem 4.3.3 Use substitution method to solve the following system of 1st-order linear equations.

$$\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 2x - 2y \\ x(0) = 1 \\ y(0) = 2 \end{cases}$$

Problem 4.3.4 Find the G.S. of the system given below with the given masses and spring constants.

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases}$$

where $m_1 = m_2 = 1, k_1 = 1, k_2 = 4$ and $k_3 = 1$

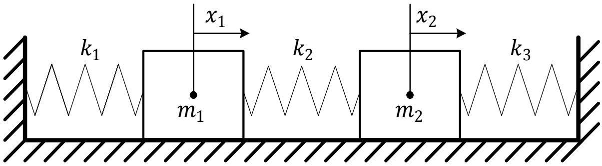

Figure 4.8 The block-spring system for Problem 4.3.4.

Problem 4.3.5 Solve the following system of DEs.

$$\begin{cases} x_1' = x_1 + 2x_2 + t \\ x_2' = 2x_1 + x_2 + t^2 \end{cases}$$

Problem 4.3.6 Solve the following system of DEs.

$$\begin{cases} x' = y \\ y' = -9x + 6y \end{cases}$$

Problem 4.3.7 Solve the following system of DEs with given I.C..

$$\begin{cases} x' = x - 2y \\ y' = x - y \\ x(0) = 1 \\ y(0) = 2 \end{cases}$$

Problem 4.3.8 Solve the following system of DEs with given I.C..

$$\begin{cases} x' = 3x + 4y \\ y' = 3x + 2y \\ x(0) = 1 \\ y(0) = 1 \end{cases}$$

Problem 4.3.9 Solve the following system of DEs.

$$\begin{cases} x' = x + 2y + z \\ y' = 3x + 2y \\ z' = x + y + z \end{cases}$$

4.4 Operator Method

In this method we introduce a new operator

$$D \equiv \frac{d}{dt}$$

Properties of Operators

(1) $D(x+y) = Dx + Dy$
(2) $D(\alpha x) = \alpha Dx$
(3) $D(\alpha x + \beta y) = \alpha Dx + \beta Dy$

where α and β are constants.

Example 1

Solve the following system of ODEs by Operator Method. (This problem was solved earlier in the Substitution Method).

$$\begin{cases} x_1' = x_1 - 8x_2 \\ x_2' = -x_1 + 3x_2 \\ x_1(0) = x_2(0) = 2 \end{cases}$$

Solution

Let us rewrite the DEs by using the operator

$$\begin{cases} Dx_1 = x_1 - 8x_2 \\ Dx_2 = -x_1 + 3x_2 \end{cases}$$

That is

$$\begin{cases} (D-1)x_1 + 8x_2 = 0 & \text{(A)} \\ x_1 + (D-3)x_2 = 0 & \text{(B)} \end{cases}$$

Solve DEs (A) and (B) by (A) $-$ $(D-1) \times$ (B).

$$\big((D-3)(D-1) - 8\big)x_2 = 0$$

That is

$$(D^2 - 4D - 5)x_2 = 0$$

whose C-Eq is

$$r^2 - 4r - 5 = 0$$

$$r_1 = 5, \quad r_2 = -1$$

Therefore, we solve x_2

$$x_2 = C_1 \exp(5t) + C_2 \exp(-t)$$

Substituting x_2 into (B) to solve for x_1, we get

$$x_1 = -2C_1 \exp(5t) + 4C_2 \exp(-t)$$

After solving for the I.C. you will have $C_1 = 1$ and $C_2 = 2$.
Finally, we can get the answer

$$\begin{cases} x_1 = -2\exp(5t) + 4\exp(-t) \\ x_2 = \exp(5t) + \exp(-t) \end{cases}$$

Example 2

Let us solve Example 2 of the Substitution Method using the Operator Method

$$\begin{cases} x' = 4x - 3y \\ y' = 6x - 7y \\ x(0) = 2 \\ y(0) = -1 \end{cases}$$

Solution

We can rewrite the system by using the operator

$$\begin{cases} Dx = 4x - 3y \\ Dy = 6x - 7y \end{cases}$$

Thus, two DEs can be written as

$$\begin{cases} (D-4)x + 3y = 0 & \text{(A)} \\ 6x - (D+7)y = 0 & \text{(B)} \end{cases}$$

Operating on the two DE by $\big(3 \times (B)\big) + \big((D+7) \times (A)\big)$, we can get

$$\big((D+7)(D-4) + 18\big)x = (D^2 + 3D - 10)x = 0$$

whose C-Eq is

$$r^2 + 3r - 10 = 0$$

$$r_1 = 2 \text{ and } r_2 = -5$$

This gives

$$x = C_1 \exp(2t) + C_2 \exp(-5t)$$

Back-substitute in Eq-(A) to solve for y.

$$y = \frac{2}{3} C_1 \exp(2t) + 3C_2 \exp(-5t)$$

The G.S. for the system

$$\begin{cases} x = C_1 \exp(2t) + C_2 \exp(-5t) \\ y = \dfrac{2}{3} C_1 \exp(2t) + 3C_2 \exp(-5t) \end{cases}$$

Plugging the I.C., we have

$$\begin{cases} x(0) = C_1 + C_2 = 2 \\ y(0) = \dfrac{2}{3} C_1 + 3C_2 = -1 \end{cases}$$

This gives

$$\begin{cases} C_1 = 3 \\ C_2 = -1 \end{cases}$$

And the final solution for the problem is

$$\begin{cases} x = 3\exp(2t) - \exp(-5t) \\ y = 2\exp(2t) - 3\exp(-5t) \end{cases}$$

One application of solving system of DEs is that one can consider solving DEs of higher order by converting them into a system of 1st-order DEs.

Given a function $f(y''', y'', y', y, x) = 0$

Let

$$y_1 = y$$

$$y_2 = y'$$

$$y_3 = y''$$

$$y_4 = y'''$$

Thus, the DE becomes

$$f(y_4, y_3, y_2, y_1, x) = 0$$

Coupling with the dependent variable introduced earlier, we can form a system of 1st-order DEs

$$\begin{cases} f(y_4, y_3, y_2, y_1, x) = 0 \\ y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \end{cases}$$

Solving this system of DEs can help solve the original DE of higher order.

Problems

Problem 4.4.1 Find the G.S. of the following system of DEs.

$$\begin{cases} x' = x + 2y + z \\ y' = 6x - y \\ z' = -x - 2y - z \end{cases}$$

Problem 4.4.2 Find the G.S. of the following system of DEs.

$$\begin{cases} (D^2 + D)x + D^2 y = 2\exp(-t) \\ (D^2 - 1)x + (D^2 - D)y = 0 \end{cases}$$

Problem 4.4.3 Find the G.S. of the following system of DEs.

$$\begin{cases} x' = 3x + 9y \\ y' = 2x + 2y \\ x(0) = y(0) = 2 \end{cases}$$

Problem 4.4.4 Find the G.S. of the following system of DEs.

$$\begin{cases} x_1' = x_1 + 2x_2 + t \\ x_2' = 2x_1 + x_2 + t^2 \end{cases}$$

Problem 4.4.5 Find the G.S. of the following system of DEs.

$$\begin{cases} x' = y + z + \exp(-t) \\ y' = x + z \\ z' = x + y \end{cases}$$

Problem 4.4.6 Solve the following system of DEs by (1) substitution method and (2) operation method.

$$\begin{cases} x' = x + y \\ y' = 6x - y \end{cases}$$

Problem 4.4.7 Use operational determinant method to solve the following system of DEs.

$$\begin{cases} (D + 2)x + (D - 3)y = \exp(-2t) \\ (D - 2)x + (D + 3)y = \exp(3t) \end{cases}$$

Problem 4.4.8 Find the G.S. of the following Des system.

$$\begin{cases} x'' + 13y' - 4x = 6\sin t \\ \quad 2x' - y'' + 9y = 0 \end{cases}$$

Problem 4.4.9 Solve the following system of DEs by (1) substitution method and (2) operator method.

$$\begin{cases} x' = 3x - 2y \\ y' = 2x + y \end{cases}$$

4.5 Eigen-Analysis Method

All systems of linear DEs can be expressed in terms of matrices and vectors, which are widely used in linear algebra and multivariable calculus.

For example, we can express a system of 3 1st-order DEs below

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

by a general (dimensionally implicit) notation

$$X' = \boldsymbol{A}X + F(t)$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$$

As in the discussions of single DEs, if

$$F(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

the system is said to be homogeneous. O.W., it is inhomogeneous. Similar to the solution of single dimension DEs, homogeneous systems are much simpler to solve than the inhomogeneous system with the non-zero inhomogeneous term.

For a homogeneous system

$$X' = \boldsymbol{A}X$$

we may assume a trial solution in the form of

$$X = V \exp(\lambda t)$$

where V is a vector and λ is a scalar and both are to be determined by the matrix $\boldsymbol{A}$

$$X' = V\lambda \exp(\lambda t)$$

Plugging it into the above original homogeneous system, we get

$$V\lambda \exp(\lambda t) = \boldsymbol{A}V \exp(\lambda t)$$

or

$$(\boldsymbol{A} - \lambda \boldsymbol{I})V \exp(\lambda t) = \boldsymbol{0}$$

Since $\exp(\lambda t) \neq 0$, we must have $(\boldsymbol{A} - \lambda \boldsymbol{I})V = \boldsymbol{0}$. Considering $V = 0$ gives only trivial solution, we are only interested in the case where

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

That is, λ is the eigenvalue and V is the associated eigenvector of the matrix $\boldsymbol{A}$. The solution of the homogeneous system can be written as

$$X = \sum c_i V_{\boldsymbol{i}} \exp(\lambda_i t)$$

Now, for three different cases of the eigenvalues, we have three different solution methods.

Case I: Non-repeated real eigenvalues. The above linear combination for composing the G.S. holds.

Case II: Non-repeated complex eigenvalues. The above linear combination for composing the G.S. holds except that the term involving $\exp(\lambda_i t)$ or the eigenvector $\boldsymbol{V_i}$ or both may be complex.

Case III: Repeated real eigenvalues. Composing the second or third or more solutions is a much trickier process which is demonstrated by the following Example 2.

For inhomogeneous systems, we need to find a P.S. and then add it to the solution of the homogeneous portion of the system. To find the P.S., we can use the rule-based trial method as well as the variation of parameters method.

Example 1

Let us solve Example 2 of the Substitution method using the Eigen-Analysis method

Solution

Rewrite the system of DEs in the matrix notation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The coefficient matrix is

$$\boldsymbol{A} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}$$

Its eigenvalues can be found by solving

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{bmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{bmatrix} = 0$$

which are

$$\lambda_1 = 2, \qquad \lambda_2 = -5$$

and the associated eigenvectors are

$$V_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore, the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \exp(2t) + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \exp(-5t)$$

Example 2

Solve the following system using the Eigen-Analysis method

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution

The system has two identical eigenvalues

$$\lambda_1 = 4 = \lambda_2 = 4$$

We can find the associated eigenvector V_1 for the eigenvalue λ_1 and its solution is

$$X_1 = V_1 \exp(\lambda t)$$

For the second solution, we need to compose the L.I. solution by introducing

$$X_2 = (V_1 t + V_2) \exp(\lambda t)$$

We plug this proposed solution to the system. We get

$$X_2' = (V_1) \exp(\lambda t) + (V_1 t + V_2)\lambda \exp(\lambda t)$$

so

$$X_2' = (V_1) \exp(\lambda t) + (V_1 t + V_2)\lambda \exp(\lambda t) = \boldsymbol{A}(V_1 t + V_2) \exp(\lambda t)$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})V_2 \exp(\lambda t) = V_1 \exp(\lambda t)$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})V_2 = V_1$$

Additionally, since $(A - \lambda \boldsymbol{I})V_1 = 0$, we have

$$(\boldsymbol{A} - \lambda \boldsymbol{I})(\boldsymbol{A} - \lambda \boldsymbol{I})V_2 = (\boldsymbol{A} - \lambda \boldsymbol{I})V_1 = 0$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 V_2 = 0$$

Now, we can find V_2 with which we can compose the second solution and then the entire solution.
Now, we return to our original example,

$$(\boldsymbol{A} - \lambda \boldsymbol{I})^2 V_2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} V_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} V_2 = 0$$

which means any solution V_2 will satisfy the above equation. For convenience, we may select

$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with which we construct the eigenvector V_1

$$V_1 = (\boldsymbol{A} - \lambda \boldsymbol{I})V_2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

One can also solve

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} V_1 = 0$$

to find the eigenvector V_1.
Finally, we have the two L.I. solutions

$$X_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \exp(4t)$$

$$X_2 = \left(\begin{bmatrix} -3 \\ 3 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \exp(4t) = \begin{bmatrix} -3t \\ 3t + 1 \end{bmatrix} \exp(4t)$$

So, finally, the G.S. for the original DE is

$$X = c_1 X_1 + c_2 X_2 = c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} \exp(4t) + c_2 \begin{bmatrix} -3t \\ 3t + 1 \end{bmatrix} \exp(4t)$$

Example 3
Solve the following system of DEs.

$$\begin{cases} x' = 3x + 4y \\ y' = 3x + 2y \end{cases}$$

Solution
The original system can be written as $X' = \boldsymbol{A}X$ where

$$\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$$

We can easily find its eigenvalues by

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{vmatrix} 3 - \lambda & 4 \\ 3 & 2 - \lambda \end{vmatrix} = 0$$

with $\lambda_1 = -1,\ \lambda_2 = 6$.
Next, we find the associated eigenvectors.
For $\lambda_1 = -1$, we get

$$\begin{bmatrix} 3 - (-1) & 4 \\ 3 & 2 - (-1) \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $u_1 + v_1 = 0$, we may select

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So, we get

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \exp(-t)$$

For $\lambda_2 = 6$, we get

$$\begin{bmatrix} 3 - 6 & 4 \\ 3 & 2 - 6 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, $3u_2 - 4v_2 = 0$, we may select

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

So, we get

$$X_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \exp(6t)$$

Finally, the G.S. is

$$X = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \exp(-t) + c_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} \exp(6t)$$

Example 4

Solve the following system of DEs.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

Solution

First, we try to solve the homogeneous DEs.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

It can be written as

$$X' = \boldsymbol{A}X$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \det \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

So the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$

For $\lambda_1 = 3$, we get

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can select

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_1 = -1$, we get

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can select

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So the G.S. to homogeneous DEs is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(3t) + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp(-t)$$

The P.S. may take the following form

$$X_P = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} t^2 + \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t + \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2A_2 \\ 2B_2 \end{bmatrix} t + \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

Plugging them to the original system, we get

$$\begin{bmatrix} 2A_2 \\ 2B_2 \end{bmatrix} t + \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} t^2 + \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t + \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \right) + \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

$$\begin{cases} A_0 = -\dfrac{43}{27} \\ B_0 = \dfrac{38}{27} \\ A_1 = \dfrac{11}{9} \\ B_1 = -\dfrac{16}{9} \\ A_2 = -\dfrac{2}{3} \\ B_2 = \dfrac{1}{3} \end{cases}$$

Then the G.S. to the original DEs is

$$\begin{aligned} X(t) &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(3t) + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp(-t) + \frac{1}{3} \begin{bmatrix} -2 \\ 1 \end{bmatrix} t^2 + \frac{1}{9} \begin{bmatrix} 11 \\ -16 \end{bmatrix} t \\ &\qquad + \frac{1}{27} \begin{bmatrix} -43 \\ 38 \end{bmatrix} \end{aligned}$$

Example 5

Find the G.S. of the following.

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Solution

The original DEs can be written as

$$\begin{bmatrix} D-5 & -5 & -2 \\ 6 & D+6 & 5 \\ -6 & -6 & D-5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} D-5 & -5 & -2 \\ 6 & D+6 & 5 \\ -6 & -6 & D-5 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ D-5 & -5 & -2 \\ -6 & -6 & D-5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ 6(D-5) & -30 & -12 \\ 0 & D & D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ 0 & -30+(D+6)(5-D) & -12+5(5-D) \\ 0 & D & D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ 0 & D(-D-1) & 13-5D \\ 0 & D & D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ 0 & D & D \\ 0 & D(-D-1) & 13-5D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ 0 & D & D \\ 0 & 0 & D(D+1)+13-5D \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & D+6 & 5 \\ 0 & D & D \\ 0 & 0 & D^2-4D+13 \end{bmatrix}$$

So we can have

$$(D^2-4D+13)z(t)=0$$

C-Eq:

$$r^2-4r+13=0$$

$$r_1=2+3i, \qquad r_2=2-3i$$

So

$$z(t)=c_1\exp(2t)\cos(3t)+c_2\exp(2t)\sin(3t)$$
$$z'(t)=(2c_1+3c_2)\exp(2t)\cos(3t)+(2c_2-3c_1)\exp(2t)\sin(3t)$$

From the second row of the matrix, we can have

$$y'+z'=0$$

$$\begin{aligned} y'(t)&=-z'(t)\\ &=(-2c_1-3c_2)\exp(2t)\cos(3t)+(-2c_2+3c_1)\exp(2t)\sin(3t) \end{aligned}$$

So

$$\begin{aligned} y(t)&=\int y'(t)dt=-\int z'(t)dt=-z(t)+c_3\\ &=-c_1\exp(2t)\cos(3t)-c_2\exp(2t)\sin(3t)+c_3 \end{aligned}$$

From the first row of the matrix, we can have

$$6x+y'+6y+5z=0$$

So

$$\begin{aligned} x(t) &= \frac{-y'(t) - 6y(t) - 5z(t)}{6} \\ &= \frac{c_1 + c_2}{2}\exp(2t)\cos(3t) + \frac{c_2 - c_1}{2}\exp(2t)\sin(3t) - c_3 \end{aligned}$$

Then the solution is

$$\begin{aligned} x(t) &= \frac{c_1 + c_2}{2}\exp(2t)\cos(3t) + \frac{c_2 - c_1}{2}\exp(2t)\sin(3t) - c_3 \\ y(t) &= -c_1\exp(2t)\cos(3t) - c_2\exp(2t)\sin(3t) + c_3 \\ z(t) &= c_1\exp(2t)\cos(3t) + c_2\exp(2t)\sin(3t) \end{aligned}$$

Problems

Problem 4.5.1 Solve the following system of DEs by (1) substitution method and (2) operator method.

$$\begin{cases} x' = 3x - 2y \\ y' = 2x + y \end{cases}$$

Problem 4.5.2 Solve the following system using Eigen-Analysis method.

$$X'(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} X(t)$$

Problem 4.5.3 Solve the following system using Eigen-Analysis method.

$$X'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 3\exp(t) \\ -t^2 \end{bmatrix}$$

Problem 4.5.4 Solve the following system using Eigen-Analysis method.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\exp(-2t)$$

Problem 4.5.5 Solve the following system using Eigen-Analysis method.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} t$$

Problem 4.5.6 Solve the following system using Eigen-Analysis method.

$$\begin{cases} x' - 4x + 2y = 0 \\ y' + 4x - 4y + 2z = 0 \\ z' + 4y - 4z = 0 \end{cases}$$

Chapter 5
Laplace Transforms

5.1 Laplace Transforms

Why do we learn Laplace Transforms for a course with focus on ODEs?

In Chapters 3 and 4, we discussed how to solve linear DEs of systems of them. In certain cases, the methods in Chapter 3 can be quite tedious and such methods as Laplace transform method can provide much convenience. The Differentiation Operator D can be viewed as a transform which, when applied to the function $f(t)$, yields the new function $D\{f(t)\} = f'(t)$. Similarly, the Laplace transformation $\mathcal{L}$ involves the operation of integration and yields a new function $\mathcal{L}\{f(t)\} = F(s)$ of a new independent variable. This new function usually can be manipulated much more easily. This is the key reason that the Laplace Transform is introduced here.

After learning how to compute the Laplace transform $F(s)$ of a function$f(t)$, we will learn how the Laplace transform converts a DE with the unknown function $f(t)$ and its derivatives into an Algebraic Equation in $F(s)$. Because algebraic equations are usually easier to solve than DEs, this method simplifies the problem of finding the solution $f(t)$.

Definition

Given a function $f(t)\ \forall t \geq 0$, the Laplace Transform of the function $f(t)$ is another function $F(s)$ defined as follows.

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty \exp(-st)\, f(t) dt$$

for all values of s for which the improper integral converges.

5.2 Properties of Laplace Transforms

Computing Laplace transforms requires understanding of the key properties of the transforms. In the following, we present a few such properties.

The first and the most important property is that of linearity.

Linearity Property

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\exp(-st)\,dt = F(s)$$

and

$$\mathcal{L}\{g(t)\} = \int_0^\infty g(t)\exp(-st)\,dt = G(s)$$

The linearity of the transform states that,

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

where α and β are constants.

Proof

$$\begin{aligned}
\text{LHS} &= \mathcal{L}\{\alpha f(t) + \beta g(t)\} \\
&= \int_0^\infty \big(\alpha f(t) + \beta g(t)\big)\exp(-st)\,dt \\
&= \int_0^\infty \alpha f(t)\exp(-st)\,dt + \int_0^\infty \beta g(t)\exp(-st)\,dt \\
&= \alpha F(s) + \beta G(s) \\
&= \text{RHS}
\end{aligned}$$

5.2.1 Laplace Transforms for Polynomials

Polynomials are a class of functions of form

$$p_n(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

From the linearity, we know the Laplace transform $P_n(s) = \mathcal{L}\{p_n(t)\}$ has form

$$P_n(s) = a_0 \mathcal{L}\{1\} + a_1 \mathcal{L}\{t\} + \cdots + a_n \mathcal{L}\{t^n\}$$

Now let us compute $\mathcal{L}\{t^n\}$. We start from the simplest case $\mathcal{L}\{1\}$

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty 1 \times \exp(-st)\, dt \\
&= \int_0^\infty \exp(-st)\, dt \\
&= -\frac{1}{s} \exp(-st)\Big|_0^\infty \\
&= -\frac{1}{s} \exp(-st)\Big|_{t\to\infty} + \frac{1}{s} \exp(-st)\Big|_{t\to 0} \\
&= \lim_{t\to 0} \frac{1}{s} \exp(-st)
\end{aligned}$$

It is obvious that the integral diverges if $s \leq 0$, resulting in meaningless transform. We only consider $s > 0$ for which we have

$$\mathcal{L}\{1\} = \frac{1}{s}$$

Thus, the Laplace transform of 1 is $1/s$.

Next, we consider $\mathcal{L}\{t\}$. Starting from the definition,

$$\mathcal{L}\{t\} = \int_0^\infty t \exp(-st)\, dt$$

Using integration by parts, we have

$$\begin{aligned}
\mathcal{L}\{t\} &= -\frac{1}{s}\int_0^\infty t d(\exp(-st)) \\
&= -\frac{1}{s} t \exp(-st)\Big|_0^\infty - \frac{1}{s}\left(-\int_0^\infty \exp(-st)\, dt\right) \\
&= -\frac{1}{s}\left(-\int_0^\infty \exp(-st)\, dt\right) \\
&= \frac{1}{s} L\{1\}
\end{aligned}$$

From the previous transform $\mathcal{L}\{1\} = \frac{1}{s}\ \forall s > 0$, we then have

$$\mathcal{L}\{t\} = \frac{1}{s^2}\ \forall\ s > 0$$

Alternatively, we can derive this directly by taking derivative of

$$\mathcal{L}\{1\} = \frac{1}{s}$$

That is

$$\frac{d}{ds}\mathcal{L}\{1\} = \frac{d}{ds}\left(\frac{1}{s}\right)$$

$$\begin{aligned}
\text{LHS} = \frac{d}{ds}\mathcal{L}\{1\} &= \frac{d}{ds}\int_0^\infty \exp(-st)\, dt = \int_0^\infty \frac{\partial}{\partial s}(\exp(-st))dt \\
&= -\int_0^\infty t \exp(-st)\, dt = -L\{t\}
\end{aligned}$$

$$\text{RHS} = \frac{d}{ds}\left(\frac{1}{s}\right) = -\frac{1}{s^2}$$

By equating both sides, we get

$$-\mathcal{L}\{t\} = -\frac{1}{s^2}$$

Or finally, we get the intended transform

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Iteratively using this method, we can easily derive

$$\frac{d}{ds}\mathcal{L}\{t\} = \frac{d}{ds}\left(\frac{1}{s^2}\right)$$

$$\int_0^\infty \frac{\partial}{\partial s}[t\exp(-st)]dt = -\frac{2}{s^3}$$

$$\int_0^\infty t^2 \exp(-st)\,dt = \frac{2}{s^3}$$

That is

$$\mathcal{L}\{t^2\} = \frac{2}{s^3} \quad \forall s > 0$$

$$\dots\dots$$

It is easy to derive the general formula for a power function (and naturally for a polynomial)

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \forall s > 0$$

which can be proved by induction as follows.

Proof

For $n = 0$, we have already discussed.

Suppose the formula is true for n. This means

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \forall s > 0$$

from which we have

$$\frac{d}{ds}\mathcal{L}\{t^n\} = \frac{d}{ds}\left(\frac{n!}{s^{n+1}}\right)$$

$$\int_0^\infty \frac{\partial}{\partial s}(t^n \exp(-st))dt = -\frac{n! \cdot (n+1)}{s^{n+2}}$$
$$\int_0^\infty t^{n+1} \exp(-st)\, dt = \frac{(n+1)!}{s^{n+2}}$$

That is

$$\mathcal{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{n+2}} \quad \forall s > 0$$

which finishes the proof.

Example

If $T(\alpha) = \mathcal{L}\{t^\alpha\}$, then prove

$$\frac{d}{ds}\big(T(\alpha)\big) = -T(\alpha+1)$$

Solution

To prove this, we simply derive the formula directly

$$\begin{aligned}\text{LHS} &= \frac{d}{ds}\mathcal{L}\{t^\alpha\} \\ &= \frac{d}{ds}\int_0^\infty t^\alpha \exp(-st)\, dt \\ &= \int_0^\infty \frac{\partial}{\partial s}(t^\alpha \exp(-st))dt \\ &= -\int_0^\infty t^{\alpha+1} \exp(-st)\, dt \\ &= -\mathcal{L}\{t^{\alpha+1}\} = \text{RHS}\end{aligned}$$

5.2.2 The Translator Property

Given $\mathcal{L}\{f(t)\} = F(s)$, compute $\mathcal{L}\{f(t)\exp(at)\}$.

$$\mathcal{L}\{f(t)\exp(at)\} = \int_0^\infty f(t)\exp(at - st)\, dt$$

$$= \int_0^\infty f(t) \exp\big((a-s)t\big)\, dt$$

By substitution $\rho = s - a$, we have

$$\mathcal{L}\{f(t)\exp(at)\} = \int_0^\infty f(t)\exp(-\rho t)\, dt$$
$$= F(\rho)$$

That is

$$\mathcal{L}\{f(t)\exp(at)\} = F(s-a)$$

Similarly, we can get

$$\mathcal{L}\{f(t)\exp(-at)\} = F(s+a)$$

Combining the two, we have

$$\mathcal{L}\{f(t)\exp(\pm at)\} = F(s \mp a)$$

This formula is the translator property of Laplace transform.

Example 1

Compute $\mathcal{L}\{\exp(at)\}$

Solution

$$\mathcal{L}\{\exp(at)\} = \mathcal{L}\{\exp(at) \times 1\}$$

Let $f(t) = 1$, we know

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s}, \qquad s > 0$$

Thus, by the translator property, we have

$$\mathcal{L}\{\exp(at)\} = F(s-a) = \frac{1}{s-a}, \qquad s > a$$

Example 2

Compute $\mathcal{L}\{t\exp(at)\}$

Solution

Let $f(t) = t$, we have

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s > a$$

By the translator property, we have

$$\mathcal{L}\{t\exp(at)\} = F(s-a) = \frac{1}{(s-a)^2}, \qquad s > a$$

Example 3

Compute $\mathcal{L}\{t^2\exp(at)\}$

Solution

Let $f(t) = t^2$, we have

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} = \frac{2}{s^3}, \qquad s > a$$

By the translator property, we have

$$\mathcal{L}\{t^2\exp(at)\} = F(s-a) = \frac{2}{(s-a)^3}, \qquad s > a$$

In general, we can prove

$$\mathcal{L}\{t^n\exp(at)\} = \frac{n!}{(s-a)^{n+1}}, \qquad s > a$$

From the above Example 1, we can generalize the formula to complex plane

$$\mathcal{L}\{\exp(i\omega t)\} = \frac{1}{s - i\omega}$$
$$\mathcal{L}\{\exp(-i\omega t)\} = \frac{1}{s + i\omega}$$

Example 4

Compute $\mathcal{L}\{\cos\omega t\}$ and $\mathcal{L}\{\sin\omega t\}$

Solution

From Euler's formula, we have

$$\begin{cases} \exp(i\omega t) = \cos\omega t + i\sin\omega t \\ \exp(-i\omega t) = \cos\omega t - i\sin\omega t \end{cases}$$

we have

$$\cos\omega t = \frac{\exp(i\omega t) + \exp(-i\omega t)}{2}$$

Thus,

$$\begin{aligned}
\mathcal{L}\{\cos\omega t\} &= \mathcal{L}\left\{\frac{\exp(i\omega t)+\exp(-i\omega t)}{2}\right\} \\
&= \frac{1}{2}\mathcal{L}\{\exp(i\omega t)\} + \frac{1}{2}\mathcal{L}\{\exp(-i\omega t)\} \\
&= \frac{1}{2}\left(\frac{1}{s-i\omega}\right) + \frac{1}{2}\left(\frac{1}{s+i\omega}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right) \\
&= \frac{s}{s^2+\omega^2}
\end{aligned}$$

Therefore, we have obtained one of the most important transforms,

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

Similarly, for $\sin\omega t$, we have

$$\sin\omega t = \frac{\exp(i\omega t)-\exp(-i\omega t)}{2i}$$

and thus

$$\begin{aligned}
\mathcal{L}\{\sin\omega t\} &= \frac{1}{2i}(\mathcal{L}\{\exp(i\omega t)\} - \mathcal{L}\{\exp(-i\omega t)\}) \\
&= \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

Therefore, we have obtained another one of the most important transforms,

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

Example 5

Compute $\mathcal{L}\{\cosh\omega t\}$

Solution

From definition, we know that

$$\cosh\omega t = \frac{\exp(\omega t)+\exp(-\omega t)}{2}$$

Thus

$$\begin{aligned}
\mathcal{L}\{\cosh\omega t\} &= \mathcal{L}\left\{\frac{\exp(\omega t)+\exp(-\omega t)}{2}\right\} \\
&= \frac{1}{2}\left(\frac{1}{s-\omega} + \frac{1}{s+\omega}\right)
\end{aligned}$$

$$= \frac{s}{s^2 - \omega^2}$$

Similarly, we can have

$$\mathcal{L}\{\sinh \omega t\} = \frac{\omega}{s^2 - \omega^2}$$

Example 6

Compute $\mathcal{L}\{\exp(at) \cos \omega t\}$

Solution

Let $f(t) = \cos \omega t$. From previous discussion, we know that

$$F(s) = \mathcal{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}$$

Therefore, by the translator property, we know that

$$\mathcal{L}\{\exp(at) \cos \omega t\} = F(s - a) = \frac{s - a}{(s - a)^2 + \omega^2}$$

Example 7

Compute $\mathcal{L}\{3 \exp(2t) + 2 \sin^2 3t\}$

Solution

$$\begin{aligned}\mathcal{L}\{3 \exp(2t) + 2 \sin^2 3t\} &= \mathcal{L}\{3 \exp(2t) + 1 - \cos 6t\} \\ &= \frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2 + 36} \\ &= \frac{3s^3 + 144s - 72}{s(s-2)(s^2 + 36)}\end{aligned}$$

5.2.3 Shifting Property

Given a unit step function defined as follows and has a graph given below. (Figure 5.1)

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

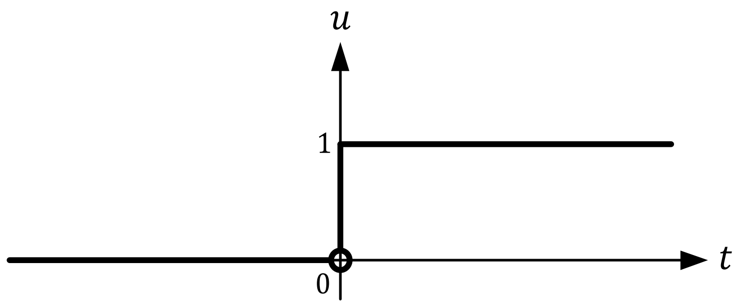

Figure 5.1 The graph of the unit step function $u(t)$.

Since $u(t) = 1$ for $t \geq 0$ and because the Laplace transform involves only the values of a function for $t \geq 0$ we see immediately that

$$\mathcal{L}\{u(t)\} = \frac{1}{s}, \qquad s \geq 0$$

Furthermore, for any function $f(t)$, $F(s) = \mathcal{L}\{f(t)\}$, we know that

$$\mathcal{L}\{f(t)u(t)\} = F(s)$$

The graph of a unit step function $u_a(t) = u(t-a)$ appears in the following figure. Its jump occurs at $t = a$ rather than at $t = 0$. So we have

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

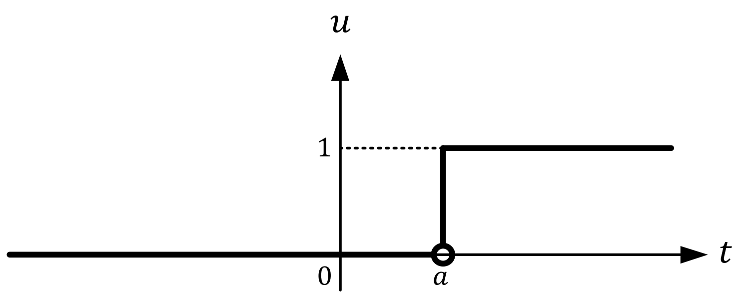

Figure 5.2 The graph of the unit step function $u(t-a)$.

To calculate the Laplace transform of $u_a(t)$, we start from the definition.

$$\begin{aligned}
\mathcal{L}\{u_a(t)\} &= \mathcal{L}\{u(t-a)\} \\
&= \int_0^\infty u(t-a)\exp(-st)\,dt \\
&= \int_0^a 0\cdot\exp(-st)\,dt + \int_a^\infty \exp(-st)\,dt \\
&= -\frac{\exp(-st)}{s}\bigg|_{t=a}^{\infty} \\
&= \frac{\exp(-as)}{s}, \qquad s>0, a>0
\end{aligned}$$

Thus, we get one important transform

$$\mathcal{L}\{u_a(t)\} = \frac{\exp(-as)}{s} \qquad \forall s, a > 0$$

Example 1

Suppose $F(s) = \mathcal{L}\{f(t)\}$, compute $\mathcal{L}\{u(t-a)f(t-a)\}$

Solution

From the definition of Laplace, we have

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\exp(-st)\,dt$$

On the other hand, we have

$$\begin{aligned}
\mathcal{L}\{u(t-a)f(t-a)\} &= \int_0^\infty u(t-a)f(t-a)\exp(-st)\,dt \\
&= \int_0^a 0dt + \int_a^\infty f(t-a)\exp(-st)\,dt \\
&= \exp(-as)\int_a^\infty f(t-a)\exp\big(-s(t-a)\big)\,dt
\end{aligned}$$

Let $\tau = t-a$, we have

$$\begin{aligned}
\mathcal{L}\{u(t-a)f(t-a)\} &= \exp(-as)\int_0^\infty f(\tau)\exp(-s\tau)\,d\tau \\
&= \exp(-as)\,\mathcal{L}\{f(\tau)\} \\
&= \exp(-as)\,F(s)
\end{aligned}$$

Example 2

Compute $\mathcal{L}\{(t-a)u(t-a)\}$

Solution

Let $f(t)=t$, we know that

$$F(s)=\mathcal{L}\{f(t)\}=\frac{1}{s^2}$$

Using the result of example 1, we have

$$\begin{aligned}\mathcal{L}\{(t-a)u(t-a)\} &= \mathcal{L}\{f(t-a)u(t-a)\}\\ &= \exp(-as)\,F(s)\\ &= \frac{\exp(-as)}{s^2}\end{aligned}$$

5.2.4 The *t*-multiplication property

The t-multiplication property, also called the frequency differentiation, is another important property. Given $\mathcal{L}\{f(t)\}=F(s)$, when taking derivative WRT s, we have

$$\frac{d}{ds}\left(\int_0^\infty f(t)\exp(-st)\,dt\right)=\frac{d}{ds}F(s)$$

By exchanging the derivative and integral operation, we have

$$\begin{aligned}\int_0^\infty \frac{d}{ds}(f(t)\exp(-st))dt &= F'(s)\\ -\int_0^\infty tf(t)\exp(-st)\,dt &= F'(s)\end{aligned}$$

Thus,

$$L\{tf(t)\}=\int_0^\infty tf(t)\exp(-st)\,dt=-F'(s)$$

Similarly, we can have

$$\mathcal{L}\{t^2 f(t)\}=F''(s)$$

Generally,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Example 1

Compute $\mathcal{L}\{t \exp(at)\}$

Solution

Method I: By t-multiplication property

Let $f(t) = \exp(at)$, we know from shifting property,

$$F(s) = \mathcal{L}\{\exp(at)\} = \frac{1}{s-a}$$

By applying the t-pultiplication property, we have

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -F'(s) \\ &= -\left(\frac{1}{s-a}\right)' \\ &= \frac{1}{(s-a)^2}\end{aligned}$$

Method II: By shifting property

Let $f(t) = t$, we know that

$$F(s) = \mathcal{L}\{t\} = \frac{1}{s^2}$$

By applying the shifting property, we have

$$\begin{aligned}\mathcal{L}\{\exp(at)\, f(t)\} &= F(s-a) \\ &= \frac{1}{(s-a)^2}\end{aligned}$$

Method III: By differentiation property

Consider $f(t) = t \exp(at)$ and denote $F(s) = \mathcal{L}\{f(t)\}$, we know that

$$f'(t) = \exp(at) + at \exp(at)$$

Perform Laplace transform on both sides, we have

$$\mathcal{L}\{f'(t)\} = \mathcal{L}\{\exp(at)\} + a\mathcal{L}\{f(t)\}$$

By applying differentiation property, we have

$$sF(s) - f(0) = \frac{1}{s-a} + aF(s)$$

Solving this algebraic equation, we can get

$$F(s) = \frac{1}{(s-a)^2} + \frac{f(0)}{s-a}$$

Since $f(0) = 0$, we final have

$$F(s) = \frac{1}{(s-a)^2}$$

Example 2

Compute $L\{t \sin \omega t\}$

Solution

Method I: By t-multiplication property

Let $f(t) = \sin \omega t$, then we know

$$F(s) = \mathcal{L}\{f(t)\} = \frac{\omega}{s^2 + \omega^2}$$

Applying the t-multiplication property, we have

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -F'(s) \\ &= -\left(\frac{\omega}{s^2 + \omega^2}\right)' \\ &= \frac{2\omega s}{(s^2 + \omega^2)^2}\end{aligned}$$

That is

$$\mathcal{L}\{t \sin \omega t\} = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

Method II: By differentiation property

Let $f(t) = t \sin \omega t$ and denote $F(s) = \mathcal{L}\{f(t)\}$. By taking derivative, we have

$$f'(t) = \sin \omega t + \omega t \cos \omega t$$

and

$$f''(t) = 2\omega \cos \omega t - \omega^2 t \sin \omega t$$

Perform Laplace transform on $f''(t)$, we can get

$$\mathcal{L}\{f''(t)\} = 2\omega L\{\cos \omega t\} - \omega^2 \mathcal{L}\{t \sin \omega t\}$$

Applying the differentiation property, this becomes

$$s^2 F(s) - sf'(0) - f(0) = \frac{2\omega s}{s^2 + \omega^2} - \omega^2 F(s)$$

It is easy to see that $f'(0) = f(0) = 0$. Thus, we can get

$$F(s) = \frac{2\omega s}{(s^2 + \omega^2)^2}$$

Remarks

Similarly, we will have,

$$\begin{aligned}\mathcal{L}\{t^2 \sin \omega t\} &= \frac{d^2}{ds^2}\left(\frac{\omega}{s^2 + \omega^2}\right) \\ &= \frac{6\omega s^2 - 2\omega^3}{(s^2 + \omega^2)^3}\end{aligned}$$

Example 3

Compute $\mathcal{L}\{t\cos\omega t\}$ and $\mathcal{L}\{t^2\cos\omega t\}$

Solution

Similar to example 2, we have

$$\begin{aligned}\mathcal{L}\{t\cos\omega t\} &= -\left(\frac{s}{s^2+\omega^2}\right)' \\ &= \frac{2s^2-s^2-\omega^2}{(s^2+\omega^2)^2}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\{t^2\cos\omega t\} &= \left(\frac{s}{s^2+\omega^2}\right)'' \\ &= \frac{2s^3-6s\omega^2}{(s^2+\omega^2)^3}\end{aligned}$$

5.2.5 Periodic Functions

For a periodic function $f(t)$ with period T that can be defined as

$$f(t) = f(t+T) \quad \forall\, t \geq 0$$

The Laplace transform can be obtained by the original definition

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty f(t)\exp(-st)\,dt \\ &= \int_0^T f(t)\exp(-st)\,dt + \int_T^{2T} f(t)\exp(-st)\,dt \\ &\qquad + \int_{2T}^{3T} f(t)\exp(-st)\,dt \ldots \\ &= \int_0^T f(t)\exp(-st)\,dt + \int_0^T f(t+T)\exp\big(-s(t+T)\big)\,dt \\ &\qquad + \int_0^T f(t+2T) + \exp\big(-s(t+2T)\big)\,dt \ldots \\ &= \int_0^T f(t)\exp(-st)\,dt\,(1+\exp(-sT)+\exp(-s2T)+\cdots)\end{aligned}$$

$$= \frac{1}{1 - \exp(-sT)} \int_0^T f(t) \exp(-st)\, dt$$

Therefore, for periodic functions, one only needs to compute the transform for the first period and then multiply the transform by a period-dependent factor given above.

5.2.6 Differentiation and Integration Property

In order to utilize Laplace transforms to solve DEs, we need to be able to perform Laplace transforms on differentiation and integration.

Let us first consider $\mathcal{L}\{f'(t)\}$, where $F(s) = \mathcal{L}\{f(t)\}$. By using integration by part, we have

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= \int_0^\infty f'(t) \exp(-st)\, dt \\
&= \exp(-st)\, f(t)|_0^\infty - \int_0^\infty f(t) d(\exp(-st)) \\
&= \exp(-s \times \infty)\, f(\infty) - \exp(-s \times 0)\, f(0) \\
&\qquad + s \int_0^\infty f(t) \exp(-st)\, dt
\end{aligned}$$

since $F(s) = \int_0^\infty f(t) \exp(-st)\, dt$, we have

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Starting from this result, we can derive the Laplace transform for higher order derivatives.

For $\mathcal{L}\{f''(t)\}$, we have

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \mathcal{L}\left\{\left(f'(t)\right)'\right\} \\
&= s\mathcal{L}\{f'(t)\} - f'(0) \\
&= s[sF(s) - f(0)] - f'(0)
\end{aligned}$$

Thus

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Similarly,

$$\begin{aligned}\mathcal{L}\{f'''(t)\} &= s\mathcal{L}\{f''(t)\} - f''(0) \\ &= s^3F(s) - s^2f(0) - sf'(0) - f''(0)\end{aligned}$$

In the same manner we can generalize for $f^{(n)}(t)$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

Laplace transform for integration, e.g. $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}$, is a little more complicated.

Consider

$$g(t) = \int_0^t f(\tau)d\tau$$

and we denote

$$G(s) = \mathcal{L}\{g(t)\}$$

and

$$F(s) = \mathcal{L}\{f(t)\}$$

From previous discussion, we know that

$$\mathcal{L}\{g'(t)\} = sG(s) - g(0)$$

Since $g'(t) = f(t)$, this means

$$sG(s) - g(0) = F(s)$$

This gives

$$G(s) = \frac{F(s) + g(0)}{s}$$

It is easy to calculate

$$g(0) = \int_0^0 f(\tau)d\tau = 0$$

Hence, we have

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

where

$$F(s) = \mathcal{L}\{f(t)\}$$

Similarly, we have

$$\mathcal{L}\left\{\iint f(t_1)dt_1\,dt_2\right\} = \frac{F(s)}{s^2}$$

and

$$\mathcal{L}\left\{\iiint f(t_1)dt_1dt_2dt_3\right\} = \frac{F(s)}{s^3}$$

Problems

Problem 5.2.1 The graph of a square wave function $g(t)$ is shown in Figure 5.3. Express $g(t)$ in terms of the function

$$f(t) = \sum_{n=0}^{\infty}(-1)^n u(t-n)$$

and hence deduce that

$$\mathcal{L}\{g(t)\} = \frac{1-\exp(-s)}{s(1+\exp(-s))} = \frac{1}{s}\tanh\frac{s}{2}$$

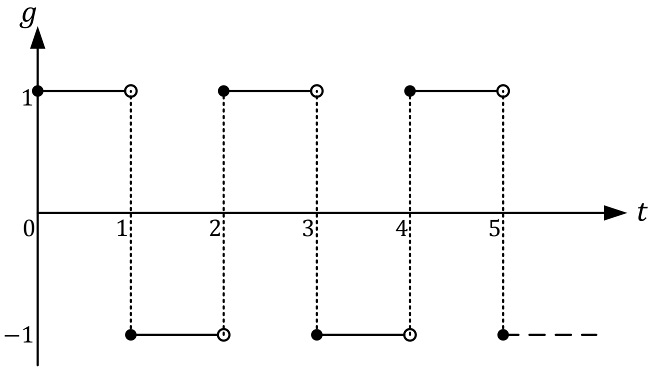

Figure 5.3 The square wave function for Problem 5.2.1.

Problem 5.2.2 Compute the Laplace Transform

$$\mathcal{L}\left\{\frac{d}{dt}\left(t^2 \exp(\alpha t)\sin(\omega t)\right)\right\}$$

Problem 5.2.3 Compute the Laplace Transform

$$\mathcal{L}\left\{\exp(t)\int_0^{\sqrt{t}} \exp(-\tau^2)\, d\tau\right\}$$

Problem 5.2.4 Compute the Laplace Transform of the following function.

$$f(t) = t^2 \sin(\omega_1 t) + \exp(\alpha t)\cos(\omega_2 t)$$

Problem 5.2.5 Compute the Laplace Transforms of the following un-related functions.

$$f_1(t) = \sum_{n=0}^{\infty} u(t-n)$$

$$f_2(t) = t - \lfloor t \rfloor$$

In both cases above, $t > 0$ and $\lfloor t \rfloor$ is the floor function of t and $u(t-n)$ is the usual step function.

Problem 5.2.6 Laplace transform is defined as

$$F(s) = \int_0^{\infty} f(t)\exp(-st)\,dt$$

Performing Laplace transform on the following functions:

(1) $f(t) = t^5$

(2) $f(t) = \exp(at)$

(3) $f(t) = \sin(\omega t)$

(4) $f(t) = \exp(at)\sin(\omega t)$

Problem 5.2.7 Compute the Laplace transform of the following triangular wave function shown in Figure 5.4.

Hint: Obviously, the function during the 1st peorid can be written as

$$f(t) = \begin{cases} 2A\left(\dfrac{t}{T}\right), & 0 \le t \le \dfrac{T}{2} \\ 2A\left(1 - \dfrac{t}{T}\right), & \dfrac{T}{2} \le t \le T \end{cases}$$

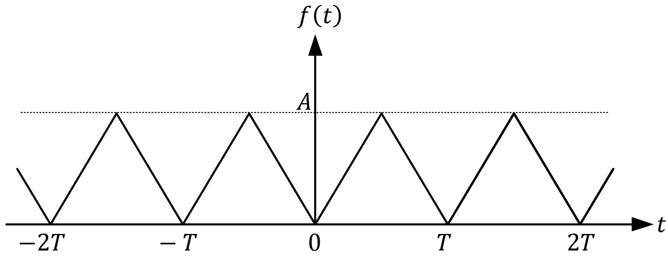

Figure 5.4 The triangular wave function for Problem 5.2.7.

5.3 Inverse Laplace Transforms

We introduce the inverse Laplace transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

where

$$F(s) = \mathcal{L}\{f(t)\}$$

Theoretically, we have to calculate the Bromwich integral (also called the Fourier-Mellin integral) to find the inverse Laplace transform.

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma - i\cdot\infty}^{\gamma + i\cdot\infty} \exp(st)\, F(s) ds$$

where $\gamma > R(s_p)$ for every singularity s_p of $F(s)$.

To actually calculate this integral, one needs to use the Cauchy residual theorem. However, for most cases, we can find out the inverse Laplace transform by simply looping up in the Laplace transform table.

Example 1

Compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s-a}\right\}$$

Solution

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s-a}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} \\ &= t - \exp(at)\end{aligned}$$

Example 2

Compute

$$\mathcal{L}^{-1}\left\{\tan^{-1}\frac{1}{s}\right\}$$

Solution

Notice that the derivative of $\tan^{-1}\frac{1}{s}$ is a simple rational function

$$\frac{d}{ds}\tan^{-1}\frac{1}{s} = -\frac{1}{s^2+1}$$

We can find the inverse Laplace transform by applying the t-multiplication property.

Denote

$$F(s) = \tan^{-1}\frac{1}{s} \text{ and } f(s) = L^{-1}\{F(s)\}$$

Since

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

We know that

$$\begin{aligned} f(t) &= -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\} \\ &= -\frac{1}{t}\mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\} \\ &= \frac{\sin t}{t} \end{aligned}$$

Example 3

Compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\}$$

Solution

Method I: Using the integration property

From the integration property, we have

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

That is

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

where

$$F(s) = \mathcal{L}\{f(t)\}$$

Now let

$$F(s) = \frac{1}{s^2(s-a)}$$

$$G(s) = \frac{1}{s(s-a)}$$

$$H(s) = \frac{1}{s-a}$$

We know that

$$\begin{aligned} h(t) &= L^{-1}\{H(s)\} \\ &= \exp(at) \end{aligned}$$

and

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\{G(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{H(s)}{s}\right\} \\ &= \int_0^t h(\tau)d\tau \\ &= \int_0^t \exp(a\tau)\,d\tau \\ &= \frac{1}{a}\exp(a\tau)\Big|_0^t \\ &= \frac{1}{a}(\exp(a\tau) - 1) \end{aligned}$$

Finally, we have

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} \\ &= \int_0^t \frac{1}{a}(\exp(a\tau) - 1)d\tau \\ &= \frac{1}{a^2}(\exp(at) - at - 1) \end{aligned}$$

<u>Method II: Using partial fractions</u>

Let

$$\frac{1}{s^2(s-a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-a}$$

We have

$$\begin{aligned} 1 &= (As+B)(s-a) + Cs^2 \\ &= (A+C)s^2 + (B-Aa)s - Ba \end{aligned}$$

This gives

$$A = -\frac{1}{a^2}, B = -\frac{1}{a} \text{ and } C = \frac{1}{a^2}$$

which means

$$\frac{1}{s^2(s-a)} = -\frac{1}{a^2}\cdot\frac{1}{s} - \frac{1}{a}\cdot\frac{1}{s^2} + \frac{1}{a^2}\cdot\frac{1}{s-a}$$

By linearity, we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} &= -\frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{a}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{a^2}\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} \\ &= -\frac{1}{a^2} - \frac{t}{a} + \frac{\exp(at)}{a^2} \\ &= \frac{\exp(at) - at - 1}{a^2}\end{aligned}$$

Problems

Problem 5.3.1 Prove that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+l^2)^2}\right\} = \frac{1}{2k^3}(\sin kt - kt\cos kt)$$

If given

$$\begin{aligned}\mathcal{L}\{k\cos kt\} &= \frac{s^2-k^2}{(s^2+k^2)^2} \\ \mathcal{L}\{\sin kt\} &= \frac{k}{s^2+k^2}\end{aligned}$$

Problem 5.3.2 Show the relationship

$$\mathcal{L}^{-1}\left\{\frac{\exp\left(-\frac{1}{s}\right)}{s}\right\} = J_0(2\sqrt{t})$$

where $J_0(t)$ is the so-called Bessel's function defined as

$$J_0(2\sqrt{t}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2\cdot n!}(\sqrt{t})^{2n}$$

Problem 5.3.3 Perform the following inverse transform.

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\}$$

Problem 5.3.4 Perform the following inverse transform.

$$\mathcal{L}^{-1}\left\{\frac{\exp(-as)}{s(1-\exp(-as))}\right\}$$

Problem 5.3.5 The following equation is one of the so-called Volterra Integral Equations containing obviously the unknown function $X(t)$:

$$x(t) = \sin(t) + 2\int_0^t \cos(t-\tau)\, x(\tau) d\tau$$

Use Laplace Transform to solve this equation by completing the following steps.

(1) Use Laplace Transform to convert the above equation to an algebraic equation containing unknown function in Laplace space $X(s)$ with given formula

$$\mathcal{L}\left\{\int_0^t \cos(t-\tau)\, x(\tau) d\tau\right\} = \frac{s}{s+1} X(s)$$

(2) Solve the algebraic equation for $X(s)$.

(3) Inverse transform to obtain the unknown function $x(t)$ as a function of *t*.

Problem 5.3.6 Solve the following Volterra Integral Equations.

$$x(t) = 2\exp(3t) - \int_0^t \exp\big(2(t-\tau)\big)\, x(\tau) d\tau$$

5.4 The Convolution of Two Functions

Consider two functions $f(t)$ and $g(t)$. Let us define a binary operation

$$f(t)\otimes g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

which is called the convolution of the functions $f(t)$ and $g(t)$. This operation has elegant yet useful property under Laplace transform. This property is called Convolution Theorem.

Convolution Theorem

Denoting $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, we have

$$\mathcal{L}\{f(t)\otimes g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

Proof

$$\begin{aligned} f(t)\otimes g(t) &= \int_0^t f(\tau)g(t-\tau)d\tau \\ &= \int_0^t f(\tau)g(t-\tau)d\tau + \int_t^\infty 0\times f(\tau)g(t-\tau)d\tau \\ &= \int_0^\infty u(t-\tau)f(\tau)g(t-\tau)d\tau \end{aligned}$$

where $u(t-\tau)$ is the unit step function. Thus, we have

$$\begin{aligned} \mathcal{L}\{f(t)\otimes g(t)\} &= \int_0^\infty \exp(-st)\, dt \int_0^\infty u(t-\tau)f(\tau)g(t-\tau)d\tau \\ &= \int_0^\infty \int_0^\infty \exp(-st)\, u(t-\tau)f(\tau)g(t-\tau)dtd\tau \end{aligned}$$

Introducing a new variable $t_1 = t-\tau$ to replace t, we have

$$\mathcal{L}\{f(t)\otimes g(t)\} = \int_0^\infty \int_0^\infty \exp[-s(t_1+\tau)]\, u(t_1)f(\tau)g(t_1)d\tau dt_1$$

$$
\begin{aligned}
&= \int_0^\infty \exp(-st_1)\ u(t_1)g(t_1)dt_1 \\
&= \int_0^\infty \exp(-st_1)\ g(t_1)dt_1 \int_0^\infty \exp(-s\tau)\, f(\tau)d\tau \\
&= F(s) \times G(s)
\end{aligned}
$$

Conversely,

$$\mathcal{L}^{-1}\{F(s) \times G(s)\} = f(t) \otimes g(t)$$

Thus we can find the inverse transform of the usual product of two functions $F(s)G(s)$ provided we can evaluate the convolution of $f(t)$ and $g(t)$.

Example 1

Compute $(\cos t) \otimes (\sin t)$

Solution

$$\cos t \otimes \sin t = \int_0^t \cos(t-\tau) \sin \tau \, d\tau$$

Applying the trigonometric identity

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

These can be written as

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

Thus,

$$
\begin{aligned}
\cos t \otimes \sin t &= \frac{1}{2}\int_0^t (\sin t - \sin(t-2\tau))d\tau \\
&= \frac{1}{2}\left(\tau \sin t - \frac{1}{2}\cos(t-2\tau)\right)\Bigg|_{\tau=0}^{t} \\
&= \frac{1}{2} t \sin t
\end{aligned}
$$

So,

$$\cos t \otimes \sin t = \frac{1}{2} t \sin t$$

Naturally,

$$\cos t \otimes \sin t \neq \cos t \times \sin t$$

Example 2

Compute

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\}$$

Solution

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s-a}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\otimes\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} \\
&= t\otimes\exp(at) \\
&= \int_0^t \exp(a\tau)\,(t-\tau)d\tau \\
&= \int_0^t (t\exp(a\tau)-\tau\exp(a\tau))d\tau \\
&= t\int_0^t \exp(a\tau)\,d\tau - \int_0^t \frac{\tau}{a}d(\exp(a\tau)) \\
&= \left(\frac{t}{a}\exp(a\tau)-\frac{\tau}{a}\exp(a\tau)+\frac{1}{a^2}\exp(a\tau)\right)\bigg|_{\tau=0}^{t} \\
&= \frac{1}{a^2}(\exp(at)-at-1)
\end{aligned}$$

Consistent with results obtained earlier in Example 3 of the previous section.

Problems

Problem 5.4.1 Use Convolution Theorem to prove the following. Identify

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)\sqrt{s}}\right\} = \exp(t)\,\mathrm{erf}\left(\sqrt{t}\right)$$

where $\mathcal{L}^{-1}$ denotes an inverse Laplace Transform and $\mathrm{erf}(x)$ is the so-called "error function" defined by

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)\,du$$

(Hint: substitute $u=\sqrt{t}$*.)*

Problem 5.4.2 Solve the following DE.

$$y(x) = x\cos(3x) - \int_0^x \exp(\tau)\, y(x-\tau)\, d\tau$$

5.5 Application of Laplace Transforms

Laplace transforms can help solve some of the linear DEs (and some special nonlinear DEs) much more conveniently. Starting from Section 5.2.4, we demonstrate such values.

First, we must use these key formulas

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$
$$\cdots \cdots$$
$$\begin{aligned}\mathcal{L}\left\{f^{(n)}(t)\right\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0) \\ &= s^n\mathcal{L}\{f(t)\} - \sum_{i=1}^{n} s^{n-i} f^{(i-1)}(0)\end{aligned}$$

Thus, for a linear DE of the following form

$$\sum_{i=0}^{n} a_i f^{(i)}(t) = \phi(t)$$

applying Laplace transforms on both sides of the equation gives

$$\sum_{i=0}^{n} a_i \mathcal{L}\left\{f^{(i)}(t)\right\} = \mathcal{L}\{\phi(t)\}$$

By applying the differentiation property, we can write the equation as

$$\mathcal{L}\{f(t)\} = \frac{\mathcal{L}\{\phi(t)\} + \sum_{i=0}^{n} a_i \sum_{j=1}^{i} s^{i-j} f^{(j-1)}(0)}{\sum_{i=0}^{n} a_i s^i}$$

where $f^{(k)}(0)$ are I.C..

Finally, the inverse Laplace transform yields the solution to the original DE.

$$f(t) = \mathcal{L}^{-1}\left\{\frac{L\{\phi(t)\} + \sum_{i=0}^{n} a_i \sum_{j=1}^{i} s^{i-j} f^{(j-1)}(0)}{\sum_{i=0}^{n} a_i s^i}\right\}$$

Example 1

Compute $x'' - x' - 6x = 0$ for $x(0) = 2$ and $x'(0) = -1$

Solution

Applying Laplace transforms to both sides of the DE, we can get

$$\mathcal{L}\{x'' - x' - 6x\} = L\{0\}$$
$$\mathcal{L}\{x''\} - L\{x'\} - 6L\{x\} = 0$$
$$\left(s^2X(s) - sx(0) - x'(0)\right) - \left(sX(s) - x(0)\right) - 6X(s) = 0$$

Plug in the I.C., and we can get

$$s^2X(s) - 2s + 1 - sX(s) + 2 - 6X(s) = 0$$
$$X(s)(s^2 - s - 6) - 2s + 3 = 0$$
$$X(s) = \frac{2s - 3}{s^2 - s - 6}$$

By partial fraction decomposition we have,

$$X(s) = \frac{7}{5} \cdot \frac{1}{s+2} + \frac{3}{5} \cdot \frac{1}{s-3}$$

Thus, we can find the solution of the original DE by inverse Laplace transform:

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{7}{5} \cdot \frac{1}{s+2} + \frac{3}{5} \cdot \frac{1}{s-3}\right\} \\ &= \frac{7}{5}\exp(-2t) + \frac{3}{5}\exp(3t) \end{aligned}$$

Example 2

Compute $x'' + 4x = \sin 3t$ for $x(0) = x'(0) = 0$

Solution

Such a problem arises in the motion of a mass and spring system with external force as shown in Figure 5.5.

