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Lectures for CH1 1st Order DEs

Lecture plan for this Chapter:

Lectures	Date	Contents	Sec.
Week01.1	01/24	Definitions of Differential Equations (DEs) Classification of DEs Direction Fields	1.1 1.2
Week01.2	01/24	Separable DE's	1.3
Week02.1	01/31	Method of Integrating Factors for 1st Linear DEs	1.4 1.5
Week02.2	02/02	Substitution methods: Polynomial sub	1.5 1.5.1
Week03.1	02/07	Sub for Homogeneous DEs	1.5.2
Week03.2	02/09	Sub for Bernoulli DEs	1.5.2
Week04.1	02/14	Riccati DEs	1.6
Week04.2	02/16	The Exact DEs	1.7

-------- Start of Lecture Week01.1 (1/24/2023) --------

(Spending ~20 minutes to discuss the Syllabus)

Basic formulas needed for this course (selected from textbook pp 551-552)

1. $e^{ix} = \cos x + i \sin x$
2. $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$
3. $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$
4. $e^{x} = \cosh x + \sinh x$
5. $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$
6. $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$
7. $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
8. $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
9. $\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$
10. $\frac{d}{dx}F\left(g(x)\right) = \frac{dF}{dg}\frac{dg}{dx}$ (Chain rule)
11. $\frac{d}{dx}\left(f(x)g(x)\right) = \frac{df}{dx}g + f\frac{dg}{dx}$ (Product rule)
12. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{1}{g^2}\left(\frac{df}{dx}g - f\frac{dg}{dx}\right)$ (Quotient rule)
13. $\frac{d}{dx}(x^n) = nx^{n-1}$ (Elementary power rule)
14. $\frac{d}{dx}(\ln x) = \frac{1}{x}$ (Logarithmic derivative)
15. $\frac{d}{dx}(e^x) = e^x$ (Exponential derivative)
16. $\frac{d}{dx}(\sin x) = \cos x$ (Sin derivative)
17. $\frac{d}{dx}(\cos x) = -\sin x$ (Cos derivative)
18. $\int udv = uv - \int vdu$
19. $\int e^x dx = e^x + C$
20. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
21. $\int \frac{dx}{x} = \ln|x| + C$
22. $\int a^x dx = \frac{a^x}{\ln a} + C$
23. $\int \sin x\, dx = -\cos x + C$
24. $\int \cos x\, dx = \sin x + C$

Ch1 First-Order Differential Equations (DEs)

Sec. 1.1 Many Definitions and Classification of DEs

(1) The general form of First-Order Differential Equations (1st.O.DEs)

$$\frac{dy}{dt} = f(t, y)$$

(2) dependent variable vs. in-dependent variable

$$y = f(t)$$

DV = dependent variable
IV = in-dependent variable

(3) ODEs vs. PDEs

# DV's→ #IV's↓	1	2+
1	ODEs	Systems of ODEs
2+	PDEs	Systems of PDEs

Let me again compose a few examples of determining the ODEs and PDEs:

E.g.1 ODE

$$\frac{dp}{dt} = \frac{1}{2}p - 450$$

E.g.2 System of ODEs

$$\begin{cases} \frac{dp_1}{dt} = a_{11}p_1 + a_{12}p_2 + b_1 \\ \frac{dp_2}{dt} = a_{21}p_1 + a_{22}p_2 + b_2 \end{cases}$$

where a_{ij} are called parameters.

E.g.3 PDE

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial x} = p(t, x)$$

E.g.4 System of PDEs

$$\begin{cases} \frac{\partial p_1}{\partial t} + \alpha \frac{\partial p_2}{\partial x} = a_{11}p_1(t, x) + a_{12}p_2(t, x) \\ \frac{\partial p_1}{\partial t} + \beta \frac{\partial p_2}{\partial x} = a_{21}p_1(t, x) + a_{22}p_2(t, x) \end{cases}$$

(4) First order DEs vs. higher (2nd and above) order DE's:

The order of a DE is determined by the highest order derivative of the DV (and sometimes we interchangeably call it unknown function.

Let me again compose a few examples of determining the orders of the following DEs.

Eg.1 1st-order (1st.O) DE:

$$x' + ax = 0$$
$$x' + ax^2 = 0$$

Eg.2 2nd-order (2nd.O) DE:

$$x'' + bx = 0$$
$$x'' + bx^5 = 0$$

Eg.3 nth-order (nth.O) DE:

$$x^{(n)} + bx'' + cx = 0$$
$$x^{(\mathrm{n})} + 5x^{(\mathrm{n}-1)} + x = 0$$

(5) Linear DEs vs. nonlinear DEs

DEs can also be classified as linear or nonlinear according to the linearity of the DVs regardless of the nature of the IVs. A DE that contains only linear terms for the DV or its derivative is a linear DE. Otherwise, a DE that contains at least one nonlinear term for the DV or its derivative(s) is a nonlinear DE.

Let me again compose a few examples of determining the linear and nonlinear DEs.

Linear	Nonlinear
$y'' + y = 0$	$y'' + y^2 = 0$
$y^{(n)} + x + y = 0$	$y^{(n)} + xy + y^3 = 0$
$y^{(n)} + x^3 + y = 0$	$\left(y^{(n)}\right)^2 + x^3 + y = 0$

-------- Start of Lecture Week01.2 (1/26/2023) --------

Goals for today:

1. Definition of solutions and types of solutions
2. One simple example (and its "cousin") in engineering
3. (Analytical) methods of solving first order DE's
    a. Separation of variables
    b. Integrating factors for 1st order linear DE's.

(6) Definition of solution:

A function that satisfies the DE.

(7) Solution methods and solution types of DEs

Solution Types:

Solution Type 1: General solution = GS

Solution Type 2: Particular solution = PS

Solution Type 3: Singular solution = SS

Example 1: an object falling in the atmosphere with constant air resistance, we can write its eq. of motion as

$$m\frac{dv}{dt} = mg - \gamma v$$

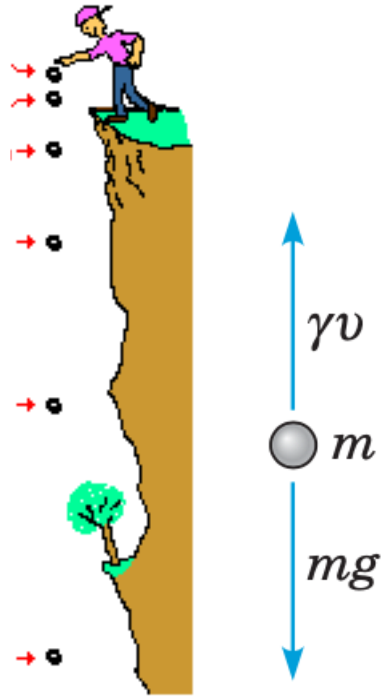

Figure 1. An object is dropped (left) and forces (right) on the object mass "m": the gravity $\boldsymbol{mg}$ pointing down and air resistance $\boldsymbol{\gamma v}$ pointing up (opposite to the direction of motion).

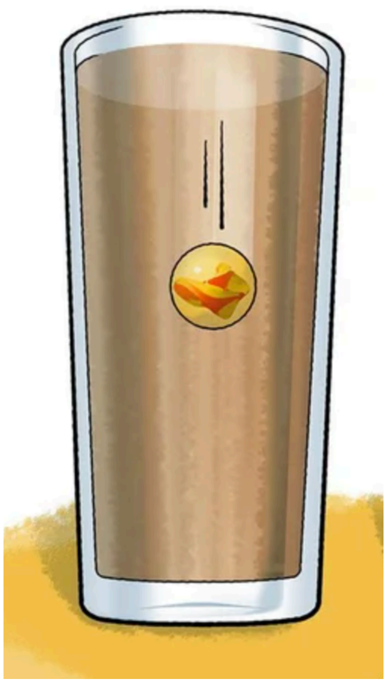

Figure 2. A marble is dropped and forces on the marble: the gravit pointing down and liquid resistance pointing up. One can easily three conditions by adjusting the liquid type (e.g., density) or by observing at different phases of the marble dropping: gravity>resistant, gravity<resistant, gravity=resistant.

(Note: This problem will be studied much more rigorously in CH2. I'm only scratching its skin now)

Before we get to the details of the **theories** and solution methods, let's play with this DE a little!

For a special case $m = 10\ kg$; $\gamma = 2\ kg/s$, and we all know $g = 9.8\ m/s^2$, we can write the DE as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

This is a simple DE comparing with a general form of 1st order DE:

$$\frac{dv}{dt} = f(t, v)$$

As you notice,

$$f(t, v) = 9.8 - \frac{v}{5}$$

does not have explicit dependance of "t". In this DE, v is the DV (dependent variable) and t IV (independent variable).

BTW, this type of DE has a good name: autonomous DEs (which have profound significance in math and physics, a different story by its own.)

The following is some geometric sketches to help solve a DE.

Now, let's plot the *direction field* (helping us to sketch the solution):

$$v = 40, \qquad \frac{dv}{dt} = 1.8$$
$$v = 50, \qquad \frac{dv}{dt} = -0.2$$
$$v = 60, \qquad \frac{dv}{dt} = -2.2$$
$$v = 49, \qquad \frac{dv}{dt} = 0$$

You can see the line segments indicating the "direction" or "slopes" at that give t-v point:

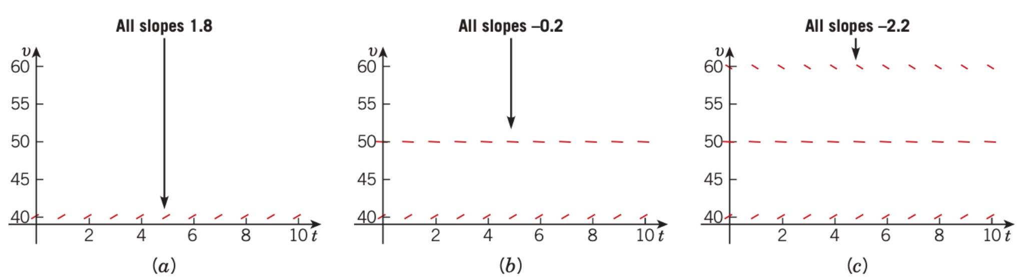

Figure 3. Slope fields at 3 different speeds v=40, 50, 60 for any time.

