UMEÅ UNIVERSITY
Department of Computing Science
Department of Physics

Examinator: Kenneth Bodin
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Exam

Course:
Visual Interactive Simulation D, 5p
TDBD22

Date: Friday Aug 26 2005 9.00 - 15.00

Social security number (persnr):

Name:
(print)

Signature

#	Done	Points	Max
1			8
2			6
3			10
4			9
5			7
6			8
Σ			48

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Exam in Visual Interactive Simulation D (TDBD22) (5p, 7.5 ECTS credits)

Sal 8 Östra paviljongerna. Bokningsnummer: 81663

Time 9.00 - 15.00 Calculator and math formulas allowed.
Always start a new assignment on a new sheet of paper and put your name on all sheets handed in! Maximum 48 credit points.

1. **Knowledge check** (8p)
   - **a.** What is the Poisson ratio of an incompressible material? If a material has a negative Poisson ratio, how will it behave when stretched? (1p)
   - **b.** What are holonomic and non-holonomic constraints and how do they affect the effective number of degrees of freedom in a rigid body system? (1p)
   - **c.** What type of problem is a Linear Complementarity Problem (LCP)? (1p)
   - **d.** Write down Newton's second law (i.e. $ma = F$) for a mass-spring system with viscous damping. What is the parameter criterion that gives the quickest damping of the amplitude and what is this case called? (1p)
   - **e.** Given $\boldsymbol{u} = (u_1, u_2, u_3)$, what is the matrix $skew(\boldsymbol{u})$ that has the property $skew(\boldsymbol{u})\boldsymbol{r} = \boldsymbol{u} \times \boldsymbol{r}$? (1p)
   - **f.** Explain in words the principles of the separating axes method. Does it apply for non-convex polyhedra? (1p)
   - **g.** What advantages are there in using quaternions in rigid body dynamics? (1p)
   - **h.** What is measured by the *Colour field*, and how can it be used to track a surface in an SPH simulation? (1p)

2. **Numerical integration** (6p)
   - **a.** Give (or derive) the expressions for updating velocity and position, respectively, in the Leap Frog method, and in the first order explicit Euler method. Why is the Leap-Frog algorithm typically much better than Euler's method, and what do we mean by "better" in a physics simulator? (2p)
   - **b.** What is the point of regularly applying Gram-Schmidt orthonormalization to a rotation matrix or a quaternion in a simulation? (1p)
   - **c.** What is the so called "gyroscopic term" and why is it hard to deal with in numerical integration? When is it important to take into account and when can it be approximated away? (2p)
   - **d.** What do we mean by a *stiff system* or *stiff differential equation*? Give an example.(1p)

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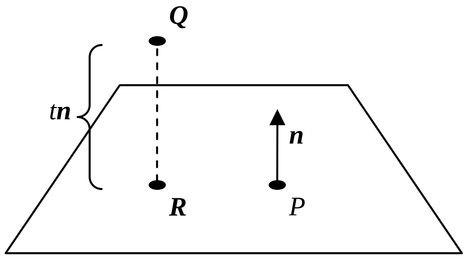
3. **Particle-plane intersection and collision response** (10p)
We can define an oriented plane by a point $\boldsymbol{P}$ and a normal $\boldsymbol{n}$, so that all points $\boldsymbol{X}$ on the plane satisfy the condition $\boldsymbol{n}\cdot(\boldsymbol{X}-\boldsymbol{P}) = 0$. Alternatively, the plane can be defined by a normal $\boldsymbol{n}$ and the distance to the origin d, so that $\boldsymbol{n}\cdot\boldsymbol{X} = d$, where $d = \boldsymbol{n}\cdot\boldsymbol{P}$.



a. Find the point $\boldsymbol{R}$ on the plane that is closest to an arbitrary point $\boldsymbol{Q}$, describing a particle. Also find the (signed) distance between the particle and the plane.
Hint: $\boldsymbol{R}$ is given by the orthonormal projection of $\boldsymbol{Q}$ onto the plane, obtained by moving $\boldsymbol{Q}$ perpendicularly toward the plane, i.e. $\boldsymbol{R} = \boldsymbol{Q} - t\boldsymbol{n}$ for some value of t. (2p)

b. Assume that $\boldsymbol{Q}$ is a particle with velocity $\boldsymbol{v}$ and that the plane is fixed in space. What are the two criteria for a colliding contact? (2p)

c. In case of a colliding contact, use Newton's impulse law to compute the post-collision velocity $\boldsymbol{v}'$ in vector notation expressed in terms of the collision normal, the pre-collision velocity $\boldsymbol{v}$ and the coefficient of restitution ε. (2p)

d. Euler's method with timestep δt is used to time-integrate the particle given it's velocity $\boldsymbol{v}'$ and that it also is subject to gravity, here given by $\boldsymbol{g} = -g\,\boldsymbol{n}$. Derive (or discuss) a criterion for which the particle is still intersecting the plane even after the time integration. What kind of problems can this give and how can it be corrected? (2p)

e. Find the corresponding post-collision velocity (as in task c.) in the presence of friction using the Newton-Coulomb impulse law. (2p)

4. **Rigid body contact** (9p)
Here we consider contact between a vertex *a* of rigid body and a static surface. The rigid body's center of mass is given by the vector **x**(*t*) at time *t*. The current rotation of the body is given by the rotation matrix *R*(*t*). The center of mass velocity is denoted **v** and angular velocity about the center of mass by **ω**. You may recall the identity: $\dot{R}(t) = \omega^* R(t)$, with ω^* being the antisymmetric matrix associated to ω. The position of the vertex *a* relative to the center of mass is **p**. See *Figure A-C* for explanations of the notations.

