Lyapunov modal analysis and participation factors

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Abstract

Lyapunov Modal Analysis (or LMA) is a novel framework, which combines the selective modal analysis with the idea of spectral decompositions of Lyapunov functions. This approach allows one to characterize the modal interactions in the dynamical systems and estimate them in connection with specific state variables. Conventional participation factors characterize the relative contribution of the system modes and state variables to the evolution of states and modes, respectively. By contrast, the proposed Lyapunov participation factors characterize similar contributions to the corresponding Lyapunov functions, which determine the integral energy associated with the states and modes on the infinite or finite time interval. This allows one to estimate modal interactions in terms of total energy produced by their mutual action over time. This approach is conceptually different from considering second-order and higher terms in the instantaneous dynamics of nonlinear model. Using two-area, four-machines power system we demonstrate that even for a linear model LMA reliably identifies resonant modal interactions, merging of modes and loss of stability and associate them with certain state variables. The Lyapunov participation factors corresponding to the selected part of the spectrum can be calculated independently and serve as a basis for the fast calculation of critical mode behavior in large-scale dynamical systems in real time.

Keywords: selective modal analysis, participation factors, Lyapunov functions, Gramian, sub-Gramians, small-signal stability, spectral decomposition, Lyapunov modal analysis.

1. Introduction

Modern technologies of Smart Grids and Micro Grids becomes the priority direction of the development of electric power systems (EPS). A critical requirement for the introduction of these technologies is the increase in the reliability of EPS and the ability to monitor and control its stability in real time. On the other hand, even for power systems with a relatively simple structure, the development of online monitoring systems is an extremely difficult task. In large EPS, weakly stable modes usually arise in groups, which cause problems of their resonance interaction and leads to the appearance of dangerous inter-area oscillations. Such oscillations may occur within the power facility, regional power grid or global power systems (Pal, Thorp (2012)). Quite frequently, these oscillations establish critical limitations of maximum transfer capability in main power transmission lines and may lead to the occurrence of the voltage avalanche and cascading failures (Weber, Al Ali (2016)). The loss of stability is accompanied by accumulation of energy in the low-frequency oscillations, which cause the resonant reaction in the system. Therefore, the small-signal stability analysis of modern EPS requires an accurate estimation and prediction of resonant interactions of weakly stable system modes with reference to specific state variables.

In conventional modal analysis of linear systems, modal interactions are not directly taken into account. In the nonlinear versions of modal analysis the modal interactions is considered through the second and higher order terms in the Taylor expansion of the system dynamics. In general, this approach imposes demanding requirements on model accuracy and amount of computation. But more importantly, the modal interactions themselves are considered in terms of instantaneous dynamics of the system. However, for the small-signal stability, the amount of the perturbation energy accumulated in the system with time may be of more importance than the instantaneous dynamics. This perturbation energy can be estimated using the spectral decompositions of Lyapunov functions proposed in (Yadykin (2010); Yadykin, Iskakov (2017)). To this purpose, present paper develops a novel framework, which combines the selective modal analysis with the idea of spectral decompositions of Lyapunov functions. This approach allows one to estimate modal interactions in terms of total energy produced in the system by their mutual action over time, rather than in terms of instantaneous dynamics.

1.1. Literature review

Modal analysis is one of the most popular methods for studying the small-signal stability of dynamical systems. A framework of *Selective Modal Analysis (SMA)* was proposed in (Pérez-Arriaga *et al.* (1982); Verghese *et al.* (1982)) and allowed an accurate identification of the elements of the system structure associated with specific eigenmodes based on the so-called *participation factors (PFs)*. For linear time-invariant systems, PFs have been defined as the relative contributions of state

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variables to the evolution of system modes, or as the relative contributions of system modes to the evolution of states. Subsequently, the concept of PFs gained widespread use in power engineering and other applied areas for stability analysis (Verghese *et al.* (1982); Song *et al.* (2019)), dynamic model reduction (Chow (2013)), optimal placement of sensors and stabilizers (Manousakis *et al.* (2011)), and solving clustering problems (Genk *et al.* (2005)). Interpretation of PFs has been expanded in connection with the sensitivity of eigenvalues (Pagola *et al.* (1989)), modal controllability and observability (Hamdan, Nayfeh (1989)), and modal mobility (Tawalbeh, Hamdan (2010)). An original interpretation of PFs was based on specially selected initial conditions. It was observed in (Hashlamoun *et al.* (2009)) that this assumption could lead to counterintuitive results and an alternative method of averaging over an uncertain set of system initial conditions was proposed. Accordingly, the original definition of PFs was retained for *the "mode-in-state" PFs*; however, a new definition was proposed for the *"state-in-mode" PFs (or SIMPFs)*. Subsequently, similar concepts of SIMPFs were considered for dynamical nonlinear systems (Hamzi, Abed (2014)) and for systems described by algebraic equations such as the power flow equations (Song *et al.* (2019)).

Attempts to take into account *nonlinear effects* and *intermodal interactions* within the framework of modal analysis developed mainly in two directions. The model-based approach is associated with taking into account the higher order terms of Taylor expansion in the system approximation and using the normal Poincaré forms (Vittal *et al.* (1991); Hamzi, Abed (2014); Tian *et al.* (2018)). A study in (Sanchez-Gasca *et al.* (2018)) showed that accounting for these terms becomes significant, for example, when studying inter-area oscillations in stressed power systems following large disturbances. These methods, however, in general require solving the highly nonlinear numerical problem and using computationally expensive algorithms. Another approach involves estimating the PFs directly from measurements. This approach, for example, can be based on the extended dynamic mode decomposition (Williams *et al.* (2015)) and Koopman mode decomposition (Netto *et al.* (2019)). The performance of these methods still requires careful verification in practical applications.

Another conceptual method in the stability analysis is associated with the use of Lyapunov equations. *The Lyapunov stability analysis* is based on a positive definite function of system state $V(\boldsymbol{x}) = \boldsymbol{x}^T \mathbf{P} \boldsymbol{x}$, where the matrix $\mathbf{P} = \mathbf{P}^T > 0$ is a solution of the corresponding Lyapunov equation, which is called *the Gramian*. In the case of linear time-invariant systems, Lyapunov functions can be associated with the integrated energy of the input or output signal. In this connection, the Gramians of controllability and observability are commonly used. In general, one can say that *the observability Gramian* characterizes the system stability in terms of its output energy limit, and *the controllability Gramian* characterizes system stability in terms of its asymptotic sensitivity to the random input disturbances. For the stable linear system the Gramians are closely related to the squared H_2 norm of its transfer function or its impulse response function. The physical interpretation of these values is that they characterize the energy amplification in the system averaged over time or frequency. The energy-based interpretation of Gramians generally holds true for the time-varying linear systems with the replacement of exponential expressions $e^{\mathbf{A}t}$ by a fundamental solution $\mathbf{\Phi}(0, t)$ of the homogeneous equation $\dot{\boldsymbol{x}} = \mathbf{A}(t)\, \boldsymbol{x}$ (Shokoohi *et al.* (1982); Verriest, Kailath (1983)). The notion of Gramians was further generalized and interpreted for the deterministic bilinear and stochastic linear systems under the name of *energy functionals* (Gray, Mesko (1998); Benner, Damm (2011)).

The spectral properties of Gramians and energy functionals were effectively used for *the model order reduction* techniques. Among them there are methods of balanced truncation (Moore (1981)), the method of using cross-Gramians (Fernando, Nicholson (1984)) and their various modifications (see the review in (Baur *et al.* (2014))). In the monograph (Antoulas (2005)), devoted to the approximation of large-scale dynamical systems, singular expansions for finite and infinite Gramians of controllability and observability were first obtained, based on diagonalization of the dynamics matrix. A more general form of *the spectral decompositions of Lyapunov functions* into components corresponding to individual eigenvalues of the system or their pairwise combinations was proposed in (Yadykin (2010); Yadykin *et al.* (2014); Yadykin, Iskakov (2017)). Each eigen-part was called *a sub-Gramian*. The sub-Gramians allow estimating the interaction between eigenmodes and were applied to the stability analysis of power system in (Yadykin *et al.* (2016)).

1.2. Main contribution

The objective of this paper is to offer a novel framework of *Lyapunov modal analysis*, which combines two approaches to the assessment of stability, namely selective modal analysis and spectral decompositions of Lyapunov functions. To this purpose, we propose the concept of *Lyapunov participation factors*, which characterize the relative contribution of the system modes $\boldsymbol{z}$ and state variables $\boldsymbol{x}$ not to the evolution of states and modes respectively, but to the corresponding Lyapunov functions, i.e. to the quantities $\boldsymbol{x}^T \mathbf{P}_x \boldsymbol{x}$ or $\boldsymbol{z}^* \mathbf{P}_z \boldsymbol{z}$. The matrices $\mathbf{P}_z$ and $\mathbf{P}_x$ here are the solutions of the Lyapunov equations, which are chosen so that their solutions measure the integrated energy associated with the particular eigen-mode or state variable. The corresponding Lyapunov functions we will call *Lyapunov energies*. These values are important when analyzing the stability of the system, since they reflect not the instantaneous values of the states or signals, but the variation of their energies over the time interval (i.e., their energy gains in the system). In terms of mechanics, Lyapunov energy corresponds not so much to the energy itself, as to *Hamilton's action*, i.e. the energy integrated over time.

The key question here is how to define the energy of states and modes. For the system state energy, we follow the definition of *stored energy*, proposed by MacFarlane (1969) for the electromechanical systems, as a quadratic function $\frac{1}{2}\, \boldsymbol{x}^T \boldsymbol{x}$ of state variables $\boldsymbol{x}$ after their suitable scaling. Then the Lyapunov energy of the k-th state variable x_k can be simply defined as

$$E_{x_k} = \int_0^t x_k^2(\tau)d\tau$$

The definition of modal energy is less obvious. The modal energy of the ith mode was defined in (Hamdan (1986)) as $\boldsymbol{x}^T\boldsymbol{u}_i(\boldsymbol{v}_i)^T\boldsymbol{x}$, where $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are the normalized right and left (column) eigenvectors of the ith mode. Unfortunately, this definition lacks physical meaning. According to it, the modal energies can be negative and unlimited in magnitude even in a very simple system (see Iskakov (2019)).

Therefore, in this paper two alternative definitions of modal energy are investigated. First, the i-th mode energy can be defined as

$$(\boldsymbol{x})^T\boldsymbol{v}_i(\boldsymbol{u}_i)^T\boldsymbol{u}_i(\boldsymbol{v}_i)^T\boldsymbol{x} = |\boldsymbol{u}_i|^2 z_i^2$$

where $z_i = (\boldsymbol{v}_i)^T\boldsymbol{x}$ is the ith component of the system mode vector $\boldsymbol{z}$. Then, choosing the normalization of eigenvectors so that $|\boldsymbol{u}_i|^2 = 1$, the Lyapunov energy of the i-th mode can be considered by analogy with E_{x_k} as

$$E_{z_i} = \int_0^t |z_i(\tau)|^2 d\tau$$

We show that this definition in practice leads to the "state-in-mode" PFs defined in (Hashlamoun *et al.* (2009)) for real eigenvalues and corrected in (Konoval, Prytula (2017); Iskakov (2019)) for the case of complex eigenvalues. Alternatively, the energy of ith mode can be defined as a modal contribution of ith mode into Lyapunov energy of states (i.e., into $\sum_k E_{x_k}$) on the basis of the spectral decompositions proposed in (Yadykin (2010); Yadykin, Iskakov (2017)). With this definition, modal contributions of some modes can be negative if there are other modes with a sufficiently large amplitude and negative correlation with a given mode. This mode correlation is determined by both the dynamic properties of the system and its current state. Based on this second definition of modal energy, we introduce *Lyapunov modal interaction energies and factors* that characterize pairwise modal interactions in terms of the Lyapunov energy produced by them in different state variables. We also offer two indicators for selecting state variables that are (i) most sensitive for identifying a specific modal interaction, and (ii) most influential for damping this interaction.

The definitions of Lyapunov energies for states and modes allow us to introduce the corresponding concepts of *Lyapunov participation factors*. We examine the theoretical properties of these indicators and show in simulation experiment with power system (Kundur (1994)) that they reliably identify resonant modal interactions, merging of modes and loss of stability and associate these events with certain state variables.

Unlike works on nonlinear modal analysis (Vittal *et al.* (1991); Hamzi, Abed (2014); Tian *et al.* (2018)), this paper proposes a new principle for evaluating modal indicators and modal interactions, which is based not on the instantaneous dynamics of variables, but on variation of their energy over the time interval. In particular, it allows identifying low-frequency oscillations dangerous for small-signal stability and detecting the effect of energy accumulation in these oscillations. Unlike works on spectral decompositions of Lyapunov functions (Yadykin *et al.* (2014, 2016); Yadykin, Iskakov (2017)), the proposed new formalism allows one to associate these decompositions with specific state variables and apply them to the problems of modal analysis. The idea of combining selective modal analysis and Lyapunov spectral expansions has been already mentioned in (Vassilyev *et al.* (2017)). In the present paper a general framework is proposed for its implementation.

1.3. Paper organization

The preliminary information on PFs and sub-Gramians is briefly summarized in Section 2. Lyapunov energies of states and modes and corresponding Lyapunov PFs are introduced in Section 3. Lyapunov modal interaction analysis is introduced and corresponding pair PFs are defined in Section 4. In Section 5 some characteristic properties of Lyapunov PFs are established. The numerical experiment, which demonstrate the potential advantages of Lyapunov modal analysis is provided in Section 6. Conclusions are drawn in Section 7.

2. Theoretical background

2.1. Participation Factors

In this subsection we remind the definition and some properties of the participation factors from (Pérez-Arriaga *et al.* (1982); Pagola *et al.* (1989); Garofalo *et al.* (2002)). Consider an autonomous linear time-invariant system

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}(t)\,, \qquad (1)$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a system state vector and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a dynamic matrix with a simple spectrum, which can be represented as

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V} = (\boldsymbol{u}_1\boldsymbol{u}_2\ldots\boldsymbol{u}_n)\begin{pmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \lambda_n\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_1^T\\ \boldsymbol{v}_2^T\\ \vdots\\ \boldsymbol{v}_n^T\end{pmatrix}, \qquad (2)$$

where $\mathbf{UV} = \mathbf{VU} = \mathbf{I}$ and $(\cdot)^T$ is an operation of transpose. Let u_i^k and v_i^l be the k-th and l-th components of eigenvectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$, respectively. Then

$$p_{ki} = u_i^k v_i^k \;\text{ and }\; p_{kil} = u_i^k v_i^l \qquad (3)$$

are called *participation factors* (PF) and *generalized participations*, respectively (Pagola *et al.* (1989)). The PF p_{ki} "weights" the participation of the i-th mode in the k-th state variable and was interpreted in (Hashlamoun *et al.* (2009)) as *mode-in-state PF*. For further presentation we remind two important properties of generalized participations from (Garofalo *et al.* (2002)).

