AMS361 (Applied Calculus IV) Spring 2023

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Lectures for CH3 Higher Order Linear DEs

Lecture plan for this Chapter:

Lecture	Date	Contents	Sec.
Week07.2	03/09	Intro to higher order linear DE's	3.1
Week08.1	03/14	**Spring Break (no lecture)**	
Week08.2	03/16	**Spring Break (no lecture)**	
Week09.1	03/21	**Test 2** (covering mostly CH2)	
Week09.2	03/23	**Type C.Homo**: Homo DEs with constant coefficients	3.3
Week10.1	03/28	**Type V.Homo**: Homo DEs with varying coefficients	3.3
Week10.2	03/30	**Type C.InHomo** InHomo DEs with constant coefficients	3.4
Week11.1	04/04	**Type C.InHomo** DEs with constant coefficients (continued)	3.4
Week11.2	04/06	**Type V.InHomo** DEs with varying coefficients	3.4
Week12.1	04/11	CH4: Systems of linear DEs (C.Homo)	4.1
Week12.2	04/13	CH4: Systems of linear DEs (V.Homo)	4.2
Week13.1	04/18	**Test 3** (2 problems in CH3 & 1 problem in CH4)	

-------- Start of Lecture Week07.2 (03/09/2023) --------

Starting CH3.

-------- Start of Lecture Week08.1 (03/14/2023) --------

Spring Break

-------- Start of Lecture Week08.2 (03/16/2023) --------

Spring Break

-------- Start of Lecture Week09.1 (03/21/2023) --------

Test2

-------- Start of Lecture Week07.2 (03/09/2023) --------

CH3 Higher Order Linear DEs

3.1 Motivations and Classification

Example 1 Find the GS to the following DE,

$$y'' - 2y' - 3y = 0$$

Example 2 Find the GS to the following DE,

$$y'' - 2y' + y = 0$$

Example 3 Find the GS to the following DE,

$$y'' - 2y' + 2y = 0$$

The Basic Theory: An **Overview:**

General form of 2nd DE's:

$$y''(x) = f(x, y, y')$$

where f is some given function. The DE is linear if,

$$f(x, y, y') = r(x) - p(x)y' - q(x)y$$

Or the DE can be written as

$$y'' + p(x)y' + q(x)y = r(x)$$

In fact, there are many ways to express it.

Or a more general (and symmetrical) form: 2nd order linear DE's

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Classification of three categories of 2nd (and higher) order DE's:

(1) linear vs. Nonlinear
(2) Homogeneous vs. In-Homogeneous
(3) Constant coefficients vs. Variable coefficients

Classification 1: Linear vs. Nonlinear and Constant coefficients vs. Variable coefficients

Linearity / Coefficients	Yes	No
Constant	$y'' + y = 0$	$y'' + y^2 = 0$
Variable	$y'' + P(x)y = 0$	$y'' + P(x)y^2 = 0$

Classification 2: Homogeneous vs. In-Homogeneous and Constant coefficients vs. Variable coefficients

Table 1. Four types of second order linear DEs (Named by me for ease of organizing)

$R(x)$ / $P(x), Q(x)$	$R(x) = 0$ **Homogeneous**	$R(x) \neq 0$ **In-Homogeneous**
Constant	$ay'' + by' + cy = 0$ (Type C.Homo)	$ay'' + by' + cy = f(x)$ (Type C.InHomo)
Variable	$A(x)y'' + B(x)y' + C(x)y = 0$ (Type V.Homo)	$A(x)y'' + B(x)y' + C(x)y = F(x)$ (Type V.InHomo)

3.2 Linear Dependance (LD) vs. Linear Independence (LI) and the Wronskian

(A concept you knew already in LA=linear algebra)

Definition: For functions y_1 and y_2, if y_1 is not a constant multiple of y_2, then, y_1 is linearly independent (LI) of y_2. That is, if one can find two constants $c_1$and c_2 that are not zero simultaneously to make $c_1 y_1 + c_2 y_2 = 0$, then, y_1 and y_2 are called linear dependent (LD). O.W., they are LI.
For the two-function case, if $\frac{y_1}{y_2}$ ≠constant, then, they are LI.

Example: Determine if y_1, y_2 are LI

Table 2. Several pair of functions to determine LI.

y_1	y_2	$\frac{y_1}{y_2}$	LI
$\sin x$	$\cos x$	$\tan x$	Y
$\sin 2x$	$\sin x$	$2 \cos x$	Y
$2 \sin x$	$\sin x$	2	N
$2e^x$	e^x	2	N
e^{2x}	e^x	e^x	Y
e^{ax}	e^{bx}	$e^{(a-b)x}$	Y $(a \neq b)$

More generally, a set of functions $\{f_1(x), f_2(x), \dots, f_n(x)\}$, defined on some given interval, is said to be LI if,

$$\sum_{i=1}^{n} \left(c_i f_i(x)\right) = 0$$

implying that $c_i = 0 \; \forall \; i \leq n$. Otherwise, the set is linearly dependent. Let's now define the differential operator $(\mathcal{D})$

$$\mathcal{D} = \frac{d}{dx}$$

Ex 1

$$\mathcal{D}y = \left(\frac{d}{dx}\right) y = \frac{dy}{dx} = y'$$
$$\mathcal{D}^2 y = \left(\frac{d}{dx}\right)^2 y = \frac{d^2y}{dx^2} = y''$$
$$\mathcal{D}^n y = \left(\frac{d}{dx}\right)^n y = \frac{d^n y}{dx^n} = y^{(n)}$$

So now, our n^{th} order DE

$$y^{(n)} + P_1(x) y^{(n-1)} + \cdots + P_n(x) y = 0$$

can be written as

$$D^n y + P_1(x) D^{n-1} y + \cdots + P_n(x) y = 0$$
$$\left(D^n + P_1(x) D^{n-1} + \cdots + P_n(x)\right) y = 0$$

Definition

A linear differential operator L is defined as

$$L = \mathcal{D}^n + P_1(x)\mathcal{D}^{n-1} + \cdots + P_n(x)$$

The general linear DE now can be written as

$$Ly = 0$$

Ex 2

Find the linear differential operator L in $y'' + y = 0$

Solution

$$\frac{d^2y}{dx^2} + y = 0$$
$$\left(\frac{d^2}{dx^2} + 1\right)y = 0$$
$$L = \mathcal{D}^2 + 1$$

Ex 3

Find the linear differential operator L in $y'' - 5y' + 6y = 0$

Solution

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$
$$\left(\frac{d^2}{dx^2} - 5\frac{d}{dx} + 6\right)y = 0$$
$$L = \mathcal{D}^2 - 5\mathcal{D} + 6$$

Superposition properties of linear differential operator

$$(L_1 + L_2)y = L_1y + L_2y$$
$$C(Ly) = L(Cy)$$

The above two identifies imply that the differential operator L are truly linear, satisfying superposition properties.

LI for More than Two Functions

For more than two functions, we have defined the relationship of LI. The methods at the beginning of the section for determining the LD of two functions are valid only for two functions. However, a set of functions that are mutually pair-wise LI does not necessarily indicate they are LI as a whole. We need to determine the dependence properties of more than two functions.

Ex 4

Given $y_1 = \sin x$, $y_2 = \cos x$ and $y_3 = \sin x + \cos x$, prove that they are mutually pairwise LI

Solution

It is easy to verify that

$$\frac{y_1}{y_2} = \tan x$$
$$\frac{y_3}{y_1} = 1 + \cot x$$
$$\frac{y_3}{y_2} = \tan x + 1$$

These prove that they are mutually LI

Definition (You have to review linear algebra and I'll just give out the def**):**
The Wronskian of n functions $y_1, y_2, \dots, y_n$ is the determinant,

$$W[y_1, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}_{n\times n}$$

The Wronskian was introduced by a Polish mathematician Jozef Maria Hoene-Wronski (1776-1853) in 1812.
If

$$W[y_1, \dots, y_n] \neq 0$$

then the functions are LI.

Ex 5
Check if $y_1 = \mathrm{e}^{\lambda_1 x}$and $y_2 = \mathrm{e}^{\lambda_2 x}$ $(\lambda_1 \neq \lambda_2)$ are LI.

Solution

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \mathrm{e}^{\lambda_1 x} & \mathrm{e}^{\lambda_2 x} \\ \lambda_1 \mathrm{e}^{\lambda_1 x} & \lambda_2 \mathrm{e}^{\lambda_2 x} \end{vmatrix} \\ &= \mathrm{e}^{\lambda_1 x} \cdot \lambda_2 \mathrm{e}^{\lambda_2 x} - \mathrm{e}^{\lambda_2 x} \cdot \lambda_1 \mathrm{e}^{\lambda_1 x} \\ &= (\lambda_2 - \lambda_1)\mathrm{e}^{(\lambda_2 - \lambda_1)x} \end{aligned}$$

Thus, $\lambda_2 - \lambda_1 \neq 0 \Longrightarrow W(y_1, y_2) \neq 0 \Longrightarrow y_1$and y_2 are LI.

Ex 6
Check if $y_1 = \mathrm{e}^{\lambda x}$ and $y_2 = x\mathrm{e}^{\lambda x}$ are LI.

Solution

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \mathrm{e}^{\lambda x} & x\mathrm{e}^{\lambda x} \\ \lambda \mathrm{e}^{\lambda x} & \mathrm{e}^{\lambda x} + \lambda x \mathrm{e}^{\lambda x} \end{vmatrix} \\ &= (1 + \lambda x)\mathrm{e}^{2\lambda x} - \lambda x \mathrm{e}^{2\lambda x} \\ &= \mathrm{e}^{2\lambda x} \neq 0 \end{aligned}$$

Thus, y_1 and y_2 are LI.

Ex 7
Check if $y_1 = \sin \omega x$ and $y_2 = \cos \omega x$ are LI, where $\omega \neq 0$ is a constant.

Solution

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} \sin \omega x & \cos \omega x \\ \omega \cos \omega x & -\omega \sin \omega x \end{vmatrix} \\ &= -\omega \sin^2 \omega x - \omega \cos^2 \omega x \\ &= -\omega(\sin^2 \omega x + \cos^2 \omega x) \\ &= -\omega \neq 0 \end{aligned}$$

Thus, y_1 and y_2 are LI.

Ex 8
Check the following forms LI:

$$\begin{aligned} y_1 &= \sin x \\ y_2 &= \cos x \\ y_3 &= \sin x + \cos x \end{aligned}$$

Solution

$$W[y_1, y_2, y_3] = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x \\ \cos x & -\sin x & \cos x - \sin x \\ -\sin x & -\cos x & -\sin x - \cos x \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x & (-1)*\sin x + \sin x + \cos x \\ \cos x & -\sin x & (-1)*\cos x + \cos x - \sin x \\ -\sin x & -\cos x & (-1)*(-\sin x) + -\sin x - \cos x \end{vmatrix}$$
$$= \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & -\sin x & -\sin x \\ -\sin x & -\cos x & -\cos x \end{vmatrix}$$
$$= \begin{vmatrix} \sin x & \cos x & 0 \\ \cos x & -\sin x & 0 \\ -\sin x & -\cos x & 0 \end{vmatrix}$$
$$= 0$$

Thus, y_1 and y_2 and y_3 are LD.

In the above, we used the very simple rule of determinants: The value of a determinant does not change when a multiple of another column is added to the column.

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} & \dots & a_{2k} & \dots & a_{2n} \\ & \vdots & & & & & \\ a_{n1} & \dots & a_{nj} & \dots & a_{nk} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1j} + \lambda\, a_{1k} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & \dots & a_{2j} + \lambda\, a_{2k} & \dots & a_{2k} & \dots & a_{2n} \\ & \vdots & & & & & \\ a_{n1} & \dots & a_{nj} + \lambda\, a_{nk} & \dots & a_{nk} & \dots & a_{nn} \end{vmatrix}$$

In fact, it's easy to see that,

$$y_1 + y_2 - y_3 = 0$$

we can make a non-zero vector,

$$(C_1, C_2, C_3) = (1, 1, -1)$$

to make

$$C_1 y_1 + C_2 y_2 + C_3 y_3 = 0$$

Thus, y_1 and y_2 and y_3 are LD.

The concept of **fundamental set (F.S.):**

(1) The Superposition Principle THM:

If y_1, y_2 are two solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then their linear combination is also the solution of the 2nd order linear DE. Further, if y_1, y_2 are LI, then, it forms the fundamental set of the solutions, and its linear combination is the GS of the DE.

(2) The **fundamental set** of solution

If y_1, y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

for IC's at some given point t_0

$$\begin{cases} y(t_0) = 1 \\ y'(t_0) = 0 \end{cases}$$

and

$$\begin{cases} y(t_0) = 0 \\ y'(t_0) = 1 \end{cases}$$

respectively, then y_1, y_2 form the fundamental set.

We will use the fundamental sets throughout the lectures of L.DEs.

Next, we concentrate on solution methods for a variety of L.DEs!

Type C.Homo: Homogeneous DEs with constant coefficients

$$ay'' + by' + cy = 0$$

Also shown in the following table:

Table 3. Type C.Homo for second order linear DEs

$R(x)$ \ $P(x), Q(x)$	$R(x) = 0$ **Homogeneous**	$R(x) \neq 0$ **In-Homogeneous**
Constant	$ay'' + by' + cy = 0$ (Type C.Homo)	$ay'' + by' + cy = f(x)$ (Type C.InHomo)
Variable	$A(x)y'' + B(x)y' + C(x)y = 0$ (Type V.Homo)	$A(x)y'' + B(x)y' + C(x)y = F(x)$ (Type V.InHomo)

We make the most trivial assumption: The solution of the equation is

$$y(x) = e^{\lambda x}$$

Since it's an assumed solution, it must satisfy the DE,

$$y'(x) = \lambda e^{\lambda x}$$
$$y''(x) = \lambda^2 e^{\lambda x}$$

Thus,

$$[a\lambda^2 + b\lambda + c]e^{\lambda x} = 0$$

Since $e^{\lambda x}$ is the assumed solution, $e^{\lambda x} \neq 0$, the only option is

$$a\lambda^2 + b\lambda + c = 0$$

the characteristic Eq. (Ch-Eq).

