Bilinearization

Original equation:

$$\dot{x} = f(x) + b(x)u$$

Bilinearized equation:

$$\dot{x} = A_0 + Ax + \sum_{i=1}^{m} \left(N_i x + B_0^i\right) u_i$$

Where:

n – Number of states

m – Number of controls

$f(x)$ – Column of n functions

$b(x)$ – Function matrix of size $n \times m$

$A_0 = f(0)$ Size: $n \times 1$

$$A = \begin{bmatrix} A_1 & A_2 & A_3 \\ A_{20} & A_{21} & A_{22} \\ 0 & A_{30} & A_{31} \end{bmatrix}$$ Size: $(n + n^2 + n^3) \times (n + n^2 + n^3)$

$$A_1 = \frac{1}{1!}\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$ Size: $n \times n$

$$A_2 = \frac{1}{2!}\begin{bmatrix} \frac{\partial^2 f_1}{\partial x_1 \partial x_1} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_n \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_n}{\partial x_1 \partial x_1} & \frac{\partial^2 f_n}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_n}{\partial x_n \partial x_n} \end{bmatrix}$$ Size: $n \times n^2$

$$A_3 = \frac{1}{3!}\begin{bmatrix} \frac{\partial^3 f_1}{\partial x_1 \partial x_1 \partial x_1} & \frac{\partial^3 f_1}{\partial x_1 \partial x_1 \partial x_2} & \cdots & \frac{\partial^3 f_1}{\partial x_n \partial x_n \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^3 f_n}{\partial x_1 \partial x_1 \partial x_1} & \frac{\partial^3 f_n}{\partial x_1 \partial x_1 \partial x_2} & \cdots & \frac{\partial^3 f_n}{\partial x_n \partial x_n \partial x_n} \end{bmatrix}$$ Size: $n \times n^3$

$A_{20} = A_0 \otimes I + I \otimes A_0$ Size: $n^2 \times n$

$A_{21} = A_1 \otimes I + I \otimes A_1$ Size: $n^2 \times n^2$

$A_{22} = A_2 \otimes I + I \otimes A_2$ Size: $n^2 \times n^3$

$A_{30} = A_0 \otimes I \otimes I + I \otimes A_0 \otimes I + I \otimes I \otimes A_0$ Size: $n^3 \times n^2$

$A_{31} = A_1 \otimes I \otimes I + I \otimes A_1 \otimes I + I \otimes I \otimes A_1$ Size: $n^3 \times n^3$

I – Identity matrix Size: $n \times n$

$\otimes$ – Kronecker product

For each control signal:

$$N_i = \begin{bmatrix} B_1^i & B_2^i & B_3^i \\ B_{20}^i & B_{21}^i & B_{22}^i \\ 0 & B_{30}^i & B_{31}^i \end{bmatrix}$$ Size: $(n + n^2 + n^3) \times (n + n^2 + n^3)$

$B_0^i = b_i(0)$ Size: $n \times 1$