The special case $v = 49$, $\frac{dv}{dt} = 0$ for which $v(t) = 49$ is the so-called **equilibrium solution.** Now, let's have a more complete **direction field** and the equilibrium solution in one figure:

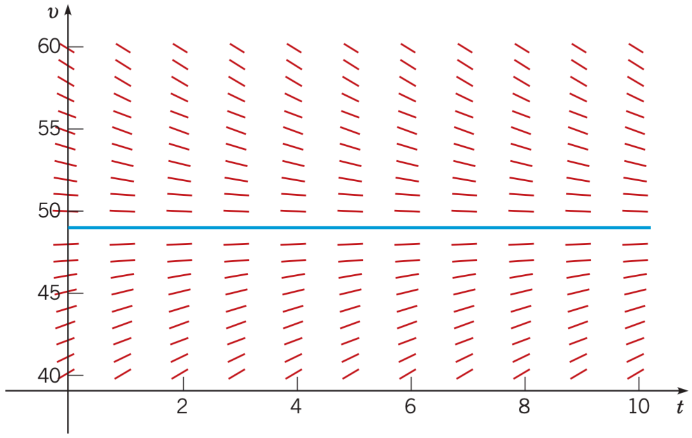

Figure 4. Full slope field with one equilibrium solution (blue).

Now, let's make a summary on how the solution show look like:

$v(t) < 49,\ v'(t) > 0,\ v(t)$ increases with time, this is because when the overall speed $v(t) < 49$, the gravity is bigger than the air resistance and the object has a net downward force and thus dropping accelerates.

$v(t) > 49,\ v'(t) < 0,\ v(t)$ decreases with time, this is because when the overall speed $v(t) > 49$, the gravity is smaller than the air resistance and the object has a net upward force and thus dropping decelerates. (But the object still falls).

$v(t) = 49,\ v'(t) = 0,\ v(t)$ does not with time, this is because when the overall speed $v(t) = 49$, the gravity precisely balances the air resistance and the net force on object is zero thus the object drops at a constant. Thinking about dropping marble ball in a tall tube of liquid (e.g., water).

In fact, the solution is

$$v(t) = c_1 e^{-\frac{t}{5}} + 49$$

Still, we have things undefined: (1) solution; (2) constant c_1.

Let's play with a few initial cases with given initial condition (IC):

Case 1: $v(t = 0) = 0$

Case 1: If the object starts from rest, i.e., $v(t=0)=0$.

From the general solution (GS):

$$v(t) = c_1 e^{-\frac{t}{5}} + 49$$

Applying the initial condition (IC) $v(t=0)=0$, we get

$$0 = c_1 e^{-\frac{0}{5}} + 49$$

or

$$c_1 = -49$$

Then, the solution (which we call particular solution: PS):

$$v(t) = -49 e^{-\frac{t}{5}} + 49$$

The solution graphic profile:

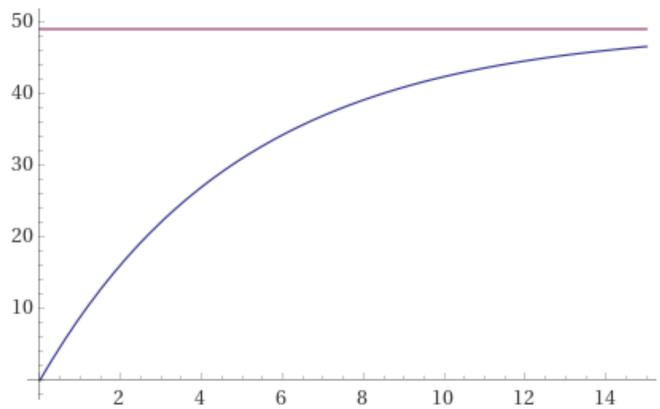

Figure 5. Speed $v(t)$ as function of time t with $v(0)=0$.

This is a very interesting curve:

1. The speed of the falling object increases from its initial zero.
2. When the speed reaches a critical point (in this case, it's $v_T = 49$), the speed stays constant.

These make perfect sense!

Case 2: If the object starts moving downward (You add a little speed to drop it). Let's assume $v(t = 0) = 9 > 0$,

Applying the initial condition (IC) $v(t = 0) = 9$, we get

$$9 = c_1 e^{-\frac{0}{5}} + 49$$

or

$$c_1 = 9 - 49 = -40$$

Then, the solution (which we call particular solution: PS):

$$v(t) = -40e^{-\frac{t}{5}} + 49$$

The solution graphic profile:

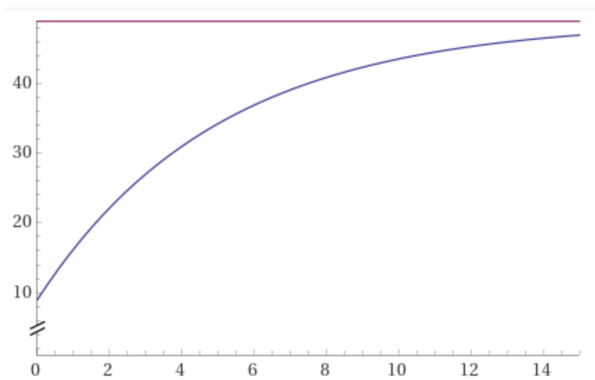

Figure 6. Speed $v(t)$ as function of time t with $v(0) = 9$.

Case 3: If the object starts moving upward (You add a little speed to throw it straight up), i.e., $v(t=0)<0$ because we made the convention, the velocity is positive when it goes down.

Let's assume $v(t=0)=-9$,

Applying the initial condition (IC) $v(t=0)=-9$, we get

$$-9 = c_1 e^{-\frac{0}{5}} + 49$$

or

$$c_1 = -9 - 49 = -58$$

Then, the solution (which we call particular solution: PS):

$$v(t) = -58e^{-\frac{t}{5}} + 49$$

The solution graphic profile:

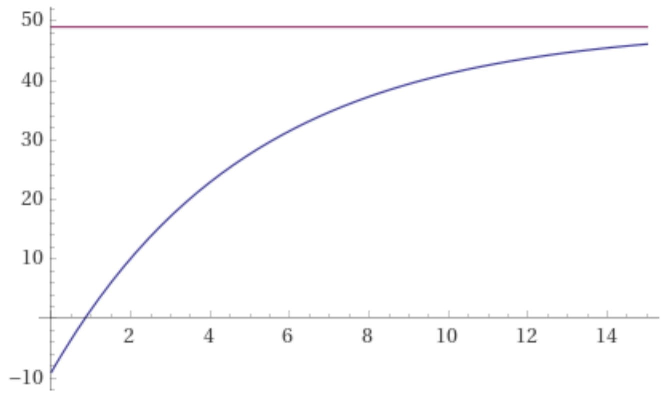

Figure 7. Speed $v(t)$ as function of time t with $v(0)=-9$.

In this Case 3, the velocity changes sign at $t\sim 1$ as it first goes up and then goes down.

Remark: In all 3 cases, you notice one interesting property: regardless of initial speed, the final speed is always 49. Now, let's examine the general solution

$$v(t) = c_1 e^{-\frac{t}{5}} + 49$$

Taking the limit $t\to\infty$, we get

$$\begin{aligned}\lim_{t\to\infty} v(t) &= \lim_{t\to\infty}\left(c_1 e^{-\frac{t}{5}} + 49\right)\\ &= 0 + 49\\ &= 49\end{aligned}$$

We will study this problem much more rigorously.

Solution Methods:

Solution Method 1: Analytical methods (the bulk of our lectures) for exact/analytical solutions

Solution Method 2: Series methods (another course) for approximating solutions

Solution Method 3: Numerical methods (another course) for approximating solutions

Sec. 1.2 Six (Analytical) Solution Methods.

Note: There are a dozen methods for a dozen different types of DEs!

Method 1: Method of Separable DEs

The **general form** of First Order DEs

$$\frac{dy}{dx} = f(x, y)$$

can also take the following form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

If M and N contains on variable each, i.e.,

function $M(x, y)$ is a function of a single variable $M(x)$
function $N(x, y)$ is a function of a single variable $N(y)$

$$M(x) + N(y)\frac{dy}{dx} = 0$$

It's called **Separable DE** and it can be solved by integrating individually. This is the most elementary method.

Of course, the following will also lead to **Separable DEs:**

function $M(x, y)$ is a function of a single variable $M(y)$
function $N(x, y)$ is a function of a single variable $N(x)$

Note: All you need is to make the functions $M(x, y)$ and $N(x, y)$ in the general 1st order DE single-variable functions.

Example 1 Solve DE

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

Solution:

Note: This is a NON-linear DE but separable!

The DE becomes

$$(1-y^2)dy = x^2 dx$$

Integrating both sides, we get

$$\int (1-y^2)dy = \int x^2 dx \tag{1}$$

Thus,

$$y - \frac{y^3}{3} + C_1 = \frac{x^3}{3} + C_2$$

$$x^3 + y^3 - 3y = C$$

This is so-called implicit general solution (GS). The earlier solutions we obtained were all explicit solution where you can express it as $y = f(x, C)$.

Note: Writing your solution in explicit form is unnecessary (and very often impossible).

Direction field and integral curves are:

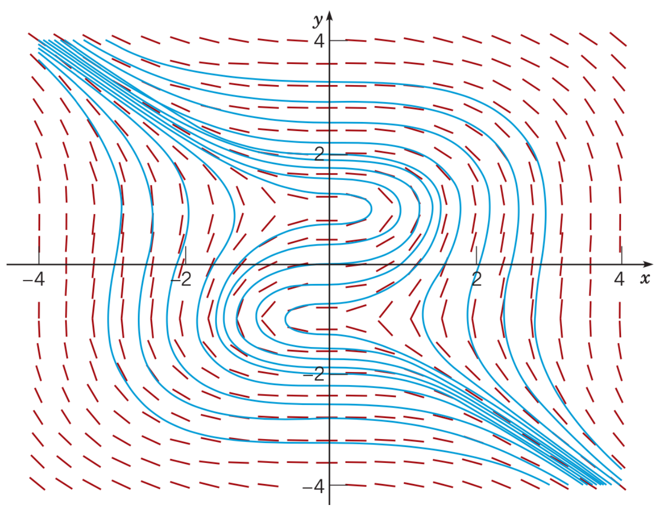

Figure 8. The slope field (red) and solution curves (blue).

Example 2 Solve DE

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Solution:

Note: This is a NON-linear DE but separable!

$$2(y-1)dy = (3x^2 + 4x + 2)dx$$

Integrating both sides, we get

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

which is the GS of the DE in implicit form!

The integral curves are:

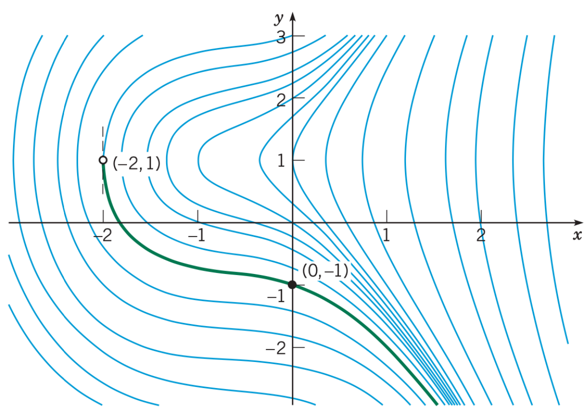

Figure 9. The solution curves with one PS (Green) for IC: $y(x=0) = -1$.