Apply Laplace transforms on both sides of the DE

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{\sin 3t\}$$

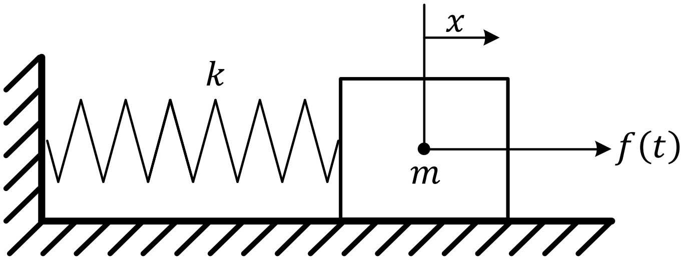

Figure 5.5 A block-spring system with an external force $f(t)$.

$$\left(s^2X(s) - sx(0) - x'(0)\right) + 4X(s) = \frac{3}{s^2+3^2}$$

Plugging in the I.C., we can get

$$X(s)(s^2+4) = \frac{3}{s^2+9}$$
$$X(s) = \frac{3}{(s^2+4)(s^2+9)}$$

Finding the partial fraction

$$X(s) = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

Solving for A, B, C and D gives

$$A = C = 0, \qquad B = \frac{3}{5}, \text{ and } D = -\frac{3}{5}$$

Thus, we get

$$\begin{aligned} X(s) &= \frac{3}{5}\cdot\frac{1}{s^2+4} - \frac{3}{5}\cdot\frac{1}{s^2+9} \\ &= \frac{3}{10}\cdot\frac{2}{s^2+2^2} - \frac{1}{5}\cdot\frac{3}{s^2+3^2} \end{aligned}$$

Finally, inverse Laplace transform generates the solution for the original DE

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{3}{10}\cdot\frac{2}{s^2+2^2} - \frac{1}{5}\cdot\frac{3}{s^2+3^2}\right\} \\ &= \frac{3}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} \\ &= \frac{3}{10}\sin 2t - \frac{1}{5}\sin 3t \end{aligned}$$

Example 3

Solve IVP

$$\begin{cases} 2x'' = -6x + 2y \\ y'' = 2x - 2y + 40\sin 3t \\ x(0) = x'(0) = y(0) = y'(0) = 0 \end{cases}$$

which can be concluded from the following spring system. (Figure 5.6)

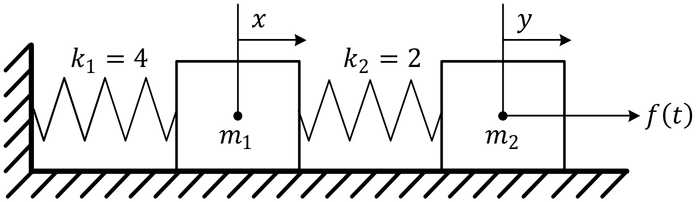

Figure 5.6 A system of two blocks connected by springs and an external force on $f(t)$.

Solution

From the I.C. we can get

$$\mathcal{L}\{x''(t)\} = s^2X(s)$$
$$\mathcal{L}\{y''(t)\} = s^2Y(s)$$

Performing Laplace transform on both sides of the original DEs, we have,

$$2s^2X(s) = -6X(s) + 2Y(s)$$
$$s^2Y(s) = 2X(s) - 2Y(s) + 40\left(\frac{3}{s^2+9}\right)$$

where we have used

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

Resulting two algebraic equations defined in the s-space:

$$\begin{cases} (s^2+3)X(s) - Y(s) = 0 \\ -2X(s) + (s^2+2)Y(s) = \dfrac{120}{s^2+9} \end{cases}$$

Substituting the first equation

$$Y(s) = (s^2+3)X(s)$$

To the second, we get

$$X(s) = \frac{120}{(s^2+9)\big((s^2+2)(s^2+3)-2\big)}$$
$$= \frac{120}{(s^2+1)(s^2+4)(s^2+9)}$$

Plugging this back to the first equation, we have

$$Y(s) = \frac{120(s^2+3)}{(s^2+1)(s^2+4)(s^2+9)}$$

Next, express $X(s)$ and $Y(s)$ using partial fractions, for example for $X(s)$ we let

$$\frac{120}{(s^2+1)(s^2+4)(s^2+9)} = \frac{A}{s^2+1} + \frac{B}{s^2+4} + \frac{C}{s^2+9}$$

Substitution of $s^2 = -1$ in the above equation gives $A = 5$. Similarly, we can get $B = -8$ and $C = 3$. Thus,

$$\begin{aligned} X(s) &= \frac{5}{s^2+1} - \frac{8}{s^2+4} + \frac{3}{s^2+9} \\ &= 5 \cdot \frac{1}{s^2+1} - 4 \cdot \frac{2}{s^2+4} + \frac{3}{s^2+9} \end{aligned}$$

Inverse Laplace transform produces

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} \\ &= 5\sin t - 4\sin 2t + \sin 3t \end{aligned}$$

Similarly,

$$y(t) = 10\sin t + 4\sin 2t - 6\sin 3t$$

Example 4

Solve IVP for the Bessel's equation of order 0.

$$\begin{cases} tx'' + x' + tx = 0 \\ x(0) = 1 \ \text{ and } x'(0) = 0 \end{cases}$$

Solution

The I.C. give us

$$\mathcal{L}\{x'(t)\} = sX(s) - 1$$
$$\mathcal{L}\{x''(t)\} = s^2X(s) - s$$

Because x and x'' are each multiplied by t, by applying the t-multiplication property, we can get the transformed equation as

$$-\frac{d}{ds}(s^2X(s) - s) + (sX(s) - 1) - \frac{d}{ds}\big(X(s)\big) = 0$$

The result of differentiation and simplification is the DE

$$(s^2+1)X'(s) + sX(s) = 0$$

Since the DE is separable we have,

$$\frac{X'(s)}{X(s)} = -\frac{s}{s^2+1}$$

and its G.S. is

$$X(s) = \frac{C}{\sqrt{s^2+1}}$$

Remarks

(1) Here the constant C in the G.S. of $X(s)$ is actually not an arbitrary number. Let $s = 0$, we have

$$X(0) = C$$

On the other hand, from definition of Laplace transform,

$$X(0) = \int_0^\infty x(t)dt$$

This means C is an normalization factor.

(2) The solution to the original DE, i.e., the inverse Laplace transform for

$$X(s) = \frac{1}{\sqrt{s^2+1}}$$

is a function called zero order Bessel function of first kind. A general form of α order Bessel function is defined as

$$J_0(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Example 5

Solve IVP.

$$\begin{cases} x'' + 4x = 2\exp(t) \\ x(0) = x'(0) = 0 \end{cases}$$

Solution

Take Laplace Transform on both sides of the DE

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{2\exp(t)\}$$

From the I.C., we have

$$(s^2+4)X(s) = \frac{2}{s-1}$$

$$X(s) = \left(\frac{2}{s-1}\right)\left(\frac{1}{s^2+4}\right)$$

Therefore, we know that the solution to the original DE is

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} \\ &= \mathcal{L}^{-1}\left\{\left(\frac{1}{s-1}\right)\left(\frac{2}{s^2+4}\right)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \otimes \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \exp(t) \otimes \sin 2t \end{aligned}$$

Since

$$\exp(t) \otimes \sin 2t = \int_0^t \exp(t-\tau)\sin 2\tau\, d\tau$$

$$= \exp(t) \int_0^t \exp(-\tau) \sin 2\tau \, d\tau$$

Let

$$I = \int_0^t \exp(-\tau) \sin 2\tau \, d\tau$$

By applying integration by part repeatedly, we have

$$\begin{aligned} I &= -\int_0^t \sin 2\tau \, d(\exp(-\tau)) \\ &= -\exp(-\tau) \sin 2\tau|_0^t + 2\int_0^t \exp(-\tau) \cos 2\tau \, d\tau \\ &= -\exp(-\tau) \sin 2t - 2\int_0^t \cos 2\tau \, d(\exp(-\tau)) \\ &= -\exp(-\tau) \sin 2t - 2\exp(-\tau) \cos 2\tau \,|_0^t + 4\int_0^t \exp(-\tau) \sin 2\tau \, d\tau \\ &= -\exp(-t) \sin 2t - 2\exp(-t) \cos 2t + 2 - 4I \end{aligned}$$

This gives

$$I = -\frac{1}{5}\exp(-t) \sin 2t - \frac{2}{5}\exp(-t) \cos 2t + \frac{2}{5}$$

and finally

$$\begin{aligned} x(t) &= \exp(-t) \cdot I \\ &= \frac{2}{5}\exp(-t) - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t \end{aligned}$$

Problems

Problem 5.5.1 Apply the convolution theorem to derive the indicated solution $x(t)$ of the given DE with I.C. $x(0) = x'(0) = 0$.

$$x'' + 4x = f(t); \quad x(t) = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau d\tau$$

Problem 5.5.2 Solve the following integral-DE and you may not leave the solution at a convolution form.

$$\begin{cases} x'(t) + 2x(t) - 4\int_0^t \exp(t-\tau)\, x(\tau) d\tau = \sin t \\ x(0) = 0 \end{cases}$$

Problem 5.5.3 Find the G.S. of the following DEs.

$$\begin{cases} x'' = -4x + \sin t \\ y'' = 4x - 8y \end{cases}$$

Problem 5.5.4 Use Laplace Transform to find the P.S. of the following IVP.

$$\begin{cases} x''' + x'' - 6x' = 0 \\ x(0) = 0 \\ x'(0) = x''(0) = 1 \end{cases}$$

Problem 5.5.5 Solve the IVP

$$\begin{cases} x'' + 4x = -5\delta(t-3) \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$$

where $\delta(t-a)$ is the so-called Delta-function which is defined as

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a \end{cases}$$

with given properties

$$\int_{\infty}^{\infty} \delta(t-a)dt = 1$$

and

$$\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$$

Problem 5.5.6 Solve the following IVP.

$$\begin{cases} x'' + 6x' + 8x = -\delta(t-2) \\ x(0) = 1 \\ x'(0) = 0 \end{cases}$$

Problem 5.5.7 Solve the following IVP.

$$\begin{cases} x'' + 2x' + x = \delta(t) - \delta(t-2) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Problem 5.5.8 Solve the following IVP.

$$\begin{cases} x'' + \omega^2 x = \sum_{n=0}^{\infty} \delta(t - 2nt_0) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Problem 5.5.9 Solve the following IVP.

$$\begin{cases} x'' + 2x' + x = u(t-a) + \delta(t-b) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Problem 5.5.10 Solve the following system of IVP.

$$\begin{cases} x' = x + 2y \\ y' = 2x - 2y \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

Problem 5.5.11 Solve the following system of IVP.

$$\begin{cases} x_1'' = -2x_1 + x_2 + \delta(t - \tau) \\ x_2'' = x_1 - x_2 + \delta(t - 2\tau) \\ x_1(0) = x_1'(0) = 0 \\ x_2(0) = x_2'(0) = 0 \end{cases}$$

Problem 5.5.12 Solve the following systems of IVP.

$$\begin{cases} x'(t) + 4x(t) + 6x(t) \otimes \exp(t) = \sin(\omega t) \\ x(0) = 0 \end{cases}$$

Problem 5.5.13 Solve the following system of DEs.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t \\ \exp(t) \end{bmatrix}$$

Problem 5.5.14 Use Laplace Transform to find the P.S. of the following systems.

$$\begin{cases} x'' - 6x' + 8x = 2 \\ x(0) = x'(0) = 0 \end{cases}$$

Problem 5.5.15 Solve the following DE.

$$\begin{cases} tx'' + 2(t-1)x' - 2x = 0 \\ x(0) = x'(0) = 0 \end{cases}$$

Problem 5.5.16 Solve the following DE.

$$\begin{cases} tx'' + x' + tx = 0 \\ x(0) = \alpha = \text{constant} \\ x'(0) = 0 \end{cases}$$

Problem 5.5.17 Solve the following DE.

$$\begin{cases} x'' + 2x' + x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

Problem 5.5.18 Use Laplace Transform to find the P.S. of the following IVP.

$$\begin{cases} x'' + 4x' + 13x = t\exp(-t) \\ x(0) = 0 \\ x'(0) = 2 \end{cases}$$

Problem 5.5.19 Use Laplace Transform and another method to solve the following DE and compare the results.

$$\begin{cases} x'' - x' - 12x = \sin 4t + \exp(3t) \\ x(0) = 2, x'(0) = 1 \end{cases}$$

Problem 5.5.20 Solve the following DE (where x_0, v_0 and ω are given constants) by: (1) Variation of Parameters; (2) Laplace Transform.

$$\begin{cases} x'' + \omega^2 x = f(t) \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

Problem 5.5.21 Solve the following IVP

$$\begin{cases} x'' + \omega_1^2 x = \sin(\omega_2 t) \\ x(0) = 0, x'(0) = 0 \end{cases}$$

by

(1) Any method of your choice except Laplace Transform.

(2) The method of Laplace Transform.

Problem 5.5.22 Solve the following DE using two different methods:

$$\begin{cases} x' - x = 1 - (t-1)u(t-1) \\ x(0) = 0 \end{cases}$$

where $u(t-1)$ is the so-called Step Function defined as

$$u(t-1) = \begin{cases} 0, t \le 1 \\ 1, t \ge 1 \end{cases}$$

(1) Laplace Transform method.

(2) Any other method of your choice.

Problem 5.5.23 Solve the following IVP

$$\begin{cases} x''(t) + 4x(t) = \big(1 - u(t-2\pi)\big)\cos 2t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Problem 5.5.24 Solve the following DE by your favorite method:

$$\begin{cases} x''(t) + x(t) = (-1)^{[\![t]\!]} \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

where $[\![t]\!]$ denotes the greatest integer not exceeding $[\![t]\!]$, e.g., $[\![0.918]\!] = 0$; $[\![1.234]\!] = 1$; $[\![1989.64]\!] = 1989$. Your final answer may not contain the convolution operator.

Problem 5.5.25 The motion of a particle in the plane can be described by

$$\begin{cases} x'' + \omega^2 x = b_0 \sin\omega_0 t \\ y'' + \omega^2 y = b_0 \cos\omega_0 t \end{cases}$$

where ω, ω_0, b_0 are all constants. Initially, the particle is placed at the origin at rest. Find the trajectories of the particle for

(1) $\omega = \omega_0$

(2) $\omega \neq \omega_0$

Problem 5.5.26 A block of mass m is attached to a mass-less spring of spring constant k, and they are placed on a horizontal and perfectly smooth bench. During the first $\frac{1}{2}\tau$ time, we add a constant force f to the block from left to right. During the second $\frac{1}{2}\tau$ time, the force direction is reversed but its constant magnitude retains. Repeat this process till eternity. Find the displacement of the block as a function of time. The initial displacement and speed can be set to zero.

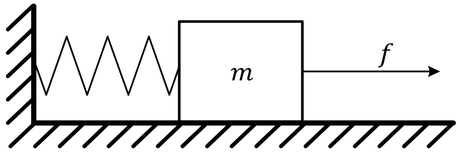

Figure 5.7 The block-spring system for Problem 5.5.26 & Problem 5.5.27.

Problem 5.5.27 A block is attached to a mass-less spring of spring and they are placed on a horizontal and perfectly smooth bench. We add a force $f(t) = \cos 2t$ to the block during $t \in [0, 2\pi]$ and remove it at all other times (The spring is still there). The equation of motion of the block is

$$\begin{cases} x'' + 4x = f(t) \\ x(0) = x'(0) = 0 \end{cases}$$

Solve the equation.

Problem 5.5.28 A block of mass m is attached to two mass-less springs of spring constants k_1 and k_2 and they are placed on a horizontal and perfectly smooth bench. During $\tau/2$ time, we add a constant force f to the block from left to right. At the end of the $\tau/2$ time, immediately, we reverse the direction of the force (but keep the magnitude) and act on the block for another $\tau/2$ time. Then, we reverse the force and keep repeating this process forever. Find the displacement of the block as a function of time.

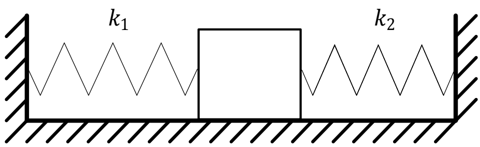

Figure 5.8 The block-spring system for Problem 5.5.28.

Problem 5.5.29 Two blocks (A & B) of the same mass m are attached to three identical mass-less springs (as shown). The assembly is placed on a horizontal and perfectly smooth bench. Initially, springs stay at their natural lengths and both blocks are at rest. During two brief moments, two forces were added to the blocks respectively:

$$f_A(t) = \begin{cases} f_0 & t \in [0, \tau/2] \\ 0 & \text{O.W.} \end{cases} \quad \text{and} \quad f_B(t) = \begin{cases} f_0 & t \in [\tau/2, \tau] \\ 0 & \text{O.W.} \end{cases}$$

Compute the displacements of the blocks as a function of time. All mentioned parameters, including the spring constant k, are given. You may not leave the solution at a convolution form

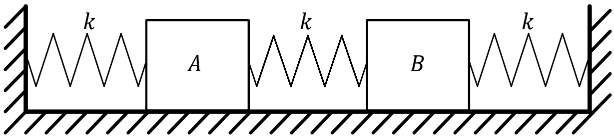

Figure 5.9 The block-spring system for Problem 5.5.29.

Problem 5.5.30 Two blocks (A & B) of the same mass m are attached to two identical mass-less springs (as shown). The assembly is placed on a horizontal and perfectly smooth bench. Initially, springs stay at their natural lengths and both blocks are at rest. During two brief moments, two forces were added to the blocks respectively:

$$f_A(t) = \begin{cases} f_0 & t \in [0, \tau/2] \\ 0 & \text{O.W.} \end{cases} \quad \text{and} \quad f_B(t) = \begin{cases} f_0 & t \in [\tau/2, \tau] \\ 0 & \text{O.W.} \end{cases}$$

Compute the displacements of the blocks as a function of time. All mentioned parameters, including the spring constant, are given. You may not leave the solution in convolution form.

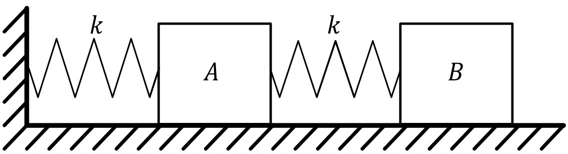

Figure 5.10 The block-spring system for Problem 5.5.30 & Problem 5.5.31.

Problem 5.5.31 Two blocks (A & B) of the same mass m are attached to two identical mass-less springs (as shown). The assembly is placed on a horizontal and perfectly smooth bench. Initially, springs stay at their natural lengths, and both blocks are at rest. During two instances $t = t_0$ and $2t_0$, two forces were added to the blocks respectively. The equation of motion of the two blocks can be written as

$$\begin{cases} mx_1'' = -kx_1 + k(x_2 - x_1) + f_0\delta(t - t_0) \\ mx_2'' = -k(x_2 - x_1) + f_0\delta(t - 2t_0) \\ x_1(0) = x_1'(0) = 0 \\ x_2(0) = x_2'(0) = 0 \end{cases}$$

Compute the displacements of the blocks as a function of time. All parameters, including m, k, f_0, t_0, are given. You may leave the solution at a convolution form if you need to.

Problem 5.5.32 Consider a system of two masses m_1 and m_2 (from left to right) connected to three massless springs whose spring constants are k_1, k_2, and k_3 (from left to right). The entire space is placed on a frictionless leveled surface as shown below.

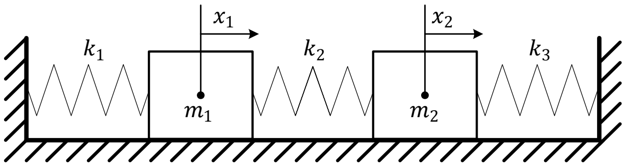

Figure 5.11 The block-spring system for Problem 5.5.32.

(1) Derive the DEs of motion for the two masses.
(2) Solve the DEs you derived above with the following I.C. and simplified parameters $m_1 = m_2 = 1, k_1 = 1, k_2 = 2, k_3 = 3$,

$$\begin{cases} x_1(0) = x_2(0) = 0 \\ x_1'(0) = 0 \\ x_2'(0) = v \end{cases}$$

Appendix A
Solutions to Selected Problems

Chapter 1 First-Order DEs

1.1 Definition of DEs

Problem 1.1.1

$$y_1 = \cos x - \cos 2x \Rightarrow \begin{cases} y_1' = -\sin x + 2\sin 2x \\ y_1'' = -\cos x + 4\cos 2x \end{cases}$$

$$\begin{aligned} LHS = y_1'' + y_1 &= -\cos x + 4\cos 2x + \cos x - \cos 2x \\ &= 3\cos 2x \ = \text{RHS} \end{aligned}$$

y_1 is a solution of the DE.

$$y_2 = \sin x - \cos 2x \Rightarrow \begin{cases} y_2' = \cos x + 2\sin 2x \\ y_2'' = -\sin x + 4\cos 2x \end{cases}$$

$$\begin{aligned} \text{LHS} = y_2'' + y_2 &= -\sin x + 4\cos 2x + \sin x - \cos 2x \\ &= 3\cos 2x = \text{RHS} \end{aligned}$$

y_2 is a solution of the DE.

Problem 1.1.2

$$y_1 = x\cos(\ln x)$$

$$\Rightarrow \begin{cases} y_1' = \cos(\ln x) - \sin(\ln x) \\ y_1'' = -\dfrac{\sin(\ln x) + \cos(\ln x)}{x} \end{cases}$$

$$\begin{aligned} \text{LHS} &= x^2 y_1'' - x y_1' + 2y_1 \\ &= -x(\sin(\ln x) + \cos(\ln x)) - x(\cos(\ln x) - \sin(\ln x)) \\ &\qquad + 2x\cos(\ln x) \\ &= 0 = \text{RHS} \end{aligned}$$

y_1 is a solution of the DE.

$$y_2 = x\sin(\ln x)$$

$$\Rightarrow \begin{cases} y_2' = \sin(\ln x) + \cos(\ln x) \\ y_2'' = \dfrac{\cos(\ln x) - \sin(\ln x)}{x} \end{cases}$$

$$\begin{aligned} \text{LHS} &= x^2 y_2'' - x y_2' + 2y_2 \\ &= -x(\sin(\ln x) - \cos(\ln x)) - x(\cos(\ln x) + \sin(\ln x)) \\ &\qquad + 2x\sin(\ln x) \\ &= 0 = \text{RHS} \end{aligned}$$

y_2 is a solution of the DE.

Problem 1.1.3

From the orthogonal relation, for two function curves $f(x)$ and $g(x)$ that are orthogonal to each other, we have

$$f' \cdot g' = -1$$

Here we have

$$f(x) = x^2 + k$$
$$f'(x) = 2x$$

Thus, we have

$$g'(x) = -\frac{1}{2x}$$

$$g(x) = \int -\frac{1}{2x}\,dx = -\frac{1}{2}\ln x + C$$

Problem 1.1.4

By observation, we can find that

$$(xy)' = xy' + x'y = xy' + y = \text{LHS}$$

and

$$(x^3)' = 3x^2 = \text{RHS}$$

This gives

$$xy = x^3$$

That means $y = x^2$ is a solution.

Problem 1.1.5

Guess

$$y = \cos x$$
$$y' = -\sin x$$
$$y'' = -\cos x$$
$$\text{LHS} = y'' + y = -\cos x + \cos x = 0 = \text{RHS}$$

$y = \cos x$ is a solution of the DE.

Problem 1.1.6

$$y = \frac{1}{1+x^2} \Rightarrow y' = -\frac{2x}{(1+x^2)^2}$$
$$\text{LHS} = y' + 2xy^2 = 0 = \text{RHS}$$
$$y = \frac{1}{1+x^2}$$

is a solution of the DE.

Problem 1.1.7

$$y(x) = C\exp(-x^3) \Rightarrow y'(x) = -3Cx^2\exp(-x^3)$$
$$\text{LHS} = y' + 3x^2y = -3Cx^2\exp(-x^3) + 3x^2(C\exp(-x^3))$$
$$= 0 = \text{RHS}$$

$y = C\exp(-x^3)$ is a solution of the DE.

Plugging $y(0) = 7$ into the G.S., we have

$$C\exp(0) = 7$$

gives $C = 7$

Therefore

$$y(x) = 7\exp(-x^3)$$

Problem 1.1.8

$$y = (x + C)\cos x \Rightarrow y' = \cos x - (x + C)\sin x$$

$$\begin{aligned}\text{LHS} &= \cos x - (x + C)\sin x + (x + C)\cos x \tan x \\ &= \cos x - (x + C)\sin x + (x + C)\sin x \\ &= \cos x = \text{RHS}\end{aligned}$$

Plugging $y(\pi) = 0$ into the G.S., we have

$$\begin{aligned}y(\pi) &= (\pi + C)\cos \pi \\ &= -(\pi + C) = 0\end{aligned}$$

gives $C = -\pi$

Therefore

$$y(x) = (x - \pi)\cos x$$

Problem 1.1.9

$$y' = (\tan(x^3 + C))'$$

Formula

$$(\tan x)' = \frac{1}{\cos^2 x}$$

and chain rule give

$$y' = \frac{3x^2}{\cos^2(x^3 + C)}$$

On the other hand

$$\begin{aligned}y' &= 3x^2(y^2 + 1)y' \\ &= 3x^2\left(\frac{\sin^2(x^3 + C)}{\cos^2(x^3 + C)} + 1\right) \\ &= 3x^2\frac{\sin^2(x^3 + C) + \cos^2(x^3 + C)}{\cos^2(x^3 + C)} \\ &= 3x^2\frac{1}{\cos^2(x^3 + C)}\end{aligned}$$

So given function satisfies the DE.

If $y(x) = \tan(x^3 + C)$ then $y(0) = 1$ gives the equation $\tan C = 1$. Hence the value of C is $C = \pi/4$ (as is this value plus any integral multiple of π).

Problem 1.1.10

$$y(x) = \frac{1}{4}x^5 + \frac{C}{x^3}$$
$$y'(x) = \frac{5}{4}x^4 - \frac{3C}{x^4}$$
$$\text{LHS} = \frac{5}{4}x^5 - \frac{3C}{x^3} + \frac{3}{4}x^5 + \frac{3C}{x^3}$$
$$= 2x^5 = \text{RHS}$$

Plugging $y(2) = 1$ into the G.S., we have

$$y(2) = \frac{1}{4} \times 2^5 + \frac{C}{2^3} = 1$$
$$C = -56$$

Problem 1.1.11

$$y_1 = \exp(3x)$$
$$y_1' = 3\exp(3x), \qquad y_1'' = 9\exp(3x) = 9y_1$$
$$y_2 = \exp(-3x)$$
$$y_2' = -3\exp(-3x), \qquad y_2'' = 9\exp(-3x) = 9y_2$$

Problem 1.1.12

$$y = \ln(x + C)$$
$$y' = \frac{1}{x + C}$$
$$\text{LHS} = \exp(\ln(x + C)) \cdot \frac{1}{x + C}$$
$$= (x + C) \cdot \frac{1}{x + C} = 1 = \text{RHS}$$

Plugging $y(0) = 0$ into the G.S., we have

$$y(0) = \ln C = 0$$
$$C = 1$$

Problem 1.1.13

$$y = C\exp(-x^n)$$
$$y' = -nx^{n-1}C\exp(-x^n)$$

$$\begin{aligned}\text{LHS} &= -nx^{n-1}C\exp(-x^n) + nx^{n-1}C\exp(-x^n) \\ &= 0 = \text{RHS}\end{aligned}$$

Plugging $y(0) = 2014$ into the G.S., we have

$$y(0) = C = 2014$$

1.2 Mathematical Models

Problem 1.2.1

From the given condition, we have the acceleration function $a(t) = -k$, where k is a positive constant, thus

$$\frac{dv}{dt} = -k \Rightarrow v(t) = \int -k dt$$

$$v(t) = -kt + C$$

Plug the condition $v(0) = 88$, we can get $C = 88$. Thus, we have velocity function

$$v(t) = -kt + 88$$

The car will stop skidding once the velocity is zero. That is

$$v(t) = -kt + 88 = 0$$

This gives

$$t_{\text{stop}} = \frac{88}{k}$$

The car will stop skidding when $t = 88/k$. On the other hand

$$v(t) = \frac{dx}{dt} \Rightarrow x(t) = \int v dt$$

This gives

$$x(t) = -\frac{k}{2}t^2 + 88t + C$$

Considering $x(0) = 0$, we can have

$$C = 0$$

Thus, we have the displacement function

$$x(t) = -\frac{k}{2}t^2 + 88t$$

Plugging the condition that

$$x(t_{\text{stop}}) = 176$$

gives

$$-\frac{k}{2}\left(\frac{88}{k}\right)^2 + 88\frac{88}{k} = 176$$

That is $k = 22$ ft/s and $t_{\text{stop}} = 4$ sec.
So the car provides a constant deceleration of 22ft/s^2 and the car continues to skid for 4 seconds.

Problem 1.2.2

$$\frac{dv}{dt} = a(t) = 50\sin 5t$$
$$v(t) = \int 50\sin 5t\,dt$$
$$= -10\cos 5t + C$$
$$v(0) = -10 \Rightarrow -10 + C = -10 \Rightarrow C = 0$$
$$v(x) = -10\cos 5t$$
$$\frac{dx}{dt} = v(t)$$
$$x(t) = \int v(t)dt$$
$$= \int -10\cos 5t\,dt$$
$$= -2\sin 5t + C$$
$$x(0) = 8 \Rightarrow C = 8$$
$$x(t) = 8 - 2\sin 5t$$

Problem 1.2.3

$$v(t) = \int a(t)dt$$
$$= \int \frac{1}{(t+1)^n}dt$$
$$= -\frac{1}{n-1}\cdot\frac{1}{(t+1)^{n-1}} + C$$

$v(0) = 0$ gives

$$C = \frac{1}{n-1}$$

Hence

$$v(t) = \frac{1}{n-1}\left(1 - \frac{1}{(t+1)^{n-1}}\right)$$

and

$$\begin{aligned} x(t) &= \int v(t)dt \\ &= \int \frac{1}{n-1}\left(1 - \frac{1}{(t+1)^{n-1}}\right)dt \\ &= \frac{1}{n-1}t + \frac{1}{(n-1)(n-2)} \cdot \frac{1}{(t+1)^{n-2}} + C \end{aligned}$$

$x(0) = 0$ gives

$$C = -\frac{1}{(n-1)(n-2)}$$

Thus

$$x(t) = \frac{1}{n-1}t + \frac{1}{(n-1)(n-2)} \cdot \frac{1}{(t+1)^{n-2}} - \frac{1}{(n-1)(n-2)}$$

Problem 1.2.4

(1) Establish the DE of draining process

$$\begin{cases} \dfrac{A(y)dy}{dt} = -k\sqrt{y} \\ y(t=0) = y_0 \end{cases}$$

$$A(y) = \pi r^2 = \pi(R^2 - (R-y)^2) = \pi y(2R - y)$$

$$\frac{A(y)dy}{dt} = \frac{\pi y(2R-y)dy}{dt} = -k\sqrt{y}$$

$$\pi\sqrt{y}(2R - y)dy = -kdt$$

$$\pi\int_{2R}^{0}\sqrt{y}(2R-y)dy = -\int_0^t kdt$$

$$\pi\left(\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_{2R}^{0} = -kt$$

$$\frac{4}{3}\pi R\left(0 - (2R)^{\frac{3}{2}}\right) - \frac{2}{5}\pi\left(0 - (2R)^{\frac{5}{2}}\right) = -kt$$

$$t = \frac{28\sqrt{2}}{15}\frac{\pi}{k}R^{\frac{5}{2}}$$

(2) If we double the radius of the container,

$$A(y) = \pi(4R^2 - (2R - y)^2) = \pi y(4R - y)$$

$$\frac{A(y)dy}{dt} = \frac{\pi y(4R - y)dy}{dt} = -k\sqrt{y}$$

$$\pi\int_{4R}^{0}\sqrt{y}(4R - y)dy = -\int_{0}^{T}kdt$$

$$\frac{8}{3}\pi R\left(0 - (4R)^{\frac{3}{2}}\right) - \frac{2}{5}\pi\left(0 - (4R)^{\frac{5}{2}}\right) = -kT$$

$$T = \frac{128}{64}\frac{\pi}{k}R^{\frac{5}{2}}$$

Problem 1.2.5

The equation can be written as

$$\begin{cases} \dfrac{\pi y(2R - y)dy}{dt} = -k\sqrt{y} \\ y(t = 0) = 2R \end{cases}$$

Solving this IVP, we get

$$\left(y^{\frac{3}{2}} - 2Ry^{\frac{1}{2}}\right)dy = \frac{k}{\pi}dt$$

Thus,

$$\int_{2R}^{y}\left(y^{\frac{3}{2}} - 2Ry^{\frac{1}{2}}\right)dy = \frac{k}{\pi}\int_{0}^{t}dt$$

We have

$$\frac{\pi}{4}\left(\frac{4(2R)^{\frac{5}{2}}}{15} + 2y^{\frac{3}{2}}\left(\frac{y}{5} - \frac{2R}{3}\right)\right) = t$$

Now, we insert $y = R$ to find T_1,

$$T_1 = \frac{\pi}{k}\left(\frac{4(2R)^{\frac{5}{2}}}{15} + 2y^{\frac{3}{2}}\left(\frac{y}{5} - \frac{2R}{3}\right)\right) = \frac{\pi}{k}\left(\frac{4(2R)^{\frac{5}{2}}}{15} - \frac{14(R)^{\frac{5}{2}}}{15}\right)$$

Next, we find T_2 by setting $y = 0$

$$T_2 = \frac{\pi}{k}\left(\frac{4(2R)^{\frac{5}{2}}}{15}\right)$$

Finally, we get $T_2 > T_1$.

Problem 1.2.6

Let y be the vertical distance from the hole. Then we have

$$r^2 = R^2\left(\frac{19}{20} + \frac{y}{10H}\right)$$

Thus

$$\pi R^2\left(\frac{19}{20} + \frac{y}{10H}\right)y' = -k\sqrt{y}$$

$$\pi R^2\left(\frac{19}{20} + \frac{y}{10H}\right)dy = -k\sqrt{y}dt$$

$$\int_{\frac{H}{2}}^{y}\frac{\left(\frac{19}{20} + \frac{1}{10H}y\right)dy}{\sqrt{y}} = \int_0^t -\frac{k}{\pi R^2}dt$$

$$\frac{19}{10}y^{\frac{1}{2}} + \frac{1}{15H}y^{\frac{3}{2}} - \frac{29\sqrt{2}}{30}H^{\frac{1}{2}} = -\frac{kt}{\pi R^2}$$

When $y = 0$

$$t = \frac{29\sqrt{2H}\pi R^2}{30k}$$

Now turn the cup upside down

$$r^2 = R^2\left(\frac{19}{20} - \frac{y}{10H}\right)$$

$$\pi R^2\left(\frac{19}{20} - \frac{y}{10H}\right)y' = -k\sqrt{y}$$

$$\int_{\frac{H}{2}}^{y}\frac{\left(\frac{19}{20} - \frac{1}{10H}y\right)dy}{\sqrt{y}} = \int_0^t -\frac{k}{\pi R^2}dt$$

$$\frac{19}{10}y^{\frac{1}{2}} - \frac{1}{15H}y^{\frac{3}{2}} - \frac{14\sqrt{2}}{15}H^{\frac{1}{2}} = -\frac{kt}{\pi R^2}$$

When $y = 0$

$$t = \frac{14\sqrt{2H}\pi R^2}{15k}$$

Problem 1.2.7

For draining the upper half:

$$\frac{dV}{dt} = -k\sqrt{y} = \frac{A(y)dy}{dt}$$

$$r^2 + (y - R)^2 = R^2$$

$$A(y) = \pi r^2 = \pi(R^2 - (y - R)^2) = \pi y(2R - y)$$

$$\frac{\pi y(2R - y)dy}{dt} = -k\sqrt{y}$$

$$\pi\sqrt{y}(2R - y)dy = -kdt$$

$$\pi \int_{2R}^{R} \left(2Ry^{\frac{1}{2}} - y^{\frac{3}{2}}\right) dy = -k \int_{0}^{t_1} dt$$

$$\pi \left(\frac{2Ry^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^{\frac{5}{2}}}{\frac{5}{2}} \right)\Bigg|_{2R}^{R} = -kt_1$$

$$-\frac{\pi}{k}\left(\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_{2R}^{R} = t_1$$

$$-\frac{\pi}{k}\left(\frac{4}{3}R\left(R^{\frac{2}{3}}\right) - \frac{2}{5}R^{\frac{5}{2}} - \left(\frac{4}{3}R\ (2R)^{\frac{3}{2}} - \frac{2}{5}(2R)^{\frac{5}{2}}\right)\right) = t_1$$

$$-\frac{\pi}{k}\left(\frac{14 - 16\sqrt{2}}{15}R^{\frac{5}{2}}\right) = t_1$$

For draining the lower half:

$$r^2 + (R - y)^2 = R^2$$

$$A(y) = \pi r^2 = \pi(R^2 - (R - y)^2) = \pi y(2R - y)$$

$$\pi\sqrt{y}(2R - y)dy = -k_1 dt$$

$$\pi \int_{R}^{0} \left(2Ry^{\frac{1}{2}} - y^{\frac{3}{2}}\right) dy = -k_1 \int_{0}^{t_2} dt$$

$$\pi\left(\frac{2Ry^{\frac{3}{2}}}{\frac{3}{2}}-\frac{y^{\frac{5}{2}}}{\frac{5}{2}}\right)\Bigg|_R^0=-k_1t_2$$

$$-\frac{\pi}{k_1}\left(\frac{4}{3}Ry^{\frac{3}{2}}-\frac{2}{5}y^{\frac{5}{2}}\right)\Big|_R^0=t_2$$

$$-\frac{\pi}{k_1}\left(-\left(\frac{4}{3}R\left(R^{\frac{3}{2}}\right)-\frac{2}{5}R^{\frac{5}{2}}\right)\right)=t_2$$

$$\frac{\pi}{k_1}\left(\frac{4}{3}R^{\frac{5}{2}}-\frac{2}{5}R^{\frac{5}{2}}\right)=t_2$$

$$\frac{\pi}{k_1}\left(\frac{14}{15}R^{\frac{5}{2}}\right)=t_2$$

We want $t_2 = t_1$.

$$\frac{\pi}{k_1}\left(\frac{14}{15}R^{\frac{5}{2}}\right)=-\frac{\pi}{k}\left(\frac{14-16\sqrt{2}}{15}R^{\frac{5}{2}}\right)$$

$$\frac{14}{k_1}=-\frac{\left(14-16\sqrt{2}\right)}{k}$$

$$k_1=-\frac{14}{14-16\sqrt{2}}$$

$$k_1=\frac{7}{8\sqrt{2}-7}k\approx 1.623k$$

Problem 1.2.8

Torricelli's law tells us that

$$\frac{dV}{dt}=-k\sqrt{y}$$

$$dV=-k\sqrt{y}dt$$

$$A(y)dy=-k\sqrt{y}dt$$

$$-\frac{A(y)}{k\sqrt{y}}=dt$$

Then the time it takes to drain the container from full ($y = 2R$) to a certain height y_m is

$$\int_{2R}^{y_m} -\frac{A(y)}{k\sqrt{y}} = \int_0^{t_m} dt = t_m$$

We seek the y_m such that the time it takes to drain the container from height *2R* to height y_m is equal to the time it takes to drain the container from height y_m to height 0, i.e.,

$$\int_{2R}^{y_m} -\frac{A(y)}{k\sqrt{y}} dy = \int_{y_m}^{0} -\frac{A(y)}{k\sqrt{y}} dy = \frac{T}{2}$$

where T is the time to drain the container fully. As shown in the textbook, cross-section area for any liquid surface height is

$$A(y) = \pi r^2 = \pi(R^2 - (R-y)^2) = \pi y(2R - y)$$

Thus we may find T:

$$\int_{2R}^{0} -\frac{\pi y(2R-y)}{k\sqrt{y}} dy = \int_0^T dt$$

$$-\frac{\pi}{k}\int_{2R}^{0}\left(2R\sqrt{y} - y^{\frac{3}{2}}\right) dy = T$$

$$-\frac{\pi}{k}\left(\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_{2R}^{0} = T = \frac{\pi}{k}\left(\frac{4}{3}R(2R)^{\frac{3}{2}} - \frac{2}{5}(2R)^{\frac{5}{2}}\right)$$

$$T = \frac{\pi}{k}\left(\frac{2^{\frac{7}{2}}}{3}R^{\frac{5}{2}} - \frac{2^{\frac{7}{2}}}{5}R^{\frac{5}{2}}\right) = \frac{\pi 2^{\frac{9}{2}}}{k}R^{\frac{5}{2}}$$

And we want y_m such that

$$\int_{2R}^{y_m} -\frac{\pi y(2R-y)}{k\sqrt{y}} dy = \int_{y_m}^{0} -\frac{\pi y(2R-y)}{k\sqrt{y}} dy = \frac{T}{2} = \frac{\pi 2^{\frac{7}{2}}}{k}R^{\frac{5}{2}}$$

$$-\int_{y_m}^{0} \sqrt{y}(2R-y)dy = 2^{\frac{7}{2}}R^{\frac{5}{2}}$$

$$\left(\frac{4}{3}Ry^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)\Bigg|_{0}^{y_m} = 2^{\frac{7}{2}}R^{\frac{5}{2}}$$

$$\frac{4}{3}Ry_m^{\frac{3}{2}} - \frac{2}{5}y_m^{\frac{5}{2}} = 2^{\frac{7}{2}}R^{\frac{5}{2}}$$

Then the volume of the leftover liquid is given by

$$\int_0^{y_m} A(y)dy = \int_0^{y_m} \pi(2Ry - y^2)dy$$

Problem 1.2.9

(1) Then total mass for a boat with n boaters is given by $M + nm$. The total force propelling the boat is nf_0. Water resistance is $\mu_0 v$, which acts in the opposite direction. Thus, we have the following DE

$$(M + nm)\frac{dv}{dt} = nf_0 - \mu_0 v$$

(2) This is a separable equation.
Hence we have

$$\int (M + nm)\frac{dv}{nf_0 - \mu_0 v} = \int dt + c$$

$$-\frac{(M + nm)}{\mu_0}\ln(nf_0 - \mu_0 v) = t + c$$

With $v(t = 0) = 0$, one can get

$$c = -\frac{(M + nm)}{\mu_0}\ln(nf_0)$$

Plugging in this value of c in the solution, we get

$$v(t) = \frac{nf_0}{\mu_0}\left(1 - \exp\left(-\frac{\mu_0 t}{M + nm}\right)\right)$$

(3) The total travel time T follows the following equation,

$$L = \int_0^T v(t)\, dt = \int_0^T \frac{nf_0}{\mu_0}\left(1 - \exp\left(-\frac{\mu_0 t}{M + nm}\right)\right) dt$$

$$L = \frac{nf_0}{\mu_0}\left(T + \frac{M + m}{\mu_0}\exp\left(-\frac{\mu_0 t}{M + nm}\right) - \frac{M + nm}{\mu_0}\right)$$

Differentiate T WRT n (assuming real variable instead of integer for convenience) and found out that $\frac{dT}{dn} < 0$ and thus the more boaters, the shorter the time. The answer is then "To win the race, put as many boaters as practical."

Alternate argument, one can compute the boat speed WRT n.

$$\frac{dv}{dn} = \frac{f_0}{\mu_0}\left(1 - \exp\left(-\frac{\mu_0 t}{M + nm}\right)\right) + \frac{nmf_0\mu_0 t}{\mu_0(M + nm)^2}\exp\left(-\frac{\mu_0 t}{M + nm}\right)$$

Now

$$\frac{f_0}{\mu_0}\left(1 - \exp\left(-\frac{\mu_0 t}{M + nm}\right)\right) \geq 0 \ \forall \ t \geq 0$$

and

$$\frac{nmf_0\mu_0 t}{\mu_0(M + nm)^2}\exp\left(-\frac{\mu_0 t}{M + nm}\right) > 0$$

Thus

$$\frac{dv}{dn} > 0 \ \forall \ t \geq 0$$

In other words v is an increasing function of n. So, at any instant, the more boaters, the faster the boat travels. “The added weight of the boater has not slowed down the boat, as long as the added boater does not pedal backward.”

1.3 Separation of Variables

Problem 1.3.1

This is a separable DE, and by separation of variables, we have

$$\frac{3xdx}{x^2+1} = -\frac{dy}{y-2} \Rightarrow \int \frac{3xdx}{x^2+1} = -\int \frac{dy}{y-2}$$

$$\frac{3}{2}\ln(x^2+1) = -\ln(y-2) + C$$

$$y = 2 + \frac{C}{(x^2+1)^{\frac{3}{2}}}$$

Problem 1.3.2

$$\frac{dy}{dx} = 4x^{\frac{1}{3}}y^{\frac{1}{3}}$$

$$y^{-\frac{1}{3}}dy = 4x^{\frac{1}{3}}dx$$

$$\int y^{-\frac{1}{3}}dy = \int 4x^{\frac{1}{3}}dx$$

$$\frac{3}{2}y^{\frac{2}{3}} = 3x^{\frac{4}{3}} + C_1$$

$$y = \left(2x^{\frac{4}{3}} + C\right)^{\frac{3}{2}}$$

Problem 1.3.3

We can rewrite the DE as

$$\frac{dy}{dx} = (1+x)(1+y)$$

and this is a separable DE. Hence we have

$$\frac{dy}{1+y} = (1+x)dx$$

$$\int \frac{dy}{1+y} = \int (1+x)dx$$

$$\ln|1+y| = x + \frac{1}{2}x^2 + C$$

Problem 1.3.4

Since

$$\frac{dy}{dx} = x\exp(-x)$$

we have $dy = x\exp(-x)\,dx$. That is

$$\begin{aligned} y &= \int x\exp(-x)\,dx \\ &= -\int xd(\exp(-x)) \\ &= -x\exp(-x) + \int \exp(-x)\,dx \\ &= -x\exp(-x) - \exp(-x) + C \end{aligned}$$

From the I.C. $y(0) = 1$, we can get

$$-1 + C = 1$$

This gives $C = 2$, and the solution to the DE is

$$y(x) = -x\exp(-x) - \exp(-x) + 2$$

Problem 1.3.5

$$\frac{dy}{dx} = -2\cos 2x$$

$$\begin{aligned} y &= \int -2\cos 2x\,dx \\ &= -\sin 2x + C \end{aligned}$$

I.C. $y(0) = 2014$ gives

$$-\sin 0 + C = 2014$$
$$C = 2014$$

Problem 1.3.6

$$\sec^2 y \cdot y' = \frac{1}{2\sqrt{x}}$$

$$\int \sec^2 y\, dy = \int \frac{dx}{2\sqrt{x}}$$
$$\tan y = \sqrt{x} + C$$

The G.S. is

$$y = \tan^{-1}\left(\sqrt{x} + C\right)$$

Plug the I.C. $y(4) = \frac{\pi}{4}$ in to the G.S., we can get

$$\frac{\pi}{4} = \tan^{-1}(2 + C)$$
$$C = -1$$

The P.S. of the DE is

$$y = \tan^{-1}(\sqrt{x} - 1)$$

Problem 1.3.7

$$\frac{y'}{y} = 2x + 3x^2 \exp(x^3)$$
$$\int \frac{dy}{y} = \int (2x + 3x^2 \exp(x^3))dx$$
$$\ln y = x^2 + \exp(x^3) + C$$

The G.S. is

$$y = \exp(x^2 + \exp(x^3) + C)$$

Given the I.C. $y(0) = 5$, we have

$$5 = \exp(0 + 1 + C) \Rightarrow C = \ln 5 - 1$$

Thus the P.S. for the DE is

$$\begin{aligned} y &= \exp(x^2 + \exp(x^2) + \ln 5 - 1) \\ &= 5\exp(x^2 + \exp(x^2) - 1) \end{aligned}$$

Problem 1.3.8

$$\frac{y^3}{y^4 + 1} y' = \cos x$$
$$\int \frac{y^3}{y^4 + 1} dy = \int \cos x\, dx$$

$$\frac{1}{4}\int \frac{d(y^4+1)}{y^4+1} = \int \cos x\, dx$$
$$\frac{1}{4}\ln(y^4+1) = \sin x + C$$

Problem 1.3.9

The DE can be written as

$$\frac{y'}{y^2} = 5x^4 - 4x$$

which is a separable DE. Thus,

$$\int \frac{dy}{y^2} = \int (5x^4 - 4x)dx$$
$$-\frac{1}{y} = x^5 - 2x^2 + C$$

Problem 1.3.10

$$\frac{dy}{dx} = 2xy^2 + 3x^2y^2 = y^2(2x + 3x^2)$$

This is also separable equation

$$\int \frac{dy}{y^2} = \int (2x + 3x^2)\, dx$$
$$-\frac{1}{y} = x^2 + x^3 + C$$
$$y = \frac{-1}{x^2 + x^3 + C}$$

Then substitution of $x = 1$ gives

$$-1 = \frac{-1}{1 + 1 + C}$$

So $C = -1$ and

$$y = \frac{-1}{x^2 + x^3 - 1}$$

Problem 1.3.11

$$y' \cos y = 2x$$
$$\int \cos y \, dy = \int 2x dx$$
$$\sin y = x^2 + C$$

Problem 1.3.12

$$\sin x \frac{dy}{dx} = y \cos x$$
$$\frac{dy}{y} = \frac{\cos x \, dx}{\sin x}$$
$$\ln |y| = \ln | \sin x | + C$$
$$y = C \sin x$$
$$y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

gives

$$\frac{\pi}{2} = C \sin \frac{\pi}{2}$$
$$C = \frac{\pi}{2}$$

The explicit P.S. is

$$y(x) = \frac{\pi}{2} \sin x$$

Problem 1.3.13

$$(x^2 + 1) \tan y \frac{dy}{dx} = x$$
$$\tan y \, dy = \frac{x}{x^2 + 1} dx$$
$$-\ln|\cos y| = \frac{1}{2} \ln(x^2 + 1) + C$$

Problem 1.3.14

$$(x + 3)^3 \left(\frac{dy}{dx}\right) = (y - 2)^2$$

$$(y-2)^2 dy = (x+3)^{-3} dx$$
$$\int (y-2)^2 dy = \int (x+3)^{-3} dx$$
$$-(y-2)^{-1} = \left(-\frac{1}{2}\right)(x+3)^{-2} + A$$
$$(y-2)^{-1} = \left(\frac{1}{2}\right)(x+3)^{-2} + B$$
$$(y-2)^{-1} = \frac{1 + 2B(x+3)^2}{2(x+3)^2}$$
$$1 = \frac{(y-2)(1 + C(x+3)^2)}{2(x+3)^2}$$
$$\frac{2(x+3)^2}{1 + C(x+3)^2} = y - 2$$
$$y = 2 + \frac{2(x+3)^2}{1 + C(x+3)^2}$$

Problem 1.3.15

$$xy' - x^2 y = 3xy$$
$$xy' = x^2 y + 3xy$$
$$xy' = xy(x+3)$$
$$\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = x + 3$$
$$\int \left(\frac{1}{y}\right) dy = \int (x+3) dx$$
$$\ln y = \frac{(x+3)^2}{2} + C_1$$
$$y = \exp\left(\frac{(x+3)^2}{2} C_1\right)$$
$$y = C_2 \exp\left(\frac{(x+3)^2}{2}\right)$$
$$y(1) = 1 = C_2 \exp\left(\frac{(1+3)^2}{2}\right) = C_2 \exp(8)$$
$$C_2 = \exp(-8)$$

$$y = \exp\left(\frac{(x+3)^2}{2} - 8\right)$$

Problem 1.3.16

The equation can be written as

$$\big((1+x)y\big)' = \cos x$$

Integrate on both sides,

$$(1+x)y = \int \cos x \, dx + C = \sin x + C$$

Thus,

$$y = \frac{\sin x + C}{x+1}$$

We find $C = 1$ by plugging in the I.C.. Finally,

$$y = \frac{\sin x + 1}{x+1}$$

Problem 1.3.17

$$\frac{dy}{\beta y(\alpha - \ln y)} = dt$$

$$\frac{d\ln y}{\beta(\alpha - \ln y)} = dt$$

$$\frac{-d(\alpha - \ln y)}{\beta(\alpha - \ln y)} = dt$$

$$\frac{-d\ln(\alpha - \ln y)}{\beta} = dt$$

$$\int \frac{-d\ln(\alpha - \ln y)}{\beta} = \int dt$$

So the G.S. is

$$-\frac{\ln(\alpha - \ln y)}{\beta} = t + C$$

$$\alpha - \ln y = \exp\big(-(\beta t + C_1)\big)$$

$$y = \exp\Big(\alpha - \exp\big(-(\beta t + C_1)\big)\Big)$$

If $y(0) = y_0$

$$\alpha - \ln y_0 = \exp(-C_1)$$

If $y(\infty) = y_\infty$

$$y_\infty = \exp(\infty)\,(\beta > 0) \Rightarrow \alpha = \ln y_\infty$$

If $(\beta < 0)$ $y_\infty = 0$

1.4 Linear First-Order DEs

Problem 1.4.1

Divided (x^2+1) by both sides of the DE, we have

$$y' + \frac{3x^3}{x^2+1}y = \frac{6x\exp\left(-\frac{3}{2}x^2\right)}{x^2+1}$$

Let

$$\begin{aligned}\rho(x) &= \exp\left(\int \frac{3x^3}{x^2+1}dx\right)\\ &= \exp\left(\int \frac{3x(x^2+1)-3x}{x^2+1}dx\right)\\ &= \exp\left(\frac{3}{2}x^2 - \frac{3}{2}\ln(x^2+1)\right)\\ &= (x^2+1)^{-\frac{3}{2}}\exp\left(\frac{3}{2}x^2\right)\end{aligned}$$

We have

$$(\rho(x)y)' = 6x(x^2+1)^{-\frac{5}{2}}$$

Thus,

$$\begin{aligned}y &= \frac{1}{\rho(x)}\int 6x(x^2+1)^{-\frac{5}{2}}dx\\ &= (x^2+1)^{\frac{3}{2}}\exp\left(-\frac{3}{2}x^2\right)\left(-2(x^2+1)^{-\frac{3}{2}}+C\right)\\ &= -2\exp\left(-\frac{3}{2}x^2\right) + C(x^2+1)^{\frac{3}{2}}\exp\left(-\frac{3}{2}x^2\right)\end{aligned}$$

which is the G.S.. Given $y(0)=1$, we have

$$1 = -2 + C \Rightarrow C = 3$$

Hence the P.S. of the DE is

$$y = -2\exp\left(-\frac{3}{2}x^2\right) + 3(x^2+1)^{\frac{3}{2}}\exp\left(-\frac{3}{2}x^2\right)$$

Problem 1.4.2

(a) We know that

$$\begin{aligned} y_c' &= \left(C\exp\left(-\int P(x)dx\right)\right)' \\ &= C\exp\left(-\int P(x)dx\right)\left(-\int P(x)dx\right)' \\ &= -C\exp\left(-\int P(x)dx\right)P(x) \\ &= -P(x)y_c \end{aligned}$$

It is easy to see that

$$y_c' + P(x)y_c = -P(x)y_c + P(x)y_c = 0$$

This proves it is a G.S. of the original DE.

(b) We have that

$$\begin{aligned} & y_p'(x) \\ &= \left(\left\{\exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)\right\}'\right) \\ &= \left(\exp\left(-\int P(x)dx\right)\right)'\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right) \\ &+ \exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right)' \\ &= -P(x)\exp\left(-\int P(x)dx\right)\left(\int Q(x)\exp\left(\int P(x)dx\right)dx\right) \\ &\qquad + \exp\left(-\int P(x)dx\right)Q(x)\exp\left(\int P(x)dx\right) \\ &= -P(x)y_p(x) + Q(x) \end{aligned}$$

Now it is easy to see

$$\begin{aligned} \text{LHS} &= y_p' + P(x)y_p \\ &= -P(x)y_p + Q(x) + P(x)y_p \\ &= Q(x) = \text{RHS} \end{aligned}$$

(c) Since $y_c(x)$ is a G.S. of $y' + P(x)y = 0$, we know that

$$y_c' = -P(x)y_c$$

On the other hand, we know that

$$y_p' = Q(x) - P(x)y_p$$

Then for $y = y_c + y_p$, we have

$$\begin{aligned} y' &= \left(y_c + y_p\right)' \\ &= y_c' + y_p' \\ &= -P(x)y_c + Q(x) - P(x)y_p \\ &= -P(x)\left(y_c + y_p\right) + Q(x) \\ &= -P(x)y + Q(x) \end{aligned}$$

That means

$$y' + P(x)y = Q(x)$$

which means it is a G.S. to the DE.