Property 1*. The generalized participation p_{kil} is the sensitivity of the i-th eigenvalue λ_i with respect to the element a_{lk} of $\mathbf{A}$, i.e.

$$p_{kil} = \frac{\partial\lambda_i}{\partial a_{lk}}\,,\quad p_{ki} = \frac{\partial\lambda_i}{\partial a_{kk}}\,. \qquad (4)$$

Define the residue matrices $\mathbf{R}_i$ as the coefficients in the expansion of the resolvent of the matrix $\mathbf{A}$:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{\mathbf{R}_1}{s-\lambda_1} + \frac{\mathbf{R}_2}{s-\lambda_2} + \cdots + \frac{\mathbf{R}_n}{s-\lambda_n} \,. \tag{5}$$

Property 2^*. The generalized participations p_{kil} are the coefficients of the corresponding residue matrix $\mathbf{R}_i$, i.e.

$$p_{kil} = \mathbf{e}_k^T\, \mathbf{R}_i\, \mathbf{e}_l \,, \;\; \mathbf{R}_i = \sum_{k,l} p_{kil}\, \mathbf{e}_k \cdot \mathbf{e}_l^T \,, \tag{6}$$

where $\mathbf{e}_k$ and $\mathbf{e}_l$ are the k-th and l-th columns of the identity matrix.

By applying to (1) the diagonalizing transformation

$$z = \mathbf{V}\boldsymbol{x} \,, \tag{7}$$

the evolution of the system mode vector $z(t)$ can be specified as

$$\dot{z}(t) = \mathbf{V}\,\mathbf{A}\boldsymbol{x}(t) = \mathbf{\Lambda}\mathbf{V}\boldsymbol{x}(t) = \mathbf{\Lambda} z(t) \,. \tag{8}$$

For the analysis of $z(t)$ it was proposed in (Hashlamoun *et al.* (2009)) to use *state-in-mode PFs*, which, taking into account the correction in (Konoval, Prytula (2017)), were defined as

$$\pi_{ki} = \frac{v_i^k\,(v_i^k)^*}{\boldsymbol{v}_i\,(\boldsymbol{v}_i)^*} \;\;. \tag{9}$$

2.2. Gramians and sub-Gramians

In this subsection we remind the definition and some properties of the Gramians and sub-Gramians from (Yadykin (2010); Yadykin *et al.* (2014); Yadykin, Iskakov (2017)). An infinite Gramian is a positive definite solution $\mathbf{P} = \mathbf{P}^* > 0$ of the matrix Lyapunov equation:

$$\mathbf{A}^*\,\mathbf{P} + \mathbf{P}\,\mathbf{A} = -\,\mathbf{Q}\,, \;\; \mathbf{Q} = \mathbf{Q}^* > 0\,, \tag{10}$$

where $(\cdot)^*$ denotes the conjugate transpose of a matrix. For simplicity, we assume further that the matrix $\mathbf{A}$ has a simple spectrum. In this case the solution of (10) can be written as (Yadykin, Iskakov (2017))

$$\mathbf{P} = -\sum_{i,j=1}^{n} \frac{\mathbf{R}_i^*\,\mathbf{Q}\,\mathbf{R}_j}{\lambda_i^* + \lambda_j} \,, \tag{11}$$

where $\mathbf{R}_i$ and $\mathbf{R}_j$ are the residue matrices defined by (5) corresponding to $\lambda = \lambda_i$ and $\lambda = \lambda_j$ respectively.

It follows from (5) that

$$\sum_{j=1}^{n} \frac{\mathbf{R}_j}{-\lambda_i^* - \lambda_j} = (-\lambda_i^*\mathbf{I} - \mathbf{A})^{-1} \,.$$

Substituting this in (11), we obtain another form of the spectral decomposition

$$\mathbf{P} = -\sum_{i=1}^{n} \mathbf{R}_i^*\,\mathbf{Q}\,(\lambda_i^*\mathbf{I} + \mathbf{A})^{-1} = -\sum_{j=1}^{n} (\lambda_i\mathbf{I} + \mathbf{A}^*)^{-1}\mathbf{Q}\,\mathbf{R}_j \,, \tag{12}$$

Hermitian parts of matrices in the spectral decompositions (11) and (12)

$$\widetilde{\mathbf{P}}_i = -\left\{\mathbf{R}_i^*\,\mathbf{Q}\,(\lambda_i^*\mathbf{I} + \mathbf{A})^{-1}\right\}_H, \;\; \mathbf{P}_{ij} = -\left\{\frac{\mathbf{R}_i^*\,\mathbf{Q}\,\mathbf{R}_j}{\lambda_i^* + \lambda_j}\right\}_H \;\;. \tag{13}$$

have been named *sub-Gramians*. Here by $\{\cdot\}_H$ we denote the Hermitian part of a matrix. Using sub-Gramians (SG) the decompositions (11) and (12) can be written as

$$\mathbf{P} = \sum_{i=1}^{n} \widetilde{\mathbf{P}}_i = \sum_{i,j=1}^{n} \mathbf{P}_{ij} \,, \;\; \widetilde{\mathbf{P}}_i = \sum_{j=1}^{n} \mathbf{P}_{ij} \,. \tag{14}$$

The SG $\widetilde{\mathbf{P}}_i$ and $\mathbf{P}_{ij}$ characterize the contribution of eigenmodes or their pairs into the asymptotic variation of the perturbation energy in the system defined by the Gramian $\mathbf{P}$.

It should be noted that the sub-Gramirans (13) also satisfy the Lyapunov equations, namely

$$\begin{aligned} \mathbf{A}^*\,\widetilde{\mathbf{P}}_i + \widetilde{\mathbf{P}}_i\,\mathbf{A} &= -\,\frac{1}{2}\Big(\mathbf{R}_i^*\mathbf{Q} + \mathbf{Q}\,\mathbf{R}_i\Big)\,, \\ \mathbf{A}^*\,\mathbf{P}_{ij} + \mathbf{P}_{ij}\,\mathbf{A} &= -\,\frac{1}{2}\Big(\mathbf{R}_i^*\mathbf{Q}\,\mathbf{R}_j + \mathbf{R}_j^*\mathbf{Q}\,\mathbf{R}_i\Big)\,. \end{aligned} \tag{15}$$

These equations can be verified by the direct substitution of (13) into them.

2.3. Relation between PF and SG

Now we establish the relation between PF and SG. Although PF and SG are clearly different by their conceptual meaning, it is easy to see that both quantities are calculated using the matrix residues. Substituting expression (6) for the matrix residues $\mathbf{R}_i$ and $\mathbf{R}_j$ into (13) we obtain

$$\widetilde{\mathbf{P}}_i = -\sum_{k,l=1}^{n} \left\{p_{lik}^*\; \mathbf{e}_k\mathbf{e}_l^T \cdot \mathbf{Q} \cdot (\lambda_i^*\mathbf{I} + \mathbf{A})^{-1}\right\}_H \,, \tag{16}$$

$$\mathbf{P}_{ij} = -\sum_{k,l,m,r=1}^{n} \left\{\frac{p_{lik}^*\; p_{rjm}\; \mathbf{e}_k\mathbf{e}_l^T \cdot \mathbf{Q} \cdot \mathbf{e}_r\mathbf{e}_m^T}{\lambda_i^* + \lambda_j}\right\}_H \,. \tag{17}$$

These new formulas allow calculating SG via PF. The SG $\widetilde{\mathbf{P}}_i$ is the sum of terms proportional to the PF corresponding to i-th eigenmode. The pairwise SG $\mathbf{P}_{ij}$ is a quadratic form of PF corresponding to i-th and j-th eigenmodes. Formulas similar to (16) were obtained in (Antoulas (2005), p.149).

Substituting (3) into (16) and (17) we obtain

$$\widetilde{\mathbf{P}}_i = -\left\{(\boldsymbol{v}_i^T)^*\boldsymbol{u}_i^* \cdot \mathbf{Q} \cdot (\lambda_i^*\mathbf{I} + \mathbf{A})^{-1}\right\}_H \,, \tag{18}$$

$$\mathbf{P}_{ij} = -\left\{\frac{(\boldsymbol{v}_i^T)^*\boldsymbol{u}_i^* \cdot \mathbf{Q} \cdot \boldsymbol{u}_j\,\boldsymbol{v}_j^T}{\lambda_i^* + \lambda_j}\right\}_H \,. \tag{19}$$

These formulas allow calculating SG using corresponding eigenvectors when matrix $\mathbf{A}$ has a simple spectrum. They demonstrate that the calculation of the individual sub-Gramians does not require knowledge of the entire spectrum, but only of the eigenvalues and eigenvectors corresponding to the particular sub-Gramian.

Substituting (4) into (16) and (17) we obtain

$$\widetilde{\mathbf{P}}_i = -\sum_{k,l=1}^{n} \left\{ \left(\frac{\partial\lambda_i}{\partial a_{kl}}\right)^* \mathbf{e}_k\mathbf{e}_l^T \cdot \mathbf{Q} \cdot (\lambda_i^*\mathbf{I}+\mathbf{A})^{-1} \right\}_H ,$$
$$\mathbf{P}_{ij} = -\sum_{k,l,m,r=1}^{n} \left\{ \frac{1}{\lambda_i^*+\lambda_j} \left(\frac{\partial\lambda_i}{\partial a_{kl}}\right)^* \frac{\partial\lambda_j}{\partial a_{mr}} \mathbf{e}_k\mathbf{e}_l^T\mathbf{Q}\,\mathbf{e}_r\mathbf{e}_m^T \right\}_H .$$

These formulas allow calculating sub-Gramians using sensitivity of the corresponding eigenvalue with respect to the elements of matrix $\mathbf{A}$.

3. Lyapunov energies and participation factors

In this section we introduce new indicators for selective modal analysis. Unlike conventional PFs, the proposed Lyapunov PFs characterize the relative contribution of the system modes $\boldsymbol{z}$ and state variables $\boldsymbol{x}$ not to the evolution of states and modes respectively, but to the corresponding *Lyapunov energies*, i.e. to the quantities $\boldsymbol{x}^T\mathbf{P}_{\boldsymbol{x}}\,\boldsymbol{x}$ or $\boldsymbol{z}^*\mathbf{P}_z\,\boldsymbol{z}$, where *the Gramians* $\mathbf{P}_{\boldsymbol{x}}$ and $\mathbf{P}_z$ are the solutions of the corresponding Lyapunov equations. For this purpose, we first consider the concept of *Lyapunov energies*, which characterize the squared state variables and system modes integrated over time. Notice, that in terms of physical meaning, Lyapunov energy corresponds not so much to energy itself, as to *action*, i.e. the energy integrated over time.

3.1. Lyapunov energies of states and modes

Consider a stable dynamical system in the form:

$$\dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x}\,, \quad \boldsymbol{y} = \mathbf{C}\boldsymbol{x}\,, \tag{20}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is a state vector and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a real matrix, which has n distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and can be represented through its eigenvectors as (2). In the further analysis, we also assume that the eigenvectors associated with real eigenvalues are real, and the eigenvectors associated with a complex conjugate pair of non-real eigenvalues λ_i, λ_i^* are complex conjugates:

$$\begin{aligned} \mathrm{Re}\{\boldsymbol{u}_{i*}\} = \mathrm{Re}\{\boldsymbol{u}_i\},\ \ \mathrm{Im}\{\boldsymbol{u}_{i*}\} = -\mathrm{Im}\{\boldsymbol{u}_i\},\\ \mathrm{Re}\{\boldsymbol{v}_{i*}\} = \mathrm{Re}\{\boldsymbol{v}_i\},\ \ \mathrm{Im}\{\boldsymbol{v}_{i*}\} = -\mathrm{Im}\{\boldsymbol{v}_i\}, \end{aligned} \tag{21}$$

where $i*$ is an index of the eigenvector associated with λ_i^*. Even taking into account the normalization and conditions (21), the eigenvectors in the representation (2) are not uniquely determined, but up to an arbitrary constant by which for any i one can multiply the left eigenvector $\boldsymbol{v}_i$ and divide the right eigenvector $\boldsymbol{u}_i$. Therefore, to avoid this ambiguity, we impose an additional normalization condition

$$\forall i : \quad |\boldsymbol{u}_i|^2 = 1\,. \tag{22}$$

The integrated output signal energy in the stable system (20) created by the initial state $\boldsymbol{x}_0$ can be defined as

$$E = \int_0^\infty \boldsymbol{x}_0^T\, e^{\mathbf{A}^* t}\, \mathbf{C}^*\mathbf{C}\, e^{\mathbf{A}t}\boldsymbol{x}_0\, dt = \boldsymbol{x}_0^*\,\mathbf{P}\,\boldsymbol{x}_0\,, \tag{23}$$

where the Gramian $\mathbf{P} = \int_0^\infty e^{\mathbf{A}^* t} \cdot \mathbf{C}^*\mathbf{C} \cdot e^{\mathbf{A}t} dt$ is the solution of the Lyapunov equation $\mathbf{A}^*\mathbf{P} + \mathbf{P}\,\mathbf{A} = -\,\mathbf{C}^*\mathbf{C}$.