For a given IC, you can identify one curve (PS) from the family of curves (GS).

Example 3 Solve DE

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

Solution:

Note: This is a NON-linear DE but separable!

Rewriting the DE, we get

$$(4 + y^3)dy = (4x - x^3)dx$$

Integrating both sides, we get

$$y^4 + 16y + x^4 - 8x^2 = c,$$

The integral curves are:

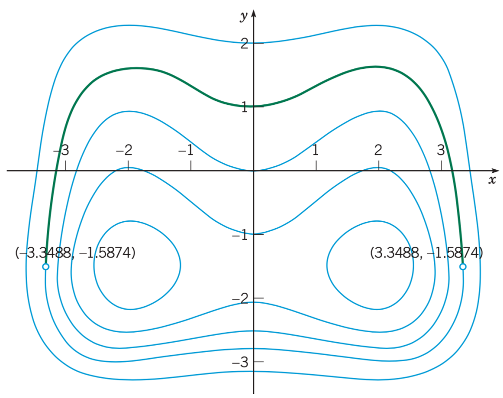

Figure 10. The solution curves with one PS (Green) for IC: $y(x = 0) = 1$.

Important remarks about this method:

Remark 1: Another form of writing the separable DE

$$\frac{dy}{dx} = P(x)Q(y)$$

In order to solve it, you write it as

$$\frac{dy}{Q(y)} = P(x)dx$$

In fact, this conversion (dividing by $Q(y)$) is not entirely mathematically legal! It breaks math when $Q(y) = 0$ at which

$$y = y_0 = \text{constant}$$

Because if

$$y = \text{constant}$$
$$\frac{dy}{dx} = 0$$

This $y = y_0$ is a solution and this solution is singular!

Example 4 Solve DE

$$x^2y' + y^2 = 0$$

Solution:

Note: This is a NON-linear DE but separable!

$$\frac{dy}{y^2} = -\frac{dx}{x^2}$$

Integrating both sides, we get (Note: I wrote the constant in such a format on purpose)

$$-\frac{1}{y} = \frac{1}{x} - \frac{1}{C}$$

Or we write the GS as

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{C}$$

It turns out $y = 0$ is also a solution (the SS) because it satisfies the DE.

The slope field and solution curves:

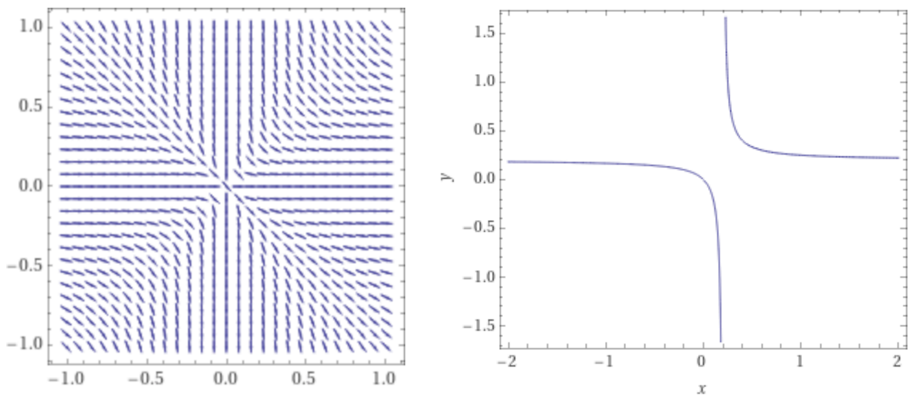

Figure 11. The slope field (left) and solution curves (right).

*** End of Method 1: Method of Separable DEs***

-------- Start of Lecture Week02.1 (1/31/2023) --------

Method 2: Method of Integrating Factors for 1st Order Linear DEs

The general form of First-Order Linear DEs (**1st.O.L DEs**)

$$A(x)\frac{dy}{dx}+B(x)y=C(x) \tag{1}$$

where

$$A(x)\neq 0$$

Otherwise, there is no DE!

Thus, we can define

$$P(x)=\frac{B(x)}{A(x)}$$
$$Q(x)=\frac{C(x)}{A(x)}$$

Thus, it can be simplified as

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{2}$$

Two cases:

(1) Constant coefficients

$$\frac{dy}{dx}+ay=b$$

(2) Variable coefficients

$$y'+P(x)y=Q(x)$$

Now, we derive the (**Super important**) Method of Integrating Factors (IF). This method is used throughout the entire DE solutions, widely!

The key idea is to absorb the $P(x)y$ term to a whole derivative by using an IF (Integrating Factor) $\rho(x)$ whose precise composition will be introduced shortly.

$$\rho(x)(y' + P(x)y) = Q(x)\rho(x)$$
$$\rho(x)y' + \rho(x)P(x)y = Q(x)\rho(x)$$

If we can make the LHS of the above

$$\rho(x)y' + \rho(x)P(x)y = \frac{d}{dx}(\rho(x)y)$$

a whole derivative, then the DE can be solved easily in terms of $\rho(x)y$.

We have

$$\rho(x)P(x)y = \rho'(x)y$$
$$\rho(x)P(x) = \rho'(x)$$

To find $\rho(x)$, we do the following

$$\frac{d\rho}{\rho} = P(x)dx$$
$$\int \frac{d\rho}{\rho} = \int P(x)dx$$
$$\ln \rho = \int P(x)dx$$

Therefore, we have

$$\rho(x) = \mathrm{e}^{\int P(x)dx + C_1} = C\mathrm{e}^{\int P(x)dx}$$

Since this function $\rho(x)$ can be multiplied by any non-zero constant to serve as a factor, with no loss of generality while retaining simplicity, we set $C = 1$. The resulting factor

$$\rho(x) = \mathrm{e}^{\int P(x)dx}$$

is called the integrating factor (IF) of DE.

$$\frac{d}{dx}(\rho(x)y) = Q(x)\rho(x)$$

i.e.,

$$d(\rho(x)y) = Q(x)\rho(x)dx$$

Integrating both sides, we get

$$\rho(x)y = \int Q(x)\rho(x)dx$$

Thus, the GS is

$$y(x) = \frac{1}{\rho(x)}\left(\int Q(x)\rho(x)dx + C\right)$$

Replacing the IF $\rho(x)$ by the given function $P(x)$, we find the GS to the general 1st order linear DE as

$$y(x) = \mathrm{e}^{-\int P(x)dx}\left(\int Q(x)\mathrm{e}^{\int P(x)dx}dx + C\right)$$

In fact, we can make many remarks for this GS.

Remark 1: If $P(x) = 0$, the original DE becomes

$$\frac{dy}{dx} + 0 = Q(x)$$

whose GS is easily obtained as

$$y_G(x) = \int Q(x)dx + C$$

Both the solution formula and quick solution are consistent.

Remark 2: If $Q(x) = 0$, the original DE becomes

$$\frac{dy}{dx} + P(x)y = 0$$

whose GS is easily obtained as

$$\begin{aligned} y_G(x) &= \mathrm{e}^{-\int P(x)dx}\left(\int 0 * \mathrm{e}^{\int P(x)dx}dx + C\right) \\ &= \mathrm{e}^{-\int P(x)dx}(0 + C) \\ &= C\mathrm{e}^{-\int P(x)dx} \end{aligned}$$

as expected, when we solve it directly.

Remark 3: If $P(x) = Q(x) = 0$, the original DE becomes

$$\frac{dy}{dx} = 0$$

Well, we can easily solve this DE,

$$y_G(x) = C$$

Using the formula, we get

$$\begin{aligned} y_G(x) &= \mathrm{e}^{-\int 0dx}\left(\int 0 * \mathrm{e}^{\int 0dx}dx + C\right) \\ &= 1(0 + C) \\ &= C \end{aligned}$$

This special case works too!

Example 5 Solve DE

$$(t^2+4)\frac{dy}{dt}+2ty=4t$$

Solution:

Note: In this example, my IV is "t" not the usual "x" for some historical reasons. This should not impact your understanding of the solution methods and I'll not remind you of such alternations in the future.

Using product rule for the LHS (for simple cases, this is convenient), we get

$$\frac{d}{dt}[(t^2+4)y]=4t$$

Or

$$d[(t^2+4)y]=4tdt$$

Integrate both sides (using indefinite integral)

$$\int d[(t^2+4)y]=\int 4tdt$$

Thus,

$$(t^2+4)y+C_1=2t^2+C_2$$

Merging the constants, we get

$$(t^2+4)y=2t^2+C$$

Thus,

$$y_G(t)=\frac{2t^2+C}{t^2+4}$$

which is the GS of the DE with a constant C to be determined by some other (initial) condition.

In general, we must use the IF method (of course, they lead to the same solution by, almost, the same procedure):

The DE is converted to

$$\frac{dy}{dt}+\frac{2t}{t^2+4}y=\frac{4t}{t^2+4}$$

Thus,

$$P(t)=\frac{2t}{t^2+4}$$
$$Q(t)=\frac{4t}{t^2+4}$$

Thus,

$$\begin{aligned}\rho(t)&=\mathrm{e}^{\int P(t)dt}\\&=\mathrm{e}^{\int\frac{2t}{t^2+4}dt}\\&=t^2+4\end{aligned}$$

You multiply back the converted DE (noticing you return to the original DE; this is an exception not the common case!)

$$(t^2+4)\frac{dy}{dt}+2ty=4t$$

Now, everything is history (already done before :=)

A bit more analysis of the solution:

Slope field:

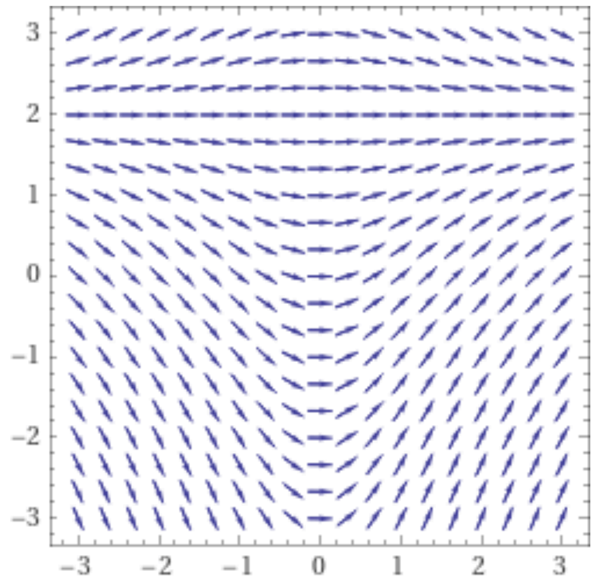

Figure 12. The slope fields.