Problem 1.4.3

The DE can be written as

$$y' + \frac{2}{x}y = 7x$$

Let

$$\begin{aligned} \rho(x) &= \exp\left(\int \frac{2}{x}dx\right) \\ &= \exp(2\ln x) \\ &= x^2 \end{aligned}$$

We have that

$$(\rho(x)y)' = 7x^3$$

That gives

$$\begin{aligned} y &= x^{-2}\left(\int 7x^3 dx\right) \\ &= x^{-2}\left(\frac{7}{4}x^4 + C\right) \\ &= \frac{7}{4}x^2 + Cx^{-2} \end{aligned}$$

Given $y(2) = 5$, we have

$$5 = 7 + \frac{C}{4} \Rightarrow C = -8$$

Thus the P.S. is

$$y = \frac{7}{4}x^2 - 8x^{-2}$$

Problem 1.4.4

Let

$$\rho(x) = \exp\left(\int \cot x \, dx\right)$$
$$= \exp(\ln \sin x)$$
$$= \sin x$$

We can have

$$(\rho(x)y)' = \sin x \cos x$$

which gives

$$y = \frac{1}{\sin x}\left(\int \sin x \cos x \, dx\right)$$
$$= \frac{\sin^2 x + C}{2 \sin x}$$

Problem 1.4.5

Method I

This DE can be written as

$$y' - 3x^2 y = 21x^2$$

which is a linear DE. Let

$$\rho(x) = \exp\left(\int -3x^2 dx\right)$$
$$= \exp(-x^3)$$

We know that

$$(\exp(-x^3)\, y)' = 21x^2 \exp(-x^3)$$

Thus

$$y = \exp(x^2) \int 21x^2 \exp(-x^3) \, dx$$
$$= -7 + C \exp(x^3)$$

Method II

The DE can be written as

$$\frac{y'}{y + 7} = 3x^2$$

which is a separable DE. Thus, we have

$$\int \frac{dy}{y+7} = \int 3x^2 dx$$
$$\ln(y+7) = x^3 + C$$
$$y = C\exp(x^3) - 7$$

Problem 1.4.6

The DE can be written as

$$y' - \frac{2}{x}y = x^2 \cos x$$

Let

$$\rho(x) = \exp\left(\int -\frac{2}{x}dx\right)$$
$$= \exp(-2\ln x)$$
$$= x^{-2}$$

We can have

$$(\rho(x)y)' = x^2 \cos x\, \rho(x)$$

That gives

$$y = \frac{1}{\rho(x)} \int x^2 \cos x\, \rho(x) dx$$
$$= x^2 \int \cos x\, dx$$
$$= x^2(\sin x + C)$$

Thus, the G.S. to the original DE is

$$y = x^2 \sin x + Cx^2$$

Problem 1.4.7

$$y' + \frac{1}{2x+1}y = (2x+1)^{\frac{1}{2}}$$

Let

$$\rho(x) = \exp\left(\int \frac{dx}{2x+1}\right)$$
$$= (2x+1)^{\frac{1}{2}}$$

We have

$$\left((2x+1)^{\frac{1}{2}}y\right)' = 2x+1$$

Thus

$$y = (2x+1)^{-\frac{1}{2}}\int(2x+1)dx$$
$$= (2x+1)^{-\frac{1}{2}}(x^2+x+C)$$

Problem 1.4.8

Method I: Separation

The equation can be written as

$$y' = \frac{2x(y+1)}{x^2+1}$$
$$\frac{y'}{y+1} = \frac{2x}{x^2+1}$$
$$\ln(y+1) = \ln(x^2+1) + C$$
$$y = Cx^2 + C - 1$$

Method II: Linear 1st-order

Rewriting the equation in the following form, we find that it is linear

$$y' - \frac{2x}{x^2+1}y = \frac{2x}{x^2+1}$$

Let

$$\rho(x) = \exp\left(\int -\frac{2x}{x^2+1}dx\right)$$
$$= \frac{1}{x^2+1}$$

Thus we know,

$$\left(\frac{1}{x^2+1}\cdot y\right)' = \frac{2x}{(x^2+1)^2}$$
$$y = (x^2+1)\int\frac{2xdx}{(x^2+1)^2}$$
$$= (x^2+1)\left(-\frac{1}{x^2+1}+C\right)$$

$$= -1 + C(x^2 + 1)$$

Problem 1.4.9

To solve this DE, we can regard x as the dependent variable and y as the independent variable. Thus, we have

$$(1 + 2xy)\frac{dy}{dx} = 1 + y^2$$
$$\frac{dy}{dx} = \frac{1 + y^2}{1 + 2xy}$$
$$\frac{dx}{dy} = \frac{1 + 2xy}{1 + y^2}$$
$$\frac{dx}{dy} - \frac{2y}{1 + y^2}x = \frac{1}{1 + y^2}$$

This is a 1st-order Linear DE of x WRT y. Let

$$\begin{aligned}\rho(y) &= \exp\left(\int -\frac{2y}{1 + y^2}dy\right) \\ &= \exp\left(-\int \frac{d(1 + y^2)}{1 + y^2}\right) \\ &= \exp[-\ln(1 + y^2)] \\ &= \frac{1}{1 + y^2}\end{aligned}$$

We have that

$$(\rho(y)x)' = \frac{1}{1 + y^2}\rho(y)$$

which gives

$$\begin{aligned}x &= \frac{1}{\rho(y)}\int \frac{1}{1 + y^2}\rho(y)dy \\ &= (1 + y^2)\int \frac{1}{(1 + y^2)^2}dy \\ &= (1 + y^2)\left(\frac{1}{2}\tan^{-1} y + \frac{1}{2}\frac{y}{1 + y^2} + C\right)\end{aligned}$$

Note: Let $y = \tan\theta$, we have $1 + y^2 = \sec^2\theta$ and $dy = \sec^2\theta\, d\theta$. Thus, the last integral becomes

$$\begin{aligned}
\int \frac{1}{(1+y^2)^2} dy &= \int \frac{1}{\sec^4\theta} \sec^2\theta\, d\theta \\
&= \int \cos^2\theta\, d\theta \\
&= \frac{1}{2}\int (1 - \cos 2\theta) d\theta \\
&= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C \\
&= \frac{1}{2}(\theta - \sin\theta\cos\theta) + C \\
&= \frac{1}{2}\left(\theta - \frac{\tan\theta}{\sec^2\theta}\right) + C \\
&= \frac{1}{2}\tan^{-1} y + \frac{1}{2}\cdot\frac{y}{1+y^2} + C
\end{aligned}$$

Problem 1.4.10

We can write the DE as

$$y' + \frac{1}{2x} y = 5x^{-\frac{1}{2}}$$

Let

$$\begin{aligned}
\rho(x) &= \exp\left(\int \frac{1}{2x} dx\right) \\
&= \exp\left(\frac{1}{2}\ln x\right) \\
&= x^{\frac{1}{2}}
\end{aligned}$$

Thus we have

$$(\rho(x)y)' = 5x^{-\frac{1}{2}}\rho(x)$$

This gives

$$y = \frac{1}{\rho(x)}\int 5x^{-\frac{1}{2}}\rho(x) dx$$

$$= x^{-\frac{1}{2}}(5x + C)$$
$$= 5\sqrt{x} + \frac{C}{\sqrt{x}}$$

Problem 1.4.11

The DE can be written as

$$\frac{dx}{dy} - x = y\exp(y)$$

which can be considered as a DE of x WRT y. Thus, Let

$$\rho(y) = \exp\left(\int -1dy\right)$$
$$= \exp(-y)$$

We can have

$$(\rho(y)x)' = y\exp(y)\,\rho(y)$$

which gives

$$x = \frac{1}{\rho(y)}\int y\exp(y)\,\rho(y)dy$$
$$= \exp(y)\int ydy$$
$$= \exp(y)\left(\frac{1}{2}y^2 + C\right)$$
$$= \frac{1}{2}y^2\exp(y) + C\exp(y)$$

Problem 1.4.12

$$y' + \frac{2}{x+1}y = 3$$

Let

$$\rho(x) = \exp\left(\int \frac{2}{x+1}dx\right)$$
$$= (x+1)^2$$

Thus

$$((x+1)^2 y)' = 3(x+1)^2$$

$$y = (x+1)^{-2}((x+1)^3 + C)$$
$$= x + 1 + \frac{C}{(x+1)^2}$$

Problem 1.4.13

Converting the DE

$$y' + \frac{1}{1+x}y = \frac{\sin x}{1+x}$$

The integrating factor is

$$\rho(x) = \exp\left(\int \frac{1}{1+x}dx\right) = 1 + x$$

Then,

$$(\rho(x)y)' = \rho(x)\frac{\sin x}{1+x}$$
$$\big((1+x)y\big)' = \sin x$$
$$(1+x)y = \int \sin x\, dx = -\cos x + C$$
$$y = \frac{-\cos x + C}{1+x}$$

Since $y(0) = 1$,

$$y(0) = \frac{C-1}{1} = C - 1 = 1$$
$$C = 2$$

The solution is

$$y = \frac{2 - \cos x}{1+x}$$

Problem 1.4.14

Substitute $v = \frac{dy}{dx}$

$$v' = y''$$
$$x^2v' + 3xv = 4x^4$$
$$v' + \frac{3v}{x} = 4x^2$$
$$P(x) = \frac{3}{x}, \qquad Q(x) = 4x^2$$

$$\rho(x) = \exp\left(\int \frac{3}{x}dx\right) = x^3$$

$$\frac{d}{dx}(\rho(x)v) = \rho(x)Q(x)$$

$$x^3v = \int x^3(4x^2)dx$$

$$v = \frac{4}{6}x^3 + \frac{c}{x^3}$$

Since $v = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{4}{6}x^3 + \frac{c}{x^3}$$

$$y = \frac{1}{6}x^4 - \frac{cx^{-2}}{2} + c_1$$

1.5 Substitution Methods

Problem 1.5.1

We can write the DE as

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

Let

$$u = \frac{y}{x}$$

We have

$$y = ux \text{ and } \frac{dy}{dx} = u + x\frac{du}{dx}$$

Thus,

$$u + x\frac{du}{dx} = u + u^2$$

$$\frac{du}{u^2} = \frac{dx}{x}$$

$$-\frac{1}{u} = \ln x + C$$

That is

$$y = -\frac{x}{\ln x + C}$$

Problem 1.5.2

Rewrite the DE as

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{4\left(\frac{y}{x}\right)^2 + 1}$$

Let

$$u = \frac{y}{x}$$

We have

$$y = ux \text{ and } \frac{dy}{dx} = u + x\frac{du}{dx}$$

Thus, we have

$$u + x\frac{du}{dx} = u + \sqrt{4u^2 + 1}$$

$$\frac{du}{\sqrt{4u^2 + 1}} = \frac{dx}{x}$$

$$\frac{1}{2}\ln\left(2u + \sqrt{4u^2 + 1}\right) = \ln x + C_1$$

$$\left(2u + \sqrt{4u^2 + 1}\right)^{\frac{1}{2}} = Cx$$

$$\left(2\frac{y}{x} + \sqrt{4\left(\frac{y}{x}\right)^2 + 1}\right)^{\frac{1}{2}} = Cx$$

$$2y + \sqrt{4y^2 + x^2} = Cx^3$$

Problem 1.5.3

Since $v = \ln y$, we have

$$y = \exp(v) \;\text{ and }\; \frac{dy}{dx} = \exp(v)\frac{dv}{dx}$$

Put this back to the DE, we have

$$\exp(v)\, v' + P(x)\exp(v) = Q(x)v\exp(v)$$

It is easy to see that $\exp(v) > 0$, so we can have

$$v' + P(x) = Q(x)v(x)$$

Problem 1.5.4

$$5y^4y' - x^2y' = 2xy$$

$$\left(\frac{dy}{dx}\right)(5y^4 - x^2) = 2xy$$

$$x' = \frac{5y^4 - x^2}{2xy}$$

$$x' = \frac{5y^3}{2x} - \frac{x}{2y}$$

$$x' + \left(\frac{1}{2y}\right)x = \frac{5y^3}{2}x^{-1}$$
$$v = x^2 \Rightarrow v' = 2xx'$$
$$x = v^{\frac{1}{2}} \Rightarrow x' = \left(\frac{1}{2}\right)v^{-\frac{1}{2}}v'$$
$$\left(\frac{1}{2}\right)v^{-\frac{1}{2}}v' + \left(\frac{1}{2}\right)yv^{\frac{1}{2}} = \left(\frac{5}{2}\right)y^3\left(v^{\frac{1}{2}}\right)^{-1}$$
$$v' + \frac{v}{y} = 5y^3$$
$$\rho(y) = \exp\left(\int\left(\frac{1}{y}\right)dy\right) = \exp(\ln y) = y$$
$$(yv')' = 5y^4$$
$$yv = \int 5y^4 dy$$
$$yv = \frac{5y^5}{5} + C$$
$$v = y^4 + \frac{C}{y}$$
$$x^2 = y^4 + \frac{C}{y}$$
$$x = \left(y^4 + \frac{C}{y}\right)^{\frac{1}{2}}$$

Problem 1.5.5

Let $v = \ln y$, we have $y = \exp(v)$ and $y' = \exp(v)\,v'$. Thus,

$$x\exp(v)\,v' - 4x^2\exp(v) + 2\exp(v)\,v = 0$$
$$v' + \frac{2}{x}v = 4x$$

which is a linear DE. Let

$$\rho(x) = \exp\left(\int\frac{2}{x}dx\right) = x^2$$

We have

$$v = x^{-2}\int 4x^3 dx$$

$$= x^2 + Cx^{-2}$$

Thus, we have

$$y = \exp(x^2 + Cx^{-2})$$

Problem 1.5.6

$$tx' - (m+1)t^m x + 2x\ln x = 0$$

$$\frac{tx'}{x} - (m+1)^{tm} + 2\ln x = 0$$

Substitute

$$y = \ln x, \qquad y' = \frac{x'}{x}$$

$$ty' - (m+1)t^m + 2y = 0$$

$$y' + \frac{2}{t}y = (m+1)t^m$$

Using 1st-order linear

$$\rho(t) = \exp\left(\int \frac{2}{t}dt\right) = t^2$$

$$t^2y' + 2ty = (m+1)t^{m+1}$$

$$t^2y = \int (m+1)t^{m+1}dt$$

$$t^2y = \frac{m+1}{m+2}t^{m+2} + c$$

$$y = \frac{m+1}{m+2}t^m + ct^{-2}$$

$$\ln x = \frac{m+1}{m+2}t^m + ct^{-2}$$

$$x = \exp\left(\frac{m+1}{m+2}t^m + ct^{-2}\right)$$

Problem 1.5.7

Let $v = y^{\frac{1}{3}}$, then

$$y = v^3, \qquad y' = 3v^2v'$$

The DE becomes

$$v' - \frac{2}{x}v = 4x^3$$

Let

$$\rho(x) = \exp\left(\int -\frac{2}{x}dx\right) = x^{-2}$$
$$(v\rho)' = 4x$$
$$vx^{-2} = 2x^2 + C$$
$$v = 2x^4 + Cx^2$$

Finally,

$$y = (2x^4 + Cx^2)^3$$

Problem 1.5.8

Let $v = y'$ and consider v as a function of y. Now we have

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

Thus, the DE becomes

$$yv\frac{dv}{dy} = 3v^2$$
$$\frac{dv}{dy} = \frac{3v}{y}$$
$$v = C_1y^3$$

That is

$$\frac{dy}{dx} = C_1y^3$$
$$-\frac{1}{2}y^2 = C_1x + C_2$$

Problem 1.5.9

<u>Method I: Separable</u>

This DE can be write as

$$\frac{y'}{y^3 - y} = x$$

Thus

$$\frac{1}{2}\ln(y^2-1)-\ln y=\frac{1}{2}x^2+C_1$$
$$y^{-2}(y^2-1)=C\exp(x^2)$$
$$1-y^{-2}=C\exp(x^2)$$

Method II: Bernoulli

Let $v=y^{-2}$. Thus $v'=-2y^{-3}y'$. The DE becomes

$$-\frac{1}{2}v'+xv=x$$
$$v'-2xv=-2x$$

This is a linear DE, so let

$$\rho(x)=\exp\left(\int -2xdx\right)$$
$$=\exp(-x^2)$$

Thus, we know that

$$(\exp(-x^2)\,v)'=-2x\exp(-x^2)$$
$$v=\exp(x^2)\,(\exp(-x^2)+C)$$
$$=1+C\exp(x^2)$$

Put $v=y^{-2}$ back, we have

$$y^{-2}-1=C\exp(x^2)$$

Problem 1.5.10

Let

$$u=\frac{y}{x}$$

We have

$$y=ux \text{ and } \frac{dy}{dx}=u+x\frac{du}{dx}$$

Thus, the DE becomes

$$u+x\frac{du}{dx}=-\frac{ux(2x^3-u^3x^3)}{x(2u^3x^3-x^3)}$$

That is

$$x\frac{du}{dx}=-\frac{u^4+u}{2u^3-1}$$

$$\ln(u+1) - \ln u + \ln(u^2 - u + 1) = -\ln x + C$$
$$\frac{u^3+1}{u} = \frac{C}{x}$$
$$\frac{(y^3+x^3)}{xy} = C$$

which gives

$$y^3 + x^3 = Cxy$$

Problem 1.5.11

The DE can be written as

$$y' = \frac{y}{x} + \exp\left(\frac{y}{x}\right)$$

Let

$$u = \frac{y}{x}$$

We have

$$y = ux \text{ and } \frac{dy}{dx} = u + x\frac{du}{dx}$$

The DE becomes

$$u + x\frac{du}{dx} = u + \exp(u)$$
$$\exp(-u)\,du = \frac{dx}{x}$$
$$-\exp(-u) = \ln x + C$$

The solution of the DE is

$$-\exp\left(-\frac{y}{x}\right) = \ln x + C$$

Problem 1.5.12

Let $u = x + y$. We have $y' = u' - 1$. Thus

$$uu' - u = 1$$
$$u' = \frac{1}{u} + 1$$

$$\frac{du}{\frac{1}{u}+1} = dx$$

$$u - \ln(u+1) = x + C$$

That is

$$x + y - \ln(x+y+1) = x + C$$

$$y - \ln(x+y+1) = C$$

Problem 1.5.13

Let $v = \sin y$, then $v' = y' \cos y$. The DE becomes

$$2xvv' = 4x^2 + v^2$$

$$v' - \frac{1}{2x}v = 2xv^{-1}$$

Let $u = v^2$. Then $u' = 2vv'$

$$u - \frac{1}{x}u = 4x$$

We have

$$\rho(x) = \exp\left(\int -\frac{1}{x}dx\right) = x^{-1}$$

Then,

$$(x^{-1}u)' = 4$$

$$x^{-1}u = 4x + C$$

$$u = 4x^2 + Cx$$

Solution is

$$\sin^2 y = 4x^2 - Cx$$

Problem 1.5.14

Let $v = y'$. Then $y'' = v'$ and the DE becomes

$$v' = (x+v)^2$$

Let $u = x + v$. Then $v' = u' - 1$

$$u' - 1 = u^2$$

$$\frac{du}{u^2+1} = dx$$

$$x = \arctan u + C_1$$
$$u = \tan(x - C_1)$$

Thus

$$y' = v = u - x$$
$$= \tan(x - C_1) - x$$

Finally,

$$y = -\ln(\cos(x - C_1)) - \frac{1}{2}x^2 + C_2$$

Problem 1.5.15

Let $v = \exp(y)$. Thus $y = \ln v$ and we have

$$y' = \frac{1}{v}v'$$

Thus the DE becomes

$$(x + v)\frac{1}{v}v' = \frac{x}{v} - 1$$
$$(x + v)v' = x - v$$

Let $u = x + v$. Then we have $v' = u' - 1$ and

$$uu' - u = -u + 2x$$
$$uu' = 2x$$
$$\frac{1}{2}u^2 = x^2 + C_1$$
$$u^2 = 2x^2 + C$$

Put back $u = x + v$ gives

$$v^2 - x^2 + 2xv = C$$

and finally

$$\exp(2y) - x^2 + 2x\exp(y) = C$$

Problem 1.5.16

We can rewrite the DE as

$$y' = \left(x^{-\frac{3}{2}} - 9x^{\frac{1}{2}}\right)y^2$$

which is a Bernoulli's equation. Let $v = y^{-1}$ and we have

$$v' = -y^{-2}y'$$

Thus, the DE becomes

$$v' = 9x^{\frac{1}{2}} - x^{-\frac{3}{2}}$$

It is easy to get

$$v = 6x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C$$

Therefore

$$y = \left(6x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C\right)^{-1}$$

Problem 1.5.17

We can rewrite the DE as

$$y' + \frac{1}{x}y = \frac{1}{\sqrt{1+x^4}}y^{-2}$$

This is a Bernoulli's equation. Let $v = y^3$, we have

$$v' + \frac{3}{x}v = \frac{3}{\sqrt{1+x^4}}$$

Let

$$\rho(x) = \exp\left(\int \frac{3}{x}dx\right)$$
$$= x^3$$

Thus,

$$(x^3v)' = \frac{3x^3}{\sqrt{1+x^4}}$$
$$v = x^{-3}\int \frac{3x^3dx}{\sqrt{1+x^4}}$$
$$= x^{-3}\left(\frac{3}{2}\sqrt{1+x^4} + C\right)$$

That is

$$x^3y^3 = \left(\frac{3}{2}\sqrt{1+x^4} + C\right)$$
$$y^2 + C_1 = C_2\exp(x)$$

Problem 1.5.18

Let $v = \exp(y)$, $v' = \exp(y)\, y'$. Thus, the DE becomes

$$v' - \frac{2}{x} v = 2x^2 \exp(2x)$$

Let

$$\rho(x) = \exp\left(\int -\frac{2}{x} dx\right) = x^{-2}$$

$$(x^{-2}v)' = 2\exp(2x)$$

$$v = x^2 \int 2\exp(2x)\, dx$$

$$= x^2(\exp(2x) + C)$$

Thus

$$y = 2\ln x + \ln(\exp(2x) + C)$$

Problem 1.5.19

Let

$$v = \ln y \ \Rightarrow y = \exp(v) \Rightarrow \frac{dy}{dx} = \exp(v)\frac{dv}{dx}$$

Substituting dy/dx and y by the above, we have

$$\frac{\exp(v)\frac{dv}{dx} - 4xv\exp(v) + 2\frac{\exp(v)}{x} v^n = 0}{\exp(v)}$$

$$\Rightarrow \frac{dv}{dx} - 4xv = -\frac{2}{x} v^n$$

which is Bernoulli's equation. Let

$$u = v^{1-n}$$

We have

$$\frac{du}{dx} - 4(1-n)xu = -(1-n)\frac{2}{x}$$

This is a 1st-order DE of u WRT x, the integrating factor is

$$\rho(x) = \exp\left(\int -4(1-n)x dx\right) = \exp(2(n-1)x^2)$$

$$u(x) = \exp(-2(n-1)x^2)\left(\int \frac{2(n-1)}{x}\exp(2(n-1)x^2)\,dx + C\right)$$

Back substitute twice to get the solution $y(x)$.

Problem 1.5.20

Let $v = y'$ and consider v as a function of y. Now we have

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

Thus, the DE becomes

$$yv\frac{dv}{dy} + v^2 = yv$$

$$y\frac{dv}{dy} + v = y$$

$$\frac{d}{dy}(yv) = y$$

$$yv = \frac{1}{2}y^2 + C$$

$$v = \frac{1}{2}y + \frac{C}{y}$$

Put $v = y'$ back, we have

$$y' = \frac{1}{2}y + \frac{C}{y}$$

$$\frac{2yy'}{y^2 + C_1} = 1$$

$$\ln(y^2 + C_1) = x + \bar{C}$$

Problem 1.5.21

It is easy to observe that $y = 0$ is a solution. For $y \neq 0$, we can rewrite the DE as

$$6\left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^2 + \left(9 + 8\left(\frac{y}{x}\right)\right)y' = 0$$

Let

$$u = \frac{y}{x}$$

We know that $y' = u + u'x$. Thus,

$$6u + 2u^2 + (9 + 8u)(u + u'x) = 0$$

$$\frac{9 + 8u}{3u + 2u^2} u' = -\frac{5}{x}$$

which is a separable DE. We can get

$$3 \ln u + \ln(3 + 2u) = -5 \ln x + C$$

$$u^3(3 + 2u) = Cx^{-5}$$

Put $y = ux$ back, we have

$$3x^2y^3 + 2x^4y = C$$

Problem 1.5.22

(1) From $y = x^{-1} + u$, we can have

$$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{du}{dx}$$

Substituting y and y' in the original DE,

$$-\frac{1}{x^2} + \frac{du}{dx} + 7\left(\frac{1}{x} + u\right)\frac{1}{x} - 3\left(\frac{1}{x} + u\right)^2 = \frac{3}{x^2}$$

$$-\frac{1}{x^2} + \frac{du}{dx} + \frac{7}{x^2} + \frac{7u}{x} - 3\left(\frac{1}{x^2} + \frac{2u}{x} + u^2\right) = \frac{3}{x^2}$$

$$\frac{du}{dx} + \frac{u}{x} - 3u^2 = 0$$

$$u' + x^{-1}u = 3u^2$$

which is a Bernoulli equation of u in terms of x, where $P(x) = x^{-1}$, $Q(x) = 3$, $n = 2$.

(2) Let $v = u^{1-n} = u^{1-2} = u^{-1}$, we have

$$\frac{dv}{dx} + (1 - n)P(x)v = (1 - n)Q(x)$$

$$v' - \frac{v}{x} = -3$$

This is a 1st-order linear DE, where integrating factor is

$$\rho(x) = \exp\left(\int -\frac{1}{x}dx\right) = \frac{1}{x}$$

Then,

$$v = \frac{1}{\rho}\left(\int \rho(x)(-3)dx + C\right) = x\left(-\int \frac{3}{x}dx + C\right)$$
$$= x(-3\ln x + C) = -3x\ln x + Cx$$

Substituting back v with u^{-1} and u with $y - x^{-1}$, we have

$$-3x\ln x + Cx = \frac{1}{u} = \frac{1}{y - x^{-1}}$$
$$y = \frac{1}{-3x\ln x + Cx} + \frac{1}{x}$$

Problem 1.5.23

Let $v = y'$, thus

$$y'' = v'$$

and the original DE becomes

$$x^2v' + 3xv = 4$$
$$(x^3v)' = 4x$$
$$x^3v = 2x^2 + C_1$$
$$v = \frac{2}{x} + \frac{C_1}{x^3}$$

Put $v = y'$ back, we have

$$y' = \frac{2}{x} + \frac{C_1}{x^3}$$
$$y = \int \left(\frac{2}{x} + \frac{C_1}{x^3}\right)dx$$
$$= 2\ln x - \frac{C_1}{2x^2} + C_2$$

Problem 1.5.24

Convert the DE to the form of Bernoulli DE.

$$x' - 1000t^{999}x = -2x^2t^{-1}$$

Let $v = x^{-1}, v' = -x^{-2}x'$. Plug into the above DE,

$$-x^2 v' - 1000t^{999} x^2 v = -2x^2 t^{-1}$$
$$v' + 1000t^{999} v = 2t^{-1}$$

The integrating factor is

$$\rho = \exp\left(\int 1000t^{999}\, dt\right) = \exp(t^{1000})$$

Multiply the integrating factor to both sides,

$$(\exp(t^{1000})\, v)' = 2t^{-1} \exp(t^{1000})$$
$$v = \exp(-t^{1000}) \int 2t^{-1} \exp(t^{1000})\, dt$$
$$x = \exp(t^{1000}) \left(\int 2t^{-1} \exp(t^{1000})\, dt\right)^{-1}$$

Problem 1.5.25

$$\frac{dy}{dx} = \frac{x(x+y)}{y(x+y)} = \frac{x}{y}$$

(when $x + y \neq 0$, dividing the equation by $x + y$)
This equation is simply homogeneous and separable.
Introducing substitution

$$v = -\frac{y}{x}, y' = v + xv'$$

Plug this into original DE, we have

$$v' + \frac{v}{x} = \frac{1}{xv}$$

which is a Bernoulli's equation with $n = -1$. Let

$$u = v^{1-n} = v^2$$

So that

$$v = u^{\frac{1}{2}}, \qquad v' = \frac{1}{2} \mathrm{u}^{-\frac{1}{2}} u'$$

Then

$$v' + \frac{v}{x} = \frac{1}{xv}, \qquad u' + \frac{2u}{x} = \frac{2}{x}$$

$$\rho(x) = \exp\left(\int p(x) dx\right) = \exp\left(\int \frac{2}{x} dx\right) = \exp(2\ln x) = x^2$$

$$u = \exp\left(-\int p(x)dx\right)\left(\int Q(x)\exp\left(\int p(x)dx\right) + C\right)$$
$$= \frac{1}{x^2}\left(\int \left(\frac{2}{x}x^2\right)dx + C\right)$$
$$= \frac{1}{x^2}(x^2 + C)$$

After back substitution

$$u = \left(\frac{y}{x}\right)^2$$

One gets the G.S.

$$y^2 = x^2 + C$$

Problem 1.5.26

$$y' + \frac{1}{x}y + \frac{1}{2x^2}y^{-1} = 0$$

(Dividing the equation with $2x^2y$)

$$y' + \frac{1}{x}y = -\frac{1}{2x^2}y^{-1}$$

Let

$$v = y^{1-n} = y^2$$

So that

$$y = v^{\frac{1}{2}}, \qquad y' = \frac{1}{2}v^{-\frac{1}{2}}v'$$

Plug these to equation

$$2x^2yy' + 2xy^2 + 1 = 0$$

We have

$$x^2v' + 2xv' + 1 = 0$$

(dividing the equation with x^2)

$$v' + \frac{2}{x}v' = -\frac{1}{x^2}, \qquad P(x) = \frac{2}{x}, \qquad Q(x) = -\frac{1}{x^2}$$

$$\rho(x) = \exp\left(\int p(x)dx\right) = \exp\left(\int \frac{2}{x}dx\right) = \exp(2\ln x) = x^2$$

$$v = \left(\int Q(x)\exp\left(\int p(x)dx\right) + C\right)$$

$$= \frac{1}{x^2}\left(\int\left(-\frac{1}{x^2}x^2\right)dx + C\right)$$
$$= \frac{1}{x^2}(-x + C)$$

Therefore the G.S. is

$$y^2 = \frac{1}{x^2}(-x + C)$$

Problem 1.5.27

Method I: Separation of variables

$$(x^2+1)\frac{dy}{dx} = 2x(y+1)$$
$$\int \frac{1}{y+1}dy = \int \frac{2x}{x^2+1}dx$$
$$\ln(y+1) = \ln(x^2+1) + \ln(C)$$
$$\mathrm{y} = C(x^2+1) - 1$$

Method II: Integration factor

$$\frac{dy}{dx} - \frac{2x}{x^2+1}y = \frac{2x}{x^2+1}$$

Find the integration factor,

$$\rho(x) = \exp\left(-\int \frac{2x}{x^2+1}dx\right) = \frac{1}{x^2+1}$$

Multiply the integration factor and take an integral,

$$\frac{d}{dx}\left(\frac{y}{x^2+1}\right) = \frac{2x}{(x^2+1)^2}$$
$$\frac{y}{x^2+1} = \int d\left(\frac{y}{x^2+1}\right) = \int \frac{2x}{(x^2+1)^2}dx = -\frac{1}{x^2+1} + \mathrm{C}$$
$$y = C(x^2+1) - 1$$

Problem 1.5.28

Using substitution $v = y/x$, we get

$$xv' = -b\sqrt{1+v^2}$$

Solving this equation by separation of variables (x and v), we get

$$y = \frac{a}{2}\left(\left(\frac{x}{a}\right)^{1-b} - \left(\frac{x}{a}\right)^{1+b}\right)$$

(1) If $b < 1$, the solution y can be zero (answer to (4)).
(2) If $b = 1$, we have

$$y = \frac{a}{2}\left(1 - \left(\frac{x}{a}\right)^2\right)$$

Thus, at $x = 0$, we have $y = a/2$.
(3) If $b > 1$, y blows up easily.

Problem 1.5.29

Let

$$x = \frac{1}{t} + u, \qquad \left(\frac{dx}{dt} = -\frac{1}{t^2} + \frac{du}{dt}\right)$$

$$-\frac{1}{t^2} + \frac{du}{dt} = 3\left(\frac{1}{t} + u\right)^2 - \frac{8}{t}\left(\frac{1}{t} + u\right) + \frac{4}{t^2}$$

$$= -\frac{1}{t^2} - \frac{2u}{t} + 3u^2$$

$$\frac{du}{dt} = -\frac{2u}{t} + \ 3u^2$$

$$\frac{du}{dt} + \frac{2u}{t} = 3u^2 \ \text{(Bernoulli equation)}$$

$$P(t) = \frac{2}{t}, \qquad q(t) = 3, \qquad n = 2$$

Let

$$v = u^{-1}, \frac{dv}{dt} + (1-n)P(t)v = (1-n)q(t)$$

$$v' - \frac{2}{t}v = -3, \qquad \rho(t) = \exp\left(\int -\frac{2}{x}dx\right) = x^{-2}$$

$$v(t) = \frac{1}{\rho(t)}\int \rho(t)\,(-3)dt$$

$$= t^2\int -\frac{3}{t^2}dt + c$$

$$= u^{-1}(t)$$

$$= \frac{1}{x(t) - \frac{1}{t}}$$

Therefore

$$x(t) = \frac{1}{3t + Ct^2} + \frac{1}{t}$$

Problem 1.5.30

The original DE can be converted if we let

$$y = \beta + z$$

$$\begin{aligned} y' &= (K(x) + y + \beta)(y - \beta) \\ &= (K(x) + 2\beta + z)z \\ &= (K(x) + 2\beta)z + z^2 \end{aligned}$$

or

$$z' - (K(x) + 2\beta)z = z^2$$

which is a perfect Bernoulli equation with

$$P(x) = -(K(x) + 2\beta)z$$

$$Q(x) = 1$$

$$n = 2$$

Problem 1.5.31

$$tx' - x = \beta x' x + \beta t$$

$$tx' - \beta x' x = \beta t + x$$

$$x'(t - \beta x) = \beta t + x$$

$$x' = \frac{\beta t + x}{t - \beta x} = \frac{\left(\beta + \frac{x}{t}\right)}{1 - \frac{\beta x}{t}}$$

Let

$$u = \frac{x}{t}$$

Then

$$\frac{dx}{dt} = t \cdot \frac{du}{dt} + u, \qquad t\frac{du}{dt} + u = \frac{\beta + u}{1 - \beta u}$$

$$t\frac{du}{dt} = \frac{\beta + u}{1-u} - u, \qquad t\frac{du}{dt} = \frac{\beta + \beta u^2}{1-\beta u}$$

$$\int \frac{1-\beta u}{\beta + \beta u^2} du = \int \frac{dt}{t}$$

$$\frac{1}{\beta}\int \frac{1}{1+u^2} du - \int \frac{u}{1+u^2} du = \ln t + C$$

$$\frac{1}{\beta}\arctan(u) + \frac{1}{2}\ln|1+u^2| = \ln t + C$$

$$\arctan(u) = \frac{\beta}{2}\ln\frac{t^2}{1+u^2} + c_1$$

So

$$u = \tan\left(\frac{\beta}{2}\ln\frac{t^2}{1+u^2} + c_1\right)$$

Back substituting, one can get the final solution

$$x = t\tan\left(\frac{\beta}{2}\ln\frac{t^2}{1+u^2} + c_1\right)$$

Problem 1.5.32

For $n = 0$

$$y' = by^2 + c$$

$$\frac{dy}{dx} = by^2 + c$$

$$\frac{dy}{by^2 + c} = dx$$

$$\int \frac{dy}{b\left(y^2 + \frac{c}{b}\right)} = \int dx$$

If $bc > 0$

$$\frac{1}{b}\int \frac{dy}{y^2 + \left(\sqrt{\frac{c}{b}}\right)^2} = x$$

$$\left(\frac{1}{b}\right)\left(\frac{1}{\sqrt{\frac{c}{b}}}\right)\tan^{-1}\left(\frac{y}{\sqrt{\frac{c}{b}}}\right)=x+A$$

$$\frac{1}{\sqrt{bc}}\tan^{-1}\sqrt{\frac{b}{c}}\,y=x+A$$

$$\tan^{-1}\sqrt{\frac{b}{c}}y=\sqrt{bc}(x+A)$$

$$\sqrt{\frac{b}{c}}y=\tan\left(\sqrt{bc}(x+A)\right)$$

$$y=\sqrt{\frac{c}{b}}\tan\left(\sqrt{bc}(x+A)\right)$$

If $bc<0$

$$\frac{1}{b}\int\frac{dy}{y^2-\left(\sqrt{-\frac{c}{b}}\right)^2}=x+D$$

$$\frac{1}{2\sqrt{-bc}}\int\left(\frac{1}{y-\sqrt{-\frac{c}{b}}}-\frac{1}{y+\sqrt{-\frac{c}{b}}}\right)dy=x+D$$

$$\frac{1}{2\sqrt{-bc}}\ln\left|\frac{y-\sqrt{-\frac{c}{b}}}{y+\sqrt{-\frac{c}{b}}}\right|=x+E$$

$$\frac{by-\sqrt{-bc}}{by+\sqrt{-bc}}=F\ \exp\left(2\sqrt{-bc}x\right)$$

For $n=-2$

$$y'=by^2+cx^{-2}$$

$$v=\frac{1}{y}\quad v'=-y^{-2}y'$$

$$y = \frac{1}{v} \quad y' = -v^{-2}v'$$
$$-v^{-2}v' = bv^{-2} + cx^{-2}$$
$$v' = -b - cx^{-2}v^2$$

This is now a simple homogeneous equation and one can solve it easily by a substitution.

$$z = \frac{v}{x}$$

We have

$$(xz)' = -b - cz^2$$
$$xz' = -cz^2 - z - b$$
$$\int \frac{1}{-cz^2 - z - b} dz = \int \frac{1}{x} dx$$

Case (1): $1 - 4bc > 0$

Let z_1, z_2 be the roots of the equation $-cz^2 - z - b = 0$. Then we have

$$\int \frac{1}{c(z_2 - z_1)} \left(\frac{1}{z - z_1} - \frac{1}{z - z_2} \right) dz = \int \frac{1}{x} dx$$
$$\frac{1}{c(z_2 - z_1)} \ln \left| \frac{z - z_1}{z - z_2} \right| = \ln |x| + G_1$$

where G_1 is a constant.

Case (2): $1 - 4bc < 0$

$$\int \frac{1}{-c\left(z + \frac{1}{2c}\right)^2 + \frac{4bc - 1}{-4c}} dz = \int \frac{1}{x} dx$$

Let

$$A = -c, B = \frac{4bc - 1}{-4c}, t = z + \frac{1}{2c}$$

Then by assumption, we have $AB > 0$. So

$$\frac{1}{\sqrt{AB}} \int \frac{1}{\left(\sqrt{\frac{A}{B}} t\right)^2 + 1} d\sqrt{\frac{A}{B}} t = \int \frac{1}{x} dx$$
$$\sqrt{\frac{1}{AB}} tan^{-1} \left(\sqrt{\frac{A}{B}} t \right) + G_2 = ln|x|$$

$$\sqrt{\frac{1}{AB}}\arctan\sqrt{\frac{A}{B}}t + G_2 = \ln|x|$$

where G_2 is another constant.
Case (3): $1 - 4bc = 0$
Then

$$-\frac{1}{c}\int\frac{1}{\left(z+\frac{1}{2c}\right)^2}dz = \int\frac{1}{x}dx$$

or

$$x = H_3\exp\left(\frac{2xy}{2c+xy}\right)$$

The G.S. can be expressed differently depending on the values of $1 - 4bc$.

1.6 The Exact DEs

Problem 1.6.1

From the DE, we have

$$M(x,y) = 1 + \ln(xy) \quad \text{and} \quad N(x,y) = \frac{x}{y}$$

We can evaluate

$$M_y(x,y) = \frac{1}{y}$$

and

$$N_x(x,y) = \frac{1}{y}$$

which gives $M_y = N_x$. This means the DE is exact. Now we can solve the DE by exact DE method.

$$\begin{aligned} F(x,y) &= \int N(x,y)dy \\ &= \int \frac{x}{y}dy \\ &= x\ln y + g(x) \end{aligned}$$

and

$$\frac{\partial F}{\partial x} = M(x,y)$$

$$\ln y + g'(x) = 1 + \ln(xy)$$

gives

$$g'(x) = 1 + \ln x$$

Thus,

$$\begin{aligned} g(x) &= x + x\ln x - x \\ &= x\ln x \end{aligned}$$

$$F(x,y) = x\ln y + x\ln x$$

and finally, we have the G.S. to the DE

$$x\ln y + x\ln x = C$$

Problem 1.6.2

(1) Multiplying the integrating factor on both side of the original DE

$$\exp\left(\int P(x)dx\right)A(x,y)dx + \exp\left(\int P(x)dx\right)B(x,y)dy = 0$$

$$M(x,y) = \exp\left(\int P(x)dx\right)A(x,y)$$

$$N(x,y) = \exp\left(\int P(x)dx\right)B(x,y)$$

$$\begin{aligned}\frac{\partial M}{\partial y} - \frac{\partial B}{\partial x} &= \frac{\partial A}{\partial y}\exp\left(\int Pdx\right) - \frac{\partial B}{\partial y}\exp\left(\int Pdx\right) \\ &\quad - BP\exp\left(\int Pdx\right) \\ &= \left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial y} - BP\right)\exp\left(\int Pdx\right) \\ &= 0\end{aligned}$$

The new DE is an exact DE.

(2) According to $\left(\frac{\partial A(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial x}\right)/B(x,y) = P(x)$,

$$P(x) = \frac{\left(\frac{\partial A(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial x}\right)}{B(x,y)} = \frac{-2y - y}{xy} = -\frac{3}{x}$$

The integrating factor is

$$\rho(x) = \exp\left(\int P(x)dx\right) = \exp\left(\int -\frac{3}{x}dx\right) = x^{-3}$$

The new exact DE is

$$x^{-3}(2x - y^2)dx + x^{-3}xydy = 0$$

$$M(x,y) = 2x^{-2} - x^{-3}y^2, \qquad N(x,y) = x^{-2}y$$

$$F(x,y) = \int Mdx + g(y) = -2x^{-1} + \frac{1}{2}x^{-2}y^2 + g(y)$$

With $\partial F/\partial y = N$, we have

$$\frac{\partial F}{\partial y} = x^{-2}y + g'(y) = N = x^{-2}y$$

$$g'(y) = 0$$

$$g(y) = C$$

The solution of the DE is

$$2x^{-1} + \frac{1}{2}x^{-2}y^2 = C_1$$

Problem 1.6.3

(1)
$$\alpha(x)dy + \big(\beta(x)y + \gamma(x)\big)dx = 0$$
$$M = \beta(x)y + \gamma(x),\ \ N = \alpha(x)$$

If $M_y = N_x$ (i.e., the DE is exact), then the solution is

$$F(x,y) = \int Mdx + \int Ndy - \int\left(\frac{d}{dy}\left(\int Mdx\right)\right)dy = C$$

If $M_y \neq N_x$, check if

$$\frac{M_y - N_x}{N} = f(x)$$

is a function of pure x, or

$$\frac{M_y - N_x}{M} = g(y)$$

is a function of pure y.

Choose the easiest

$$\rho(x) = \exp\left(\int f(x)dx\right)$$

or

$$\rho(y) = \exp\left(-\int g(y)dy\right)$$

Let $\bar{M} = \rho(x)M$ and $\bar{N} = \rho(x)N,$ replace the $M$and N in the solution above with $\bar{M}$ and $\bar{N}$.

(2)

$$y' + \frac{\beta(x)}{\alpha(x)}y = -\frac{\gamma(x)}{\alpha(x)}, P(x) = \frac{\beta(x)}{\alpha(x)}$$

and

$$Q(x) = -\frac{\gamma(x)}{\alpha(x)}$$

The solution is

$$y = \exp\left(-\int P(x)dx\right)\int Q(x)\,dx + c$$

Problem 1.6.4

(1) Check if this DE is Exact.

$$M(x,y) = \left(P(x)y - Q(x)\right),\;\; N(x,y) = \;1$$

$$P(x) = \frac{\partial M(x,y)}{\partial y} \neq \;\frac{\partial N(x,y)}{\partial x} = 0$$

So, this DE is not an Exact DE.

(2) If not, convert it to an Exact DE.

For converting given DE into Exact DE, we need to calculate two ratios.

$$f(x) = \frac{\dfrac{\partial M(x,y)}{\partial y} - \dfrac{\partial N(x,y)}{\partial x}}{N(x,y)} = P(x)$$

Because $f(x)$ is a function of x, we have the integration factor

$$\rho(x) = \exp\left(\int f(x)dx\right) = \;\exp\left(\int p(x)dx\right)$$

By multiplying the integration factor into given DE, we can obtain the Exact DE.

$$\rho(x)\left(P(x)y - Q(x)\right)dx + \rho(x)dy = 0$$

$$\exp\left(\int p(x)dx\right)\left(P(x)y - Q(x)\right)dx + \exp\left(\int p(x)dx\right)dy = 0$$

So, we can get new $M(x,y)$ and $N(x,y)$ as following

$$M(x,y) = \exp\left(\int p(x)dx\right)\left(P(x)y - Q(x)\right)$$

$$N(x,y) = \;\exp\left(\int p(x)dx\right)$$

$$\exp\left(\int p(x)dx\right)P(x) = \frac{\partial M(x,y)}{\partial y} = \;\frac{\partial N(x,y)}{\partial x}$$

$$= \exp\left(\int p(x)dx\right)P(x)$$

(3) Solve the DE using the Exact Equation method and your solution may be expressed in terms of the functions $P(x)$ and $Q(x)$.

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) = \exp\left(\int p(x)dx\right)\left(P(x)y - Q(x)\right)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \exp\left(\int p(x)dx\right)$$

By integrating first equation, we can get

$$F(x,y) = \int M(x,y)dx + g(y)$$
$$= y\exp\left(\int p(x)dx\right)$$
$$- \int Q(x)\exp\left(\int p(x)dx\right)\left(\int p(x)dx\right)dx$$
$$+ g(y)$$

By integrating second equation, we can get

$$F(x,y) = \int N(x,y)dy + f(x) = y\exp\left(\int p(x)dx\right) + f(x)$$

By comparing the two results, we get the solution.

$$F(x,y) = y\exp\left(\int p(x)dx\right)$$
$$- \int Q(x)\exp\left(\int p(x)dx\right)\left(\int p(x)dx\right)dx + C$$

(4) If $P(x) = \frac{1}{x}$ and $Q(x) = \frac{\cos x}{x}$, get the specific solution.

We can apply the result of previous problem.

$$F(x,y) = y\exp\left(\int p(x)dx\right) - \int Q(x)\exp\left(\int p(x)dx\right)dx$$
$$+ C$$
$$= y\exp\left(\int \frac{1}{x}dx\right) - \int \frac{\cos x}{x}\exp\left(\int \frac{1}{x}dx\right)dx + C$$
$$= xy - \sin x + C$$

$$F = xy - \sin x + C$$

Problem 1.6.5

First, let us check if the DE is exact. We have

$$M(x,y) = 2x - y^2 \text{ and } N(x,y) = xy$$
$$M_y = -2y \text{ and } N_x = y$$

$M_y \neq N_x$, the DE is not exact. Since

$$M_y - N_x = -3y$$

We have

$$\frac{M_y - N_x}{N} = -\frac{3}{x}$$

which is a single-variable function. Let

$$I(x) = \exp\left(\int -\frac{3}{x}dx\right)$$
$$= x^{-3}$$

We know that

$$\frac{2x - y^2}{x^3}dx + x^{-2}ydy = 0$$

is exact.

$$F(x,y) = \int \frac{y}{x^2}dy$$
$$= \frac{y^2}{2x^2} + g(x)$$

and

$$\frac{\partial}{\partial x}F = M$$

gives

$$\frac{2x - y^2}{x^3} = -y^2x^{-3} + g'(x)$$
$$g(x) = -\frac{2}{x}$$

Therefore, the solution to the DE is

$$\frac{y^2}{2x^2} - \frac{2}{x} = C$$

Problem 1.6.6

Method I: Substitution

The DE can be written as

$$y' = -\frac{3}{4}\frac{x}{y} - \frac{y}{2x}$$

Let

$$u = \frac{y}{x}$$

We know that

$$y' = xu' + u$$

The DE becomes

$$xu' + u = -\frac{3}{4u} - \frac{u}{2}$$

$$\frac{4uu'}{2u^2 + 1} = -\frac{3}{x}$$

$$\ln(2u^2 + 1) = -3\ln x + C_1$$

$$2u^2 + 1 = Cx^{-3}$$

Plug $u = \frac{y}{x}$ back, we have

$$2\left(\frac{y}{x}\right)^2 + 1 = Cx^{-3}$$

$$2y^2x + x^3 = C$$

Method II: Exact DE

The DE can be written as

$$(3x^2 + 2y^2)dx + 4xydy = 0$$

Thus we have

$$M = 3x^2 + 2y^2 \text{ and } N = 4xy$$

Since

$$M_y = 4y \text{ and } N_x = 4y$$

We know that it is an exact DE. Hence we have

$$F(x,y) = \int N dy$$
$$= 2xy^2 + g(x)$$

On the other hand

$$\frac{\partial F}{\partial x} = M$$

We have

$$2y^2 + g'(x) = 3x^2 + 2y^2$$
$$g'(x) = 3x^2$$
$$g(x) = x^3$$

Thus, the G.S. of the DE is

$$2xy^2 + x^3 = C$$

Problem 1.6.7

<u>Method I: Exact DE</u>

We can write the DE as

$$(x + 3y)dx + (3x - y)dy = 0$$

where $M = x + 3y$ and $N = 3x - y$. Thus, we have

$$M_y = N_x = 3$$

Hence the DE is exact. Now we can get

$$F(x,y) = \int M(x,y)dx$$
$$= \frac{1}{2}x^2 + 3xy + g(y)$$

Put it back, we have

$$N = \frac{\partial F}{\partial y}$$
$$3x + g'(y) = 3x - y$$
$$g'(y) = -y$$
$$g(y) = -\frac{1}{2}y^2$$

Thus, we have G.S.

$$\frac{1}{2}x^2 + 3xy - \frac{1}{2}y^2 = C$$

Method II: Substitution

We can write the DE as

$$y' = \frac{1 + 3\frac{y}{x}}{\frac{y}{x} - 3}$$

Let $v = \frac{y}{x}$. Thus $y = vx$ and $y' = v + xv'$. This gives

$$v + xv' = \frac{1 + 3v}{v - 3}$$

$$\frac{v - 3}{1 + 6v - v^2}v' = \frac{1}{x}$$

$$-\frac{1}{2}\ln(1 + 6v - v^2) = \ln x + C_1$$

$$1 + 6v - v^2 = Cx^{-2}$$

Put $v = y/x$ back, we have

$$1 + 6\frac{y}{x} - \left(\frac{y}{x}\right)^2 = Cx^{-2}$$

$$x^2 + 6xy - y^2 = C$$

Problem 1.6.8

We have

$$M(x, y) = y \text{ and } N(x, y) = 2x + y^4$$

By checking

$$M_y = 1 \text{ and } N_x = 2$$

we know the DE is not exact. Since

$$M_y - N_x = -1$$

We can find

$$f(y) = \frac{M_y - N_x}{M} = -\frac{1}{y}$$

which is a single-variable function of y. Let

$$I(y) = \exp\left(-\int -\frac{1}{y}dy\right) = y = y$$

Now we can have an exact DE

$$y^2dx + (2xy + y^5)dy = 0$$

To solve this, we have

$$F(x,y) = \int y^2 dx$$
$$= y^2x + g(y)$$

Since

$$\frac{\partial F}{\partial y} = N$$

we can get

$$2xy + g'(y) = 2xy + y^5$$
$$g(y) = \frac{1}{6}y^6$$

Thus, the solution to the DE is

$$y^2x + \frac{y^6}{6} = C$$

Problem 1.6.9

Rearrange the original equation to yield

$$(2x - y^2)dx + \big(-(2xy + 1)\big)dy = 0$$

Then two parameter functions are given below.

$$M(x,y) = (2x - y^2)$$
$$N(x,y) = -2xy - 1$$

Calculate the difference.

$$M_y - N_x = (-2y) - (-2y) = 0$$

This DE is exact.

Expand the original equation. Using the formula $d(uv) = udv + vdu$ to trace back the solution.

$$2xdx - y^2dx - 2xydy - dy = 0$$

One more step,

$$dx^2 - y^2dx - xdy^2 - dy = 0$$

or

$$d(x^2 - xy^2 - y) = 0$$

or

$$x^2 - xy^2 - y = C$$

Problem 1.6.10

Multiplying the DE by dx makes the equation

$$y^3 dx + (xy^2 - 1)dy = 0$$

The DE can be made exact by multiplying both sides by the integration factor

$$\rho(y) = \exp\left(-\int \frac{3y^2 - y^2}{y^3} dy\right) = y^{-2}.$$

$$ydx + (x - y^{-2})dy = 0$$

$$M(x, y) = y, \qquad N(x, y) = x - y^{-2}$$

$$F(x, y) = \int ydx = xy + g(y)$$

$$\frac{\partial F}{\partial y} = x + g'(y) = x - y^{-2}$$

$$g'(y) = -y^{-2} \Rightarrow g(y) = y^{-1}$$

The solution is then found to be

$$F(x, y) = xy + \frac{1}{y} = C$$

Problem 1.6.11

$$M(x, y) = \cos x + \ln y$$

$$N(x, y) = \frac{x}{y} + \exp(y)$$

$$\frac{\partial M}{\partial y} = \frac{1}{y}, \qquad \frac{\partial N}{\partial x} = \frac{1}{y}$$

It is an exact equation since $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0.$

$$F(s, y) = \int M(x, y)dx + g(y)$$

$$= \int (\cos x + \ln y)dx + g(y)$$
$$= \sin x + x \ln y + g(y)$$
$$\frac{\partial F}{\partial y} = \frac{x}{y} + g' = \frac{x}{y} + \exp(y)$$
$$g' = \exp(y)$$
$$g(y) = \exp(y)$$

Thus, the solution is $F(x, y) = \sin x + x \ln y + \exp(y) = C$.

Problem 1.6.12

Method I: Exact DE

$$(\exp(y) + y \cos x)dx + (x \exp(y) + \sin x)dy = 0$$
$$M = \exp(y) + y \cos x$$
$$N = x \exp(y) + \sin x$$
$$M_y = \exp(y) + \cos x$$
$$N_x = \exp(y) + \cos x$$
$$M = \frac{\partial F}{\partial x} = \exp(y) + y \cos x$$
$$F(x, y) = \int M\, dx = x \exp(y) + y \sin x + A(y)$$
$$\frac{\partial F}{\partial y} = x \exp(y) + \sin x + A'(y) = N = x \exp(y) + \sin x$$
$$x \exp(y) + \sin x + A'(y) = x \exp(y) + \sin x$$
$$A'(y) = 0$$
$$A(y) = C$$
$$F(x, y) = x \exp(y) + y \sin x + C = C_2$$
$$x \exp(y) + y \sin x = C_1$$

Method II: Substitution

$$\exp(y) + y \cos x + (x \exp(y) + \sin x)y' = 0$$
$$v = x \exp(y) + y \sin x$$
$$v' = \exp(y) + x \exp(y)\, y' + y' \sin x + y \cos x$$
$$v = (x \exp(y) + \sin x)y' + \exp(y) + y \cos x$$
$$v' = 0$$
$$v = C$$
$$x \exp(y) + y \sin x = C$$

1.7 Riccati DEs

Problem 1.7.1

Since

$$y = y_1 + \frac{1}{v}$$

We have

$$y' = y_1' - \frac{1}{v^2} v'$$

Put these back to the DE, we have

$$y_1' - \frac{1}{v^2} v' = A(x)\left(y_1^2 + \frac{2y_1}{v} + \frac{1}{v^2}\right) + B(x)\left(y_1 + \frac{1}{v}\right) + C(x)$$

Since y_1 is a solution of the DE, we know $y_1' = A(x)y_1^2 + B(x)y_1 + C(x)$. This gives

$$-\frac{1}{v^2} v' = 2A(x)\frac{y_1}{v} + \frac{A(x)}{v^2} + \frac{B(x)}{v}$$

That is

$$v' + (2Ay_1 + B)v = -A$$

Problem 1.7.2

Let

$$y = x + \frac{1}{v}$$

We have that

$$y' = 1 - \frac{1}{v^2} v'$$

Thus, the DE becomes

$$1 - \frac{1}{v^2} v' + x^2 + \frac{2x}{v} + \frac{1}{v^2} = 1 + x^2$$

That is

$$v' - 2xv = 1$$

Let

$$\rho(x) = \exp\left(\int -2x dx\right)$$

$$= \exp(-x^2)$$

We have that

$$(\rho(x)v)' = \rho(x)$$

This gives

$$v = \exp(x^2) \int \exp(-x^2)\, dx$$

$$= \exp(x^2) \left(\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + C \right)$$

Finally, we can have

$$y = x + \frac{1}{v}$$

$$= x + \exp(-x^2) \left(\frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + C \right)^{-1}$$

Note: The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$$

Problem 1.7.3

Substitute

$$-\frac{1}{v} = x - y$$

$$\frac{dv}{v^2 dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{y}{dx} = 1 - \frac{dv}{v^2 dx}$$

$$1 - \frac{dv}{v^2 dx} = 1 + \frac{1}{4}\left(-\frac{1}{v}\right)^2$$

$$-\frac{dv}{v^2 dx} = \frac{1}{4v^2} \Rightarrow \quad -4dv = dx$$

Integrate

$$-4 \int dv = \int dx$$

$$-4v = x + c \Rightarrow \quad v = -\frac{x + c}{4}$$

Since

$$\frac{1}{v} = y - x \Rightarrow \quad v = \frac{1}{y - x}$$
$$\frac{1}{y - x} = -\frac{x + c}{4}$$
$$y = -\frac{4}{x + c} + x$$

Problem 1.7.4

Substitute

$$y = x + \frac{1}{v}$$

and

$$y' = 1 - \frac{v'}{v^2}$$

The original DE becomes

$$1 - \frac{v'}{v^2} - 13\left(x^2 + \left(x + \frac{1}{v}\right)^2\right) + 26x\left(x + \frac{1}{v}\right) = 1$$
$$\frac{v'}{v^2} + \frac{13}{v^2} = 0, \qquad v' = -13, \qquad v = -13x + C$$

Put this back, and the solution to the original DE is

$$y = x + \frac{1}{C - 13x}$$

Problem 1.7.5

Substituting

$$y = a + \frac{1}{z}$$

We get

$$\frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

Substituting into the original DE, we get

$$\frac{dz}{dx} + (f(x) + 2a)z + 1 = 0$$

which is a 1st-order linear DE and solvable with the integrating factor method

$$\rho(x) = \exp\left(\int (f(x) + 2a)dx\right)$$

Apply the integrating factor to the both sides of the $z$function, and we will have:

$$\rho(x)\frac{dz}{dx} + (f(x) + 2a)\rho(x)z + \rho(x) = 0$$

Using the formula $d(uv) = udv + vdu$, we get

$$\frac{d}{dx}(\rho(x)z) = -\rho(x)$$

whose G.S. is

$$z = -\frac{1}{\rho(x)}\int \rho(x)dx + C$$

Finally, the G.S. for the original DE is

$$y = a + \frac{1}{z} = a - \frac{\rho(x)}{\int \rho(x)dx - C\rho(x)}$$

where

$$\rho(x) = \exp\left(\int (f(x) + 2a)dx\right)$$

Problem 1.7.6

$$y' = y^2 + 4x^3y - (a^2 + 4ax^3)$$

This is a Riccati DE. Substitution $y = a + \frac{1}{v}$, we have

$$-\frac{1}{v^2}\frac{dv}{dx} = \left(a + \frac{1}{v}\right)^2 + 4x^3\left(a + \frac{1}{v}\right) - (a^2 + 4ax^3)$$

Simplifying it, we have

$$v' + (2a + 4x^3)v = -1 \tag{1}$$

This is a 1st-order linear DE, where

$$P(x) = 2a + 4x^3, Q(x) = -1$$

$$\rho(x) = \exp\left(\int (2a + 4x^3)dx\right) = \exp(2ax + x^4)$$

Multiplying on both side of Eq-(1) by $\rho(x)$, we have

$$(\rho(x)v)' = -\rho(x)$$

$$v(x) = -\frac{\exp(\int(2ax + x^4)\,dx) + C}{\exp(2ax + x^4)}$$

So the G.S. for original DE is

$$y = a + \frac{1}{v} = a - \frac{\exp(2ax + x^4)}{\exp(\int(2ax + x^4)\,dx) + C}$$

Problem 1.7.7

The DE is a Ricatti Equation with the given solution $y_1(x) = \sin x$, So the necessary substitution is

$$y = y_1(x) + \frac{1}{v} = \sin x + \frac{1}{v}$$

$$y' = \cos x - \frac{v'}{v^2}$$

Substituting back into the original DE gives

$$\cos x - \frac{v'}{v^2} = \frac{2\cos^2 x - \sin^2 x + \left(\sin x + \frac{1}{v}\right)^2}{2\cos x}$$

$$= \frac{2\cos^2 x - \sin^2 x + \sin^2 x + \frac{2\sin x}{v} + \frac{1}{v^2}}{2\cos x}$$

$$= \cos x + \frac{\tan x}{v} + \frac{\frac{1}{v^2}}{2\cos x}$$

The DE can then be rewritten as

$$v' + \tan x\, v = -\frac{1}{2}\sec x$$

Multiply both sides by the integration factor

$$\rho(x) = \exp\left(\int \tan x\, dx\right) = \exp(\ln|\sec x|) = \sec x$$

$$\sec x\, v' + \sec x \tan x\, v = -\frac{1}{2}\sec^2 x$$

$$\int \frac{d}{dx}(\sec x\, v)dx = -\frac{1}{2}\int \sec^2 x\, dx$$

$$\sec x\, v = -\frac{1}{2}\tan x + C$$

$$v = -\frac{1}{2}\sin x + C\cos x$$

Back substituting for

$$v = (y - \sin x)^{-1}$$

gives

$$\frac{1}{y - \sin x} = -\frac{1}{2}\sin x + C\cos x$$

$$y = \frac{2}{c\cos x - \sin x} + \sin x$$

Problem 1.7.8

It is Riccati DE and we know one P.S. is $\boldsymbol{y_1 = x^2}$. We substitute:

$$y = y_1 + \frac{1}{v} = x^2 + \frac{1}{v}$$

$$y' = 2x - \frac{1}{v^2}\frac{dv}{dx}$$

The original DE becomes

$$2x - \frac{1}{v^2}\frac{dv}{dx} = \left(x^2 + \frac{1}{v}\right)^2 + \alpha(x)\left(x^2 + \frac{1}{v} - x^2\right) + 2x - x^4$$

or

$$\frac{dv}{dx} + (\alpha(x) + 2x^2)v = -1$$

If $\alpha(x) \equiv -2x^2$, for any x, then $\frac{dv}{dx} = -1, v(x) = -x + C$. So the solution is

$$y = x^2 + \frac{1}{-x + C}$$

If $\alpha(x) \neq -2x^2$, the integrating factor is

$$\rho(x) = \exp\left(\int (\alpha(x) + 2x^2)dx\right) = \exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)$$

Multiplying on both sides by the integrating factor, we have

$$\frac{d}{dx}\left(\exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)v\right) = -\exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)$$

$$\exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)v$$
$$= -\int \exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)dx + C$$

$$v = -\exp\left(-\int \alpha(x)dx - \frac{2}{3}x^3\right)\int \exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)dx$$
$$+ C\exp\left(-\int \alpha(x)dx - \frac{2}{3}x^3\right)$$

Substituting back, we have

$$y = x^2 + \left(-\exp\left(-\int \alpha(x)dx - \frac{2}{3}x^3\right)\int \exp\left(\int \alpha(x)dx + \frac{2}{3}x^3\right)dx + C\exp\left(-\int \alpha(x)dx - \frac{2}{3}x^3\right)\right)^{-1}$$

Problem 1.7.9

The DE is another Riccati DE with a given solution of

$$y = x^2$$

Using the substitution

$$y = x^2 + \frac{1}{v}$$
$$y' = 2x - \frac{v'}{v^2}$$

The DE then becomes

$$x^3\left(2x - \frac{v'}{v^2}\right) + x^2\left(x^2 + \frac{1}{v}\right) - \left(x^2 + \frac{1}{v}\right)^2 = 2x^4$$

$$2x^4 - \frac{x^3}{v^2}v' + x^4 + \frac{x^2}{v} - x^4 - \frac{2x^2}{v} - \frac{1}{v^2} = 2x^4$$

$$-\frac{x^3}{v^2}v' - \frac{x^2}{v} = \frac{1}{v^2}$$

$$v' + \frac{v}{x} = -\frac{1}{x^3}.$$

Multiplying both sides by the integration factor

$$\rho(x) = \exp\left(\int \frac{1}{x}dx\right) = x$$

gives

$$xv' + v = -\frac{1}{x^2}$$

$$\int \frac{d}{dx}(xv)dx = -\int \frac{1}{x^2}dx$$

$$xv = \frac{1}{x} + C$$

$$v = \frac{1}{x^2} + \frac{C}{x} = \frac{1 + Cx}{x^2}.$$

Back substitute

$$v = (y - x^2)^{-1}$$

$$\frac{1}{y - x^2} = \frac{1 + Cx}{x^2}$$

$$y = x^2 + \frac{x^2}{1 + Cx}$$

Chapter 2 Mathematical Models

2.1 Population Model

Problem 2.1.1

From the population model, we can get

$$\begin{aligned}\frac{dP}{dt} &= (\beta - \delta)P \\ &= (k_1 P - k_2 P)P \\ &= (k_1 - k_2)P^2\end{aligned}$$

where k_1 and k_2 are both constant. Let $k = k_1 - k_2$, which is also a constant.