As the matrix $\mathbf{C}$ let us choose a unit vector with a unit k-state component and other components equal zero

$$\mathbf{C} = \mathbf{e}_k^T = (0, \ldots, 0, 1_k, 0, \ldots, 0)\,.$$

Then $y = x_k$ is a k-th component of the state vector $\boldsymbol{x}$, and the expression (23) characterizes the integrated energy produced in the k-th state. Therefore, we define *Lyapunov energy produced in the k-th state* as

$$\begin{aligned} E_{x_k} = \boldsymbol{x}^T\mathbf{P}_{x_k}\boldsymbol{x} = \int_0^\infty x_k^2(t)\, dt\,, \quad \text{where} \\ \mathbf{A}^*\,\mathbf{P}_{x_k} + \mathbf{P}_{x_k}\,\mathbf{A} = -\,\mathbf{e}_k\mathbf{e}_k^T\,. \end{aligned} \tag{24}$$

Total Lyapunov energy of all state variables is

$$E_x = \sum_{k=1}^{n} E_{x_k} = \boldsymbol{x}^T\mathbf{P}_{\boldsymbol{x}}\,\boldsymbol{x}\,, \text{ where } \mathbf{P}_{\boldsymbol{x}} = \sum_{k=1}^{n} \mathbf{P}_{x_k} \tag{25}$$

is the solution of the Lyapunov equation

$$\mathbf{A}^*\,\mathbf{P}_{\boldsymbol{x}} + \mathbf{P}_{\boldsymbol{x}}\,\mathbf{A} = -\,\mathbf{I}\,. \tag{26}$$

By applying to the state vector $\boldsymbol{x}$ the diagonalizing transformation (7) we obtain the system mode vector $\boldsymbol{z} = \mathbf{V}\boldsymbol{x} \in \mathbb{C}^n$, for which the equation of dynamics (20) takes the diagonal form $\dot{\boldsymbol{z}} = \mathbf{\Lambda}\boldsymbol{z}$. By analogy with (24) we define *Lyapunov energy of the i-th mode* as

$$\begin{aligned} E_{z_i} = \int_0^\infty z_i^*(t)\, z_i(t)\, dt = \boldsymbol{z}^*\,\mathbf{P}_{z_i}\,\boldsymbol{z}\,, \quad \text{where} \\ \mathbf{\Lambda}^*\,\mathbf{P}_{z_i} + \mathbf{P}_{z_i}\,\mathbf{\Lambda} = -\,\mathbf{e}_i\mathbf{e}_i^T\,, \end{aligned} \tag{27}$$

where z_i is the i-th component of the mode vector $\boldsymbol{z} \in \mathbb{C}^n$. Total Lyapunov energy of the mode vector $\boldsymbol{z}$ is

$$E_z = \int_0^\infty \boldsymbol{z}^*(t)\, \boldsymbol{z}(t)\, dt = \sum_{i=1}^{n} E_{z_i} = \boldsymbol{z}^*\mathbf{P}_z\,\boldsymbol{z}\,, \tag{28}$$

where $\mathbf{P}_z = \sum_{i=1}^n \mathbf{P}_{z_i}$ is the solution of the Lyapunov equation

$$\mathbf{\Lambda}^*\,\mathbf{P}_z + \mathbf{P}_z\,\mathbf{\Lambda} = -\,\mathbf{I}\,. \tag{29}$$

Note that the quantities E_k and E_z in (25) and (28) are defined in the spaces of state variables and modal vectors, respectively. Therefore, in general, $E_k \neq E_z$.

Remark 1. Without the additional normalization condition (22), the invariant definition of Lyapunov modal energy takes the form

$$E_{z_i} = |\boldsymbol{u}_i|^2 \cdot \int_0^\infty z_i^*(t)\, z_i(t)\, dt\,.$$

On the basis of definitions (24) and (27) of Lyapunov energies of the states and modes we will further introduce a novel concept of Lyapunov participation factors.

3.2. Mode-in-state Lyapunov PF

The Lyapunov energy of each state E_{x_k} in (24) can be partitioned into parts associated with individual eigenvalues using the decomposition (13) and (14) of Gramian $\mathbf{P}_{x_k}$ into the sub-Gramians

$$E_{x_k} = \sum_{i=1}^{n} E_{x_k i}\ , \ \text{where}\ E_{x_k i} = \boldsymbol{x}^T \widetilde{\mathbf{P}}_{x_k i}\, \boldsymbol{x}\ ,\ \mathbf{P}_{x_k} = \sum_{i=1}^{n} \widetilde{\mathbf{P}}_{x_k i}\ , \quad (30)$$

and the sub-Gramians $\widetilde{\mathbf{P}}_{x_k i}$, according to (15), satisfy the following Lyapunov equations

$$\mathbf{A}^* \widetilde{\mathbf{P}}_{x_k i} + \widetilde{\mathbf{P}}_{x_k i}\, \mathbf{A} = -\frac{1}{2}\left(\mathbf{R}_i^* \mathbf{e}_k \mathbf{e}_k^T + \mathbf{e}_k \mathbf{e}_k^T \mathbf{R}_i\right).$$

According to (18)-(19), for these sub-Gramians we obtain

$$\begin{aligned}\widetilde{\mathbf{P}}_{x_k i} &= -\left\{(\boldsymbol{v}_i^T)^* (u_i^k)^* \mathbf{e}_k^T (\lambda_i^* \mathbf{I} + \mathbf{A})^{-1}\right\}_H \\ &= -\left\{\sum_{j=1}^{n} \frac{(\boldsymbol{v}_i^T)^* (u_i^k)^* u_j^k \boldsymbol{v}_j^T}{\lambda_i^* + \lambda_j}\right\}_H . \end{aligned} \quad (31)$$

Therefore, Lyapunov energies of the states (24) can be expressed through the eigenvectors and state variables as

$$\begin{aligned} E_{x_k} = \boldsymbol{x}^T \mathbf{P}_{x_k} \boldsymbol{x} &= -\boldsymbol{x}^T \left\{\sum_{i=1}^{n} (\boldsymbol{v}_i^T)^* (u_i^k)^* \mathbf{e}_k^T (\lambda_i^* \mathbf{I} + \mathbf{A})^{-1}\right\}_H \boldsymbol{x} \\ &= -\boldsymbol{x}^T \left\{\sum_{i,j=1}^{n} \frac{(\boldsymbol{v}_i^T)^* (u_i^k)^* u_j^k \boldsymbol{v}_j^T}{\lambda_i^* + \lambda_j}\right\}_H \boldsymbol{x}\ . \end{aligned} \quad (32)$$

This expression can also be obtained directly from the solution of the system (1)

$$x_k(t) = \sum_{i=1}^{n} u_i^k\, z_i(0) \exp(\lambda_i t),\ \text{where}\ z_i(0) = \boldsymbol{v}_i^T \boldsymbol{x}_0,\ \boldsymbol{x}_0 = \boldsymbol{x}(0).$$

Indeed, substituting this in the definition (24), we obtain the same result as in (32)

$$\begin{aligned} E_{x_k} &= \int_0^\infty x_k^*(t) x_k(t) dt = \\ &= \int_0^\infty \Big(\sum_{i=1}^{n} \boldsymbol{x}_0^T (\boldsymbol{v}_i^T)^* (u_i^k)^* \exp(\lambda_i^* t)\Big) \cdot \Big(\sum_{j=1}^{n} u_j^k \boldsymbol{v}_j^T \boldsymbol{x}_0 \exp(\lambda_j t)\Big) dt \\ &= \boldsymbol{x}_0^T \left(\sum_{i,j=1}^{n} (\boldsymbol{v}_i^T)^* (u_i^k)^* u_j^k \boldsymbol{v}_j^T \int_0^\infty \exp((\lambda_i^* + \lambda_j) t)\, dt\right) \boldsymbol{x}_0\ . \end{aligned}$$

On the basis of (30) we introduce the following definition.

Definition 1. *The mode-in-state Lyapunov participation factor (MISLPF)* characterizes the relative participation of the i-th mode in the Lyapunov energy E_{x_k} of the k-th state (24), that is

$$e_{ki} = \frac{E_{x_k i}}{E_{x_k}} = \frac{\boldsymbol{x}_0^T\, \widetilde{\mathbf{P}}_{x_k i}\, \boldsymbol{x}_0}{\boldsymbol{x}_0^T\, \mathbf{P}_{x_k} \boldsymbol{x}_0}\ . \quad (33)$$

The MISLPFs defined in (33) clearly depend on the initial state vector $\boldsymbol{x}_0$ and the correlation among its components. Two different ways to take into account the initial conditions have been employed in the conventional definitions of participation factors. The original definition (3) of PFs and generalized participations (GPs) in (Pérez-Arriaga *et al.* (1982)) used specially selected initial conditions of the form

$$\boldsymbol{x}_0 = \mathbf{e}_k\ \text{ and }\ \boldsymbol{x}_0 = \mathbf{e}_l,\ l \neq k\ . \quad (34)$$

In particular, for the first initial condition, we get the PF p_{ki}, which "weights" the participation of the i-th mode and initial k-th state in the dynamics of the k-th state. For the second initial condition, we get the GP p_{kil}, which "weights" the participation of the i-th mode and initial l-th state in the dynamics of the k-th state

$$x_k(t) = \sum_i p_{ki}\, x_k(0) \exp(\lambda_i t) + \sum_{i,\, l \neq k} p_{kil}\, x_l(0) \exp(\lambda_i t)\ .$$

Under initial conditions (34), by analogy with the p_{ki} and p_{kil}, the corresponding mode-in-state Lyapunov PFs and GPs can be calculated as

$$e_{ki} = \frac{(\widetilde{\mathbf{P}}_{x_k i})_{kk}}{\sum_{i'=1}^{n} (\widetilde{\mathbf{P}}_{x_k i'})_{kk}}\ ,\ e_{kil} = \frac{(\widetilde{\mathbf{P}}_{x_k i})_{ll}}{\sum_{i'=1}^{n} (\widetilde{\mathbf{P}}_{x_k i'})_{ll}}\ . \quad (35)$$

The difference from p_{ki} and p_{kil} is that the participations in (35) are taken not in relation to the state variable itself, but in relation to its Lyapunov energy.

Another approach was proposed in (Hashlamoun *et al.* (2009)), which was based on averaging over an uncertain set of initial conditions. The corresponding formula for calculation of PFs was proposed for the spherically symmetric distribution of the initial conditions with respect to zero. Following this approach, from Definition 1 we obtain the following formula for calculating the MISLPF:

$$e_{ki} = \frac{\text{trace}\,(\widetilde{\mathbf{P}}_{x_k i})}{\text{trace}\,(\mathbf{P}_{x_k})}\ . \quad (36)$$

Substituting here from (31), we obtain

$$\begin{aligned} e_{ki} &= \frac{\mathrm{Re}\left\{\text{trace}\left((\boldsymbol{v}_i^T)^* (u_i^k)^* \mathbf{e}_k^T (\lambda_i^* \mathbf{I} + \mathbf{A})^{-1}\right)\right\}}{\mathrm{Re}\left\{\text{trace}\left(\sum_{i'=1}^{n} (\boldsymbol{v}_{i'}^T)^* (u_{i'}^k)^* \mathbf{e}_k^T (\lambda_{i'}^* \mathbf{I} + \mathbf{A})^{-1}\right)\right\}} \\ &= \frac{\mathrm{Re}\left\{\sum_{j=1}^{n} \frac{\boldsymbol{v}_i^* \boldsymbol{v}_j\, (u_i^k)^* u_j^k}{\lambda_i^* + \lambda_j}\right\}}{\mathrm{Re}\left\{\sum_{i',j=1}^{n} \frac{\boldsymbol{v}_{i'}^* \boldsymbol{v}_j (u_{i'}^k)^* u_j^k}{\lambda_{i'}^* + \lambda_j}\right\}}\ . \end{aligned} \quad (37)$$

3.3. State-in-mode Lyapunov PF

The initial Lyapunov energy of the i-th mode (27) can be expressed through the eigenvectors and initial state variables

$$E_{z_i} = \boldsymbol{x}_0^T \mathbf{V}^* \mathbf{P}_{zi} \mathbf{V} \boldsymbol{x}_0 = \frac{\boldsymbol{x}_0^T (\boldsymbol{v}_i^T)^*\, \boldsymbol{v}_i^T \boldsymbol{x}_0}{-2\,\mathrm{Re}\{\lambda_i\}},\ \text{where}\ \mathbf{P}_{zi} = \frac{-\,\mathbf{e}_i\, \mathbf{e}_i^T}{2\,\mathrm{Re}\{\lambda_i\}}\ .$$

This Lyapunov energy can be partitioned into the parts corresponding to the state variables so that the contribution from each two state variables be divided in half between them. Then the contribution in E_{z_i} from the k-th state is

$$E_{z_i k} = \frac{1}{2}\boldsymbol{x}_0^T\Big(\mathbf{V}^*\mathbf{P}_{zi}\mathbf{V}\,\mathbf{e}_k\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_k^T\mathbf{V}^*\mathbf{P}_{zi}\mathbf{V}\Big)\boldsymbol{x}_0 \,. \\ = \frac{\boldsymbol{x}_0^T(\boldsymbol{v}_i^T)^* v_i^k x_k^0 + (v_i^k)^* x_k^0\, \boldsymbol{v}_i^T\boldsymbol{x}_0}{-4\,\mathrm{Re}\{\lambda_i\}}\,, \tag{38}$$

where v_i^k and x_k^0 are the k-th components of $\boldsymbol{v}_i$ and $\boldsymbol{x}_0$, respectively. Introducing the notation

$$E_{z_i k} = z_0^*\,\widetilde{\mathbf{P}}_{zik}\,z_0\,, \quad \widetilde{\mathbf{P}}_{zik} = \mathbf{P}_{zi}\mathbf{V}\,\mathbf{e}_k\mathbf{e}_k^T\mathbf{U} + \mathbf{U}^*\mathbf{e}_k\mathbf{e}_k^T\mathbf{V}^*\mathbf{P}_{zi}\,,$$

we can formulate the following definition.

Definition 2. *The state-in-mode Lyapunov participation factor (SIMLPF)* characterizes the relative participation of the k-th state in the Lyapunov energy E_{z_i} of the i-th mode (27), that is

$$\varepsilon_{ki} = \frac{E_{z_i k}}{E_{z_i}} = \frac{z_0^*\,\widetilde{\mathbf{P}}_{zik}\,z_0}{z_0^*\,\mathbf{P}_{zi}\,z_0} = \frac{\boldsymbol{x}_0^T(\boldsymbol{v}_i^T)^* v_i^k x_k^0 + (v_i^k)^* x_k^0\,\boldsymbol{v}_i^T\boldsymbol{x}_0}{2\,\boldsymbol{x}_0^T(\boldsymbol{v}_i^T)^*\,\boldsymbol{v}_i^T\boldsymbol{x}_0}\,. \tag{39}$$

This definition coincides with the modified definition for conventional SIMPF proposed in (Iskakov (2019)). In this paper, it was shown that Definition 2 in general reproduces the results of the definition of SIMPFs proposed in (Hashlamoun *et al.* (2009)) for real eigenvalues, and rectifies the results in the case of complex eigenvalues. In particular, for the spherically symmetric distribution of the initial conditions with respect to zero, we obtain from Definition 2

$$\varepsilon_{ki} = \frac{v_i^k\,(v_i^k)^*}{(\boldsymbol{v}_i^T)^*\,\boldsymbol{v}_i^T}\,. \tag{40}$$

This coincides with (9) obtained in (Hashlamoun *et al.* (2009)) for real eigenvalues and corrected in (Konoval, Prytula (2017)) for the case of complex eigenvalues. The obtained coincidence can be explained as follows. The amplitude of the i-th mode and all its components (into whatever parts it is divided) change in time with the same exponential rate. Therefore, the ratio of the corresponding Lyapunov energies coincides with the ratio of the same "instantaneous" energies at the initial moment. And thus, the state-in-mode LPFs coincide with the corresponding conventional PFs. In contrast, the mode-in-state LPFs in (33) are characterized by Lyapunov energies in the state space, which are determined by the exponential terms with different rates. Therefore, in contrast to (39), there is a characteristic dependence in (37) on pair combinations of eigenvalues.