Solution family:

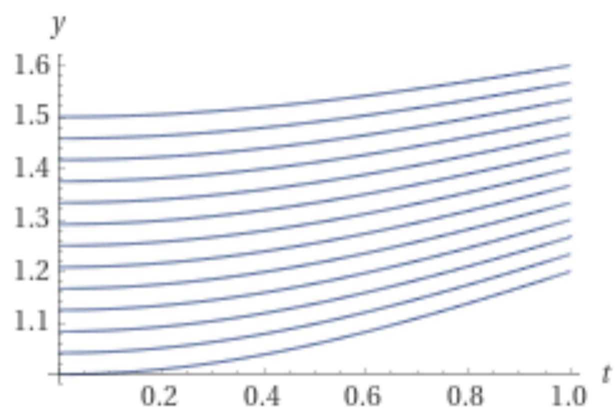

Figure 13. The solution family with different IC's.

Example 6 Solve DE

$$\frac{dy}{dt}+\frac{1}{2}y=\frac{1}{2}e^{\frac{t}{3}}$$

Solution:
Using the Method of Integrating Factors, we first find the IF

$$\begin{aligned}\rho(t)&=\mathrm{e}^{\int P(t)dt}\\&=\mathrm{e}^{\int\frac{1}{2}dt}\\&=\mathrm{e}^{\frac{1}{2}t}\end{aligned}$$

Thus, using the formula, we get the GS as

$$\mathrm{e}^{\frac{1}{2}t}y(t)=\frac{3}{5}\mathrm{e}^{\frac{5}{6}t}+C$$

Or

$$y(t)=\frac{3}{5}\mathrm{e}^{\frac{1}{3}t}+C\mathrm{e}^{-\frac{1}{2}t}$$

The slope field and integral curves (solution family) are:

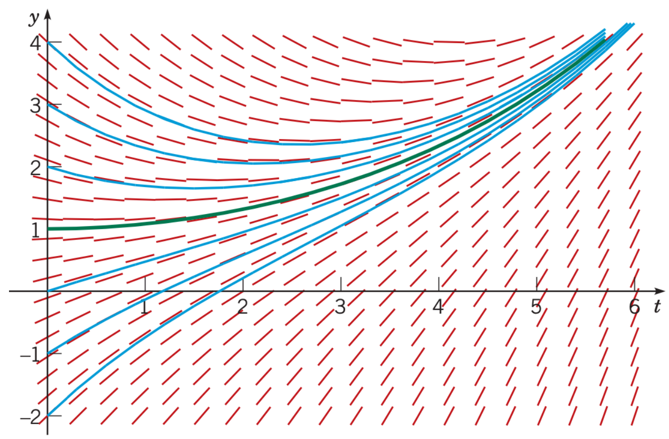

Figure 14. The direction field (red) and integral curves (blue). Green is one PS $y(t=0)=1$.

Example 7 Solve DE

$$\frac{dy}{dt} - 2y = 4 - t$$

Solution:

Using the Method of Integrating Factors, we first find the IF

$$\begin{aligned}\rho(t) &= \mathrm{e}^{\int P(t)dt} \\ &= \mathrm{e}^{\int (-2)dt} \\ &= \mathrm{e}^{-2t}\end{aligned}$$

Thus,

$$\frac{d}{dt}\left(e^{-2t}y\right) = 4e^{-2t} - te^{-2t}$$

Integrating both sides, we get

$$e^{-2t}y = -2e^{-2t} + \frac{1}{2}te^{-2t} + \frac{1}{4}e^{-2t} + c.$$

Thus, the GS is

$$y = -\frac{7}{4} + \frac{1}{2}t + ce^{2t}$$

The direction field and integral curves (solution family) are:

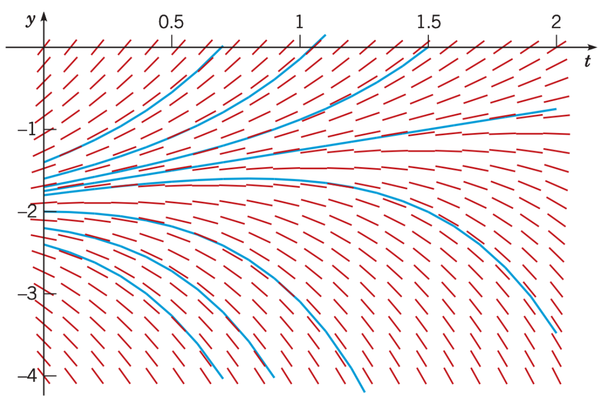

Figure 15. The direction field (red) and integral curves (blue).

Example 8 Solve DE

$$t\frac{dy}{dt} + 2y = 4t^2$$

with a few initial conditions (IC):

IC-1: $y(1) = 1$
IC-2: $y(1) = 2$
IC-3: $y(1) = 0$

I must tell you this DE has a super-rich solution profile and it's why I'm lecturing it!

Solution:

Let's convert the DE into

$$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = 4t$$

Thus, we have

$$P(t) = \frac{2}{t}$$
$$Q(t) = 4t$$

Using the Method of Integrating Factors, we first find the IF

$$\rho(t) = \mathrm{e}^{\int P(t)dt}$$
$$= \mathrm{e}^{\int \left(\frac{2}{t}\right)dt}$$
$$= t^2$$

Thus,

$$t^2 y = \int 4t^3\, dt = t^4 + c$$

Finally, the GS is

$$y_G(t) = t^2 + \frac{C}{t^2}$$

The slope field is

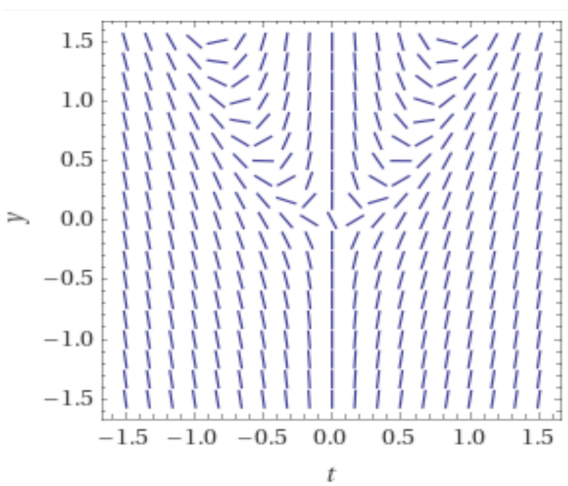

Figure 16.The slope field.

Applying IC (=initial condition), we can find three families of PS's!

Family 1 (for Case 1 IC example): $C = 0$

Family 2 (for Case 2 IC example): $C > 0$

Family 3 (for Case 3 IC example): $C < 0$

Family 1 (the red curve, see figure)

$$y(1) = 1$$

We get

$$1 = 1^2 + \frac{C}{1^2}$$

Thus,

$$C = 0$$

The PS is

$$y_P(t) = t^2$$

Family 2 (the green curve and some blue curves, see figure)

$$y(1) = 2$$

We get

$$2 = 1^2 + \frac{C}{1^2}$$

Thus,

$$C = 1$$

The PS is

$$y_P(t) = t^2 + \frac{1}{t^2}$$

The integral curves = solution family are:

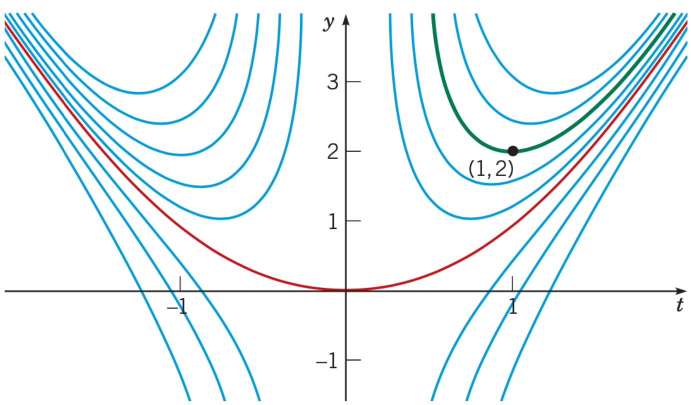

Figure 17. Solution curves with 3 different IC's (red, green, blue curves).

Of course, we can consider another one:

Family 3

$$y(1) = 0$$

We get

$$0 = 1^2 + \frac{C}{1^2}$$

Thus,

$$C = -1$$

The PS is

$$y_P(t) = t^2 - \frac{1}{t^2}$$

So, we have one of the strange looking solution curves:

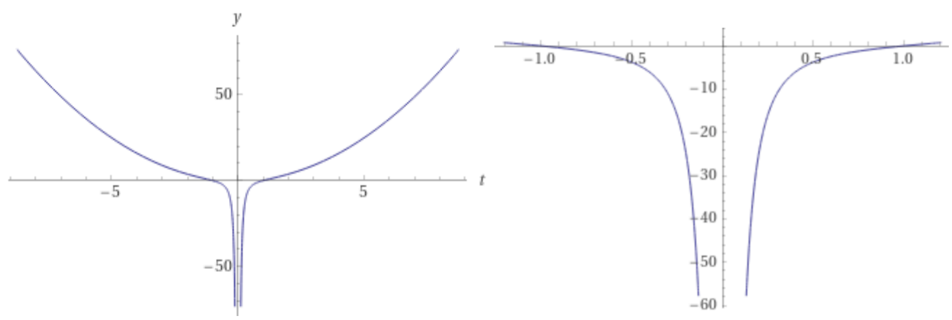

Figure 18. Solution curves with focused look at t~0 (right).

-------- Start of Lecture Week02.2 (2/2/2023) --------

A new concept in Homo and In-homo linear DEs (to be re-taught in CH3 much more seriously).