We solve this Ch-eq to get

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Since $a \neq 0$, there are three possible cases for $\lambda_{1,2}$ depending on the discriminant $b^2 - 4ac$:

(1) $b^2 - 4ac > 0$
(2) $b^2 - 4ac = 0$
(3) $b^2 - 4ac < 0$

Case 1: $\Delta = b^2 - 4ac > 0$

The Ch-Eq has two distinct real roots $\lambda_1 \neq \lambda_2$ and the two solutions are $y_{1,2}(x) = \mathrm{e}^{\lambda_{1,2}x}$ which are LI and, thus, the GS for the DE is

$$y_{GS}(x) = C_1\mathrm{e}^{\lambda_1 x} + C_2\mathrm{e}^{\lambda_2 x}$$

Example 1 Find the GS to the following DE

$$y'' - 2y' - 3y = 0$$

Solution

Solve the Ch-Eq:

$$\lambda^2 - 2\lambda - 3 = 0$$

whose roots are,

$$\lambda_{1,2} = -1, 3$$

We can compose the F.S. as

$$\{e^{-x}, e^{3x}\}$$

Thus, the GS is,

$$y_{GS}(x) = C_1 e^{-x} + C_2 e^{3x}$$

In fact, there exist infinitely many F.S.'s. Let's find a different F.S. (using the original def of FS):

Let's try IC's to find $y_1(x)$:

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

With $y(0) = 1$, we have,

$$y(0) = [C_1 e^{-x} + C_2 e^{3x}]|_{x=0} = C_1 + C_2 = 1$$

With $y'(0) = 0$, we have,

$$y'(0) = [-C_1 e^{-x} + 3C_2 e^{3x}]|_{x=0} = -C_1 + 3C_2 = 0$$

Thus,

$$y_1(x) = \frac{3}{4} e^{-x} + \frac{1}{4} e^{3x}$$

Similarly,

Let's try IC's to find $y_2(x)$:

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

With $y(0) = 0$, we have,

$$y(0) = [C_1 e^{-x} + C_2 e^{3x}]|_{x=0} = C_1 + C_2 = 0$$

With $y'(0) = 1$, we have,

$$y'(0) = [-C_1 e^{-x} + 3C_2 e^{3x}]|_{x=0} = -C_1 + 3C_2 = 1$$

Thus,

$$y_2(x) = -\frac{1}{4} e^{-x} + \frac{1}{4} e^{3x}$$

Alternate F.S.:

$$\{y_1(x), y_2(x)\} = \left\{\frac{3}{4} e^{-x} + \frac{1}{4} e^{3x}, -\frac{1}{4} e^{-x} + \frac{1}{4} e^{3x}\right\}$$

Sample individual solutions $y_1(x)$ and $y_2(x)$:

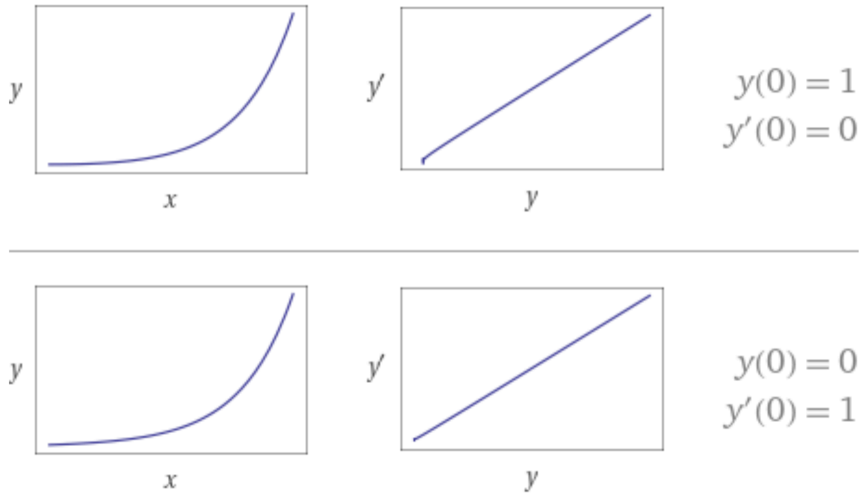

Figure 1. Sample individual solutions.

Sample solution family:

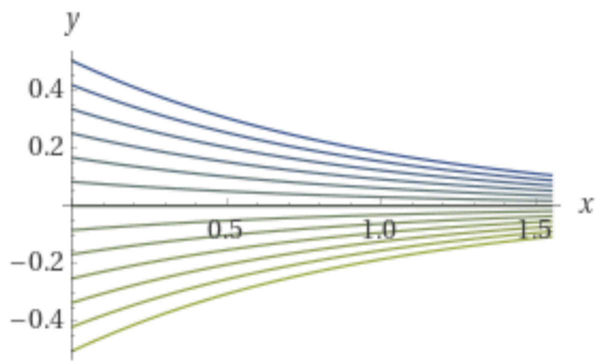

Figure 2. Sample solution family.

-------- Start of Lecture Week09.2 (03/23/2023) --------

Case 2: $\Delta = b^2 - 4ac = 0$
The Ch-Eq has two equal real roots $\lambda_1 = \lambda_2$, life here is much harder.

Example 2 Find the GS to the following DE

$$y'' - 2y' + y = 0$$

Solution

Generally, when you have two identical roots for your Ch-Eq such as this case,

$$y_1(x) = \mathrm{e}^x$$

You need to find the $y_2(x)$ that's LI of $y_1(x)$ and is also a solution of the original DE. For satisfying these, I propose

$$y_2(x) = H(x)y_1(x) = H(x)\, e^x$$
$$y_2'(x) = H'(x)e^x + H(x)\, e^x$$
$$y_2''(x) = H''(x)e^x + 2H'(x)e^x + H(x)\, e^x$$

Thus,

$$\begin{aligned} y_2''(x) - 2y_2'(x) + y_2(x) &= \underbrace{H''(x)e^x + 2H'(x)e^x + H(x)e^x}_{y_2''(x)} - 2\underbrace{(H'(x)e^x + H(x)\, e^x)}_{y_2'(x)} + \underbrace{H(x)e^x}_{y_2(x)} \\ &= H''(x)e^x \\ &= 0 \end{aligned}$$

Thus, we get

$$H''(x) = 0$$

This means

$$H(x) = C_0 + C_1 x$$

Thus, the 2nd solution can be any of the following

$$y_2(x) = (C_0 + C_1 x)\, e^x$$

Well, $(C_0)\, e^x$ belongs to $y_1(x)$ and constant C_1 is not important. We can compose the F.S. as

$$\{\mathrm{e}^x, \mathrm{e}^x x\}$$

Thus, the GS is

$$y_{GS}(x) = C_1\mathrm{e}^x + C_2\mathrm{e}^x x$$

Sample individual solutions:

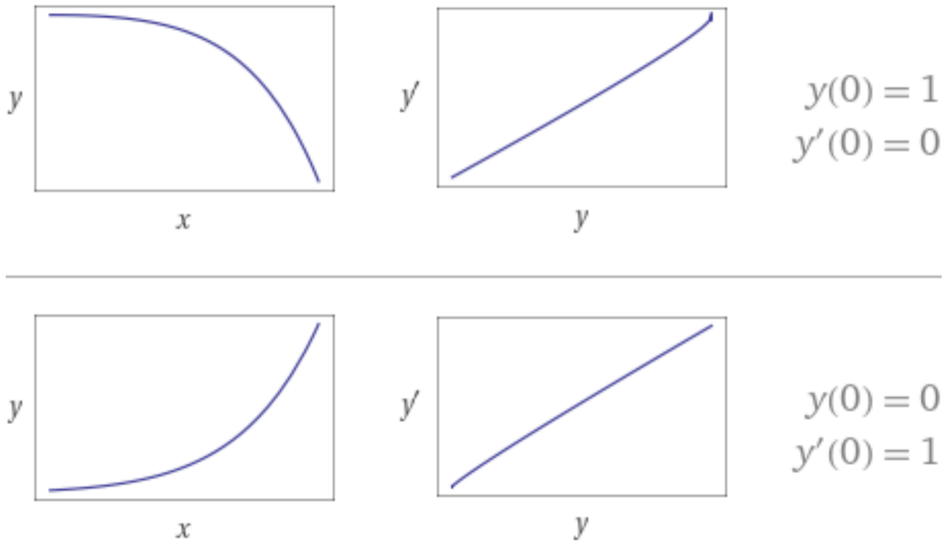

Figure 3. Sample individual solutions.

Sample solution family:

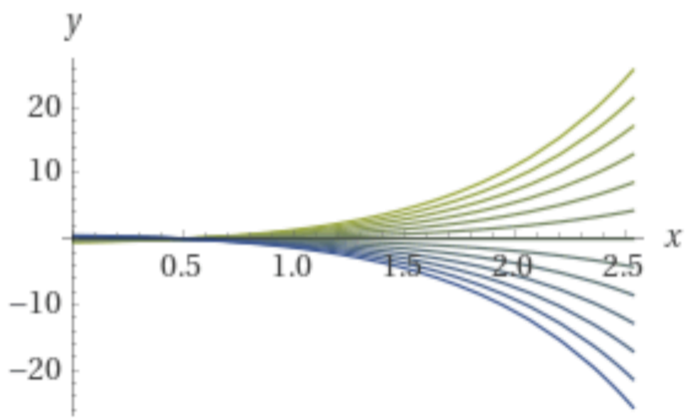

Figure 4. Sample solution family.

Case 3: $\Delta = b^2 - 4ac < 0$
The Ch-Eq has two equal real roots $\lambda_1 = \lambda_2$, life here is much harder.

Example 3 Find the GS to the following DE

$$y'' - 2y' + 2y = 0$$

Solution

Solve the Ch-Eq:

$$\lambda^2 - 2\lambda + 2 = 0$$

whose roots are

$$\lambda_{1,2} = 1 \pm i$$

We can compose the F.S. as

$$\left\{e^{(1+i)x}, e^{(1-i)x}\right\}$$

One can also simplify it as

$$\{e^x \cos x, e^x \sin x\}$$

where we have used the famous Euler formula

$$e^{ix} = \cos x + i \sin x$$

Thus, the GS is

$$y_{GS}(x) = (C_1 \sin x + C_2 \cos x)e^x$$

Sample individual solutions:

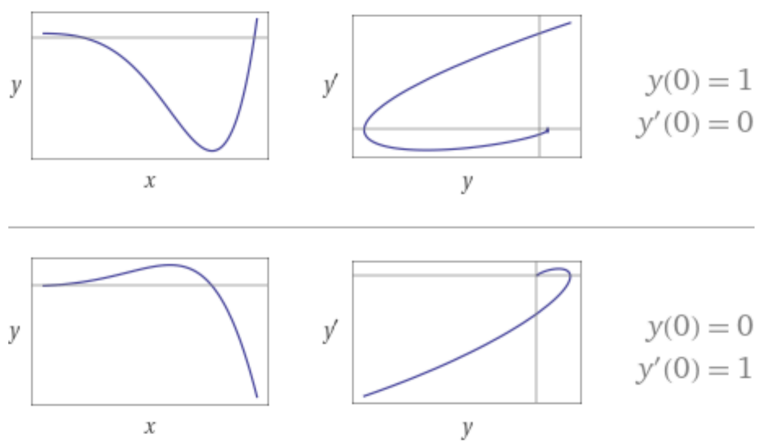

Figure 5. Sample individual solutions.

Sample solution family:

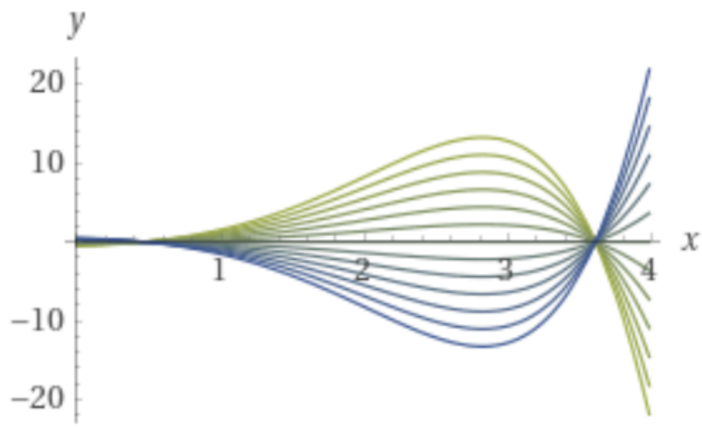

Figure 6. Sample solution family.

A brief review of operator based DEs of Type C.Homo (and it's fully generalizable):

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

Type C.Homo

$$ay'' + by' + cy = 0$$

that can be written as

$$(aD^2 + bD + c)y = 0$$

which has much convenience including a more similar looking CH-Eq

$$a\lambda^2 + b\lambda + c = 0$$

Cases 1 and 3

$$\Delta = b^2 - 4ac \begin{cases} < 0 \\ > 0 \end{cases}$$

You may write your DE as

$$(D - \lambda_1)(D - \lambda_2)y = 0$$

where $\lambda_1 \neq \lambda_2$ and can be real or complex.

Case 2 (degenerate):

$$\Delta = b^2 - 4ac = 0$$

The DE can be written as

$$(D - \lambda_1)^2 y = 0$$

Thus, I encourage you to use operators when you can.

Ex1: Now, we can easily solve problems like this,

$$(D - \lambda_1)^3 y = 0$$

whose CH-Eq has 3 identical roots:

$$\lambda_1, \lambda_1, \lambda_1$$

The F.S. is

$$\{e^{\lambda_1 x}, x\, e^{\lambda_1 x}, x^2 e^{\lambda_1 x}\}$$

Ex2: Now, we can easily solve problems like this,

$$(D - \lambda_1)(D - \lambda_2)(D - \lambda_3)y = 0$$

The F.S. is

$$\{e^{\lambda_1 x}, e^{\lambda_2 x}, e^{\lambda_3 x}\}$$

Ex3: Now, we can easily solve problems like this,

$$(D - \lambda_1)(D - \lambda_2)(D - \lambda_3)(D - \lambda_4)^4 y = 0$$

The F.S. is

$$\{e^{\lambda_1 x}, e^{\lambda_2 x}, e^{\lambda_3 x}, e^{\lambda_4 x}, x\, e^{\lambda_4 x}, x^2 e^{\lambda_4 x}, x^3 e^{\lambda_4 x}\}$$

Type V.Homo: Homo DEs with varying coefficients

$$A(x)y'' + B(x)y' + C(x)y = 0$$

This is a very interesting type as only a few can be found solutions!

Table 4. Type V.Homo: second order linear DEs

$R(x)$ \ $P(x), Q(x)$	$R(x) = 0$ **Homogeneous**	$R(x) \neq 0$ **In-Homogeneous**
Constant	$ay'' + by' + cy = 0$ (Type C.Homo)	$ay'' + by' + cy = f(x)$ (Type C.InHomo)
Variable	$A(x)y'' + B(x)y' + C(x)y = 0$ (Type V.Homo)	$A(x)y'' + B(x)y' + C(x)y = F(x)$ (Type V.InHomo)

Remark: In general, nobody knows how to find the solution of this DE! But there are some special cases that we can solve.

The Abel's THM: As a start, let's prove this interesting THM:

If y_1 and y_2 are solutions of the second-order linear differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c\exp\left(-\int p(t)\,dt\right), \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W[y_1, y_2](t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Proof: Very simple!

Since y_1 and y_2 are solutions, they satisfy the DE,

$$y_1'' + p(t)y_1' + q(t)y_1 = 0,$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Multiplying Eq(1) by y_2 and Eq(2) by y_1 and subtract, we get,

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0.$$

Well, we know

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Thus,

$$W'[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$$

Then, we have

$$W' + p(t)W = 0$$

and then

$$W(t) = c\exp\left(-\int p(t)\,dt\right)$$

QED.

Next, let's solve one simple case:

The methods for Cauchy-Euler DEs for this type (and the next):

The following special DE is called Cauchy-Euler DE's,
aka,

Euler-Cauchy DE:
Euler DE:
Equal-dimensional DE:

$$a_2x^2y'' + a_1xy' + a_0y = 0$$

A more general form

$$\sum_{m=0}^{n} a_m x^m y^{(m)} = 0$$

But the method for solving the above 2nd order Cauchy-Euler DEs can be generalized to higher order DE's. So, we focus on dealing with 2nd order!

For Type **V.Homo** DE's, we have three methods which can be interchangeable for special cases, but some methods are more general than others.