Remark 2. Although Definition 2 is invariant with respect to a change of units of the system state variables, the expression (40) is not, because it includes the assumption of spherical symmetry of the initial conditions. The expression (40), however, justifies its meaning when all state variables are independent and scaled so that they have the same variance.

4. Modal analysis of Lyapunov energy of states

4.1. Modal contributions to Lyapunov energy of states

The Lyapunov modal energy E_z in (28) is defined in the space of modal vectors $z \in \mathbb{C}^n$. Nevertheless, the Lyapunov energy of states E_x in (25) can also be considered from the point of view of modes, if we apply the transformation (7), that is

$$E_x = \boldsymbol{x}^T\mathbf{P}_x\,\boldsymbol{x} = \boldsymbol{x}^T\mathbf{V}^*\mathbf{U}^*\mathbf{P}_x\,\mathbf{U}\,\mathbf{V}\,\boldsymbol{x} = z^*\,\overline{\mathbf{P}}_z\,z = \overline{E}_z\,, \tag{41}$$

where the matrix $\overline{\mathbf{P}}_z = \mathbf{U}^*\mathbf{P}_x\,\mathbf{U}$, and the modal vector $z = \mathbf{V}\boldsymbol{x}$. Substituting the representation (2) of the matrix $\mathbf{A}$ into (26) and multiplying the resulting equation from the right by the matrix $\mathbf{U}$ and from the left by the matrix $\mathbf{U}^*$, we obtain that $\overline{\mathbf{P}}_z$ satisfies the following Lyapunov equation:

$$\boldsymbol{\Lambda}^*\,\overline{\mathbf{P}}_z + \overline{\mathbf{P}}_z\,\boldsymbol{\Lambda} = -\,\mathbf{U}^*\mathbf{U}\,. \tag{42}$$

The joint contribution to E_x from ith and jth modes can be divided equally between these modes. Then a contribution of the i-th mode to E_x can be defined as

$$\overline{E}_{z_i} = \frac{1}{2}\sum_{j=1}^{n}\Big(z_i^*\,(\overline{\mathbf{P}}_z)_{ij}\,z_j + z_j^*\,(\overline{\mathbf{P}}_z)_{ji}\,z_i\Big)\,, \tag{43}$$

where $(\overline{\mathbf{P}}_z)_{ij}$ and $(\overline{\mathbf{P}}_z)_{ji}$ are the elements of the matrix $\overline{\mathbf{P}}_z$. Substituting their values from (42), we obtain

$$\overline{E}_{z_i} = -\,\mathrm{Re}\left\{\sum_{j=1}^{n}\frac{\boldsymbol{u}_i^*\,\boldsymbol{u}_j}{\lambda_i^* + \lambda_j}z_i^*\,z_j\right\}\,. \tag{44}$$

On the other hand, one can apply to (42) the decomposition (13-14) of $\overline{\mathbf{P}}_z$ into the sub-Gramians.

$$\overline{\mathbf{P}}_z = \sum_{i=1}^{n}\overline{\mathbf{P}}_{zi}\,, \quad \text{where}$$

$$\overline{\mathbf{P}}_{zi} = -\Big\{\mathbf{e}_i\mathbf{e}_i^T\mathbf{U}^*\mathbf{U}(\lambda_i^*\mathbf{I} + \boldsymbol{\Lambda})^{-1}\Big\}_H = -\left\{\sum_{j=1}^{n}\frac{\mathbf{e}_i\,\boldsymbol{u}_i^*\boldsymbol{u}_j\,\mathbf{e}_j^T}{\lambda_i^* + \lambda_j}\right\}_H\,. \tag{45}$$

Therefore, comparing this with (44), we can define *modal contribution (MC) of the i-th mode to the Lyapunov energy of states* using the sub-Gramian $\overline{\mathbf{P}}_{zi}$ as

$$\overline{E}_{z_i} = z^*\overline{\mathbf{P}}_{zi}\,z\,, \quad \text{where} \quad z = \mathbf{V}\boldsymbol{x}\,, \\ \boldsymbol{\Lambda}^*\,\overline{\mathbf{P}}_{zi} + \overline{\mathbf{P}}_{zi}\,\boldsymbol{\Lambda} = -\frac{1}{2}\Big(\mathbf{U}^*\mathbf{U}\,\mathbf{e}_i\mathbf{e}_i^T + \mathbf{e}_i\mathbf{e}_i^T\mathbf{U}^*\mathbf{U}\Big)\,. \tag{46}$$

Values E_{z_i} in (27) characterize modal energies in the space of modal vectors. By definition, they are always positive. In contrast, the quantities $\overline{E}_{z_i}$ in (46) determine the contributions of the modes to the Lyapunov energy accumulated in state variables. As can be seen from (44), they can be negative if there are modes with a sufficiently large amplitude and negative correlation with a given mode. It follows from (43) that

$$\overline{\mathbf{P}}_{zi} = \frac{1}{2}\Big(\overline{\mathbf{P}}_z\,\mathbf{e}_i\mathbf{e}_i^T + \mathbf{e}_i\mathbf{e}_i^T\,\overline{\mathbf{P}}_z\Big)\,. \tag{47}$$

Because the matrix $\overline{\mathbf{P}}_{zi}$ is self-adjoint, then $(\overline{E}_{z_i})^* = \overline{E}_{z_i}$, i.e. the MCs are always real. In addition, MCs of the modes corresponding to a pair of complex conjugate eigenvalues are the same.

Proposition 1. Let i and $i*$ be the indexes associated with eigenvalues λ_i and λ_i^*, respectively. Then, the corresponding modal contributions are the same

$$\overline{E}_{z_{i*}} = \overline{E}_{z_i}$$

Proof. For arbitrary j, we consider in the expression (46) for $\overline{E}_{z_i}$ a term

$$z_i^* \, \overline{\mathbf{P}}_{zi} \, z_j + z_j^* \, \overline{\mathbf{P}}_{zi} \, z_i = -\frac{1}{2} \left(\frac{z_i^* \boldsymbol{u}_i^* \boldsymbol{u}_j z_j}{\lambda_i^* + \lambda_j} + \frac{z_j^* \boldsymbol{u}_j^* \boldsymbol{u}_i z_i}{\lambda_j^* + \lambda_i} \right)_H . \tag{48}$$

Choose $j*$ so that if j corresponds to a real eigenvalue, then $j* = j$, and if j corresponds to a complex eigenvalue λ_j, then $j*$ corresponds to a complex eigenvalue λ_j^*. Then one can verify that in the expression for $\overline{E}_{z_{i*}}$ there is the term

$$z_{i*}^* \, \overline{\mathbf{P}}_{zi*} \, z_{j*} + z_{j*}^* \, \overline{\mathbf{P}}_{zi*} \, z_{i*} \,, \tag{49}$$

which equals to (48) under the assumption (21). By the one-to-one correspondence of terms (48) and (49), we obtain that $\boldsymbol{z}^* \, \overline{\mathbf{P}}_{zi} \, \boldsymbol{z} = \boldsymbol{z}^* \, \overline{\mathbf{P}}_{zi*} \, \boldsymbol{z}$. □

The total contribution of all modes is equal to the Lyapunov energy of all states:

$$\sum_{i=1}^{n} \overline{E}_{z_i} = \boldsymbol{z}^* \overline{\mathbf{P}}_z \, \boldsymbol{z} = E_x \,, \text{ where } \overline{\mathbf{P}}_z = \sum_{i=1}^{n} \overline{\mathbf{P}}_{zi} \tag{50}$$

is the solution of Lyapunov equation (42). From (45) and (31) one can also verify that the modal contribution of the i-th mode is a sum of its contributions to all states

$$\overline{E}_{z_i} = \boldsymbol{x}^T \mathbf{V}^* \overline{\mathbf{P}}_{z_i} \mathbf{V} \boldsymbol{x} = \sum_{k=1}^{n} \boldsymbol{x}^T \widetilde{\mathbf{P}}_{x_k i} \boldsymbol{x} = \sum_{k=1}^{n} E_{x_k i} \,.$$

4.2. Lyapunov modal interactions

Unlike state variables or signals, their Lyapunov energies can be partitioned not only into parts corresponding to individual eigenvalues, but also into parts corresponding to pair combinations of eigenvalues, i.e. the solution of Lyapunov equation (10) can be decomposed as (13)-(14)

$$\mathbf{P} = \sum_{i,j=1}^{n} \mathbf{P}_{ij} = -\sum_{i,j=1}^{n} \left\{ \frac{\mathbf{R}_i^* \, \mathbf{Q} \, \mathbf{R}_j}{\lambda_i^* + \lambda_j} \right\}_H .$$

This allows us to characterize the pairwise modal interactions in the system by the Lyapunov energies produced in the system states by the corresponding pairwise mode combinations.

Definition 3. *The Lyapunov modal interaction energy (LMIE)* of the i-th and j-th modes in the system (1) is

$$\begin{aligned} \overline{E}_{z\,ij} = \boldsymbol{x}^T \mathbf{P}_{(ij)} \, \boldsymbol{x} \,, \text{ where } \mathbf{P}_{(ij)} &= -\frac{1}{2} \left\{ \frac{\mathbf{R}_i^* \, \mathbf{R}_j}{\lambda_i^* + \lambda_j} + \frac{\mathbf{R}_i^T \, \mathbf{R}_j}{\lambda_i + \lambda_j} \right\}_H \\ &= -\frac{1}{2} \left\{ \frac{(\boldsymbol{v}_i^T)^* \boldsymbol{u}_i^* \, \boldsymbol{u}_j \, \boldsymbol{v}_j^T}{\lambda_i^* + \lambda_j} + \frac{\boldsymbol{v}_i \, \boldsymbol{u}_i^T \, \boldsymbol{u}_j \, \boldsymbol{v}_j^T}{\lambda_i + \lambda_j} \right\}_H . \end{aligned} \tag{51}$$

Because the matrix $\mathbf{P}_{(ij)}$ is self-adjoint, then the LMIEs are always real, i.e. $\overline{E}_{z\,ij} = \boldsymbol{x}^T \mathrm{Re}\left\{\mathbf{P}_{(ij)}\right\} \boldsymbol{x}$. In addition, for any matrix $\mathbf{C}$ the equality $\mathrm{Re}\{\{\mathbf{C}\}_H\} = \mathrm{Re}\{\{\mathbf{C}\}_{SYM}\}$ holds, where $\{\mathbf{C}\}_{SYM} = \frac{1}{2}(\mathbf{C}+\mathbf{C}^T)$. Therefore, the definition of LMIE implies that

$$\begin{aligned} \overline{E}_{z\,ji} &= -\frac{1}{2} \boldsymbol{x}^T \mathrm{Re} \left\{ \left\{ \frac{\mathbf{R}_j^* \, \mathbf{R}_i}{\lambda_j^* + \lambda_i} \right\}_H + \left\{ \frac{\mathbf{R}_j^T \, \mathbf{R}_i}{\lambda_j + \lambda_i} \right\}_{SYM} \right\} \boldsymbol{x} \\ &= -\frac{1}{2} \boldsymbol{x}^T \mathrm{Re} \left\{ \left\{ \frac{\mathbf{R}_i^* \, \mathbf{R}_j}{\lambda_i^* + \lambda_j} \right\}_H + \left\{ \frac{\mathbf{R}_i^T \, \mathbf{R}_j}{\lambda_i + \lambda_j} \right\}_{SYM} \right\} \boldsymbol{x} = \overline{E}_{z\,ij} \,. \end{aligned}$$

Furthermore, the symmetrization in (51) ensures that the LMIEs associated with the complex conjugate eigenvalues λ_i and λ_i^* are always the same, that is

$$\overline{E}_{z\,i*j} = \overline{E}_{z\,ij} \,, \tag{52}$$

where i and $i*$ are the indexes associated with eigenvalues λ_i and λ_i^*, respectively. The matrix $\mathbf{P}_{(ij)}$ in (51) is the solution of the Lyapunov equation

$$\mathbf{A}^* \, \mathbf{P}_{(ij)} + \mathbf{P}_{(ij)} \, \mathbf{A} = -\frac{1}{4} \left(\mathbf{R}_i^* \mathbf{R}_j + \mathbf{R}_i^T \mathbf{R}_j + \mathbf{R}_j^* \mathbf{R}_i + \mathbf{R}_j^* (\mathbf{R}_i^T)^* \right).$$

It follows from (44), (45), and (51) that

$$\frac{1}{2} \left(\overline{E}_{z_i} + \overline{E}_{z_{i*}} \right) = \sum_{j=1}^{n} \overline{E}_{z\,ij} \,, \;\; \frac{1}{2} \mathbf{V}^* \left(\overline{\mathbf{P}}_{zi} + \overline{\mathbf{P}}_{zi*} \right) \mathbf{V} = \sum_{j=1}^{n} \mathbf{P}_{(ij)} \,.$$

In view of Proposition 1, this implies that the MC of the mode is a sum of Lyapunov energies corresponding to its modal interactions with other modes, that is

$$\overline{E}_{z_i} = \sum_{j=1}^{n} \overline{E}_{z\,ij} \,, \;\; \mathbf{V}^* \overline{\mathbf{P}}_{zi} \mathbf{V} = \sum_{j=1}^{n} \mathbf{P}_{(ij)} \,.$$

When analyzing small-signals in a linearized system it would be convenient to have general indicators that would characterize the participation of the modes and their interaction in terms of accumulated Lyapunov energy, but would not depend on state variables, in which the initial perturbation was made. For this purpose, we consider a probabilistic description of the uncertainty in the initial condition and assume that the components of the initial condition vector $x_1^0, x_2^0, \ldots, x_n^0$ are distributed independently with zero mean and unit variance, that is

$$\mathrm{E}\,\{x_k^0 x_{k'}^0\} = \delta_{kk'} \,, \tag{53}$$

where the expectation operator $\mathrm{E}\{\cdot\}$ is evaluated using some assumed joint probability function $f(\boldsymbol{x}_0)$ for the initial condition uncertainty, and $\delta_{kk'}$ is the Kronecker delta.