(1) From the problem, we know that

$$\beta = \frac{k_1}{\sqrt{P}} \text{ and } \delta = \frac{k_2}{\sqrt{P}}$$

where k_1 and k_2 are both constant. Thus

$$\begin{aligned}\frac{dP}{dt} &= (\beta - \delta)P \\ &= \left(\frac{k_1}{\sqrt{P}} - \frac{k_2}{\sqrt{P}}\right)P \\ &= (k_1 - k_2)\sqrt{P}\end{aligned}$$

Let $k = k_1 + k_2$ which is also a constant. We have

$$\frac{dP}{\sqrt{P}} = kdt$$

$$2\sqrt{P} = kt + C_1$$

$$P(t) = \left(\frac{1}{2}kt + C\right)^2$$

Plugging $P(0) = P_0$ into it, we can get $C^2 = P_0$. Thus

$$P(t) = \left(\frac{1}{2}kt + \sqrt{P_0}\right)^2$$

(2) Given $P_0 = 100$ and $P(t = 6) = 169$, we can get

$$169 = \left(\frac{1}{2} \times 6k + \sqrt{100}\right)^2$$

This gives $k = 1$. Thus, it is straightforward to evaluate $P(12) = 16^2 = 256$.

Problem 2.1.2

(1) From the problem, we can get

$$\beta = k_1 P \text{ and } \delta = k_2 P$$

$$\frac{dP}{dt} = kP^2$$

Solve this separable DE, we can get

$$\frac{1}{P} = -kt + C$$

That is

$$P(t) = \frac{1}{C - kt}$$

Plugging $P(0) = P_0$ into the solution, we can get

$$P_0 = \frac{1}{C}$$

Thus, the population is

$$P(t) = \frac{1}{\frac{1}{P_0} - kt}$$

$$= \frac{P_0}{1 - kP_0 t}$$

(2) When $t \to 1/kP_0$ we know that $kP_0 t \to 1$. Thus, $(1 - kP_0 t) \to 0$ and

$$P(t) = \frac{P_0}{1 - kP_0 t} \to \infty$$

(3) Given $P_0 = 6$ and $P(10) = 9$, we can get

$$9 = \frac{6}{1 - k \cdot 6 \cdot 10}$$

$$k = \frac{1}{180}$$

Thus, the doomsday occurs at

$$t = \frac{1}{kP_0} = 30 \text{ months}$$

(4) Suppose $\beta < \delta$, we know that $k < 0$. Thus, we know that

$$1 - kP_0 t > 0, \qquad t > 0$$

and

$$P(t) \to 0, \qquad t \to \infty$$

Problem 2.1.3

According to the discussion in textbook, we know that we can always write the logistic equation into form

$$\frac{dP}{dt} = kP(M - P)$$

where

$$b = k, \text{ and } M = \frac{a}{b}$$

Since for $k = b > 0$, the limiting population will be M, we can easily calculate that

$$M = \frac{a}{b} = \frac{B_0/P_0}{D_0/P_0^2} = \frac{B_0 P_0}{D_0}$$

Problem 2.1.4

(1) This is a 1st-order separable DE.

$$\int \frac{dP}{P(\ln M - \ln P)} = \int k dt + C$$

Let $\ln P = Z$, then $PdZ = dP$. Substitute P by Z and dZ,

$$\int \frac{PdZ}{P(\ln M - Z)} = \int \frac{dZ}{\ln M - Z} = \int k dt + C$$

$$-\ln(\ln M - Z) = kt + C$$

$$\ln(\ln M - \ln P) = -kt - C$$

$$\ln M - \ln P = K \exp(-kt)$$

$$P(t) = \exp(\ln M - K \exp(-kt))$$

$$P(t = 0) = P_0$$

$$K = \ln M - \ln P_0$$

$$P(t) = \exp(\ln M - (\ln M - \ln P_0)\exp(-kt))$$

(2) Since $P_0 > M$,

$$\ln M - \ln P_0 < 0$$

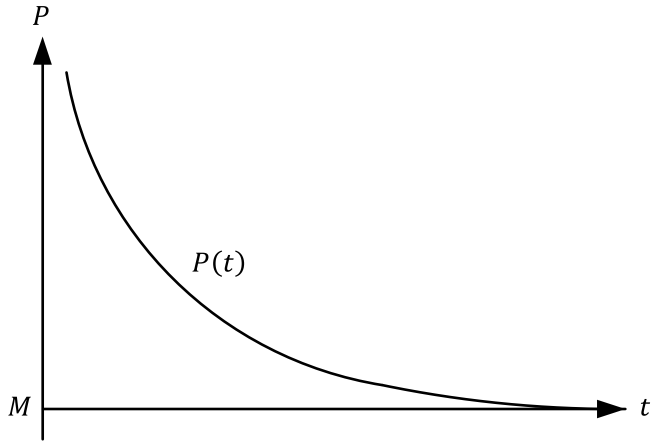

Figure A.1 If $P_0 > M$, population will decrease and approach value M as $t \to \infty$.

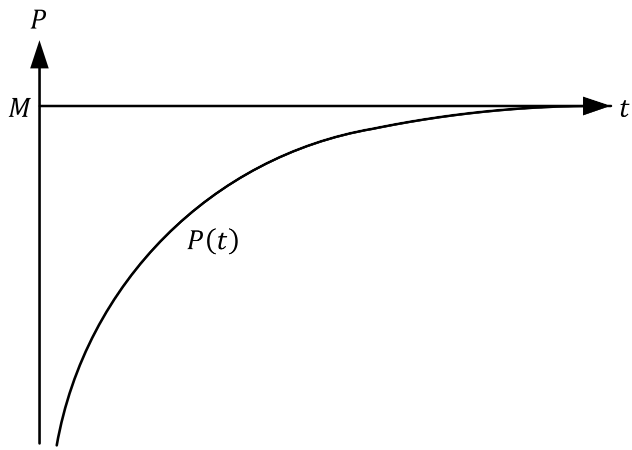

Figure A.2 If $P_0 < M$, population will increase and approach value M as $t \to \infty$.

$$(\ln M - \ln P_0)\exp(-kt) < 0$$
$$\frac{dP}{dt} = kP(\ln M - \ln P_0)\exp(-kt) < 0$$

This means $P(t)$ is a monotonically decreasing function of t. When $t \to \infty$, $P(t) \to M$.

(3) Since $P_0 < M$,

$$\ln M - \ln P_0 > 0$$
$$(\ln M - \ln P_0)\exp(-kt) > 0$$
$$\frac{dP}{dt} = kP(\ln M - \ln P_0)\exp(-kt) > 0$$

This means $P(t)$ is a monotonically increasing function of t. When $t \to \infty$, $P(t) \to M$.

Problem 2.1.5

The DE is

$$\frac{dP}{dt} = \beta_0 \exp(-\alpha t)\, P$$

which is a separable DE. We have

$$\frac{dP}{P} = \beta_0 \exp(-\alpha t)\, dt$$
$$\ln P = -\frac{\beta_0}{\alpha}\exp(-\alpha t) + C$$

Since $P(0) = P_0$, we can get

$$C = \ln P_0 + \frac{\beta_0}{\alpha}$$

Thus,

$$\ln P = \ln P_0 + \frac{\beta_0}{\alpha}(1 - \exp(-\alpha t))$$
$$P(t) = P_0 \exp\left(\frac{\beta_0}{\alpha}(1 - \exp(-\alpha t))\right)$$

Now it is clear that for $t \to \infty$, $\exp(-\alpha t) \to 0$ and hence

$$P(t) \to P_0 \exp\left(\frac{\beta_0}{\alpha}\right)$$

Problem 2.1.6

Since $k > 0$, $M > 0$ and $P > 0$, for the DE

$$\frac{dP}{dt} = kP(M - P)$$

we consider the following two cases.

(1) If $M > P$, we have

$$\frac{dP}{dt} = kP(M - P) > 0$$

Thus the function $P(t)$ is monotonically increasing WRT t.

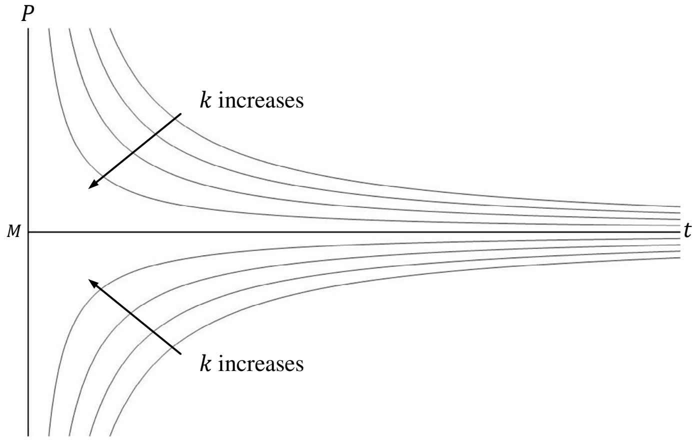

Figure A.3 Function $P(t)$ is close to value M as k increases.

Since dP/dt is the slope of the graph, as k increases, the slope of the curve increase, then it will go closer to M.

(2) If $M < P$, we know

$$\frac{dP}{dt} = kP(M - P) < 0$$

We know $P(t)$ is a monotonically decreasing function in terms of t., see fig.1(Blue lines). As k increases, the absolute value of dP/dt, say $|dP/dt|$ increases, and the slope of the curve is larger, thus the curve goes closer to M.

From above, we can conclude, as k increases, the curve goes closer to the limiting population M.

Problem 2.1.7

Birth rate: $\beta = 0$, and death rate: $\delta = k/\sqrt{p}$, where k is a constant

By general population function, we have

$$\begin{aligned}\frac{dP}{dt} &= (\beta - \delta)P \\ &= \left(0 - \frac{k}{\sqrt{P}}\right)P \\ &= -k\sqrt{P}\end{aligned}$$

with I.C. $P(0) = P_0$. Solving this 1st-order separable DE, we have

$$\frac{dP}{\sqrt{P}} = -kdt$$

$$2\sqrt{P} = -kt + C$$

$$P(0) = P_0 \Rightarrow C = 2\sqrt{P_0}$$

Thus

$$P(t) = \left(-\frac{kt}{2} + \sqrt{P_0}\right)^2$$

From the I.C., we know $P(0) = P_0 = 900$, and $P(6) = 441$, thus

$$441 = \left(-\frac{6k}{2} + \sqrt{900}\right)^2 \Rightarrow k = 3$$

Therefore

$$P(t) = \left(-\frac{3t}{2} + 30\right)^2$$

All the fish being dead means $P(t) = 0$, then we have

$$\left(-\frac{3t}{2} + 30\right)^2 = 0 \Rightarrow t = 20 \text{ weeks}$$

Problem 2.1.8

From the I.C., we have

$$a = \frac{B(0)}{P(0)} = \frac{8}{120} \text{ and } b = \frac{D(0)}{P^2(0)} = \frac{6}{120^2}$$

To solve this DE, we should write the DE into standard logistic equation form

$$\frac{dP}{dt} = kP(M - P)$$

where

$$k = b, \text{ and } M = \frac{a}{b}$$

Since we know the solution to the logistic equation should be

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)\exp(-kMt)}$$

Thus, for $P(t) = .95M$, we can solve for t

$$0.95M = \frac{MP_0}{P_0 + (M - P_0)\exp(-kMt)}$$

gives $t = 27.69$

Problem 2.1.9

With $\alpha = k_1 P$, $\beta = k_2 P$ and $k = k_2 - k_1$, population model is given by

$$\frac{dP}{dt} = (\beta - \alpha)P = (k_2 - k_1)P^2 = kP^2$$

Take the anti-derivative and gives

$$-\frac{1}{P} = kt + C$$

(1) Coupling with I.C., where $(\beta - \alpha) = kP > 0$, and locate the solution as

$$P = \left(\frac{1}{P_0} - kt\right)^{-1}$$

(2) Calculate the doomsday formula.

The denominator is zero, which satisfies the definition of doomsday. So, time is given below as

$$T = \frac{1}{P_0 k}$$

(3) Calculate the doomsday time.

First, calculate the difference between the birth rate and death rate.

$$P(12) = \left(\frac{1}{P_0} - (\beta - \alpha)t\right)^{-1} = \left(\frac{1}{2011} - 12k\right)^{-1} = 4027$$

$$k \approx 2.0745 \cdot 10^{-5}$$

Second, calculate the time.

$$T = \frac{1}{P_0 k} = \frac{1}{2011(2.0745 \cdot 10^{-5})} \approx 23.970$$

(4) Calculate the population.

Due to $(\alpha - \beta) = -kP > 0$, we have $(-k) > 0$. So,

$$P = \left(\frac{1}{P_0} - kt\right)^{-1} \to 0, t \to +\infty$$

As time goes to infinity, the population goes to zero.

Problem 2.1.10

The population modeling is given by

$$\frac{dP}{dt} = (\beta - \alpha)P = \left(0 - \frac{k}{\sqrt{P}}\right)P = -kP^{\frac{1}{2}}$$

The solution is followed by

$$2\sqrt{P} = -kt + C$$

When we couple with the I.C., we get

$$C = 2\sqrt{4000}$$

$$2\sqrt{P} = -kt + 2\sqrt{4000}$$

When we plug in the population value of 2014 WRT time 11, the unknown coefficient is

$$k = \frac{2\sqrt{4000} - 2\sqrt{2014}}{11} \approx 3.3396$$

So, the time for all the fish in the lake to die (let $P(t) = 0$)

$$T = \frac{C}{k} = 37.876$$

If the population does not change with time, $k = 0$. So we need 4000 rather than 2014.

2.2 Acceleration-Velocity Model

Problem 2.2.1

(1) This is a separable DE which can be written as

$$\frac{dv}{v} = -kdt$$
$$\ln v = -kt + C$$
$$v = C\exp(-kt)$$

From the I.C., we know that $v(0) = v_0$. This gives $C = v_0$. Hence the velocity function is

$$v(t) = v_0 \exp(-kt)$$

Also, we know

$$v = \frac{dx}{dt}$$

thus

$$dx = v_0 \exp(-kt)\, dt$$
$$x = -\frac{v_0}{k}\exp(-kt) + C$$

Since $x(0) = x_0$, we have

$$C = x_0 + \frac{v_0}{k}$$

Thus

$$x(t) = x_0 + \frac{v_0}{k}(1 - \exp(-kt))$$

(2) As $t \to \infty$, we know that $e^{-kt} \to 0$. Thus

$$x \to x_0 + \frac{v_0}{k}, \qquad t \to \infty$$

Problem 2.2.2

$$\frac{dv}{dt} = -kv^{\frac{3}{2}}$$
$$\int_{v_0}^{v} \frac{dv}{v^{3/2}} = \int_0^t -kdt$$

$$-2v^{-\frac{1}{2}} + 2{v_0}^{-\frac{1}{2}} = -kt$$

$$2v^{-\frac{1}{2}} = \frac{\sqrt{v_0}kt + 2}{\sqrt{v_0}}$$

$$v(t) = \frac{4v_0}{\left(\sqrt{v_0}kt + 2\right)^2}$$

and

$$\frac{dx}{dt} = v(t) = \frac{4v_0}{\left(\sqrt{v_0}kt + 2\right)^2}$$

$$x(t) = -\frac{4\sqrt{v_0}}{k\left(2 + kt\sqrt{v_0}\right)} + C$$

Putting the I.C. $x(0) = x_0$ into it, we can have

$$C = x_0 + \frac{2\sqrt{v_0}}{k}$$

Thus

$$x(t) = x_0 + \frac{2\sqrt{v_0}}{k}\left(1 - \frac{2}{kt\sqrt{v_0} + 2}\right)$$

As $t \to \infty$, we know that

$$\frac{2}{kt\sqrt{v_0} + 2} \to 0, \qquad t \to \infty$$

Thus, it is easy to know the body only goes a finite distance before it stops.

Note

As the power is greater than $3/2$, the unknown t will be found in the numerator part of function $x(t)$, which means as $t \to \infty$, $x \to \infty$. You can check it by yourself.

Problem 2.2.3

Let

$$v = \frac{dr}{dt}$$

and consider v to be an function of r. We can get

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} = v\frac{dv}{dr}$$

Thus, the DE becomes

$$v\frac{dv}{dr} = -\frac{GM_{\mathrm{e}}}{r^2} + \frac{GM_m}{(S-r)^2}$$

By separation of variables, we have

$$\int v dv = \int \left(-\frac{GM_{\mathrm{e}}}{r^2} + \frac{GM_m}{(S-r)^2}\right) dt$$

$$\frac{v^2}{2} = \frac{GM_{\mathrm{e}}}{r} + \frac{GM_m}{S-r} + C$$

Since $r'(0) = v(0) = v_0$ and $r(0) = 0$, we know $v(r = 0) = v_0$. This gives

$$C = \frac{v_0^2}{2} - \frac{GM_{\mathrm{e}}}{R} - \frac{GM_m}{S-R}$$

Hence the velocity function is

$$v = \sqrt{2GM_{\mathrm{e}}\left(\frac{1}{r} - \frac{1}{R}\right) + 2GM_m\left(\frac{1}{S-r} - \frac{1}{S-R}\right) + v_0^2}$$

The earth and moon attractions balance at the point where the right-hand side in the acceleration function vanished, which means

$$-\frac{GM_{\mathrm{e}}}{r^2} + \frac{GM_m}{(S-r)^2} = 0$$

solve this equation for r, we have

$$r = \frac{\sqrt{M_{\mathrm{e}}}S}{\sqrt{M_{\mathrm{e}}} - \sqrt{M_m}}$$

If we substitute this value of r, $M_m = 7.35 \times 10^{22}$ (kg), $S = 3.844 \times 10^8$ (m), $M_{\mathrm{e}} = 5.975 \times 10^{24}$ (kg), $R = 6.378 \times 10^6$ (m), and set $v = 0$(to just reach the balancing point), we can solve the resulting equation for $v_0 = 11109$ m/s. Note that this is only 71 m/s less than the earth escape velocity of 11180 m/s, so the moon really does not help much.

Problem 2.2.4

(1) The DE for this motion is

$$m\frac{dv}{dt} = -mg - kv$$

$$v = \frac{dy}{dt} \Rightarrow m\frac{dv}{dt} = m\frac{dv}{dy}\frac{dy}{dt} = mv\frac{dv}{dy} = -mg - kv$$

$$\Rightarrow v\frac{dv}{dy} = -g - \frac{k}{m}v$$

This is a separable DE, and by separation of variables, we have

$$\int \frac{vdv}{-g - \frac{k}{m}v} = \int dy + C$$

Let

$$w = -g - \frac{k}{m}v \Rightarrow v = -\frac{mw + mg}{k} \Rightarrow dv = -\frac{m}{k}dw$$

$$\int \frac{mw + mg}{kw}\frac{m}{k}dw = \left(\frac{m}{k}\right)^2 \int \left(1 + \frac{g}{w}\right) dw$$

$$= \left(\frac{m}{k}\right)^2 (w + gln|w|)$$

$$\Rightarrow \left(\frac{m}{k}\right)^2 \left(-g - \frac{k}{m}v + g\,ln\left|-g - \frac{k}{m}v\right|\right) = y + C$$

$$\begin{cases} y(0) = 0 \\ v(0) = v_0 \end{cases} \Rightarrow C = \left(\frac{m}{k}\right)^2 \left(-g - \frac{k}{m}v_0 + g\,ln\left|-g - \frac{k}{m}v_0\right|\right)$$

$$\Rightarrow y(t) = \left(\frac{m}{k}\right)^2 \left(-\frac{k}{m}(v - v_0) + g\,ln\left|\frac{mg + kv}{mg + kv_0}\right|\right)$$

At highest point y_{max} we have $v = 0$

$$y_{max} = \left(\frac{m}{k}\right)^2 \left(\frac{k}{m}v_0 + g\,ln\left|\frac{mg}{mg + kv_0}\right|\right)$$

(2)

$$m\frac{dv}{dt} = -mg - kv \Rightarrow \frac{dv}{dt} = -g - \frac{kv}{m}$$

which is a separable DE, we have

$$\int \frac{dv}{-g-\frac{k}{m}v} = \int dt$$

$$t + C = -\frac{m}{k} ln\left|-g-\frac{k}{m}v\right| = -\frac{m}{k} ln\left|\frac{-mg-kv}{m}\right|$$

$$v(0) = v_0 \Rightarrow C = -\frac{m}{k} ln\left|\frac{-mg-kv_0}{m}\right|$$

$$y(t) = -\frac{m}{k} ln\left|\frac{-mg-kv}{m}\right| + \frac{m}{k} ln\left|\frac{-mg-kv_0}{m}\right|$$

$$= \frac{m}{k} ln\left|\frac{mg+kv_0}{mg+kv}\right|$$

At highest point, we have $v = 0$

$$t_{up} = \frac{m}{k} ln\left|\frac{mg+kv_0}{m}\right| = \frac{m}{k} ln\left|1+\frac{kv_0}{mg}\right|$$

Problem 2.2.5

(1) The DE is

$$\frac{dv}{dt} = -kv$$

$$\ln v = -kt + C$$

$$v = C\exp(-kt)$$

Since $v(0) = 40$ and $v(10) = 20$, we can solve that

$$C = 40 \text{ and } k = \frac{\ln 2}{10}$$

which gives

$$v = 40\exp(-kt)$$

and similarly,

$$\frac{dx}{dt} = v$$

gives

$$\int_0^x dx = \int_0^t 40\exp(-kt)\,dt$$

$$x = \frac{40}{k}(1 - \exp(-kt))$$

when $t \to \infty$, we have

$$x = \frac{400}{\ln 2} \approx 577 \text{ ft}$$

(2) The DE becomes

$$\frac{dv}{dt} = -kv^2$$

This gives

$$\frac{1}{v} = kt + C$$

which is

$$v = \frac{1}{kt + C}$$

Given $v(0) = 40$ and $v(10) = 20$, we can get

$$C = \frac{1}{40} \text{ and } k = \frac{1}{400}$$

Thus

$$v = \frac{40}{1 + 40kt}$$

And finally

$$\frac{dx}{dt} = v(t) = \frac{40}{1 + \frac{t}{10}}$$

We can get

$$x = 400 \ln\left(1 + \frac{t}{10}\right) + C$$

Since $x(0) = 0$, we get $C = 0$. Thus

$$x(t) = 400 \ln\left(1 + \frac{t}{10}\right)$$

and $x(60) = 400 \ln\left(1 + \frac{60}{10}\right) \approx 778$ ft.

Problem 2.2.6

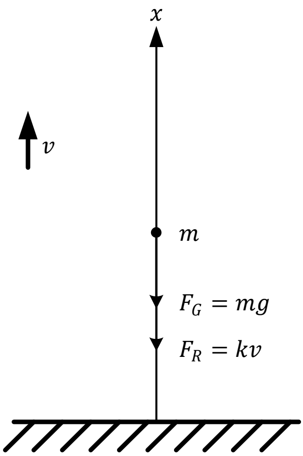

Figure A.4 A system of moving up object.

(1) The total force on this object is $F = -(F_R + F_G)$, by Newton's law, $F = ma = m\frac{dv}{dt}$ thus $F = -mg - kv = m\frac{dv}{dt}$ which is a 1st-order separable DE.

Solving this DE with I.C. $v(0) = v_0$, we get

$$v(t) = \left(v_0 + \frac{mg}{k}\right) exp\left(-\frac{k}{m}t\right) - \frac{mg}{k}$$

(2) At the point it reverses, the velocity of the object vanishes at $v(t) = 0$, thus

$$\left(v_0 + \frac{mg}{k}\right) \exp\left(-\frac{k}{m}t\right) - \frac{mg}{k} = 0$$

$$t = -\frac{m}{k}\ln\frac{mg}{mg + kv_0}$$

It depends on the initial velocity v_0 (the bigger it is, it takes longer to reverse the direction).

(3) When the object keeps a constant speed, the acceleration vanishes. Thus, $a = -mg - kv = 0 \Rightarrow v = -\frac{mg}{k}$, which is independent on the initial velocity v_0.

Problem 2.2.7

$$v_x = -v_0 \cos(\alpha)$$

gives

$$\cos(\alpha) = \frac{x}{\sqrt{x^2 + y^2}}$$

and

$$v_y = -v_0 \sin(\alpha) - w$$

gives

$$\sin(\alpha) = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{dy}{dx} = \frac{-v_0 \sin(\alpha) - w}{-v_0 \cos(\alpha)} = \frac{\sin(\alpha)}{\cos(\alpha)} + \frac{w}{v_0 \cos(\alpha)}$$

$$= \frac{y}{x} + \frac{w}{v_0}\left(\frac{\sqrt{x^2 + y^2}}{x}\right)$$

$$= \frac{y}{x} + \frac{w}{v_0}\frac{\left(x^2\sqrt{1 + \frac{y^2}{x^2}}\right)}{x}$$

$$= \frac{y}{x} + \frac{xw}{v_0}\sqrt{1 + \frac{y^2}{x^2}}$$

Substitute $u = \frac{y}{x}$

$$ux = y$$

$$\frac{dy}{dx} = u + \frac{du}{dx}x$$

$$u + \frac{du}{dx}x = u + \frac{xw}{v_0}\sqrt{1 + u^2}$$

$$\frac{du}{dx} = \frac{w}{v_0}\sqrt{1 + u^2}$$

$$\ln\left(u + \sqrt{1 + u}\right) = -\frac{w}{v_0}\ln(x) + c$$

$$y(x) = \frac{M}{2}\left(\left(\frac{x}{M}\right)^{1-\frac{w}{v_0}} - \left(\frac{x}{M}\right)^{1+\frac{w}{v_0}}\right)$$

when $w \geq w_c$

If $w > v_0$

$$\lim_{x \to 0}\left(\frac{x}{M}\right)^{1-\frac{w}{v_0}} \to \infty$$

When $w < w_c$

$$y(x) = \frac{M}{2}\left(\left(\frac{x}{M}\right)^{1-\frac{w}{v_0}} - \left(\frac{x}{M}\right)^{1+\frac{w}{v_0}}\right)$$

$$L = \int_0^M \sqrt{dx^2 + dy^2} = \int_0^M \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

When $w = 0$

$$y(x) = \frac{M}{2}\left(\left(\frac{x}{M}\right)^{1-\frac{0}{v_0}} - \left(\frac{x}{M}\right)^{1+\frac{0}{v_0}}\right) = 0$$

$$L = M$$

Problem 2.2.8

Let t_1 be the time you accelerate, $t_2$be the time you decelerate. $t_1 + t_2 = t$, which is the time you travel from one stop sign to another. Note that

$$a_1 t_1 = a_2 t_2$$

Since your initial velocity and final velocity are both zero.

So

$$\frac{1}{2}a_1 t_1^2 + \frac{1}{2}a_2 t_2^2 = L$$

plug in

$$t_2 = \frac{a_1}{a_2}t_1$$

and we have

$$\frac{1}{2}a_1 t_1^2 + \frac{1}{2}\frac{a_1^2}{a_2^2}t_1^2 = L$$

which gives

$$t_1 = \sqrt{\frac{2a_2 L}{a_1 a_2 + a_1^2}} \text{ and } t_2 = \sqrt{\frac{2a_1 L}{a_1 a_2 + a_2^2}}$$

If $v_m \geq a_1 t_1 (= a_2 t_2)$, the result stays the same.

If $v_m < a_1 t_1$, then you should accelerate to v_m (need time t_1). Then travel at a constant velocity v_m for time t_3. Finally decelerate from v_m to velocity 0 (need time t_2).The total time is $t = t_1 + t_2 + t_3$. Here

$$t_1 = \frac{v_m}{a_1} \text{ gives } L_1 = \frac{v_m}{2a_1^2}$$

$$t_2 = \frac{v_m}{a_2} \text{ gives } L_2 = \frac{v_m}{2a_2^2}$$

and

$$L_3 = L - L_1 - L_2 \text{ gives } t_3 = \frac{L - \frac{v_m}{2a_1^2} - \frac{v_m}{2a_2^2}}{v_m}$$

Finally

$$t = \frac{v_m}{a_1} + \frac{v_m}{a_2} + \frac{L - \frac{v_m}{2a_1^2} - \frac{v_m}{2a_2^2}}{v_m}$$

Problem 2.2.9

From the problem, we know that with closed parachute, we have

$$m\frac{dv}{dt} = mg - kv$$

This is a linear DE and we have

$$v = \frac{mg}{k} + C_1 \exp\left(-\frac{kt}{m}\right)$$

Given $v(0) = 0$, we know that

$$C_1 = -\frac{mg}{k}$$

Hence

$$v = \frac{mg}{k}\left(1 - \exp\left(-\frac{kt}{m}\right)\right)$$

And we can get

$$x = \int v dt$$

$$= \frac{mg}{k}t + \frac{m^2 g}{k^2}\exp\left(-\frac{kt}{m}\right) + C_2$$

Since $x(0) = 0$, we have

$$C_2 = -\frac{m^2 g}{k^2}$$

Thus

$$x(t) = \frac{mg}{k}\left(t + \frac{m}{k}\exp\left(-\frac{kt}{m}\right) - \frac{m}{k}\right)$$

Suppose he open the parachute at $t = t_1$, we know that

$$x_1 = \frac{mg}{k}\left(t_1 + \frac{m}{k}\exp\left(-\frac{kt_1}{m}\right) - \frac{m}{k}\right)$$

and

$$v_1 = \frac{mg}{k}\left(1 - \exp\left(-\frac{kt_1}{m}\right)\right)$$

Starting from t_1, the parachute will be open and thus, the DE becomes

$$mv' = mg - nkv$$

which we can get

$$v = \frac{mg}{nk} + C_3 \exp\left(-\frac{nkt}{m}\right)$$

where from $v(t_1) = v_1$, we have

$$C_3 = \frac{n-1}{n}\frac{mg}{k}\exp\left(\frac{nkt_1}{m}\right) - \frac{mg}{k}\exp\left(\frac{(n-1)kt_1}{m}\right)$$

For x, we have

$$x = \int v dt$$

$$= \frac{mg}{nk}t - \frac{C_3 m}{nk}\exp\left(-\frac{nkt}{m}\right) + C_4$$

where C_4 satisfies $x(t_1) = x_1$. After finding C_4, we can have

$$x_2 = H - x_1$$

Solving for t_2 such that $x(t_2) = x_2$ and make $v(t_2) = v_0$, we can find the optimal t_1.

Problem 2.2.10

We start with the usual force equation

$$m\frac{dv}{dt} = m\frac{dv}{dt}\frac{dy}{dy} = m\frac{dy}{dt}\frac{dv}{dy} = mv\frac{dv}{dy} = -\frac{GMm}{(y+R)^2}$$

where M is the mass on the Earth, m is the mass of the projectile, y is the height from the earth's surface and $r = y + R$.

Hence we will separate this equation and integrate

$$\int_{v_0}^{v} v dv = \int_0^y -\frac{GM}{(y+R)^2}dy$$

to obtain

$$v^2 = {v_0}^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

We found the velocity to be

$$v(y) = \sqrt{{v_0}^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right)} = \sqrt{{v_0}^2 - 2GM\left(\frac{1}{R} - \frac{1}{r}\right)}$$

The velocity will become imaginary when

$${v_0}^2 < 2GM\left(\frac{1}{R} - \frac{1}{r}\right)$$

Problem 2.2.11

Consider v to be a function of y. Thus,

$$y'' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$$

and the original DE can be written as

$$v\frac{dv}{dy} = -\frac{GM}{(y+R)^2} - \beta\exp(-y)$$
$$\int v dv = \int \left(-\frac{GM}{(y+R)^2} - \beta\exp(-y)\right) dy$$
$$\frac{v^2}{2} = \frac{GM}{y+R} + \beta\exp(-y) + C$$

Plugging I.C. $y(0) = 0$ and $y'(0) = v_0$, we have

$$\frac{v_0^2}{2} = \frac{GM}{R} + \beta + C$$
$$C = \frac{v_0^2}{2} - \frac{GM}{R} - \beta$$
$$v^2 = v_0^2 - \frac{2GMy}{(y+R)R} + 2\beta(\exp(-y) - 1)$$

$y_{\max}$ occurs when $v = 0$, which gives the implicit solution:

$$\frac{2GMy_{\max}}{(y_{\max}+R)R} - 2\beta\exp(-y_{\max}) = v_0^2 - 2\beta$$

For the total time T, we should consider

$$v = \frac{dy}{dt}$$

and it gives

$$\sqrt{v_0^2 - \frac{2GMy}{(y+R)R} + 2\beta(\exp(-y) - 1)} = \frac{dy}{dt}$$

$$T = \int_0^{y_{\max}} \left(v_0^2 - \frac{2GMy}{(y+R)R} + 2\beta(\exp(-y) - 1)\right)^{-\frac{1}{2}} dy$$

Problem 2.2.12

Before opening

In vertical (y-) direction:

$$m\frac{dv_y}{dt} = mg$$

$$\int_0^{v_y} dv_y = \int_0^t gdt\,, v_y = gt$$

$$\frac{dy}{dt} = gt, \int_0^y dy = \int_0^t gtdt$$

$$y = \frac{1}{2}gt^2 \;\Rightarrow\; \frac{H}{2} = \frac{1}{2}g{t_1}^2, t_1 = \sqrt{\frac{H}{g}}, v_{y1} = gt_1 = \sqrt{gH}$$

In horizontal (x-) direction:
There is no force, constant velocity v_0

$$x = v_0 t \Rightarrow\; t = \frac{x}{v_0}, y = \frac{1}{2}gt^2 = \frac{1}{2}g(\frac{x}{v_0})^2$$

$$\frac{dy}{d} = \frac{g}{{v_0}^2}x, x_1 = v_0 t_1 = v_0\sqrt{\frac{H}{g}}$$

So the length

$$L_1 = \int_0^{x_1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{v_0\sqrt{\frac{H}{g}}}\sqrt{1+\left(\frac{g}{{v_0}^2}x\right)^2}\,dx$$

$$= \frac{1}{2}\sqrt{\frac{{v_0}^2 H}{g} + H^2} + \frac{{v_0}^2}{2g}\ln\left(\frac{\sqrt{gH}}{v_0} + \sqrt{\frac{gH}{{v_0}^2}+1}\right)$$

After opening
In vertical (y-) direction:

$$m\frac{dv_y}{dt} = mg - \alpha v_y$$

$$\int_{v_{y1}}^{v_y}\frac{mdv_y}{mg-\alpha v_y} = \int_0^t dt\,, v_y$$

$$= \frac{mg}{\alpha} + \left(\sqrt{gH} - \frac{mg}{\alpha}\right)\exp\left(-\frac{\alpha t}{m}\right)$$

$$\frac{dy}{dt} = \frac{mg}{\alpha} + \left(\sqrt{gH} - \frac{mg}{\alpha}\right)\exp\left(-\frac{\alpha t}{m}\right), \int_0^y dy$$

$$= \int_0^t\left(\frac{mg}{\alpha} + \left(\sqrt{gH} - \frac{mg}{\alpha}\right)\exp\left(-\frac{\alpha t}{m}\right)\right)dt$$

$$y = \frac{mg}{\alpha}t - \frac{m}{\alpha}\left(\sqrt{gH} - \frac{mg}{\alpha}\right)\exp\left(-\frac{\alpha t}{m}\right) + \frac{m}{\alpha}(\sqrt{gH} - \frac{mg}{\alpha})$$

$$\frac{H}{2}=\frac{mg}{\alpha}t_2-\frac{m}{\alpha}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)\exp\left(-\frac{\alpha t_2}{m}\right)+\frac{m}{\alpha}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)$$

We could get t_2 implictitly.

In horizontal (x-) direction:

$$m\frac{dv_x}{dt}=-\alpha v_x$$

$$-\int_{v_0}^{v_x}\frac{m}{\alpha v_x}dv_x=\int_0^t dt\,, v_x=v_0\exp\left(-\frac{\alpha t}{m}\right)$$

$$\frac{dx}{dt}=v_0\exp\left(-\frac{\alpha t}{m}\right), \int_0^x dx=\int_0^t v_0\exp\left(-\frac{\alpha t}{m}\right)dt$$

$$x=-\frac{mv_0}{\alpha}\exp\left(-\frac{\alpha t}{m}\right)+\frac{mv_0}{\alpha}\Rightarrow\ t=-\frac{m}{\alpha}\ln(1-\frac{\alpha}{mv_0}x)$$

$$\begin{aligned}y=&\ \frac{mg}{\alpha}t-\frac{m}{\alpha}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)\exp\left(-\frac{\alpha}{m}\right)+\frac{m}{\alpha}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)\\ =&-\frac{m^2g}{\alpha^2}\ln\left(1-\frac{\alpha}{mv_0}x\right)-\frac{m}{\alpha}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)(1-\frac{\alpha}{mv_0}x)\\ &+\frac{m}{\alpha}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)\end{aligned}$$

$$\frac{dy}{dx}=\frac{mg}{\alpha v_0-\frac{\alpha^2}{m}x}+\frac{1}{v_0}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)$$

$$x_2=-\frac{mv_0}{\alpha}\exp\left(-\frac{\alpha t_2}{m}\right)+\frac{mv_0}{\alpha}$$

So the length

$$\begin{aligned}L_2=&\int_0^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ =&\int_0^{x_2}\left(1+\left(\frac{mg}{\alpha v_0-\frac{\alpha^2}{m}x}+\frac{1}{v_0}\left(\sqrt{gH}-\frac{mg}{\alpha}\right)\right)^2\right)^{\frac{1}{2}}dx\end{aligned}$$

The length of the trajectory is $L=L_1+L_2$

Problem 2.2.13

From the problem, we know that the resistance in vector form is

$$\vec{R} = -\alpha|\vec{v}|\frac{\vec{v}}{|\vec{v}|}$$
$$= -\alpha\vec{v}$$

Now it is clear to see the resistance in x and y directions are independent from each other. Thus, we only need to consider the y axis. Before the parachute opens, we have

$$\frac{dv_y}{dt} = -g$$

which gives

$$v_y = -gt + C_1$$

given $v_y(0) = 0$, $C_1 = 0$ we have

$$v_y = -gt$$
$$y = -\frac{1}{2}gt^2 + C_2$$

Plugging in $y(0) = H$, $C_2 = H$

$$y = H - \frac{1}{2}gt^2$$

If the man opens the parachute at $y = \frac{1}{2}H$, the time it takes from start to open the parachute is

$$\frac{1}{2}H = H - \frac{1}{2}gt_1^2$$
$$t_1 = \sqrt{\frac{H}{g}}$$

and at that time,

$$v_y = -\sqrt{gH}$$

After opening the parachute, the equation of motion becomes

$$\frac{dv_y}{dt} = -g - \frac{\alpha}{m}v_y$$

which is a separable DE. The G.S. is

$$-\frac{m}{\alpha}\ln\left(\frac{\alpha}{m}v_y + g\right) = t + C_3$$

Given $v_y(0) = -\sqrt{gH}$, we have

$$C_3 = -\frac{m}{\alpha}\ln\left(g - \frac{\alpha}{m}\sqrt{gH}\right)$$

Thus,

$$\frac{m}{\alpha}\ln\frac{g - \frac{\alpha}{m}\sqrt{gH}}{g + \frac{\alpha}{m}v_y} = t$$

$$\left(mg - \alpha\sqrt{gH}\right)\exp\left(-\frac{\alpha}{m}t\right) - mg = \alpha v_y$$

$$v_y = \left(\frac{mg}{\alpha} - \sqrt{gH}\right)\exp\left(-\frac{\alpha}{m}t\right) - \frac{mg}{\alpha}$$

and

$$y = -\frac{m}{\alpha}\left(\frac{mg}{\alpha} - \sqrt{gH}\right)\exp\left(-\frac{\alpha}{m}t\right) - \frac{mg}{\alpha}t + C_4$$

Given $y(0) = 0$, we have

$$C_4 = \frac{m^2 g}{\alpha^2} - \frac{m\sqrt{gH}}{\alpha}$$

and solve for $y(t_2) = \frac{1}{2}H$ gives an implicit equation

$$\frac{1}{2}\frac{\alpha H}{mg} = \left(\sqrt{\frac{H}{g}} - \frac{m}{\alpha}\right)\left(\exp\left(-\frac{\alpha}{m}t\right) - 1\right) - t_2$$

The total time is $t = t_1 + t_2$.

In the case he opens the parachute at $y = \frac{1}{4}H$, we have

$$t_1 = \sqrt{\frac{3H}{2g}}$$

and at that time

$$v_y(t_1) = -\sqrt{\frac{3gH}{2}}$$

The equation after opening the parachute then becomes

$$v_y = \left(\frac{mg}{\alpha} - \sqrt{\frac{3}{2}gH}\right)\exp\left(-\frac{\alpha}{m}t\right) - \frac{mg}{\alpha}$$

and the implicit expression for t_2 is

$$\frac{3}{4}\frac{\alpha H}{mg} = \left(\sqrt{\frac{3H}{2g}} - \frac{m}{\alpha}\right)\left(\exp\left(-\frac{\alpha}{m}t_2\right) - 1\right) - t_2$$

Problem 2.2.14

This is a problem of Newtonian mechanics $F_g + F_d = ma$. Total weight of the man and his parachute m with acceleration a, gravitational force F_g and a drag force F_d. Let x be the distance above the earth's surface, with positive direction downward, $= \frac{dv}{dt}$, where $v = \frac{dx}{dt}$ is the velocity and $F_g = -mg$.

Since the drag force is assumed to be proportional to velocity,

$F_d = -kv$ (with closed paracute) and $F_d = -nkv$ (with open parachute).

The deployment occurs at time t_0,

During initial fall with closed parachute,

$$m\frac{dv}{dt} = mg - kv, v(0) = 0$$

This is an IVP problem. Solve this separable DE. We obtain,

$$\frac{dv}{dt} = g - \frac{kv}{m}$$

or

$$\frac{dv}{g - \frac{kv}{m}} = dt$$

$$-\frac{m}{k}\;\frac{d\left(g - \frac{kv}{m}\right)}{\left(g - \frac{kv}{m}\right)} = dt$$

$$-\frac{m}{k}\ln\left(g-\frac{kv}{m}\right)=t+C$$

$$g-\frac{kv}{m}=C_1\exp\left(-\frac{kt}{m}\right)\Rightarrow v(t)=\frac{m}{k}\left(g-C_1\exp\left(-\frac{kt}{m}\right)\right)$$

$$t=0, v(0)=0$$

$$C_1=g\frac{dx}{dt}=v=\frac{m}{k}g\left(1-\exp\left(-\frac{kt}{m}\right)\right)$$

$$x=\frac{m}{k}g\left(t+\frac{m}{k}\exp\left(-\frac{kt}{m}\right)\right)+C_2$$

$$x(0)=0\Rightarrow C_2=-\frac{m^2}{k^2}g$$

So the distance of close parachute fall is

$$x_1=\frac{m}{k}g\left(t_0+\frac{m}{k}\exp\left(-\frac{kt_0}{m}\right)\right)-\frac{m^2}{k^2}g$$

At t_0,

$$V(t_0)=\frac{m}{k}\left(1-\exp\left(-\frac{kt_0}{m}\right)\right)$$

Starting with the opening parachute t_0 time, the DE becomes

$$m\frac{dv}{dt}=mg-nkv$$

$$m\frac{dv}{dx}\frac{dx}{dt}=mg-nkv$$

$$m\frac{dv}{dx}v=mg-nkv$$

whose solution is

$$v(t)=\frac{m}{nk}\left(g-C_3\exp\left(-\frac{nkt}{m}\right)\right)$$

Since

$$v(t_0)=\frac{m}{k}g\left(1-\exp\left(-\frac{kt_0}{m}\right)\right)$$

$$C_3=\exp\left(\frac{nkt_0}{m}\right)ng\left(\frac{1}{n}-1+\exp\left(-\frac{kt_0}{m}\right)\right)$$

$$\frac{dx}{dt}=\frac{m}{nk}\left(g-C_3\exp\left(-\frac{nkt}{m}\right)\right)$$

$$x_2 = \frac{m}{nk} g\left(t + C_3 \frac{m}{nk} \exp\left(-\frac{kt}{m}\right)\right) + C_4$$
$$x_2(t_0) = x_1$$

which means

$$\frac{m}{nk} g\left(t_0 + C_3 \frac{m}{nk} \exp\left(-\frac{kt_0}{m}\right)\right) + C_4$$
$$= \frac{m}{k} g\left(t_0 + \frac{m}{k} \exp\left(-\frac{kt_0}{m}\right)\right) - \frac{m^2}{k^2} g$$

One can easily find the constant C_4

$$H = x_1 + x_2$$
$$x_2 = \frac{m}{nk} g\left(t_1 + C_3 \frac{m}{nk} \exp\left(-\frac{kt_1}{m}\right)\right) + C_4$$
$$= H - \frac{m}{k} g\left(t_0 + \frac{m}{k} \exp\left(-\frac{kt_0}{m}\right)\right) - \frac{m^2}{k^2} g$$

The falling time t_1 is in principle solvable from the above equation, but it is tedious and may require a numerical solution technique. (Reaching this point without the actual solutions for C_4 and t_1 is sufficient to gain full marks.)

Therefore the total falling time is t_1+t_0, the speed he hits the ground is

$$v(t_1) = \frac{m}{nk}\left(g - C_3 \exp\left(-\frac{nkt}{m}\right)\right)$$

We can adjust the t_0 to enable the quickest fall and lightest hit on the ground.

Problem 2.2.15

Suppose the speed of the jet is v_0. And the jet needs to know which angle to go in, call it α. Then, without the wind the jet's velocity will be $[v_0 \cos(\alpha), v_0 \cos(\alpha)]$. With the wind the jet's velocity can be $[v_0 \cos(\alpha), v_0 \cos(\alpha) v_w]$.

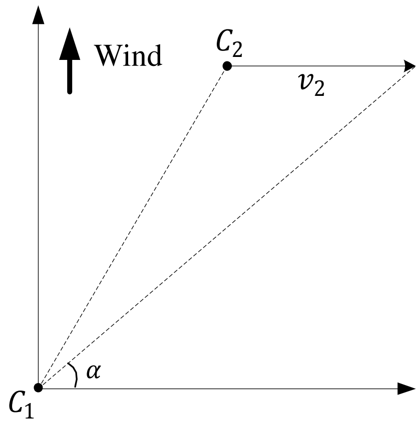

Figure A.5 The jet landing model.

$$v_x = \frac{dx}{dt} = v_0 \cos\alpha = v_0 \frac{x}{\sqrt{x^2+y^2}} \tag{1}$$

$$v_y = \frac{dy}{dt} = v_0 \sin(\alpha) + v_w = v_0 \frac{y}{\sqrt{x^2+y^2}} + v_w \tag{2}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{dy}{dx} = \frac{v_0 \frac{y}{\sqrt{x^2+y^2}} + v_w}{v_0 \frac{x}{\sqrt{x^2+y^2}}} \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{v_w}{v_0}\sqrt{1+\left(\frac{y}{x}\right)^2}$$

Problem 2.2.16

According to Newton's Law, we have

$$m\frac{dv}{dt} = \alpha v + \beta v^2$$

Since

$$v = \frac{dx}{dt} = \frac{dx}{dv}\frac{dv}{dt}$$

then

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

Plugging this back

$$mv\frac{dv}{dx} = \alpha v + \beta v^2$$
$$m\frac{dv}{dx} = \alpha + \beta v$$
$$\frac{m}{\alpha + \beta v}dv = dx$$
$$\frac{m}{\beta}\ln(\alpha + \beta v) = x + C$$

At $x = 0$, it gives $v = v_0$. So,

$$C = \frac{m}{\beta}\ln(\alpha + \beta v_0)$$

Therefore,

$$x(v) = \frac{m}{\beta}\ln\left(\frac{\alpha + \beta v}{\alpha + \beta v_0}\right)$$

The maximal distance is

$$x_m(v = 0) = \frac{m}{\beta}\ln\left(\frac{\alpha}{\alpha + \beta v_0}\right)$$

If we double the initial speed,

$$x(v) = \frac{m}{\beta}\ln\left(\frac{\alpha + \beta v}{\alpha + 2\beta v_0}\right)$$
$$x_m(v = 0) = \frac{m}{\beta}\ln\left(\frac{\alpha}{\alpha + 2\beta v_0}\right)$$

Problem 2.2.17

Assume the following speeds.

v_0 and v_1 are entering and exiting Medium-1, respectively.

v_1 and v_2 are entering and exiting Medium-2, respectively.

v_2 and v_3 are entering and exiting Medium-3, respectively.

For Medium-1: the equation of motion is

$$m\frac{dv}{dt} = -kv \quad \text{or} \quad m\frac{dv}{dt}\frac{dx}{dx} = mv\frac{dv}{dx} = -kv$$

Integrating both sides, we get:

$$\int_{v_0}^{v_1} dv = \int_0^L -\frac{k}{m}dx$$

Thus, the solution is given by

$$v_1 = v_0 - \frac{k}{m}L$$

For Medium-2, treat it using the same strategy for Medium-1.

$$m\frac{dv}{dt} = -kv^2$$

or

$$m\frac{dv}{dt}\frac{dx}{dx} = mv\frac{dv}{dx} = -kv^2$$

Integrating both sides, we get

$$\int_{v_0}^{v_1} \frac{1}{v}dv = \int_0^L -\frac{k}{m}dx$$

Thus, the solution is given by

$$v_2 = v_1 \exp\left(-\frac{k}{m}L\right) = (v_0 - \frac{k}{m}L)\exp\left(-\frac{k}{m}L\right)$$

For Medium-3, we know easily (from Medium-1)

$$v_3 = v_2 - \frac{k}{m}L$$

Force $v_3 = 0$ and we have

$$v_3 = v_2 - \frac{k}{m}L = 0$$

or

$$v_2 = \frac{k}{m}L$$

Trace back and find the final solution

Problem 2.2.18

$$\boldsymbol{v_0 = \frac{k}{m}L\left(1 + \exp\left(\frac{k}{m}L\right)\right)}$$

<u>Method I:</u>

For all three cases, we have the following DE.

$$\begin{cases} m\dfrac{dv}{dt} = m\dfrac{dv}{dx}\dfrac{dx}{dt} = m\dfrac{dv}{dx}v = -kv^\alpha \\ v(x=0) = v_0 \end{cases}$$

We can easily get the following.

$$\int_{v_0}^{0} v^{1-\alpha} dv = -\int_{0}^{x_M} \frac{k}{m} dx$$

The bullet stops at the max distance x_M.

$$x_M = \frac{m}{k}\int_{0}^{v_0} v^{1-\alpha} dv = \frac{m}{k}\frac{v_0^{2-\alpha}}{2-\alpha}$$

For Case 1, $\alpha = 1$,

$$x_M = \frac{m}{k}\frac{v_0^1}{1}$$

For Case 2, $\alpha = 3/2$,

$$x_M = \frac{m}{k}\frac{v_0^{1/2}}{1/2} = 2\frac{m}{k}v_0^{1/2}$$

For Case 3, $\alpha = 2$,

$$x_M \to \infty$$

<u>Method II:</u>

(i)

$$m\frac{dv}{dt} = -k_1 v$$

$$\frac{dv}{v} = -\left(\frac{k_1}{m}\right)dt$$

$$\ln v = -\left(\frac{k_1}{m}\right)dt$$

$$v = \exp\left(-\frac{k_1}{m}t + C\right)$$

$$v = C_1 \exp\left(-\frac{k_1}{m}t\right)$$

$$v(0) = v_0 = C_1$$

So we have

$$v(t) = v_0 \exp\left(-\frac{k_1}{m}t\right)$$

$$\frac{dx}{dt} = v_0 \exp\left(-\frac{k_1}{m}t\right)$$

$$dx = v_0 \exp\left(-\frac{k_1}{m}t\right)dt$$

$$x = \frac{v_0 \exp\left(-\frac{k_1}{m}t\right)}{-\frac{k_1}{m}} + C_2$$
$$x = -\frac{k_1 v_0}{m} \exp\left(-\frac{k_1}{m}t\right) + C_2$$
$$x(0) = x_0 = -\frac{k_1 v_0}{m} + C_2$$
$$C_2 = x_0 + \frac{k_1 v_0}{m}$$

So we have

$$x(t) = x_0 + \frac{k_1 v_0}{m}\left(1 - \exp\left(-\frac{k_1}{m}t\right)\right)$$

As t goes to infinity, $\exp\left(-\frac{k_1}{m}t\right)$ goes to 0. Thus x goes to

$$x_0 + \frac{k_1 v_0}{m}$$

So the total distance travelled is x at infinity minus x_0, or

$$\frac{k_1 v_0}{m}$$

(ii)

$$m\frac{dv}{dt} = -k_2 v^{\frac{3}{2}}$$
$$\frac{dv}{dt} = -\frac{k_2}{m} v^{\frac{3}{2}}$$
$$v^{-\frac{3}{2}} dv = -\frac{k_2}{m} dt$$
$$\frac{v^{-\frac{1}{2}}}{-\frac{1}{2}} = -\frac{k_2}{m} t + C$$
$$v^{-\frac{1}{2}} = \frac{k_2}{2m} t + \frac{-C}{2}$$
$$v = \left(C_1 + \frac{k_2}{2m} t\right)^{-2}$$
$$v(0) = v_0 = C_1^{-2}$$
$$C_1 = \frac{1}{\sqrt{v_0}}$$

$$v = \left(\frac{1}{\sqrt{v_0}} + \frac{k_2}{2m}t\right)^{-2}$$

$$x = \int v\,dt$$

$$u = \frac{1}{\sqrt{v_0}} + \frac{k_2}{2m}t$$

$$du = \frac{k_2}{2m}dt$$

$$+\frac{2m}{k_2}du = dt$$

So we have

$$x = \frac{2m}{k_2}\int u^{-2}du$$

$$x = -\frac{2m}{k_2}\left(\frac{1}{\sqrt{v_0}} + \frac{k_2}{2m}t\right)^{-1} + C_3$$

Putting in I.C. $x(0) = x_0$:

$$x_0 = -\frac{2m}{k_2}\sqrt{v_0} + C_3$$

$$x(t) = \frac{2m}{k_2}\left(\frac{1}{\sqrt{v_0}} + \frac{k_2}{2m}t\right)^{-1} + x_0 + \frac{2m}{k_2}\sqrt{v_0}$$

$$x(t) = \frac{2m}{k_2}\left(\frac{2m}{2m\sqrt{v_0}} + \frac{k_2 t\sqrt{v_0}}{2m\sqrt{v_0}}\right)^{-1} + x_0 + \frac{2m}{k_2}\sqrt{v_0}$$

$$x(t) = \frac{2m}{k_2}\left(\frac{2m\sqrt{v_0}}{2m + k_2 t\sqrt{v_0}}\right) + x_0 + \frac{2m}{k_2}\sqrt{v_0}$$

So as t goes to infinity, x goes to

$$x_0 + \frac{2m}{k_2}\sqrt{v_0}$$

Thus the total distance is this value minus x_0, or

$$\frac{2m}{k_2}\sqrt{v_0}$$

(iii)

$$m\frac{dv}{dt} = -k_3 v^2$$
$$v^{-2}dv = -\frac{k_3}{m}dt$$
$$-v^{-1} = -\frac{k_3}{m}t + C$$
$$v = \left(\frac{k_3}{m}t + C_2\right)^{-1}$$
$$v(0) = v_0 = \frac{1}{C_2}$$
$$v(t) = \left(\frac{k_3}{m}t + \frac{1}{v_0}\right)^{-1}$$
$$x = \int v dt$$
$$x = \frac{m}{k_3}\ln\left(\frac{k_3}{m}t + \frac{1}{v_0}\right) + C_3$$
$$x(0) = x_0 = \frac{m}{k_3}\ln\frac{1}{v_0} + C_3 = -\frac{m}{k_3}\ln v_0 + C_3$$
$$x(t) = \frac{m}{k_3}\ln\left(\frac{k_3}{m}t + \frac{1}{v_0}\right) + x_0 + \frac{m}{k_3}\ln v_0$$

As t goes to infinity, $x(t)$ goes to infinity. So in this situation the bullet does not stop.

Problem 2.2.19

The motion equation is given below.

$$\frac{dv}{dt} = -\frac{\alpha(\beta + v^2)}{m}$$

(1) Suppose the distance position function is given by X, which is a function of t, only.

Rearrange the motion equation.

$$\frac{dv}{dt}\frac{dX}{dX} = -\frac{\alpha(\beta + v^2)}{m}$$

We know the fact that

$$\frac{dX}{dt} = v$$

So, the rearranged motion equation becomes

$$\left(\frac{dX}{dt}\right)\frac{dv}{dX} = v\frac{dv}{dX} = -\frac{\alpha(\beta + v^2)}{m}$$

Use the separation of variable to get

$$\ln(v^2 + \beta) = -\frac{2\alpha}{m}X + C$$

Coupled with I.C. to locate the unknown coefficient C.

$$\ln(v_0^2 + \beta) = C$$

So, the expression for velocity and position is given by

$$\ln(v^2 + \beta) = -\frac{2\alpha}{m}X + \ln(v_0^2 + \beta)$$

When the velocity decreases to zero, the bullet reaches the farthest.

$$\ln\beta = -\frac{2\alpha}{m} + \ln(v_0^2 + \beta)$$

The maximum distance is

$$X = \frac{m}{2\alpha}\ln\frac{(v_0^2 + \beta)}{\beta}$$

(2) Coupling the I.C. with original motion equation to give

$$\begin{cases} \dfrac{dv}{dt} = -\dfrac{\alpha(\beta + v^2)}{m} \\ v(t = 0) = v_0 \end{cases}$$

Using the integral table to gain

$$\frac{1}{\sqrt{\beta}}\tan^{-1}\frac{v}{\sqrt{\beta}} = C - \frac{\alpha}{m}t$$

The explicit expression of velocity is

$$v = \sqrt{\beta}\tan\left(\sqrt{\beta}\left(C - \frac{\alpha}{m}t\right)\right)$$

Apply the I.C. to locate the unknown coefficient.

$$v(t = 0) = v_0 = \sqrt{\beta}\tan\left(\sqrt{\beta}C\right)$$

The explicit expression of C is

$$C = \frac{\tan^{-1}\left(\frac{v_0}{\sqrt{\beta}}\right)}{\sqrt{\beta}}$$

When the velocity becomes zero, time reaches the maximum.

$$\frac{\alpha}{m}T = \frac{\tan^{-1}\left(\frac{v_0}{\sqrt{\beta}}\right)}{\sqrt{\beta}}$$

Finally, time is

$$T = \frac{m \cdot \tan^{-1}\left(\frac{v_0}{\sqrt{\beta}}\right)}{\alpha\sqrt{\beta}}$$

As long as time is in the interval $t \in [0, T]$, the bullet is in motion.