Let me give a summary of methods for solving Type **V.Homo** DE's (before I fully introduce them):

Table 5. A summary of methods for solving Type V.Homo DE's:

Methods	Applicable DEs	Resulting Solvable DEs or AEs
Method 1: Simple Trial Solution: $y(x) = x^\lambda$	$a_2x^2y'' + a_1xy' + a_0y = 0$	$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0$
Method 2: Change of Variables: $x = e^t$	$a_2x^2y'' + a_1xy' + a_0y = 0$	$a_2\ddot{y} + (a_1 - a_2)\dot{y} + a_0y = 0$
Method 3: Order Reduction: $y_2(x) = v(x)y_1(x)$	$A(x)y'' + B(x)y' + C(x)y = 0$	$Ay_1v'' + (2Ay_1' + By_1)v' = 0$

Method 1: A trivial way is to assume a trial solution,

$$y(x) = x^{\lambda}$$

The rest is quite easy,

$$y'(x) = \lambda x^{\lambda-1}$$
$$y''(x) = \lambda(\lambda-1)x^{\lambda-2}$$

Thus,

$$a_2\lambda(\lambda-1)x^{\lambda}+a_1\lambda x^{\lambda}+a_0x^{\lambda}=0$$

The Ch-Eq is

$$a_2\lambda(\lambda-1)x^{\lambda}+a_1\lambda x^{\lambda}+a_0x^{\lambda}=0$$

Or

$$a_2\lambda^2+(a_1-a_2)\lambda+a_0=0$$

Solving this will get us the λ values and then one can compose the F.S.

Example 4 Find the GS to the following DE,

$$2x^2y''+3xy'-y=0$$

Solution: This C-E DE gives us

$$a_2=2$$
$$a_1=3$$
$$a_0=-1$$

The Ch-Eq becomes,

$$a_2\lambda^2+(a_1-a_2)\lambda+a_0=0$$

Or

$$2\lambda^2+(3-2)\lambda+(-1)=0$$
$$2\lambda^2+\lambda-1=0$$
$$(2\lambda-1)(\lambda+1)=0$$

Thus,

$$\lambda_{1,2}=-1,\quad \frac{1}{2}$$

The two solutions are,

$$x^{-1},\quad x^{\frac{1}{2}}$$

These two solutions are LI and thus, the FS is

$$\left\{x^{-1},\ x^{\frac{1}{2}}\right\}$$

Note: We will revisit this example when discussing Method 2. This method has some flaws!

-------- Start of Lecture Week10.1 (03/28/2023) --------

Method 2: Change of variables (Cauchy sub).

$$x = \mathrm{e}^t$$

with which we get

$$\frac{dy}{dx} = \frac{dt}{dt}\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

Thus,

$$xy' = \frac{dy}{dt} \equiv \dot{y}$$

$$x^2y'' = \ddot{y} - \dot{y}$$

Therefore, the C-E DE will become,

$$\begin{aligned} a_2x^2y'' + a_1xy' + a_0y &= a_2(\ddot{y} - \dot{y}) + a_1\dot{y} + a_0y \\ &= a_2\ddot{y} + (a_1 - a_2)\dot{y} + a_0y = 0 \end{aligned}$$

A constant-coefficient DE (Type C.Homo) is generated,

$$a_2\ddot{y} + (a_1 - a_2)\dot{y} + a_0y = 0$$

Now, you turned your variable coefficient DE into a constant-coefficient DE.
Naturally, we can write the Ch-Eq as

$$a_2\lambda^2 + (a_1 - a_2)\lambda + a_0 = 0$$

And, as usual, it has **three possible cases**:

Case 1: 2 distinct real roots λ_1 and λ_2

The GS (in t) is

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

In terms of FS

$$\left\{e^{\lambda_1 t}, \quad e^{\lambda_2 t}\right\}$$

The GS (in x) is

$$y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$$

In terms of FS

$$\left\{x^{\lambda_1}, \quad x^{\lambda_2}\right\}$$

Solution family for different e-values looks:

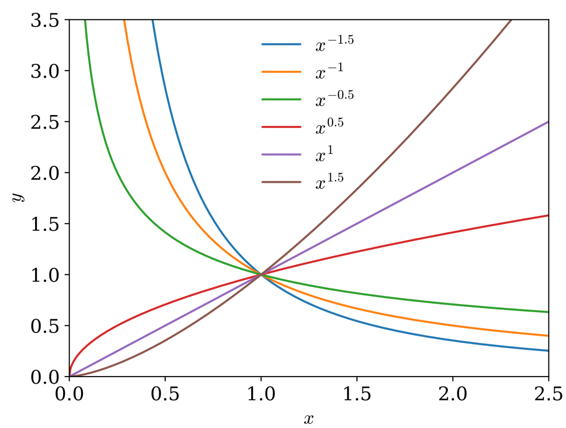

Figure 7. Solution family for equal dim DEs with distinct real roots.

Case 2: 2 distinct complex roots $\lambda_{1,2} = \alpha \pm i\beta$

The GS (in t) is

$$\begin{aligned} y(t) &= c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha+i\beta)t} \\ &= e^{\alpha t}(A\cos\beta t + B\sin\beta t) \end{aligned}$$

In terms of FS

$$\{e^{\alpha t}\cos\beta t\,, \qquad e^{\alpha t}\sin\beta t\}$$

The GS (in x) is

$$y(x) = x^{\alpha}(A\cos(\beta\ln x) + B\sin(\beta\ln x))$$

In terms of FS

$$\{x^{\alpha}\cos(\beta\ln x)\,, \qquad x^{\alpha}\sin(\beta\ln x)\}$$

Solution family for different e-values looks:

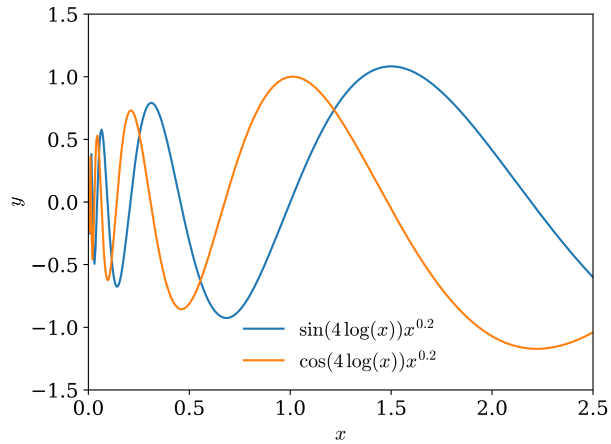

Figure 8. Solution family for equal-dim DEs with distinct complex roots.

Case 3: 2 identical roots $\lambda_{1,2} = \lambda$

The GS (in t) is

$$y(t) = e^{\lambda t}(c_1 + c_2 t)$$

In terms of FS

$$\{e^{\lambda t}, \quad te^{\lambda t}\}$$

The GS (in x) is

$$y(x) = x^{\lambda}(c_1 + c_2 \ln x)$$

In terms of FS

$$\{x^{\lambda}, \quad (\ln x)x^{\lambda}\}$$

Solution family for different e-values looks:

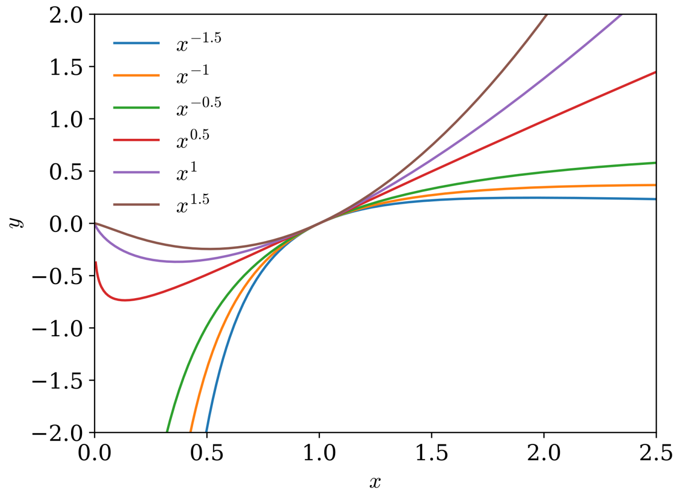

Figure 9. Solution family for equal-dim DEs with identical roots.

Example 4A Find the GS to the following DE,

$$2x^2y'' + 3xy' - y = 0$$

Solution: This C-E DE gives us

$$a_2 = 2$$
$$a_1 = 3$$
$$a_0 = -1$$

Using any one of the above methods, e.g., using the change of variable,

$$x = \mathrm{e}^t$$

we transform our DE into

$$2\ddot{y} + (3-2)\dot{y} - y = 0$$
$$2\ddot{y} + \dot{y} - y = 0$$

whose Ch-Eq is

$$2\lambda^2 + \lambda - 1 = 0$$
$$(2\lambda - 1)(\lambda + 1) = 0$$

Thus,

$$\lambda_{1,2} = -1, \frac{1}{2}$$

Thus,

$$y(t) = c_1 e^{-t} + c_2 e^{\frac{1}{2}t}$$

Now, back sub

$$y(x) = c_1 x^{-1} + c_2 x^{\frac{1}{2}}$$

Sample solution:

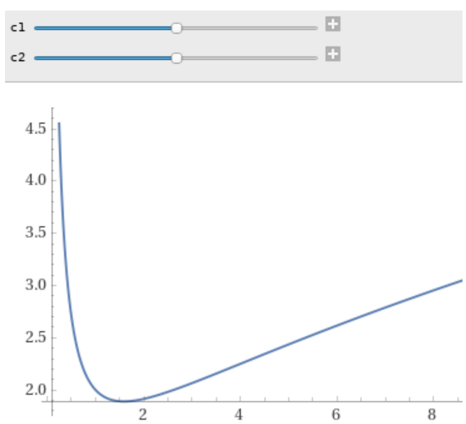

Figure 10. A PS with one given IC for this example.

Example 5 Find the GS to the following DE,

$$x^2y'' - xy' + y = 0$$

Solution: This C-E DE gives us

$$a_2 = 1$$
$$a_1 = -1$$
$$a_0 = 1$$

Using any one of the above methods, e.g., using the change of variable,

$$x = \mathrm{e}^t$$

we transform our DE into

$$\ddot{y} + (-1-1)\dot{y} + y = 0$$
$$\ddot{y} - 2\dot{y} + y = 0$$

whose Ch-Eq is

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$

Thus,

$$\lambda_{1,2} = 1, 1$$

Thus,

$$y(t) = c_1e^t + c_2te^t$$

Now, back sub

$$y(x) = c_1x + c_2x\ln x$$

Sample solution:

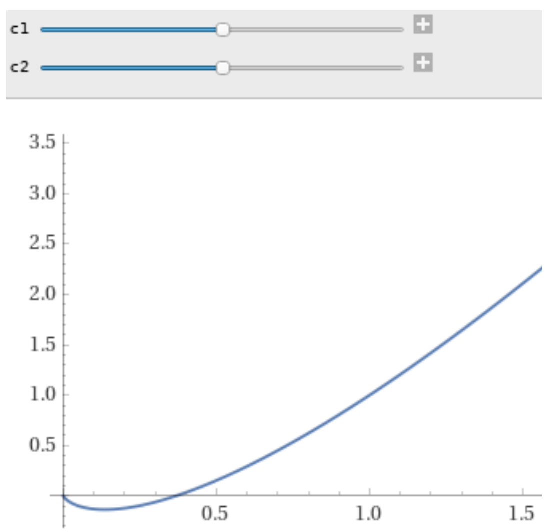

Figure 11. A PS with one given IC for this example.

Method 3: The method of reduction of order

This applies to a more general set of coefficients.

Still remember how we got the 2nd solution,

$$ay'' + by' + cy = 0$$

When,

$$\Delta = b^2 - 4ac = 0$$

This means,

$$c = \frac{b^2}{4a}$$

Or the DE is,

$$ay'' + by' + \frac{b^2}{4a}y = 0$$

For this DE, its Ch-Eq

$$a\lambda^2 + b\lambda + \frac{b^2}{4a} = 0$$

has 2 identical roots,

$$\lambda_{1,2} = -\frac{b}{2a}$$

Thus, we found one solution,

$$y_1(x) = e^{-\frac{b}{2a}x}$$

To find another one, we set

$$\begin{aligned} y_2(x) &= v(x)y_1(x) \\ &= v(x)e^{-\frac{b}{2a}x} \end{aligned}$$

Plugging this into the DE, we get

$$v''(x)y_1(x) = 0$$

Thus,

$$v''(x) = 0$$

whose solution is

$$v(x) = c_1x + c_2$$

Thus, the GS of the original DE is

$$\begin{aligned} y_G(x) &= (c_1x + c_2)y_1(x) \\ &= (c_1x + c_2)e^{-\frac{b}{2a}x} \end{aligned}$$

In fact, the above method can be generalized!

Note: Please recall Riccati DE (in CH1):

$$y' = A_0(x) + A_1(x)y + A_2(x)y^2$$

where we made Riccati sub

$$y(x) = y_1(x) + \frac{1}{u(x)}$$

But what we are doing here is quite different (yet, connected)

If you know one solution (anyone) of the DE, I call it a pilot solution $y_1(x)$, then you may assume the other solution as

$$y(x) = v(x)y_1(x)$$

Now, let's solve for $v(x)$. The best part is the DE can be in the following form:

$$A(x)y'' + B(x)y' + C(x)y = 0$$

If $y_1(x)$ is a solution, then,

$$A(x)y_1'' + B(x)y_1' + C(x)y_1 = 0$$

Now, let's try to insert the following to the DE,

$$y = vy_1$$
$$y' = v'y_1 + vy_1'$$
$$y'' = v''y_1 + vy_1'' + 2v'y_1'$$

Now, let's multiply $C(x), B(x), A(x)$ respectively,

$$Cy = Cvy_1$$
$$By' = \mathrm{B}v'y_1 + \mathrm{B}vy_1'$$
$$Ay'' = \mathrm{A}v''y_1 + \mathrm{A}vy_1'' + A2v'y_1'$$

Let's add up both sides,

$$\begin{aligned} Ay'' + By' + Cy &= \underbrace{(\mathrm{A}v''y_1 + \mathrm{A}vy_1'' + A2v'y_1')}_{Ay''} + \underbrace{(\mathrm{B}v'y_1 + \mathrm{B}vy_1')}_{By'} + Cvy_1 \\ &= (\mathrm{A}v''y_1 + \mathrm{A}vy_1'' + A2v'y_1') + (\mathrm{B}v'y_1 + \mathrm{B}vy_1') + Cvy_1 \\ &= (\mathrm{A}v''y_1 + A2v'y_1') + \mathrm{B}v'y_1 \\ &= \mathrm{A}y_1v'' + (2Ay_1' + By_1)v' = 0 \\ &= y_1\left[\mathrm{A}v'' + \left(2A\frac{y_1'}{y_1} + B\right)v'\right] = 0 \\ &= y_1\left[\mathrm{A}v'' + \left(2A(\ln y_1)' + B\right)v'\right] = 0 \end{aligned}$$

Thus, we can reduce the original Homo DE into

$$\mathrm{A}v'' + (2A(\ln y_1)' + B)v' = 0$$

If we set $u = v'$, the above DE becomes,

$$\mathrm{A}u' + (2A(\ln y_1)' + B)u = 0$$

Now, it's a first order DE! Therefore, it's called **reduction of order**.

This u DE is a simple one and easy to solve!

$$\begin{aligned} \frac{u'}{u} &= -\frac{2A(\ln y_1)' + B}{A} \\ &= -2(\ln y_1)' - \frac{B}{A} \end{aligned}$$

Summary of the order reduction method:

Given a DE

$$A(x)y'' + B(x)y' + C(x)y = 0$$

Step 1: Find or be given solution $y_1(x)$.

Step 2: Assuming another solution $y_2(x) = \boldsymbol{y_1(x)v(x)}$, **converting** the DE governing the other solution as

$$\mathrm{A}v'' + (2A(\ln y_1)' + B)v' = 0$$

Or

$$\frac{v''}{v'} = -2\left(\ln y_1\,(x)\right)' - \frac{B(x)}{A(x)}$$

Step 3: Solving this first order DE, one gets the solution for $v(x)$ and then,

$$y_2(\boldsymbol{x}) = \boldsymbol{y_1(x)v(x)}$$

Step 4: Composing the FS:

$$\{y_1(x), \qquad y_1(x)v(x)\}$$

The key to the method is to find $v(x)$!