Proposition 2. Under assumption (53), the averaged Lyapunov MCs $\overline{E}_{z_i}$ and modal interaction energies $\overline{E}_{z\,ij}$ can be computed as

$$\mathrm{E}\left\{\overline{E}_{z_i}\right\} = -\operatorname{Re}\left\{\operatorname{trace}\left(\mathbf{R}_i^* \left(\lambda_i^* \mathbf{I} + \mathbf{A}\right)^{-1}\right)\right\},$$

$$\mathrm{E}\left\{\overline{E}_{z\,ij}\right\} = -\frac{1}{2}\operatorname{Re}\left\{\frac{\operatorname{trace}\left(\mathbf{R}_i^*\mathbf{R}_j\right)}{\lambda_i^* + \lambda_j} + \frac{\operatorname{trace}\left(\mathbf{R}_i^T\mathbf{R}_j\right)}{\lambda_i + \lambda_j}\right\}. \quad (54)$$

Proof. Under assumption (53), the first expression follows from (45), (46), and the formula $\mathbf{R}_i = \boldsymbol{u}_i\boldsymbol{v}_i^T$ for the residue of the resolvent of the matrix $\mathbf{A}$. The second expression follows directly from the definition (51). We also take into account the equality $\operatorname{trace}\{\mathbf{C}\}_H = \operatorname{Re}\{\operatorname{trace}\mathbf{C}\}$. □

Based on the result of Proposition 2, the following indicator can be proposed as a convenient measure of the relative participation of the modes in each other.

Definition 4. *The Lyapunov modal interaction factor (LMIF)* of the j-th mode in the i-th mode is

$$LMIF_{ij} = \frac{\mathrm{E}\left\{\overline{E}_{z\,ij}\right\}}{\sum_{j'=1}^{n}\left|\mathrm{E}\left\{\overline{E}_{z\,ij'}\right\}\right|}, \quad (55)$$

where $\mathrm{E}\left\{\overline{E}_{z\,ij}\right\}$ is defined in (54).

These indicators make practical sense if the state variables can be scaled in such a way that their disturbances have approximately the same importance from the point of view of analyzing small signals in the system. This somewhat specific interpretation is compensated, however, by the fact that LMIFs do not have an explicit dependence on individual state variables.

4.3. Lyapunov pair PFs

In order to analyze modal interactions in connection with specific state variables, it is necessary to introduce Lyapunov energies and participation factors that correspond not only to individual eigenvalues, but also to their pair combinations. According to (13), (14), (15), and (24), *Lyapunov energy of the k-th state variable associated with a pair of i-th and j-th modes* can be defined as

$$E_{x_k ij} = \boldsymbol{x}^T\mathbf{P}_{x_k ij}\,\boldsymbol{x}, \quad \mathbf{P}_{x_k} = \sum_{i,j=1}^{n}\mathbf{P}_{x_k ij}, \quad \text{where}$$

$$\mathbf{P}_{x_k ij} = -\frac{1}{2}\left\{\frac{\mathbf{R}_i^*\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}_j}{\lambda_i^* + \lambda_j} + \frac{\mathbf{R}_i^T\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}_j}{\lambda_i + \lambda_j}\right\}_H$$

$$= -\frac{1}{2}\left\{\frac{(\boldsymbol{v}_i^T)^*(u_i^k)^* u_j^k \boldsymbol{v}_j^T}{\lambda_i^* + \lambda_j} + \frac{\boldsymbol{v}_i\, u_i^k\, u_j^k \boldsymbol{v}_j^T}{\lambda_i + \lambda_j}\right\}_H, \quad (56)$$

and the symmetrized sub-Gramians $\mathbf{P}_{x_k ij}$ satisfy the Lyapunov equations

$$\mathbf{A}^*\,\mathbf{P}_{x_k ij} + \mathbf{P}_{x_k ij}\,\mathbf{A} =$$
$$-\frac{1}{4}\left(\mathbf{R}_i^*\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}_j + \mathbf{R}_j^*\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}_i + \mathbf{R}_i^T\mathbf{e}_k\mathbf{e}_k^T\mathbf{R}_j + \mathbf{R}_j^*\mathbf{e}_k\mathbf{e}_k^T(\mathbf{R}_i^T)^*\right).$$

By this definition we have $E_{x_k ij} = E_{x_k ji}$ and $E_{x_k i^* j} = E_{x_k ij}$. Moreover, from (51) and (56) it follows that

$$\sum_{k=1}^{n} E_{x_k ij} = \overline{E}_{z\,ij}. \quad (57)$$

Similar to (56), Lyapunov energy of the i-th mode associated with a pair of k-th and l-th states can be defined as

$$E_{z_i kl} = \boldsymbol{z}^*\mathbf{P}_{z_i kl}\,\boldsymbol{z}$$
$$= \frac{1}{2}\boldsymbol{x}^T\left(\mathbf{e}_l\mathbf{e}_l^T\mathbf{V}^*\mathbf{P}_{zi}\mathbf{V}\,\mathbf{e}_k\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_k^T\mathbf{V}^*\mathbf{P}_{zi}\mathbf{V}\,\mathbf{e}_l\mathbf{e}_l^T\right)\boldsymbol{x}$$
$$= \frac{(v_i^l)^* x_l\, v_i^k x_k + (v_i^k)^* x_k\, v_i^l x_l}{-4\operatorname{Re}\{\lambda_i\}}. \quad (58)$$

On the basis of Lyapunov energies (56) and (58) we can introduce the corresponding Lyapunov PFs.

Definition 5. *The pair MISLPF* characterizes the relative pairwise participation of i-th and j-th modes in the Lyapunov energy (24) of the k-th state

$$\tilde{e}_{k(ij)} = \frac{E_{x_k ij}}{E_{x_k}} = \frac{\boldsymbol{x}^T\mathbf{P}_{x_k ij}\,\boldsymbol{x}}{\boldsymbol{x}^T\mathbf{P}_{x_k}\boldsymbol{x}}. \quad (59)$$

The pair SIMLPF characterizes the relative participation of k-th and l-th states in the Lyapunov energy (27) of the i-th mode

$$\tilde{\varepsilon}_{i(kl)} = \frac{E_{z_i kl}}{E_{z_i}} = \frac{\boldsymbol{z}^*\mathbf{P}_{z_i kl}\,\boldsymbol{z}}{\boldsymbol{z}^*\mathbf{P}_{z_i}\boldsymbol{z}}. \quad (60)$$

The first indicator (59) shows which state variables have Lyapunov energy which is most sensitive to a particular modal interaction. The second indicator (60) shows which pairs of state variables produce the particular mode. Definition 5 is consistent with the previous definitions in the sense that the following relations hold between the Lyapunov PFs:

$$\sum_{j=1}^{n}\tilde{e}_{k(ij)} = e_{ki}, \quad \sum_{l=1}^{n}\tilde{\varepsilon}_{i(kl)} = \varepsilon_{ki}, \quad \sum_{i,j=1}^{n}\tilde{e}_{k(ij)} = \sum_{k,l=1}^{n}\tilde{\varepsilon}_{i(kl)} = 1.$$

Under conventional initial conditions (34) the corresponding pair mode-in-state Lyapunov PFs and GPs can be calculated by analogy with (35) as

$$\tilde{e}_{k(ij)} = \frac{(\mathbf{P}_{x_k ij})_{kk}}{(\mathbf{P}_{x_k})_{kk}}, \quad \tilde{e}_{k(ij)l} = \frac{(\mathbf{P}_{x_k ij})_{ll}}{(\mathbf{P}_{x_k})_{ll}}. \quad (61)$$

The calculation formulas for pair Lyapunov PFs can be also easily obtained for the spherically symmetric distribution of the initial conditions with respect to zero using (56), (58), (27), and (32):

$$\tilde{e}_{k(ij)} = \frac{\mathrm{trace}\,(\mathbf{P}_{x_k ij})}{\mathrm{trace}\,(\mathbf{P}_{x_k})} = \frac{\dfrac{1}{2}\mathrm{Re}\left\{\dfrac{\boldsymbol{v}_i^* \boldsymbol{v}_j\,(u_i^k)^* u_j^k}{\lambda_i^* + \lambda_j} + \dfrac{\boldsymbol{v}_i \boldsymbol{v}_j\, u_i^k\, u_j^k}{\lambda_i + \lambda_j}\right\}}{\mathrm{Re}\left\{\sum\limits_{i',j'=1}^{n} \dfrac{\boldsymbol{v}_{i'}^* \boldsymbol{v}_{j'}\,(u_{i'}^k)^* u_{j'}^k}{\lambda_{i'}^* + \lambda_{j'}}\right\}}, \quad (62)$$

$$\tilde{\varepsilon}_{i(kl)} = \delta_{kl}\varepsilon_{ki}\ .$$

Although the pair SIMLPF (60) is in some sense dual to the pair MISPF (59), it does not seem to be a meaningful indicator. As a more conceptual indicator can be considered the state participation in LMIEs, which characterizes the relative participation of the k-th state in the LMIE (51) of the i-th and j-th modes as

$$\overline{e}_{k(ij)} = \frac{\overline{E}_{z_{ij}k}}{\overline{E}_{z\,ij}} = \frac{\boldsymbol{x}^T \mathbf{P}_{(ij)k}\,\boldsymbol{x}}{\boldsymbol{x}^T \mathbf{P}_{(ij)}\,\boldsymbol{x}}\,,\quad \text{where} \quad (63)$$

$$\overline{E}_{z_{ij}k} = \boldsymbol{x}^T \mathbf{P}_{(ij)k}\,\boldsymbol{x}\ ,\ \mathbf{P}_{(ij)} = \sum_{k=1}^{n} \mathbf{P}_{(ij)k}\ ,\ \overline{E}_{z\,ij} = \sum_{k=1}^{n} \overline{E}_{z_{ij}k}\ ,$$

$$\mathbf{P}_{(ij)k} = \frac{1}{2}\left(\mathbf{P}_{(ij)}\,\mathbf{e}_k\mathbf{e}_k^T + \mathbf{e}_k\mathbf{e}_k^T\mathbf{P}_{(ij)}\right).$$

This indicator shows, which state variables produce a given modal interaction. Indicator (59) can be useful for selecting state variables that are most sensitive for identifying a specific interaction, while indicator (63) can be used to select state variables for damping this interaction.

5. Lyapunov PFs in the selective modal analysis

In this section we formulate some characteristic properties of Lyapunov PFs that highlight the potential advantages of Lyapunov modal analysis.

5.1. Relation of Lyapunov PFs and conventional PFs

We compare conventional PFs (3) and MISLPFs from Definition 1 under conventional initial conditions (34). In this case the conventional coefficients p_{ki} and p_{kil} correspond to the Lyapunov PFs from (35) and pair Lyapunov PFs from (61).

Property 1. *Under conventional initial conditions* (34), *MISLPFs from Definition* 1 (33) *and pair MISLPFs from Definition* 5 (59) *can be represented through the conventional PFs* (3) *and corresponding eigenvalues as follows*

$$e_{ki} = -\frac{1}{\mathbf{e}_k^T \mathbf{P}_{x_k}\mathbf{e}_k}\left\{p_{ki}^* \mathbf{e}_k^T (\lambda_i^* \mathbf{I} + \mathbf{A})^{-1}\mathbf{e}_k\right\}_H\ ,$$

$$\tilde{e}_{k(ij)} = -\frac{1}{2\,\mathbf{e}_k^T \mathbf{P}_{x_k}\mathbf{e}_k}\left\{\frac{p_{ki}^* p_{kj}}{\lambda_i^* + \lambda_j} + \frac{p_{ki} p_{kj}}{\lambda_i + \lambda_j}\right\}_H. \quad (64)$$

Proof. We substitute (31) and (56) into the definitions (35) and (61), for the MISLPF and pair MISLPF, respectively, obtained under the conventional initial conditions (34). Then we take into account the definition of the conventional PFs (3) □

While the conventional PFs characterize the participation of modes in *the instantaneous dynamics* of the state variables, the MISLPFs and pair MISLPFs characterize their participation in *the integrated energy* accumulated in the state variables. Thus, in contrast to the conventional PFs, new indicators take into account the time response of specific devices, which is reflected as the dependence on the eigenvalues in the denominators of formulas in (64).

5.2. LPFs indicate the distance from the stability boundary

Suppose that the matrix $\mathbf{A}$ smoothly changes depending on the parameter γ so that the following assumption is satisfied.

Assumption 1. All conventional PFs (3) remain limited, i.e.

$$\forall\ k, i\ \ \exists\, C \in \mathbb{R}:\quad |p_{ki}| < C\ . \quad (65)$$

Since the conventional PFs represent the sensitivities of eigenvalues, this assumption means that the spectrum of the dynamics matrix changes smoothly. Then the characteristic property of LPFs is established by the following proposition.

Property 2. *Let the matrix* $\mathbf{A}$ *of the stable system* (20) *changes under Assumption* 1 *in such a way that the i-th eigenvalue tends to the imaginary axis from the left, while other eigenvalues are limited*

$$\mathrm{Re}\,(\lambda_i) \to -0\ ,\ \ \mathrm{Re}\,(\lambda_j) < -\alpha < 0,\ \ \lambda_j \neq \lambda_i, \lambda_i^*. \quad (66)$$

Then under conventional initial conditions (34)*,* $E_{zi} \to +\infty$*, and there is at least one state variable k such that the following limiting relations are satisfied for MISLPFs:*

$$\begin{aligned} &\text{if } \lambda_i \text{ is real}: \ E_{x_k i} \to +\infty,\ \ e_{ki} \to 1\ ,\ \ e_{kj} \to 0\ ,\ \ j \neq i\ ;\\ &\qquad \text{if } \lambda_i \text{ is complex}: \ E_{x_k i}, E_{x_k i^*} \to +\infty, \qquad (67)\\ &\qquad e_{ki}, e_{ki^*} \to 0.5\ ,\ \ e_{kj} \to 0\ ,\ \ j \neq i,\ i^*\ . \end{aligned}$$

Proof. Consider the case when λ_i is real. Choose a state variable k such that $|p_{ki}| \neq 0$. Note that under conventional initial conditions (34), $E_{x_k} = \mathbf{e}_k^T \mathbf{P}_{x_k}\mathbf{e}_k$. Then it follows from (64) that

$$E_{x_k}\tilde{e}_{kii} = -\left\{\frac{p_{ki}^* p_{ki}}{\lambda_i^* + \lambda_i}\right\}_H = \frac{|p_{ki}|^2}{-2Re(\lambda_i)} \to +\infty\ ,$$

$$\forall j \neq i:\ |E_{x_k}\tilde{e}_{kij}| < \frac{|p_{ki}^*|\,|p_{kj}|}{-Re(\lambda_i) - Re(\lambda_j)} < \frac{C^2}{\alpha}\ ,$$

$$\forall j, j' \neq i:\ |E_{x_k}\tilde{e}_{kjj'}| < \frac{|p_{kj}^*|\,|p_{kj'}|}{-Re(\lambda_{j'}) - Re(\lambda_j)} < \frac{C^2}{2\alpha}\ .$$

From here we obtain

$$E_{x_k i} = E_{x_k} e_{ki} = E_{x_k}\sum_{j'=1}^{n}\tilde{e}_{kij'} \to +\infty\ ,$$

$$\forall j \neq i:\ E_{x_k j} = E_{x_k} e_{kj} = E_{x_k}\sum_{j'=1}^{n}\tilde{e}_{kjj'} < \frac{(n+1)\,C^2}{2\alpha}\ ,$$

$$e_{ki} = \frac{E_{x_k i}}{\sum_{j'=1}^{n} E_{x_k j'}} \to 1,\ \ \forall j \neq i: e_{kj} = \frac{E_{x_k j}}{\sum_{j'=1}^{n} E_{x_k j'}} \to 0.$$

The case of complex λ_i is treated similarly. □

It follows from Property 2 that, in contrast to conventional PFs, the values of LPFs depend on the distance of the corresponding eigenvalues from the stability boundary, i.e. the proposed indicators take into account the risk of stability loss associated with individual weakly stable modes. In expression (64) this is reflected by the dependence of the denominator on λ_i. The closer i-mode is to the stability boundary, the smaller the value of the real part of λ_i, and the greater the value of the corresponding LPFs and Lyapunov energies.