Problem 2.2.20

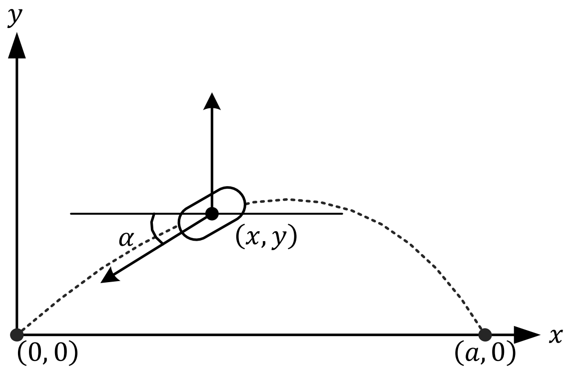

Figure A.6 The ferryboat model.

(1) Establish the Des for the boat's trajectory,

$$\begin{cases} v_x = \dfrac{dx}{dt} = -V_0 \cos\alpha & (1) \\ v_y = \dfrac{dy}{dt} = W(x) - V_0 \sin\alpha & (2) \end{cases}$$

We know that

$$\begin{cases} \sin\alpha = \dfrac{y}{\sqrt{x^2+y^2}} \\ \cos\alpha = \dfrac{x}{\sqrt{x^2+y^2}} \end{cases}$$

Eq-(2) divided by Eq-(1), we get

$$\frac{dy}{dx} = \frac{W(x) - V_0 \sin\alpha}{V_0 \cos\alpha} = \frac{W(x)}{-V_0\cos\alpha} + \tan\alpha$$

$$= -\frac{\frac{w_0 x(a-x)}{a^2}}{V_0\cos\alpha} + \frac{y}{x} = -\frac{w_0 x(a-x)}{V_0 a^2}\sqrt{1+\left(\frac{y}{x}\right)^2} + \frac{y}{x}$$

$$= -\frac{x(a-x)}{b^2}\sqrt{1+\left(\frac{y}{x}\right)^2} + \frac{y}{x}$$

where

$$\frac{1}{b^2} = \frac{w_0}{V_0 a^2}$$

(2) By substitution $v = y/x$, we can transform the equation as

$$v + xv' = -\frac{x(a-x)}{b^2}\sqrt{z+v^2} + v$$

$$v' = -\frac{a-x}{b^2}\sqrt{1+v^2}$$

By separation of variables, we have

$$\int \frac{dv}{\sqrt{1+v^2}} = -\int \frac{a-x}{b^2}dx$$

$$\ln\left(v+\sqrt{1+v^2}\right) = \frac{x^2-2ax}{2b^2} + C$$

With the I.C. $y(\alpha) = 0$, we find $C = a^2/(2b^2)$. Finally, we have

$$\ln\left(v+\sqrt{1+v^2}\right) = \frac{1}{2}\left(\frac{x-a}{b}\right)^2$$

$$v + \sqrt{1+v^2} = \exp\left(\frac{(x-a)^2}{2b^2}\right)$$

Substituting back v with y/x,

$$\frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = \exp\left(\frac{(x-a)^2}{2b^2}\right)$$

(3) If the river flows slowly relative to the boat, $b \to \infty$, the result becomes $y \to 0$, which is a straight line.

Problem 2.2.21

The net force

$$F = mg - \frac{mg}{\rho} - \mu v$$

(positive sign means the force is downwards).

So

$$\frac{dv}{dt} = \frac{F}{m} = \frac{mg\left(1 - \frac{1}{\rho}\right) - \mu v}{m} = \frac{\mu\left(\frac{mg}{\mu} - \frac{mg}{\mu\rho} - v\right)}{m}$$

$$\int \frac{m}{\mu\left(\frac{mg}{\mu} - \frac{mg}{\mu\rho} - v\right)} dv = \int dt$$

$$v(t) = \exp\left(-\frac{\mu t}{m}\right)\left(\frac{mg}{\mu\rho} - \frac{mg}{\mu}\right) + \left(\frac{mg}{\mu} - \frac{mg}{\mu\rho}\right)$$

We also know that

$$\frac{dy}{dt} = v(t) \Rightarrow \int_0^H dy = \int_0^T v(t)\, dt$$

$$H = \exp\left(-\frac{\mu}{m}T\right)\left(\frac{mg}{\mu\rho} - \frac{mg}{\mu}\right)\left(-\frac{m}{\mu}\right) + \left(\frac{mg}{\mu} - \frac{mg}{\mu\rho}\right)T - \left(\frac{mg}{\mu\rho} - \frac{mg}{\mu}\right)\left(-\frac{m}{\mu}\right)$$

Once we solve for T, which is the time that the egg hits the bottom, we can just plug T into $v(t)$to get the impact speed (at time T).

Alternative Method

$$m\frac{dv}{dt} = m\frac{dv}{dy}\frac{dy}{dt} = m\frac{dv}{dy}v = mg - \frac{mg}{\rho} - \mu v$$

Or

$$\frac{dv}{dy}v = g\left(1 - \frac{1}{\rho}\right) - \frac{\mu}{m}v$$

Define

$$\alpha = g\left(1 - \frac{1}{\rho}\right), \qquad \beta = -\frac{\mu}{m}$$

We have separable equation

$$\frac{dv}{dy}v = \alpha + \beta v$$

Next, solve the equation

$$\int_0^{v_f} \frac{vdv}{\alpha + \beta v} = \int_0^H dy$$

where v_f is the impact speed. So

$$H = \frac{v}{\beta} - \frac{\alpha\ln(\alpha + \beta v)}{\beta^2}\bigg|_0^{v_f} = \frac{v_f}{\beta} - \frac{\alpha\ln\left(\alpha + \beta v_f\right)}{\beta^2} + \frac{\alpha\ln(\alpha)}{\beta^2}$$

You can solve for v_f now.

2.3 An example in Finance

Problem 2.3.1

(1) Establish the DE

$$dZ = Zrdt - Q_0 tdt$$

Which gives

$$\frac{dZ}{dt} = Zr - Q_0 t$$

I.C.: $Z(0) = Z_0$

(2) We can rewrite the DE

$$Z' - Zr = -Q_0 t$$

which is a linear DE. Thus

$$(Z\exp(-rt))' = -Q_0 t\exp(-rt)$$

$$Z\exp(-rt) = -Q_0\left(-\frac{\exp(-rt)}{r^2} - \frac{t\exp(-rt)}{r}\right) + C$$

$$Z = \frac{Q_0}{r^2} + \frac{Q_0 t}{r} + C\exp(-rt)$$

Plug in the I.C. $Z(0) = Z_0$, we have

$$Z_0 = \frac{Q_0}{r^2} + C$$

$$C = Z_0 - \frac{Q_0}{r^2}$$

Finally,

$$Z(t) = \frac{Q_0}{r}t + \frac{Q_0}{r^2} + \left(Z_0 - \frac{Q_0}{r^2}\right)\exp(-rt)$$

Problem 2.3.2

(1) The change of loan dx after time interval dt

$$dx = xrdt - Ddt$$

Thus, we have an equation

$$\frac{dx}{dt} = xr - D$$

with $x(0) = Z$.

(2) The above DE can be solved easily by separation of variables.

$$\int \frac{dx}{xr - D} = \int dt + C$$

$$\frac{1}{r}\ln(xr - D) = t + C$$

$$\left.\begin{array}{c} x(t) = \frac{1}{r}\big(D + \exp\big(r(t + C)\big)\big) \\ x(0) = Z \end{array}\right\} \Rightarrow C = \frac{1}{r}\ln(rZ - D)$$

$$x(t) = \frac{1}{r}(D + \exp(rt + \ln(rZ - D)))$$

$$= \frac{1}{r}(D + (rZ - D)\exp(rN))$$

(3) Obviously, if $rZ - D > 0$, the loan will grow with time. O.W., it will decrease. Thus, the critical point is $rZ - D = 0$, i.e., $D = rZ$ (The Youngs only pays off the interest).

(4) Pay off in N days,

$$x(N) = \frac{1}{r}(D + (rZ - D)\exp(rN)) = 0$$

$$N = -\frac{1}{r}\ln\left(1 - \frac{rZ}{D}\right)$$

(5) Pay off in $N/2$ days,

$$\frac{N}{2} = -\frac{1}{r}\ln\left(1 - \frac{rZ}{E}\right)$$

where E is the daily payment to pay off loan in half time. We also know

$$N = -\frac{1}{r}\ln\left(1 - \frac{rZ}{E}\right)$$

$$\frac{N}{2} = -\frac{1}{2r}\ln\left(1 - \frac{rZ}{E}\right) = -\frac{1}{r}\ln\sqrt{1 - \frac{rZ}{E}}$$

Comparing it with previous equation, we have

$$\sqrt{1-\frac{rZ}{D}}=1-\frac{rZ}{E}$$

Let $Y=rZ/D$ and $rZ=YD$, then

$$\sqrt{1-Y}=1-\frac{YD}{E}$$

$$E=\frac{YD}{1-\sqrt{1-Y}}$$

Problem 2.3.3

(1) The equation can be written as $dZ=Zrdt-Wdt$ which can be solved easily by separation of variables

$$\int_{Z_0}^{Z}\frac{dZ}{Z-\frac{W}{r}}=\int_0^t rdt$$

We can then get $Z(t)=\frac{W}{r}+\left(Z_0-\frac{W}{r}\right)\exp(rt)$

Now, the pay-off time is when

$$Z(T)=\frac{W}{r}+\left(Z_0-\frac{W}{r}\right)\exp(rt)=0$$

$$T=-\frac{1}{r}\ln\left(1-\frac{rZ_0}{W}\right)$$

(2) According to the result above,

$$T_1=T\left(Z_0,W,r\to 0\right)$$

$$=\lim_{x\to 0}\left(-\frac{1}{r}ln\left(1-\frac{rZ_0}{W}\right)\right)=-\frac{1}{r}\left(-\frac{rZ_0}{W}\right)=\frac{Z_0}{W}$$

$$T_2=T\left(Z_0,W,r=\frac{W}{2Z_0}\right)=\frac{2Z_0}{W}ln\,2=\frac{Z_0}{W}(2\,ln\,2).$$

(3) With equation in (1) we have

$$T = -\frac{1}{r} ln\left(1 - \frac{rZ_0}{W}\right)$$

Now to pay in half time, we need to have

$$\frac{T}{2} = -\frac{1}{r} ln\left(1 - \frac{rZ_0}{W_2}\right)$$

Solving these two algebraic equations, we get

$$W_2 = W + \sqrt{W(W - rZ_0)}$$

The additional amount for each payment is now

$$W_2 = W + \sqrt{W(W - rZ_0)}$$

Problem 2.3.4

Mr. Wyze

Borrow = z_0

Payment = w_0

Interest rate = r_w= r

$z_w(t)$= amt owed to bank at time t

dz_w= decrement of loan due to payments made

$$dz_w = z_w(t)rdt - w_0dt$$

$$z_w' = z_w r - w_0$$

$$\int_{z_0}^{z_w} \frac{z_w'}{z_w - \frac{w_0}{r}} = \int_{t=0}^{t} r\,dt$$

$$z_w(t) = \frac{w_0}{r} + \left(z_0 - \frac{w_0}{r}\right)\exp(rt)$$

Mr. Fulesch

Borrow = z_0

Payment = w_0

Interest rate = r_f= $\frac{1}{5}r(1 + t)$

$z_f(t)$= amt owed to bank at time t

dz_f= decrement of loan due to payments made

$$dz_f = z_f(t)r_f dt - w_0dt$$

$$z_f' = z_f \frac{r}{5}(1+t) - w_0 \tag{1}$$

It is in the form of a linear equation.

$$y' + P(x)y = Q(x)$$

$$P(t) = -\frac{r}{5}(1+t)$$

$$\rho(t) = \exp\left(-\frac{r}{5}\int_0^t (1+t)\,dt\right) = C\exp\left(-\frac{r}{5}\left(t+\frac{t^2}{2}\right)\right)$$

$$Q(t) = -w_0$$

$$\frac{d}{dt}\big(z_f(t)\rho(t)\big) = Q(t)\rho(t)$$

$$z_f(t) = z_0 - w_0\rho(t)\int_0^t \rho(t)\,dt \tag{2}$$

Solve for t with Eq-(1) and Eq-(2).

Problem 2.3.5

With the new periodic payment $(1+\alpha)W_0$,

$$dz = zrdt - (1+\alpha)W_0 dt$$

$$\frac{dz}{z - \frac{(1+\alpha)W_0}{r}} = rdt$$

$$\int_{Z_0}^{0} \frac{dz}{z - \frac{(1+\alpha)W_0}{r}} = \int_0^{T_\alpha} rdt$$

$$\ln\left(\frac{-\frac{(1+\alpha)W_0}{r}}{Z_0 - \frac{(1+\alpha)W_0}{r}}\right) = rT_\alpha$$

$$T_\alpha = \frac{1}{r}\ln\left(\frac{(1+\alpha)W_0}{(1+\alpha)W_0 - Z_0 r}\right)$$

Similarly, we can find the payoff time with periodic payment W_0 is

$$T = \frac{1}{r}\ln\left(\frac{W_0}{W_0 - Z_0 r}\right)$$

At periodic payment W_0, the total interests paid to the bank is

$$Z_0 rT = Z_0 \ln\left(\frac{W_0}{W_0 - Z_0 r}\right)$$

At periodic payment $(1+\alpha)W_0$, the total interests paid to the bank is

$$Z_0 rT_\alpha = Z_0 \ln\left(\frac{(1+\alpha)W_0}{(1+\alpha)W_0 - Z_0 r}\right)$$

Problem 2.3.6

(1) With double periodic payment, we have

$$dz = zrdt - 2Wdt$$

$$\frac{dz}{z - \frac{2W}{r}} = rdt$$

$$\int_{Z_0}^{0} \frac{dz}{z - \frac{2W}{r}} = \int_0^{T_{2w}} rdt$$

$$\ln\left(\frac{-\frac{2W}{r}}{Z_0 - \frac{2W}{r}}\right) = rT_{2w}$$

$$T_{2w} = \frac{1}{r}\ln\left(\frac{2W}{2W - Z_0 r}\right)$$

(2) With double rate, we have

$$dz = 2zrdt - W_{2r}dt$$

$$\frac{dz}{z - \frac{W_{2r}}{2r}} = 2rdt$$

$$\int_{Z_0}^{0} \frac{dz}{z - \frac{W_{2r}}{2r}} = \int_0^{T_1} 2rdt$$

$$\ln\left(\frac{-\frac{W_{2r}}{2r}}{Z_0 - \frac{W_{2r}}{2r}}\right) = 2rT_1$$

$$W_{2r} = \frac{2Z_0 r}{1 - \exp(-2rT_1)}$$

Problem 2.3.7

(1) With periodic payment nW, we have

$$dz = zrdt - nWdt$$

$$\frac{dz}{z - \frac{nW}{r}} = rdt$$

$$\int_{Z_0}^{0} \frac{dz}{z - \frac{nW}{r}} = \int_0^{T_w} rdt$$

$$\ln\left(\frac{-\frac{nW}{r}}{Z_0 - \frac{nW}{r}}\right) = rT_w$$

$$T_w = \frac{1}{r}\ln\left(\frac{nW}{nW - Z_0 r}\right)$$

(2) With interest rate br, we have

$$dz = bzrdt - W_r dt$$

$$\frac{dz}{z - \frac{W_r}{br}} = brdt$$

$$\int_{Z_0}^{0} \frac{dz}{z - \frac{W_r}{br}} = \int_0^{T} brdt$$

$$\ln\left(\frac{-\frac{W_r}{br}}{Z_0 - \frac{W_r}{br}}\right) = brT$$

$$W_r = \frac{brZ_0}{1 - \exp(-brT)}$$

Problem 2.3.8

$$dW = Wrdt - W_0 dt$$

and

$$\int_{Z_0}^{0} \frac{dW}{Wr - W_0} = \int_{0}^{T_0} dt = T_0$$

or

$$T_0 = -\frac{1}{r} \ln\left(1 - \frac{rZ_0}{W_0}\right)$$

If payment is αr

$$dW = W\alpha r dt - W_0 dt$$

and

$$\int_{Z_0}^{0} \frac{dW}{W\alpha r - W_0} = \int_{0}^{T_1} dt = T_1$$

or

$$T_1 = -\frac{1}{\alpha r} \ln\left(1 - \frac{\alpha r Z_0}{\alpha W_0}\right)$$

Let

$$T(r) = -\frac{1}{r} \ln\left(1 - \frac{rZ_0}{W_0}\right) = f(r)g(r)$$

where

$$f(r) = \frac{1}{r}$$

and

$$g(r) = -\ln\left|1 - \frac{rZ_0}{W_0}\right|$$

T is always greater than zero, we have $T = f(r)g(r) \geq 0$

$$f(r) = \frac{1}{r} > 0 \Rightarrow g(r) = -\ln\left|1 - \frac{rZ_0}{W_0}\right| > 0 \Rightarrow \left|1 - \frac{rZ_0}{W_0}\right| < 1$$

$$T_0 = T(r) = f(r)g(r)$$
$$T_1 = T(\alpha r)g(\alpha r)$$

Let

$$A = \frac{Zr}{W_0}$$

Then

$$\begin{cases} T_0 = -\dfrac{\ln|1-A|}{r} \\ T_1 = -\dfrac{\ln|1-\alpha A|}{\alpha r} \end{cases}$$

(1) If $a > 1$

$$\alpha r > r \Rightarrow \frac{1}{\alpha r} < \frac{1}{r} \Rightarrow f(\alpha r) < f(r) \qquad \text{(I)}$$

One case.

$$g(r) = -\ln(1 - \mathrm{A}) \text{ and } g(\alpha r) = -\ln(1 - \alpha A)$$
$$1 > \alpha A > A > 0 \Rightarrow 0 < 1 - \alpha A < 1 - A < 1$$
$$-\ln(1 - \alpha\mathrm{A}) > -\ln(1 - \mathrm{A}) > 0$$
$$g(\alpha r) > g(r) \qquad \text{(II)}$$

Another case.

$$g(r) = -\ln(A - 1) \text{ and } g(\alpha r) = -\ln(\alpha A - 1)$$

Then

$$1 > A\alpha - 1 > A - 1 > 0$$
$$0 < -\ln(\alpha A - 1) < -\ln(A - 1)$$
$$\Rightarrow g(\alpha r) < g(r) \qquad \text{(III)}$$

If (I) and (II) are satisfied, it is difficult to get a result.

If (I) and (III) are satisfied, then $T_1 < T_0$.

So the function should be

$$T_0 = -\frac{1}{r}\ln\left(\frac{Zr}{W_0} - 1\right)$$

and

$$T_1 = -\frac{1}{\alpha r}\ln\left(\frac{Z\alpha r}{W_0} - 1\right)$$

This means $Zr > W_0$

(2) If $0 < \alpha < 1$

With the similar method, we have $T_1 > T_0$

(3) If $\alpha = 0$, the DE becomes

$$dW = -W_0 dt$$

and I.C. is $W(0) = Z_0$, which is a 1st-order separable DE.

Solving it, we can get

$$\int_{Z_0}^{0} dW = \int_{0}^{T_1} -W_0 dt \Rightarrow T_1 = \frac{Z_0}{W_0}$$

Chapter 3 Linear DEs of Higher Order

3.1 Classification of DEs

Problem 3.1.1

$$\text{LHS} = y' + y^2 = -\frac{1}{x^2} + \left(\frac{1}{x}\right)^2 = 0 = \text{RHS}$$

For $y = \frac{C}{x}$ where $C \neq 1$ and $C \neq 0$, we have

$$\text{LHS} = -\frac{C}{x^2} + \left(\frac{C}{x}\right)^2 = \frac{C(C-1)}{x^2} \neq 0$$

Problem 3.1.2

From $y = x^3$, we know $y' = 3x^2$ and $y'' = 6x$. Thus

$$\begin{aligned}\text{LHS} &= yy'' \\ &= x^3 \times 6x \\ &= 6x^4 = \text{RHS}\end{aligned}$$

For $y = Cx^3$ where $C \neq 1$ we have

$$\begin{aligned}\text{LHS} &= Cx^3 \times 6Cx \\ &= 6C^2x^4 \neq 6x^4\end{aligned}$$

Problem 3.1.3

For $y_1 = 1$, we know that $y_1' = 0$ and $y_1'' = 0$. Thus

$$\begin{aligned}\text{LHS} &= y_1 y_1'' + (y_1')^2 \\ &= 0 = \text{RHS}\end{aligned}$$

For $y_2 = \sqrt{x}$, we know that

$$y_2' = \frac{1}{2\sqrt{x}} \quad \text{and} \quad y_2'' = -\frac{1}{4x\sqrt{x}}$$

Thus

$$\begin{aligned}\text{LHS} &= \sqrt{x}\left(-\frac{1}{4x\sqrt{x}}\right) + \left(\frac{1}{2\sqrt{x}}\right)^2 \\ &= 0 = \text{RHS}\end{aligned}$$

However for $y = y_1 + y_2$, we have

$$\text{LHS} = \left(\sqrt{x} + 1\right)\left(-\frac{1}{4x\sqrt{x}}\right) + \left(\frac{1}{2\sqrt{x}}\right)^2$$
$$= -\frac{1}{4x\sqrt{x}} \neq 0$$

Problem 3.1.4

(a) y_c is the G.S. of the homogeneous DE, thus

$$y_c'' + py_c' + qy_c = 0$$

y_p is a P.S. of the inhomogeneous DE, then

$$y_p'' + py_p' + qy_p = f(x)$$

Let $y(x) = y_c + y_p$, we have

$$\begin{aligned}\text{LHS} &= \left(y_c + y_p\right)'' + p\left(y_c + y_p\right)' + q\left(y_c + y_p\right) \\ &= (y_c'' + py_c' + qy_c) + \left(y_p'' + py_p' + qy_p\right) \\ &= 0 + f(x) = \text{RHS}\end{aligned}$$

(b) From part (a), we know

$$y = y_c + y_p = 1 + C_1 \cos x + C_2 \sin x$$

is a solution of the given DE, and

$$y(0) = -1 = 1 + C_1 \text{ and } y'(0) = -1 = C_2$$

gives $C_1 = -2$ and $C_2 = -1$. The P.S. is

$$y(x) = 1 - 2\cos x - \sin x$$

3.2 Linear Independence

Problem 3.2.1

Part 1

$$\begin{aligned} L_1L_2x(t) &= L_1\big((D^2+\alpha_2 D+\beta_2)x\big) \\ &= L_1(x''+\alpha_2x'+\beta_2x) \\ &= (D^2+\alpha_1D+\beta_1)(x''+\alpha_2x'+\beta_2x) \\ &= x^{(4)}+(\alpha_1+\alpha_2)x'''+(\alpha_1\alpha_2+\beta_1+\beta_2)x'' \\ &\qquad +(\alpha_1\beta_2+\alpha_2\beta_1)x'+\beta_1\beta_2x \\ L_2L_1x(t) &= L_2\,\big((D^2+\alpha_1D+\beta_1)x\big) \\ &= L_2(x''+\alpha_1x'+\beta_1x) \\ &= (D^2+\alpha_2D+\beta_2)(x''+\alpha_1x'+\beta_1x) \\ &= x^{(4)}+(\alpha_1+\alpha_2)x'''+(\alpha_1\alpha_2+\beta_1+\beta_2)x'' \\ &\qquad +(\alpha_1\beta_2+\alpha_2\beta_1)x'+\beta_1\beta_2x \end{aligned}$$

$$L_1L_2x(t)=L_2L_1x(t)$$

Part 2

$$L_1\equiv D+t,\qquad L_2\equiv tD+1$$

$$\begin{aligned} L_1L_2x(t) &= L_1\big((tD+1)x\big) \\ &= L_1(tx'+x) \\ &= (D+t)(tx'+x) \\ &= tx''+(t^2+2\,)x'+tx \end{aligned}$$

$$\begin{aligned} L_2L_1x(t) &= L_2\big((D+t)x\big) \\ &= L_2(x'+tx) \\ &= (tD+1)(x'+tx) \\ &= tx''+(t^2+1)x'+(t+1)x \end{aligned}$$

$$L_1L_2x(t)\neq L_2L_1x(t)$$

Problem 3.2.2

Note that

$$\cosh x=\frac{\exp(x)+\exp(x)}{2};\ \text{ and }\ \sinh x=\frac{\exp(x)-\exp(x)}{2}$$

we know that

$$\cosh x + \sinh x = \exp(x)$$

That means we are able to find a linear combination of the three functions is identically zero. Hence the three functions are L.D..

Problem 3.2.3

Let $c_1 = 1$, $c_2 = c_3 = 0$, then we have

$$c_1 \cdot 0 + c_2 \sin x + c_3 \exp(x) = 0$$

So these functions are L.D..

Note: 0 is always L.D. with any other function.

Problem 3.2.4

Let $c_1 = 15, c_2 = -16, c_3 = -6$, then we have

$$15 \cdot 2x - 16 \cdot 3x^2 - 6(5x - 8x^2) = 0$$

So these functions are L.D..

Problem 3.2.5

$f(x) = \pi$ gives $f'(x) = 0$, and $g(x) = \cos^2 x + \sin^2 x = 1$ gives $g'(x) = 0$. Hence the Wronskian

$$W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

$$= \begin{vmatrix} \pi & 1 \\ 0 & 0 \end{vmatrix} = 0$$

Thus $f(x)$ and $g(x)$ are not L.I..

Problem 3.2.6

(1) Since both y_1 and y_2 are the solution to the DE

$$A(x)y'' + B(x)y' + C(x)y = 0$$

we know that

$$A(x)y_1'' = -B(x)y_1' - C(x)y_1$$

and

$$A(x)y_2'' = -B(x)y_2' - C(x)y_2$$

The Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= y_1 y_2' - y_2 y_1'$$

Thus, we have

$$A(x)\frac{dW}{dx} = A(x)(y_1 y_2' - y_2 y_1')'$$
$$= A(x)(y_1 y_2'' - y_2 y_1'')$$
$$= y_1(Ay_2'') - y_2(Ay_1'')$$

Substituting Ay_1'' and Ay_2'', we have

$$A(x)\frac{dW}{dx} = y_1(-By_2' - Cy_2) - y_2(-By_1' - Cy_1)$$
$$= -B(y_1 y_2' - y_2 y_1')$$
$$= -B(x)W(x)$$

(2) To solve this equation, we notice it is a separable DE

$$\frac{dW}{W} = -\frac{B}{A}dx$$
$$\ln W = \int -\frac{B(x)}{A(x)}dx + K$$
$$W = K\exp\left(-\int \frac{B(x)}{A(x)}dx\right)$$

Problem 3.2.7

For $f_i(x) = x^i$, we know that

$$f_i^{(i)}(x) = i! \text{ and } f_i^{(j)}(x) = 0, \qquad j > i$$

Thus, the Wronskian

$$W(f_0, f_i, \dots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ f_0' & f_1' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix}$$

is actually an upper triangular matrix with diagonal elements

$$f_i^{(i)} = i!$$

Thus, we have

$$W(f_0, f_1, \dots, f_n) = \prod_{i=0}^{n} i! \neq 0$$

We know functions $f_0, f_1, \dots, f_n$ are L.I.

Problem 3.2.8

Suppose $x_1, \dots, x_n$ are linear independent solutions for the homogeneous DE.

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1}(t) & x_2^{n-1}(t) & \cdots & x_n^{n-1}(t) \end{vmatrix}$$

Taking the derivative of $W(t)$ we get

$$\frac{W(t)}{dt} = \frac{d}{dt}\begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1}(t) & x_2^{n-1}(t) & \cdots & x_n^{n-1}(t) \end{vmatrix}$$

$$= \begin{vmatrix} {x_1}'(t) & {x_2}'(t) & \cdots & {x'}_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1}(t) & x_2^{n-1}(t) & \cdots & x_n^{n-1}(t) \end{vmatrix}$$

$$+ \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1''(t) & x_2''(t) & \cdots & x_n''(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1}(t) & x_2^{n-1}(t) & \cdots & x_n^{n-1}(t) \end{vmatrix} + \cdots$$

$$+ \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n}(t) & x_2^{n}(t) & \cdots & x_n^{n}(t) \end{vmatrix}$$

All the terms except the last one are zero.

$$\frac{W(t)}{dt} = \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n}(t) & x_2^{n}(t) & \cdots & x_n^{n}(t) \end{vmatrix}$$

Plugging $x_1, \dots, x_n$ back to the homogeneous DE, we have

$$x_1^n = P_1(t)x_1^{n-1} - \cdots - P_n(t)x_1$$
$$x_2^n = P_1(t)x_2^{n-1} - \cdots - P_n(t)x_2$$
$$\vdots$$
$$x_n^n = P_1(t)x_n^{n-1} - \cdots - P_n(t)x_n$$

Put this into the DE wrt W

$$\frac{dW(t)}{dt} =$$

$$\begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1^{'}(t) & x_2^{'}(t) & \cdots & x_n^{'}(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n(t) & x_2^n(t) & \cdots & x_n^n(t) \end{vmatrix}$$

$$\times \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1^{'}(t) & x_2^{'}(t) & \cdots & x_n^{'}(t) \\ \vdots & \vdots & \ddots & \vdots \\ -p_1(t)x_1^{n-1}(t) & -p_1(t)x_2^{n-1}(t) & \cdots & -p_1(t)x_n^{n-1}(t) \end{vmatrix}$$

$$+\begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ -p_2(t)x_1^{n-2}(t) & -p_2(t)x_2^{n-2}(t) & \cdots & -p_2(t)x_n^{n-2}(t) \end{vmatrix}+\ldots$$

$$\ldots + \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ -p_n(t)x_1^{n-1}(t) & -p_n(t)x_2^{n-1}(t) & \cdots & -p_n(t)x_n^{n-1}(t) \end{vmatrix}$$

Now everything except the first term cancels out

$$\frac{dW(t)}{dt}$$
$$= \begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ -P_1(t)x_1^{n-1}(t) & -P_1(t)x_2^{n-1}(t) & \cdots & -P_1(t)x_n^{n-1}(t) \end{vmatrix}$$

$$= -P_1(t)\begin{vmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1}(t) & x_2^{n-1}(t) & \cdots & x_n^{n-1}(t) \end{vmatrix}$$

$$= -P_1(t)W(t)$$
$$\frac{dW(t)}{dt} = -P_1(t)W(t)$$

Thus, $W(t)$ is a function of $P_1(t)$. Easily, we can solve the above equation to get

$$W(t) = C \exp\left(-\int P_1(t)\,dt\right)$$

Problem 3.2.9

First of all, we know that $\exp\big((r_1 + r_2 + r_3)x\big) > 0$, $\forall x, r_1, r_2$ and r_3. Notice that

$$\begin{vmatrix} 1 & 1 & 1 \\ r_! & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2)$$

is a Vandermonde determinant which we know that for distinct r_1, r_2 and r_3, it is non-zero. Therefore, the Wronskian

$$W = \exp\big((r_1 + r_2 + r_3)x\big)\begin{vmatrix} 1 & 1 & 1 \\ r_! & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} \neq 0$$

and hence $\exp(r_1 x)$, $\exp(r_2 x)$ and $\exp(r_2 x)$ are L.I..

Problem 3.2.10

First, if $\exists\ 1 \le i, j \le n, i \ne j$ such that $r_i = r_j$, from the property of determinants we know that $V = 0$. Now suppose r_i's are mutually distinct. Some simple calculations suggest

$$V = \prod_{1 \le j < i \le n} \left(r_i - r_j\right)$$

We will prove this by induction. It is easy to see this is true for $n = 2$. Suppose it is true for $n = k$, then for $n = k + 1$, we have

$$V_{k+1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^k & r_2^k & \cdots & r_{k+1}^k \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ r_1 - r_{k+1} & r_2 - r_{k+1} & \cdots & r_k - r_{k+1} & r_{k+1} \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ r_1^k - r_{k+1}^k & r_2^k - r_{k+1}^k & \cdots & r_k^k - r_{k+1}^k & r_{k+1}^k \end{vmatrix}$$

Similarly without changing the value of V, we can subtract, in order, r_{k+1} times row k from row $k+1$, r_{k+1} times row $k-1$ from row k, and so on, till we subtract r_{k+1} times row 2 from row 3.

$$V_{k+1} = \begin{vmatrix} 0 & \cdots & 0 & 1 \\ (r_1 - r_{k+1})r_1^0 & \cdots & (r_k - r_{k+1})\, r_k^0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ (r_1 - r_{k+1})r_1^{k-1} & \cdots & (r_k - r_{k+1})r_k^{k-1} & 0 \end{vmatrix}$$

$$= (-1)^k V_k \prod_{i=1}^{k} (r_i - r_{k+1})$$

$$= V_k \prod_{i=1}^{k} (r_{k+1} - r_i)$$

$$= \prod_{1 \le j < i \le k+1} (r_i - r_j)$$

Here finishes the proof.

Problem 3.2.11

We can get the Wronskian

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

$$= \begin{vmatrix} \exp(r_1x) & \exp(r_2x) & \cdots & \exp(r_nx) \\ r_1\exp(r_1x) & r_2\exp(r_2x) & \cdots & r_n\exp(r_nx) \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1}\exp(r_1x) & r_2^{n-1}\exp(r_2x) & \cdots & r_n^{n-1}\exp(r_nx) \end{vmatrix}$$

$$= \exp\big((r_1+r_2+\cdots+r_n)x\big)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

Since $\exp\big((r_1+r_2+\cdots+r_n)x\big) > 0$, and the second part is a Vandermonde determinant, we know that for $r_1, r_2, \dots, r_n$ that are distinct, the Wronskian is non-zero. Hence f_i are L.I..

3.3 Constant Coefficient Homogeneous DEs

Problem 3.3.1

(1) We have

$$x_{1,2} = \frac{-i \pm \sqrt{i^2 - 8}}{2}$$

This gives $x_1 = i$ and $x_2 = -2i$

(2) We have

$$x_{1,2} = \frac{2i \pm \sqrt{(2i)^2 - 12}}{2}$$

This gives $x_1 = 3i$ and $x_2 = -i$

Problem 3.3.2

(a) We know that any complex number z can be written as $z = a + ib$, where a and b are the real part and imaginary part respectively, and this number can also be written in polar-coordinate as

$$z = r(\cos\theta + i\sin\theta), \text{ where } r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\frac{b}{a}$$

By Euler's law $\exp(i\theta) = \cos\theta + i\cos\theta$, so we proved that any complex number can be written in the form $r\exp(i\theta)$.

(b)

$$4 = 4 + i0 \Rightarrow 4 = 4\exp(i0)$$
$$-2 = -2 + i0 = 2\exp(i\pi)$$
$$3i = 0 + 3i = 3\exp\left(i\frac{\pi}{2}\right)$$
$$1 + i \Rightarrow r = \sqrt{2} \text{ and } \theta = \frac{\pi}{4} \Rightarrow 1 + i = \sqrt{2}\exp\left(i\frac{\pi}{4}\right)$$
$$-1 + i\sqrt{3} \Rightarrow r = 2 \text{ and } \theta = \frac{2}{3}\pi \Rightarrow -1 + i\sqrt{3} = 2\exp\left(i\frac{2\pi}{3}\right)$$

(c) We have

$$2 - 2i\sqrt{3} = 4\exp\left(-i\frac{\pi}{3}\right) \text{ and } -2 + 2i\sqrt{3} = 4\exp\left(i\frac{2\pi}{3}\right)$$

Hence

$$2 - 2i\sqrt{3} = \left(\pm 2\exp\left(-i\frac{\pi}{6}\right)\right)^2 \text{ and } -2 + 2i\sqrt{3}$$
$$= \left(\pm 2\exp\left(i\frac{\pi}{3}\right)\right)^2$$

Problem 3.3.3

Let $t = \exp(u)$, then $u = \ln t$

$$tx' = \frac{dx}{du}$$
$$t^2x'' = \frac{d^2x}{du^2} - \frac{dx}{du}$$
$$t^3x''' = \frac{d^3x}{du^3} - 3\frac{d^2x}{du^2} + 2\frac{dx}{du}$$

Here $x' = \frac{dx}{du}$

$$x''' - 3x'' + 2x' + 6(x'' - x') + 7x' + x = 0$$
$$x''' + 3x'' + 3x' + x = 0$$
$$r^3 + 3r^2 + 3r + 1 = 0$$
$$(r+1)^3 = 0$$
$$r_1 = r_2 = r_3 = -1$$

Therefore, the G.S. is

$$x(u) = C_1\exp(-u) + C_2u\exp(-u) + C_3u^2\exp(-u)$$
$$x(t) = C_1t^{-1} + C_2t^{-1}\ln t + C_3t^{-1}\ln^2 t$$
$$x_1 = t^{-1}, \qquad x_2 = t^{-1}\ln t, \qquad x_3 = t^{-1}\ln^2 t$$

$W(x_1, x_2, x_3) =$

$$\begin{vmatrix} t^{-1} & -t^{-2} & 2t^{-3} \\ t^{-1}\ln(t) & t^{-2}(1 - \ln(t)) & t^{-3}(2\ln(t) - 3) \\ t^{-1}\ln^2(t) & t^{-2}(2\ln(t) - \ln^2(t)) & t^{-3}(2 - 6\ln(t) + 2\ln^2(t)) \end{vmatrix}$$

$$= t^{-1}t^{-2}t^{-3}\begin{vmatrix} 1 & \ln(t) & \ln^2(t) \\ -1 & 1 - \ln(t) & 2\ln(t) - \ln^2(t) \\ 2 & 2\ln(t) - 3 & 2 - 6\ln(t) + 2\ln^2(t) \end{vmatrix}$$

$$= t^{-6}\begin{vmatrix} 1 & \ln(t) & \ln^2(t) \\ 0 & 1 & 2\ln(t) \\ 0 & -3 & 2-6\ln(t) \end{vmatrix} = t^{-6}\begin{vmatrix} 1 & 2\ln(t) \\ -3 & 2-6\ln(t) \end{vmatrix}$$
$$= t^{-6}(2-6\ln(t)+6\ln(t)) = 2t^{-6}$$
$$W(x_1, x_2, x_3) = \frac{2}{t^6}$$

Problem 3.3.4

This is a 2nd-order DE with constant coefficients, C-Eq is
$$r^3 + 3r + 2 = 0$$
$$(r+1)(r+2) = 0$$
which gives
$$r_1 = -1, \qquad r_2 = -2$$
Thus the G.S. is
$$y = C_1 \exp(-x) + C_2 \exp(-2x)$$
Given I.C. $y(0) = 1, y'(0) = 6$, we know that
$$\begin{cases} C_1 + C_2 = 1 \\ -C_1 - 2C_2 = 6 \end{cases}$$
This gives
$$\begin{cases} C_1 = 8 \\ C_2 = -7 \end{cases}$$
Therefore
$$y(x) = 8\exp(-x) - 7\exp(-2x)$$
At the highest point, the velocity vanishes, which means
$$y'(x) = -8\exp(-x) + 14\exp(-2x) = 0$$
Solving this, we get
$$x = \ln\frac{7}{4}$$
and
$$y\left(\ln\frac{7}{4}\right) = \frac{16}{7}$$
Thus, the highest point is

$$\left(\ln\frac{7}{4},\frac{16}{7}\right)$$

Problem 3.3.5

This is a 2nd-order DE with constant coefficients, C-Eq is

$$r^3 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

which gives

$$r_1 = -1, \qquad r_2 = -2$$

Thus the G.S. is

$$y = C_1 \exp(-x) + C_2 \exp(-2x)$$

From the given I.C., we have for y_1

$$\begin{cases} \quad C_1 + C_2 = 3 \\ -C_1 - 2C_2 = 1 \end{cases}$$

This gives

$$\begin{cases} C_1 = 7 \\ C_2 = -4 \end{cases}$$

$$y_1 = 7\exp(-x) - 4\exp(-2x)$$

Similarly, we have for y_2

$$\begin{cases} \quad C_1 + C_2 = 0 \\ -C_1 - 2C_2 = 1 \end{cases}$$

This gives

$$\begin{cases} C_1 = 1 \\ C_2 = -1 \end{cases}$$

$$y_2 = \exp(-x) - \exp(-2x)$$

To find their intersection, let

$$y_1(x) = y_2(x)$$

This gives

$$7\exp(-x) - 4\exp(-2x) = \exp(-x) - \exp(-2x)$$

$$x = -\ln 2$$

Thus

$$y_{1,2}(-\ln 2) = -2$$

Hence the intersection is

$$(-\ln 2\,, -2)$$

Problem 3.3.6

The C-Eq is

$$3r^3 + 2r^2 = 0$$

It is easy to solve

$$r_1 = r_2 = 0, \qquad r_3 = -\frac{2}{3}$$

Thus, we have G.S.

$$y = C_1 + C_2 x + C_3 \exp\left(-\frac{2}{3}x\right)$$

Given the I.C., we can get

$$\begin{cases} C_1 + C_3 = -1 \\ C_2 - \dfrac{2}{3}C_3 = 0 \\ \dfrac{4}{9}C_3 = 1 \end{cases}$$

which gives

$$\begin{cases} C_1 = -\dfrac{13}{4} \\ C_2 = \dfrac{3}{2} \\ C_3 = \dfrac{9}{4} \end{cases}$$

Therefore, the P.S. is

$$y = \frac{1}{4}\left(-13 + 6x + 9\exp\left(-\frac{2}{3}x\right)\right)$$

Problem 3.3.7

From the solution we know that the C-Eq has 3 repeated roots at both $2i$ and $-2i$. That means

$$(r + 2i)^3 (r - 2i)^3 = 0$$
$$r^6 + 12r^4 + 48r^2 + 64 = 0$$

Therefore, the DE is

$$y^{(6)} + 12y^{(4)} + 48y'' + 64 = 0$$

Problem 3.3.8

The C-Eq is

$$r^3 = 1$$
$$(r-1)(r^2 + r + 1) = 0$$

We can get

$$r_1 = 1, r_{2,3} = \frac{-1 \pm \sqrt{3}i}{2}$$

Thus, the G.S. is

$$y = C_1 \exp(x) + C_2 \exp\left(\frac{-1+\sqrt{3}i}{2}x\right) + C_3 \exp\left(\frac{-1-\sqrt{3}i}{2}x\right)$$

From the I.C., we have

$$\begin{cases} C_1 + C_2 + C_3 = 1 \\ C_1 + \dfrac{-1+\sqrt{3}i}{2}C_2 + \dfrac{-1-\sqrt{3}i}{2}C_3 = 0 \\ C_1 + \dfrac{-1-\sqrt{3}i}{2}C_2 + \dfrac{-1+\sqrt{3}i}{2}C_3 = 0 \end{cases}$$

This gives

$$C_1 = C_2 = C_3 = \frac{1}{3}$$

Hence we have

$$y = \frac{1}{3}\exp(x) + \frac{1}{3}\exp\left(\frac{-1+\sqrt{3}i}{2}x\right) + \frac{1}{3}\exp\left(\frac{-1-\sqrt{3}i}{2}x\right)$$

which can be written as

$$y = \frac{1}{3}\exp(x) + \frac{2}{3}\exp\left(-\frac{1}{2}x\right)\cos\frac{\sqrt{3}}{2}x$$

Problem 3.3.9

For the 4th order DE with constant coefficients, the C-Eq is

$$r^4 - r^3 - r^2 - r - 2 = 0$$
$$(r+1)(r-2)(r^2+1) = 0$$

The G.S. is

$$y = C_1 \exp(-x) + C_2 \exp(2x) + C_3 \cos x + C_4 \sin x$$

From the given I.C., we have

$$\begin{cases} C_1 + C_2 + C_3 = 0 \\ -C_1 + 2C_2 + C_4 = 0 \\ C_1 + 4C_2 - C_3 = 0 \\ -C_1 + 8C_2 - C_4 = 30 \end{cases}$$

Solving for the I.C. we have,

$$C_1 = -5, C_2 = 2, C_3 = 3 \text{ and } C_4 = -9$$

Thus, the P.S. is

$$y = -5\exp(-x) + 2\exp(2x) + 3\cos x - 9\sin x$$

Problem 3.3.10

If $x > 0$, then the DE is $y'' + y = 0$ with the G.S. $y = A\cos x + B\sin x$

If $x < 0$ the DE becomes $y'' - y = 0$ with G.S. $y = C\exp(x) + D\exp(-x)$

To satisfy the I.C. $y_1(0) = 1$ and $y_1'(0) = 0$, we have

$$\begin{cases} A = 1 \\ B = 0 \end{cases} \text{ and } \begin{cases} C + D = 1 \\ C - D = 0 \end{cases} \Rightarrow C = D = \frac{1}{2}$$

Thus,

$$y = \begin{cases} \cos x, & x \geq 0 \\ \cosh x, & x < 0 \end{cases}$$

You can try the similar ways to gety_2,

$$y = \begin{cases} \sin x, & x \geq 0 \\ \sinh x, & x < 0 \end{cases}$$

Problem 3.3.11

This is a higher order DE with constant coefficients, the corresponding C-Eq is

$$r^3(r-2)(r+3)(r^2+1) = 0$$

Thus we have

$$r_1 = r_2 = r_3 = 0, r_4 = 2, r_5 = -3, r_{6,7} = \pm i$$

So the solution is

$$y = C_1 + C_2 x + C_3 x^2 + C_4 \exp(2x) + C_5 \exp(-3x) + C_6 \sin x + C_7 \cos x$$

Problem 3.3.12

The C-Eq is

$$r^2 - 2r + 2 = 0$$
$$r_{1,2} = 1 \pm i$$

The G.S. is

$$y = \exp(x)\,(C_1 \sin x + C_2 \cos x)$$

From the given I.C., we have

$$C_2 = 0, C_1 = 5$$

Thus, we have

$$y = 5\exp(x)\sin x$$

Problem 3.3.13

The C-Eq is

$$r^3 + 9r = 0$$
$$r_1 = 0, r_{2,3} = \pm 3i$$

The G.S. is

$$y = C_1 + C_2 \sin 3x + C_3 \cos 3x$$

From the given I.C., we have

$$\begin{cases} C_1 + C_3 = 3 \\ 3C_2 = -1 \\ -9C_3 = 2 \end{cases}$$

This gives

$$\begin{cases} C_1 = \dfrac{29}{9} \\ C_2 = -\dfrac{1}{3} \\ C_3 = -\dfrac{2}{9} \end{cases}$$

Thus, the solution is

$$y = \frac{29}{9} - \frac{1}{3}\sin 3x - \frac{2}{9}\cos 3x$$

Problem 3.3.14

The C-Eq is

$$(r-1)^3(r-2)^2(r-3)(r^2+9) = 0$$
$$r_1 = r_2 = r_3 = 1, r_4 = r_5 = 2, r_6 = 3, r_{7,8} = \pm 3i$$

Thus, the G.S. is

$$y = (C_1 + C_2 x + C_3 x^2)\exp(x) + (C_4 + C_5 x)\exp(2x) + C_6\exp(3x) + C_7\cos 3x + C_8\sin 3x$$

Problem 3.3.15

This is a 2nd-order DE with constant coefficients, the C-Eq is

$$r^2 - r - 15 = 0$$

Solve it, we have

$$r_1 = \frac{1+\sqrt{61}}{2}, \qquad r_2 = \frac{1-\sqrt{61}}{2}$$
$$y_1 = \exp\left(\frac{1+\sqrt{61}}{2}x\right), \qquad y_2 = \exp\left(\frac{1-\sqrt{61}}{2}x\right)$$

Thus the G.S. is

$$y = c_1\exp\left(\frac{1+\sqrt{61}}{2}x\right) + c_2\exp\left(\frac{1-\sqrt{61}}{2}x\right)$$

Problem 3.3.16

The C-Eq is

$$9r^2 - 12r + 4 = 0$$
$$\Rightarrow (3r-2)^2 = 0$$
$$\Rightarrow r = \frac{2}{3}$$
$$y_1(x) = \exp\left(\frac{2x}{3}\right); \;\; y_2(x) = x\exp\left(\frac{2x}{3}\right)$$
$$y(x) = c_1\exp\left(\frac{2x}{3}\right) + c_2 x\exp\left(\frac{2x}{3}\right)$$

Problem 3.3.17

(1) This is a k[th] order DE with constant coefficients, and the C-Eq is $(r-r_1)^{k_1}=0$. Since $r=r_1$ has k_1 repeated roots, we have

$$y(x)=\left(C_1+C_2x+C_3x^2+\cdots+C_{k_1}x^{k_1-1}\right)\exp(r_1x)$$

(2) The C-Eq for this DE is

$$(r-r_1)^{k_1}(r-r_2)^{k_2}\dots(r-r_n)^{k_n}=0$$

Thus $r=r_1$ is k_1 repeated root, $r=r_2$ is k_2 repeated root, …, $r=r_n$ is k_n repeated root.

$$\begin{aligned}y(x)=&\left(C_{11}+C_{12}x+\cdots+C_{1k}x^{k_1-1}\right)\exp(r_1x)\\&+\left(C_{12}+C_{22}x+\cdots+C_{2k_2}x^{k_2-1}\right)\exp(r_2x)\\&+\cdots\\&+\left(C_{n1}+C_{n2}x+\cdots+C_{nk_n}x^{k_n-1}\right)\exp(r_nx)\end{aligned}$$

where C_{ij} means constant.

Problem 3.3.18

First find the G.S. of corresponding homogenous DE:

$$\left(x\frac{d}{dx}-\alpha\right)^n y(x)=0$$

Let $x=\exp(t)$, we have

$$\left(\exp(t)\frac{d}{\exp(t)\,dt}-\alpha\right)^n y=0$$

$$\left(\frac{d}{dt}-\alpha\right)^n y=0$$

The C-Eq is$(r-\alpha)^n=0$. There are n identical real roots.

So the G.S. for the homogenous DE is

$$\begin{aligned}y_c(t)&=\exp(\alpha t)\left(C_1+C_2t+C_3t^2+\cdots+C_nt^{n-1}\right)\\&=\exp(\alpha t)\sum_{k=1}^{n}C_kt^{k-1}\end{aligned}$$

Then plugging x back, we have

$$y_c(x) = x^{\alpha} \sum_{k=1}^{n} C_k (\ln x)^{k-1}$$

Finding the P.S. consists of three cases:

Case I: $\alpha = 0$

The original DE becomes

$$y^{(n)} = \exp(t)$$

whose P.S. is

$$y_p = \exp(t)$$

The G.S. is

$$y = y_c + y_p = (C_1 + C_2 t + C_3 t^2 + \cdots + C_n t^{n-1}) + \exp(t)$$

Or in terms of x

$$y = (C_1 + C_2 \ln x + C_3 (\ln x)^2 + \cdots + C_n (\ln x)^{n-1}) + x$$

Case II: $\alpha = 1$, we select the trial P.S. as

$$y_p = At^n \exp(t)$$

Thus,

$$\begin{aligned}
\text{LHS} &= \left(\frac{d}{dt} - 1\right)^{n} (At^n \exp(t)) \\
&= \left(\frac{d}{dt} - 1\right)^{n-1} \left(\frac{d}{dt} - 1\right) (At^n \exp(t)) \\
&= \left(\frac{d}{dt} - 1\right)^{n-1} \left(\frac{d}{dt} (At^n \exp(t)) - At^n \exp(t)\right) \\
&= \left(\frac{d}{dt} - 1\right)^{n-1} (Ant^{n-1} \exp(t)) \\
&= \left(\frac{d}{dt} - 1\right)^{n-2} \left(\frac{d}{dt} - 1\right) Ant^{n-1} \exp(t) \\
&= \left(\frac{d}{dt} - 1\right)^{n-2} \left(\frac{d}{dt} (Ant^{n-1} \exp(t)) - Ant^{n-1} \exp(t)\right) \\
&= \left(\frac{d}{dt} - 1\right)^{n-2} An(n-1)t^{n-2} \exp(t) \\
&= \cdots = An! \exp(t)
\end{aligned}$$

Plugging in to the original DE, we get

$$\left(\frac{d}{dt} - 1\right)^{n} (At^n \exp(t)) = \exp(t)$$

Or

$$A(n!)\exp(t) = \exp(t)$$

Thus,

$$A = \frac{1}{n!}$$

The G.S. for case $\alpha = 1$ is

$$\begin{aligned} y &= y_c + y_p \\ &= \exp(t)\,(C_1 + C_2 t + C_3 t^2 + \cdots + C_n t^{n-1}) + \frac{1}{n!} t^n \exp(t) \end{aligned}$$

Or in terms of x

$$\begin{aligned} y &= y_c + y_p \\ &= x(C_1 + C_2 \ln x + C_3 (\ln x)^2 + \cdots + C_n (\ln x)^{n-1}) \\ &\quad + \frac{1}{n!}(\ln x)^n x \end{aligned}$$

Case III: $\alpha \neq 1$

We select trial P.S.

$$y_p = A\exp(t)$$

Thus,

$$\begin{aligned} \text{LHS} &= \left(\frac{d}{dt} - \alpha\right)^{n-1} \left(\frac{d}{dt} - \alpha\right) A\exp(t) \\ &= \left(\frac{d}{dt} - \alpha\right)^{n-1} (1-\alpha)A\exp(t) \\ &= \left(\frac{d}{dt} - \alpha\right)^{n-2} \left(\frac{d}{dt} - \alpha\right)(1-\alpha)A\exp(t) \\ &= \left(\frac{d}{dt} - \alpha\right)^{n-2} (1-\alpha)^2 A\exp(t) \\ &= \cdots = (1-\alpha)^n A\exp(t) \end{aligned}$$

Thus

$$\left(\frac{d}{dt} - \alpha\right)^{n} A\exp(t) = \exp(t)$$

Or

$$A(1-\alpha)^n \exp(t) = \exp(t)$$

i.e.,

$$A = \frac{1}{(1-\alpha)^n}$$

The G.S. is

$$y = y_c + y_p$$

$$= \exp(\alpha t)\,(C_1 + C_2 t + C_3 t^2 + \cdots + C_n t^{n-1}) + \frac{1}{(1-\alpha)^n}\exp(t)$$

Or in terms of x

$$y = y_c + y_p$$
$$= x^{\alpha}(C_1 + C_2 \ln x + C_3(\ln x)^2 + \cdots + C_n(\ln x)^{n-1}) + \frac{1}{(1-\alpha)^n}x$$

One can see that this case covers the simple case of $\alpha = 0$.