Example 6 Find the GS to the following DE,

$$x^2y'' - 2y = 0$$

with one given solution $y_1(x) = x^2$

Solution:

Method 1: the equal-dim method (This DE is equal-dim!)

$$a_2 = 1$$
$$a_1 = 0$$
$$a_0 = -2$$

Using any one of the above methods, e.g., change of variables,

$$x = \mathrm{e}^t$$

we transform our DE into

$$\ddot{y} + (0-1)\dot{y} - 2y = 0$$
$$\ddot{y} - \dot{y} - 2y = 0$$

whose Ch-Eq is,

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda+1)(\lambda-2) = 0$$

Thus,

$$\lambda_{1,2} = -1, 2$$

Thus,

$$y(t) = c_1e^{-t} + c_2e^{2t}$$

Now, back sub

$$y(x) = c_1x^{-1} + c_2x^2$$

In terms of FS,

$$\{x^{-1}, \quad x^2\}$$

This is how it looks:

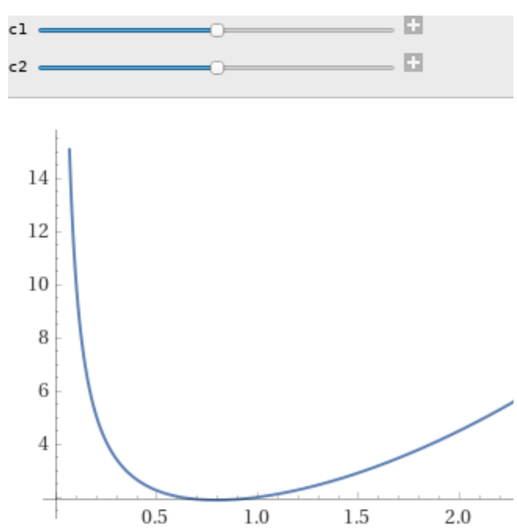

Figure 12. A PS with one given IC for this example.

Method 2: use the reduction of order method:

$$\begin{aligned} y(x) &= v(x)y_1(x) \\ &= vx^2 \end{aligned}$$

with which we have

$$\begin{aligned} y' &= v'x^2 + v(2x) \\ y'' &= v''x^2 + v'(4x) + v(2) \end{aligned}$$

Thus,

$$\begin{aligned} x^2y'' - 2y &= x^2\left(\underbrace{v''x^2 + v'(4x) + v(2)}_{y''}\right) - 2\left(\underbrace{vx^2}_{y}\right) \\ &= x^2\left(v''x^2 + v'(4x) + v(2)\right) - 2vx^2 \\ &= x^2\left(v''x^2 + v'(4x)\right) \\ &= x^3\left(v''x + v'(4)\right) \end{aligned}$$

Thus,

$$v''x + v'(4) = 0$$

This can be solved easily.
Step 1:

$$v'' = \frac{dv'}{dx} = -\frac{4}{x}v'$$

Thus,

$$v' = C_1x^{-4}$$

Step 2:

$$v = \frac{C_1}{-3}x^{-3} + c_2 = c_1x^{-3} + c_2$$

Thus, the GS for the original DE is

$$\begin{aligned} y(x) &= v(x)y_1(x) \\ &= (c_1x^{-3} + c_2)x^2 \\ &= c_1x^{-1} + c_2x^2 \end{aligned}$$

In terms of FS,

$$\{x^{-1}, \quad x^2\}$$

Fully corroborating the results obtained in Method 1.

Alternate to Method 2:

For the given DE

$$x^2y'' - 2y = 0$$

with one given solution $y_1(x) = x^2$.

We can identify the following by comparing with the standard form: $A(x)y'' + B(x)y' + C(x)y = 0$,

$$\begin{aligned} A(x) &= x^2 \\ B(x) &= 0 \\ C(x) &= -2 \end{aligned}$$

Well, if we use $y_1(x) = x^2$ and try,

$$y_2(x) = v(x)y_1(x) = vx^2$$

and use the following formula:

$$\frac{v''}{v'} = -2\big(\ln y_1(x)\big)' - \frac{B(x)}{A(x)}$$

we get the DE for $v(x)$:

$$\frac{v''}{v'} = -2(\ln x^2)' - \frac{0}{x^2}$$
$$\frac{v''}{v'} = -\frac{4}{x}$$

Thus,

$$v' = C_1 x^{-4}$$
$$v = \frac{C_1}{3} x^{-3} + c_2$$

Or

$$\begin{aligned} y_2(x) &= v(x) y_1(x) \\ &= \left(\frac{C_1}{3} x^{-3} + c_2\right) x^2 \\ &= c_1 x^{-1} + c_2 x^2 \end{aligned}$$

The result is consistent with that obtained before!

-------- Start of Lecture Week10.2 (03/30/2023) --------

Example 7 Find the GS to the following DE,

$$xy'' - (2x+1)y' + (x+1)y = 0$$

with one given solution $y_1(x) = e^x$

Solution:

Note: This DE is no longer an equal-dim DE and thus the two earlier methods that are applicable to equal-dim DE's can't be used. We need to use the method of **order reduction**.

Let

$$y = ve^x$$
$$y' = v'e^x + ve^x$$
$$y'' = v''e^x + 2v'e^x + ve^x$$

Thus,

$$xy'' - (2x+1)y' + (x+1)y = (xv'' - v')e^x = 0$$

Thus,

$$xv'' - v' = 0$$

This DE is easy to solve,

$$v' = c_1 x$$

Thus,

$$v = \frac{c_1}{2}x^2 + C_2$$

Or

$$v = C_1 x^2 + C_2$$

Thus,

$$y_G(x) = (C_1 x^2 + C_2)e^x$$

The FS can be,

$$\{x^2 e^x, \quad e^x\}$$

One of the PS's looks

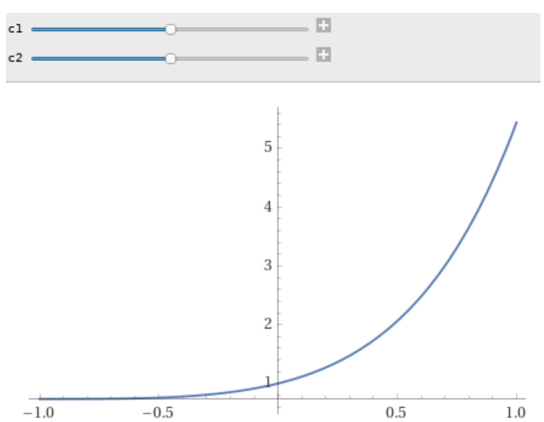

Figure 13. One sample PS.

More sample PS's:

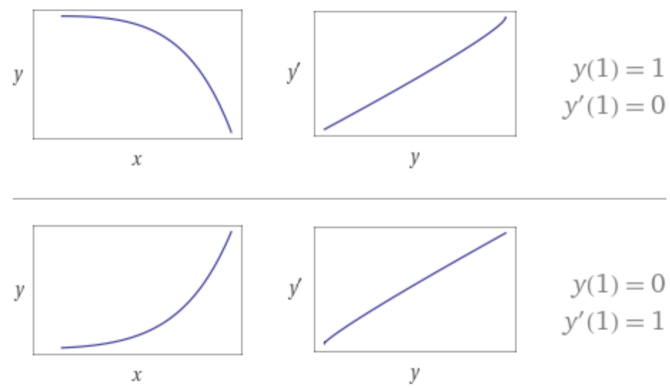

Figure 14. More sample PS's.

Sample solution family:

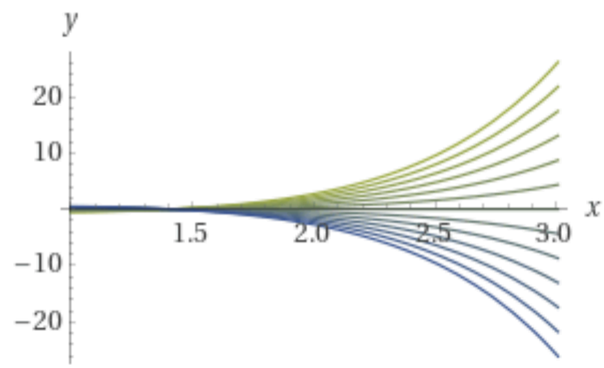

Figure 15. Sample solution family.

Summary remark: Order reduction method seems powerful to solve "general V.Homo", it has one major flaw: One must know the "pilot" solution!

Alternate to the above Method by direct use of formulas (less algebra, same math!)

For the given DE

$$xy'' - (2x+1)y' + (x+1)y = 0$$

with one given solution $y_1(x) = e^x$.

We can identify the following by comparing with the standard form: $A(x)y'' + B(x)y' + C(x)y = 0$,

$$A(x) = x$$
$$B(x) = -(2x+1)$$
$$C(x) = x+1$$

Well, if we use $y_1(x) = e^x$ and try,

$$y_2(x) = v(x)y_1(x) = ve^x$$

and use the following formula:

$$\frac{v''}{v'} = -2\big(\ln y_1(x)\big)' - \frac{B(x)}{A(x)}$$

we get the DE for $v(x)$:

$$\frac{v''}{v'} = -2(\ln e^x)' - \frac{-(2x+1)}{x}$$
$$\frac{v''}{v'} = -2 + \left[2 + \frac{1}{x}\right]$$
$$= \frac{1}{x}$$

Thus,

$$v' = C_1 x$$
$$v = \frac{C_1}{2}x^2 + c_2$$

Or

$$y_2(x) = v(x)y_1(x)$$
$$= \left(\frac{C_1}{2}x^2 + c_2\right)e^x$$
$$= c_1 x^2 e^x + c_2 e^x$$

The FS is,

$$\{x^2 e^x, \qquad e^x\}$$

Example 8 Find the GS to the following DE,

$$xy'' - (x+1)y' + y = 0$$

Solution:

With some magical trials, we found one solution $y_1(x) = e^x$. This can be easily verified!

Thus, we make a Cauchy method for the 2nd solution:

$$y_2(x) = y_1(x)v(x) = e^x v(x)$$
$$y_2'(x) = \left(e^x v(x)\right)' = e^x(v + v')$$
$$y_2''(x) = (ve^x + v'e^x)' = e^x(v + 2v' + v'')$$

Thus,

$$x\,\underbrace{e^x(v + 2v' + v'')}_{y_2''} - (x+1)\,\underbrace{e^x(v + v')}_{y_2'} + e^x v = 0$$

Or (after dividing e^x)

$$x[v + 2v' + v''] - [xv + xv' + v + v'] + v = 0$$
$$x[2v' + v''] - [xv' + v'] = 0$$
$$xv'' + [x-1]v' = 0$$

Thus,

$$v(x) = c_1 e^{-x}(x+1) + c_2$$

Finally, the GS for the original DE is,

$$y_G(x) = c_1(x+1) + c_2 e^x$$

The FS is (quite uncommonly),

$$\{x+1, \qquad e^x\}$$

As before, we can directly use the formulas:
For the given DE

$$xy'' - (x+1)y' + y = 0$$

with one given solution $y_1(x) = e^x$.

We can identify the following by comparing with the standard form: $A(x)y'' + B(x)y' + C(x)y = 0$,

$$A(x) = x$$
$$B(x) = -(x+1)$$
$$C(x) = 1$$
$$y_1(x) = e^x$$

We assume,

$$y_2(x) = v(x)y_1(x) = ve^x$$

we get the DE for $v(x)$:

$$\frac{v''}{v'} = -2\left(\ln y_1(x)\right)' - \frac{B(x)}{A(x)}$$
$$= -2(\ln e^x)' - \frac{-(x+1)}{x}$$
$$\frac{v''}{v'} = -2 + \left[1 + \frac{1}{x}\right]$$
$$= \frac{1}{x} - 1$$

Thus,

$$\ln v' = (\ln x) - x + c_1$$

Or

$$\begin{aligned} v' &= e^{(\ln x)-x+c_1} \\ &= C_1 x\, e^{-x} \end{aligned}$$

Integrating this, we get,

$$\begin{aligned} v &= \int C_1 x\, e^{-x}\, dx \\ &= C_1(x+1)\, e^{-x} + c_2 \end{aligned}$$

Thus,

$$\begin{aligned} y_2(x) &= v(x) y_1(x) \\ &= [C_1(x+1)\, e^{-x} + c_2] e^{x} \\ &= C_1(x+1) + c_2 e^{x} \end{aligned}$$

The FS is,

$$\{x+1, \qquad e^{x}\}$$

I can easily generalize this DE to many other variants:

"Similar" DE-1:

$$xy'' + (x+1)y' + y = 0$$

How about trying $y_1(x) = e^{-x}$?

"Similar" DE-2:

$$xy'' + (x-1)y' + y = 0$$

How about trying $y_1(x) = x^2 e^{-x}$?

"Similar" DE-3:

$$xy'' + (x+1)y' - y = 0$$

How about trying $y_1(x) = x+1$?

"Similar" DE-4:

$$x^2 y'' + (x-1)y' - y = 0$$

How about trying $y_1(x) = x+1$?

Example 8A (skip in spring 2023): Find the GS to the following DE,

$$x^2y'' - x(x-1)y' + (x-1)y = 0$$

with one given solution $y_1(x) = x$

Solution:

$$\begin{aligned} y(x) &= v(x)y_1(x) \\ &= vx \end{aligned}$$

Thus,

$$\begin{aligned} y' &= v'\mathrm{x} + v \\ y'' &= v''\mathrm{x} + 2v' \end{aligned}$$

Thus,

$$\begin{aligned} x^2y'' - x(x-1)y' + (x-1)y &= \mathrm{x}^2(v''\mathrm{x} + 2v') - x(x-1)(v'\mathrm{x} + v) + (x-1)vx \\ &= \mathrm{x}^2(v''\mathrm{x} + 2v') - x(x-1)(v'\mathrm{x} + v) + (x-1)vx \\ &= \mathrm{x}^2(v''\mathrm{x} + 2v') - x(x-1)v'\mathrm{x} \\ &= \mathrm{x}^3v'' + (3x^2 - x^3)v' = 0 \end{aligned}$$

Thus,

$$v'' + \left(\frac{3}{x} - 1\right)v' = 0$$

Let $u = v'$, we get

$$\frac{du}{u} = -\left(\frac{3}{x} - 1\right)dx$$

Thus,

$$\ln u = -3\ln x + x + c_1$$

$$u = C_1\frac{e^x}{x^3}$$

Thus,

$$v = C_1\int\frac{e^x}{x^3}dx + C_2$$

Finally,

$$\begin{aligned} y_G &= v(x)y_1 \\ &= \left(C_1\int\frac{e^x}{x^3}dx + C_2\right)x \\ &= \left(\frac{C_1}{2}\left(\mathrm{Ei}(x) - \frac{e^x(x+1)}{x^2}\right) + C_2\right)x \end{aligned}$$

One may leave the y_G to contain an integral form $\int\frac{e^x}{x^3}dx$ because this integral does not have a closed form expression, i.e., $\int\frac{e^x}{x^3}dx$ is not integrable.