In fact, this kind of first order of linear DEs can teach us much more… and even spur the birth of the concept in higher order linear DEs!
The general form of first order of linear DEs:

$$\underbrace{\underbrace{\frac{dy}{dx}+P(x)y}_{\text{Homo Portion}}=\boldsymbol{Q(x)}}_{\text{In-Homo DE}}$$

The GS can be fused from 2 pieces (This statement needs proof, but we take it for granted for now):

$$y_G(x)=y_c(x)+y_P(x) \tag{1}$$

$y_c(x)$ satisfies

$$\frac{dy}{dx}+P(x)y=0$$

It's a piece of cake to find $y_c(x)$ because it's a simple separable DE,

$$\frac{dy}{y}=-P(x)dx$$

$$\ln y=-\int P(x)dx+c_0$$

Thus,

$$y_C(x)=C_1e^{-\int P(x)dx} \tag{2}$$

$y_P(x)$ is **any one** particular solution (unlike the PS we mentioned before) satisfying

$$\frac{dy}{dx}+P(x)y=Q(x)$$

It turns out (taking a ton of proof to arrive at this and we will do in Ch3),

$$y_P(x)=v(x)e^{-\int P(x)dx}$$

$$y_P'(x)=v'(x)e^{-\int P(x)dx}-P(x)v(x)e^{-\int P(x)dx}$$

Plugging the above two to the In-Homo DE, we get

$$v'(x)e^{-\int P(x)dx}=Q(x)$$

Thus,

$$v'(x)=Q(x)e^{\int P(x)dx}$$

$$v(x)=\int Q(x)e^{\int P(x)dx}\,dx+c_0$$

Since one "particular" solution is needed, I can force $c_0=0$ to simplify life without "losing" it.
Thus, the $y_P(x)$ can be

$$y_p(x)=e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}\,dx\right] \tag{2}$$

Thus, the GS of the whole DE is

$$y_G(x)=y_c(x)+y_P(x)$$

i.e.,

$$\boxed{y_G(x) = C_1 e^{-\int P(x)dx} + e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}\,dx\right]}$$

Example 8A Solve DE

$$t\frac{dy}{dt} + 2y = 4t^2$$

Solution: (This problem is the same as **Example 8A**, but I would use a different method to solve it)
Let's convert the DE into

$$\underbrace{\underbrace{\frac{dy}{dt} + \left(\frac{2}{t}\right)y}_{\text{Homo DE}} = 4t}_{\text{Ih-Homo DE}}$$

Thus, we have

$$P(t) = \frac{2}{t}$$
$$Q(t) = 4t$$

Now, let's use the new method to solve this problem:

Step 1. Solve the Homo DE for $y_c(t)$:

$$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = 0$$

Easily,

$$\frac{dy}{y} = -\left(\frac{2}{t}\right)dt$$
$$\int \frac{dy}{y} = -\int \left(\frac{2}{t}\right)dt$$
$$\ln y = -2\ln t + c_1$$

Thus,

$$y_C(t) = \frac{C}{t^2}$$

Step 2. Solve the In-Homo DE for $y_P(t)$:

$$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = 4t$$

All we need is to find ONE legit $y_P(t)$... by setting

$$y_P(t) = v(t)e^{-\int P(t)dt} = v(t)t^{-2}$$

Thus, we can form a new DE in terms of $v(t)$ vs. t:

$$v'(t) = \underbrace{Q(t)}_{4t}\, e^{\int P(t)dt}$$
$$= 4t\, e^{\int \frac{2}{t}dt} = 4t\, e^{2\ln t} = 4t \times t^2 = 4t^3$$

Thus,

$$v(t) = t^4$$

Thus,

$$y_P(t) = v(t)e^{-\int P(t)dt} = v(t)t^{-2} = t^2$$

(Easily, we can verify t^2 satisfies the In-Homo DE!)

Step 3. Compose the GS for the original DE:

$$y_G(t) = y_C(t) + y_P(t) = \frac{C}{t^2} + t^2$$

*** End of Method 2: Method of Integrating Factors for 1st Order Linear DEs***

Method 3: The Substitution Methods

Substitution methods (S-methods) are those introducing one or more new variables to represent the variables in the original DE. The procedure will convert the original DE to a separable or other more easily solvable DE, e.g., one can change one variable x to u,

$$F_1(x, y) = 0 \xRightarrow{f(x)\to u} F_2(u, y)$$

One may also consider changing both variables.

$$G_1(x, y) = 0 \xRightarrow[h(x,y)\to v]{g(x,y)\to u} G_2(u, v)$$

Let's discover the power of the S-methods by solving several families of DEs.

Note: Sub method is difficult because you have hardly any "formulas" on what to sub for what. Only a few special cases may have a pattern to follow. Some may require you to fuse ("gang up") a few substitution methods to create a solution method.

Are you responsible for those unknown patterns? Probably no.

Method 3A: The Substitution Methods: Polynomial Substitution

Solve

$$y' = F(\underbrace{ax + by + c}_{v})$$

where a, b, and c are constants.

This DE is called Polynomial Substitutable DE.

Step 1: Introducing a new variable v.

$$v = ax + by + c$$

Transform the original DE $y' = F(ax + by + c)$ into a new DE of v.

$$v' = a + by'$$
$$y' = \frac{1}{b}v' - \frac{a}{b}$$
$$y' = F(v)$$

Or

$$\frac{1}{b}v' - \frac{a}{b} = F(v)$$

Or

$$\frac{1}{b}v' = F(v) + \frac{a}{b}$$

which is now separable.

Example 8 Solve DE

$$y' = (x + y + 3)^2$$

Solution:

Let

$$v = x + y + 3$$
$$v' = 1 + y'$$
$$y' = v' - 1$$

After the substitution, the original DE becomes

$$v' - 1 = v^2$$
$$\frac{dv}{v^2 + 1} = dx$$
$$\int \frac{dv}{v^2 + 1} = \int dx$$
$$\tan^{-1} v = x + C$$
$$v = \tan(x + C)$$

Substituting v back into the above DE, we obtain

$$x + y + 3 = \tan(x + C)$$

Thus, (It's perfectly fine to leave it in implicit form),

$$y_G = \tan(x + C) - x - 3$$

The following is how the GS looks like (and it takes a while to figure it out):

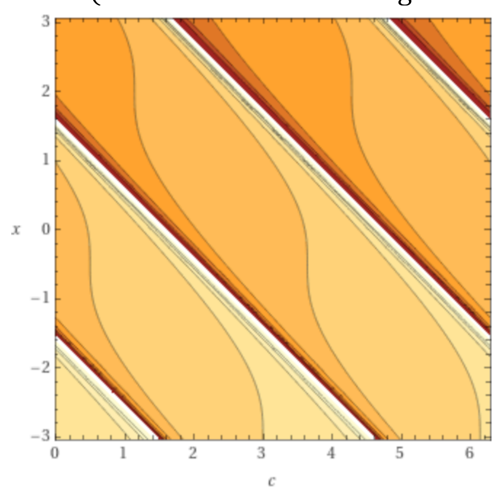

Figure 19. The solution family for a variety of "c" where the colors are for y values.

PS for $c = 1$, $c = 3$, $c = 5$:

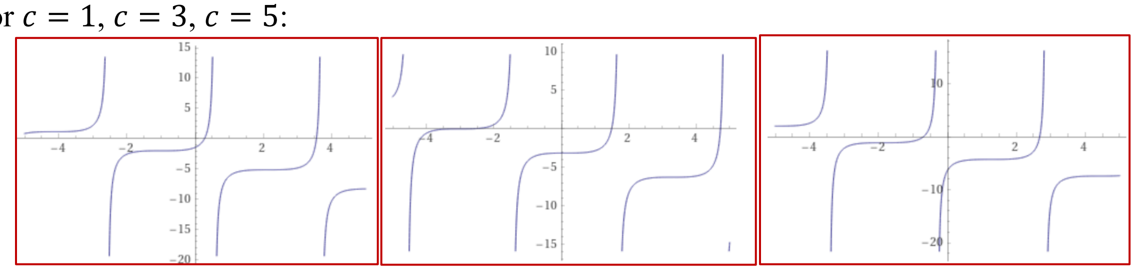

Figure 20. The PS's with c=1 (left), c=3 (middle), and c=5 (right).

Method 3B: The Substitution Methods: For Homogeneous DEs

Note: Please notice one more use of the word "Homogeneous" for a different meaning. In math, "homo" is badly overused word.

For the 1st.O homogeneous DEs:

$$y' = F\left(\frac{y}{x}\right)$$

Apply substitution

$$v = \frac{y}{x}$$

with which $y = xv$ and $y' = xv' + v$,

$$xv' + v = F(v)$$

Leading to

$$v' = \frac{F(v) - v}{x}$$

which is separable and, thus, easily solved!

Example 9 Solve DE

$$y' = \frac{2x}{y} + \frac{3y}{2x}$$

Solution:

$$v + xv' = \frac{2}{v} + \frac{3}{2}v$$

$$v' = \frac{1}{x}\left(\frac{2}{v} + \frac{v}{2}\right)$$

$$\frac{2vdv}{4+v^2} = \frac{dx}{x}$$

$$\int \frac{d(4+v^2)}{4+v^2} = \int \frac{dx}{x}$$

$$\ln(4+v^2) = \ln x + C$$

$$4 + v^2 = Cx$$

Back sub, we have

$$4 + \left(\frac{y}{x}\right)^2 = Cx$$

$$y^2 + 4x^2 = Cx^3$$

The PS's for a few IC's:

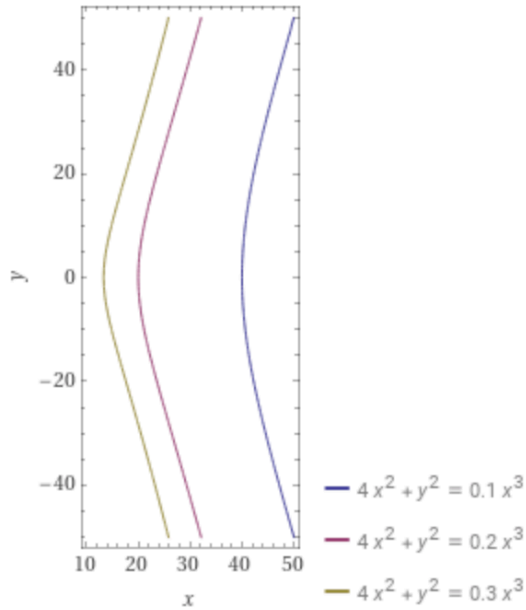

Figure 21. The PS's with 3 given IC's.

-------- Start of Lecture Week03.1 (2/7/2023) --------

Method 3C: The Substitution Methods: For Bernoulli DEs

Motivations: Can you use the learned methods to solve this DE?

Example: Find the GS of

$$2x + y^2 + 2xyy' = 0$$

Solution: Let's first classify by first writing it as

$$y' + \frac{1}{2x}y = -\frac{1}{y}$$

Or

$$y' = -\frac{1}{2x}y - \frac{1}{y}$$

- 1st-order linear?
- Inseparable?

Now we are stuck!

DEs like this

$$y' + P(x)y = Q(x)y^n$$

are called **Bernoulli**. Of course, Bernoulli DE is the general form for many DE types.

Case 1 ($n < 0$):
a general 1st.O nonlinear DE.

Case 2 ($n = 0$):
the 1st.O.L DE $y' + P(x)y = Q(x)$ whose solution method was introduced before.

Case 3 ($n = \frac{1}{2}$):
a general 1st.O nonlinear DE.