3.4 Cauchy-Euler DEs

Problem 3.4.1

The original DE can be written as the following form

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0$$

Let $t=\ln x$, then

$$\frac{dy}{dx}=\exp(-t)\frac{dy}{dt}$$

$$\begin{aligned}
\frac{d^2y}{dx^2}&=\frac{d}{dx}\left(\frac{dy}{dx}\right)\\
&=\frac{d}{dx}\left(\exp(-t)\exp(t)\frac{dy}{dt}\right)\\
&=\frac{d}{dx}(\exp(-t))\frac{dy}{dt}+\exp(-t)\frac{d}{dx}\left(\frac{dy}{dt}\right)\\
&=-\exp(-t)\frac{dt}{dx}\frac{dy}{dt}+\exp(-t)\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}\\
&=-\exp(-2t)\frac{dy}{dt}+\exp(-2t)\frac{d^2y}{dt^2}\\
&=\exp(-2t)\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)
\end{aligned}$$

Plugging these back to the DE, we have

$$a\exp(2t)\left(\exp(-2t)\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right)+b\exp(t)\left(\exp(-t)\frac{dy}{dt}\right)$$
$$+cy=0$$
$$a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=0$$

which is a linear DE WRT t. The C-Eq is

$$ar^2+(b-a)r+c=0$$
$$r_{1,2}=\frac{a-b\pm\sqrt{(b-a)^2-4ac}}{2a}$$

If $(b-a)^2-4ac>0$, we have two distinct real roots.

$$y(t) = C_1 \exp(r_1 t) + C_2 \exp(r_2 t)$$

Put $t = \ln x$ back, we have

$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

If $(b-a)^2 - 4ac = 0$, we have repeated root r, thus

$$y(t) = (C_1 + C_2 t) \exp(rt)$$

Put $t = \ln x$ back, we have

$$y(x) = (C_1 + C_2 \ln x) x^r$$

If $(b-a)^2 - 4ac < 0$, we have two complex conjugated roots

$$r_{1,2} = r \exp(\pm i\theta)$$

Thus

$$y(t) = (C_1 \sin \theta t + C_2 \cos \theta t) \exp(rt)$$

Put $t = \ln x$ back, we have

$$y(x) = (C_1 \sin \theta \ln x + C_2 \cos \theta \ln x) x^r$$

Problem 3.4.2

We use the substitution $v = \ln x$, we get the DE

$$v = \ln x$$

$$v = \ln x \Rightarrow dv = \frac{dx}{x}$$

$$\frac{dx}{dv} = x \Rightarrow xy' = \frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\left(\frac{d^2y}{dv^2}\right)$$

$$x^2 y'' = -\frac{dy}{dv} + \left(\frac{d^2y}{dv^2}\right)$$

$$y_v'' - 9y_v = 0$$

$$r^2 - 9 = 0$$

$$r = \pm 3$$

$$y_1 = \exp(3t) = x^3$$

$$y_2 = \exp(-3t) = x^{-3}$$

$$y_c = C_1 x^3 + C_2 x^{-3}$$

Problem 3.4.3

We use the substitution

$$v = \ln x$$

$$v = \ln x \Rightarrow dv = \frac{dx}{x}$$

$$\frac{dx}{dv} = x \Rightarrow xy' = \frac{dy}{dv}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\left(\frac{d^2y}{dv^2}\right)$$

$$x^2y'' = -\frac{dy}{dv} + \left(\frac{d^2y}{dv^2}\right)$$

$$y_v'' + (b-1)y_v' + cy = 0$$

C-Eq:

$$r^2 + (b-1)r + c = 0$$

$$r = \frac{-(b-1) \pm \sqrt{(b-1)^2 - 4c}}{2}$$

$$y_1 = \exp\left(\frac{-(b-1) + \sqrt{(b-1)^2 - 4c}}{2}v\right)$$

$$y_2 = \exp\left(\frac{-(b-1) - \sqrt{(b-1)^2 - 4c}}{2}v\right)$$

Problem 3.4.4

Let $t = \ln x$, then

$$\frac{dy}{dx} = \exp(-t)\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \exp(-2t)\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

Thus, the DE becomes

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

Its C-Eq is

$$r^2 - 3r + 2 = 0$$
$$r_1 = 1, r_2 = 2$$

Thus

$$y(t) = C_1 \exp(t) + C_2 \exp(2t)$$

Put x back, we have

$$y(x) = C_1 x + C_2 x^2$$

From the given I.C., we can get

$$\begin{cases} C_1 + C_2 = 3 \\ C_1 + 2C_2 = 1 \end{cases}$$

This gives

$$\begin{cases} C_1 = 5 \\ C_2 = -2 \end{cases}$$

Thus the IVP has solution

$$y(x) = 5x - 2x^2$$

Problem 3.4.5

Let $y = (x+2)^\lambda$, we have

$$y' = \lambda(x+2)^{\lambda-1}$$
$$y'' = (\lambda-1)\lambda(x+2)^{\lambda-2}$$

Substitute into DE

$$(x+2)^\lambda(\lambda^2 - 2\lambda + 1) = 0$$
$$(\lambda-1)^2 = 0$$
$$\lambda_1 = 1, \qquad \lambda_2 = 1$$

The G.S. is

$$y = C_1(x+2) + C_2(x+2)\ln|x+2|$$

Problem 3.4.6

Let $y = x^\lambda$, $y' = \lambda x^{\lambda-1}, y'' = \lambda(\lambda-1)x^{\lambda-2}$

Thus,

$$x^2\lambda(\lambda-1)x^{\lambda-2} - 2x\lambda x^{\lambda-1} - 10y = 0$$
$$x^\lambda(\lambda(\lambda-1) - 2\lambda - 10) = 0$$
$$\lambda^2 - 3\lambda - 10 = 0$$

There are two real and different roots $\lambda_1 = 5, \lambda_2 = -2$
So the G.S. is

$$y = C_1 x^5 + C_2 x^{-2}$$

Problem 3.4.7

Substitute

$$y = x^r$$

we have

$$y' = rx^{r-1}, y'' = r(r-1)x^{r-2}, y''' = r(r-1)(r-2)x^{r-3}$$

Then

$$r(r-1)(r-2) + r(r-1) - r + 1 = 0$$
$$r^3 - 2r^2 + 1 = 0$$
$$(r-1)(r^2 - r - 1) = 0$$

which has three distinct roots, resulting three L.I. solutions.

$$r = 1, \quad \frac{1-\sqrt{5}}{2}, \quad \frac{1+\sqrt{5}}{2}$$
$$y(x) = c_1 x^1 + c_2 x^{\frac{1-\sqrt{5}}{2}} + c_3 x^{\frac{1+\sqrt{5}}{2}}$$

3.5 Inhomogeneous Higher Order DEs

Problem 3.5.1

This is a 2nd-order DE with constant coefficients, and the corresponding C-Eq is

$$r^2 - 2r - 8 = (r+2)(r-4) = 0$$
$$r_1 = -2, \qquad r_2 = 4$$

Thus, $y_c(x) = C_1 \exp(-2x) + C_2 \exp(4x)$

$$f(x) = \exp(4x)$$

Trial solution is

$$y_p(x) = Ax^s \exp(4x)$$

Compare terms in y_c and y_p, we notice $s = 1$.

$$y_p(x) = Ax \exp(4x) \Longrightarrow \begin{cases} y_p' = A4x \exp(4x) + A \exp(4x) \\ y_p'' = A8 \exp(4x) + A16x \exp(4x) \end{cases}$$

Plugging into original DE,

$$A8 \exp(4x) + a16x \exp(4x) - 2(A4x \exp(4x) + A \exp(4x)) \\ - 8Ax \exp(4x) = \exp(4x)$$
$$A6 \exp(4x) = \exp(4x)$$
$$A = \frac{1}{6} \Longrightarrow y_p = \frac{x}{6} \exp(4x)$$

So the G.S. is

$$y(x) = C_1 \exp(-2x) + \mathrm{C}_2 \exp(4x) + \frac{x}{6} \exp(4x)$$

Problem 3.5.2

C-Eq for homogeneous portion of the DE is

$$r^4 - 1 = 0$$
$$r_1 = 1, r_2 = -1, r_3 = i, r_4 = -i$$
$$y_c(x) = c_1 \exp(x) + c_2 \exp(-x) + c_3 \cos x + c_4 \sin x$$

Trial P.S.

$$y_p = A$$

So

$$y_p' = 0 = y_p'' = y_p''' = y_p''''$$

After substitution, we get

$$0 - A = 1$$
$$A = -1$$
$$y_p = -1$$

The G.S. is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 \exp(x) + c_2 \exp(-x) + c_3 \cos x + c_4 \sin x - 1 \end{aligned}$$

Problem 3.5.3

We first consider the homogeneous equation

$$y'''' - y = 0$$

The C-Eq is

$$\begin{aligned} r^4 - 1 &= (r^2 + 1)(r^2 - 1) \\ &= (r + i)(r - i)(r + 1)(r - 1) \\ &= 0 \end{aligned}$$

The roots are $r_1 = -i\,, r_2 = i, r_3 = -1, r_4 = 1.$

Then the complementary solution can be written as

$$y_c = C_1 \cos x + C_2 \sin x + C_3 \exp(x) + C_4 \exp(-x)$$

Next we assume the P.S. y_p has the form

$$y_p = Ax \exp(x)$$

Its derivatives are

$$\begin{aligned} y_p' &= \exp(x) + Ax \exp(x) \\ y_p'' &= 2A \exp(x) + Ax \exp(x) \\ y_p''' &= 3A \exp(x) + Ax \exp(x) \\ y_p'''' &= 4A \exp(x) + Ax \exp(x) \end{aligned}$$

Substituting them in the inhomogeneous ODE, we have

$$4A \exp(x) = 4 \exp(x)$$

Solving the equation gives us

$$A = 1$$
$$y_p = x \exp(x)$$

Finally the G.S. of the original ODE is

$$y = y_p + y_c = x\exp(x) + C_1\cos x + C_2\sin x + C_3\exp(x) + C_4\exp(-x)$$

Problem 3.5.4

Solve the corresponding homogeneous DE. The C-Eq is

$$r^4 - r^3 - r^2 - r - 2 = 0$$
$$(r+1)(r^2+1)(r-2) = 0$$
$$r_1 = -1, r_2 = 2, r_{3,4} = \pm i$$

The G.S. to the homogeneous DE is

$$y_c = C_1\exp(-x) + C_2\exp(2x) + C_3\sin x + C_4\cos x$$

From the RHS $f(x) = 18x^5$, we can assume the P.S.

$$y_p = x^s(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5)$$

Since there has no polynomial in the homogeneous solution, we can let $s = 0$. Thus

$$y_p = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5$$

and

$$y_p' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4$$
$$y_p'' = 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3$$
$$y_p''' = 6A_3 + 24A_4x + 60A_5x^2$$
$$y_p^{(4)} = 24A_4 + 120A_5x$$

Plugging this into the DE, we have

$$y_p^{(4)} - y_p''' - y_p'' - y_p' - 2y_p = 18x^5$$

Equating the terms, we have

$$\begin{cases} 24A_4 - 6A_3 - 2A_2 - A_1 - 2A_0 = 0 \\ 120A_5 - 24A_4 - 6A_3 - 2A_2 - 2A_1 = 0 \\ -60A_5 - 12A_4 - 3A_3 - 2A_2 = 0 \\ -20A_5 - 4A_4 - 2A_3 = 0 \\ -5A_5 - 2A_4 = 0 \\ -2A_5 = 18 \end{cases}$$

Solving this gives

$$\begin{cases} A_0 = \dfrac{135}{4} \\ A_1 = \dfrac{135}{2} \\ A_2 = \dfrac{135}{2} \\ A_3 = 45 \\ A_4 = \dfrac{45}{2} \\ A_5 = -9 \end{cases}$$

Thus, the solution is

$$y_p = \frac{135}{4} + \frac{135}{2}x + \frac{135}{2}x^2 + 45x^3 + \frac{45}{2}x^4 - 9x^5$$

Finally, the G.S. is

$$y = C_1 \exp(-x) + C_2 \sin x + C_3 \cos x + C_4 \exp(2x) + \frac{135}{4} + \frac{135}{2}x + \frac{135}{2}x^2 + 45x^3 + \frac{45}{2}x^4 - 9x^5$$

Problem 3.5.5

We can assume the P.S. has form

$$y_p = x^s(A_0 + A_1x + A_2x^2 + A_3x^3)$$

To determine s, we have to find the G.S. to the homogenous equation. The C-Eq is

$$r^3 + r^2 + r + 1 = 0$$
$$(r + 1)(r^2 + 1) = 0$$
$$r_1 = -1, r_{2,3} = \pm i$$

This means the polynomials are not part of the y_c. Hence $s = 0$. Now we have

$$y_p = A_0 + A_1x + A_2x^2 + A_3x^3$$

and

$$y_p' = A_1 + 2A_2x + 3A_3x^2$$
$$y_p'' = 2A_2 + 6A_3x$$

$$y_p''' = 6A_3$$

Plugging this back to the DE

$$y_p''' + y_p'' + y_p' + y_p = x^3 + x^2$$

Equating the terms, we have

$$\begin{cases} A_0 + A_1 + 2A_2 + 6A_3 = 0 \\ A_1 + 2A_2 + 6A_3 = 0 \\ A_2 + 3A_3 = 1 \\ A_3 = 1 \end{cases}$$

which gives

$$\begin{cases} A_0 = 0 \\ A_1 = -2 \\ A_2 = -2 \\ A_3 = 1 \end{cases}$$

Thus, we have a P.S. of the DE

$$y_p = -2x - 2x^2 + x^3$$

Problem 3.5.6

We can assume the P.S. has form

$$y_p = A\exp(3x) + B\exp(2x) + C\exp(x) + D$$

Thus

$$y_p' = 3A\exp(3x) + 2B\exp(2x) + C\exp(x)$$

and

$$y_p'' = 9A\exp(3x) + 4B\exp(2x) + C\exp(x)$$
$$y_p''' = 27A\exp(3x) + 8B\exp(2x) + C\exp(x)$$

Putting this back to the DE, equating the terms and we can get

$$\begin{cases} 27A + 9A + 3A + A = 1 \\ 8B + 4B + 2B + B = 1 \\ C + C + C + C = 1 \\ D = 1 \end{cases}$$

this gives

$$\begin{cases} A = \dfrac{1}{40} \\ B = \dfrac{1}{15} \\ C = \dfrac{1}{4} \\ D = 1 \end{cases}$$

Thus, the P.S. is

$$y_p = \frac{1}{40}\exp(3x) + \frac{1}{15}\exp(2x) + \frac{1}{4}\exp(x) + 1$$

Problem 3.5.7

The C-Eq of the homogeneous DE is

$$r^4 + \omega^2 r^2 = 0$$

$$r_1 = r_2 = 0, r_{3,4} = \pm\omega i$$

Thus the G.S. for the homogeneous DE is

$$y_c = A_1 + A_2 x + A_3 \sin\omega x + A_4 \cos\omega x$$

Since $\sin\omega x$ and $\cos\omega x$ are part of the G.S., we can assume the P.S. has form

$$\begin{aligned} y_p = {} & C_1 x^2 \sin(\omega x) + C_2 x^2 \cos(\omega x) + C_3 x \sin\omega x \\ & + C_4 x \cos\omega x \end{aligned}$$

$$\begin{aligned} y_p'' = {} & (2C_1 - 2\omega C_4 - 4\omega x C_2 + \omega^2 x C_3 - \omega^2 x^2 C_1)\sin\omega x \\ & +(2C_2 + 2\omega C_3 + 4\omega x C_1 - \omega^2 x C_4 - \omega^2 x^2 C_2)\cos\omega x \end{aligned}$$

$$\begin{aligned} y_p^{(4)} = {} & (4C_4\omega^3 - 12C_1\omega^2 + C_4\omega^4 x + 8C_2\omega^3 x \\ & + C_1\omega^4 x^2)\sin\omega x \\ & + (-4C_3\omega^3 - 12C_2\omega^2 + C_3\omega^4 x - 8C_1\omega^3 x \\ & + C_2\omega^4 x^2)\cos\omega x \end{aligned}$$

Putting these back and equating the terms, we can get

$$\begin{cases} C_1 = 0 \\ C_2 = \dfrac{1}{4\omega^3} \\ C_3 = -\dfrac{5}{4\omega^4} \\ C_4 = \dfrac{1}{\omega^3} \end{cases}$$

Finally, we have $y = y_c + y_p$

Problem 3.5.8

The homogeneous G.S. is

$y_c = c_1\cos(\omega t) + c_2\sin(\omega t)$If $\omega = \omega_{0,}$ suppose the P.S.:

$$y_p = Bx\exp(i\omega_0 x)$$

and find y_p''

Plug $y_{p\,,}$ y_p'' to $y'' + \omega^2 y = \exp(i\omega_0 x)$, then find $B = \frac{1}{2i\omega_0}$.

$$y_p = \frac{1}{2i\omega_0}x\exp(i\omega_0 x)$$

So, the G.S. is

$$y = y_c +_p = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{1}{2i\omega_0}x\exp(i\omega_0 x)$$

If $\omega \neq \omega_0$, suppose the P.S. $y_p = A\exp(i\omega_0 x)$ and find y_p''

Plug $y_{p\,,}$ y_p'' to

$$y'' + \omega^2 y = \exp(i\omega_0 x)$$

then

$$A = \frac{1}{\omega^2 - \omega_0^2}$$

$$y_p = \frac{1}{\omega^2 - \omega_0^2}\exp(i\omega_0 x)$$

So, the G.S. is

$$y = y_c + y_p = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{1}{\omega^2 - \omega_0^2}\exp(i\omega_0 x)$$

Problem 3.5.9

The C-Eq of the homogeneous DE is

$$r^2 + 9 = 0$$
$$r_{1,2} = \pm 3i$$

Thus the G.S. for the homogeneous DE is

$$y_c = A_1 \cos 3x + A_2 \sin 3x$$

Since $\sin 3x$ and $\cos 3x$ are part of the G.S.,
we can assume the P.S. has form

$$y_p = x(C_1 \sin 3x + C_2 \cos 3x)$$

Thus,

$$y_p' = C_1 \sin 3x + C_2 \cos 3x$$
$$+3xC_1 \cos 3x - 3xC_2 \sin 3x$$

and

$$y_p'' = (6C_1 - 9xC_2)\cos 3x + (-6C_2 - 9xC_1)\sin 3x$$

Putting these back and equating the terms, we can get

$$C_1 = \frac{1}{6}, \qquad C_2 = 0$$

Thus, the P.S. is

$$y_p = \frac{1}{6} x \sin 3x$$

and the G.S. for the original DE is

$$y = A_1 \cos 3x + A_2 \sin 3x + \frac{1}{6} x \sin 3x$$

Problem 3.5.10

First we find the complementary solution, i.e., the solution to

$$y'' + 2y' + y = 0$$

The C-Eq is $r^2 + 2r + 1 = 0$ whose two identical roots $r = -1$. So

$$y_c(x) = c_1 \exp(-x) + c_2 x \exp(-x)$$

Now we must solve for the P.S., $y_p(x)$. Because we see that the inhomogeneous term is a $\sin x$ and $\cos x$, we will use the method of undetermined coefficients with our guess of y_p as,

$$y_p = A\cos x + B\sin x$$

$$y_p' = -A\sin x + B\cos x, \quad y_p'' = -A\cos x - B\sin x$$

Substituting these values into the DE, we obtain that

$$-A\cos x - B\sin x - 2A\sin x + 2B\cos x + A\cos x + B\sin x$$
$$= 5\sin x + 5\cos x$$

Hence we find that

$$-2A\sin x + 2B\cos x = 5\sin x + 5\cos x$$

Therefore

$$A = -\frac{5}{2}, \qquad B = \frac{5}{2}$$

The final solution is then

$$y(x) = y_p + y_c = \frac{5}{2}(\sin x - \cos x)$$
$$+ c_1\exp(-x) + c_2 x\exp(-x)$$

Problem 3.5.11

The C-Eq is

$$r^3 + r^2 + r + 1 = 0$$

$$r_1 = -1, \qquad r_{2,3} = \pm i$$

$$y_c = C_1\exp(-x) + C_2\sin x + C_3\cos x$$

We can assume the particular has form

$$y_p = A + x(B\cos x + C\sin x) + D\cos 2x$$
$$+E\sin 2x + xF\exp(-x)$$

$$y_p' = B\cos x + C\sin x + x(-B\sin x + C\cos x)$$
$$-2D\sin 2x + 2E\cos 2x + F\exp(-x) - xF\exp(-x)$$

$$y_p'' = -2B\sin x + 2C\cos x - x(B\cos x + C\sin x)$$
$$-4D\cos x - 4E\sin 2x - 2F\exp(-x) + xF\exp(-x)$$

$$y_p''' = -3(B\cos x + C\sin x)$$

$$+x(B\sin x - C\cos x) + 8D\sin 2x - 8E\cos 2x + (3-x)F\exp(-x)$$

Plugging this back and comparing the terms, we have

$$\begin{cases} A = 1 \\ 2C - 2B = 1 \\ -2B - 2C = 0 \\ -3D - 6E = 0 \\ -3E + 6D = 1 \\ 2F = 1 \end{cases}$$

Solving this gives

$$\begin{cases} A = 1 \\ B = -\dfrac{1}{4} \\ C = \dfrac{1}{4} \\ D = \dfrac{2}{15} \\ E = -\dfrac{1}{15} \\ F = \dfrac{1}{2} \end{cases}$$

The P.S. is

$$y_p = 1 - \frac{1}{4}x\cos x + \frac{1}{4}x\sin x + \frac{2}{15}\cos 2x - \frac{1}{15}\sin 2x + \frac{1}{2}x\exp(-x)$$

Problem 3.5.12

Let $x = \exp(t)$, then $t = \ln x$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 4\frac{dy}{dt} + 6y = \exp(3t)$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = \exp(3t)$$

C-Eq of homogeneous portion:

$$r^2 - 5r + 6 = 0$$
$$(r-3)(r-2) = 0$$
$$r_1 = 3, \qquad r_2 = 2$$
$$y_c(t) = C_1 \exp(3t) + C_2 \exp(2t)$$

Assume the P.S. has form

$$y_p = At \exp(3t)$$
$$y_p' = A \exp(3t) + 3At \exp(3t)$$
$$y_p'' = 6A \exp(3t) + 9At \exp(3t)$$

Plugging these back, we have

$$A \exp(3t) = \exp(3t)$$

Thus,

$$A = 1$$
$$y_p = t \exp(3t)$$

The G.S. is

$$y(t) = C_1 \exp(3t) + C_2 \exp(2t) + t \exp(3t)$$
$$y(x) = C_1 x^3 + C_2 x^2 + x^3 \ln x$$

Problem 3.5.13

Use the trial function $y = x^\alpha$ and plug into the homogeneous equation first.

$$x^6(x^\alpha)'' + 2x^5(x^\alpha)' - 12x^4x^\alpha = (\alpha(\alpha-1) + 2\alpha - 12)x^{4+\alpha}$$
$$= (\alpha^2 + \alpha - 12)x^{4+\alpha} = 0$$
$$(\alpha+4)(\alpha-3) = 0$$
$$\alpha = -4,3$$

Thus,

$$y_C(x) = c_1 x^{-4} + c_2 x^3$$

To find P.S. use trial function

$$y_P = Ax^{-4} \ln x$$

$$x^6(Ax^{-4}\ln x)'' + 2x^5(Ax^{-4}\ln x)' - 12x^4(Ax^{-4}\ln x) = 1$$
$$x^6(20Ax^{-6}\ln x - 9Ax^{-6}) + 2x^5(-4Ax^{-5}\ln x + Ax^{-5}) - 12A\ln x = 1$$
$$(20 - 8 - 12)\ln x + A(-9 + 2) = 1$$
$$A = -\frac{1}{7}$$

The G.S. is then

$$y = c_1x^{-4} + c_2x^3 - \frac{1}{7}x^{-4}\ln x$$

Problem 3.5.14

Let $x = \exp(t)$, then $t = \ln x$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$
$$x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$
$$\ddot{y} + \dot{y} - 6y = 72\exp(5t)$$

C-Eq:

$$r^2 + r - 6 = 0$$
$$(r + 3)(r - 2) = 0$$
$$r_1 = -3, \qquad r_2 = 2$$
$$y_c(t) = C_1\exp(-3t) + C_2\exp(2t)$$

Assume the P.S. has form

$$y_p = A\exp(5t)$$
$$\dot{y_p} = 5A\exp(5t)$$
$$\ddot{y_p} = 25A\exp(5t)$$

Plugging these back, we have

$$25A\exp(5t) + 5A\exp(5t) - 6A\exp(5t) = 72\exp(5t)$$

Thus,

$$A = 3$$
$$y_p = 3\exp(5t)$$

The G.S. is

$$y(t) = C_1 \exp(-3t) + C_2 \exp(2t) + 3 \exp(5t)$$
$$y(x) = C_1 x^{-3} + C_2 x^2 + 3x^5$$

Problem 3.5.15

We use the substitution $t = \ln x$, we have

$$xy' = \dot{y}$$
$$x^2 y'' = \ddot{y} - \dot{y}$$

The equation becomes

$$4\ddot{y} - 4\dot{y} - 4\dot{y} + 3y = 8 \exp\left(\frac{4}{3}t\right)$$
$$4\ddot{y} - 8\dot{y} + 3y = 8 \exp\left(\frac{4}{3}t\right)$$

which is a constant coefficient inhomogeneous linear DE WRT t. The C-Eq to the associated homogeneous DE is

$$4r^2 - 8r + 3 = 0$$
$$r_1 = \frac{3}{2}, \qquad r_2 = \frac{1}{2}$$
$$y_c(t) = C_1 \exp\left(\frac{3}{2}t\right) + C_2 \exp\left(\frac{1}{2}t\right)$$

Assume the P.S. has the form $y_p(t) = A \exp\left(\frac{4}{3}t\right)$, then

$$\frac{64}{9} A \exp\left(\frac{4}{3}t\right) - \frac{32}{3} A \exp\left(\frac{4}{3}t\right) + 3A \exp\left(\frac{4}{3}t\right) = 8 \exp\left(\frac{4}{3}t\right)$$
$$A = -\frac{72}{5}$$

Thus,

$$y(t) = C_1 \exp\left(\frac{3}{2}t\right) + C_2 \exp\left(\frac{1}{2}t\right) - \frac{72}{5} \exp\left(\frac{4}{3}t\right)$$

That is

$$y(x) = C_1 x^{\frac{3}{2}} + C_2 x^{\frac{1}{2}} - \frac{72}{5} x^{\frac{4}{3}}$$

Problem 3.5.16

Using Euler's Equation approach.

$$x = \exp(t)\,, \quad t = \ln x$$

$$xy' = \frac{dy}{dt}, \qquad x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + y = t$$

$$\frac{d^2y}{dt^2} + y = t$$

The homogeneous portion of the solution

$$y_c(t) = C_1 \cos t + C_2 \sin t$$

Try P.S.

$$y_p(t) = At + B$$

We get $A = 1, B = 0$. The final solution is

$$y(t) = C_1 \cos t + C_2 \sin t + t$$

$$y(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x) + \ln x$$

Problem 3.5.17

We use the substitution $t = \ln x$, we have

$$xy' = \dot{y}$$

$$x^2y'' = \ddot{y} - \dot{y}$$

$$x^3y''' = 2\dot{y} - 3\ddot{y} + \dddot{y}$$

The equation reduces to

$$\dddot{y} - 2\ddot{y} + 2\dot{y} - y = 1 + \exp(t) + \exp(2t)$$

The C-Eq for the homogeneous equation is

$$r^3 - 2r^2 + 2r - 1 = 0$$

$$(r-1)(r^2 - r + 1) = 0$$

$$r_1 = 1, r_{2,3} = \frac{1 \pm i\sqrt{3}}{2}$$

Thus, the G.S. to the homogeneous DE is

$$y_c(t) = C_1 \exp(t) + \left(C_2 \cos\frac{\sqrt{3}}{2}t + C_3 \sin\frac{\sqrt{3}}{2}t\right)\exp\left(\frac{1}{2}t\right)$$

Since e^t is part of the G.S., we can assume the P.S. has form

$$y_p(t) = A_1 t\exp(t) + A_2\exp(2t) + A_3$$

Thus,

$$\dot{y}_p = A_1(t+1)\exp(t) + 2A_2\exp(2t)$$
$$\ddot{y}_p = A_1(t+2)\exp(t) + 4A_2\exp(2t)$$
$$\dddot{y}_p = A_1(t+3)\exp(t) + 8A_2\exp(2t)$$

Plugging these into the DE, we have

$$A_1\exp(t) + 3A_2\exp(2t) - A_3 = \exp(t) + \exp(2t) + 1$$

This gives

$$A_1 = 1, \qquad A_2 = \frac{1}{3}, \qquad A_3 = -1$$

Thus,

$$y_p = t\exp(t) + \frac{1}{3}\exp(2t) - 1$$

and

$$y(t) = t\exp(t) + \frac{1}{3}\exp(2t) - 1 + C_1\exp(t) + \left(C_2\cos\frac{\sqrt{3}}{2}t + C_3\sin\frac{\sqrt{3}}{2}t\right)\exp\left(\frac{1}{2}t\right)$$

Finally, we have

$$y(x) = x\ln x + \frac{1}{3}x^2 - 1 + C_1 x + \left(C_2\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + C_3\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right)\sqrt{x}$$

Problem 3.5.18

Plug in test solution $y = x^\alpha$ to solve homogeneous equation:

$$[\alpha(\alpha-1) + \alpha - 4]x^\alpha = 0$$
$$\alpha^2 - 4 = 0$$
$$\alpha = \pm 2$$

So $y_C = C_1x^2 + C_2x^{-2}$ is the homogeneous solution. For inhomogeneous part, the trial solution can be

$$y_P = Ax^2 \ln x + Bx^{-2} \ln x$$

where A and B are constants. Plugging into DE gives

$$\begin{aligned} &x^2(Ax^2 \ln x + Bx^{-2} \ln x)'' + x(Ax^2 \ln x + Bx^{-2} \ln x)' \\ &\quad - 4(Ax^2 \ln x + Bx^{-2} \ln x) \\ &= x^2\big(A(2 \ln x + 3) + Bx^{-4}(6 \ln x - 5)\big) \\ &\quad + x\big(Ax(2 \ln x + 1) + Bx^{-3}(-2 \ln x + 1)\big) \\ &\quad - 4(Ax^2 \ln x + Bx^{-2} \ln x) \\ &= 4Ax^2 - 4Bx^{-2} \end{aligned}$$

$$A = \frac{1}{4}, B = -\frac{1}{4}$$

The G.S. is then

$$y = C_1x^2 + C_2x^{-2} + \frac{1}{4}(x^2 - x^{-2}) \ln x$$

Problem 3.5.19

Given two L.I. solutions $y_1(x)$ and $y_2(x)$, which imply

$$\begin{cases} y_1'' + \alpha(x)y_1' + \beta(x)y_1 = 0 & (1) \\ y_2'' + \alpha(x)y_2' + \beta(x)y_2 = 0 & (2) \end{cases}$$

$(1) \times y_2 - (2) \times y_1$ gives

$$\alpha(x)(y_1' y_2 - y_1y_2') = y_2y_1'' - y_2y_2''$$

The Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$\frac{dW}{dx} = y_1y_2'' - y_2y_1''$$

This gives

$$\alpha(x)W(x) = -\frac{dW}{dx}$$

$$\frac{dW}{W} = -\alpha(x)dx$$

$$\ln W = -\int \alpha(x)dx$$

$$W = \exp\left(-\int \alpha(x)dx\right)$$

This is the required relation between the Wronskian and $\alpha(x)$.

The variation of parameters formula gives:

$$y_p = \int^x \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W} f(t)dt$$

Substitute this value of W in the above variation of parameters formulate find a P.S.

$$y_p = \int^x \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{\exp\left(-\int^{t} \alpha(x)dx\right)} f(t)dt$$

Since y_1 and y_2 are the two L.I. solutions of the homogeneous portion of $y'' + \alpha(x)y' + \beta(x)y = f(x)$, the G.S. is

$$y_c(x) = C_1y_1(x) + C_2y_2(x)$$

$$y(x) = y_c(x) + y_p(x)$$

$$= C_1y_1(x) + C_2y_2(x) + \int^x \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{\exp\left(-\int^{t} \alpha(x)dx\right)} f(t)dt$$

Problem 3.5.20

C-Eq of homogeneous portion:

$$x'' + \omega^2 x = 0$$

$$r^2 + \omega^2 = 0 \ \rightarrow \ \ r_1, r_2 = \pm i\omega$$

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

(1) Function $f(t)$ is a general form

$$x_p(t) = \int K(t,\tau)f(\tau)d\tau$$

$$x(t) = x_c + x_p$$

$$= c_1 \cos(\omega t) + c_2 \sin(\omega t) + \int_{\tau_0}^{\tau} K(t,\tau) \int f(\tau)\, d\tau$$

where

$$K(t,\tau) = \frac{\cos\omega t \sin\omega\tau - \cos\omega\tau \sin\omega t}{\omega}$$

(2) Function $f(t)$ is a specific function $f(t) = \exp(i\omega t)$. Naturally, $f(t)$ is a solution of the homogeneous equation and thus, the trial solution should be

$$x_p(t) = At \exp(i\omega t)$$

Thus

$$x_p'(t) = A\exp(i\omega t) + Ati\omega\exp(i\omega t)$$
$$x_p''(t) = -A\,t\omega^2\exp(i\omega t) + 2Ai\omega\exp(i\omega t)$$

Thus, we have

$$-A\,t\omega^2\exp(i\omega t) + 2Ai\omega\exp(i\omega t) \ + At\omega^2\exp(i\omega t)$$
$$= \exp(i\omega t)$$

From this, we get

$$A = \frac{1}{2i\omega}$$

Thus, the P.S. should be

$$x_p(t) = \frac{1}{2i\omega}\exp(i\omega t)$$

$$x(t) = x_c + x_p = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{1}{2i\omega}\exp(i\omega t)$$

(3) Function $f(t)$ is a specific function $f(t) = \exp(\omega t)$. For $\omega \ \neq 0$

$$x_p = A\exp(\omega t)$$
$$x_p' = A\omega\exp(\omega t)$$
$$x_p'' = A\omega^2\exp(\omega t)$$

Since $x'' + \omega^2 x = \exp(\omega t)$

$$A\omega^2\exp(\omega t) + A\omega^2\exp(\omega t) = \exp(\omega t)$$
$$A = \frac{1}{\omega^2}$$
$$x_p = \frac{1}{\omega^2}\exp(\omega t)$$
$$x(t) = x_c + x_p = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{1}{\omega^2}\exp(\omega t)$$

Remarks

When $\omega = 0$

$$x'' = 1$$
$$x'' - 1 = 0$$

$$(r-1)(r+1)=0$$
$$r_1=1, r_2=-1$$
$$x(t)=x_c=c_1\exp(t)+c_2\exp(-t)$$

3.6 Variation of Parameters

Problem 3.6.1

(1) This is a 2nd-order DE with constant coefficients, the C-Eq is

$$r^2 - (r_1 + r_2)r + r_1 r_2 = 0$$

whose solution are r_1 and r_2. Thus, the two solutions are

$$y_1 = \exp(r_1 x), \quad y_2 = \exp(r_2 x)$$

(2) Let $y_p = u_1 y_1 + u_2 y_2$, and the trial solutions are

$$u_1 = -\int^x \frac{y_2(t)}{W(t)} f(t)dt = -\int^x \frac{\exp(r_2 x)}{W(t)} f(t)dt$$

$$u_2 = -\int^x \frac{y_1(t)}{W(t)} f(t)dt = -\int^x \frac{\exp(r_1 x)}{W(t)} f(t)dt$$

where

$$W(t) = \begin{vmatrix} \exp(r_1 x) & \exp(r_2 x) \\ r_1 \exp(r_1 x) & r_2 \exp(r_2 x) \end{vmatrix}$$
$$= (r_2 - r_1)\exp\big((r_1 + r_2)t\big)$$

(3) The G.S. for the whole equation

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \frac{1}{r_2 - r_1}\int^x \Big(\exp\big(r_2(x-t)\big) - \exp\big(r_1(x-t)\big)\Big) f(t)dt$$

Thus the G.S. is

$$y = C_1 \exp(r_1 x) + C_2 \exp(r_2 x)$$
$$+ \frac{1}{r_2 - r_1}\int^x \Big(\exp\big(r_2(x-t)\big)$$
$$- \exp\big(r_1(x-t)\big)\Big) f(t)dt$$

Problem 3.6.2

First we must take all necessary derivatives of $y_2(x)$,

$$y_2'(x) = Z y_1' + Z' y_1$$
$$y_2''(x) = Z y_1'' + 2Z' {y_1}' + Z'' y_1$$

Plugging these values into the original DE, we obtain

$$Zy_1'' + 2Z'y_1' + Z''y_1 + P(x)(Zy_1' + Z'y_1) + Q(x)Zy_1 = 0$$

Now note that we can collect terms to get

$$Z(y_1'' + P(x)y_1' + Q(x)y_1) + Z''y_1 + Z'(2y_1' + P(x)y_1) = 0$$

Since y_1 is a solution

$$Z(y_1'' + P(x)y_1' + Q(x)y_1) = 0$$

We get

$$Z''y_1 + Z'(2y_1' + P(x)y_1) = 0$$

Now we will perform the following substitution to cast the equation into a 1st-order DE form

$$u = Z' \;=>\; u' = Z''$$

We obtain

$$u' + u\left(2\frac{y_1'}{y_1} + P(x)\right) = 0$$

Using the integrating factor method, we find that

$$u(x) = \exp\left(-\int\left(2\frac{y_1'}{y_1} + P(x)\right)dx\right)$$
$$= \exp\left(-2\ln y_1 - \int P(x)dx\right)$$

Thus,

$$u(x) = \frac{\exp(-\int P(x)dx)}{y_1^2(x)}$$

Now we must solve for Z.

$$Z(x) = \int \frac{\exp(-\int P(x)dx)}{y_1^2(x)}dx$$

The second L.I. solution is

$$y_2(x) = y_1(x)\int \frac{\exp(-\int P(x)dx)}{y_1^2(x)}dx$$

Problem 3.6.3

The corresponding homogeneous DE is

$$y'' + y = 0$$

The C-Eq is

$$r^2 + 1 = 0$$
$$r_{1,2} = \pm i$$

So the G.S. to the homogeneous DE is

$$y_c = C_1 \sin x + C_2 \cos x$$

By using the variation of parameters method, denote

$$y_1 = \sin x\,, y_2 = \cos x \text{ and } f = \cot x$$

Then

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$= \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$
$$= -1$$

and we can get the P.S.

$$y_p = -y_1 \int \frac{y_2 f}{W} dx + y_2 \int \frac{y_1 f}{W} dx$$
$$= \sin x \int (\csc x - \sin x) dx - \cos \int \cos x\, dx$$
$$= \sin x \ln|\csc x - \cot x|$$

Problem 3.6.4

The homogeneous equation is

$$y''' + y'' + y' + y = 0$$

The C-Eq is

$$r^3 + r^2 + r + 1 = 0$$
$$(r + 1)(r^2 + 1) = 0$$
$$r_1 = -1, r_{2,3} = \pm i$$

We have

$$y_1 = \exp(-x)\,, \qquad y_2 = \sin x\,, \qquad y_3 = \cos x$$

Let

$$y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$$

Then

$$y_p' = (u_1 y_1' + u_2 y_2' + u_3 y_3') + (u_1' y_1 + u_2' y_2 + u_3' y_3)$$

Since y_p is not unique, we can assume

$$u_1'y_1 + u_2'y_2 + u_3'y_3 = 0$$

Thus

$$y_p' = u_1y_1' + u_2y_2' + u_3y_3'$$

Similarly, we have

$$y_p'' = (u_1y_1'' + u_2y_2'' + u_3y_3'') + (u_1'y_1' + u_2'y_2' + u_3'y_3')$$

and we can also assume

$$u_1'y_1' + u_2'y_2' + u_3'y_3' = 0$$

Thus,

$$y_p'' = u_1y_1'' + u_2y_2'' + u_3y_3''$$

Finally,

$$y_p''' = (u_1y_1''' + u_2y_2''' + u_3y_3''') + (u_1'y_1'' + u_2'y_2'' + u_3'y_3'')$$

Since y_1 , y_2 and y_3 are solution of the corresponding homogeneous DE, thus

$$\begin{cases} y_1''' = -y_1'' - y_1' - y_1 \\ y_2''' = -y_2'' - y_2' - y_2 \\ y_3''' = -y_3'' - y_3' - y_3 \end{cases}$$

Plugging these into the equation for y_p''', we can get

$$y_p''' = -y_p'' - y_p' - y_p + (u_1'y_1'' + u_2'y_2'' + u_3'y_3'')$$

Therefore, from the original DE, we have

$$u_1'y_1'' + u_2'y_2'' + u_3'y_3'' = f(x)$$

Now we have

$$\begin{cases} u_1'y_1 + u_2'y_2 + u_3'y_3 = 0 \\ u_1'y_1 + u_2'y_2 + u_3'y_3 = 0 \\ u_1'y_1'' + u_2'y_2'' + u_3'y_3'' = f(x) \end{cases}$$

or we can write them in the matrix form

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f(x) \end{bmatrix}$$

Solving this, we can get

$$u_1 = \int \frac{W(y_2, y_3) f(x)}{W(y_1, y_2, y_3)} dx$$
$$u_2 = \int \frac{W(y_3, y_1) f(x)}{W(y_1, y_2, y_3)} dx$$
$$u_3 = \int \frac{W(y_1, y_2) f(x)}{W(y_1, y_2, y_3)} dx$$

Problem 3.6.5

The homogeneous equation is
$$y'' + 9y = 0$$
whose C-Eq is
$$r^2 + 9 = 0$$
$$r_{1,2} = \pm 3i$$
Thus, we have
$$y_1 = \cos 3x \quad \text{and} \quad y_2 = \sin 3x$$
The P.S. has the form
$$y_p = u_1 y_1 + u_2 y_2$$
where u_1 and u_2 satisfies
$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = f(x)$$
where
$$f(x) = \sin x \tan x$$
and we know that the Wronskian
$$W(y_1, y_2) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$
$$= 3$$
We get
$$u_1 = -\int \frac{y_2 f}{W} dx$$
$$= -\frac{1}{3} \int \sin 3x \sin x \tan x \, dx$$

$$= -\frac{1}{3}\left(\sin^4 x + \frac{1}{2}\sin^2 x + \ln\cos x\right)$$

and

$$u_2 = \int \frac{y_1 f}{W} dx$$
$$= \frac{1}{3}\int \cos 3x \cos x \cot x \, dx$$
$$= \frac{1}{3}(-\sin x \cos^3 x + 2\cos x \sin x - x)$$

The P.S. is

$$y_p = u_1 y_1 + u_2 y_2$$

Problem 3.6.6

If $\omega = 0$, we have $y'' = 0$ where $y = x$ is a P.S..

Now consider $\omega \neq 0$. It is easy to get the G.S. for the homogeneous equation

$$y_c = C_1 \cos\omega x + C_2 \sin\omega x$$

Thus, we have

$$y_p = \int \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W\big(y_1(t), y_2(t)\big)} f(t)dt$$

where $y_1(x) = \cos\omega x$, $y_2(x) = \sin\omega x$, $f(t) = \sin\omega t \tan\omega t$ and the Wronskian $W = \omega$.

We can get

$$y_p = \frac{1}{2\omega^2}(\omega x \sin\omega x + 2\cos\omega x \ln\cos\omega x)$$

Problem 3.6.7

The C-Eq of the homogeneous equation is

$$r^2 + a = 0$$

which gives

$$r_1, r_2 = \pm i\sqrt{a}$$

The G.S. for the homogeneous equation is

$$y_c = C_1 \cos(\sqrt{a}x) + C_2 \sin(\sqrt{a}x)$$

Thus, we have

$$y_p = \int \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{W\big(y_1(t), y_2(t)\big)} f(t)dt$$

where $y_1(x) = C_1 \cos(\sqrt{a}x)$, $y_2(x) = C_2 \sin(\sqrt{a}x)$, $f(t) = \tan(bt)$ and

$$W\big(y_1(t), y_2(t)\big) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \sqrt{a}$$

Problem 3.6.8

(1) The associated homogeneous DE is $y'' + y = 0$, and the C-Eq for this equation is

$$r^2 + 1 = 0$$

$$y_c(x) = C_1 \sin x + C_2 \cos x$$

The two L.I. solutions for the homogeneous DE is

$$y_1 = \sin x, \quad y_2 = \cos x$$

and $f(x) = 2 \sin x$.

(2) The Wronskian is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

$$u_1 = -\int \frac{y_2(x)f(x)}{W(x)} dx = -\int \frac{\cos x \cdot 2 \sin x}{-1} dx = -\frac{\cos x}{2}$$

$$u_2 = \int \frac{y_1(x)f(x)}{W(x)} dx = \int \frac{\sin x \cdot 2 \sin x}{-1} dx = \frac{\sin x}{2} - x$$

$$\begin{aligned} y_p(x) &= y_1 u_1 + y_2 u_2 \\ &= -\frac{\cos 2x \sin x}{2} + \frac{\sin 2x \cos x}{2} - x \cos x \end{aligned}$$

(3) The G.S. of the DE is

$$y(x) = y_c + y_p = C_1 \sin x + C_2 \cos x + \sin x - x \cos x$$

Problem 3.6.9

With $y_1 = x$, $y_2 = 1/x$, and $f(x) = 72x^3$, the variation of parameters takes the form

$$\begin{cases} xu_1' + \dfrac{u_2'}{x} = 0 \\ u_1' - \dfrac{u_2'}{x^2} = 72x^3 \end{cases}$$

Upon multiplying the second equation by x and then adding, we readily solve first for u_1

$$u_1' = 36x^3$$

$$u_1 = \int 36x^3 dx = 9x^4$$

And then

$$u_2' = -x^2 u_1' = -36x^5$$

$$u_2 = \int -36x^5 dx = -6x^6$$

Then it follows that: a P.S. can be written as

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= x \cdot 9x^4 - \frac{1}{x} \cdot 6x^6 \\ &= 3x^5 \end{aligned}$$

Problem 3.6.10

The homogenous portion is

$$x'' - 3x' - 4x = 0$$

C-Eq:

$$r^2 - 3r - 4 = 0$$

$$r_1 = 4, \qquad r_2 = -1$$

$$x_c(t) = C_1 \exp(4t) + C_2 \exp(-t)$$

<u>Method I: Trail Solution</u>

Trial P.S. is $x_p(t) = A \exp(4t) + B \exp(-t)$

Since each term in the trial P.S. $x_p(t)$ depends on terms in $x_c(t)$, we modify terms in $x_p(t)$, i.e., multiply each term in $x_p(t)$ with *t* we get

$$x_p(t) = At \exp(4t) + Bt \exp(-t)$$

Evaluating

$x_p' = A\exp(4t) + 4A\exp(4t) + B\exp(-t) - Bt\exp(-t)$

$x_p'' = 8A\exp(4t) + 16At\exp(4t) - 2B\exp(-t) + Bt\exp(-t)$

Substituting and equating the coefficients

$$x' - 3x' - 4x = 15\exp(4t) + 5\exp(-t)$$

Produces $A = 3, B = -1$

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) \\ &= C_1\exp(4t) + C_2\exp(-t) + 3t\exp(4t) - t\exp(-t) \end{aligned}$$

Method II: Variation of Parameters

From $x_c(t)$ we have $x_1(t) = \exp(4t), x_2(t) = \exp(-t)$ and $x_1'(t) = 4\exp(4t), x_2'(t) = -\exp(-t)$

Evaluate the Wronskian of x_1, x_2.

$$W(t) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} \exp(4t) & \exp(-t) \\ 4\exp(4t) & -\exp(-t) \end{vmatrix} = -5\exp(3t)$$

We have

$$x_p = \int^t \frac{x_2(t)x_1(s) - x_1(t)x_2(s)}{W(s)} f(s)ds$$

where

$x_1(s) = \exp(4s), x_2(s) = \exp(-s), W(s) = -5\exp(3s)$

We get

$$x_p = 3t\exp(4t) + \frac{3}{5}\exp(4t) - t\exp(-t) - \frac{1}{5}\exp(-t)$$

$$\begin{aligned} x(t) &= x_c(t) + x_p(t) \\ &= C_1\exp(4t) + C_2\exp(-t) + 3t\exp(4t) + \frac{3}{5}\exp(4t) \\ &\qquad - t\exp(-t) - \frac{1}{5}\exp(-t) \\ &= C_3\exp(4t) + C_4\exp(-t) + 3t\exp(4t) - t\exp(-t) \end{aligned}$$

where $C_3 = C_1 + \frac{3}{5}, C_4 = C_2 - \frac{1}{5}$

Problem 3.6.11

The Wronskian is

$$W(x_1, x_2, x_3) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}$$

Taking the derivative, we have

$$\frac{dW}{dt} = \frac{d}{dt}\begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix}$$

$$= \begin{vmatrix} x_1' & x_2' & x_3' \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1'' & x_2'' & x_3'' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1''' & x_2''' & x_3''' \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1''' & x_2''' & x_3''' \end{vmatrix}$$

x_1, x_2 and x_3 are solutions of the DE

$$x_1''' + p_1x_1 + p_2x_1 + p_3x_1 = 0$$
$$x_2''' + p_1x_2 + p_2x_2 + p_3x_2 = 0$$
$$x_3''' + p_1x_3 + p_2x_3 + p_3x_3 = 0$$

$$\Rightarrow \quad \begin{aligned} x_1''' &= -p_1x_1'' - p_2x_1' - p_3x_1 = a \\ x_2''' &= -p_1x_2'' - p_2x_2' - p_3x_2 = b \\ x_3''' &= -p_1x_3'' - p_2x_3' - p_3x_3 = c \end{aligned}$$

$$\frac{dW}{dt} = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -p_1x_1'' & -p_1x_2'' & -p_1x_3'' \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -p_2x_1' & -p_2x_2' & -p_2x_3' \end{vmatrix}$$

$$+ \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ -p_3x_1 & -p_3x_2 & -p_3x_3 \end{vmatrix}$$

$$= -p_1 \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{vmatrix} = -p_1W$$

$$W = K \exp\left(\int -p_1(t)dt\right)$$

$$W(x_1, x_2) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = x_1 x_2' - x_2 x_1'$$

Similarly

$$W(x_2, x_3) = x_2 x_3' - x_3 x_2', \qquad W(x_3, x_1) = x_3 x_1' - x_1 x_3'$$

$$u_1(t) = \int \frac{W(x_2, x_3) f(t)}{W(x_1, x_2, x_3)} dt$$

$$\Rightarrow \quad u_2(t) = \int \frac{W(x_3, x_1) f(t)}{W(x_1, x_2, x_3)} dt$$

$$u_3(t) = \int \frac{W(x_1, x_2) f(t)}{W(x_1, x_2, x_3)} dt$$

Finally, substituting $u_1(t)$, $u_2(t)$, $u_3(t)$, $x_1(t)$, $x_2(t)$, $x_3(t)$ into

$$x_p = u_1(t) x_1(t) + u_2(t) x_2(t) + u_3(t) x_3(t)$$

Chapter 4 Systems of Linear DEs

4.2 First-Order Systems and Applications

Problem 4.2.1

(1) Consider the forces on the first object: the spring k_1 is stretched x_1 units and spring k_2 by $x_2 - x_1$units. Stretched force is pointed away from the object. By Newton's law, we have

$$mx_1'' = k_2(x_2 - x_1) - k_1x_1 = -(k_1 + k_2)x_1 + k_2x_2$$

For the second object, the spring k_2 is stretched by $(x_2 - x_1)$ units, and the spring k_3 is compressed by x_2 units. Again, by Newton's law, we have

$$\begin{aligned} mx_2'' &= -k_2(x_2 - x_1) - k_3x_2 \\ &= k_2x_1 - (k_2 + k_3)x_2 \end{aligned}$$

(2) If the spring k_2 is broken, then $k_2 = 0$, the DEs are

$$\begin{cases} mx_1'' = -k_1x_1 \\ mx_2'' = -k_3x_2 \end{cases}$$

(3) If $k_1 = k_2 = k_3 = k$

$$\begin{cases} mx_1'' = -2kx_1 + kx_2 \Rightarrow x_2 = \dfrac{mx_1''}{k + 2x_1} \\ \qquad mx_2'' = kx_1 - 2kx_2 \end{cases}$$

$$\frac{mx_1^{(4)}}{k + 2x_1''} = kx_1 - 2k\left(\frac{mx_1''}{k + 2x_1}\right)$$

$$\frac{mx_1^{(4)}}{k + 2x_1''} = kx_1 - 2mx_1'' - 4kx_1$$

$$x_1^{(4)} + \frac{2k + 2km}{m}x_1'' + \frac{3k^2}{m}x_1 = 0$$

which is a 4th order DE with constant coefficient. Solving the C-Eq, and after we find the characteristic roots, we can easily get the generally solution of $x_1(t)$. $x_2(t)$ can be attained with the similar method.

Problem 4.2.2

The force can be show as the figure on the top of the next page. From Newton's law, we have:

Force in x direction: $F_x = -F\cos\theta = mx''$

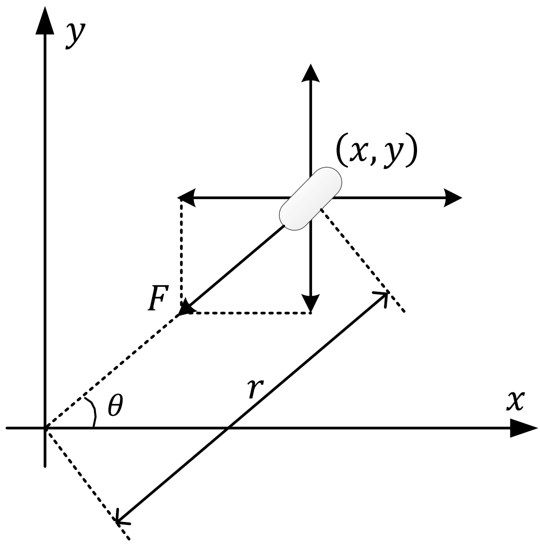

Figure A.7 The particle moves under the influence of force F.

Force is y direction: $F_y = -F\sin\theta = my''$

From the given condition, we know:

$$r = \sqrt{x^2 + y^2} \text{ and } F = \frac{k}{x^2 + y^2} = \frac{k}{r^2}$$

$$\sin\theta = \frac{y}{r} \text{ and } \cos\theta = \frac{x}{r}$$

By substitution of $\sin\theta$, $\cos\theta$, F, we have

$$\begin{cases} mx'' = -\dfrac{kx}{r^3} \\ my'' = -\dfrac{ky}{r^3} \end{cases}$$

Problem 4.2.3

Forces on this object can be shown in the following figure. By Newton's law:

Forces in x-direction: $F_x = -F\cos\theta = mx''$

Forces in y directions: $F_y = -F\sin\theta - F_G = my''$

Also from the given condition, we have

$$\cos\theta = \frac{v_x}{v} = \frac{x'}{v}$$

$$\sin\theta = \frac{v_y}{v} = \frac{y'}{v}$$

$$F_G = mg \text{ and } F = kv^2$$

By substitution, we have

$$\begin{cases} mx'' = -kvx' \\ my'' = -kvy' - mg \end{cases}$$

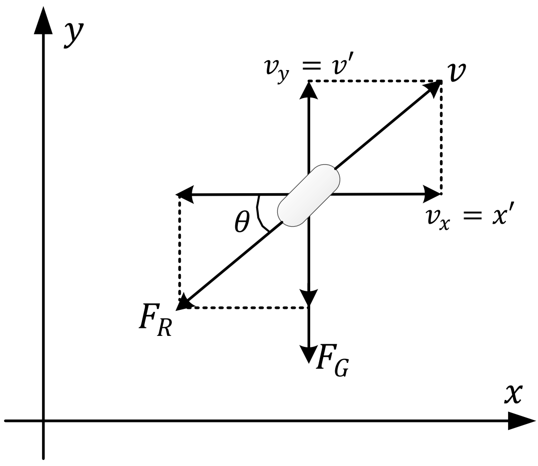

Figure A.8 A projectile model moving with speed v.

Problem 4.2.4

Assuming the displacements of blocks from the nature positions $x_1(t)$ and $x_2(t)$. The DEs of motion for the two blocks are

$$\begin{cases} m_1x_1'' = -k_1x_1 + k_2(x_2 - x_1) + m_1g \\ m_2x_2'' = -k_2(x_2 - x_1) + m_2g \end{cases}$$

By substitution, we get

$$m_1m_2x_2^{(4)} + [(k_1 + k_2)m_2 + k_2m_1]x_2'' - k_1k_2x_2$$
$$= (k_1 + k_2)m_2g$$

This is a constant coefficient inhomogeneous DE that can be solved with C-Eq method.

$$x_2(t) = c_1\cos\omega_1t + c_2\sin\omega_1t$$
$$+c_3\cos\omega_2t + c_4\sin\omega_2t + x_{2P}(t)$$

where ω_1 and ω_2 are two characteristic frequencies. Back substitute can generate the solution for $x_1(t)$.

Problem 4.2.5

Let $x_i(t)$ be the displacement of block-i from its initial position, $i = 1,2,3$.

The DEs for the system are

$$\begin{cases} m_1x''_1 = -k_1x_1 + k_2(x_2 - x_1) + m_1g \\ m_2x''_2 = -k_2(x_2 - x_1) + k_3(x_3 - x_2 - x_1) + m_2g \\ m_3x''_3 = -k_3(x_3 - x_2 - x_1) + m_2g \end{cases}$$

Also, we have the I.C.

$$x_1(0) = x'_1(0) = x_2(0) = x'_2(0) = x_3(0) = x'_3(0) = 0$$

The above system of DEs together with the I.C. describes the motion of three blocks.

Let

$$w_1{}^2 = \frac{k_1}{m_1}, w_2{}^2 = \frac{k_2}{m_2}, w_3{}^2 = \frac{k_3}{m_3}, w_{12}{}^2 = \frac{k_2}{m_1}, w_{23}{}^2 = \frac{k_3}{m_2},$$

We can have

$$\begin{cases} x_1{}'' = -(w_1{}^2 + w_{12}{}^2)x_1 + w_{12}{}^2 + g \\ x_2'' = -(w_2{}^2 + w_{23}{}^2)x_2 - w_{23}{}^2x_1 + w_{23}{}^2x_3 + g \\ x_3'' = -w_3{}^2x_3 + w_3{}^2x_1 + w_3{}^2x_2 + g \\ x_1(0) = x'_1(0) = x_2(0) = x'_2(0) = x_3(0) = x'_3(0) = 0 \end{cases}$$

Problem 4.2.6

On the ball, the force can be decomposed into two components, one along the direction of the spring x and another θ along which we can establish the following two DEs.

$$\begin{cases} mx'' = mg - kx & (1) \\ m[\theta(x + L_0)]'' = -mg\theta & (2) \end{cases}$$

Along the extension of the spring direction, the gravity is

$$mg\cos\theta \approx mg \times 1$$

Because the angle is small so we can make the approximation given above.

Eq-(1) is trivial to solve, one can write the solution as

$$x(t) = \sin(\omega t) + \frac{g}{\omega^2}$$

and $\frac{g}{\omega^2}$ can be absorbed to L_0 where $\omega^2 = \frac{k}{m}$. Inserting such solution to Eq-(2) results in

$$\theta''(x + L_0) + 2\theta' x' + \theta x'' = -g\theta$$

$$\theta''(\sin\omega t + L_0) + 2\theta'\omega\cos\omega t + \theta\omega^2\sin\omega t = -g\theta$$

Now, we combine with I.C. to get

$$\begin{cases} \theta''(\sin\omega t + L_0) + 2\theta'\omega\cos\omega t + \theta\omega^2\sin\omega t = -g\theta \\ \theta(t = 0) = \theta_0, \;\; \theta'(t = 0) = 0 \end{cases}$$

We can now perform Laplace transform on the above equation to solve it or we can use other methods.