However, the y_G can be written (unnecessary to do this for credits if it appears in HW problem or tests) with the special function, $\text{Ei}(x)$, called exponential integral,

$$\text{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$$

that has a peculiar look:

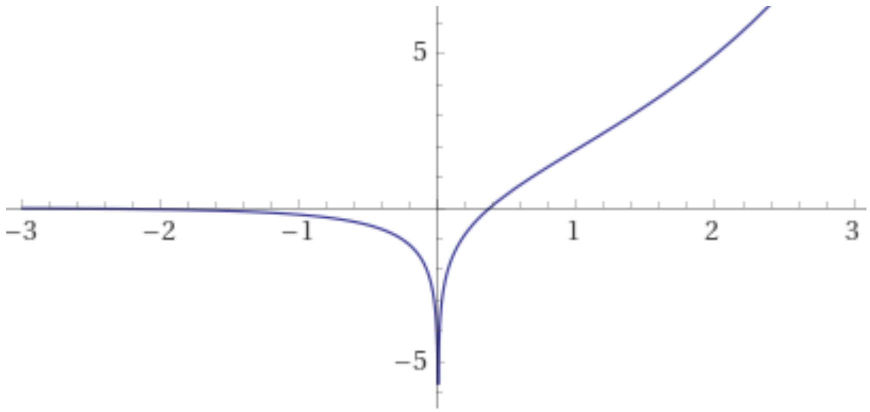

Figure 16. The exponential integral Ei(x).

Type C.InHomo: In-Homogeneous DEs with **Constant** coefficients

$$L[y] = ay'' + \mathrm{by}' + \mathrm{cy} = \mathrm{f(x)}$$

Table 6. Type C.InHomo for second order linear DEs

$R(x)$ \ $P(x), Q(x)$	$R(x) = 0$ **Homogeneous**	$R(x) \neq 0$ **In-Homogeneous**
Constant	$ay'' + by' + cy = 0$ (Type C.Homo)	$ay'' + by' + cy = f(x)$ (Type C.InHomo)
Variable	$A(x)y'' + B(x)y' + C(x)y = 0$ (Type V.Homo)	$A(x)y'' + B(x)y' + C(x)y = F(x)$ (Type V.InHomo)

Important THM:
If y_C is the GS to $L(x)y_C = 0$ and y_P is a PS to $L(x)y(x) = f(x)$, $y_G = y_C + y_P$ is the GS to the InH.DE $L(x)y_G(x) = f(x)$ where the linear operator for 2nd order DE with variable coefficients is defined as,

$$L(x) = A(x)\frac{d^2}{dx^2} + B(x)\frac{d}{dx} + C(x)$$

Proof: (Abb.: InH.DE=In-Homogeneous DE**)**
Given

$$Ly_C(x) = 0$$
$$Ly_P(x) = f(x)$$

Thus,

$$\begin{aligned} L[y_G(x)] &= L[y_C(x)+y_P(x)] \\ &= Ly_C(x) + Ly_P(x) \\ &= 0 + f(x) \\ &= f(x) \end{aligned}$$

The second step in the above is possible only if L is a linear operator!

Thus, $y_G(x)$ satisfies the original DE $L(x)y(x) = f(x)$ and is a solution. And, additionally, we claim $y_G(x)$ is the GS because the complementary solution $y_C(x)$ is the GS of the homo portion.

With this THM, the task of finding the GS of a linear InHomo DE is divided into finding the GS (=the complementary solution) of the Homo portion and one (anyone) particular solution of the whole InHomo DE, i.e.,

$$y_G(x) = y_C(x) + y_P(x)$$

This THM applies to both **Types C.InHomo** & **V.InHomo** if the In-Homo DE is linear. NOT applicable for non-linear DE's.

Example 9 Find a GS of

$$y'' + 3y' - 4y = 4x + 1$$

Example 10 Find a GS of

$$y'' + 3y' - 4y = 14\mathrm{e}^{3x}$$

Example 11 Find a GS of

$$y'' + 3y' - 4y = 34\cos x$$

3.5A Method on undetermined coefficients

The corresponding "trial" solutions for a given right-hand-side:

Table 7. The corresponding "trial" solutions for a given f(x).

$f(x)$	$y_p(x)$
$P_m(x) = b_0 + b_1 x + \cdots + b_m x^m$	$x^s(A_0 + A_1 x + A_2 x^2 + \cdots + A_m x^m)$
$a \cos kx + b \sin kx$	$x^s(A \cos kx + B \sin kx)$
e^{rx}	$x^s(A\mathrm{e}^{rx})$
$\mathrm{e}^{rx}(a \cos kx + b \sin kx)$	$\mathrm{e}^{rx}(A \cos kx + B \sin kx)$
$P_m(x)\mathrm{e}^{rx}$	$\mathrm{e}^{rx}(A_0 + A_1 x + \cdots + A_m x^m)$
$P_m(x)(a \cos kx + b \sin kx)$	$(A \cos kx + B \sin kx)(A_0 + A_1 x + \cdots + A_m x^m)$

Note: There are many other methods, but this is the simplest.

Example 9 Find a PS of

$$y'' + 3y' - 4y = 4x + 1$$

Solution:

It is obvious the complementary solution (fundamental) set is,

$$y_C = c_1 e^{-4x} + c_2 e^x$$

Now, for finding y_P, we select the following trial since $f(x) = 4x + 1$,

$$y_P = Ax + B$$

Therefore,

$$y_P' = A$$
$$y_P'' = 0$$

Substituting these in the given DE, we have

$$\begin{aligned} y_P'' + 3y_P' - 4y_P &= 0 + 3A - 4(Ax + B) \\ &= 4x + 1 \end{aligned}$$

Thus, matching coefficients of the polynomials leads to

$$\begin{cases} -4A = 4 \\ 3A - 4B = 1 \end{cases}$$
$$A = -1, \qquad B = -1$$

Therefore,

$$y_P = -x - 1$$

The GS for the original InH.DE is

$$\begin{aligned} y_G &= y_C + y_P \\ &= c_1 e^{-4x} + c_2 e^x - x - 1 \end{aligned}$$

Sample individual solutions:

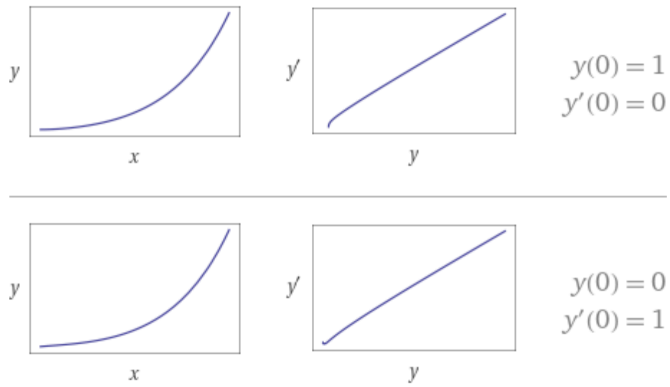

Figure 17. Sample individual solutions.

Sample solution family:

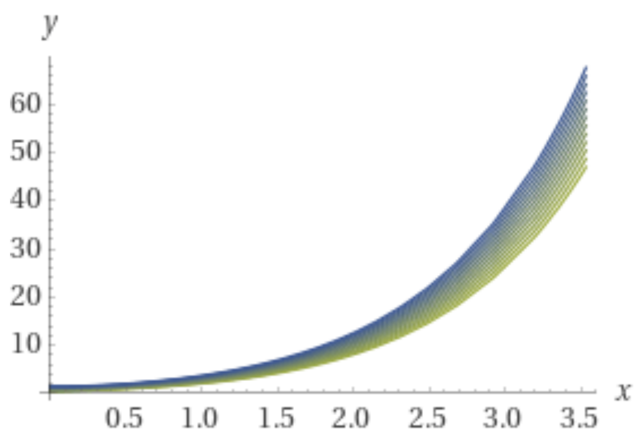

Figure 18. Sample solution family.

Example 10 Find a PS of

$$y'' + 3y' - 4y = 14e^{3x}$$

Solution:

Since any derivative of e^{3x} is a constant multiple of e^{3x}, we select the TS,

$$y_P = Ae^{3x}$$

Thus,

$$y_P' = 3Ae^{3x}$$
$$y_P'' = 9Ae^{3x}$$

Substituting it into the given DE, we have

$$\begin{aligned} y'' + 3y' - 4y &= 9Ae^{3x} + 3(3Ae^{3x}) - 4Ae^{3x} \\ &= 14Ae^{3x} \\ &= 14e^{3x} \\ A &= 1 \end{aligned}$$

Therefore,

$$y_P = e^{3x}$$

The GS for the original InH.DE is

$$y_G = y_C + y_P$$

$$= c_1 e^{-4x} + c_2 e^{x} + e^{3x}$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

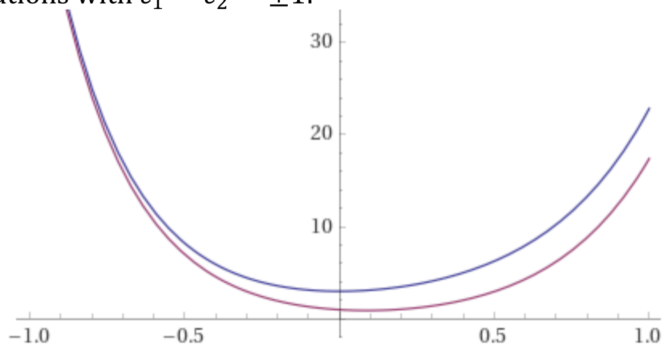

Figure 19. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = -c_2 = 1$.

Example 11 Find a GS of

$$y'' + 3y' - 4y = 34\cos x$$

Solution

Though our first guess might be $y_P = A\cos x$, the presence of y' on the LHS shows that we will have to include a term of $\sin x$ as well. Selecting the TS

$$y_P = A\cos x + B\sin x$$

we get,

$$y_P' = -A\sin x + B\cos x$$
$$y_P'' = -A\cos x - B\sin x$$

Substituting these into the given DE, we have

$$(-A\cos x - B\sin x) + 3(-A\sin x + B\cos x) - 4(A\cos x + B\sin x)$$
$$= (-5A + 3B)\cos x - (3A + 5B)\sin x = 34\cos x$$

Equating the terms of $\cos x$ and $\sin x$, we have

$$\begin{cases} -5A + 3B = 34 \\ -(3A + 5B) = 0 \end{cases}$$

Solving the above two simultaneous equations we have

$$A = -5$$
$$B = 3$$

Thus,

$$y_P = -5\cos x + 3\sin x$$

The GS for the original InH.DE is

$$y_G = y_C + y_P$$
$$= c_1 e^{-4x} + c_2 e^{x} - 5\cos x + 3\sin x$$

Sample individual solutions:

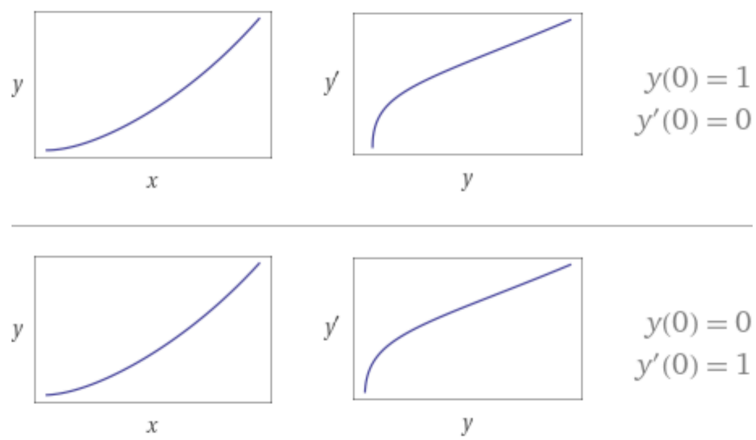

Figure 20. Sample individual solutions.

Sample solution family:

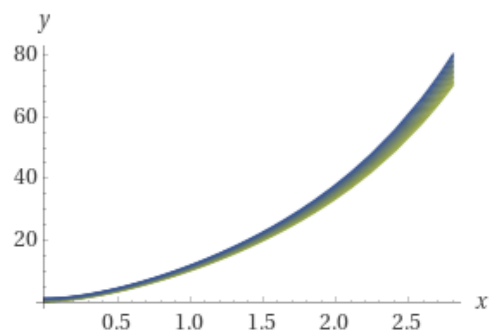

Figure 21. Sample solution family.

-------- Start of Lecture Week11.1 (04/04/2023) --------

Example 10A Find a PS of

$$y'' + 3y' - 4y = 5e^x$$

Solution:

Following the previous example, we select the TS,

$$y_P = Ae^x$$
$$y_P' = Ae^x$$
$$y_P'' = Ae^x$$

Substituting it into the given DE, we have

$$y'' + 3y' - 4y = Ae^x + 3(Ae^x) - 4Ae^x = 0$$

Thus, we can't find "A" to enable it to be equal to the RHS: $5e^x$. This is because Ae^x is a solution of the Homo DE of the InH.DE and, thus, it will always make the LSH 0.

To overcome this, we may select the following trial:

$$y_P = Axe^x$$
$$y_P' = Ae^x + Axe^x$$
$$y_P'' = 2Ae^x + Axe^x$$

Thus,

$$\begin{aligned} y'' + 3y' - 4y &= (2Ae^x + Axe^x) + 3(Ae^x + Axe^x) - 4(Axe^x) \\ &= 5Ae^x \\ &= RHS = 5e^x \end{aligned}$$

Thus,

$$A = 1$$

Therefore,

$$y_P = xe^x$$

The GS for the original InH.DE is

$$\begin{aligned} y_G &= y_C + y_P \\ &= c_1 e^{-4x} + c_2 e^x + xe^x \end{aligned}$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

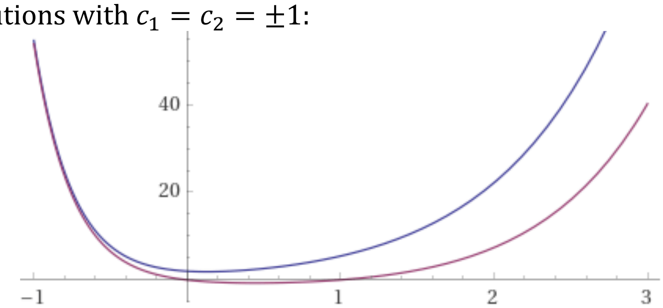

Figure 22. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = -c_2 = 1$.

Example 12 Find a GS of

$$y'' + 3y' - 4y = 50e^{-x}\cos x$$

Solution

This is different from the previous Examples.