Case 4 ($n = 1$):
1st.O.L DE $y' + \big(P(x) - Q(x)\big)y = 0$ which is separable.

Case 5 ($n = 2$):
a special 1st.O nonlinear DE, *i.e.*, the Riccati DE (to be discussed next)

Case 6 ($n > 2$):
a general 1st.O nonlinear DE.

To sum up, for $n \neq 0, 1$, we know that the Bernoulli DE is a 1st.O nonlinear DE and we have not learned any methods to solve the DE for such cases. Now, let's use a substitution to solve the Bernoulli DEs for $n \neq 0, 1$.

The most important part is the **Bernoulli Sub**:

$$v = y^{1-n}$$

With math magic
(*I'll elaborate using the following space*):

we get

$$\left(\frac{1}{1-n}\right)v' + P(x)v = Q(x) \tag{1}$$

Now, it's a PQ equation with x=IV and v = DV.

Note: This process has a better name: **linearization**!

Example 10 Solve DE

$$y' + \frac{6}{x}y = 3y^{\frac{4}{3}}$$

Solution:

$$P(x) = \frac{6}{x}$$
$$Q(x) = 3$$
$$n = \frac{4}{3}$$

Applying Bernoulli Sub:

$$v = y^{1-n} = y^{-\frac{1}{3}}$$
$$y = v^{-3}$$

Walking through in class,

we get

$$v' - \frac{2v}{x} = -1$$

This PQ DE is easy to solve (walking through):

$$v_G = x + Cx^2$$

Back sub:

$$y_G = \frac{1}{x^3(1 + Cx)^3}$$

Some **profiles** of the PS's:

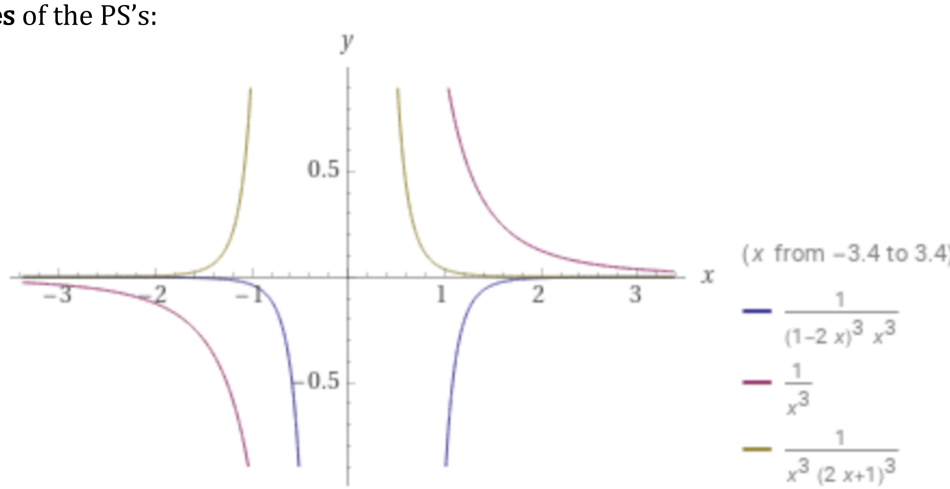

Figure 22. The PS's with c=-2 (blue), c=0 (red), and c=2 (yellow).

End of Method 3: The Substitution Methods ***

Method 4: Methods for Riccati DEs

DEs that can be expressed as follows are called Riccati DEs, named after the Italian mathematician J. F. Riccati (1676–1754),

$$y' = A_0(x) + A_1(x)y + A_2(x)y^2$$

Two special cases:

1) If $A_0(x) = 0$, Riccati DE reduces to a Bernoulli DE for which you may make a simple substitution introduced for Bernoulli DE $v(x) = y^{1-2}$ to transform it into a 1st.O.L DE.

$$y' = \cancel{A_0(x) +} A_1(x)y + A_2(x)y^2$$

Or

$$y' \underbrace{-A_1(x)}_{P(x)} y = \underbrace{A_2(x)}_{Q(x)} y^2$$

becoming

$$y' + P(x)y = Q(x)y^2$$

2) If $A_2(x) = 0$, the Riccati DE reduces to a 1st.O.L DE

$$y' = A_0(x) + A_1(x)y \cancel{+ A_2(x)y^2}$$

Or

$$y' \underbrace{-A_1(x)}_{P(x)} y = \underbrace{A_0(x)}_{Q(x)}$$

becoming

$$y' + P(x)y = Q(x)$$

These two special cases can be solved by the known methods. But the general case (including $A_1(x) = 0$) requires development of new methodologies!

The method was introduced by Riccati!

The **"exotic" method** for the general case: One needs a pilot solution $y_1(x)$.

This pilot solution $y_1(x)$ is usually obtained by intuition or by guessing and it's extremely difficult!

Assuming one finds a pilot solution $y_1(x)$ with which one finds the GS, according to Riccati,

$$y(x) = y_1(x) + \frac{1}{u(x)}$$

(I group this method substitution, reluctantly.)

Thus,

$$y' = y_1' - \frac{u'}{u^2}$$
$$y^2 = y_1^2 + \frac{1}{u^2} + \frac{2y_1}{u}$$

Thus,

$$\underbrace{y_1' - \frac{u'}{u^2}}_{y'(x)} = A_0 + A_1 \underbrace{\left(y_1 + \frac{1}{u}\right)}_{y(x)} + A_2 \underbrace{\left(y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}\right)}_{y^2(x)}$$

Finally (after a ton of manipulations, See textbook p 38)

$$u' + (A_1(x) + 2A_2(x)y_1(x))u = -A_2(x)$$

$$u' + P(x)u = Q(x)$$

where

$$P(x) = A_1(x) + 2A_2(x)y_1(x)$$
$$Q(x) = -A_2(x)$$

Many interesting remarks one can make!

The original DE

$$y' = A_0(x) + A_1(x)y + A_2(x)y^2$$

Found a pilot:

$$y_1(x)$$

Make a Riccati sub:

$$y(x) = y_1(x) + \frac{1}{u(x)}$$

Turning the Riccati DE into

$$u' + P(x)u = Q(x)$$

where

$$P(x) = A_1(x) + 2A_2(x)y_1(x)$$
$$Q(x) = -A_2(x)$$

Remark 1: If $A_0(x) = 0$, the Riccati DE becomes

$$y' = \cancel{A_0(x) +} A_1(x)y + A_2(x)y^2$$

Or

$$y' - A_1(x)y = A_2(x)y^2$$

We can solve this Bernoulli DE by the Bernoulli sub

$$u(x) = y^{1-2}$$

which can be viewed as

$$u = y^{-1}$$

Or

$$y = \frac{1}{u}$$

In fact, I can even write it as

$$y = 0 + \frac{1}{u}$$

It turns out

$$y_1(x) = 0$$

is a pilot solution of

$$y' = \cancel{A_0(x) +} A_1(x)y + A_2(x)y^2$$

One can verify $y_1(x) = 0$ can be regarded as a pilot solution!

Remark 2: If $A_1(x) = 0$, the Riccati DE becomes

$$y' = A_0(x) \cancel{+ A_1(x)y} + A_2(x)y^2$$

We have no easy way to handle this DE

$$y' - A_0(x) = A_2(x)y^2$$

We can't easily find a pilot, nor can we use anything we have learned so far.

But we can use the Riccati Sub to get to the following (assuming we know the pilot):

$$u' + P(x)u = Q(x)$$

where

$$P(x) = \cancel{A_1(x) +} 2A_2(x)y_1(x)$$
$$Q(x) = -A_2(x)$$

Thus, our new 1st order linear DE is

$$u' + 2A_2(x)y_1(x)u = -A_2(x)$$

The fundamental here is to find $y_1(x)$!

Remark 3: If $A_2(x) = 0$, the Riccati DE becomes

$$y' = A_0(x) + A_1(x)y \cancel{+ A_2(x)y^2}$$

We have no easy way to handle this DE

$$y' - A_1(x)y = A_0(x)$$

Well, we need to do nothing to solve this DE.

(Will skip this):

However, if you really want to use the Riccati sub, you expect to get a new DE

$$u' + P(x)u = Q(x)$$

with

$$P(x) = A_1(x) \cancel{+ 2A_2(x)y_1(x)}$$
$$Q(x) = \cancel{-A_2(x)}$$

meaning that, the DE becomes

$$u' + A_1(x)u = 0$$

One can also explain the method.

-------- Start of Lecture Week03.2 (2/9/2023) --------

Example 11 Solve DE

$$y' + y^2 = \frac{2}{x^2}$$

Solution:

This is a simple Riccati DE. Let's first find the Pilot! (Some magic here ☹)

We got two

$$y_1 = -\frac{1}{x}$$
$$y_1 = \frac{2}{x}$$

Let's use $y_1 = -\frac{1}{x}$, we get

$$u' + \frac{2}{x}u = 1$$

Whose GS is

$$u = \frac{x}{3} + \frac{C}{x^2}$$

Back sub:

$$y(x) = -\frac{1}{x} + \frac{3x^2}{x^3 + C_1}$$

The GS family looks

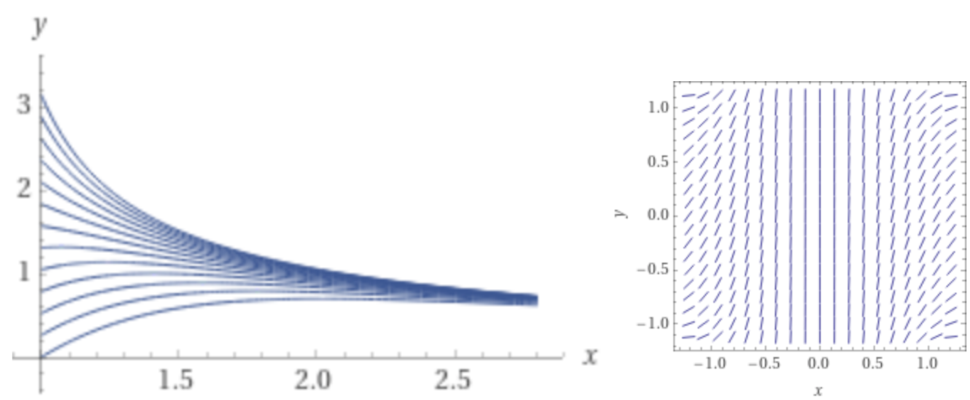

Figure 23. The GS family (left) and slope field (right) for this example.