4.3 Substitution Method

Problem 4.3.1

Sum up two equations,

$$x'' = \exp(-t)$$

whose solution is

$$x(t) = \exp(-t) + C_1 t + C_2$$

Substitute into second equation

$$\exp(-t) + y'' - \exp(-t) - C_1 t - C_2 = 0$$
$$y'' = C_1 t + C_2$$
$$y = \frac{C_1}{6} t^3 + \frac{1}{2} C_2 t^2 + C_3 t + C_4$$

Problem 4.3.2

From the first equation $x' = -y$, we know that $y = -x'$.
Substituting this into the second DE, we get

$$(-x')' = 10x + 7x' \Rightarrow x'' + 7x' + 10x = 0$$

This is a 2nd-order DE with constant coefficients and the C-Eq is

$$r^2 + 7r + 10 = 0$$
$$r_1 = -5, r_2 = -2$$

Therefore,

$$x = C_1 \exp(-5t) + C_2 \exp(-2t)$$

and thus, we can have

$$\begin{aligned} y &= -x' \\ &= 5C_1 \exp(-5t) + 2C_2 \exp(-2t) \end{aligned}$$

From the given I.C., we know that

$$\begin{cases} C_1 + C_2 = 2 \\ 5C_1 + 2C_2 = -7 \end{cases}$$

Solving this, we can get

$$C_1 = -\frac{11}{3}, \qquad C_2 = \frac{17}{3}$$

Finally, the solutions to the DEs are

$$\begin{cases} x = -\dfrac{11}{3}\exp(-5t) + \dfrac{17}{3}\exp(-2t) \\ y = -\dfrac{55}{3}\exp(-5t) + \dfrac{34}{3}\exp(-2t) \end{cases}$$

Problem 4.3.3

From the first DE

$$y = \frac{x' - x}{2}$$
$$y' = \frac{x'' - x'}{2}$$

Plugging into the second DE, we have

$$\frac{x'' - x'}{2} = 2x - 2\frac{x' - x}{2}$$
$$x'' + x' - 6x = 0$$

This is a 2nd-order DE with constant coefficients, and the C-Eq is

$$r^2 + r - 6 = (r + 3)(r - 2) = 0$$
$$r_1 = -3, \quad r_2 = 2$$
$$x(t) = C_1\exp(-3t) + C_2\exp(2t)$$
$$y(t) = \frac{x' - x}{2} = -2C_1\exp(-3t) + \frac{1}{2}C_2\exp(2t)$$

Insert I.C., we get $C_1 = -3/5$ and $C_2 = 8/5$. Thus the G.S. of the system is

$$\begin{cases} x(t) = \dfrac{8\exp(2t) - 3\exp(-3t)}{5} \\ y(t) = \dfrac{4\exp(2t) + 6\exp(-3t)}{5} \end{cases}$$

Problem 4.3.4

As per the information given in the above problem we have,

$$\begin{cases} m_1x_1'' = -(k_1 + k_2)x_1 + k_2x_2 \\ m_2x_2'' = k_2x_1 - (k_2 + k_3)x_2 \end{cases}$$

Given $m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 4$ and $k_3 = 1$, we have

$$\begin{cases} x_1'' = -5x_1 + 4x_2 \\ x_2'' = 4x_1 - 5x_2 \end{cases}$$

From the second equation, we have

$$x_1 = \frac{x_2''}{4} + \frac{5x_2}{4}$$

Put it back to the first equation, we can get

$$\left(\frac{x_2''}{4} + \frac{5x_2}{4}\right)'' = -5\left(\frac{x_2''}{4} + \frac{5x_2}{4}\right) + 4x_2$$

$$x_2^{(4)} + 10x_2'' + 9x_2 = 0$$

The C-Eq is

$$r^4 + 10r^2 + 9 = 0$$

$$r_{1,2} = \pm 3i, \qquad r_{3,4} = \pm i$$

This gives the G.S. for x_2:

$$x_2 = C_1 \sin 3t + C_2 \cos 3t + C_3 \sin t + C_4 \cos t$$

and put it back for x_1:

$$\begin{aligned} x_1 &= \frac{x_2''}{4} + \frac{5}{4}x_2 \\ &= -C_1 \sin 3t - C_2 \cos 3t + C_3 \sin t + C_4 \cos t \end{aligned}$$

Problem 4.3.5

From

$$x_2' = 2x_1 + x_2 + t^2$$

We can have

$$x_1 = \frac{x_2' - x_2 - t^2}{2}, \qquad x_1' = \frac{x_2'' - x_2' - 2t}{2}$$

Substitute into

$$x_1' = x_1 + 2x_2 + t$$

We can have

$$x_2'' - 2x_2' - 3x_2 = -t^2 + 4t$$

C-Eq:

$$r^2 - 2r - 3 = 0$$

$$r_1 = 3, r_2 = -1$$

$$x_{2c} = c_1 \exp(3t) + c_2 \exp(-t)$$

Then we can have the P.S. like the following form

$$x_{2p} = A_2 t^2 + A_1 t + A_0$$
$$x'_{2p} = 2A_2 t + A_1$$
$$x''_{2p} = 2A_2$$

Substitute into the original equation.

$$2A_2 - 2(2A_2 t + A_1) - 3(A_2 t^2 + A_1 t + A_0) = -t^2 + 4t$$

Then

$$A_0 = \frac{38}{27}, A_1 = -\frac{16}{9}, A_2 = \frac{1}{3}$$

So the G.S. is

$$x_2 = c_1 \exp(3t) + c_2 \exp(-t) + \frac{1}{3}t^2 - \frac{16}{9}t + \frac{38}{27}$$
$$x'_2 = 3c_1 \exp(3t) - c_2 \exp(-t) + \frac{2}{3}t - \frac{16}{9}$$

Substitute into

$$x_1 = \frac{{x_2}' - x_2 - t^2}{2}$$

We can have

$$x_1 = c_1 \exp(3t) - c_2 \exp(-t) - \frac{2}{3}t^2 + \frac{11}{9}t - \frac{43}{27}$$

So the solution is

$$x_1 = c_1 \exp(3t) - c_2 \exp(-t) - \frac{2}{3}t^2 + \frac{11}{9}t - \frac{43}{27}$$
$$x_2 = c_1 \exp(3t) + c_2 \exp(-t) + \frac{1}{3}t^2 - \frac{16}{9}t + \frac{38}{27}$$

Problem 4.3.6

$$x' = y$$
$$x'' = y' = -9x + 6x'$$
$$x'' - 6x' + 9x = 0$$
$$r^2 - 6r + 9 = 0$$
$$r_1 = r_2 = 3$$

Hence

$$x(t) = (A + Bt)\exp(3t)$$
$$y(t) = (3A + B + 3Bt)\exp(3t)$$

Problem 4.3.7

$$\begin{cases} x' = x - 2y & (1) \\ y' = x - y & (2) \end{cases}$$

From Eq-(2)

$$x = y' + y$$
$$x' = y'' + y'$$

Plugging this into Eq-(1)

$$x' = x - 2y$$
$$y'' + y' = y' + y - 2y$$
$$y'' + y = 0$$

The C-Eq is

$$r^2 + 1 = 0$$
$$r_{1,2} = \pm i$$
$$y(t) = C_1 \cos t + C_2 \sin t$$

Put this back, and

$$x(t) = (C_2 - C_1)\sin t + (C_1 + C_2)\cos t$$

Plugging in the I.C., we have

$$\begin{cases} 1 = C_1 + C_2 \\ 2 = C_1 \end{cases}$$
$$\begin{cases} C_1 = 2 \\ C_2 = -1 \end{cases}$$

Thus

$$\begin{cases} x(t) = \cos t - 3\sin t \\ y(t) = 2\cos t - \sin t \end{cases}$$

Problem 4.3.8

$$\begin{cases} x' = 3x + 4y & (1) \\ y' = 3x + 2y & (2) \end{cases}$$

From Eq-(2)

$$x = \frac{y' - 2y}{3}$$
$$x' = \frac{y'' - 2y'}{3}$$

Plugging this into Eq-(1)

$$\frac{y'' - 2y'}{3} = y' - 2y + 4y$$
$$y'' - 5y' - 6y = 0$$

The C-Eq is

$$r^2 - 5r - 6 = 0$$
$$(r+1)(r-6) = 0$$
$$r_1 = -1, \quad r_2 = 6$$
$$y(t) = C_1 \exp(-t) + C_2 \exp(6t)$$
$$y'(t) = -C_1 \exp(-t) + 6C_2 \exp(6t)$$

Put this back, and

$$x(t) = -C_1 \exp(-t) + \frac{4}{3} C_2 \exp(6t)$$

Plugging in the I.C., we have

$$\begin{cases} 1 = -C_1 + \frac{4}{3} C_2 \\ 1 = C_1 + C_2 \end{cases}$$
$$\begin{cases} C_1 = \frac{1}{7} \\ C_2 = \frac{6}{7} \end{cases}$$

Thus

$$\begin{cases} x(t) = -\frac{1}{7} \exp(-t) + \frac{8}{7} \exp(6t) \\ y(t) = \frac{1}{7} \exp(-t) + \frac{6}{7} \exp(6t) \end{cases}$$

Problem 4.3.9

Using Eq-(1)

$$z = x' - x - 2y$$

$$z' = x'' - x' - 2y'$$

Substitute into Eq-(3)

$$x'' - x' - 2y' = x + y + x' - x - 2y$$

$$x'' - 2x' = -y + 2y' \text{ (4)}$$

Using Eq-(2)

$$x = \frac{y' - 2y}{3}$$

$$x' = \frac{y'' - 2y'}{3}$$

$$x'' = \frac{y''' - 2y''}{3}$$

Substitute above DEs to Eq-(4)

$$\frac{y''' - 2y''}{3} - \frac{2(y'' - 2y')}{3} = -y + 2y'$$

$$y''' - 4y'' - 2y' + 3y = 0$$

Find the C-Eq

$$r^3 - 4r^2 - 2r + 3 = 0$$

$$r_1 = -1, \qquad r_2 = \frac{5 + \sqrt{13}}{2}, \qquad r_3 = \frac{5 - \sqrt{13}}{2}$$

$$y(t) = b_1 \exp(-t) + b_2 + b_3$$

$$y'(t) = -b_1 \exp(-t) + \frac{5 + \sqrt{13}}{2} b_2 + \frac{5 - \sqrt{13}}{2} b_3$$

Using Eq-(2)

$$y' - 2y = 3x$$

$$-b_1 + \frac{5 + \sqrt{13}}{2} b_2 + \frac{5 - \sqrt{13}}{2} b_3 - 2b_1 - 2b_2 - 2b_3$$

$$= 3a_1 + 3a_2 + 3a_3$$

$$b_1 = -3a_1$$

$$b_2 = -\frac{6 - 6\sqrt{13}}{12} a_2$$

$$b_3 = -\frac{6 + 6\sqrt{13}}{12} a_3$$

$$x(t) = a_1 \exp(-t) + a_2 + a_3$$

$$x'(t) = -a_1 \exp(-t) + \frac{5+\sqrt{13}}{2}\, a_2 + \frac{5-\sqrt{13}}{2}\, a_3$$

Using Eq-(1)

$$x' - x - 2y = z$$

$$-a_1 + \frac{5+\sqrt{13}}{2}\, a_2 + \frac{5-\sqrt{13}}{2}\, a_3 - a_1 - a_2 - a_3 + 2a_1 + \frac{12-12\sqrt{13}}{12} a_2 + \frac{12+12\sqrt{13}}{12} a_3 = c_1 + c_2 + c_3$$

$$c_1 = 0, \qquad c_2 = \frac{5-\sqrt{13}}{2} a_2, \qquad c_3 = \frac{5+\sqrt{13}}{2} a_3$$

$$x(t) = a_1 \exp(-t) + a_2 + a_3$$

$$y(t) = -3a_1 \exp(-t) - \frac{6-6\sqrt{13}}{12} a_2 - \frac{6+6\sqrt{13}}{12} a_3$$

$$z(t) = \frac{5-\sqrt{13}}{2} a_2 + \frac{5+\sqrt{13}}{2} a_3$$

4.4 Operator Method

Problem 4.4.1

From the problem

$$\begin{cases}(D-1)x-2y-z=0\\ -6x+(D+1)y=0\\ x+2y+(D+1)z=0\end{cases}$$

gives

$$\begin{bmatrix}D-1 & -2 & -1\\ -6 & D+1 & 0\\ 1 & 2 & D+1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The operational determinant is:

$$\begin{vmatrix}D-1 & -2 & -1\\ -6 & D+1 & 0\\ 1 & 2 & D+1\end{vmatrix}=D(D+4)(D-3)$$

So x, y, z all satisfy a third-order homogeneous linear DE with C-Eq

$$r(r+4)(r-3)=0$$

the corresponding G.S. are

$$\begin{cases}x=a_1+a_2\exp(-4t)+a_3\exp(3t)\\ x=b_1+b_2\exp(-4t)+b_3\exp(3t)\\ x=c_1+c_2\exp(-4t)+c_3\exp(3t)\end{cases}$$

If we substitute $x(t)$ and $y(t)$ in the second DE $y'=6x-y$, and collect the coefficients of like terms, we have:

$$\begin{aligned}&-4b_2\exp(-4t)+3b_3\exp(3t)\\ &=6a_1+6a_2\exp(-4t)+6a_3\exp(3t)-b_1-b_2\exp(-4t)\\ &\qquad -b_3\exp(3t)\end{aligned}$$

This gives

$$b_1=6a_1, b_2=-2a_2, b_3=\frac{3}{2}a_3$$

Similarly, we can substitute $x(t)$, $y(t)$ and $z(t)$ in the first or third DE, and compare the coefficients of like terms, we have

$$c_1=-12a_1, c_2=-a_2, c_3=-a_3$$

thus the final G.S. are:

$$\begin{cases} x = a_1 + a_2 \exp(-4t) + a_3 \exp(3t) \\ y = 6a_1 - 2a_2 \exp(-4t) + \dfrac{3}{2} a_3 \exp(3t) \\ z = -12a_1 - a_2 \exp(-4t) - a_3 \exp(3t) \end{cases}$$

Problem 4.4.2

By the following operations

$$(D^2 - D) \times \text{Eq-(1)} - D^2 \times \text{Eq-(2)}$$

we have:

$$(D^2 - D)(D^2 + D)x - D^2(D^2 - 1)x = (D^2 - D)2 \exp(-t)$$

which gives

$$0 = 4 \exp(-t)$$

This is impossible, so the DEs have no solution.

Problem 4.4.3

$$(d - 3)x - 9y = 0$$
$$-2x + (D - 2)y = 0$$
$$\Rightarrow \begin{vmatrix} D-3 & -9 \\ -2 & D-2 \end{vmatrix} x = 0$$
$$\Rightarrow (D^2 - 5D - 12)x = 0$$

The C-Eq is

$$r^2 - 5r - 12 = 0$$
$$r_{1,2} = \frac{5 \pm \sqrt{73}}{2}$$
$$x(t) = A_1 \exp(r_1 t) + B_1 \exp(r_2 t)$$
$$y(t) = A_2 \exp(r_1 t) + B_2 \exp(r_2 t)$$

Apply the I.C. $x(0) = y(0) = 2$ gives

$$A_1 + B_1 = A_2 + B_2 = 2$$

Problem 4.4.4

The original DEs can be written as

$$\begin{bmatrix} D-1 & -2 \\ -2 & D-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t^2 \end{bmatrix}$$

$$\begin{bmatrix} D-1 & -2 & t \\ -2 & D-1 & t^2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & D-1 & t^2 \\ D-1 & -2 & t \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} -2 & D-1 & t^2 \\ 2(D-1) & -4 & 2t \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} -2 & D-1 & t^2 \\ 0 & D^2-2D-3 & -t^2+4t \end{bmatrix}$$

So we can have

$$(D^2-2D-3)x_2 = -t^2+4t$$

C-Eq:

$$r^2-2r-3=0$$
$$r_1=3, r_2=-1$$
$$x_{2c} = c_1 \exp(3t) + c_2 \exp(-t)$$

Then we can have the P.S. like the following form

$$x_{2p} = A_2t^2 + A_1t + A_0$$
$$x'_{2p} = 2A_2t + A_1$$
$$x''_{2p} = 2A_2$$

Substitute into the original equation

$$2A_2 - 2(2A_2t + A_1) - 3(A_2t^2 + A_1t + A_0) = -t^2 + 4t$$

Then

$$A_0 = \frac{38}{27}, A_1 = -\frac{16}{9}, A_2 = \frac{1}{3}$$

So the G.S. is

$$x_2 = c_1 \exp(3t) + c_2 \exp(-t) + \frac{1}{3}t^2 - \frac{16}{9}t + \frac{38}{27}$$

From the first row of the matrix

$$-2x_1 + (D-1)x_2 = t^2$$
$$x_1 = \frac{x'_2 - x_2 - t^2}{2}$$
$$= c_1 \exp(3t) - c_2 \exp(-t) - \frac{2}{3}t^2 + \frac{11}{9}t - \frac{43}{27}$$

So the solution is

$$x_1 = c_1 \exp(3t) - c_2 \exp(-t) - \frac{2}{3}t^2 + \frac{11}{9}t - \frac{43}{27}$$

$$x_2 = c_1 \exp(3t) + c_2 \exp(-t) + \frac{1}{3}t^2 - \frac{16}{9}t + \frac{38}{27}$$

Problem 4.4.5

$$Dx - 1y - 1z = \exp(-t)$$
$$-1x + Dy - 1z = 0$$
$$-1x - 1y + Dz = 0$$

$$\begin{vmatrix} D & -1 & -1 \\ -1 & D & -1 \\ -1 & -1 & D \end{vmatrix} = D(D^2 - 1) + 1(-D - 1) - 1(1 + D)$$
$$= D^3 - D - D - 1 - 1 - D$$
$$= D^3 - 3D - 2$$
$$= (D + 1)^2(D - 2) = 0$$

$$x(t) = a_1 \exp(2t) + a_2 \exp(-t) + a_3 \exp(-t)$$
$$y(t) = b_1 \exp(2t) + b_2 \exp(-t) + b_3 \exp(-t)$$
$$z(t) = c_1 \exp(2t) + c_2 \exp(-t) + c_3 \exp(-t)$$

$$\begin{cases} x = \frac{1}{3}C_1 \exp(2t) - (C_2 + C_3) \exp(-t)\,, \\ y = -\frac{1}{2}\exp(t) + \frac{1}{3}C_1 \exp(2t) + C_2 \exp(-t)\,, \\ z = -\frac{1}{2}\exp(t) + \frac{1}{3}C_1 \exp(2t) + C_3 \exp(-t)\,. \end{cases}$$

Problem 4.4.6

Method I: Substitution

By the first equation, we have $y = x' - x$, and $y' = x'' - x'$. Return it to the second equation, we have $x'' - x' = 6x - (x' - x)$.

$x'' = 7x$. The C-Eq is $r^2 = 7$. $r_1 = \sqrt{7}, r_2 = -\sqrt{7}$. Therefore

$$\begin{cases} x(t) = c_1 \exp(\sqrt{7}t) + c_2 \exp(-\sqrt{7}t) \\ y(t) = (\sqrt{7} - 1)c_1 \exp(\sqrt{7}t) - (\sqrt{7} + 1)c_2 \exp(-\sqrt{7}t) \end{cases}$$

Method II: Operator

The DEs are written as

$$\begin{cases} (D-1)x - y = 0 & (3) \\ -6x + (D+1)y = 0 & (4) \end{cases}$$

By $(D+1)\times$(3)+(4), we have

$$(D+1)(D-1)x - 6x = 0$$
$$(D^2 - 7)x = 0$$

The C-Eq is

$$r^2 - 7 = 0$$
$$r_1 = \sqrt{7}, r_2 = -\sqrt{7}$$

Therefore

$$\begin{cases} x(t) = c_1 \exp\left(\sqrt{7}t\right) + c_2 \exp\left(-\sqrt{7}t\right) \\ y(t) = (\sqrt{7} - 1)c_1 \exp\left(\sqrt{7}t\right) - (\sqrt{7} + 1)c_2 \exp\left(-\sqrt{7}t\right) \end{cases}$$

Problem 4.4.7

We add and subtract the two DEs respectively, and we have

$$D(x+y) = \frac{1}{2}(\exp(-2t) + \exp(3t)) \qquad (1)$$
$$2x - 3y = \frac{1}{2}(\exp(-2t) - \exp(3t)) \qquad (2)$$

From (2) we have $x = \frac{1}{4}(\exp(-2t) - \exp(3t)) + \frac{3}{2}y$.

We return x to (1) and

$$\left(\frac{1}{4}(\exp(-2t) - \exp(3t)) + \frac{5}{2}y\right)' = \frac{1}{2}(\exp(-2t) - \exp(3t)).$$

Then

$$y' = \frac{2}{5}\exp(-2t) + \frac{1}{2}\exp(3t)$$

and

$$y(t) = -\frac{1}{5}\exp(-2t) + \frac{1}{6}\exp(3t) + C_1.$$

Therefore $x(t) = -\frac{1}{20}\exp(-2t) + \frac{3}{2}C_{1.}$

The solution is

$$\begin{cases} x(t) = -\dfrac{1}{20}\exp(-2t) + \dfrac{3}{2}C_1 \\ y(t) = -\dfrac{1}{5}\exp(-2t) + \dfrac{1}{6}\exp(3t) + C_1 \end{cases}$$

Problem 4.4.8

The DE can be rewritten into operator form

$$\begin{cases} (D^2-4)x + 13Dy = 6\sin t & (1) \\ 2Dx + (-D^2+9)y = 0 & (2) \end{cases}$$

From the equation

$$\text{Eq-(1)} \times 2D - \text{Eq-(2)} \times (D^2-4)$$

we get

$$D^4y + 13D^2y + 36y = 12\cos t$$

C-Eq:

$$r^4 + 13r^2 + 36 = 0$$

$$r_{1,2} = \pm 2i, \qquad r_{3,4} = \pm 3i$$

$$y_c = C_1\sin 2t + C_2\cos 2t + C_3\sin 3t + C_4\cos 3t$$

Consider the DE has only even number of derivatives, we can assume

$$y_p = A\cos t$$

$$y_p'' = -A\cos t\,, \qquad y_p^{(4)} = A\cos t$$

This gives

$$A = \frac{1}{2}$$

So

$$y(t) = C_1\sin 2t + C_2\cos 2t + C_3\sin 3t + C_4\cos 3t + \frac{1}{2}\cos t$$

Put this back into (2) and we can get

$$\begin{aligned} x' &= \frac{1}{2}(y'' - 9y) \\ &= \frac{1}{2}(-13C_1\sin 2\mathrm{t} - 13C_2\cos 2\mathrm{t} - 18C_3\sin 3\mathrm{t} \\ &\qquad - 18C_4\cos 3\mathrm{t} - 5\cos t) \end{aligned}$$

$$x = \frac{1}{2}\left(\frac{13}{2}C_1 \cos 2\mathrm{t} - \frac{13}{2}C_2 \sin 2\mathrm{t} + 6C_3 \cos 3\mathrm{t} - 6C_4 \sin 3\mathrm{t} + 5 \sin t\right)$$

Problem 4.4.9

Method I: Substitution

From the second equation

$$x = \frac{y' - y}{2}$$
$$x' = \frac{y'' - y'}{2}$$

Substitute these into the first equation

$$\frac{y'' - y'}{2} = 3\frac{y' - y}{2} - 2y$$
$$y'' - 4y' + 7y = 0$$

C-Eq:

$$r^2 - 4r + 7 = 0$$
$$r_1 = 2 + \sqrt{3}i, \qquad r_2 = 2 - \sqrt{3}i$$

So

$$y(t) = c_1 \exp(2t)\cos(\sqrt{3}t) + c_2 \exp(2t)\sin(\sqrt{3}t)$$
$$y'(t) = \left(2c_1 + \sqrt{3}c_2\right) \exp(2t)\cos\left(\sqrt{3}t\right) + \left(2c_2 - \sqrt{3}c_1\right)\exp(2t)\sin\left(\sqrt{3}t\right)$$
$$x(t) = \frac{y' - y}{2}$$
$$= \frac{1}{2}\left(c_1 + \sqrt{3}c_2\right) \exp(2t)\cos(\sqrt{3}t) + \frac{1}{2}\left(c_2 - \sqrt{3}c_1\right)\exp(2t)\sin(\sqrt{3}t)$$

So

$$y(t) = c_1 \exp(2t)\cos(\sqrt{3}t) + c_2 \exp(2t)\sin(\sqrt{3}t)$$

Method II: Operator

The original DEs can be written as

$$\begin{bmatrix} D-3 & 2 \\ -2 & D-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} D-3 & 2 \\ -2 & D-1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & D-1 \\ D-3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & D-1 \\ 2D-6 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & D-1 \\ 0 & 4+(D-1)(D-3) \end{bmatrix}$$

So we can have

$$\big(4+(D-1)(D-3)\big)y(t) = (D^2-4D+7)y(t) = 0$$

C-Eq:

$$r^2 - 4r + 7 = 0$$

$$r_1 = 2+\sqrt{3}i, \qquad r_2 = 2-\sqrt{3}i$$

So

$$y(t) = c_1 \exp(2t)\cos(\sqrt{3}t) + c_2 \exp(2t)\sin(\sqrt{3}t)$$

$$y'(t) = \left(2c_1 + \sqrt{3}c_2\right) \exp(2t)\cos\left(\sqrt{3}t\right) + \left(2c_2 - \sqrt{3}c_1\right)\exp(2t)\sin\left(\sqrt{3}t\right)$$

From the first row of the matrix, we can have

$$-2x + y' - y = 0$$

$$x(t) = \frac{y'-y}{2} = \frac{1}{2}\cos(\sqrt{3}t) + \frac{1}{2}\sin(\sqrt{3}t)$$

$$y(t) = c_1 \exp(2t)\cos(\sqrt{3}t) + c_2 \exp(2t)\sin(\sqrt{3}t)$$

4.5 Eigen-Analysis Method

Problem 4.5.1

$$\det\begin{bmatrix}1-\lambda & -3\\ 3 & 7-\lambda\end{bmatrix} = (1-\lambda)(7-\lambda)+9$$
$$= (\lambda-4)^2 = 0$$

The system has two identical eigenvalues

$$\lambda_1 = \lambda_2 = 4$$

We can find the associated eigenvector V_1 for the eigenvalue λ_1 and its solution is

$$X_1 = V_1 \exp(\lambda t)$$

For the second solution, we need to compose the L.I. solution by introducing

$$X_2 = (V_1 t + V_2)\exp(\lambda t)$$

We plug this proposed solution to the system, we get

$$X_2' = (V_1)\exp(\lambda t) + (V_1 t + V_2)\lambda\exp(\lambda t)$$

so

$$X_2' = (V_1)\exp(\lambda t) + (V_1 t + V_2)\lambda\exp(\lambda t)$$
$$= A(V_1 t + V_2)\exp(\lambda t)$$

or

$$(A-\lambda)V_2\exp(\lambda t) = V_1\exp(\lambda t)$$

or

$$(A-\lambda)V_2 = V_1$$

Additionally, since $(A-\lambda)V_1 = 0$, we have

$$(A-\lambda)(A-\lambda)V_2 = (A-\lambda)V_1 = 0$$

or

$$(A-\lambda)^2 V_2 = 0$$

Now, we can find V_2 with which we can compose the second solution and then the entire solution.

Now, we return to our original example,

$$(A-\lambda)^2 V_2 = (A-4)^2 V_2$$
$$= \begin{bmatrix}-3 & -3\\ 3 & 3\end{bmatrix}\begin{bmatrix}-3 & -3\\ 3 & 3\end{bmatrix}V_2$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} V_2 = 0$$

which means any solution V_2 will satisfy the above equation. For convenience, we may select

$$V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with

$$V_1 = (A - \lambda)V_2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

which we construct the eigenvector V_1 and solve

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} V_1 = 0$$

to find the eigenvector V_1.

Finally, we have the two L.I. solutions

$$X_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \exp(4t)$$

$$X_2 = \left(\begin{bmatrix} -3 \\ 3 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \exp(4t) = \begin{bmatrix} -3t \\ 3t + 1 \end{bmatrix} \exp(4t)$$

So, finally, the G.S. for the original DE is

$$X = c_1 X_1 + c_2 X_2 = c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} \exp(4t) + c_2 \begin{bmatrix} -3t \\ 3t + 1 \end{bmatrix} \exp(4t)$$

Problem 4.5.2

$$X'(t) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 3\exp(t) \\ -t^2 \end{bmatrix}$$

First, we try to solve the homogeneous DEs

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix} = (1 - \lambda)^2 - 4 = 0$$

So the eigenvalues are $\lambda_1 = 3,\ \lambda_2 = -1$

For $\lambda_1 = 3$, we get

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can select

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_1 = -1$, we get

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can select

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So the G.S. to homogeneous DEs is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(3t) + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp(-t)$$

The P.S. may take the following form

$$X_P = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \cdot 3\exp(t) - \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \cdot t^2 + \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = 3 \begin{bmatrix} 2A_2 \\ 2B_2 \end{bmatrix} \exp(t) - 2 \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t$$

Plugging them to the original system, we get

$$3 \begin{bmatrix} 2A_2 \\ 2B_2 \end{bmatrix} \exp(t) - 2 \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t$$
$$= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(3 \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} \exp(t) - \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t^2 + \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \right) + \begin{bmatrix} 3\exp(t) \\ -t^2 \end{bmatrix}$$

Solve for $A_0, A_1, A_2, B_0, B_1, B_2$ and plug them into $X(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(3t) + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp(-t)$$
$$+ \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} 3\exp(t) - \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} t^2 + \begin{bmatrix} A_0 \\ B_0 \end{bmatrix}$$

Problem 4.5.3

Eigen-Analysis method for the homogeneous portion:

$$\frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4 = 0$$

whose roots are

$$\lambda_1 = 2 \quad \lambda_2 = -2$$

Find eigenvectors. For $\lambda_1 = 2$

$$\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}\right] \Rightarrow -x + 3y = 0$$
$$V_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

For $\lambda_2 = -2$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 1 & 1 & 0 \end{array}\right] \Rightarrow 3x + 3y = 0$$
$$V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So

$$X_c = C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \exp(2t) + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp(-2t)$$

Let P.S. be

$$X_p = B\exp(-2t) + Et\exp(-2t)$$
$$X_p' = -2B\exp(-2t) + E\exp(-2t) - 2Et\exp(-2t)$$

Plugging into the original system, we have

$$-2B\exp(-2t) + E\exp(-2t) - 2Et\exp(-2t)$$
$$= A(B\exp(-2t) + t\exp(-2t)) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \exp(-2t)$$

Grouping like terms, we get

$$\begin{cases} -2B + E = AB + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ -2E = AE \end{cases}$$

Solve the 2nd equation

$$E = \begin{bmatrix} e \\ -e \end{bmatrix}$$

where e is a constant.

Then the first equation becomes

$$-2B + \begin{bmatrix} \mathrm{c} \\ -\mathrm{c} \end{bmatrix} = AB + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which allows us to construct the following two equations for B vector

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$\begin{cases} b_1 + 3b_2 = -2b_1 + e - 1 \\ -4b_2 = -2b_2 + 2b_1 - 2e + 1 \end{cases}$$

or

$$\begin{cases} b_1 + b_2 = e/3 - 1/3 \\ b_1 + b_2 = e - 1/2 \end{cases}$$

Thus

$$\frac{e}{3} - \frac{1}{3} = e - \frac{1}{2}$$

or

$$e = \frac{1}{4}$$

Thus

$$E = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Also

$$b_1 + b_2 = -\frac{1}{4}$$

We may choose

$$B = \frac{1}{4}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Then

$$X_p = \frac{1}{4}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\exp(-2t) + \frac{1}{4}\begin{bmatrix} 1 \\ -1 \end{bmatrix} t\exp(-2t)$$

And the G.S. is

$$X = X_c + X_p = C_1\begin{bmatrix} 3 \\ 1 \end{bmatrix}\exp(2t) + C_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}\exp(-2t) + \frac{1}{4}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\exp(-2t) + \frac{1}{4}\begin{bmatrix} 1 \\ -1 \end{bmatrix} t\exp(-2t)$$

Problem 4.5.4

First find solution to corresponding homogeneous system:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$

To find eigenvalues set $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - 8 = 0$$

$$\lambda_1 = 3 \quad \lambda_2 = -3$$

Find eigenvector for $\lambda_1 = 3$

$$\begin{bmatrix} 1-3 & 2 \\ 4 & -1-3 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2u_1 + 2v_1 = 0$$

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = -3$

$$\begin{bmatrix} 1-(-3) & 2 \\ 4 & -1-(-3) \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4u_2 + 2v_2 = 0$$

$$V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So the complementary solution is:

$$X_c(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(3t) + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \exp(-3t)$$

The P.S. has form:

$$X_p = Bt + D = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$X_p' = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Plugging back into original system, we get:

$$\begin{aligned} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \left(\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \\ &= \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \\ &= \begin{bmatrix} b_1 + 2b_2 \\ 4b_1 - b_2 \end{bmatrix} t + \begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} d_1 + 2d_2 \\ 4d_1 - d_2 \end{bmatrix} \\ &= \begin{bmatrix} b_1 + 2b_2 - 1 \\ 4b_1 - b_2 + 1 \end{bmatrix} t + \begin{bmatrix} d_1 + 2d_2 \\ 4d_1 - d_2 \end{bmatrix} \end{aligned}$$

We can solve first for *B*:

$$\begin{bmatrix} b_1 + 2b_2 - 1 \\ 4b_1 - b_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e.,

$$\begin{cases} b_1 + 2b_2 = 1 \\ 4b_1 - b_2 = -1 \end{cases}$$

Adding twice the second equation to the first equation yields:

$$9b_1 = -1$$

$$b_1 = -\frac{1}{9}$$

Then the first equation gives

$$b_2 = \frac{5}{9}$$

From our above equation we have:

$$\begin{bmatrix} d_1 + 2d_2 \\ 4d_1 - d_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

i.e.,

$$\begin{cases} d_1 + 2d_2 = -\dfrac{1}{9} \\ 4d_1 - d_2 = \dfrac{5}{9} \end{cases}$$

Solving this system finds:

$$d_1 = \frac{1}{9} \qquad d_2 = -\frac{1}{9}$$

So the P.S. is:

$$X_p = \frac{1}{9}\begin{bmatrix} -1 \\ 5 \end{bmatrix} t + \frac{1}{9}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

And the G.S. is:

$$X = X_c + X_p$$

$$= C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(3t) + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \exp(-3t) + \frac{1}{9}\begin{bmatrix} -1 \\ 5 \end{bmatrix} t + \frac{1}{9}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Problem 4.5.5

This system may be written as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -4 & 4 & -2 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Let us find the eigenvalues and eigenvectors of the coefficient matrix.

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

$$\begin{vmatrix} 4-\lambda & -2 & 0 \\ -4 & 4-\lambda & -2 \\ 0 & -4 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)((4-\lambda)^2 - 8) + 2\big(-4(4-\lambda)\big) = 0$$

$$\lambda(\lambda-4)(\lambda-8)=0$$
$$\lambda_1=0,\lambda_2=4,\lambda_3=8$$

The corresponding eigenvectors can be found from the equation $Av_i=\lambda_i v_i$, where v_i is an eigenvector corresponding to λ_i. Hence,

$$v_1=\begin{bmatrix}1\\2\\2\end{bmatrix},\qquad v_2=\begin{bmatrix}1\\0\\-2\end{bmatrix},\qquad v_3=\begin{bmatrix}1\\-2\\2\end{bmatrix}$$

The solution of the system is given by

$$X=c_1v_1\exp(\lambda_1 t)+c_2v_2\exp(\lambda_2 t)+c_3v_3\exp(\lambda_3 t)$$

where c_1,c_2,c_3 are arbitrary constants and $X=\begin{bmatrix}x\\y\\z\end{bmatrix}$.

Hence

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=c_1\begin{bmatrix}1\\2\\2\end{bmatrix}\exp(0t)+c_2\begin{bmatrix}1\\0\\-2\end{bmatrix}\exp(4t)+c_3\begin{bmatrix}1\\-2\\2\end{bmatrix}\exp(8t)$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}c_1+c_2\exp(4t)+c_3\exp(8t)\\2c_1-2c_3\exp(8t)\\2c_1-2c_2\exp(4t)+2c_3\exp(8t)\end{bmatrix}$$

Chapter 5 Laplace Transforms

5.2 Properties of Laplace Transforms

Problem 5.2.1

From the figure we write the square wave function as

$$g(t) = \begin{cases} 1, & 2N \le t < 2N+1 \\ -1, & 2N+1 < t \le 2N+2 \end{cases}, \qquad N = 0,1,2,\ldots$$

We can write the above function as

$$\begin{aligned} g(t) &= 2u(t) - u(t) - 2u(t-1) + 2u(t-2) - 2u(t-3) \\ &\quad + \cdots \\ &= 2\sum_{n=0}^{\infty}(-1)^n u(t-n) - u(t) \\ &= 2f(t) - u(t) \end{aligned}$$

Taking Laplace on both sides we get

$$\begin{aligned} \text{LHS} &= \mathcal{L}\{g(t)\} \\ &= \mathcal{L}\{2f(t) - u(t)\} \\ &= 2\sum_{n=0}^{\infty}(-1)^n \mathcal{L}\{u(t-n)\} - \mathcal{L}\{u(t)\} \\ &= 2\sum_{n=0}^{\infty}(-1)^n \frac{\exp(-ns)}{s} - \frac{1}{s} \\ &= \frac{2}{s}\sum_{n=0}^{\infty}(-1)^n \exp(-ns) - \frac{1}{s} \\ &= \frac{2}{s}\left(\frac{1}{1+\exp(-s)}\right) - \frac{1}{s} \\ &= \frac{1}{s}\left(\frac{1-\exp(-s)}{1+\exp(-s)}\right) \\ &= \frac{1}{s}\tanh\left(\frac{s}{2}\right) = \text{RHS} \end{aligned}$$

Problem 5.2.2

$$
\begin{aligned}
F(s) &= \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 \exp(\alpha t)\sin(\omega t)\} = sF(s) - f(0) \\
&= s(-1)^2 F(s)'' - f(0) \\
&= s\left(\frac{\omega}{(s-\alpha)^2+\omega^2}\right)'' - 0 \\
&= s\left(-\frac{2(s-\alpha)\omega}{((s-\alpha)^2+\omega^2)^2}\right)' \\
&= s\,\frac{(-2\omega)(\,(s-\alpha)^2+\omega^2)+8(s-\alpha)^2\omega}{((s-\alpha)^2+\omega^2)^3} \\
&= s\left(\frac{6\omega(s-\alpha)^2-2\omega^3}{((s-\alpha)^2\ +\omega^2)^3}\right) \\
&= \frac{6\omega s(s-\alpha)^2-2s\omega^3}{((s-\alpha)^2\ +\omega^2)^3}
\end{aligned}
$$

Problem 5.2.3

$$
\begin{aligned}
&\mathcal{L}\left\{\exp(t)\int_0^{\sqrt{t}} \exp(-\tau^2)\,d\tau\right\} \\
&= \mathcal{L}\left\{\frac{\sqrt{\pi}}{2}\,\frac{2\exp(t)}{\sqrt{\pi}}\int_0^{\sqrt{t}} \exp(-\tau^2)\,d\tau\right\} \\
&= \mathcal{L}\left\{\frac{\exp(t)}{\sqrt{\pi}}\int_0^{\sqrt{t}} \frac{1}{\tau}\exp(-\tau^2)\,2\tau d\tau\right\} \\
&= \mathcal{L}\left\{\int_0^{t} \frac{1}{\sqrt{\pi x}}\exp(t-x)\,dx\right\} \\
&= \frac{\sqrt{\pi}}{2}\mathcal{L}\left\{\exp(t)\,\frac{1}{\sqrt{\pi x}}\right\} \\
&= \mathcal{L}\{\exp(t)\}\,\mathcal{L}\left\{\frac{1}{\sqrt{\pi x}}\right\} \\
&= \frac{1}{(s-1)\sqrt{s}}
\end{aligned}
$$

$$\mathcal{L}\{f(t)\} = \frac{2A}{s^2T}\tanh\frac{sT}{4}$$

Problem 5.2.4

Apply Laplace transform to the function, and use the property of Laplace transform to split it into two parts.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\sin(\omega_1 t)\} + \mathcal{L}\{\exp(\alpha t)\cos(\omega_2 t)\}$$

For the first term $\mathcal{L}\{t^2\sin(\omega_1 t)\}$, we use the property of t-multiplication

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

In this case, $n = 2$. $f(t) = \sin(\omega_1 t)$. Then

$$F(s) = \frac{\omega_1}{s^2 + \omega_1^2}$$

$$\mathcal{L}\{t^2\sin(\omega_1 t)\} = (-1)^2 F^{(n)}(s) = \frac{6\omega_1 s^2 - 2\omega_1^3}{(s^2 + \omega_1^2)^3}$$

For the second term $\mathcal{L}\{\exp(\alpha t)\cos(\omega_2 t)\}$, we use the property of translator

$$\mathcal{L}\{\exp(\alpha t)\cos(\omega_2 t)\} = F(s - \alpha)$$

In this case, $f(t) = \cos(\omega_2 t)$. Then

$$F(s) = \frac{s}{s^2 + \omega_2^2}$$

$$\mathcal{L}\{\exp(\alpha t)\cos(\omega_2 t)\} = \frac{s - \alpha}{(s - \alpha)^2 + \omega_2^2}$$

Finally, after applying Laplace transform to the function $f(t)$, we will have

$$\mathcal{L}\{f(t)\} = \frac{6\omega_1 s^2 - 2\omega_1^3}{(s^2 + \omega_1^2)^3} + \frac{s - \alpha}{(s - \alpha)^2 + \omega_2^2}$$

Problem 5.2.5

1)

$$f_1(t) = \sum_{n=0}^{\infty} u(t - n)$$

By taking the Laplace Transform, we get

$$\begin{aligned}\mathcal{L}\{f_1(t)\} &= \mathcal{L}\left\{\sum_{n=0}^{\infty} u(t-n)\right\} \\ &= \sum_{n=0}^{\infty} \mathcal{L}\{u(t-n)\} \\ &= \sum_{n=0}^{\infty} \frac{\exp(-ns)}{s} \\ &= \frac{1}{\mathrm{s}} \sum_{n=0}^{\infty} \exp(-ns) \\ &= \frac{1}{s(1-\exp(-s))}\end{aligned}$$

So the solution is

$$\mathcal{L}\{f_1(t)\} = \frac{1}{s(1-\exp(-s))}$$

2)

$$f_2(t) = t - \lfloor t \rfloor$$

In this case, the floor function can be expressed as

$$\lfloor t \rfloor = \sum_{n=1}^{\infty} u(t-n)$$

So, if we perform the same process of (1), then

$$\begin{aligned}\mathcal{L}\{\lfloor t \rfloor\} &= \mathcal{L}\left\{\sum_{n=1}^{\infty} u(t-n)\right\} \\ &= \sum_{n=1}^{\infty} \mathcal{L}\{u(t-n)\} \\ &= \sum_{n=1}^{\infty} \frac{\exp(-ns)}{s}\end{aligned}$$

$$= \frac{1}{s}\sum_{n=1}^{\infty}\exp(-ns)$$
$$= \frac{\exp(-s)}{s(1-\exp(-s))}$$

So, we obtain the following solution

$$\mathcal{L}\{f_2(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{\lfloor t\rfloor\} = \frac{1}{s^2} - \frac{\exp(-s)}{s(1-\exp(-s))}$$

Problem 5.2.6

(1) $F(s) = \mathcal{L}\{t^5\} = \int_0^\infty t^5 \exp(-st)\, dt$

Let $u = st$, then $t = u/s$ and $dt = du/s$.

$$F(s) = \int_0^\infty \left(\frac{u}{s}\right)^5 \exp(-u)\frac{du}{x}$$
$$= \frac{1}{s^6}\int_0^\infty u^5 \exp(-u)\, du$$
$$= \frac{1}{s^6}\int_0^\infty u^{6-1} \exp(-u)\, du$$
$$= \frac{5!}{s^6}$$

Note

Gamma function is defined as

$$\Gamma(x) = \int_0^\infty \exp(-t)\, t^{x-1} dt$$

and for an integer n, we have

$$\Gamma(n+1) = n\Gamma(n) = n!$$

(2) $$F(s) = \mathcal{L}\{\exp(at)\}$$
$$= \int_0^\infty \exp(at)\exp(-st)\, dt$$
$$= \int_0^\infty \exp\big((a-s)t\big)\, dt$$

(3) When $s > a$,

$$\left.\frac{\exp\big((a-s)t\big)}{a-s}\right|_0^\infty = \frac{1}{s-a}$$

(4) $$F(s) = \mathcal{L}\{\sin(\omega t)\}$$
$$= \int_0^\infty \exp(at)\exp(-st)\,dt$$
$$= -\frac{1}{s}\int_0^\infty \sin(\omega t)\,d(\exp(-st))$$

By product rule,

$$F(s) = \left.-\frac{\exp(-st)}{s}\sin(\omega t)\right|_0^\infty + \frac{1}{s}\int_0^\infty \exp(-st)\,d(\sin(\omega t))$$
$$= 0 - 0 + \frac{\omega}{s}\int_0^\infty \cos(\omega t)\exp(-st)\,dt$$
$$= -\frac{\omega}{s^2}\int_0^\infty \cos(\omega t)\,d(\exp(-st))$$
$$= \left.-\frac{\omega}{s^2}\cos(\omega t)\exp(-st)\right|_0^\infty$$
$$+ \frac{\omega}{s^2}\int_0^\infty \exp(-st)\,d(\cos(\omega t))$$
$$= \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\int_0^\infty \exp(-t)\sin(\omega t)\,dt$$
$$= \frac{\omega}{s^2} - \frac{\omega^2}{s^2}F(s)$$
$$\left(1 + \frac{\omega^2}{s^2}\right)F(s) = \frac{\omega}{s^2}$$
$$F(s) = \frac{\omega}{s^2+\omega^2}$$
$$F(s) = \mathcal{L}\{\exp(at)\sin(\omega t)\}$$
$$= \int_0^\infty \exp(at)\sin(\omega t)\exp(-st)\,dt$$
$$= \int_0^\infty \exp(-(s-a)t)\sin(\omega t)\,dt$$

Substitute $(s-a)$ with s_1, we can change the integral to the same form as part (3). Thus,

$$F(s) = \frac{\omega}{s_1^2 + \omega^2} = \frac{\omega}{(s-a)^2 + \omega^2}$$

Problem 5.2.7

This is a periodic function (of period T) whose Laplace transform can be expressed as

$$\mathcal{L}\{f(t)\} = \frac{1}{1-\exp(-sT)} \int_0^T f(t)\exp(-st)\,dt$$

where the integral portion can be carried out as

$$\begin{aligned}
&\int_0^T f(t)\exp(-st)\,dt \\
&= \int_0^{\frac{T}{2}} 2A\left(\frac{t}{T}\right)\exp(-st)\,dt + \int_{\frac{T}{2}}^{T} 2A\left(1-\frac{t}{T}\right)\exp(-st)\,dt \\
&= \left(-\frac{A}{s}\exp\left(-\frac{sT}{2}\right) - \frac{2A}{s^2T}\exp\left(-\frac{sT}{2}\right) + \frac{2A}{s^2T}\right) \\
&\qquad + \left(\frac{2A}{s^2T}\exp(-sT) + \frac{A}{s}\exp\left(-\frac{sT}{2}\right) - \frac{2A}{s^2T}\exp\left(-\frac{sT}{2}\right)\right) \\
&= \frac{2A}{s^2T}\left(-2\exp\left(-\frac{sT}{2}\right) + 1 + \exp(-sT)\right) \\
&= \frac{2A}{s^2T}\left(1-\exp\left(-\frac{sT}{2}\right)\right)^2
\end{aligned}$$

Therefore, the transform is

$$\mathcal{L}\{f(t)\} = \frac{1}{1-\exp(-sT)}\frac{2A}{s^2T}\left(1-\exp\left(-\frac{sT}{2}\right)\right)^2$$

$$= \frac{2A}{s^2T}\frac{1 - \exp\left(-\frac{sT}{2}\right)}{1 + \exp\left(-\frac{sT}{2}\right)}$$

$$= \frac{2A}{s^2T}\left(\frac{\exp\left(\frac{sT}{4}\right) - \exp\left(-\frac{sT}{4}\right)}{\exp\left(\frac{sT}{4}\right) + \exp\left(-\frac{sT}{4}\right)}\right)$$

$$= \frac{2A}{s^2T}\frac{\sinh\frac{sT}{4}}{\cosh\frac{sT}{4}} = \frac{2A}{s^2T}\tanh\frac{sT}{4}$$

5.3 Inverse Laplace Transforms

Problem 5.3.1

Taking Laplace on the RHS of the given problem, we get

$$\begin{aligned}
&\mathcal{L}\left\{\frac{1}{2k^3}(\sin kt - kt\cos kt)\right\} \\
&= \frac{1}{2k^3}(\mathcal{L}\{\sin kt\} - k\mathcal{L}\{t\cos kt\}) \\
&= \frac{1}{2k^3}\left(\frac{k}{s^2+k^2} - \frac{k(s^2-k^2)}{(s^2+k^2)^2}\right) \\
&= \frac{1}{2k^3}\left(\frac{2k^3}{(s^2+k^2)^2}\right) \\
&= \frac{1}{(s^2+k^2)^2} \\
&= \mathcal{L}\{\text{LHS}\}
\end{aligned}$$

Hence the above is proven.

Problem 5.3.2

We have

$$\begin{aligned}
\mathcal{L}\{J_0(2\sqrt{t})\} &= \mathcal{L}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n}{2\cdot n!}\left(\sqrt{t}\right)^{2n}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2\cdot n!}\mathcal{L}\{t^n\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\cdot s^{n+1}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{s}\right)^n}{n!} \\
&= \frac{\exp\left(-\frac{1}{s}\right)}{s}
\end{aligned}$$

which finishes the proof.

Problem 5.3.3

Let us suppose that the solution has the form of $x(t) = tf(t)$. Then we have a Laplace transformation for this function.

$$\mathcal{L}\{x(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\} = \frac{2s}{(s^2-1)^2}$$

So, we have

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$

By taking the inverse Laplace transform, we get

$$f(t) = \frac{1}{2}(\exp(t) - \exp(-t))$$

Consequently, the solution is

$$x(t) = tf(t) = \frac{1}{2}t[\exp(t) - \exp(-t)]$$

Problem 5.3.4

Let $p = a$ and $f(t) = \frac{t}{a}$,

$$F(s) = \frac{1}{1-\exp(-st)}\int_0^p \exp(-st)\, f(t)dt$$

Let $u = -st$

$$\int u\exp(u) = (u-1)\exp(u)$$

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{1}{a(1-\exp(-as))}\int_0^{-as} \exp(-st)\, tdt \\
&= \frac{1}{a(1-\exp(-as))}\int_0^{-as} \exp(u)\left(-\frac{u}{s}\right)\left(-\frac{du}{s}\right) \\
&= \frac{1}{as^2(1-\exp(-as))}\int_0^{-as} \exp(u)\, udu \\
&= \frac{1}{as^2(1-\exp(-as))}\big((u-1)\exp(u)\big)\Big|_0^{-as}
\end{aligned}$$

$$= \frac{1}{as^2(1-\exp(-as))}\big((-as-1)\exp(-as)+1\big)$$
$$= \frac{1}{as^2} - \frac{\exp(-as)}{s(1-\exp(-as))}$$

Since

$$\mathcal{L}\left\{-\frac{t}{a} - g(t)\right\} = \frac{1}{as^2} - G(s)$$
$$\mathcal{L}^{-1}\left\{\frac{\exp(-as)}{s(1-\exp(-as))}\right\} = at - \frac{t}{a}$$

Problem 5.3.5

Performing Laplace Transform on both sides of the equation, we get

$$\mathcal{L}\left\{\int_0^t \cos(t-\tau)\,x(\tau)d\tau\right\}$$
$$= \mathcal{L}\{\sin(t)\} + 2\mathcal{L}\left\{\int_0^t \cos(t-\tau)\,x(\tau)d\tau\right\}$$
$$X = \frac{1}{s^2+1} + \frac{2s}{s^2+1}X$$
$$X(s) = \frac{1}{(s-1)^2}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\} = t\exp(t)$$

Problem 5.3.6

Take Laplace on each side of the equation:

$$\mathcal{L}\{x\} = \mathcal{L}\{2\exp(3t)\} - \mathcal{L}\left\{\int_0^t \exp\big(2(t-\tau)\big)\,x(\tau)d\tau\right\}$$

Let $\mathcal{L}\{x\} = X(s)$. By the convolution of two functions, we have

$$\mathcal{L}\left\{\int_0^t \exp\big(2(t-\tau)\big)\,x(\tau)d\tau\right\} = \mathcal{L}\{\exp(2t)\}\mathcal{L}\{x(t)\} = \frac{X(s)}{s-2}$$

Substitution of the above equations, we have

$$X(s) = \frac{2}{s-3} - \frac{X(s)}{s-2}$$

This gives

$$X(s) = \frac{1}{s-1} + \frac{1}{s-3}$$

Thus

$$x(t) = \exp(t) + \exp(3t)$$

5.4 The Convolution of Two Functions

Problem 5.4.1

$$\mathcal{L}^{-1}\left\{\frac{1}{s-1\sqrt{s}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\frac{1}{\sqrt{s}}\right\}$$

Let

$$F(s) = \frac{1}{s-1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = \exp(t)$$

$$G(s) = \frac{1}{\sqrt{s}} = s^{-\frac{1}{2}}$$

$$g(t) = \mathcal{L}^{-1}\left\{s^{-\frac{1}{2}}\right\} = \frac{t^{-\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}$$

$$\xrightarrow{\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}} g(t) = \frac{1}{\sqrt{\pi t}}$$

By convolution

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= \int_0^t exp(t-\tau)\cdot\frac{1}{\sqrt{\pi\tau}}d\tau$$

$$= \frac{exp(t)}{\sqrt{\pi}}\int_0^t \frac{exp(-\tau)}{\sqrt{\tau}}d\tau$$

Let $u = \sqrt{\tau} \Longrightarrow \tau = u^2 \Longrightarrow d\tau = 2udu$,

$$\frac{exp(t)}{\sqrt{\tau}}\int_0^t \frac{exp(-\tau)}{\sqrt{\tau}}d\tau = \frac{exp(t)}{\sqrt{\tau}}\int_0^{\sqrt{\tau}} \frac{exp(-u^2)}{u}2u\,du$$

$$= exp(t)\frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\tau}} exp(-u^2)\,du$$

$$= exp(t)\,erf\left(\sqrt{\tau}\right)$$

Problem 5.4.2

$$\mathcal{L}\{\cos(wx)\} = \frac{s}{s^2 + w^2}$$
$$\mathcal{L}\{\cos(3x)\} = \frac{s}{s^2 + 9}$$
$$\mathcal{L}\{xf(x)\} = -\frac{dF(s)}{ds}$$
$$\mathcal{L}\{x\cos 3x\} = -\frac{d}{ds}\left(\frac{s}{s^2 + 9}\right) = \frac{s^2 - 9}{(s^2 + 9)^2}$$
$$\mathcal{L}\{y(x)\} = Y(s)$$
$$\int_0^x \exp(\tau)\, y(x - \tau) d\tau = \exp(x) \otimes y(x)$$
$$\mathcal{L}\{\exp(x) \otimes y(x)\} = \mathcal{L}\{\exp(x)\}\mathcal{L}\{y(x)\} = \frac{1}{s - 1} Y(s)$$
$$Y(s) = \frac{s^2 - 9}{(s^2 + 9)^2} - \frac{Y(s)}{s - 1}$$

Solving the above equation, one gets

$$Y(s) = \left(1 - \frac{1}{s}\right)\frac{s^2 - 9}{(s^2 + 9)^2}$$

Assuming

$$\frac{s^2 - 9}{(s^2 + 9)^2} = G(s)$$

Inverse transforming the above, one gets

$$\frac{s^2 - 9}{(s^2 + 9)^2} = \frac{s^2 + 9 - 18}{(s^2 + 9)^2}$$
$$G(s) = \frac{1}{s^2 + 9} - \frac{18}{(s^2 + 9)^2}$$
$$g(t) = L^{-1}\{G(s)\}$$
$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + k^2)^2}\right\} = \frac{1}{2k^3}(\sin kt - kt\cos kt)$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + k^2}\right\} = \frac{1}{k}\sin kt$$

$$g(t) = \frac{1}{3}\sin(3x) - \frac{18}{54}(\sin(3x) - 3x\cos(3x))$$

$$Y(s) = G(s) - \frac{1}{s}G(s)$$

$$y(t) = g(x) - \int_0^x g(\tau)d\tau$$

$$\mathcal{L}\left\{\int_0^x g(\tau)d\tau\right\} = \frac{G(s)}{s}$$

$$g(x) = x\cos(3x)$$

$$\int_0^x g(\tau)d\tau = \frac{\tau\sin(3\tau)}{3}\bigg|_0^x - \int_0^x \frac{\sin(3\tau)}{3}d\tau$$

$$= \frac{x\sin(3x)}{3} + \frac{\cos(3x)}{9} - \frac{1}{9}$$

$$y(t) = t\cos(3t) - \frac{x\sin(3x)}{3} - \frac{\cos(3x)}{9} + \frac{1}{9}$$

5.5 Application of Laplace Transforms

Problem 5.5.1

Let us apply Laplace Transform to the equation. Since I.C. are 0

$$s^2X(s) + 4X(s) = F(s)$$

where $F(s)$ is the Laplace Transform of $f(s)$. We got algebraic equation for $X(s)$

$$X(s) = \frac{F(s)}{s^2 + 4}$$

Because

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2}\sin 2t$$

we can rewrite the equation on term of two Laplace Transforms

$$X(s) = \mathcal{L}\{f(t)\}\mathcal{L}\left\{\frac{1}{2}\sin 2t\right\}$$

Then the Convolution Property yields

$$x(t) = f(t)\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau d\tau$$

Problem 5.5.2

Solve the integral-DE

$$x'(t) + 2x(t) - 4\int_0^t \exp(t-\tau)\, x(\tau)d\tau = \sin(t), \quad x(0) = 0.$$

Taking the Laplace Transform of each side of the equation we find,

$$\big(sX(s) - x(0)\big) + 2X(s) - 4\mathcal{L}\{\exp(t) \otimes x(t)\} = \frac{1}{s^2 + 1}$$

$$sX(s) + 2X(s) - 4\left(\frac{1}{s-1}\right)X(s) = \frac{1}{s^2 + 1}$$

and solving for $X(s)$, we obtain (and performing partial fraction)

$$X(s) = \frac{s-1}{(s^2+1)(s^2+s-6)} = \frac{1}{25}\left(\frac{4-3\mathrm{s}}{s^2+1} + \frac{1}{\mathrm{s}-2} + \frac{2}{\mathrm{s}+3}\right)$$

Taking the inverse transform, we get

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{25}\mathcal{L}^{-1}\left\{\frac{4-3s}{s^2+1} + \frac{1}{s-2} + \frac{2}{s+3}\right\}$$
$$= \frac{1}{25}(4\sin(t) - 3\cos(t) + \exp(2t) + 2\exp(-3t))$$

Problem 5.5.3

Take Laplace on each side of two DEs, we have

$$\begin{cases} \mathcal{L}\{x''\} = -4\mathcal{L}\{x\} + \mathcal{L}\{\sin t\} \\ \mathcal{L}\{y''\} = 4\mathcal{L}\{x\} - 8\mathcal{L}\{y\} \end{cases}$$

Let $\mathcal{L}\{x\} = X(s)$. We know that

$$\mathcal{L}\{x''\} = s^2X(s) - x(0)s - x'(0)$$

Plugging into the first DE we get

$$s^2\,X(s) - x(0)s - x'(0) = -4X(s) + \frac{1}{s^2+1}$$

This gives

$$X(s) = \frac{x(0)s^3 + x'(0)s^2 + x(0)s + x'(0) + 1}{(s^2+4)(s^2+1)}$$

Thus, the G.S. of $x(t)$ is $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

Similarly, for y, let $\mathcal{L}\{y\} = Y(s)\}$. This gives

$$\mathcal{L}\{y''\} = s^2Y(s) - y(0)s - y'(0)$$

The second DE can be written as

$$s^2Y(s) - y(0)s - y'(0) = 4X(s) - 8Y(s)$$

This gives

$$Y(s) = \frac{4X(s) + y(0)s + y'(0)}{s^2+8}$$

Plugging $X(s)$ into the above equation, we can get $Y(s)$ in terms of s, then the G.S. of $y(t)$ is: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Problem 5.5.4

Take Laplace on each side of the equation, we have

$$\mathcal{L}\{x'''\} + \mathcal{L}\{x''\} - 6\mathcal{L}\{x'\} = 0$$

Let $\mathcal{L}\{x\} = X(s)$, then

$$\mathcal{L}\{x'\} = sX(s) - x(0) = sX(s)$$

$$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - x'(0) = s^2X(s) - 1$$
$$\mathcal{L}\{x'''\} = s^3X(s) - s^2x(0) - sx'(0) - x''(0)$$
$$= s^3X(s) - s - 1$$

By substitution, we have

$$s^3X(s) - s - 1 + s^2X(s) - 1 - 6sX(s) = 0$$
$$X(s) = \frac{s+2}{s(s+3)(s-2)}$$
$$= -\frac{1}{3}\cdot\frac{1}{s} - \frac{1}{15}\cdot\frac{1}{s+3} + \frac{2}{5}\cdot\frac{1}{s-2}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$
$$= -\frac{1}{3} - \frac{1}{15}\exp(-3t) + \frac{2}{5}\exp(2t)$$

Problem 5.5.5

Take Laplace on each side of the equation:

$$\mathcal{L}\{x''\} + 4L\{x\} = -5\mathcal{L}\{\delta(t-3)\}$$
$$s^2X(s) - x(0)s - x'(0) + 4X(s) = -5\exp(3s)$$

Given $x(0) = 1, x'(0) = 0$, we have

$$s^2X(s) - s + 4X(s) = -5\exp(3s)$$
$$X(s) = -5\cdot\frac{s}{s^2+4}\cdot\frac{\exp(3s)}{s} + \frac{s}{s^2+2^2}$$

Thus

$$x(t) = -5\int_0^t \cos 2(t-\tau)\,\delta(\tau+3)d\tau + \cos 2t$$

Note

It can be proved that

$$\delta(t) = u'(t)$$

This means

$$\mathcal{L}\{\delta(t)\} = 1$$

and

$$\mathcal{L}\{\delta(t-a)\} = \exp(-as)$$

Problem 5.5.6

Apply Laplace transforms on both sides of the DE,

$$\mathcal{L}\{x'' + 6x' + 8x\} = \mathcal{L}\{-\delta(t-2)\}$$
$$s^2X(s) - s + 6sX(s) - 6 + 8X(s) = -\exp(-2s)$$
$$X(s) = \frac{s + 6 - \exp(-2s)}{s^2 + 6s + 8} = \frac{s + 6 - \exp(-2s)}{(s+2)(s+4)}$$
$$= \frac{2}{s+2} - \frac{1}{s+4} - \frac{1}{2}\exp(-2s)\left(\frac{1}{s+2} - \frac{1}{s+4}\right)$$
$$x(t) = \mathcal{L}^{-1}\left\{\frac{2}{s+2} - \frac{1}{s+4} - \frac{1}{2}\exp(-2s)\left(\frac{1}{s+2} - \frac{1}{s+4}\right)\right\}$$
$$= 2\exp(-2t) - \exp(-4t) - \frac{1}{2}\delta(t-2) \otimes \exp(-2t)$$
$$+ \frac{1}{2}\delta(t-2) \otimes \exp(-4t)$$

Problem 5.5.7

$$x'' + 2x' + x = \delta(t) - \delta(t-2)$$
$$x(0) = 0$$
$$x'(0) = 0$$

Apply Laplace transform on both sides.

$$\mathcal{L}\{\delta(t)\} = 1$$
$$\mathcal{L}\{\delta(t-a)\} = \exp(-as)\}$$
$$\mathcal{L}\{x'' + 2x' + x\} = \mathcal{L}\{\delta(t) - \delta(t-2)\}$$
$$\left(s^2X(s) - sx(0) - x'(0)\right) + 2\left(sX(s) - x(0)\right) + X(s)$$
$$= 1 - \exp(-2s)$$
$$x(0) = x'(0) = 0$$

We get

$$X(s) = \frac{1 - \exp(-2s)}{s^2 + 2s + 1} = \frac{1 - \exp(-2s)}{(s+1)^2}$$
$$= \frac{1}{(s+1)^2} - \frac{\exp(-2s)}{(s+1)^2}$$

Since

$$\mathcal{L}\{f(t-\tau)u(t-\tau)\} = \exp(-\tau s)\,F(s)$$

Inversely transform leads to

$$x(t) = t\exp(-t) - (t-2)\exp\left(-(t-2)\right)u(t-2)$$

Problem 5.5.8

$$\mathcal{L}\{\delta_a(t)\} = \mathcal{L}\{\delta(t-a)\} = \exp(-as)$$

$$\mathcal{L}\{x''\} + \mathcal{L}\{\omega^2 x\} = \sum_{n=0}^{\infty} \mathcal{L}\{\delta(t-2nt_0)\}$$

$$s^2 X(s) - sx(0) - x'(0) + \omega^2 X(s) = \sum_{n=0}^{\infty} \exp(-2nt_0 s)$$

$$(s^2+\omega^2)X(s) = \sum_{n=0}^{\infty} \exp(-2nt_0 s)$$

$$X(s) = \sum_{n=0}^{\infty} \frac{\exp(-2nt_0 s)}{s^2+\omega^2}$$

Note that

$$\mathcal{L}^{-1}\{\exp(-as)\,F(s)\} = u(t-a)f(t-a)$$

$$x(t) = \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty} \frac{\exp(-2nt_0 s)}{s^2+\omega^2}\right\} = \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{\exp(-2nt_0 s)}{s^2+\omega^2}\right\}$$

$$= \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left\{\frac{1}{\omega}\frac{\omega\exp(-2nt_0 s)}{s^2+\omega^2}\right\}$$

$$= \frac{1}{\omega}\sum_{n=0}^{\infty} u(t-2nt_0)\sin\omega(t-2nt_0)$$

Problem 5.5.9

Case (I) $a > 0,\ b > 0$

$$\mathcal{L}\{u(t-a)\} = \frac{\exp(-as)}{s}$$

$$\mathcal{L}\{\delta(t-b)\} = \int_0^{\infty} \delta(t-b)\exp(-st)\,dt$$

$$= \int_0^{\infty} \delta(t-b)\exp(-st)\,dt$$

$$= \exp(-bs)$$

Applying Laplace to both sides of the equation

$$s^2X(s) - sx(0) - x'(0) + 2\big(sX(s) - x(0)\big) + X(s)$$
$$= \frac{\exp(-as)}{s} + \exp(-bs)$$

Substitute I.C.