5.3. Pair LPFs indicate resonant modal interactions

Consider a stable system (20) with two close eigenfrequencies $\omega_1 \approx \omega_2$ corresponding to the eigenvalues $\lambda_{1,1^*} = -\alpha_1 \pm i\omega_1$ and $\lambda_{2,2^*} = -\alpha_2 \pm i\omega_2$. Suppose that the matrix $\mathbf{A}$ smoothly changes depending on the parameter γ so that the following assumptions are satisfied.

Assumption 2. The damping for λ_1, λ_2 is small, i.e

$$\alpha_1 + \alpha_2 << \omega_1 + \omega_2 \,. \tag{68}$$

Remark. This assumption is fulfilled in many practical situations. For example, if we consider all modes with a damping less than unity and a frequency greater than 1 Hz (i.e. $\alpha < 1$ and $\omega > 2\pi$), then this assumption is practically satisfied.

Assumption 3. The eigen-frequencies ω_1 and ω_2 change with γ much faster that the corresponding conventional PFs, i.e. for each k, l:

$$\frac{1}{\omega_1}\frac{d\omega_1}{d\gamma}, \; \frac{1}{\omega_2}\frac{d\omega_2}{d\gamma} >> \frac{1}{p_{k1l}}\frac{dp_{k1l}}{d\gamma}, \; \frac{1}{p_{k2l}}\frac{dp_{k2l}}{d\gamma} \,. \tag{69}$$

The characteristic property of LMIEs and pair LPFs is established by the following proposition.

Property 3. *Let the matrix* $\mathbf{A}$ *changes under assumptions* 1, 2, 3 *in such a way that it passes through the point, where* $\omega_1 = \omega_2$. *Then the LMIE* $\overline{E}_{zij}$ *in* (51) *and Lyapunov energies* $E_{x_k 12}(\gamma)$ *in* (56) *associated with a pair of eigenvalues* λ_1 *and* λ_2 *reach the local maximum in the neighborhood of point* $\omega_1 \approx \omega_2$ *both under conventional* (34) *and spherically symmetrical* (53) *initial conditions.*

Proof. Choose a state variable k such that $|p_{ki}| \neq 0$, $|p_{kj}| \neq 0$. Under the conventional initial conditions (34), according to (64) we have

$$E_{x_k 12} = E_{x_k}\tilde{e}_{k(12)} = \frac{1}{4}\sum_{i=1,1^*}\sum_{j=2,2^*} \mathrm{Re}\left\{-\frac{p^*_{ki}p_{kj}}{\lambda^*_i + \lambda_j}\right\} \,. \tag{70}$$

Let introduce the notations $\alpha = \alpha_1 + \alpha_2$, $\omega = \omega_1 + \omega_2$, $\Delta\omega = \omega_2 - \omega_1$. According to definition (3) we have $p_{k1^*} = p^*_{k1}$ and $p_{k2^*} = p^*_{k2}$. Using this and combining the terms in (70) we obtain

$$E_{x_k 12} = \frac{1}{2}\frac{\alpha C^k_R + \Delta\omega C^k_I}{\alpha^2 + \Delta\omega^2} + \frac{1}{2}\frac{\alpha D^k_R + \omega D^k_I}{\alpha^2 + \omega^2} \,, \tag{71}$$

where the coefficients $C^k_R = \mathrm{Re}\{p_{k1}p^*_{k2}\}$, $C^k_I = \mathrm{Im}\{p_{k1}p^*_{k2}\}$, $D^k_R = \mathrm{Re}\{p_{k1}p_{k2}\}$, $D^k_I = \mathrm{Im}\{p_{k1}p_{k2}\}$ have the same order of magnitude. According to assumption (68), $\alpha << \omega$ and the second term in (71) is negligible

$$\alpha \cdot E_{x_k 12} = \frac{1}{2}\frac{C^k_R + \Delta\tilde{\omega}\, C^k_I}{1 + \Delta\tilde{\omega}^2} + O(\epsilon) \,, \tag{72}$$
$$\text{where} \quad \Delta\tilde{\omega} = \Delta\omega/\alpha, \;\; \epsilon = \alpha/\omega << 1 \,.$$

According to assumption (69), the dependence of the coefficients C^k_R and C^k_I on γ can be neglected. Then the maximum absolute value of (72) is reached approximately at $C_I\Delta\tilde{\omega} \approx \sqrt{C^2_R + C^2_I} - C_R$, or returning to the original notation at

$$|\Delta\omega| \approx \alpha\frac{|p_{k1}p^*_{k2}| - Re\{p_{k1}p^*_{k2}\}}{|\mathrm{Im}\{p_{k1}p^*_{k2}\}|} < \alpha_1 + \alpha_2 << \omega_1 + \omega_2 \,.$$

Under the spherically symmetrical initial conditions (53), the proof is similar with the replacement of C^k_R and C^k_I by

$$\widetilde{C}^k_R = \sum_{l=1}^{n} \mathrm{Re}\{p_{k1l}p^*_{k2l}\} \quad \text{and} \quad \widetilde{C}^k_I = \sum_{l=1}^{n} \mathrm{Im}\{p_{k1}p^*_{k2}\}.$$

According to (57), $\overline{E}_{z12} = \sum_{k=1}^{n} E_{x_k 12}$. Therefore, the function $\overline{E}_{z12}$ under the spherically symmetrical initial conditions (53) has the same structure as (72)

$$\overline{E}_{z12} = \sum_{k=1}^{n} E_{x_k 12} \approx \frac{1}{2\alpha}\frac{(\sum_k \widetilde{C}^k_R) + \Delta\tilde{\omega}\,(\sum_k \widetilde{C}^k_I)}{1 + \Delta\tilde{\omega}^2} \,,$$

This function reaches its local maximum approximately at

$$\Delta\omega \approx \alpha\left(\sqrt{\sigma^2 + 1} - \sigma\right) < \alpha_1 + \alpha_2 << \omega_1 + \omega_2 \,,$$

where $\sigma = \sum_k \widetilde{C}^k_R \big/ \sum_k \widetilde{C}^k_I$. □

Remark. It follows from Property 3 that if the Lyapunov energies corresponding to other mode pairs change in the neighborhood of point $\omega_1 \approx \omega_2$ rather slowly, then the functions $LMIF_{12}\gamma$ in (55) and MISLPF $\tilde{e}_{k(12)}(\gamma)$ in (59) also reach a local maximum in the neighborhood of point $\omega_1 \approx \omega_2$.

It follows from Property 3 that the values of pair Lyapunov energies and pair LPFs allow one to identify the resonant interactions between lightly damped oscillating modes. In general one can see that in (64) the value of LPF $\epsilon_{ki} = \sum_j \tilde{\epsilon}_{kij}$ is not simply proportional to $|p_{ki}|^2$, but involves the interaction of i-mode with other modes j. In accordance with Property 3, the closer j-mode is in frequency to i-mode, the greater its contribution $\tilde{\epsilon}_{kij}$ to ϵ_{ki}, which characterizes *the resonance interaction*. In a similar way, the smaller $|Re(\lambda_j)|$, the greater j-mode contribution $\tilde{\epsilon}_{kij}$ to ϵ_{ki}, which characterizes *the interaction with a weakly stable mode*.

5.4. Fast calculation of Lyapunov PFs

The possibility of fast calculation of Lyapunov energies and LPFs for the purposes of selective modal analysis is based on formulas (31, 35, 37, 56, 61, 62), which can be summarized as follows.

Property 4. *The Lyapunov energies of states associated with particular modes and unnormalized LPFs be calculated through the corresponding eigenvalues and eigenvectors, namely under conventional initial conditions* (34)

$$E_{x_k i} = E_{x_k} e_{ki} = -\left\{(u_i^k)^*(v_i^k)^*\ \mathbf{e}_k^T(\lambda_i^*\mathbf{I}+\mathbf{A})^{-1}\mathbf{e}_k\right\}_H\ ,$$

$$E_{x_k i} = E_{x_k}\tilde{e}_{kij} = -\frac{1}{2}\left\{\frac{(v_i^k)^*(u_i^k)^* u_j^k v_j^k}{\lambda_i^*+\lambda_j} + \frac{v_i^k u_i^k u_j^k v_j^k}{\lambda_i+\lambda_j}\right\}_H\ ,$$

and under spherically symmetric initial conditions (53)

$$E_{x_k i} = E_{x_k} e_{ki} = -\mathrm{Re}\left\{\mathrm{trace}\left((\boldsymbol{v}_i^T)^*(u_i^k)^*\mathbf{e}_k^T(\lambda_i^*\mathbf{I}+\mathbf{A})^{-1}\right)\right\}\ ,$$

$$E_{x_k i} = E_{x_k}\tilde{e}_{kij} = -\frac{1}{2}\mathrm{trace}\left\{\frac{(\boldsymbol{v}_i^T)^*(u_i^k)^* u_j^k \boldsymbol{v}_j^T}{\lambda_i^*+\lambda_j} + \frac{\boldsymbol{v}_i u_i^k u_j^k \boldsymbol{v}_j^T}{\lambda_i+\lambda_j}\right\}_H\ .$$

Property 4 implies that in order to analyze the dynamics of Lyapunov energies and unnormalized LPFs in the critical part of the spectrum, it is not required to know the entire spectrum of the system matrix. It is enough to know only the eigenvalues and eigenvectors in the critical part of the spectrum, which can be obtained using the well-known algorithms of selective modal analysis, such as the modified Arnoldi method or simultaneous iterations. Therefore, the LMA can be performed quickly for the critical modes and serve as a basis for the fast calculation of their behavior in large-scale dynamical systems in real time.

Remark. Formulas of Proposition 4 are valid only for the systems with a simple spectrum. When two eigenvalues approach each other, these formulas start to become ill-conditioned. When a multiple root appears, the eigenvectors lose their uniqueness. In this case, all the formulas must be modified, but this is beyond the scope of this paper.

5.5. LPFs and eigenvalue sensetivities

Property 5. *LPFs and corresponding Lyapunov energies can be represented using sensitivities of the corresponding eigenvalues with respect to the elements of the dynamics matrix* **A**.

$$E_{x_k}\, e_{ki} = -\left\{\left(\frac{\partial\lambda_i}{\partial a_{kk}}\right)^* \mathbf{e}_k^T(\lambda_i^*\mathbf{I}+\mathbf{A})^{-1}\mathbf{e}_k\right\}_H\ ,$$

$$E_{x_k}\,\tilde{e}_{k(ij)} = -\frac{1}{2}\left\{\frac{1}{\lambda_i^*+\lambda_j}\left(\frac{\partial\lambda_i}{\partial a_{kk}}\right)^*\frac{\partial\lambda_j}{\partial a_{kk}} + \frac{1}{\lambda_i+\lambda_j}\frac{\partial\lambda_i}{\partial a_{kk}}\frac{\partial\lambda_j}{\partial a_{kk}}\right\}_H\ .$$

These formulas are obtained by substitution of (4) into the formulas of Proposition 1. In practical applications, the corresponding sensitivity coefficients can be identified using various measurement methods.

Based on the above considerations, one can expect that the LPFs may provide additional advantages for the purposes of the small-signal stability analysis, while the PFs retain their importance in assessing the instantaneous dynamics of the system.

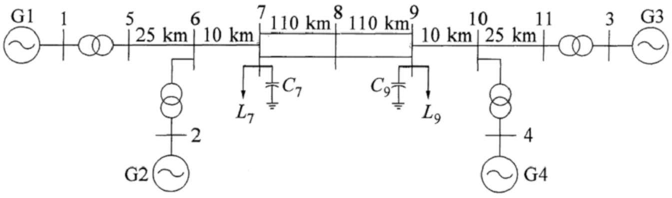

Figure 1: Two-area power system with four generators from (Kundur (1994)).

Mode	Initial eigenvalue	Type and location
S_1	−0.096	aperiodic rotor angle mode (mostly between G3 and G4)
S_2	−0.117	aperiodic rotor angle mode (mostly between G1 and G2)
S_3	$-0.111 \pm 3.43j$	inter-area rotor oscillation (between G1, G2, G3, G4)
S_4	−0.265	local inter-machine flux linkage mode (between G3 and G4)
S_5	−0.276	local inter-machine flux linkage mode (between G1 and G2)
S_6	$-0.492 \pm 6.82j$	local inter-machine oscillation (between G1 and G2)
S_7	$-0.506 \pm 7.02j$	local inter-machine oscillation (between G3 and G4)

Table 1: Modes and initial eigenvalues.

6. Simulation experiment

In this section, we present a fairly simple numerical experiment, which nevertheless is able to demonstrate the potential advantages of Lyapunov modal analysis.

6.1. Experiment description

For carrying out a simulation experiment we use the two-area power system with four generators that was considered by Kundur [Power System Stability and Control, 1994] and is shown in Figure 1). It consists of two similar areas connected by a weak tie. Each area consists of two coupled stations. The third station G3 was considered a swing bus. For dynamical analysis, all generators of the power system are represented by sixth order models. The speed controllers were ignored. Model parameters were chosen the same as in the textbook (Kundur (1994), Example 12.6, p.813). To avoid zero eigenvalues in the dynamic matrix all rotor angle and speed deviations were taken with respect to that of reference generator G3.