Let me make a contour plot for your viewing pleasure:

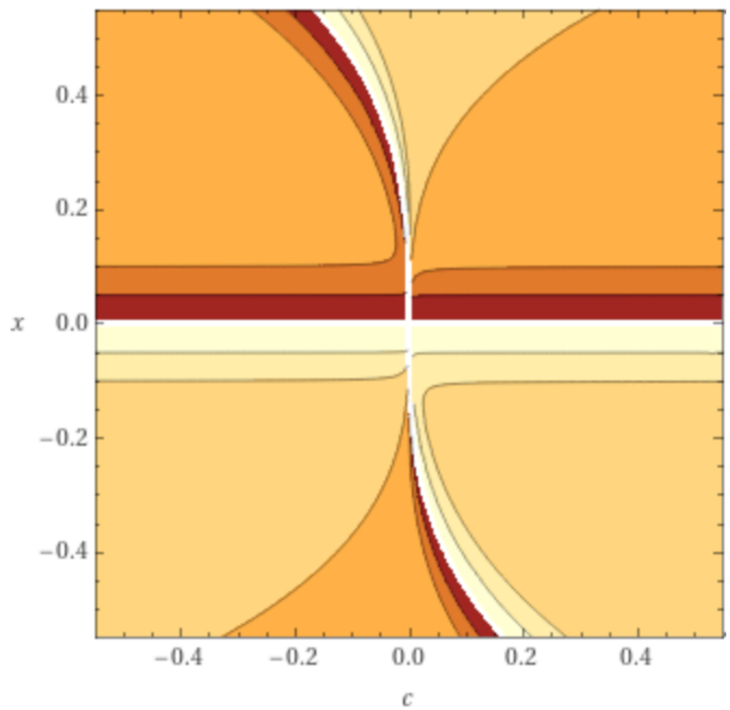

Figure 24. The contour plot of solution family.

The following is a few samples of PS's:

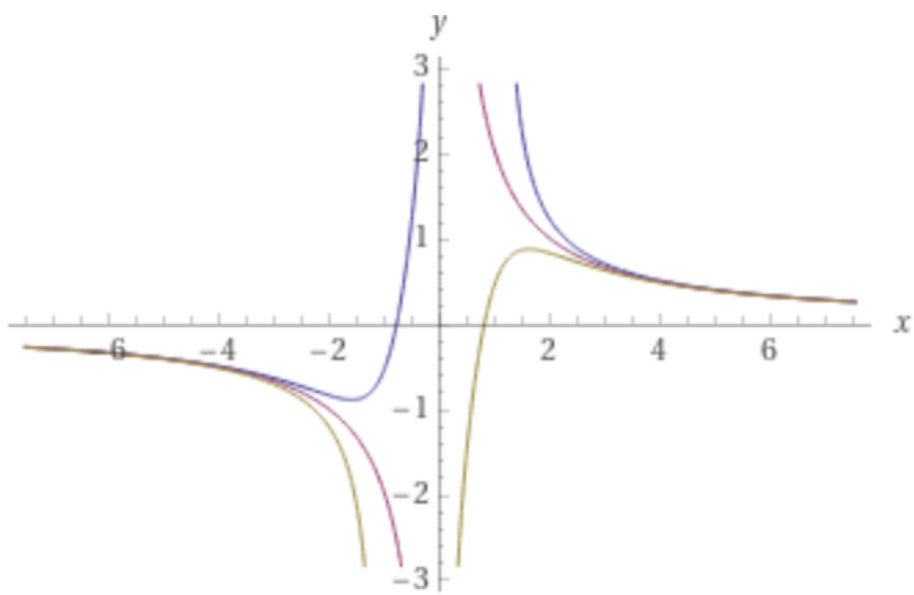

Figure 25. The samples of PS's at c=-1 (blue), 0 (red), 1 (yellow).

For your information, the following DE I just made up does not appear to have a convenient pilot but the solution profile looks similar:

Alternate A to Example 11

$$y' + y^2 = \frac{2}{x^{10}}$$

The solution profile:

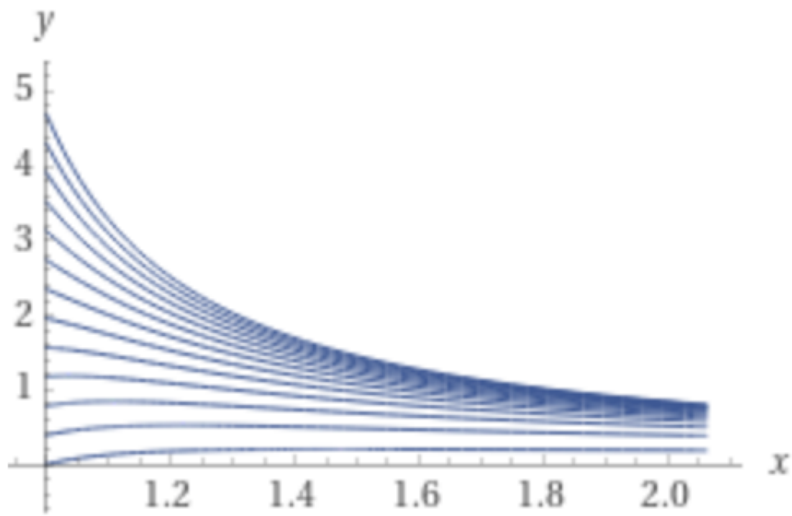

Figure 26. The GS family for this Alternate A example.

Alternate B to Example 11

$$y' + y^3 = \frac{2}{x^2}$$

Note: This DE is no longer Riccati! I'm including it here for showing the similarity of solution profile.

The solution profile:

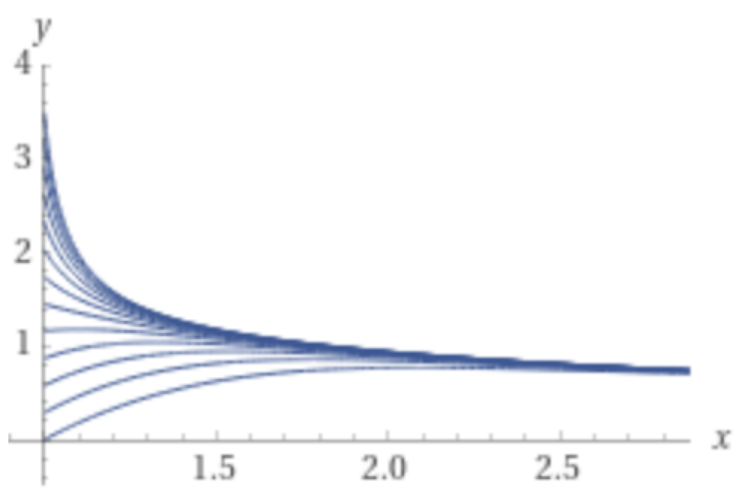

Figure 27. The GS family for this Alternate B example.

*** End of Method 4: Methods for Riccati DEs ***

Method 5: The Exact DE Methods

Motivations: How to deal with these DEs?

Example M12A: Find the GS of

$$(y\cos x + 2x\, e^{y}) + (\sin x + x^2 e^{y} - 1)y' = 0$$

Solution: Let's first classify

- 1st-order linear?
- Inseparable?

Example M12B: Find the GS of

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Solution: Let's first classify

- 1st-order linear?
- Inseparable?

Example M12C: Find the GS of

$$(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$$

Solution: Let's first classify

- 1st-order linear?
- Inseparable?

We must learn some new tricks!

In fact, all the above DE's have the following general form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Or

$$M(x,y)dx + N(x,y)dy = 0$$

It turns out all first order DE's can be written in such a form. There is a subset of them that are natively Exact DE's and others, with some conditions, can be turned to Exact DE's. What's the big deal of Exact DE's? Well, you can solve it easily!

By the way, the process of turning them to Exact DEs is to find the integrating factor!

Now, let's study the details.

Necessary and sufficient conditions for

$$\boldsymbol{M(x,y)dx + N(x,y)dy = 0}$$

to be exact is

$$\boldsymbol{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}$$

Let's work on a few examples.

Example 13: Check if the DE is exact

$$2x + y^2 + 2xyy' = 0$$

Solution: The DE can be rewritten as

$$(2x + y^2)dx + (2xy)dy = 0$$

Thus,

$$M(x,y) = 2x + y^2$$
$$N(x,y) = 2xy$$

We have

$$\frac{\partial M(x,y)}{\partial y} = 2y$$
$$\frac{\partial N(x,y)}{\partial x} = 2y$$

Therefore,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Thus,

$$2x + y^2 + 2xyy' = 0$$

is exact!

Well, exact DE's can be solved easily. Wait for me to develop the method.

-------- Start of Lecture Week04.1 (2/14/2023) --------
Part I (to be completed within ~45 minutes)

Example 14A: Check if the DE is exact

$$ydx + 3xdy = 0$$

Solution: The DE has

$$M(x,y) = y$$
$$N(x,y) = 3x$$

We have

$$\frac{\partial M(x,y)}{\partial y} = 1$$
$$\frac{\partial N(x,y)}{\partial x} = 3$$

Therefore,

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$

Thus,

$$ydx + 3xdy = 0$$

is not exact!

Note: This example demonstrates the following fact: This DE has perfect solution. So, you can't infer that non-exact DE's do not have solutions. They most likely have solutions except that they can't be found by the Exact DE method!

	Non-Exact DEs	**Exact DEs**
GS Exists?	Yes or no	Yes or no
GS can be found if exist	Unsure (Likely no)	Yes (surely)

Performing the following will change our fate in solving a DE:

Non-Exact DEs→ Exact DEs

In each case, one conversion (assuming method exists) corresponding to one integrating factor (IF).

Now, let's explore these concepts!

Example 14B: Check if the DE is exact

$$y^2(ydx + 3xdy) = 0$$

Solution: The DE has

$$M(x,y) = y^3$$
$$N(x,y) = 3xy^2$$

We have

$$\frac{\partial M(x,y)}{\partial y} = 3y^2$$
$$\frac{\partial N(x,y)}{\partial x} = 3y^2$$

Therefore,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Thus,

$$y^2(ydx + 3xdy) = 0$$

is exact!

Note: This example demonstrates the following fact: A non-exact DE can be converted to exact DE without impacting its solution. As you know **Example 14A** and **Example 14B** have the same solution**.**

In fact, in the following example, **Example 14C's** solution is also the same!

This y^2 is actually an integrating factor!

Example 14C: Check if the DE is exact

$$x^{-\frac{2}{3}}(ydx + 3xdy) = 0$$

Solution: The DE has

$$M(x, y) = x^{-\frac{2}{3}}y$$
$$N(x, y) = x^{-\frac{2}{3}}(3x)$$

We have

$$\frac{\partial M(x, y)}{\partial y} = x^{-\frac{2}{3}}$$
$$\frac{\partial N(x, y)}{\partial x} = x^{-\frac{2}{3}}$$

Therefore,

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

Thus,

$$x^{-\frac{2}{3}}(ydx + 3xdy) = 0$$

is exact!