We need to introduce a new trial PS:

$$\begin{aligned}
y_P &= e^{-x}(A\cos x + B\sin x) \\
y_P' &= \mathrm{e}^{-x}\big((-\mathrm{A}+\mathrm{B})\cos x - (\mathrm{A}+\mathrm{B})\sin x\big) \\
y_P'' &= \mathrm{e}^{-x}\big((-2\mathrm{B})\cos x + (2\mathrm{A})\sin x\big)
\end{aligned}$$

$$\begin{aligned}
y'' + 3y' - 4y &= e^{-x}\big((-2B + 3(-A+B) - 4A)\cos x + (2A + 3(-A-B) - 4B)\sin x\big) \\
&= e^{-x}\big((B-7A)\cos x + (-A-7B)\sin x\big) \\
&= 50e^{-x}\cos x
\end{aligned}$$

Thus,

$$\begin{cases} B - 7A = 50 \\ -(A+7B) = 0 \end{cases}$$

Thus,

$$\begin{aligned} A &= -7 \\ B &= 1 \end{aligned}$$

Thus,

$$y_P = e^{-x}(-7\cos x + \sin x)$$

The GS for the original InH.DE is

$$\begin{aligned}
y_G &= y_C + y_P \\
&= c_1e^{-4x} + c_2e^{x} + e^{-x}(-7\cos x + \sin x)
\end{aligned}$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

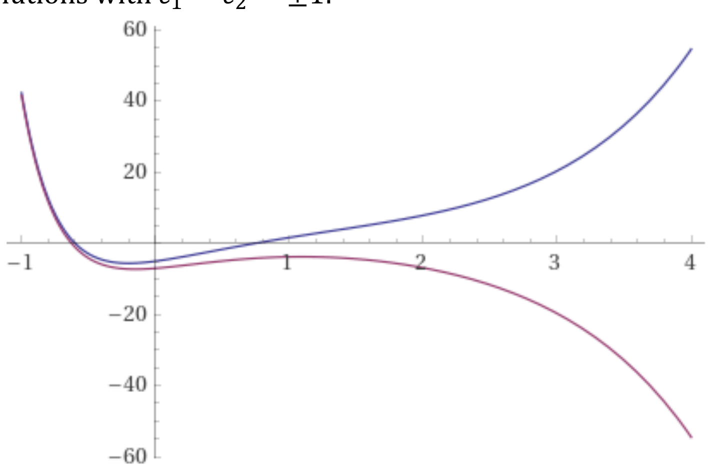

Figure 23. Two typical solutions (blue for + and red for -) $c_1 = c_2 = \pm 1$.

Example 12A Find a GS of

$$y'' + 3y' - 4y = 26e^x \cos x$$

Solution:

The method (procedure) is similar, and we can find the GS for this InH.DE as

$$\begin{aligned} y_G &= y_C + y_P \\ &= c_1 e^{-4x} + c_2 e^x + e^x(-\cos x + 5\sin x) \end{aligned}$$

In the solution of this InH.DE, we have a serious issue of e^x is a solution of the Homo DE when finding y_P. Which is the trial solution (TS)?

TS1: $y_P = e^x(A\cos x + B\sin x)$

TS2: $y_P = xe^x(A\cos x + B\sin x)$

It turns out, TS1 is correct because each term in TS1 $e^x(A\cos x)$ and $e^x(B\sin x)$ will not be solution of the Homo DE. So, they are sufficient to be considered as the TS.

It turns out, TS2 is not a correct trial!

Now, let's try to find A and B:

$$\begin{aligned} y_P &= e^x(A\cos x + B\sin x) \\ y_P' &= e^x\big((B + A)\cos x + (B - A)\sin x\big) \\ y_P'' &= e^x(2B\cos x - 2A\sin x) \end{aligned}$$

In fact, the above derivatives are quite easy to compute (no need to go through detailed steps).

Now,

$$\begin{aligned} y_P'' + 3y_P' - 4y_P &= e^x\big((5B - A)\cos x - (5A + B)\sin x\big) \\ &= 26e^x \cos x \end{aligned}$$

Matching coefficients yields

$$\begin{cases} 5B - A = 26 \\ -(5A + B) = 0 \end{cases}$$

Thus,

$$A = -1$$
$$B = 5$$

Thus, we get

$$y_P = e^x(-\cos x + 5\sin x)$$

Thus,

$$\begin{aligned} y_G &= y_C + y_P \\ &= c_1 e^{-4x} + c_2 e^x + e^x(-\cos x + 5\sin x) \end{aligned}$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

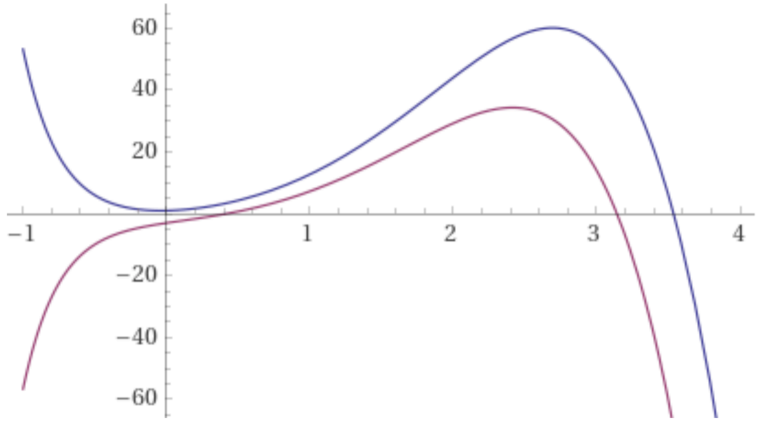

Figure 24. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = -c_2 = 1$.

Note: there are several other methods for finding the y_P of the Type C.InHomo DEs including the D operator method. Let me not lecture on these but type up one important example with two solution methods to corroborate.

Additionally, I would type up the solution using the D operator method for problems in our HW set.

Ex1 Find the y_P of the following:

$$y'' + 3y' - 4y = x^2$$

Solution method 1 (the D operator method, not require for AMS 361):

Rewrite the DE in terms of D operator:

$$(D^2 + 3D - 4)y = x^2$$

Thus,

$$\begin{aligned}
y_P(x) &= \frac{1}{D^2 + 3D - 4}(x^2) \\
&= \frac{1}{5}\left(\frac{1}{D-1} - \frac{1}{D+4}\right)(x^2) \\
&= -\frac{1}{5}\left(\frac{1}{1-D} + \frac{1}{4}\frac{1}{1+\frac{D}{4}}\right)(x^2) \\
&= -\frac{1}{5}\left((1 + D + D^2 + D^3 + \cdots) + \frac{1}{4}\left(1 - \frac{1}{4}D + \left(\frac{1}{4}D\right)^2 - \left(\frac{1}{4}D\right)^3 + \cdots\right)\right)(x^2) \\
&= -\frac{1}{5}\left((x^2 + 2x + 2 + 0 + 0 + \cdots) + \frac{1}{4}\left(x^2 - \frac{1}{4} * 2x + \frac{1}{16} * 2 - 0 + \cdots\right)\right) \\
&= -\frac{1}{5}\left((x^2 + 2x + 2) + \frac{1}{4}\left(x^2 - \frac{1}{2}x + \frac{1}{8}\right)\right) \\
&= -\frac{1}{5}\left(\frac{5}{4}x^2 + \frac{15}{8}x + \frac{65}{32}\right) \\
&= -\left(\frac{1}{4}x^2 + \frac{3}{8}x + \frac{13}{32}\right)
\end{aligned}$$

Therefore,

$$y_P(x) = -\left(\frac{1}{4}x^2 + \frac{3}{8}x + \frac{13}{32}\right)$$

Note: in the above, I used the (most famous) power series for the D operator

$$\frac{1}{1-D} = 1 + D + D^2 + D^3 + \cdots$$

Remember, the power series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

where $|x| < 1$ is the condition for the series to converge. We *pretend* this is true for the D operator.

Solution method 2 (M.U.C., required for AMS 361):

Following the Rule for polynomials, we propose a TS as,

$$y_P(x) = a_2x^2 + a_1x + a_0$$
$$y_P'(x) = 2a_2x + a_1$$
$$y_P''(x) = 2a_2$$

Plugging into the original DE, we get

$$\underbrace{2a_2}_{y_P''} + 3\underbrace{(2a_2x + a_1)}_{y_P'} - 4\underbrace{(a_2x^2 + a_1x + a_0)}_{y_P} = x^2$$

which means

$$\begin{cases} -4a_2 = 1 \\ 6a_2 - 4a_1 = 0 \\ 2a_2 + 3a_1 - 4a_0 = 0 \end{cases}$$

Thus,

$$\begin{cases} a_2 = -\dfrac{1}{4} \\ a_1 = -\dfrac{3}{8} \\ a_0 = -\dfrac{13}{32} \end{cases}$$

Thus, we get the same y_P as,

$$y_P(x) = -\left(\frac{1}{4}x^2 + \frac{3}{8}x + \frac{13}{32}\right)$$

Both methods corroborate!

Ex2 Find a PS $y_P(x)$ for

$$y''' + y'' + y' + y = 1 - x + e^x - \cos x$$

Solution Method 1 (D operator method, not required in AMS 361):

$$D = \frac{d}{dx}$$

we convert the DE into

$$(D^3 + D^2 + D + 1)y = 1 - x + e^x - \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

and

$$P(D) = D^3 + D^2 + D + 1$$

We write,

$$f(x) = f_1(x) + f_2(x) = 1 - x + e^x - \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$
$$f_1(x) = 1 - x$$
$$f_2(x) = e^x - \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

We use MUC to find y_P associated with $f_1(x) = 1 - x$:

$$y_{P1} = 2 - x$$

(In principle, the corresponding y_{P1} for the InHomo term $f_1(x) = P_1(x) = 1 - x$) by the D operator method.)

We use D operator to find y_P associated with $f_2(x)$. To do so, we need

Now, we introduce the following THM (which is quite easy to prove.)

THM1:DE $P(D)y = e^{ax}$ has a PS,

$$y_P = \frac{e^{ax}}{P(a)} \text{ if } P(a) \neq 0$$

or

$$y_P = \frac{xe^{ax}}{P'(a)} \text{ if } P'(a) \neq 0$$

where $P(D)$ is a polynomial.

Using this THM, we get

$$y_{P2}(x) = \frac{e^x}{P(1)} - \frac{1}{2}\left(\frac{xe^{ix}}{P'(i)} + \frac{xe^{-ix}}{P'(-i)}\right)$$

where

$$\begin{aligned} P(1) &= 4 \\ P(i) &= i^3 + i^2 + i + 1 = 0 \\ P(-i) &= (-i)^3 + (-i)^2 + (-i) + 1 = 0 \\ P'(i) &= 3 * i^2 + 2 * i + 1 = 2i - 2 \\ P'(-i) &= 3 * (-i)^2 + 2 * (-i) + 1 = -2i - 2 \end{aligned}$$

Thus,

$$\begin{aligned} y_{P2}(x) &= \frac{e^x}{4} - \frac{1}{2}\left(\frac{xe^{ix}}{2i - 2} - \frac{xe^{-ix}}{2i + 2}\right) \\ &= \frac{1}{4}(e^x + x\cos x - x\sin x) \end{aligned}$$

Finally,

$$y_P = y_{P1} + y_{P2} = 2 - x + \frac{1}{4}(e^x + x\cos x - x\sin x)$$

Solution Method 2 (MUC, required in AMS 361):

We need to apply a lot of rules for the trial PS!

First, we find the FS by solving,

$$y''' + y'' + y' + y = 0$$

Or its CH-Eq

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

Or

$$(\lambda^2 + 1)(\lambda + 1) = 0$$

Thus,

$$\lambda_{1,2,3} = -1, i, -i$$

The FS can be,

$$\{e^{-x}, \cos x, \sin x\}$$

Now, to start MUC, we must use the rules to try,

$$y_P(x) = \underbrace{(a_0 + a_1 x)}_{\text{For } 1-x} + \underbrace{be^x}_{\text{For } e^x} + \underbrace{c\, x\cos x + s\, x\sin x}_{\text{For } \cos x}$$

Thus,

$$y_P'(x) = a_1 + be^x + c(\cos x - x\sin x) + s(\sin x + x\cos x)$$
$$y_P''(x) = be^x - c(2\sin x + x\cos x) - s(x\sin x - 2\cos x)$$
$$y_P'''(x) = be^x - c(3\cos x - x\sin x) - s(3\sin x + x\cos x)$$

Plugging all these into the DE, we get

$$\begin{aligned} y_P''' + y_P'' + y_P' + y_P &= (a_0 + a_1) + a_1 x + 4be^x + 2(s - c)\cos x - 2(s + c)\sin x \\ &= 1 - x + e^x - \cos x \end{aligned}$$

Matching coefficients, we get,

$$\begin{cases} a_0 + a_1 = 1 \\ a_1 = -1 \\ 4b = 1 \\ 2(s - c) = -1 \\ 2(s + c) = 0 \end{cases}$$

Easily, we get,

$$a_0 = 2$$
$$a_1 = -1$$
$$b = \frac{1}{4}$$
$$c = \frac{1}{4}$$
$$s = -\frac{1}{4}$$

Thus, a PS is,

$$y_P(x) = 2 - x + \frac{1}{4}e^x + \frac{1}{4}x\cos x - \frac{1}{4}x\sin x$$

perfectly corroborating the PS found by the D operator method.

Ex3 Find GS of (similar to a problem in HW6)

$$x^3y''' + 6x^2y'' + 4xy' - 4y = 1 - x - 2x^2$$

Solution: Making Cauchy sub

$$x = e^t$$

we get (See formula 3.119 on p. 180 textbook)

$$xy' = \dot{y}$$
$$x^2y'' = \ddot{y} - \dot{y}$$
$$x^3y''' = \dddot{y} - 3\ddot{y} + 2\dot{y}$$

The DE becomes,

$$(\dddot{y} - 3\ddot{y} + 2\dot{y}) + 6(\ddot{y} - \dot{y}) + 4\dot{y} - 4y = 1 - e^t - 2e^{2t}$$

Or

$$\dddot{y} + 3\ddot{y} - 4y = 1 - e^t - 2e^{2t}$$

The Ch-Eq is,

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

whose roots are

$$\lambda_{1,2,3} = 1, -2, -2$$

Thus, the FS is

$$\{e^t, e^{-2t}, te^{-2t}\}$$

Now, if we use the D operator to find the y_P, we need to introduce the operator,

$$D = \frac{d}{dt}$$

The DE now becomes,

$$(D^3 + 3D^2 - 4)y = 1 - e^t - 2e^{2t}$$

For our problem,

$$P(D) = D^3 + 3D^2 - 4$$

The RHS DE can be written as

$$f(t) = e^{0t} - e^t - 2e^{2t}$$

We also know,

$$P(0) = -4$$
$$P(1) = 0$$
$$P'(1) = 3 * 1^2 + 6 * 1 - 0 = 9$$
$$P(2) = 16$$

Thus,

$$y_P(t) = \frac{e^{0t}}{P(0)} - \frac{te^{1t}}{P'(1)} - 2\frac{e^{2t}}{P(2)}$$
$$= \frac{1}{-4} - \frac{te^t}{9} - 2\frac{e^{2t}}{16}$$
$$= -\frac{1}{4} - \frac{te^t}{9} - \frac{e^{2t}}{8}$$

Alternatively, we can find a PS by MUC.

After getting the following Type C.InHomo DE,

$$\dddot{y} + 3\ddot{y} - 4y = 1 - e^t - 2e^{2t}$$

we know the FS to be,

$$\{e^t, e^{-2t}, te^{-2t}\}$$

Thus, the TS is,

$$y_P(x) = \underbrace{(a_0)}_{\text{For } 1} + \underbrace{bte^t}_{\text{For } e^t} + \underbrace{c\,e^{2t}}_{\text{For } e^{2t}}$$

Thus,

$$\begin{aligned} y_P''' + 3y_P'' - 4y_P &= -4a_0 + 9be^t + 16ce^{2t} \\ &= 1 - e^t - 2e^{2t} \end{aligned}$$

Matching coefficients, we get,

$$\begin{cases} -4a_0 = 1 \\ 9b = -1 \\ 16c = -2 \end{cases}$$

Thus,

$$a_0 = -\frac{1}{4}$$
$$b = -\frac{1}{9}$$
$$c = -\frac{1}{8}$$

Thus,

$$y_P(t) = -\frac{1}{4} - \frac{1}{9}te^t - \frac{1}{8}e^{2t}$$

If we wish to compose the GS in t, here it is,

$$y_G(t) = c_1e^t + c_2e^{2t} + c_3te^{2t} - \frac{1}{4} - \frac{1}{9}te^t - \frac{1}{8}e^{2t}$$

If we wish to compose the GS in x, just back sub with $t = \ln x$ and get,

$$y_G(x) = c_1x + c_2x^2 + c_3x^2\ln x - \frac{1}{4} - \frac{1}{9}x\ln x - \frac{1}{8}x^2$$

Type V.InHomo: InHomo DEs with varying coefficients

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

This is the most general type!