We studied the limit of the system stability by simultaneously increasing all of the loads (P_L, Q_L, Q_C) and the active power of each generator (P_i) while keeping the ratios between these fixed. We define the power increase coefficient as $\alpha = P/P_0 = Q/Q_0$. This change of α results in further changes in system modes. List of the main ones is presented in Table 1. All oscillations listed in the table are rotor angle electro-mechanical oscillations. The aperiodic modes S1 and S2 correspond to rotor angle. The aperiodic modes S4 and S5 correspond to flux

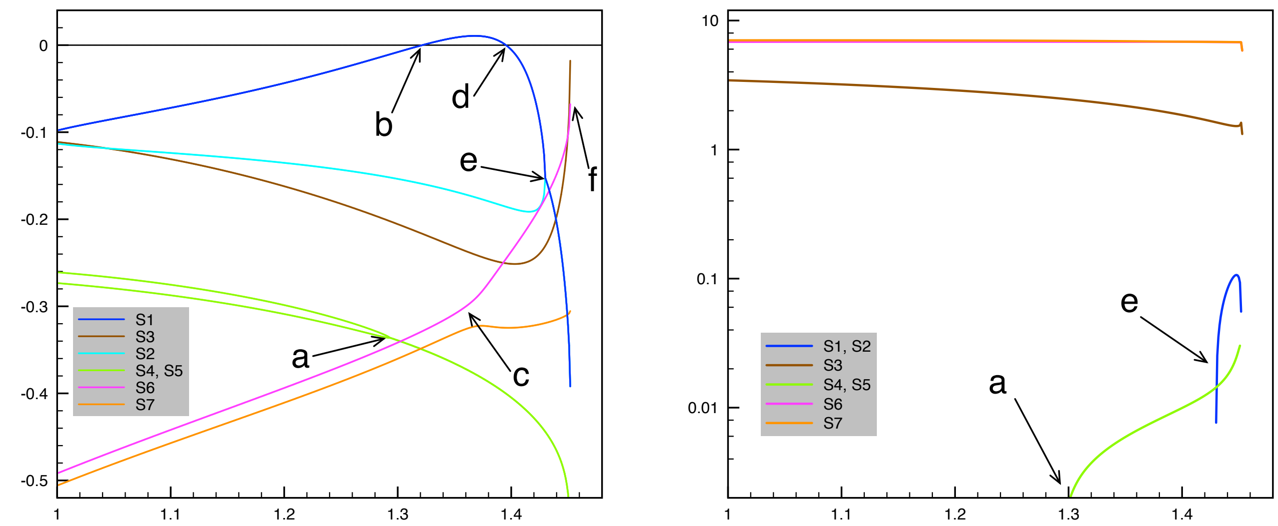

Figure 2: The trajectory of real and imaginary parts of eigenvalues in the experiment (on the right and on the left respectively).

linkage. Evolution of the real and imaginary part of eigenvalues during the test experiment is shown in Figure 2 as functions of the power increase coefficient α. Modes' names on the legend correspond to the Table 1. The simulation demonstrates the nontrivial dynamics of system modes. As the parameter α increases, the following changes are observed, marked in Figure 2.

(a) At $\alpha_{\mathrm{a}} \approx 1.293$, the aperiodic S4 and S5 modes merge into one low-frequency oscillation.

(b) When $\alpha_{\mathrm{b}} \approx 1.321$ the aperiodic inter-area angle mode S1 becomes unstable. As can be seen, however, the instability of the S1 mode does not influence the behavior of the other modes.

(c) When $\alpha_{\mathrm{c}} \approx 1.375$ there is a resonance between the S6 and S7 oscillations, after which they become inter-area oscillations.

(d) The S1 mode becomes stable again at $\alpha_{\mathrm{d}} \approx 1.395$.

(e) When $\alpha_{\mathrm{e}} \approx 1.430$ the aperiodic modes S1 and S2 merge into one low-frequency oscillation.

(f) The system becomes unstable again at $\alpha_{\mathrm{f}} \approx 1.453$. An obvious correlation can be seen in the behavior of the dangerous modes S3, S6, and S7 in the pre-fault operation.

6.2. Simulation results and discussion

The evolution of Lyapunov energies of the states and modes is shown in Figure 3 as a function of the weighting parameter

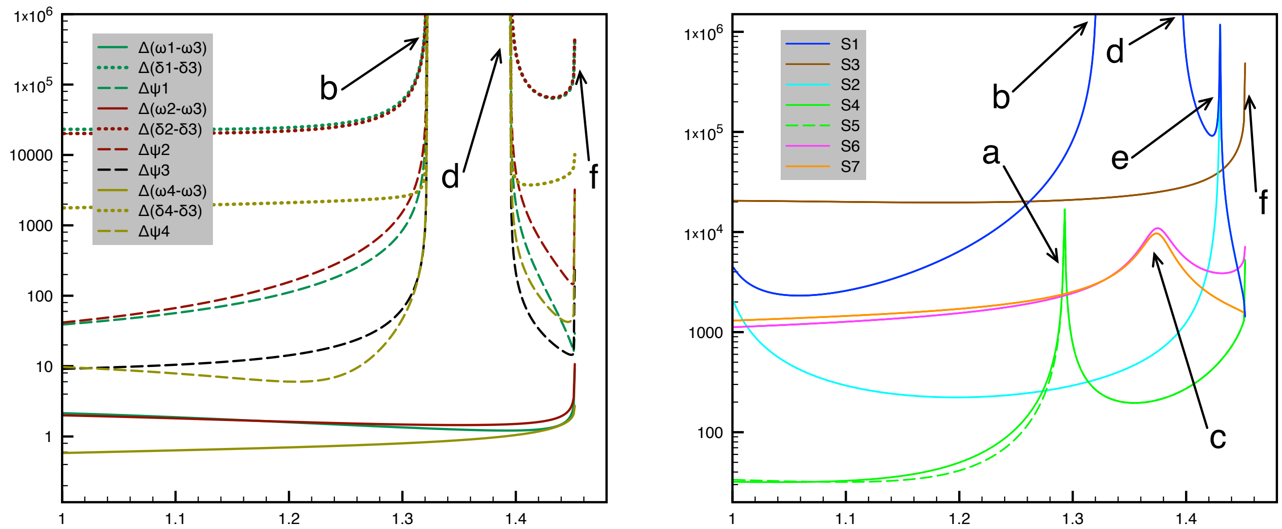

Figure 3: The evolution of Lyapunov energies of the states and modes in the experiment (on the left and on the right, respectively).

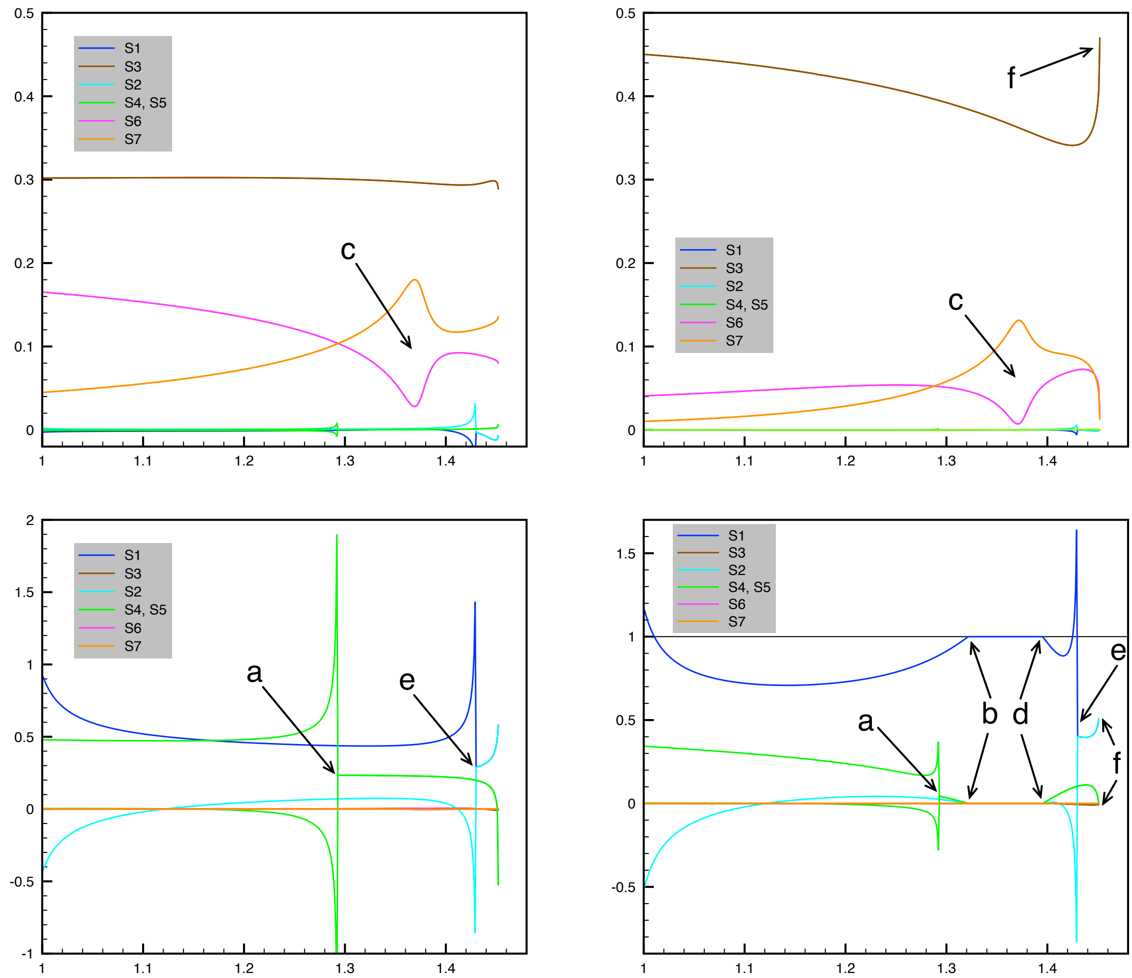

Figure 4: The behavior of the conventional PFs p_{ki} (on the left) and Lyapunov PFs e_{ki} (on the right) for the state variables $\Delta(\omega_1 - \omega_3)$ and $\Delta\Psi_1$ above and below, respectively.

α. The graphs on the left show the values of E_{x_k} calculated by (24) for the main state variables, namely for the deviations of flux linkages and rotor angles and speeds of different generators. When the system loses stability at points (b), (d) and (f), the Lyapunov energies in the corresponding variables go to infinity. However, in some other variables they remain bounded. The graphs on the right show the values of $|\boldsymbol{u}_i|^2 E_{z_i}$ which are calculated using (27) and characterize the invariant measure of Lyapunov modal energies (see Remark 1) for the modes $S_1 - S_7$ listed in Table 1. Each curve characterizing an oscillation contains two identical graphs, either of which corresponds to one of the complex conjugate eigenvalues. The graphs reflect all qualitative changes in the spectrum of the system, including the loss of stability by the corresponding modes at points (b), (d), (f), the fusion of aperiodic modes at points (a), (e), and the resonance between the oscillations S_6 and S_7 at (c).

Figure 4 shows the behavior of the conventional MISPF p_{ki} defined by (3) and MISLPF e_{ki} defined by (33) and (35) depending on the α on the left and on the right, respectively. The plots above show the modal PFs in $\Delta(\omega_1 - \omega_3)$, i.e. in the deviation of rotor angle speed of generator G1 with respect to that of generator G3. The plots below show the modal PFs in $\Delta\Psi_1$, i.e. in the deviation of flux-linkage of generator G1. The general composition of modes in the considered state variables according to both p_{ki} and e_{ki} are similar. Both coefficients identify the process of merging of aperiodic modes at points (a) and (e). However, unlike conventional PFs, Lyapunov PFs clearly identify the moments of stability loss occurring due to the corresponding mode and state variable. On the plots above, two graphs of e_{ki} characterizing the oscillation S_3, which loses stability at (f), in the sum tend to unity at α_f. On the plots below, the graph characterizing the aperiodic mode S_1, which loses stability in the interval between points (b) and (d), tends to unity on the same interval. Thus, in accordance with Property 2, the MISLPFs identify the stability loss occurring due to specific modes and state variables.

Figure 5 shows the behavior of LMIFs calculated by (55) depending on the weighting parameter α. The participations of all modes in the modes S_1, S_3, S_6 and S_7 are shown on separate tabs. In the case of unstable modes, in accordance with the

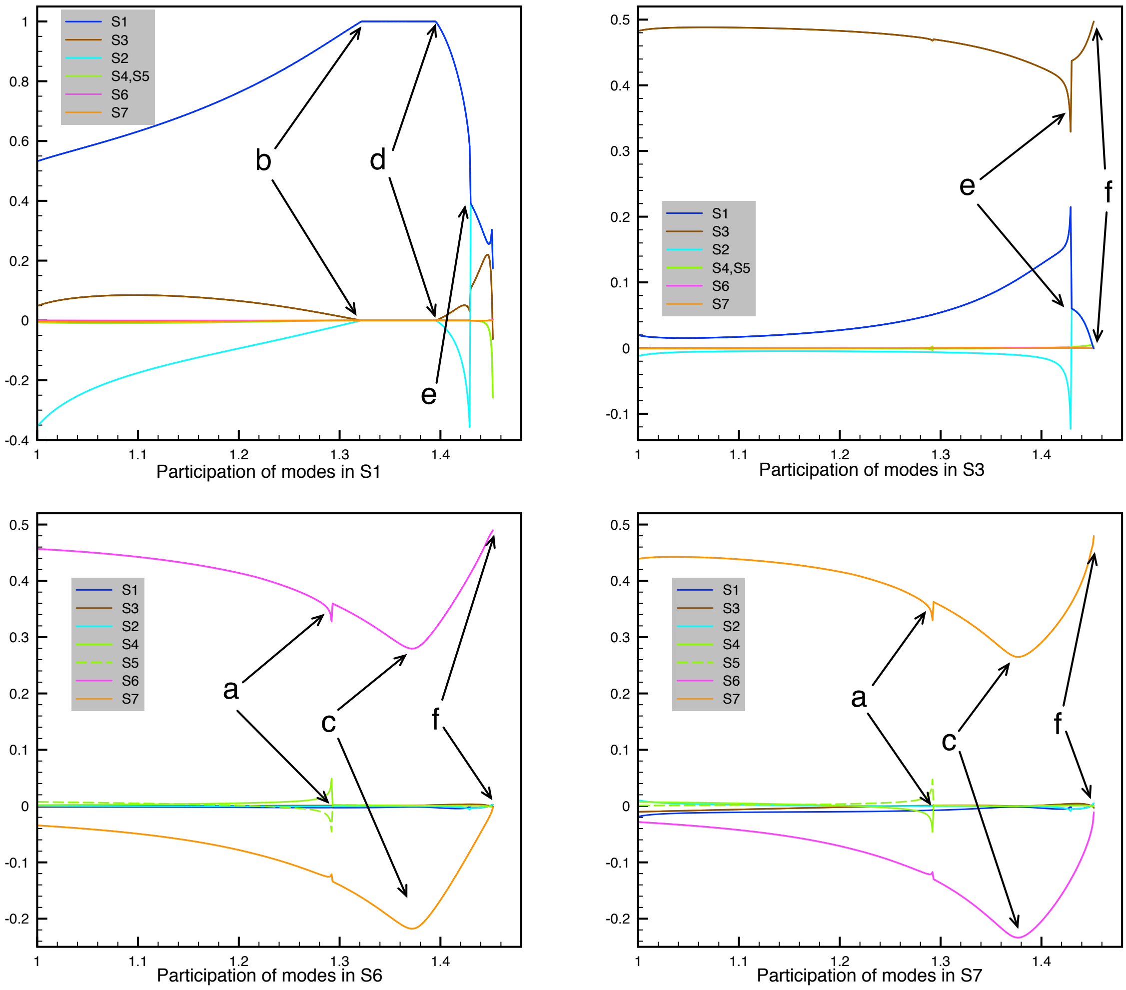

Figure 5: The Lyapunov modal interaction factors normalized to 1.