Note: This example demonstrates the following fact: A non-exact DE can be converted to exact DE without impacting its solution. As you know **Example 5A** and **Example 5B** have the same solution**.**

This $x^{-\frac{2}{3}}$ is another integrating factor!

Question: How many methods can a non-exact DE be converted to exact?

Answer(s): 0, 1, 2, 2+, infinity

Each conversion is related to one (family) integrating factor (IF)

Rephase the above question and answers:

Question: How many IF's a non-exact DE be have?

Answer(s): Some non-exact DE's have **zero** IF.
Some non-exact DE's have **one and only one** IF.
Some non-exact DE's have **two or more** IF's.
Some non-exact DE's have **infinitely many** IF's.

We now learn to find them if exist.

Theorem 1

For a given 1st.O DE $M(x, y)dx + N(x, y)dy = 0$, if

$$\frac{M_y - N_x}{N} = f(x)$$

is a function of purely x, the DE can be converted into an exact DE by multiplying the original DE by

$$\rho(x) = \mathrm{e}^{\int f(x)dx}$$

Theorem 2

For a given 1st.O DE $M(x,y)dx + N(x,y)dy = 0$, if

$$\frac{M_y - N_x}{M} = g(y)$$

is a function of purely y, the DE can be converted into an exact DE by multiplying the original DE by

$$\rho(y) = \mathrm{e}^{-\int g(y)dy}$$

Next, I'll prove these two theorems!

When they are exact, how do we use it to solve the DE? The following is the logic:

Claim 1: The GS to all DEs can be written as

$$F(x,y) = C$$

Corollary 1 of Claim 1:

$$d\big(F(x,y)\big) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = dC = 0$$

Corollary 2 of Claim 1:

$$\begin{cases} M(x,y) = \dfrac{\partial F}{\partial x} \\ N(x,y) = \dfrac{\partial F}{\partial y} \end{cases}$$

What does this mean? It means if you can find $F(x,y)$, you find the solution of

$$M(x,y)dx + N(x,y)dy = 0$$

Now, the steps of finding $F(x,y)$.

Step 1: Write your DE correctly to identify $M(x,y), N(x,y)$.

Step 2: Checking if your DE $M(x,y)dx + N(x,y)dy = 0$ is exact. If yes, go to Step 3. If not, convert it. If you can't convert, this method can't serve you.

Step 3: If $M(x,y)dx + N(x,y)dy = 0$ is exact, solve

$$\begin{cases} M(x,y) = \dfrac{\partial F}{\partial x} \\ N(x,y) = \dfrac{\partial F}{\partial y} \end{cases}$$

You first get

$$F(x,y) = \int M(x,y)\,dx + g(y)$$

Plug this into the 2nd DE, you find $g(y)$.

(Now, let me demonstrate the solutions via several examples!)

Example 15: Find the GS of

$$2x + y^2 + 2xyy' = 0$$

Solution:
Step 1:

$$M(x,y) = 2x + y^2$$
$$N(x,y) = 2xy$$

Step 2:

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Thus, the DE is exact!
Step 3:

$$\begin{cases} 2x + y^2 = \dfrac{\partial F}{\partial x} \\ 2xy = \dfrac{\partial F}{\partial y} \end{cases}$$

Integrating the 1st DE, we get

$$F(x,y) = \int (2x + y^2)\, dx + g(y)$$
$$= x^2 + xy^2 + g(y)$$

Now, plugging this into the 2nd DE, we get

$$2xy = \frac{\partial}{\partial y}(x^2 + xy^2 + g(y))$$
$$= 0 + 2xy + g'(y)$$

Thus,

$$g'(y) = 0$$
$$g(y) = C_1$$

One very important cancellation step that was made possible by Exact DE condition!

Finally,

$$F(x,y) = x^2 + xy^2 + C_1 = C_2$$

or

$$x^2 + xy^2 = C$$

is the GS!

You can verify it by taking derivative wrt "x". A few sample cases of the solution:

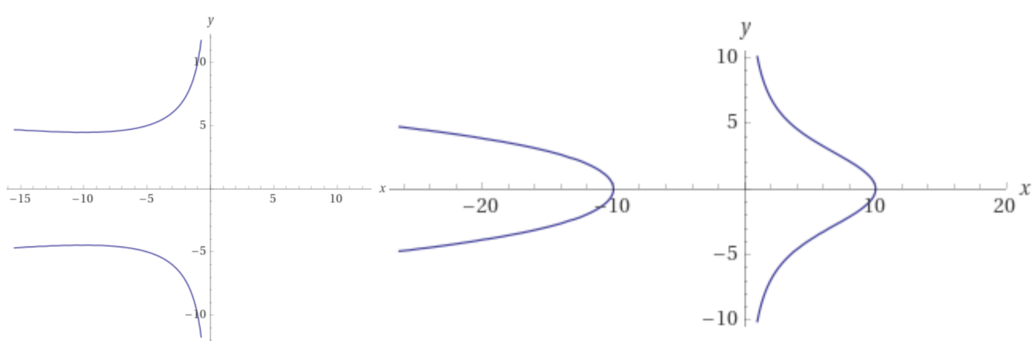

Figure 28. Two PS's with c=-100 (left) and c=100 (right).

Example 16: Find the GS of

$$(y\cos x + 2x\,e^y) + (\sin x + x^2e^y - 1)y' = 0$$

Solution:

Step 1:

$$M(x,y) = y\cos x + 2x\,e^y$$
$$N(x,y) = \sin x + x^2e^y - 1$$

Step 2:

$$\frac{\partial M(x,y)}{\partial y} = \cos x + 2x\,e^y$$
$$\frac{\partial N(x,y)}{\partial x} = \cos x + 2x\,e^y$$

Thus, the DE is exact!

Step 3:

$$\begin{cases} y\cos x + 2x\,e^y = \dfrac{\partial F}{\partial x} \\ \sin x + x^2e^y - 1 = \dfrac{\partial F}{\partial y} \end{cases}$$

Integrating the 1st DE, we get

$$F(x,y) = \int (y\cos x + 2x\,e^y)\,dx + g(y)$$
$$= y\sin x + x^2e^y + g(y)$$

Now, plugging this into the 2nd DE, we get

$$\sin x + x^2e^y - 1 = \frac{\partial}{\partial y}\left(y\sin x + x^2e^y + g(y)\right)$$
$$= \sin x + x^2e^y + g'(y)$$

Thus,

$$g'(y) = -1$$
$$g(y) = -y + C_1$$

Finally,

$$F(x,y) = y\sin x + x^2e^y - y + C_1 = C_2$$

or

$$y\sin x + x^2e^y - y = C$$

is the GS!

You can verify it by taking derivative wrt "x".

I plot the PS at C=5 and it looks very interesting (and crazy):

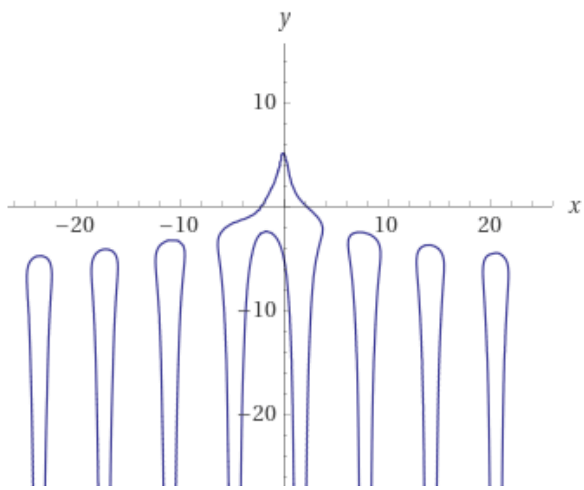

Figure 29. One PS for this example.

Example 17: Find the GS of

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Solution:

Step 1:

$$M(x,y) = 3xy + y^2$$
$$N(x,y) = x^2 + xy$$

Step 2:

$$M_y = \frac{\partial M(x,y)}{\partial y} = 3x + 2y$$
$$N_x = \frac{\partial N(x,y)}{\partial x} = 2x + y$$

Thus, the DE is non-exact! Now, let's see if we can find the integrating factor:

$$f(x) = \frac{M_y - N_x}{N}$$
$$= \frac{(3x + 2y) - (2x + y)}{x^2 + xy}$$
$$= \frac{1}{x}$$

Thus, $\rho(x) = \mathrm{e}^{\int \frac{1}{x} dx} = x$. If we multiply the DE by "x", we convert it into Exact:

$$x(3xy + y^2) + x(x^2 + xy)y' = 0$$

Step 3:

$$\begin{cases} x(3xy + y^2) = \dfrac{\partial F}{\partial x} \\ x(x^2 + xy) = \dfrac{\partial F}{\partial y} \end{cases}$$

Integrating the 1st DE, we get

$$F(x,y) = \int (3x^2y + xy^2)\, dx + g(y)$$
$$= x^3y + \left(\frac{x^2}{2}\right)y^2 + g(y)$$

Now, plugging this into the 2nd DE, we get

$$x(x^2 + xy) = \frac{\partial}{\partial y}\left(x^3y + \left(\frac{x^2}{2}\right)y^2 + g(y)\right)$$
$$= x^3 + x^2y + g'(y)$$

Thus,

$$g'(y) = 0$$
$$g(y) = C_1$$

Finally,

$$F(x,y) = x^3y + \left(\frac{x^2}{2}\right)y^2 + C_1 = C_2$$

or

$$x^3y + \left(\frac{x^2}{2}\right)y^2 = C$$

is the GS! Sample solution family:

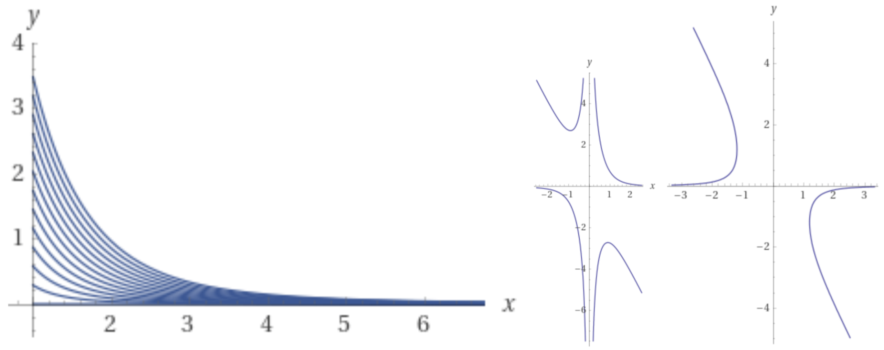

Figure 30. The GS (left) and PS's for c=1 (middle) and c=-1 (right).

*** The End of Chapter 1 ***