Table 8. Type V.InHomo: second order linear DEs

$R(x)$ / $P(x), Q(x)$	$R(x) = 0$ **Homogeneous**	$R(x) \neq 0$ **In-Homogeneous**
Constant	$ay'' + by' + cy = 0$ (Type C.Homo)	$ay'' + by' + cy = f(x)$ (Type C.InHomo)
Variable	$A(x)y'' + B(x)y' + C(x)y = 0$ (Type V.Homo)	$A(x)y'' + B(x)y' + C(x)y = F(x)$ (Type V.InHomo)

There are many methods for this type of DEs but I would not go through the theory but demonstrate through a few examples.

I would use this example to demonstrate order reduction!

Example 13 Find a GS of

$$xy'' - (2x+1)y' + (x+1)y = x^2$$

with one given solution $y_1(x) = e^x$ for the Homo DE (complementary DE)

Solution:

Using the given conditions of the Homo DE, we have

$$y_1(x) = e^x$$
$$A(x) = x$$
$$B(x) = -(2x+1)$$
$$C(x) = x+1$$

(Note: $C(x) = x + 1$ is not needed here on, but it's included for completeness.)

$$(\ln y_1(x))' = (\ln e^x)' = 1$$

Now, applying the method of order reduction with

$$y(x) = v(x)y_1(x) = v(x)e^x$$

we turn the Homo DE into

$$Av'' + (2A(\ln y_1)' + B)v' = 0$$

Or

$$xv'' + (2x * 1 - (2x+1))v' = 0$$
$$xv'' - v' = 0$$

Thus, the original InH.DE becomes,

$$e^x(xv'' - v') = x^2$$

This DE is now easily solvable, e.g., let $u = v'$, because it is now a 1st.O. linear DE (order reduction):

$$xu' - u = x^2e^{-x}$$

Or

$$u' - \left(\frac{1}{x}\right)u = xe^{-x}$$

Using 1st order linear DE method (the I.F. method), we get

$$\rho(x) = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Thus, the DE becomes,

$$\frac{1}{x}u' - \left(\frac{1}{x^2}\right)u = e^{-x}$$
$$\left(\frac{1}{x}u\right)' = e^{-x}$$

Thus,

$$\frac{1}{x}u = -e^{-x} + c_1$$

Or

$$u = -xe^{-x} + c_1x$$

Thus,

$$\begin{aligned} v &= \int u\,dx \\ &= \int (-xe^{-x} + c_1x)\,dx \\ &= (x+1)e^{-x} + \frac{c_1}{2}x^2 + c_2 \end{aligned}$$

Finally, the GS for the original DE is,

$$y_G = vy_1 = ve^x = \left((x+1)e^{-x} + \frac{c_1}{2}x^2 + c_2\right)e^x$$

$$= \frac{c_1}{2}x^2e^x + c_2e^x + x + 1$$

For appearance purposes, we can express,

$$y_G = \underbrace{C_1e^x + C_2x^2e^x}_{y_C} + \underbrace{(x+1)}_{y_P}$$

It turns out, we can show the following:

(1) The following

$$y_C = C_1e^x + C_2x^2e^x$$

is the GS of complementary (Homo) DE.

To validate the above statement, we need to prove e^x and x^2e^x satisfy the Homo DE. The first e^x is automatic because it's given. Now, let's examine $y_2 = x^2e^x$:

$$xy_2'' - (2x+1)y_2' + (x+1)y_2 = x\underbrace{(x^2e^x)''}_{y_2''} - (2x+1)\underbrace{(x^2e^x)'}_{y_2'} + (x+1)\left(\underbrace{x^2e^x}_{y_2}\right)$$
$$= xe^x(x^2+4x+2) - (2x+1)e^x(x^2+2x) + (x+1)(x^2e^x)$$
$$= e^x[x(x^2+4x+2) - (2x+1)(x^2+2x) + (x+1)(x^2)]$$
$$= e^x[(x^3+4x^2+2x) - (2x^3+5x^2+2x) + (x^3+x^2)]$$
$$= 0$$

(2) The following

$$y_P = x + 1$$

is a PS of the entire InH.DE because,

$$xy_P'' - (2x+1)y_P' + (x+1)y_P = x*0 - (2x+1)*1 + (x+1)*(x+1)$$
$$= x^2$$

where we used

$$y_P = x + 1$$
$$y_P' = 1$$
$$y_P'' = 0$$

Therefore,

$$y_G = y_C + y_P$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

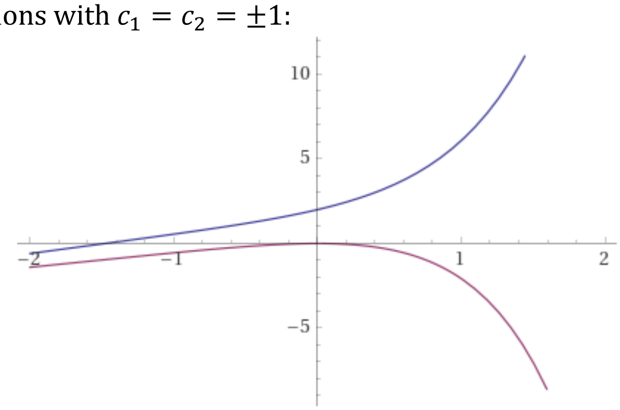

Figure 25. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = c_2 = -1$.

Remark: This method is quite picky, and the Homo portion must be capable of "order reduction"!

Example 14 Find a GS of

$$x^2y'' + xy' - y = x^2 + 1$$

Solution: We have two methods (the simple Cauchy method and order reduction method) to solve this problem:

Method 1: Cauchy method with Cauchy sub: $x = e^t$

Noticing the structure of the Homo DE (complementary DE), we make the Cauchy-like variable change

$$x = e^t$$

with which we convert the InH.DE into

$$\ddot{y} + (1-1)\dot{y} - y = e^{2t} + 1$$
$$\ddot{y} - y = e^{2t} + 1$$

Thus, the Ch-Eq is

$$\lambda^2 - 1 = 0$$

Thus, the GS of the complementary DE is

$$y_C(t) = c_1e^t + c_2e^{-t}$$

We can make a trial PS

$$y_P(t) = Ae^{2t} + B$$
$$\dot{y}_P(t) = 2Ae^{2t}$$
$$\ddot{y}_P(t) = 4Ae^{2t}$$

Thus,

$$\begin{aligned}\ddot{y} - y &= 4Ae^{2t} - (Ae^{2t} + B)\\ &= 3Ae^{2t} - B\\ &= e^{2t} + 1\end{aligned}$$

Matching coefficients, we get

$$\begin{cases}3A = 1\\ -B = 1\end{cases}$$

Thus,

$$y_P(t) = \frac{1}{3}e^{2t} - 1$$

Finally (almost), the GS for the InH.DE is

$$\begin{aligned}y_G(t) &= y_C(t) + y_P(t)\\ &= c_1e^t + c_2e^{-t} + \frac{1}{3}e^{2t} - 1\end{aligned}$$

Finally (truly), after back sub, the GS for the InH.DE is

$$y_G(x) = c_1x + c_2\frac{1}{x} + \frac{1}{3}x^2 - 1$$

Method 2: order reduction method

We notice (magically) a solution for the complementary DE (i.e., the Homo portion of the V.InHomo),

$$y_1(x) = x$$

***Note**: This solution $y_1(x)$ is only for the complementary DE (NOT for the entire DE).*

That you can easily verify by plugging it into the complementary DE,

$$x^2y'' + xy' - y = 0$$

With this, we can change the original InH.DE after setting,

$$y(x) = v(x)y_1(x) = vx$$

$$y'(x) = v'x + v$$
$$y''(x) = v''x + 2v'$$

Thus,

$$x^2y'' + xy' - y = x^2\left(\underbrace{v''x + 2v'}_{y''}\right) + x\left(\underbrace{v'x + v}_{y'}\right) - vx$$
$$= x^3v'' + 3x^2v'$$
$$= x^2 + 1$$

Thus,

$$v'' + \frac{3}{x}v' = \frac{1}{x} + \frac{1}{x^3}$$

Easily, we can turn this into a 1st order linear DE and then solve (omitting the details):

$$v = \frac{x}{3} - \frac{1}{x} + \frac{c_2}{x^2} + c_1$$

Finally, the GS for the InH.DE is

$$y = vx = \left(\frac{x}{3} - \frac{1}{x} + \frac{c_2}{x^2} + c_1\right)x$$
$$= \underbrace{c_1x + c_2\frac{1}{x}}_{y_C} + \underbrace{\frac{1}{3}x^2 - 1}_{y_P}$$

Both methods corroborate!

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

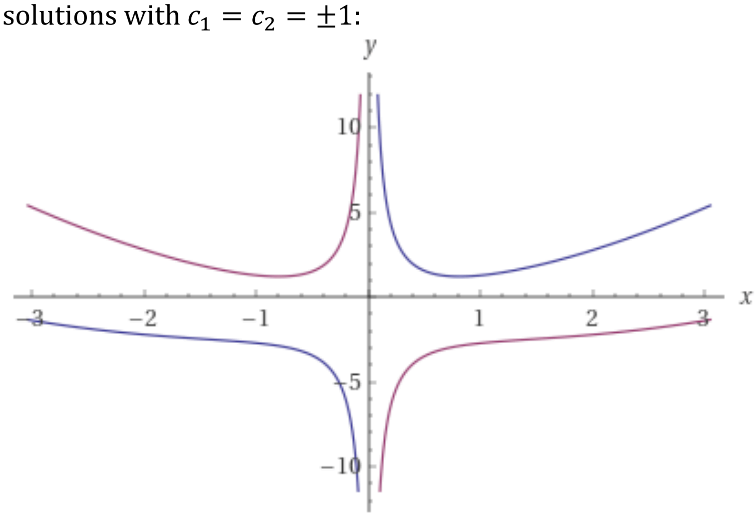

Figure 26. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = c_2 = -1$.

-------- Start of Lecture Week11.2 (04/06/2023) --------

Example 14A Find a GS of

$$x^2y'' + xy' + y = x^2 + x + 1$$

Solution: We have several methods to solve this problem:
Noticing the structure of the Homo DE, we make the Cauchy-like variable change,

$$x = e^t$$

The DE becomes,

(Space reserved to demonstrate the details during lectures)

The GS is,

$$y_G(t) = y_C(t) + y_P(t) = c_1 \cos t + c_2 \sin t + \frac{1}{5}\mathrm{e}^{2\mathrm{t}} + \frac{1}{2}\mathrm{e}^{\mathrm{t}} + 1$$

Back sub,

$$y_G(x) = c_1 \cos \ln x + c_2 \sin \ln x + \frac{1}{5}x^2 + \frac{1}{2}x + 1$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

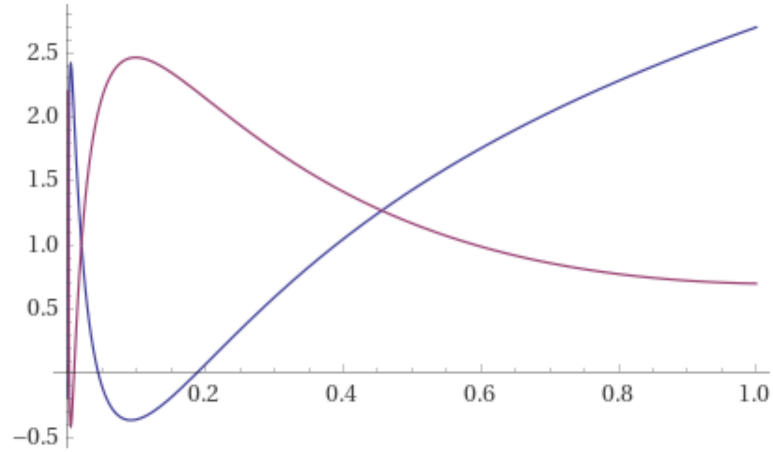

Figure 27. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = c_2 = -1$.

Example 15 Find a GS of

$$y'' + y = \frac{\sin x}{\cos x}$$

Solution: (We have, at least, two methods to solve this type of problems.)

Method 1: (Reduction of Order) The fundamental set $\{y_1(x), y_2(x)\}$ for the complementary DE

$$\{y_1(x), y_2(x)\} = \{\cos x, \sin x\}$$

Then, we set,

$$y_1(x) = \cos x$$

(Space reserved to demonstrate the details during lectures)

Finally, we reduce our original 2nd order DE to a first order one:

$$v'' \cos x - 2v' \sin x = \frac{\sin x}{\cos x}$$

and the I.F. method leads to

$$\rho(x) = e^{-2\int \frac{\sin x}{\cos x} dx} = e^{\ln \cos^2 x} = \cos^2 x$$

Thus, the DE becomes,

$$(\cos x)^2 (u' \cos x - 2u \sin x) = (\cos x)^2 \frac{\sin x}{\cos x}$$
$$\cos x\,(u' \cos x - 2u \sin x) = \sin x$$
$$\underbrace{(\cos x)^2 u' - 2 \cos x \sin x\, u}_{(u \cos^2 x)'} = \sin x$$

Or

$$(u \cos^2 x)' = \sin x$$

Thus,

$$u \cos^2 x = -\cos x + c_1$$

Therefore,

$$u(x) = c_1 \frac{1}{\cos^2 x} - \frac{1}{\cos x}$$

Integrating the above, we get

$$v(x) = \int \left(c_1 \frac{1}{\cos^2 x} - \frac{1}{\cos x} \right) dx$$
$$= c_1 \int \frac{1}{\cos^2 x} dx - \int \frac{1}{\cos x} dx$$
$$= c_1 \tan x - \ln|\sec x + \tan x| + c_2$$

In the above, we used two important formulas:

$$\int \frac{1}{\cos^2 x} dx = \tan x$$
$$\int \frac{1}{\cos x} dx = \ln|\sec x + \tan x|$$

Both are easy to prove,

$$\int \frac{1}{\cos^2 x} dx = \int d\left(\frac{\sin x}{\cos x}\right) = \frac{\sin x}{\cos x}$$

For proving $\int \frac{1}{\cos x} dx$*, introduce a sub:*

$$u = \sec x + \tan x$$
$$du = \sec x\,(\sec x + \tan x) dx$$

Or

$$\frac{du}{\sec x + \tan x} = \sec x\, dx$$

Or

$$\sec x\, dx = \frac{du}{u}$$

Then, we can integrate both sides,

$$\int \sec x\, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

Thus, the GS is

$$\begin{aligned} y_G(x) &= v(x)\cos x \\ &= (c_1 \tan x - \ln|\sec x + \tan x| + c_2)\cos x \\ &= c_1 \sin x + c_2 \cos x - \ln|\sec x + \tan x| \cos x \end{aligned}$$

$$y_G(x) = \underbrace{c_1 \sin x + c_2 \cos x}_{y_C} \underbrace{- \ln|\sec x + \tan x| \cos x}_{y_P}$$

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

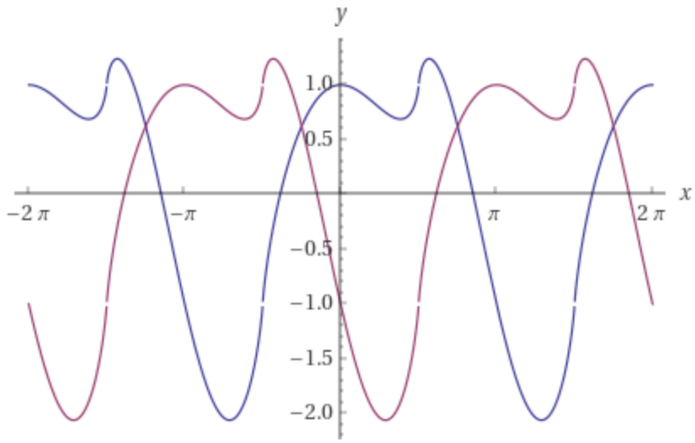

Figure 28. Two typical solutions with one set (blue) $c_1 = c_2 = 1$ and another (red) $c_1 = c_2 = -1$.

