AMS 361: Applied Calculus IV by Prof. Y. Deng

Short Test 3 Solution: 11/27/2018 Tuesday 5:30pm-6:50pm Frey 100

(1) Closed Book with 1-page (double-sided 8.5x11) self-prepared hand-written.
(2) Do any two of the three problems.
(3) If all three are attempted, the best two (and only two) will be credited.
(4) Each problem is worth 7.5 points for a total of 15 points (max).
(5) No points for solutions without appropriate intermediate steps.
(6) Partial credits are given only for steps that are relevant to the solutions.
(7) No name, no grade and no request will be answered.

SB ID		
Name		
Problems	Score	Remarks
T3-1		
T3-2		
T3-3		
Total Score		

T3-1 (7.5 Points): Use any method to find the GS of

$$y''' - y'' + y' - y = e^x + \cos x$$

Solution:
C-Eq is

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$

whose root is

$$\lambda_{1,2,3} = 1, \pm i$$

$$y_c(x) = C_1 e^x + C_2 \cos x + C_3 \sin x$$

One may also express

$$y_c(x) = C_1 e^x + C_2 e^{ix} + C_3 e^{-ix}$$

Given $f(x) = e^x + \cos x$, our TS is

$$y_p = Axe^x + Bx\cos x + Cx\sin x$$

$$y'_p = Ae^x + Axe^x + B\cos x - Bx\sin x + C\sin x + Cx\cos x$$

$$y''_p = 2Ae^x + Axe^x - 2B\sin x - Bx\cos x + 2C\cos x - C\sin x$$

$$y'''_p = 3Ae^x + Axe^x - 3B\cos x + Bx\sin x - 3C\sin x - Cx\cos x$$

$$y''' - y'' + y' - y = 2Ae^x - 2(C+B)\cos x + (2B - 2C)\sin x$$

$$= e^x + \cos x$$

$$A = \frac{1}{2}, B = -\frac{1}{4}, C = -\frac{1}{4}$$

$$y_p(x) = \frac{1}{2}xe^x - \frac{1}{4}x\cos x - \frac{1}{4}x\sin x$$

The GS is

$$y_{GS}(x) = C_1 e^x + C_2 \cos x + C_3 \sin x + \frac{1}{2} x e^x - \frac{1}{4} x \cos x - \frac{1}{4} x \sin x$$

If the following is used as the complementary solution,

$$y_c(x) = C_1 e^x + C_2 e^{ix} + C_3 e^{-ix}$$

we may select the TS

$$\begin{aligned} y_P(x) &= Axe^x + Bxe^{ix} + Cxe^{-ix} \\ &= x\left(Ae^x + Be^{ix} + Ce^{-ix}\right) \end{aligned}$$

Then,

$$\begin{aligned} y_P'(x) &= \left(Ae^x + Be^{ix} + Ce^{-ix}\right) + x\left(Ae^x + Bie^{ix} - Cie^{-ix}\right) \\ y_P''(x) &= 2\left(Ae^x + Bie^{ix} - Cie^{-ix}\right) + x\left(Ae^x - Be^{ix} - Ce^{-ix}\right) \\ y_P'''(x) &= 3\left(Ae^x - Be^{ix} - Ce^{-ix}\right) + x\left(Ae^x - Bie^{ix} + Cie^{-ix}\right) \end{aligned}$$

Now, we get

$$\begin{aligned} -y_P(x) &= -x\left(Ae^x + Be^{ix} + Ce^{-ix}\right) \\ y_P'(x) &= \left(Ae^x + Be^{ix} + Ce^{-ix}\right) + x\left(Ae^x + Bie^{ix} - Cie^{-ix}\right) \\ -y_P''(x) &= -2\left(Ae^x + Bie^{ix} - Cie^{-ix}\right) - x\left(Ae^x - Be^{ix} - Ce^{-ix}\right) \\ y_P'''(x) &= 3\left(Ae^x - Be^{ix} - Ce^{-ix}\right) + x\left(Ae^x - Bie^{ix} + Cie^{-ix}\right) \end{aligned}$$

Adding up the LHS and RHS, we get

$$\begin{aligned} y_P'''(x) - y_P''(x) + y_P'(x) - y_P(x) &= 2Ae^x - (2+2i)Be^{ix} + (-2+2i)Ce^{-ix} \\ &= e^x + \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix} \end{aligned}$$