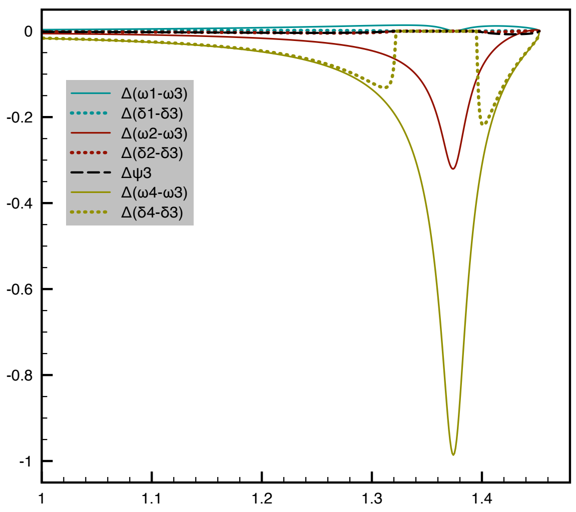

Figure 6: The behavior of pair MISLPFs $\tilde{e}_{k(ij)}$ for the interaction of the oscillations S_6 and S_7 and different state variables.

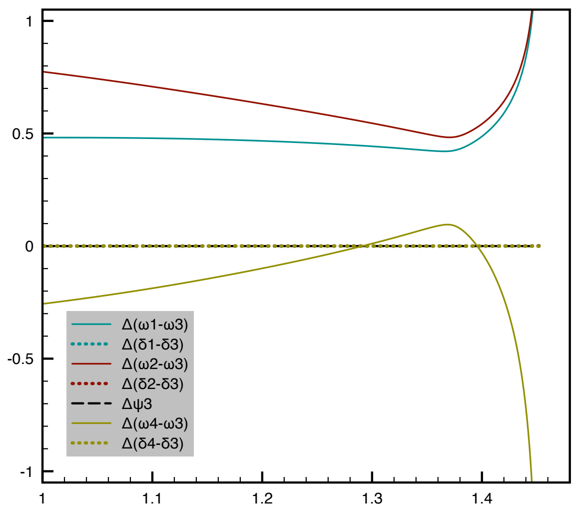

Figure 7: The behavior of state participations in LMIEs $\overline{e}_{k(ij)}$ for the interaction of the oscillations S_6 and S_7 and different state variables.

physical meaning, we assume that

$$LMIF_{ij} = +\infty\,, \quad \text{if } \operatorname{Re}\{\lambda_i^* + \lambda_j\} > 0\,.$$

Each curve showing the interaction with the oscillation contains two identical graphs, either of which corresponds to one of the complex conjugate eigenvalues. In accordance with the chosen normalization, the sum of the absolute values of the graphs on each tab is always equal to one. LMIF plots allow one to see a general structure of the modal interaction, as well as to identify the following characteristic features of the modal dynamics.

- *Loss of stability of an aperiodic or oscillatory mode.* When the S_1 mode becomes unstable at point (b), its own participation approaches 1, and the participations of the other modes in it disappear. Similarly, when the oscillatory modes S_3, S_6 and S_7 approach the stability boundary at (f), their own participations also tend to unity.
- *Merging of two aperiodic modes into one oscillation.* When aperiodic modes S_4 and S_5 merge into a single oscillation at (a), there is a noticeable increase in the participations of merging modes in other modes before the merging and a sharp increase in the participations of other modes after it. A similar phenomenon is observed when aperiodic modes S_1 and S_2 merge into a single oscillation at (e).
- *The occurrence of a resonance between two oscillations.* Oscillations S_6 and S_7 interact mainly with each other (see the tabs on the bottom of Figure 5). As their frequencies approach each other at (c) the graphs show a characteristic increase in the mutual participation of these modes in each other with the opposite sign.

Figures 6 and 7 show the behavior of pair MISLPFs $\tilde{e}_{k(ij)}$ defined in (59) and state participations in LMIEs $\overline{e}_{k(ij)}$ defined in (63), respectively, for the interaction of the oscillations S_6 and S_7 and different state variables. In Figure 6, one can indicate the state variables $\Delta(\omega_2 - \omega_3)$ and $\Delta(\omega_4 - \omega_3)$, which are sensitive to this interaction, when it becomes resonant in the neighborhood of $\alpha_c \approx 1.375$. We note that the presence in the system of the unstable mode S_1 creates large or infinite in magnitude additional terms of the Lyapunov energy in some state variables. This can make some state variables insensitive to the interaction of oscillations S_6 and S_7. In Figure 7 one can indicate the state variables $\Delta(\omega_1 - \omega_3)$, $\Delta(\omega_2 - \omega_3)$ and $\Delta(\omega_4 - \omega_3)$, which provide the main contribution to this interaction. Note that the magnitudes of $\overline{e}_{k(ij)}$ practically does not depend on the unstable mode S_1, since the Lyapunov energy of oscillations S_6 and S_7 does not depend on it.

7. Conclusion

This paper proposes a novel framework of LMA based on the concepts of the Lyapunov energies and Lyapunov PFs, which characterize the time integrated energy associated either with particular modes and state variables, or with their pairwise combinations. It was proved that, in contrast to conventional PFs, the proposed indicators have characteristic properties that allow one to identify

- the loss of stability of a particular mode,
- the resonant interactions between two modes,
- merging of two aperiodic modes into low-frequency oscillation,

and associate these phenomena with certain system state variables. The calculation of the proposed Lyapunov indicators for the critical part of the spectrum does not require knowledge of the entire spectrum of the system matrix and can be performed independently. Therefore, the LMA can be performed quickly to analyze resonant interactions of the critical modes in large-scale dynamical systems. The proposed indicators can be also calculated from the sensitivities of eigenvalues obtained directly from measurements.

Although in this paper the LMA was applied for the small-signal stability analysis of the test power system, its performance can be also tested for solving other problems of modal analysis, such as transient stability analysis, optimal placement of sensors and stabilizers, cluster analysis of electrical networks, which is the subject of our further research.

References

Antoulas, A.C. (2005). *Approximation of Large-Scale Dynamical Systems*. SIAM, Philadelphia, PA, USA.

Baur, U., Benner, P., Feng, L. (2014). Model order reduction for linear and nonlinear systems: a system-theoretic perspective. *Archives of Computational Methods in Engineering*, 21(4), 331–358.

Benner, P., and Damm, T. (2011). Lyapunov Equations, Energy Functionals, and Model Order Reduction of Bilinear and Stochastic Systems. *SIAM J. Control Optim.*, 49(2), 686–711.

Chow, J.H. (2013). *Power System Coherency and Model Reduction*. Springer, New York.

Fernando, K.V., Nicholson, H. (1984). On a fundamental property of the cross-Gramian matrix. *IEEE Trans. Circuits Syst.*, CAS-31(5), 504–505.

Garofalo, F., Iannelli, L., and Vasca, F. (2002). Participation Factors and their Connections to Residues and Relative Gain Array. *IFAC Proceedings Volumes*, 35(1), 125–130.

Genc, I., Schattler, H., and Zaborszky, J. (2005). Clustering the bulk power system with applications towards Hopf bifurcation related oscillatory instability. *Electric Power Components and Systems*, 33(2), 181–198.

Gray, W.S., and Mesko, J. (1998). Energy Functions and Algebraic Gramians for Bilinear Systems. Preprints of *4th IFAC Nonlinear Control Systems Design Symposium*. Enschede. The Netherlands. 103–108.

Hamdan, A.M.A. (1986). Coupling measures between modes and state variables in power-system dynamics. *Int. J. Control*, 43(3), 1029–1041.

Hamdan, A.M.A., and Nayfeh, A.H. (1989). Measures of Modal Controllability and Observability for First and Second order Linear Systems. *AIAA Journal: Guidance, Control, and Dynamics*, 12(3), 421–428.

Hamzi, B., Abed, E.H. (2014). Local Mode-in-State Participation Factors for Nonlinear Systems. In: *53rd IEEE Conference on Decision and Control*, Los Angeles, California, USA.

Hashlamoun, W.A., Hassouneh, M.A., and Abed, E.H. (2009). New results on modal participation factors: Revealing a previously unknown dichotomy. *IEEE Trans. Autom. Control*, 54(7), 1439–1449.

Iskakov, A.B. (2019). *Definition of State-In-Mode Participation Factors for Modal Analysis of Linear Systems*. *IEEE Trans. Autom. Control*, submitted.

Konoval, V., Prytula, R. (2017). Power system participation factors for real and complex eigenvalues cases. *Poznan University of Technology Academic Journals: Electrical Engineering*, 90, 369–381.

Kundur P. (1994). *Power Systems Stability and Control*. McGraw-Hill, New York, USA.

MacFarlane, A.G.J. (1969). Use of power and energy concepts in the analysis of multivariable feedback controllers. *Proc. IEE*, 116(8), 1449–1452.

Manousakis, N.M., Korres, G.N., and Georgilakis P.S. (2011). Optimal placement of phasor measurement units: A literature review. In: *16th International Conference on Intelligent System Applications to Power Systems*, DOI: 10.1109/ISAP.2011.6082183.

Moore, B.C. (1981). Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Trans. Automat. Control*, AC-26, 17–32.

Netto, M., Susuki, Y., Mili, L. (2019) Data-Driven Participation Factors for Nonlinear Systems Based on Koopman Mode Decomposition. *IEEE Control Systems Letters*, DOI: 10.1109/LCSYS.2018.2871887.

Pagola, F.L., Pérez-Arriaga, I.J., and Verghese, G.C. (1989). On sensitivities, residues and participations: applications to oscillatory stability analysis and control. *IEEE Transactions on Power Systems*, 4(1), 278–285.

Pal, A., Thorp, J.S. (2012). Co-ordinated control of inter-area oscillations using SMA and LMI. In: *Innovative Smart Grid Technologies (ISGT), 2012 IEEE PES*. DOI: 10.1109/ISGT.2012.6175535.

Pérez-Arriaga, I.J., Verghese, and Schweppe, F.C. (1982). Selective modal analysis with applications to electric power systems, Part I: Heuristic introduction. *IEEE Trans. Power Apparatus Syst.*, 101 (9), 3117– 3125.

Sanchez-Gasca, J.J., Vittal, V., Gibbard M.J., Messina A.R., Vowles D.J., Liu S., and Annakkage, U.D. (2005). Inclusion of higher order terms for small-signal (modal) analysis: committee report-task force on assessing the need to include higher order terms for small-signal (modal) analysis. *IEEE Trans. on Power Systems*, 20(4), 1886–1904.

Shokoohi, S., Silverman, L.M., and Van Dooren, P. (1983). Linear Time-Variable Systems: Balancing and Model Reduction. *IEEE Trans. Automat. Control*, AC-28(8), 810–822.

Song, Y., Hill, D.J., and Liu, T. (2019). State-in-mode analysis of the power flow Jacobian for static voltage stability. *Int. J. Elec. Power Energy Syst.*, 105, 671–678.

Tawalbeh, N.I., and Hamdan, A.M. (2010). Participation Factors and Modal Mobility. *Engineering Sciences*, 37(2), 226–231.

Tian, T., Kestelyn, X., Thomas, O., Amano, H., and Messina, A.R. (2018). An Accurate Third-Order Normal Form Approximation for Power System Nonlinear Analysis. *IEEE Trans. on Power Systems*, 33(2), 2128–2139.

Vassilyev S.N., Yadykin I.B., Iskakov A.B., Kataev D.E., Grobovoy A.A., Kiryanova N.G. (2017). Participation factors and sub-Gramians in the selective modal analysis of electric power systems. *IFAC-PapersOnLine*, 50 (1), 14806–14811.

Verghese, G.C., Pérez-Arriaga, I.J., and Schweppe, F.C. (1982). Selective modal analysis with applications to electric power systems, Part II: The dynamic stability problem. *IEEE Trans. Power Apparatus Syst.*, 101(9), 3126–3134.

Verriest, E., and Kailath, T. (1983). On Generalized Balanced Realizations. *IEEE Trans. Automat. Control*, AC-28(8), 833–844.

Vittal, V., Bhatia, N., and Fouad, A.A. (1991). Analysis of the inter-area mode phenomenon in power systems following large disturbances. *IEEE Trans. on Power Systems*, 6(4), 1515–1521.

Weber, H., Al Ali, S. (2016). Influence of huge renewable Power Production on Inter Area Oscillations in the European ENTSO-E-System. *IFACPapersOnLine*, 49(27), 012–017.

Williams, M.O., Kevrekidis, I.G., and Rowley, C.W. (2015). A Data?Driven Approximation of the Koopman Operator: Extending Dynamic Mode Decomposition. *Journal of Nonlinear Science*, 25(6), 1307–1346.

Yadykin, I.B (2010). On properties of gramians of continuous control systems. *Automation and Remote Control*, 71(6), 1011–1021.

Yadykin, I.B., Iskakov, A.B., Akhmetzyanov, A.V. (2014). Stability analysis of large-scale dynamical systems by sub-Gramian approach. *Int. J. Robust. Nonlin. Control*, 24, 1361–1379.

Yadykin, I.B., Kataev, D.E., Iskakov, A.B., Shipilov, V.K. (2016). Characterization of power systems near their stability boundary using the sub-Gramian method. *Control Eng. Practice*, 53, 173–183.

Yadykin, I.B., Iskakov, A.B. (2017). Spectral Decompositions for the Solutions of Sylvester, Lyapunov, and Krein Equations. *Doklady Mathematics*, 95(1),103–107.