Example 16 (HW7) Find a GS of

$$xy'' - (x+1)y' + y = x^2 e^x$$

Solution (Reduction of Order):

For the Homo portion

$$xy'' - (x+1)y' + y = 0$$

we can easily find a pilot solution,

$$y_1(x) = e^x$$

With this, we propose another solution for the original In-Homo DE,

$$y_2(x) = v(x)e^x$$

(Space reserved to demonstrate the details during lectures)

Finally, we converted our 2nd order DE to a 1st order DE,

$$v'' + \left(\frac{x-1}{x}\right)v' = x$$

Or writing it as,

$$u' + \left(\frac{x-1}{x}\right)u = x$$

whose solution is,

$$u = v' = C_1 x e^{-x} + x$$

Integrating, we get,

$$v(x) = -C_1(x+1)e^{-x} + \frac{x^2}{2} + C_2$$

Thus,

$$y_2(x) = \left[-C_1(x+1)e^{-x} + \frac{x^2}{2} + C_2\right]e^x = -C_1(x+1) + \frac{x^2}{2}e^x + C_2 e^x$$

Finally, we can write the GS of the original In-Homo DE as

$$y_G(x) = \underbrace{c_1(x+1) + c_2 e^x}_{y_C} + \underbrace{\frac{x^2}{2}e^x}_{y_P}$$

The following example is very similar to the previous one.

Example 16A Find a GS of

$$xy'' - (x+1)y' + y = x^2(x+1)e^x$$

Solution (Method of reduction of order):

For the Homo portion

$$xy'' - (x+1)y' + y = 0$$

we can easily find a pilot solution,

$$y_1(x) = e^x$$

With this, we propose another solution for the original In-Homo DE,

$$y_2(x) = v(x)e^x$$

Thus (simple algebra),

$$\begin{aligned} y_2(x) &= v(x)e^x \\ y_2'(x) &= \left(e^x v(x)\right)' = e^x(v+v') \\ y_2''(x) &= (ve^x + v'e^x)' = e^x(v+2v'+v'') \end{aligned}$$

Thus, the original In-Homo DE becomes,

$$x\,\underbrace{e^x(v+2v'+v'')}_{y_2''} - (x+1)\,\underbrace{e^x(v+v')}_{y_2'} + e^x v = x^2(x+1)e^x$$

Or (after dividing by e^x),

$$\begin{aligned} x[v+2v'+v''] - [xv + xv' + v + v'] + v &= x^2(x+1) \\ xv'' + (x-1)v' &= x^2(x+1) \end{aligned}$$

Let

$$u = v'$$

we get a simple first-order linear DE,

$$u' + \left(\frac{x-1}{x}\right)u = x(x+1)$$

At this stage, let's demonstrate a "new" method for solving this first-order linear DE.

In fact, we can find the complementary solution of the homo portion,

$$u' + \left(\frac{x-1}{x}\right)u = 0$$

by the following steps,

$$\frac{u'}{u} = -\frac{x-1}{x} = \frac{1}{x} - 1$$

Thus,

$$\begin{aligned} \ln u &= (\ln x) - x + c_1 \\ u_C &= C_1 x e^{-x} \end{aligned}$$

Now, let's assume a PS by using this u_C (thanks to Cauchy) by letting,

$$\begin{aligned} u_P &= u_C(x)w(x) = xe^{-x}w \\ u_P' &= (e^{-x} - xe^{-x})w + xe^{-x}w' \\ &= -(x-1)e^{-x}w + xe^{-x}w' \end{aligned}$$

This u_P must satisfy the first-order linear (the InHomo) DE,

$$u_P' + \left(\frac{x-1}{x}\right)u_P = x(x+1)$$

Thus,

$$\underbrace{-(x-1)e^{-x}w + xe^{-x}w'}_{u_P'} + \left(\frac{x-1}{x}\right)\underbrace{xe^{-x}w}_{u_P} = x(x+1)$$

$$w' = (x+1)e^x$$

Thus,

$$w = xe^x + d_2$$

Setting this arbitrary constant $d_2 = 0$, we get,

$$\begin{aligned} u_P &= u_C(x)w(x) \\ &= xe^{-x}(xe^x) \\ &= x^2 \end{aligned}$$

Thus,

$$u = u_C + u_P = C_1 xe^{-x} + x^2$$

After recognizing $u = v'$ and integrating, we get,

$$v(x) = -C_1(x+1)e^{-x} + \frac{x^3}{3} + C_2$$

Thus, GS to the original DE is,

$$y_2(x) = \left[-C_1(x+1)e^{-x} + \frac{x^3}{3} + C_2\right]e^x = -C_1(x+1) + \frac{x^3}{3}e^x + C_2e^x$$

Finally, we can write the GS of the original In-Homo DE as

$$y_G(x) = \underbrace{c_1(x+1) + c_2e^x}_{y_C} + \underbrace{\frac{x^3}{3}e^x}_{y_P}$$

***** End of CH3 Requirements *****

The method of variational principles

For **Type V.InHomo** DE:

$$y'' + P(x)y' + Q(x)y = f(x)$$

We assume the complementary DE has the fundamental set $\{y_1(x), y_2(x)\}$

$$\begin{cases} y_1'' + P(x)y_1' + Q(x)y_1 = 0 \\ y_2'' + P(x)y_2' + Q(X)y_2 = 0 \end{cases}$$

Compose a TS y_p as

$$y_P(x) = u_1(x)y_1 + u_2(x)y_2$$

Now, we derive equations for the set $\{u_1(x), u_2(x)\}$ after a long process,

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

Solving the above for $\{u_1(x), u_2(x)\}$ leads to $y_P(x)$

Example 15A Find a GS of

$$y'' + y = \frac{\sin x}{\cos x}$$

Note: This DE is identical to Example 15, but I use a different method (the Variational Principles)

Method 2: (Variational Principles)

Trial PS

$$y_P = u_1(x) * y_1(x) + u_2(x) * y_2(x)$$

where

$$y_1 = \cos x$$
$$y_2 = \sin x$$

We just need to determine $u_1(x)$ and $u_2(x)$ by plugging this trial PS to the original DE:

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = \tan x$$

Thus,

$$\begin{aligned} u_1' &= -\sin x \tan x \\ &= \cos x - \sec x \\ u_2' &= \cos x \tan x \\ &= \sin x \end{aligned}$$

Therefore,

$$\begin{aligned} u_1(x) &= \int^x (\cos t - \sec t) dt \\ &= \sin x - \ln|\sec x + \tan x| \\ u_2(x) &= \int^x \sin t \, dt \\ &= -\cos x \end{aligned}$$

$$\begin{aligned} y_P(x) &= u_1 y_1 + u_2 y_2 \\ &= (\sin x - \ln|\sec x + \tan x|) \cos x - \cos x \sin x \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned}$$

Note: The following material is highly generalizable from our 2nd DEs.

Will skip this portion of the lecture!

Ch3A General Higher-Order Linear DE's

3A.1 General theory of nth order linear DE's

Everything we lectured for 2nd order Linear DEs for all four types can be generalized to Higher-Order Linear DE's, naturally:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t).$$

One THM can be extended:

If $y_1(t), \ldots, y_n(t)$ form a fundamental set of solutions of the homogeneous n^{th} order linear differential equation (4)

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0$$

on an interval I, then $y_1(t), \ldots, y_n(t)$ are linearly independent on I. Conversely, if $y_1(t), \ldots, y_n(t)$ are linearly independent solutions of equation (4) on I, then they form a fundamental set of solutions on I.

We'll demonstrate these generalizations through a few examples.

4.2 Homogeneous DE's (Homo DE) with constant coefficients

Example 16 Find a GS of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

Solution:

The Ch-Eq is

$$\lambda^4 + \lambda^3 - 7\lambda^2 - \lambda^1 + 6 = 0$$

We can decompose this AE (=algebraic equation) as

$$(\lambda - 1)(\lambda + 1)(\lambda - 2)(\lambda + 3) = 0$$

Thus,

$$\lambda_{1,2,3,4} = -3, -1, 1, 2$$

The fundamental set

$$\{e^{-3x}, e^{-x}, e^{x}, e^{2x}\}$$

The linear combo forms the GS of the DE

$$y_G(x) = c_1 e^{-3x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}$$

If one is given an IC set

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \\ y''(0) = -2 \\ y'''(0) = -1 \end{cases}$$

We get the PS as

$$y_P(x) = -\frac{1}{8}e^{-3x} + \frac{5}{12}e^{-x} + \frac{11}{8}e^{x} - \frac{2}{3}e^{2x}$$

Illustrated in the following figure

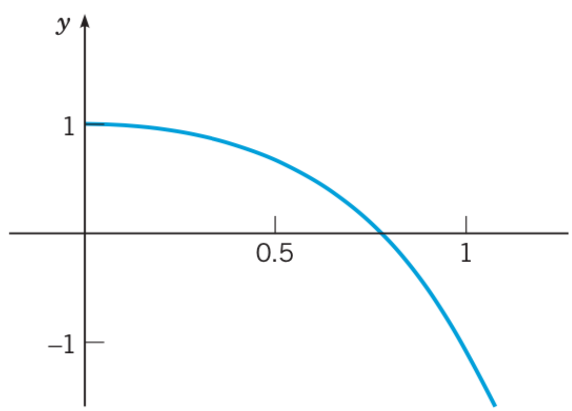

Figure 29. The PS for this example.

Example 17 and Example 18 Find a GS of

$$y^{(4)} - y = 0$$
$$y^{(4)} + y = 0$$

Solution to Example 17

The Ch-Eq is

$$\lambda^4 - 1 = 0$$

whose roots are

$$\lambda_{1,2,3,4} = -1, +1, -i, +i$$

The fundamental set

$$\{e^{-x}, e^x, \cos x, \sin x\}$$

The linear combo forms the GS of the DE

$$y_G(x) = c_1 e^{-x} + c_2 e^x + c_3 \cos x + c_4 \sin x$$

If one is given an IC set

$$\begin{cases} y(0) = \frac{7}{2} \\ y'(0) = -4 \\ y''(0) = \frac{5}{2} \\ y'''(0) = -2 \end{cases}$$

We get the PS as

$$y_P(x) = 3e^{-x} + \frac{1}{2}\cos x - \sin x$$

Illustrated in the following figure

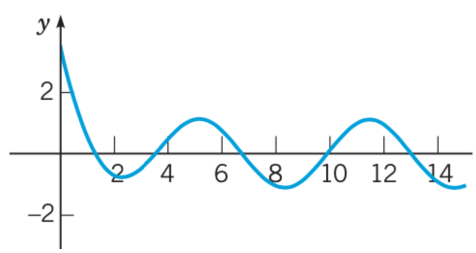

Figure 30. The PS for this example.

Example 17 and **Example 18** Find a GS of

$$y^{(4)} - y = 0$$
$$y^{(4)} + y = 0$$

Solution to Example 18:

The Ch-Eq is

$$\lambda^4 + 1 = 0$$

whose roots are

$$\lambda_{1,2,3,4} = \frac{\pm 1 \pm i}{\sqrt{2}}$$

The GS is

$$y_G(x) = e^{\frac{x}{\sqrt{2}}}\left(c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right) + e^{-\frac{x}{\sqrt{2}}}\left(c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right)$$

Example 19 Find a GS of

$$y^{(4)} + 2y'' + y = 0$$

Solution

The Ch-Eq is

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

whose roots are,

$$\lambda_{1,2,3,4} = i, -i, i, -i$$

The FS is,

$$\{\cos x, \sin x, t\cos x, t\sin x\}$$

Thus, the GS is

$$y_G(x) = c_1 \cos x + c_2 \sin x + c_3 t \cos x + c_4 t \sin x$$

4.2A Homogeneous DE's (Homo DE) variable coefficients

Example 20 Find a GS of
$$x^3y''' + x^2y'' - 2xy' + 2y = 0$$

Solution: Two similar methods.
Method 1 (using a trial):
$$y = x^\lambda$$
$$xy' = \lambda x^\lambda$$
$$x^2y'' = \lambda(\lambda-1)x^\lambda$$
$$x^3y''' = \lambda(\lambda-1)(\lambda-2)x^\lambda$$
leading us to the following Ch-Eq:
$$\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1) - 2\lambda + 2 = 0$$
$$(\lambda-1)[\lambda(\lambda-2) + \lambda - 2] = 0$$
$$(\lambda-1)(\lambda-2)(\lambda+1) = 0$$
We have 3 roots
$$\lambda_{1,2,3} = -1, 1, 2$$
Thus, we have the following GS
$$y_G = c_1x^{-1} + c_2x + c_3x^2$$

Method 2 (using Cauchy sub $x = e^t$):
$$y = e^t$$
$$xy' = \dot{y}$$
$$x^2y'' = \ddot{y} - \dot{y}$$
$$x^3y''' = \dddot{y} - 3\ddot{y} + 2\dot{y}$$
Thus, the DE becomes
$$(\dddot{y} - 3\ddot{y} + 2\dot{y}) + (\ddot{y} - \dot{y}) - 2\dot{y} + 2y$$
leading us to the following Ch-Eq (actually identical to the one in Method 1)
$$\lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1) - 2\lambda + 2 = 0$$
$$(\lambda-1)[\lambda(\lambda-2) + \lambda - 2] = 0$$
$$(\lambda-1)(\lambda-2)(\lambda+1) = 0$$
with 3 roots
$$\lambda_{1,2,3} = -1, 1, 2$$

Thus, we have the GS in t
$$y_G(t) = c_1e^{-t} + c_2e^t + c_3e^{2t}$$
Back sub leads to the same final GS.

A plot of two typical solutions with $c_1 = c_2 = \pm 1$:

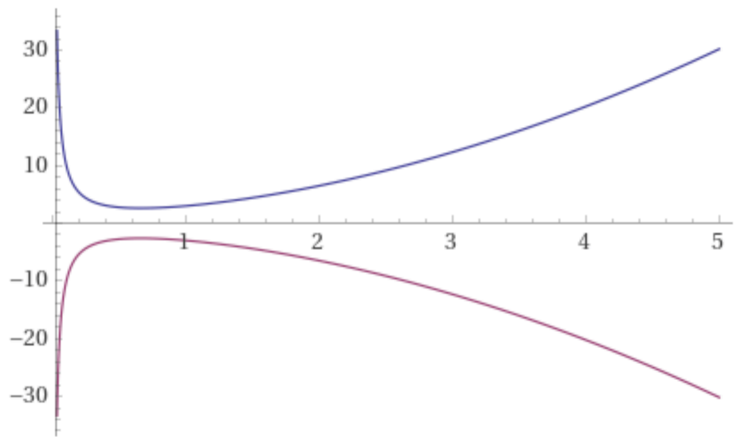

Figure 31. Two solutions: (blue) $c_1 = c_2 = c_3 = 1$ and (red) $c_1 = c_2 = c_3 = -1$.

*** The End of CH3 ***