Thus,

$$\begin{aligned} 2A &= 1 \\ -(2+2i)B &= \frac{1}{2} \\ (-2+2i)C &= \frac{1}{2} \\ A &= \frac{1}{2} \\ B &= -\frac{1}{8}(1-i) \\ C &= -\frac{1}{8}(1+i) \end{aligned}$$

Therefore, the PS is

$$\begin{aligned} y_P(x) &= \frac{1}{2}xe^x - \frac{1}{8}(1-i)xe^{ix} - \frac{1}{8}(1+i)xe^{-ix} \\ &= \frac{1}{2}xe^x - \frac{1}{4}x\cos x - \frac{1}{4}x \sin x \end{aligned}$$

Both solutions perfectly match.

T3-2 (7.5 Points): Use the eigen-method (4.5 Points) and another method of your choice (3.0 Points) to find the GS of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution:

Method 1 The E-method

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{pmatrix} = 0$$

$$\lambda_{1,2} = 3, -2$$

For $\lambda_1 = 3$, we have

$$\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} V_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda_2 = -2$, we have

$$\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} V_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The GS is

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

Method 2 Substitution method

From $y' = x - y$, we have

$$x = y' + y$$
$$(y' + y)' = 2(y' + y) + 4y$$
$$y'' - y' - 6y = 0$$
$$\lambda^2 - \lambda - 6 = 0$$
$$\lambda_{1,2} = 3, -2$$
$$y(t) = C_1 e^{3t} + C_2 e^{-2t}$$
$$y'(t) = 3C_1 e^{3t} - 2C_2 e^{-2t}$$
$$x(t) = y' + y = 4C_1 e^{3t} - C_2 e^{-2t}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

Both methods lead to the same solution.

T3-3 (7.5 Points) Use any method to find the GS of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^t \\ e^{2t} \end{pmatrix}$$

Solution:

Substitution method

$$x' = 3x - y + e^t$$
$$y' = x + y + e^{2t}$$

From the 2nd DE, we get

$$x = y' - y - e^{2t}$$

Plugging this into the 1st DE, we get

$$y'' - y' - 2e^{2t} = 3y' - 3y - 3e^{2t} - y + e^t$$

Or

$$y'' - 4y' + 4y = -e^{2t} + e^t$$

The Homo portion of the above DE has the following C-Eq

$$\lambda^2 - 4\lambda + 4 = 0$$

Resulting a complementary solution

$$y_c = C_1 e^{2t} + C_2 t e^{2t}$$

Since

$$f(t) = -e^{2t} + e^t$$

We select TS

$$\begin{aligned} y_p &= Ae^t + Bt^2 e^{2t} \\ y'_p &= Ae^t + 2Bte^{2t} + 2Bt^2 e^{2t} \\ y''_p &= Ae^t + 2Be^{2t} + 8Bte^{2t} + 4Bt^2 e^{2t} \\ Ae^t + 2Be^{2t} &= -e^{2t} + e^t \end{aligned}$$

Matching coefficients of like terms, we get

$$A = 1 \quad B = -\frac{1}{2}$$

$$\begin{aligned} y_p &= e^t - \frac{1}{2} t^2 e^{2t} \\ y &= y_c + y_p \\ y_{GS}(t) &= C_1 e^{2t} + C_2 t e^{2t} + e^t - \frac{1}{2} t^2 e^{2t} \end{aligned}$$

$$\begin{aligned} x(t) &= y' - y - e^{2t} \\ &= \left(C_1 e^{2t} + C_2 t e^{2t} + e^t - \frac{1}{2} t^2 e^{2t}\right)' - \left(C_1 e^{2t} + C_2 t e^{2t} + e^t - \frac{1}{2} t^2 e^{2t}\right) - e^{2t} \\ &= C_1 e^{2t} + C_2 e^{2t} + 2C_2 t e^{2t} - \frac{1}{2} t^2 e^{2t} - t e^{2t} - e^{2t} \end{aligned}$$

Thus,

$$\binom{x}{y} = C_1 \binom{1}{1} e^{2t} + C_2 \left[\binom{1}{0} + \binom{2}{1} t\right] e^{2t} + \binom{0}{1} e^t + \left[\binom{-1}{0} + \binom{-1}{0} t - \frac{1}{2}\binom{1}{1} t^2\right] e^{2t}$$

Reference to the Textbook (Deng-2018) is suggestive. Homework problems are not identical to those in the book but the solution methods are similar.