$$s^2X(s) + 2sX(s) + X(s) = \frac{\exp(-as)}{s} + \exp(-bs)$$

$$X(s) = \frac{\frac{\exp(-as)}{s} + \exp(-bs)}{(s+1)^2} = \frac{\frac{\exp(-as)}{s}}{(s+1)^2} + \frac{\exp(-bs)}{(s+1)^2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)^2}\exp(-as)\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\exp(-bs)\right\}$$

$$= \mathcal{L}^{-1}\left\{\exp(-as)\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)\right\}$$
$$+ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\exp(-bs)\right\}$$

$$= u(t-a)\big\{u(t-a) - \exp\big(-(t-b)\big)$$
$$- (t-a)\exp\big(-(t-b)\big)\big\}$$
$$+ u(t-b)(t-b)\exp\big(-(t-b)\big)$$

$$= u(t-a)\Big(1 - \exp\big(-(t-b)\big) - (t-a)\exp\big(-(t-b)\big)\Big)$$
$$+ u(t-b)(t-b)\exp\big(-(t-b)\big)$$

Case (II) $a \le 0, b > 0$

$$\mathcal{L}\{u(t-a)\} = \frac{1}{s}$$

$$\mathcal{L}\{\delta(t-b)\} = \exp(-bs)$$

Applying Laplace to both sides of the equation

$$s^2X(s) + 2sX(s) + X(s) = \frac{1}{s} + \exp(-bs)$$

$$X(s) = \frac{1}{s(s+1)^2} + \frac{\exp(-bs)}{(s+1)^2}$$

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{\exp(-bs)}{(s+1)^2}\right\} \\ &= u(t) - \exp(-t) - t\exp(-t) \\ &\qquad + u(t-b)(t-b)\exp\big(-(t-b)\big) \\ &= 1 - \exp(-t) - t\exp(-t) \\ &\qquad + u(t-b)(t-b)\exp\big(-(t-b)\big) \end{aligned}$$

Problem 5.5.10

Take Laplace on each side of both equations.

$$\begin{cases} \mathcal{L}\{x'\} = \mathcal{L}\{x\} + 2\mathcal{L}\{y\} \\ \mathcal{L}\{y'\} = 2\mathcal{L}\{x\} - 2\mathcal{L}\{y\} \end{cases}$$

Given I.C., this gives

$$\begin{cases} (s-1)X(s) = 2Y(s) + 1 \\ 2X(s) - (s+2)Y(s) \end{cases}$$

This gives

$$\begin{cases} X(s) = \dfrac{1}{5}\cdot\dfrac{1}{s+3} + \dfrac{4}{5}\cdot\dfrac{1}{s-2} \\ Y(s) = -\dfrac{2}{5}\cdot\dfrac{1}{s+3} + \dfrac{2}{5}\cdot\dfrac{1}{s-2} \end{cases}$$

Finally

$$\begin{cases} x(t) = \dfrac{4\exp(2t) + \exp(-3t)}{5} \\ y(t) = \dfrac{2(\exp(2t) - \exp(-3t))}{5} \end{cases}$$

Problem 5.5.11

$$\begin{cases} L\{x_1''\} = -2L\{x_1\} + L\{x_2\} + \exp(-\tau s) \\ L\{x_2''\} = L\{x_1\} - L\{x_2\} + \exp(-2\tau s) \end{cases}$$

$$\begin{cases} s^2 X_1(s) = -2X_1(s) + X_2(s) + \exp(-\tau s) \\ s^2 X_2(s) = X_1(s) - X_2(s) + \exp(-2\tau s) \end{cases}$$

$$\begin{cases} X_2(s) = (s^2+2)X_1(s) - \exp(-\tau s) \\ (s^2+1)X_2(s) = X_1(s) + \exp(-2\tau s) \end{cases}$$

Insert first equation into second.

$$(s^2+1)\big((s^2+2)X_1(s) - \exp(-\tau s)\big) = X_1(s) + \exp(-2\tau s)$$

$$(s^4+3s^2+1)X_1(s) = \exp(-\tau s)\,(\exp(-\tau s) + s^2 + 1)$$

$$x_1(t) = \mathcal{L}^{-1}\left\{\frac{\exp(-\tau s)\,(\exp(-\tau s)+s^2+1)}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)}\right\}$$

$$= u(t-\tau)\mathcal{L}^{-1}\left\{\frac{\exp(-\tau s)}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)} + \frac{s^2+1}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)}\right\}$$

Using partial fractions we find

$$\frac{1}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)} = \frac{B}{s^2+\frac{3+\sqrt{5}}{2}} + \frac{D}{s^2+\frac{3-\sqrt{5}}{2}}$$

Where

$$B = \frac{1}{\sqrt{5}}$$

and

$$D = -\frac{1}{\sqrt{5}}$$

and

$$\frac{s^2 + 1}{\left(s^2 + \frac{3+\sqrt{5}}{2}\right)\left(s^2 + \frac{3-\sqrt{5}}{2}\right)} = \frac{F}{s^2 + \frac{3+\sqrt{5}}{2}} + \frac{H}{s^2 + \frac{3-\sqrt{5}}{2}}$$

where

$$F = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

and

$$H = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

Then

$$x_1(t) = u(t-\tau)\left(u(t-\tau)\left(\frac{1}{\sqrt{5}}\sqrt{\frac{2}{3+\sqrt{5}}}\sin\sqrt{\frac{2}{3+\sqrt{5}}}t\right.\right.$$
$$\left.-\frac{1}{\sqrt{5}}\sqrt{\frac{2}{3-\sqrt{5}}}\sin\sqrt{\frac{2}{3-\sqrt{5}}}t\right)$$
$$+\frac{1+\sqrt{5}}{2\sqrt{5}}\sqrt{\frac{2}{3+\sqrt{5}}}\sin\sqrt{\frac{2}{3+\sqrt{5}}}t$$
$$\left.+\frac{\sqrt{5}-1}{2\sqrt{5}}\sqrt{\frac{2}{3-\sqrt{5}}}\sin\sqrt{\frac{2}{3-\sqrt{5}}}t\right)$$

We also have

$$(s^2 + 1)X_2(s) = X_1(s) + \exp(-2\tau s)$$

$$X_2(s) = \frac{1}{s^2+1}\left(\frac{\exp(-\tau s)\,(\exp(-\tau s)+s^2+1)}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)} + \exp(-2\tau s)\right)$$

$$= \exp(-\tau s)\left(\frac{\exp(-\tau s)}{(s^2+1)\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)} + \frac{1}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)} + \frac{1}{s^2+1}\right)$$

By partial fractions we have

$$\frac{1}{(s^2+1)\left(s^2+\frac{3+\sqrt{5}}{2}\right)\left(s^2+\frac{3-\sqrt{5}}{2}\right)} = \frac{A}{s^2+1} + \frac{B}{\left(s^2+\frac{3+\sqrt{5}}{2}\right)} + \frac{C}{\left(s^2+\frac{3-\sqrt{5}}{2}\right)}$$

When

$$A = \frac{1}{\sqrt{5}+1}, B = \frac{2}{5+\sqrt{5}}, C = \frac{2}{5-\sqrt{5}}$$

Then

$$x_2(t) = u(t-\tau)\left(u(t-\tau)\left(\frac{1}{\sqrt{5}+1}\sin t\right.\right.$$
$$+\frac{2}{5+\sqrt{5}}\sqrt{\frac{2}{3+\sqrt{5}}}\sin\sqrt{\frac{3+\sqrt{5}}{2}}t$$
$$\left.+\frac{2}{5-\sqrt{5}}\sqrt{\frac{2}{3-\sqrt{5}}}\sin\sqrt{\frac{3-\sqrt{5}}{2}}t\right)$$
$$+\frac{1}{\sqrt{5}}\sqrt{\frac{2}{3+\sqrt{5}}}\sin\sqrt{\frac{2}{3+\sqrt{5}}}t$$
$$\left.-\frac{1}{\sqrt{5}}\sqrt{\frac{2}{3-\sqrt{5}}}\sin\sqrt{\frac{2}{3-\sqrt{5}}}t\right)$$

Problem 5.5.12

Apply Laplace transform on both sides we have

$$sX(s) + 4X(s) + 6X(s)\frac{1}{s-1} = \mathcal{L}\{\sin\omega t\}$$
$$X(s)\left(s+4+\frac{6}{s-1}\right) = \mathcal{L}\{\sin\omega t\}$$
$$X(s) = \mathcal{L}\{\sin\omega t\}\frac{s-1}{(s+1)(s+2)}$$

Using partial fractions, we get

$$\frac{s-1}{(s+1)(s+2)} = \frac{3}{s+2} - \frac{2}{s+1}$$

Then $X(s)$ can be expressed as

$$X(s) = \mathcal{L}\{\sin\omega t\}\left(\frac{3}{s+2} - \frac{2}{s+1}\right)$$

Finally apply inverse Laplace transform and use convolution theorem

$$\mathcal{L}^{-1}\{X(s)\} = \sin\omega t \otimes \mathcal{L}^{-1}\left\{\frac{3}{s+2}\right\} - \sin\omega t \otimes \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\}$$
$$= 3\sin\omega t \otimes \exp(-2t) - 2\sin\omega t \otimes \exp(-t)$$

Problem 5.5.13

By applying Laplace transform to given a system of DE, we get following.

$$\begin{cases} x'(t) = 2x(t) + y(t) + t \\ y'(t) = x(t) + 2y(t) + \exp(t) \end{cases}$$

$$\begin{cases} sX(s) - x(0) = 2X(s) + Y(s) + \dfrac{1}{s^2} \\ sY(s) - y(0) = X(s) + 2Y(s) + \dfrac{1}{s-1} \end{cases}$$

$$\begin{cases} X(\quad) = \dfrac{x(0)(s-2)}{(s-1)(s-3)} + \dfrac{s-2}{s^2(s-1)(s-3)} \\ \qquad + \dfrac{y(0)}{(s-1)(s-3)} + \dfrac{1}{(s-1)^2(s-3)} \\ Y(s) = \dfrac{x(0)}{(s-1)(s-3)} + \dfrac{1}{s^2(s-1)(s-3)} \\ \qquad + \dfrac{y(0)(s-2)}{(s-1)(s-3)} + \dfrac{(s-2)}{(s-1)^2(s-3)} \end{cases}$$

We need partial fractions for each term in the above DEs.

$$\begin{cases} \dfrac{x(0)(s-2)}{(s-1)(s-3)} = \dfrac{x(0)}{2}\left(\dfrac{1}{s-1} + \dfrac{1}{s-3}\right) \\ \dfrac{s-2}{s^2(s-1)(s-3)} = -\dfrac{5}{9s} - \dfrac{2}{3s^2} + \dfrac{1}{2(s-1)} + \dfrac{1}{18(s-3)} \\ \dfrac{y(0)}{(s-1)(s-3)} = \dfrac{y(0)}{2}\left(\dfrac{-1}{s-1} + \dfrac{1}{s-3}\right) \\ \dfrac{1}{(s-1)^2(s-3)} = -\dfrac{1}{4(s-1)} - \dfrac{1}{2(s-1)^2} + \dfrac{1}{4(s-3)} \end{cases}$$

$$\begin{cases} \dfrac{(0)}{(s-1)(s-3)} = \dfrac{x(0)}{2}\left(\dfrac{-1}{s-1} + \dfrac{1}{s-3}\right) \\ \dfrac{1}{s^2(s-1)(s-3)} = \dfrac{4}{9s} + \dfrac{1}{3s^2} - \dfrac{1}{2(s-1)} + \dfrac{1}{18(s-3)} \\ \dfrac{y(0)(s-2)}{(s-1)(s-3)} = \dfrac{y(0)}{2}\left(\dfrac{1}{s-1} + \dfrac{1}{s-3}\right) \\ \dfrac{(s-2)}{(s-1)^2(s-3)} = -\dfrac{1}{4(s-1)} + \dfrac{1}{2(s-1)^2} + \dfrac{1}{4(s-3)} \end{cases}$$

Using results above, we perform an inverse Laplace transform, or directly use the convolution.

Problem 5.5.14

$$\mathcal{L}\{x''\} - 6\mathcal{L}\{x'\} + 8\mathcal{L}\{x\} = 2\mathcal{L}\{1\}$$

From given I.C., we have

$$s^2X(s) - 6sX(s) + 8X(s) = \frac{2}{s}$$

$$X(s) = \frac{2}{s(s-2)(s-4)}$$

$$= \frac{1}{4}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s-2} + \frac{1}{4}\cdot\frac{1}{s-4}$$

Therefore

$$x(t) = \frac{1}{4} - \frac{1}{2}\exp(2t) + \frac{1}{4}\exp(4t)$$

Problem 5.5.15

$$\mathcal{L}\{tx''\} + 2\mathcal{L}\{(t-1)x'\} - 2\mathcal{L}\{x\} = 0$$

$$-\left(s^2X(s)\right)' + 2\left\{-\left(sX(s)\right)' - sX(s)\right\} - 2X(s) = 0$$

$$\frac{X'(s)}{X(s)} = -\frac{4(s+1)}{s(s+2)}$$

$$X(s) = C(s^2+2s)^{-2}$$

$$x(t) = \frac{C}{2}\exp(-t)\,(t\cdot\cosh t - \sinh t)$$

Problem 5.5.16

Perform Laplace Transform on equation

$$\mathcal{L}\left\{t\frac{d^2x}{dt^2}\right\}+\mathcal{L}\left\{\frac{dx}{dt}\right\}+\mathcal{L}\{tx\}=0$$

$$-\frac{d}{ds}(s^2X-s\alpha)+(sX-\alpha)-\frac{dX}{ds}=0$$

$$\Rightarrow (s^2+1)X'+sX=0$$

which is a separable DE. We have

$$\int\frac{dX}{X}=-\frac{sds}{s^2+1}$$

$$ln\,x=-\frac{1}{2}ln(s^2+1)$$

$$X(s)=\frac{1}{\sqrt{s^2+1}}$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}=\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\}$$

Problem 5.5.17

$$\mathcal{L}\{x''\}-2\mathcal{L}\{x'\}+\mathcal{L}\{x\}=\mathcal{L}\{f(t)\}$$

$$s^2X(s)-2sX(s)+X(s)=F(s)$$

$$X(s)=\frac{F(s)}{(s^2-2s+1)}$$

$$x(t)=\mathcal{L}^{-1}\{X(s)\}$$

Problem 5.5.18

$$(s^2X(s)-2)+4sX(s)+13X(s)=\frac{1}{(s+1)^2}$$

$$X(s)=\frac{2+1/(s+1)^2}{s^2+4s+13}=\frac{2s^2+4s+13}{(s+1)^2(s^2+4s+13)}$$

$$= \frac{1}{50}\left(-\frac{1}{s+1} + \frac{5}{(s+1)^2} + \frac{s+2}{(s+2)^2+9} + 32\frac{3}{(s+2)^2+9}\right)$$

So

$$x(t) = \frac{1}{50}\left((-1+5t)\exp(-t) + \exp(-2t)\,(\cos 3t + 32\sin 3t)\right)$$

Problem 5.5.19

Method I: Laplace Transform

By applying the Laplace Transform to the given DE, we get

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 12\mathcal{L}\{x\} = \mathcal{L}\{\sin 4t + \exp(3t)\}$$

$$\left(s^2x(s) - sx(0) - x'(0)\right) - \left(sx(s) - x(0)\right) - 12x(s) = \frac{4}{s^2+16} + \frac{1}{\mathrm{s}-3}$$

From the I.C., we have

$$(s^2x(s) - 2s - 1) - (sx(s) - 2) - 12x(s) = \frac{4}{s^2+16} + \frac{1}{\mathrm{s}-3}$$

$$x(s) = \frac{2s^4 - 7s^3 + 36s^2 - 108s + 52}{(s-3)(s+3)(s-4)(s^2+16)}$$

Now, if we apply the partial fraction for the RHS, we get

$$\frac{2s^4 - 7s^3 + 36s^2 - 108s + 52}{(s-3)(s+3)(s-4)(s^2+16)} = \frac{A}{s-3} + \frac{B}{s+3} + \frac{C}{s-4} + \frac{Ds+E}{s^2+16}$$

$$\begin{cases} A+B+C+D=2 \\ -A-7B-4D+E=-7 \\ 4A+28B+7C-9D-4E=36 \\ -16A-112B+36D-9E=-108 \\ -192A+192B-144C+36E=52 \end{cases} \Rightarrow \begin{cases} \mathrm{A}=-\dfrac{1}{6} \\ \mathrm{B}=\dfrac{1051}{1050} \\ \mathrm{C}=\dfrac{65}{56} \\ \mathrm{D}=\dfrac{1}{200} \\ \mathrm{E}=-\dfrac{28}{200} \end{cases}$$

So, we get

$$x(s)=\frac{2s^4-7s^3+36s^2-108s+52}{(s-3)(s+3)(s-4)(s^2+16)}$$
$$=-\frac{1}{6}\left(\frac{1}{s-3}\right)+\frac{1051}{1050}\left(\frac{1}{s+3}\right)+\frac{65}{56}\left(\frac{1}{s-4}\right)$$
$$+\frac{1}{200}\left(\frac{s}{s^2+16}\right)-\frac{7}{200}\left(\frac{4}{s^2+16}\right)$$

By using the inverse Laplace Transform, we obtain

$$x(t)=\frac{1051}{1050}\exp(-3t)+\frac{65}{56}\exp(4t)-\frac{7}{200}\sin 4t+\frac{1}{200}\cos 4t$$
$$-\frac{1}{6}\exp(3t)$$

<u>Method II:</u>

For the common solution, we have to solve the C-Eq.

$$r^2-r-12=0, \quad r=-3,\ 4$$

The common solution is

$$x_c(t)=C_1\exp(-3t)+C_2\exp(4t)$$

For the P.S., we get it by considering the RHS of the given DE.

$$x_p(t)=A\sin 4t+B\cos 4t+C\exp(3t)$$
$$x_p'(t)=4A\cos 4t-4B\sin 4t+3C\exp(3t)$$
$$x_p''(t)=-16A\sin 4t-16B\cos 4t+9C\exp(3t)$$

$$x_p''(t)-x_p'(t)-12x_p(t)$$
$$=(-28A+4B)\sin 4t+(-4A-28B)\cos 4t-6C\exp(3t)$$

$= \sin 4t + \exp(3t)$

From this relation we get the system of equations

$$\begin{cases} -28A + 4B = 1 \\ -4A - 28B = 0 \\ -6C = 1 \end{cases} \Rightarrow \begin{cases} \mathrm{A} = -\dfrac{7}{200} \\ \mathrm{B} = \dfrac{1}{200} \\ \mathrm{C} = -\dfrac{1}{6} \end{cases}$$

So, the P.S. is

$$x_p(t) = -\frac{7}{200}\sin 4t + \frac{1}{200}\cos 4t - \frac{1}{6}\exp(3t)$$

The G.S. is

$$x(t) = x_c(t) + x_p(t)$$
$$= C_1 \exp(-3t) + C_2 \exp(4t) - \frac{7}{200}\sin 4t + \frac{1}{200}\cos 4t - \frac{1}{6}\exp(3t)$$

Because we have I.C., we can determine the coefficients of common solution.

$$x(0) = C_1 + C_2 + \frac{1}{200} - \frac{1}{6} = 2$$

$$x'(0) = -3C_1 + 4C_2 - \frac{28}{200} - \frac{3}{6} = 1$$

$$\begin{cases} x(0) = C_1 + C_2 + \dfrac{1}{200} - \dfrac{1}{6} = 2 \\ x'(0) = -3C_1 + 4C_2 - \dfrac{28}{200} - \dfrac{3}{6} = 1 \end{cases} \Rightarrow \begin{cases} \mathrm{C}_1 = \dfrac{1051}{1050} \\ \mathrm{C}_2 = \dfrac{65}{56} \end{cases}$$

Consequently, we have the G.S. with determined solution

$$x(t) = x_c(t) + x_p(t)$$
$$= \frac{1051}{1050}\exp(-3t) + \frac{65}{56}\exp(4t) - \frac{7}{200}\sin 4t + \frac{1}{200}\cos 4t - \frac{1}{6}\exp(3t)$$

Problem 5.5.20

(1) The C-Eq of the homogeneous portion is

$$r^2 + \omega^2 = 0$$
$$r = \pm i\omega$$

The two L.I. solutions are

$$x_1 = \cos\omega t, \quad x_2 = \sin\omega t$$
$$W = \begin{vmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{vmatrix} = \cos^2\omega t + \sin^2\omega t = 1$$

The trial functions are

$$u_1 = -\int \frac{y_2(t)f(t)}{W(t)}dt = -\int \cos\omega t\, f(t)dt$$
$$u_2 = \int \frac{y_1(t)f(t)}{W(t)}dt = \int \sin\omega t\, f(t)dt$$

The G.S. of the original DE is

$$x = C_1\cos\omega t + C_2\sin\omega t + u_1\cos\omega t + u_2\sin\omega t$$

(2) Take Laplace on both sides of the DE, we have

$$\mathcal{L}\{x''\} + \omega^2\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

Let $X(s) = \mathcal{L}\{x\}$

$$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - x'(0) = s^2X(s) - x_0 s - v_0$$

By substitution

$$s^2X(s) - x_0 s - v_0 + w^2X(s) = \mathcal{L}\{f(t)\}$$
$$(s) = \frac{\mathcal{L}\{f(t)\} + x_0 s + v_0}{s^2 + w^2}$$
$$= \frac{1}{w}\mathcal{L}\{f(t)\}\frac{w}{s^2 + w^2} + x_0\frac{s}{s^2 + w^2} + \frac{v_0}{w}\frac{w}{s^2 + w^2}$$
$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$
$$= \frac{1}{w}\int_0^t f(t-\tau)\sin(wt)\,dt + x_0\cos(wt) + \frac{v_0}{w}\sin(wt)$$

Problem 5.5.21

(1) The corresponding homogeneous DE is: $x'' + \omega_1^2 x = 0$ which is a 2nd-order DE with constant coefficients, the C-Eq is

$$r^2 + \omega_1^2 = 0 \Rightarrow r_{1,2} = \pm\omega_1 i$$
$$x_c(t) = C_1\cos(\omega_1 t) + C_2\sin(\omega_1 t)$$

$$\begin{cases} x_1(t) = \cos(\omega_1 t) \\ x_2(t) = \sin(\omega_1 t) \end{cases}$$

$$W(t) = \begin{vmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) \\ -\omega_1 \sin(\omega_1 t) & \omega_1 \cos(\omega_1 t) \end{vmatrix} = \omega_1$$

Let $x_p(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$, we have

$$u_1(t) = -\int \frac{y_2(t)f(t)}{W(t)}dt = -\int \frac{\sin(\omega_1 t)\sin(\omega_2 t)}{\omega_1}dt$$

$$= \frac{1}{2\omega_1}\int \Big(\cos\big((\omega_1+\omega_2)t\big) - \cos\big((\omega_1-\omega_2)t\big)\Big)\,dt$$

$$= \frac{1}{2\omega_1}\left(\frac{\sin\big((\omega_1+\omega_2)t\big)}{\omega_1+\omega_2} - \frac{\sin\big((\omega_1-\omega_2)t\big)}{\omega_1-\omega_2}\right)$$

$$u_2(t) = -\int \frac{y_1(t)f(t)}{W(t)}dt = -\int \frac{\cos(\omega_1 t)\sin(\omega_2 t)}{\omega_1}dt$$

$$= \frac{1}{2\omega_1}\int \Big(\sin\big((\omega_1+\omega_2)t\big) - \sin\big((\omega_1-\omega_2)t\big)\Big)\,dt$$

$$= \frac{1}{2\omega_1}\left(-\frac{\cos\big((\omega_1+\omega_2)t\big)}{\omega_1+\omega_2} - \frac{\cos\big((\omega_1-\omega_2)t\big)}{\omega_1-\omega_2}\right)$$

Thus a P.S. can be written as

$$x_p(x) = u_1(t)x_1(t) + u_2(t)x_2(t)$$

$$= \frac{\cos(\omega_1 t)}{2\omega_1}\left(\frac{\sin\big((\omega_1+\omega_2)t\big)}{\omega_1+\omega_2} - \frac{\sin\big((\omega_1-\omega_2)t\big)}{\omega_1-\omega_2}\right)$$
$$+ \frac{\sin(\omega_1 t)}{2\omega_1}\left(-\frac{\cos\big((\omega_1+\omega_2)t\big)}{\omega_1+\omega_2}\right.$$
$$\left.- \frac{\cos\big((\omega_1-\omega_2)t\big)}{\omega_1-\omega_2}\right)$$

$$= \frac{\cos(\omega_1 t)\sin\big((\omega_1+\omega_2)t\big) - \sin(\omega_1 t)\cos\big((\omega_1+\omega_2)t\big)}{2\omega_1(\omega_1+\omega_2)}$$
$$+ \frac{-\cos(\omega_1 t)\sin\big((\omega_1 - \quad {}_2)t\big) + \sin(\omega_1 t)\cos\big((\omega_1-\omega_2)t\big)}{2\omega_1(\omega_1-\omega_2)}$$

$$= \frac{\sin\big((\omega_1+\omega_2)t - \omega_1 t\big)}{2\omega_1(\omega_1+\omega_2)} + \frac{\sin(\omega_1 t - (\omega_1-\omega_2)t)}{2\omega_1(\omega_1-\omega_2)}$$

$$= \frac{\sin(\omega_2 t)}{\omega_1^2 - \omega_2^2}$$

Thus, the G.S. of the DE is

$$x(t) = x_c(t) + x_p(t)$$
$$= C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t) - \frac{\sin(\omega_2 t)}{\omega_1^2 - \omega_2^2}$$

With the given I.C.

$$\xrightarrow{x(0)=x'(0)=0} C_1 = 0, C_2 = \frac{\omega_2}{\omega_1(\omega_1^2 - \omega_2^2)}$$

Thus

$$x(t) = \frac{\omega_2 \sin(\omega_1 t)}{\omega_1(\omega_1^2 - \omega_2^2)} - \frac{\sin(\omega_2 t)}{(\omega_1^2 - \omega_2^2)}$$
$$= \frac{\omega_2}{\omega_1^2 - \omega_2^2}\left(\frac{\sin(\omega_1 t)}{\omega_1} - \frac{\sin(\omega_2 t)}{\omega_2}\right)$$

Note

We applied the following triangle relations.

$$sin(\alpha \pm \beta) = sin\,\alpha\, cos\,\beta \pm cos\,\alpha\, sin\,\beta$$
$$sin\,\alpha\, sin\,\beta = -\frac{1}{2}(cos(\alpha + \beta) - cos(\alpha - \beta))$$
$$sin\,\alpha\, cos\,\beta = \frac{1}{2}(sin(\alpha + \beta) + sin(\alpha - \beta))$$

(2) Take Laplace on each side of the DE, we have

$$\mathcal{L}\{\mathrm{x}''\} + \omega_1^2 \mathrm{L}\{\mathrm{x}\} = \mathcal{L}\{\sin(\omega_2 \mathrm{t})\}$$

Let

$$X(s) = L\{x\} \Rightarrow L\{x''\} = s^2 X(x) - x(0)s - x'(0) = s^2 X(s)$$

Substituting $L\{x''\}$ and $L\{x\}$ in the DE, we have

$$s^2 X(s) + \omega_1^2 X(s) = \frac{\omega_2}{s^2 + \omega_2^2}$$

$$X(s) = \frac{\omega_2}{(s^2 + \omega_2^2)(s^2 + \omega_1^2)}$$
$$= \frac{\omega_2}{\omega_2^2 - \omega_1^2}\left(\frac{1}{\omega_1}\frac{\omega_1}{s^2 + \omega_1^2} - \frac{1}{\omega_2}\frac{\omega_2}{s^2 + \omega_2^2}\right)$$

Thus

$$x(t) = \mathcal{L}^{-1}\{X(x)\} = \frac{\omega_2}{\omega_2^2 - \omega_1^2}\left(\frac{\sin(\omega_1 t)}{\omega_1} - \frac{\sin(\omega_2 t)}{\omega_2}\right)$$

is the solution.

Problem 5.5.22

(1) Take Laplace on both sides of the DE.

$$\mathcal{L}\{x'\} - \mathcal{L}\{x\} = \mathcal{L}\{1\} - \mathcal{L}\{(t-1)u(t-1)\}$$

Let $X(s) = \mathcal{L}\{x\}$, then $\mathcal{L}\{x'\} = sX(s) - x(0) = sX(s)$.

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{(t-1)u(t-1)\} = \frac{1}{s^2} exp(-s)$$

By substitution, we have

$$sX(s) - X(s) = \frac{1}{s} - \frac{1}{s^2} exp(-s) = \frac{s - exp(-s)}{s^2}$$

Note

$\mathcal{L}\{u(t-1)f(t-a)\} = F(s)\exp(-as)$ *where* $F(s) = \mathcal{L}\{f(t)\}$.

Solve the above equation for $X(s)$

$$\begin{aligned} X(s) &= \frac{s - \exp(-s)}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{\exp(-s)}{s^2(s-1)} \\ &= \frac{1}{s-1} - \frac{1}{s} - \exp(-s)\left(\frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}\right) \\ &= \frac{1}{s-1} - \frac{1}{s} - \frac{1}{e}\frac{\exp(-s)}{s-1} + \frac{\exp(-s)}{s} + \frac{\exp(-s)}{s^2} \end{aligned}$$

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\{X(s)\} \\ &= \exp(-i) - 1 - \frac{u(t-1)}{e} + u(t-1) + (t-1)u(t-1) \\ &= \exp(-i) - 1 + u(t-1)\left(t - \frac{1}{e}\right) \end{aligned}$$

(2) We divide the equation in two regions

If $t \le 1$, the DE is

$$\begin{cases} x' - x = 1 \\ x(0) = 0 \end{cases}$$

whose solution can be easily found as $x(t) = \exp(t) - 1$.

If $t > 1$, the DE is

$$\begin{cases} x' - x = 2 - t \\ x(0) = 0 \end{cases}$$

which is a 1st-order Linear DE, and

$$P(t) = -1, Q(t) = 2 - t$$

$$\rho(t) = \exp\left(-\int dt\right) = \exp(t)$$

Solving this DE, we have

$$\begin{aligned} x(t) &= \frac{1}{\rho}\left(\int \rho Q dt + C\right) \\ &= \exp(t)\left(\int \exp(-t)\,(2-t)dt + C\right) \\ &= \exp(t)\,(-2\exp(-t) + t\exp(-t) + \exp(-t) + C) \\ &= -1 + t + C\exp(t) \end{aligned}$$

I.C. yields $x(0) = 0 \Rightarrow C = 1 \Rightarrow x(t) = \exp(t) + t + 1$. Thus the final solution of this DE is

$$x(t) = \begin{cases} \exp(t) - 1 & t \leq 1 \\ \exp(t) + t - 1 & t > 1 \end{cases}$$

Problem 5.5.23

For the original DE, if $0 < t < 2\pi$ then $u(t - 2\pi) = 0$

Or

$$x''(t) + 4x(t) = \cos 2t$$

Thus

$$\begin{cases} x''(t) + 4x(t) = \cos 2t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

Using Laplace transformation method, we have

$$(s^2 + 4)X(s) = \mathcal{L}\{cos2t\}$$

So

$$\begin{aligned} x(t) &= \frac{1}{2}\sin 2t \otimes \cos 2t \\ &= \frac{1}{2}\int_0^t \sin 2\tau \cos 2(t - \tau)\,d\tau \end{aligned}$$

$$= \frac{1}{4} t \sin 2t$$

For the original DE, if $t > 2\pi$ then $u(t - 2\pi) = 1$

Or

$$x''(t) + 4x(t) = 0$$

with starting from $t = 2\pi$.

When $t = 2\pi$, from the condition of the phase I, $0 < t < 2\pi$, we know

$$x(2\pi) = 0, x'(2\pi) = 2\pi$$

After resetting the origin to $t = 2\pi$, DE system becomes

$$\begin{cases} x''(t) + 4x(t) = 0 \\ x(0) = 0 \\ x'(0) = 2\pi \end{cases}$$

whose G.S. for $t > 2\pi$ is

$$x(t) = \pi \sin 2t$$

Many also directly evaluate the convolution of the following to find the G.S.

$$x(t) = \frac{1}{2} \sin 2t \otimes \left(\left(1 - u(t - 2\pi)\right) \cos 2t\right)$$

Problem 5.5.24

Applying Laplace transformation on both sides of DE, we have

$$s^2 X(s) - s x(0) - x'(0) + X(s) = \mathcal{L}\{(-1)^{[\![t]\!]}\}$$

Plugging the IC, we get

$$(s^2 + 1) X(s) = \mathcal{L}\{(-1)^{[\![t]\!]}\}$$

Or

$$X(s) = \frac{1}{s^2 + 1} \mathcal{L}\{(-1)^{[\![t]\!]}\}$$

Applying inverse transform, we have

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1} L\left\{(-1)^{[\![t]\!]}\right\}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} \otimes (-1)^{[\![t]\!]}$$

$$= \sin t \otimes (-1)^{[\![t]\!]}$$

Now, one can compute the convolution directly by definition. One may also compute the Laplace transform of the RHS.

$$\begin{aligned}\mathcal{L}\{(-1)^{[\![t]\!]}\} &= \int_0^\infty (-1)^{[\![t]\!]} \exp(-st)\, dt \\ &= \int_0^1 \exp(-st)\, dt - \int_1^2 \exp(-st)\, dt \\ &\quad + \int_2^3 \exp(-st)\, dt \ldots \\ &= \frac{1}{s} + \frac{2}{s}\left(\sum_{n=0}^\infty (-1)^n \exp(-ns) - 1\right) \\ &= -\frac{1}{s} + \frac{2}{s}\sum_{n=0}^\infty (-1)^n \exp(-ns)\end{aligned}$$

Thus,

$$\begin{aligned}X(s) &= \frac{1}{s^2+1}\left(-\frac{1}{s} + \frac{2}{s}\sum_{n=0}^\infty (-1)^n \exp(-ns)\right) \\ &= -\frac{1}{s} + \frac{s}{s^2+1} + \frac{2}{s^2+1}\frac{1}{s}\sum_{n=0}^\infty (-1)^n \exp(-ns)\end{aligned}$$

$$x(t) = -1 + \cos t + 2\sin t \otimes \sum_{n=0}^\infty (-1)^n u(t-n)$$

$$\begin{aligned}&\sin t \otimes \sum_{n=0}^\infty (-1)^n u(t-n) \\ &= \int_0^t \sin\tau \sum_{n=0}^\infty (-1)^n u(t-\tau-n)\, d\tau \\ &= \sum_{n=0}^\infty (-1)^n \int_0^t \sin\tau\, u(t-\tau-n) d\tau \\ &= \sum_{n=0}^\infty (-1)^n \int_0^{t-n} \sin\tau\, d\tau + \sum_{n=0}^\infty (-1)^n \int_{t-n}^t 0 \sin\tau\, d\tau\end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n (1 - \cos(t - n))$$

Thus

$$x(t) = -1 + \cos t + 2 \sum_{n=0}^{\infty} (-1)^n (1 - \cos(t - n))$$

Problem 5.5.25

$$x(0) = x'(0) = y(0) = y'(0) = 0$$

(1) When $\omega = \omega_0$

$$L\{x''\} + L\{\omega^2 x\} = L\{b_0 \sin(\omega t)\}$$

$$s^2 X(s) + \omega^2 X(s) = \frac{b_0 \omega}{\omega^2 + s^2}$$

$$X(s) = \frac{b_0 \omega}{(\omega^2 + s^2)^2}$$

$$x(t) = \frac{b_0 \omega}{(\omega^2 + s^2)^2} = b_0 \omega \left(\frac{1}{2\omega^3}\right) (\sin(\omega t) - \omega t \cos \omega t)$$

$$y(t) = L^{-1} \left\{ -\frac{b_0}{2} \left(\frac{1}{\omega^2 + s^2}\right)' \right\} = \frac{b_0}{2} t \sin \omega t$$

(2) When $\omega \neq \omega_0$

$$s^2 X(s) + \omega^2 X(s) = \frac{b_0 \omega_0}{\omega_0^2 + s^2}$$

$$X(s) = \frac{b_0 \omega_0}{(\omega_0^2 + \quad ^2)(\omega^2 + s^2)}$$

$$Y(s) = \frac{b_0 s}{(\omega_0^2 + s^2)(\omega^2 + s^2)}$$

$$\frac{b_0 \omega_0}{(\omega_0^2 + s^2)(\omega^2 + s^2)} = \frac{As + B}{\omega_0^2 + s^2} + \frac{Cs + D}{\omega^2 + s^2}$$

$$b_0 \omega_0 = (As + B)(\omega^2 + s^2) + (Cs + D)(\omega_0^2 + s^2)$$

$$b_0 \omega_0 = A\omega^2 s + As^3 + B\omega^2 + Bs^2 + C\omega_0^2 s + Cs^3 + D\omega_0^2 + Ds^2$$

$$A + C = 0 \Rightarrow A\omega^2 + C\omega_0^2 = 0$$
$$B + D = 0 \Rightarrow B\omega^2 + D\omega_0^2 = b_0\omega_0$$
$$C = -A \text{ and } D = -B$$
$$A\omega^2 - A\omega_0^2 = 0 \Rightarrow A = C = 0$$
$$B\omega^2 - B\omega_0^2 = b_0\omega_0$$
$$B = \frac{b_0\omega_0}{\omega^2 - \omega_0^2}$$
$$D = \frac{b_0\omega_0}{\omega_0^2 - \omega^2}$$
$$L^{-1}\left\{\frac{b_0\omega_0}{\omega^2 - \omega_0^2}\left(\frac{1}{\omega_0^2 + s^2}\right) + \frac{b_0\omega_0}{\omega_0^2 - \omega^2}\left(\frac{1}{\omega^2 + s^2}\right)\right\}$$
$$x(t) = \frac{b_0\omega_0}{\omega^2 - \omega_0^2}\sin(\omega_0 t) + \frac{b_0\omega_0}{\omega_0^2 - \omega^2}\sin(\omega t)$$
$$\frac{b_0 s}{(\omega_0^2 + s^2)(\omega^2 + s^2)} = \frac{As + B}{\omega_0^2 + s^2} + \frac{Cs + D}{\omega^2 + s^2}$$
$$b_0 s = A\omega^2 s + As^3 + B\omega^2 + Bs^2 + C\omega_0^2 s + Cs^3 + D\omega_0^2 + Ds^2$$
$$A + C = 0 \Rightarrow A\omega^2 + C\omega_0^2 = b_0$$
$$B + D = 0 \Rightarrow B\omega^2 + D\omega_0^2 = 0$$
$$A\omega^2 - A\omega_0^2 = b_0$$
$$A = \frac{b_0}{\omega^2 - \omega_0^2}$$
$$C = \frac{b_0}{\omega_0^2 - \omega}$$
$$B = D = 0$$
$$y(t) = \frac{b_0}{\omega^2 - \omega_0^2}\cos(\omega_0 t) + \frac{b_0}{\omega_0^2 - \omega^2}\cos(\omega t)$$

Problem 5.5.26

Observe that the external force is periodic and during the first period, the external force looks like the following,

$$r(t) = \begin{cases} f & t \in [0, \frac{\tau}{2}) \\ -f & t \in [\frac{\tau}{2}, \tau) \end{cases}$$

Hence, force as a function of time can be given as

$$r(t) = r(t + \tau)$$

We have assumed that the directions towards right are positive. Let $x(0) = 0$ be the initial displacement and $x'(0) = 0$ be the initial velocity, where $x(t)$ represents the displacement. According to our sign convention, the force exerted by the spring on mass *m* is always in the negative direction.

Hence the DE which governs the motion is

$$m\frac{d^2x}{dt^2} = -kx + r(t)$$

We can write this equation as

$$\frac{d^2x}{dt^2} + \omega^2 x = q(t)$$

where

$$q(t) = \frac{1}{m} r(t)$$

and

$$\omega = \sqrt{\frac{k}{m}}$$

Take the Laplace transform on both sides. Hence

$$s^2 X(s) + \omega^2 X(s) = Q(s)$$

$$X(s) = \frac{Q(s)}{s^2 + \omega^2}$$

Taking the Inverse Laplace transform of this equation, we get

$$x(t) = \mathcal{L}^{-1}\left\{\frac{Q(s)}{s^2 + \omega^2}\right\} = \frac{1}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\}\mathcal{L}^{-1}\{Q(s)\}$$

$$= \frac{1}{\omega}\sin(\omega t)\, q(t)$$

$$x(t) = \frac{1}{m\omega}\sin(\omega t)\, r(t)$$

Problem 5.5.27

$$f(t) = \begin{cases} \cos 2t & t \in [0,2\pi] \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned}
f(t) &= \big(1 - u(t-2\pi)\big)\cos 2t \\
&= \cos 2t - u(t-2\pi)\cos 2(t-2\pi) \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{\cos 2t - u(t-2\pi)\cos 2(t-2\pi)\} \\
&= \mathcal{L}\{\cos 2t\} - \mathcal{L}\{u(t-2\pi)\cos 2(t-2\pi)\} \\
&= \frac{s(1-\exp(-2\pi s))}{s^2+4}
\end{aligned}$$

$$\mathcal{L}\{x'' + 4x\} = \mathcal{L}\{f(t)\}$$

$$s^2X(s) - sx(0) - x'(0) + 4X(s) = \frac{s(1-\exp(-2\pi s))}{s^2+4}$$

$$X(s)(s^2+4) = \frac{s(1-\exp(-2\pi s))}{s^2+4}$$

$$X(s) = \frac{s(1-\exp(-2\pi s))}{(s^2+4)^2}$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s(1-\exp(-2\pi s))}{(s^2+4)^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\sin 2t$$

$$\begin{aligned}
x(t) &= \frac{t}{4}\sin 2t \; - u(t-2\pi)\frac{1}{4}(t-2\pi)\sin 2(t-2\pi) \\
&= \begin{cases} \frac{1}{4}t\sin 2t & t \in [0, 2\pi] \\ \frac{1}{2}\pi\sin 2t & \text{o.w.} \end{cases}
\end{aligned}$$

Problem 5.5.28

By setting the naturally spot at 0 of x-axis, we can write the system as

$$mx'' = F - (k_1 + k_2)x$$

Applying Laplace transforms on both sides,

$$m\left(s^2X(s) - sx(0) - x'(0)\right) = \mathcal{L}\{F\} - (k_1 + k_2)X(s)$$

The force can be expressed as

$$F = f\left(2\sum_{n=0}^{\infty}(-1)^n u\left(t - \frac{\tau}{2}n\right) - u(t)\right)$$

$$\mathcal{L}\{F\} = \frac{f}{s}\tanh\left(\frac{\tau s}{4}\right)$$

With I.C., $x(0) = x'(0) = 0$, simplify the DE,

$$ms^2X(s) = \frac{f}{s}\tanh\left(\frac{\tau s}{4}\right) - (k_1 + k_2)X(s)$$

$$X(s) = \frac{f}{s}\frac{\tanh\left(\frac{\tau s}{4}\right)}{ms^2 + k_1 + k_2}$$

The solution is $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

Problem 5.5.29

The system of DEs

$$mx_1'' = k(x_2 - 2x_1) + f_A(t)$$
$$mx_2'' = k(x_1 - 2x_2) + f_B(t)$$

Defining $w = \sqrt{\frac{k}{m}}$, we get

The system becomes

$$x_1'' = w^2(x_2 - 2x_1) + \frac{f_A(t)}{m}$$
$$x_2'' = w^2(x_1 - 2x_2) + \frac{f_B(t)}{m}$$

Applying Laplace transform on the above system yields

$$(s^2 + 2\mathrm{w}^2)X_1 = w^2X_2 + F_A(s)$$
$$(s^2 + 2\mathrm{w}^2)X_2 = w^2X_1 + F_B(s)$$

where

$$F_A(s) = \mathcal{L}\left\{\frac{f_A(t)}{m}\right\}$$

$$F_B(s) = \mathcal{L}\left\{\frac{f_B(t)}{m}\right\}$$

Solving the above DEs, we get

$$\begin{aligned}
X_1 &= \frac{F_A(s^2+2w^2)+F_Bw^2}{s^4+4w^2s^2+3w^2} \\
&= \frac{F_As^2+(2F_A+F_B)w^2}{(s^2+w^2)(s^2+3w^2)} \\
&= \frac{F_A+F_B}{2}\frac{1}{s^2+w^2}+\frac{F_A-F_B}{2}\frac{1}{s^2+3w^2} \\
X_2 &= \frac{F_Aw^2+F_B(s^2+2w^2)}{s^4+4w^2s^2+3w^2} \\
&= \frac{F_Bs^2+(2F_B+F_A)w^2}{(s^2+w^2)(s^2+3w^2)} \\
&= \frac{F_A+F_B}{2}\frac{1}{s^2+w^2}-\frac{F_A-F_B}{2}\frac{1}{s^2+3w^2}
\end{aligned}$$

Then we have

$$\begin{aligned}
x_1 &= \mathcal{L}^{-1}\left\{\frac{F_A+F_B}{2}\frac{1}{s^2+w^2}+\frac{F_A-F_B}{2}\frac{1}{s^2+3w^2}\right\} \\
&= \frac{1}{2w}\mathcal{L}^{-1}\{F_A+F_B\}\otimes\sin wt+\frac{1}{2\sqrt{3w}}\mathcal{L}^{-1}\{F_A-F_B\} \\
&\qquad\otimes\sin\sqrt{3}\,wt \\
&= \frac{1}{2mw}\{f_A(t)+f_B(t)\}\otimes\sin wt+\frac{1}{2\sqrt{3}mw}\{f_A(t)-f_B(t)\} \\
&\qquad\otimes\sin\sqrt{3}\,wt \\
x_2 &= \mathcal{L}^{-1}\left\{\frac{F_A+F_B}{2}\frac{1}{s^2+w^2}-\frac{F_A-F_B}{2}\frac{1}{s^2+3w^2}\right\} \\
&= \frac{1}{2w}\mathcal{L}^{-1}\{F_A+F_B\}\otimes\sin wt-\frac{1}{2\sqrt{3w}}\mathcal{L}^{-1}\{F_A+F_B\} \\
&\qquad\otimes\sin\sqrt{3}\,wt \\
x_2 &= \frac{1}{2mw}\{f_A(t)+f_B(t)\}\otimes\sin wt-\frac{1}{2\sqrt{3}mw}\{f_A(t)-f_B(t)\} \\
&\qquad\otimes\sin\sqrt{3}\,wt
\end{aligned}$$

The above convolutions are trivial to compute.

Problem 5.5.30

Two driving forces can be expressed as following (Here f_0 is constant).

$$f_A(t) = f_0\left(u(t) - u\left(t - \frac{\tau}{2}\right)\right)$$
$$f_A(t) = f_0\left(u\left(t - \frac{\tau}{2}\right) - u(t - \tau)\right)$$

Now, we transform the DEs through a Laplace transform.

$$mx_1'' = -kx_1 + k(x_2 - x_1) + f_0\left(u(t) - u\left(t - \frac{\tau}{2}\right)\right)$$
$$mx_2'' = -k(x_2 - x_1) + f_0\left(u\left(t - \frac{\tau}{2}\right) - u(t - \tau)\right)$$

$$m\left(s^2X_1(s) - sx_1(0) - x_1'(0)\right)$$
$$= -2kX_1(s) + kX_2(s) + f_0\left(\frac{1}{s} - \frac{\exp\left(-\frac{\tau s}{2}\right)}{s}\right)$$

$$m\left(s^2X_2(s) - sx_2(0) - x_2'(0)\right)$$
$$= kX_1(s) - kX_2(s) + f_0\left(\frac{\exp\left(-\frac{\tau s}{2}\right)}{s} - \frac{\exp(-\tau s)}{s}\right)$$

Substitute I.C.

$$x_1(0) = x_2(0) = 0$$
$$x_1'(0) = x_2'(0) = 0$$

$$(ms^2 + 2)X_1(s) - kX_2(s) = f_0\left(\frac{1}{s} - \frac{\exp\left(-\frac{\tau s}{2}\right)}{s}\right)$$

$$-\mathrm{kX}_1(\mathrm{s}) + (ms^2 + k)X_2(s) = f_0\left(\frac{\exp\left(-\frac{\tau s}{2}\right)}{s} - \frac{\exp(-\tau s)}{s}\right)$$

$$X_1(s) = \frac{\frac{f_0}{s}\left(ms^2 + k - ms^2 \exp\left(-\frac{\tau s}{2}\right) - k\exp(-\tau s)\right)}{(ms^2 + k)(ms^2 + 2k) - k^2}$$

$$X_2(s) = \frac{\frac{f_0}{s}\left(k + (ms^2 + k)\exp\left(-\frac{\tau s}{2}\right) - (ms^2 + 2k)\exp(-\tau s)\right)}{(ms^2 + k)(ms^2 + 2k) - k^2}$$

After performing inverse Laplace transform, we get the solution.

$$x_1(t) = \mathcal{L}^{-1}\{X_1(s)\}$$
$$x_2(t) = \mathcal{L}^{-1}\{X_2(s)\}$$

Problem 5.5.31

Applying Laplace Transform the both equations:

$$ms^2 X_1(s) = -2kX_1(s) + kX_2(s) + f_0 \exp(-t_0 s)$$
$$ms^2 X_2(s) = kX_1(s) - kX_2(s) + f_0 \exp(-2t_0 s)$$

Writing these in matrix and using Gaussian Elimination to solve for $X_2(s)$.

$$\begin{bmatrix} s^2 + \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & s^2 + \frac{k}{m} \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{f_0}{m}\exp(-t_0 s) \\ \frac{f_0}{m}\exp(-2t_0 s) \end{bmatrix}$$

$$\begin{bmatrix} s^2 + \frac{2k}{m} & -\frac{k}{m} \\ 0 & \left(s^2 + \frac{k}{m}\right)\left(s^2 + \frac{2k}{m}\right) - \frac{k^2}{m^2} \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} \frac{f_0}{m}\exp(-t_0 s) \\ \left(s^2 + \frac{2k}{m}\right)\frac{f_0}{m}\exp(-2t_0 s) + \frac{kf_0}{m^2}\exp(-t_0 s) \end{bmatrix}$$

$$\left(s^2 + \frac{3 - \sqrt{5}}{2}\frac{k}{m}\right)\left(s^2 + \frac{3 + \sqrt{5}}{2}\frac{k}{m}\right)X_2(s) = \left(s^2 + \frac{2k}{m}\right)\frac{f_0}{m}\exp(-2t_0 s) + \frac{kf_0}{m^2}\exp(-t_0 s)$$

$$X_2(s) = \frac{\left((s^2 + \omega_1^2 + \omega_2^2)\frac{f_0}{m}\exp(-2t_0 s) + \frac{kf_0}{m^2}\exp(-t_0 s)\right)}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$$

$$= \frac{\frac{f_0}{m}\exp(-2t_0 s)}{s^2 + \omega_2^2}$$

$$+ \frac{\omega_2^2 \frac{f_0}{m}\exp(-2t_0 s) + \frac{k}{m^2} f_0 \exp(-t_0 s)}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$$

$$x_2(t) = \frac{1}{\omega_2}\frac{f_0}{m}\sin\omega_2 t \otimes \delta(t - 2t_0)$$

$$+ \frac{1}{\omega_1\omega_2}\left(\omega_2^2 \frac{f_0}{m}\ \delta(t - 2t_0) + \frac{k}{m^2} f_0 \delta(t - t_0)\right)$$

$$\otimes (\sin\omega_1 t \otimes \sin\omega_2 t)$$

where we have substituted

$$\omega_{1,2}^2 = \frac{3 \mp \sqrt{5}}{2}\frac{k}{m}$$

above, and the substitution

$$\omega_1^2 + \omega_2^2 = \frac{2k}{m}$$

Now solve for $X_1(s)$.

$$\left(s^2 + \frac{2k}{m}\right) X_1(s) - \frac{k}{m} X_2(s) = \frac{f_0}{m}\exp(-t_0 s)$$

$$X_1(s) = \frac{k}{m\left(s^2 + \frac{2k}{m}\right)}\left(\frac{\frac{f_0}{m}\exp(-2t_0 s)}{s^2 + \omega_2^2}\right.$$

$$+ \frac{\omega_2^2 \frac{f_0}{m}\exp(-2t_0 s) + \frac{k}{m^2} f_0 \exp(-t_0 s)}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$$

$$\left. + \frac{f_0}{k}\exp(-t_0 s)\right)$$

$$x_1(t) = \frac{k}{m}\frac{f_0}{m}\delta(t - 2t_0) \otimes \frac{1}{\omega_3}\sin\omega_3 t \otimes \frac{1}{\omega_2}\sin\omega_2 t$$
$$+ \frac{k}{m}\left(\omega_2^2\frac{f_0}{m}\delta(t - 2t_0) + \frac{k^2}{m^2}f_0\delta(t - t_0)\right)$$
$$\otimes \frac{1}{\omega_1}\sin\omega_1 t \otimes \frac{1}{\omega_2}\sin\omega_2 t \otimes \frac{1}{\omega_3}\sin\omega_3 t$$
$$+ \frac{f_0}{m}\delta(t - t_0) \otimes \frac{1}{\omega_3}\sin\omega_3 t$$

where

$$\omega_3^2 = \frac{2k}{m} = \omega_1^2 + \omega_2^2$$

Problem 5.5.32

(1) Considering the spring forces acting on the two masses, we get

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2 \end{cases}$$

(2) Plug the parameters into the DEs, we get

$$\begin{cases} x_1'' = -3x_2 + 2x_2 \\ x_2'' = 2x_1 - 5x_2 \end{cases}$$

From here, we have many ways to solve the equation. For example, substitution or Laplace method can be used. Let us use substitution method.

From the second equation, we get

$$x_1 = \frac{1}{2}(x_2'' + 5x_2)$$

Plugging into the first equation, we get

$$x_2^{(4)} + 8x_2^{(2)} + 11x_2 = 0$$

whose C-Eq is $r_2^4 + 8r_2^2 + 11 = 0$ whose solutions are

$$r_2^2 = -4 \pm \sqrt{5}$$

Next, one can find all four solutions b_1, b_2, b_3, b_4

$$+\sqrt{-4 \pm \sqrt{5}}, \qquad -\sqrt{-4 \pm \sqrt{5}}$$

So that one can compose the final solution for m_1 and then for m_2

$$x_2 = C_1\, exp(b_1 t) + C_2\, exp(b_2 t) + C_3\, exp(b_3 t) + C_4\, exp(b_4 t)$$

One can find x_1 and then determine C_1, C_2, C_3, C_4 by I.C..

Appendix B
Laplace Transforms

Selected Laplace Transforms

$\boldsymbol{f(t) = L^{-1}\{F(s)\}}$	$\boldsymbol{F(s) = L\{f(t)\}}$
$\delta(t-a)$	$\exp(-as)$
$\delta^n(t)$	s^n
$u(t-a)$	$\dfrac{\exp(-as)}{s}$
t	$\dfrac{1}{s^2}$
t^2	$\dfrac{2!}{s^3}$
$t^n,\ n \geq 0$	$\dfrac{n!}{s^{n+1}}$

$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$t \sin \omega t$	$\frac{2s\omega}{(s^2 + \omega^2)^2}$
$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
$\exp(-at)$	$\frac{1}{s + a}$
$t^n \exp(-at)$	$\frac{n!}{(s + a)^{n+1}}$
$\exp(-at) \sin \omega t$	$\frac{}{(s + a)^2 + \omega^2}$
$\exp(-at) \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$

Selected Properties of Laplace Transforms

$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
$f(t)$	$F(s)$
$cf(t)$	$cF(s)$
$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$

$\dfrac{df(t)}{dt}$	$sF(s) - f(0)$
$\dfrac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1} f(0)$ $-s^{n-2} f'(0) - s^{n-3} f''(0)$ $- \cdots - f^{(n-1)}(0)$
$\int_0^t f(\tau) d\tau$	$\dfrac{F(s)}{s}$
$\exp(-at)\, f(t)$	$F(s+a)$
$f(t-\tau)u(t-\tau)$	$\exp(-s\tau)\, F(s)$
$f(t) \otimes g(t)$	$F(s) \cdot G(s)$
$tf(t)$	$-\dfrac{dF(s)}{ds}$
$t^n f(t)$	$(-1)^n F^n(s)$
$\dfrac{f(t)}{t}$	$\int_s^t F(s) d$
$f(ct), c > 0$	$\dfrac{1}{c} F\left(\dfrac{s}{c}\right)$

Remarks

(1) $\delta(t)$ is the Dirac delta-function which is defined as

$$\delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

and also satisfied

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

(2) $u(t)$ is the unit step function which is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

(3) $\otimes$ is the convolution operator which is defined as

$$f(t) \otimes g(t) = \int_0^t f(\tau) g(t - \tau) d\tau$$

Appendix C
Derivatives & Integrals

1. $\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$
2. $\frac{d}{dx}F\big(g(x)\big) = \frac{dF}{dg} \cdot \frac{dg}{dx}$ (Chain Rule)
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
5. $\frac{d}{dx}(\exp(x)) = \exp(x)$
6. $\frac{d}{dx}(\sin x) = \cos x$
7. $\frac{d}{dx}(\cos x) = -\sin x$
8. $\frac{d}{dx}(\tan u) = \sec^2 u$

9. $uv = \int udv + \int vdu \;\Rightarrow\; \int udv = uv - \int vdu$

   (Integration by Parts)

10. $\int \exp(x)\,dx = \exp(x) + C$

11. $\int x^n dx = \frac{1}{n+1} x^{n+1} + C \qquad (n \neq -1)$

12. $\int \frac{dx}{x} = \ln|x| + C$

13. $\int a^x du = \frac{a^x}{\ln a} + C$

14. $\int \sin x\,dx = -\cos x + C$

15. $\int \cos x\,dx = \sin x + C$

16. $\int \tan x\,dx = -\ln|\cos x| + C$

17. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

18. $\int \csc x\,dx = \ln|\csc x + \cot x| + C$

19. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$

20. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

21. $\int \frac{dx}{a^2 + x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

Appendix D
Abbreviations

Abbreviation	Meaning
C	Constant, or Const
C-Eq	Characteristic Equation
DE(s)	Differential Equation(s)
Eq	Equation
G.S.	General Solution(s)
I.C.	Initial Condition(s)
IVP	Initial Value Problem(s)
L.D.	Linearly Dependent
LHS	Left Hand Side

L.I.	Linearly Independent
O.W.	Otherwise
P.S.	Particular Solution(s)
RHS	Right Hand Side
WRT	With Respect To

Appendix E
Teaching Plans

I. Ten homework sets, three quizzes, and one final exam that will contribute to a total of 100 points for the entire semester. The usual partitioning of grades:

II. During the semester, you expect 9-10 homework sets, approximately one per week.

Item	Expected Time (Hours)	Points
10 HWs	2-3	20
Quiz 1	1.5	15
Quiz 2	1.5	15
Quiz 3	1.5	15
Final	2.5	35
Total		100

(1) You usually have one week for each assignment.
(2) All homework grades will be counted.
(3) No grade for late homework.
(4) No assignments are given for quiz weeks.

III. Four tests include one final and three in-class quizzes.

(1) All tests except the final will take place in the classroom.
(2) All tests are open-book: calculators, your own lecture notes, and textbook are allowed during all tests.
(3) The final exam is cumulative.
(4) Scores will be curved.

IV. Student partitioning for the class

- 20% Like it and need it
- 15% Like it but don ot need it
- 60% Do not like it but need it
- 05% Do not like it and do not need it

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- **Web**: http://www.ams.sunysb.edu/~deng

References

1. Introduction to Numerical Analysis by J. Stoer and R. Bulirsch (Springer-Verlag, 1980.)
2. Numerical Mathematics and Computing (2nd Ed.) by Ward Cheney and David Kincaid (Books/Cole Publishing Company, Pacific Grove, CA. 1985.)
3. Introduction to Numerical Analysis by F. B. Hilderbrand (Dover, 1987. ISBN: 0-486-65363-3)
4. Computer Methods for Mathematical Computations by Forsythe, Malcolm, and Moler.
5. Numerical Methods for Engineers and Computer Scientists by Paul F. Hultquist (Benjamin/Cummings, 1988.)
6. Applied Numerical Linear Algebra by William W. Hage (Prentice-Hall, N.J., 1988.)
7. Discrete Variable Methods in Ordinary DE by Peter Henrici (John Wiley \& Sons, New York, 1962.)
8. A Scientist's and Engineer's Guide to Workstations and Supercomputers by Rubin H. Landau and Paul J. Fink, Jr. (1995)
9. Numerical Initial Value Problems in Ordinary DE by C. William Gear (Prentice-Hall, N.J., 1971.)
10. Finite Difference Methods in Partial DE by Mitchell and Griffith

11. C. H. Edwards and D. E. Penney, Differential Equations and Boundary Value Problems (4th Ed, Prentice Hall, 2007)

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