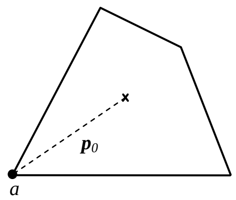


Figure A. Non-rotated body. In this frame the position of *a* relative to the center of mass is given by $\mathbf{p}_0$.

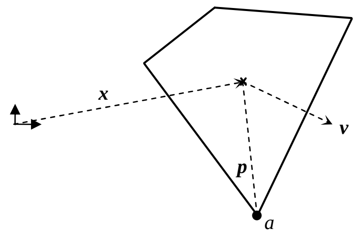


Figure B. Here the body is rotated and put in the world frame. The position of *a* relative to the center of mass is given by **p**.

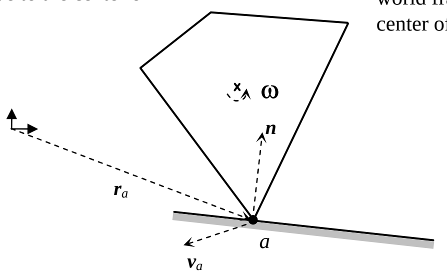


Figure C. The body collides with a plane with normal **n**. The position and velocity of the point *a* in the world frame is $\mathbf{r}_a$ and $\mathbf{v}_a$ respectively.

a. In the local body frame (prior to any rotation) the position of the vertex *a* relative to the center of mass is given by the vector $\mathbf{p}_0$, see *Figure A*. What is **p** when the body is rotated in the world frame (see *Figure B*)? (2p)

b. What is $\mathbf{r}_a$, i.e. the position of *a* in the world frame (see *Figure C*)? Express it in terms of **x**, *R* and $\mathbf{p}_0$. (2p)

c. The velocity of *a* in the world frame is $\mathbf{v}_a = \dot{\mathbf{r}}_a$. What is $\mathbf{v}_a$ in terms of **v**, **ω** and **p**. (2p)

d. There are three possible types of contacts here, namely:

1. $\mathbf{n} \cdot \mathbf{v}_a < 0$
2. $\mathbf{n} \cdot \mathbf{v}_a = 0$
3. $\mathbf{n} \cdot \mathbf{v}_a > 0$

What do these three possibilities mean? Describe what action should be taken by a *constraint based* physics engine in each of the three cases. (3p)

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5. SPH for a 1D-case (7p)

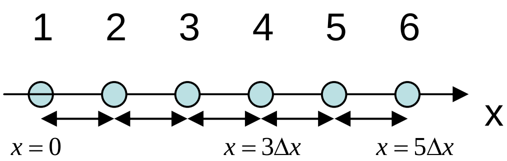
In SPH, the value of a function *f(x)* and its derivative, at particle *i*, can be approximated by

$$\langle f \rangle_i \approx \sum_j \frac{m_j}{\rho_j} f_j W(x_{ij}) \text{ and } \langle \nabla f \rangle_i \approx \sum_j \frac{m_j}{\rho_j} f_j W'(x_{ij})$$

where m_j, ρ_j, f_j is the mass, density and function value at particle *j*. The sum also includes particle *i*, and x_{ij} is the distance between particle i and j. In a 1D case we can use the following Kernel function:

$$W_{poly6}(x_{ij}) = N\left(h^2 - x_{ij}^{\;2}\right)^3, \quad W'_{poly6}(x_{ij}) = -6x_{ij} N\left(h^2 - x_{ij}^{\;2}\right)^2$$

where $N = 35/(32h^7)$, and *h* is the radius of the interaction region. Let us now consider the following uniform 1D particle distribution with particle spacing Δx:



Consider the case where all particles initially are at rest, $v_i(0) = 0, \; i = 1,2,...6$. In order to simplify the calculations we set $m = \Delta x = \Delta t = 1$. The interaction radius is chosen to be $h = 1.5 \cdot \Delta x = 1.5$. The SPH-averages then become

$$\langle f \rangle_i = \sum_j \frac{f_j}{\rho_j} W(x_{ij}) \quad \text{and} \quad \langle \nabla f \rangle_i = \sum_j \frac{f_j}{\rho_j} W'(x_{ij}).$$

a) Calculate the SPH value of the density at particle 1, 2 and 3, i.e. calculate $\langle \rho \rangle_1$, $\langle \rho \rangle_2$ and $\langle \rho \rangle_3$. (3p)

b) Calculate the velocity at $t = \Delta t$ for particle 1 and 2, using the update formula

$$v_i(\Delta t) = v_i(0) + \Delta t \cdot \frac{F_i}{m} \qquad \text{where} \qquad \frac{F_i}{m} = \left\langle \frac{dv}{dt} \right\rangle_i = -\frac{1}{\langle \rho \rangle_i} \langle \nabla \rho \rangle_i \qquad (3p)$$

c) Explain with words what happens and why. (1p)

6. **Cloth and constraints** (8p)
Cloth is typically very stiff with regard to stretch and elastic when it comes to bending. A popular model for cloth is mass-spring systems. An alternative to this is to replace the springs by distance constraints.
   a. What can be the advantage of using constraints instead of springs? (1p)

We consider a rectangular system of $N \times N$ particles and $M=2N(N-1)$ constraints as illustrated in *Figure A*. Each dashed line corresponds to a distance constraint involving the particles connected by the line (labelled by *m*). The constraint for the *m*:th line is

$$0 = c_m \equiv |\mathbf{x}_a - \mathbf{x}_b| - L$$

where $\mathbf{x}_a$ and $\mathbf{x}_b$ are the positions of particle *a* and *b* and *L* is the constant distance the particles should be fixed at.

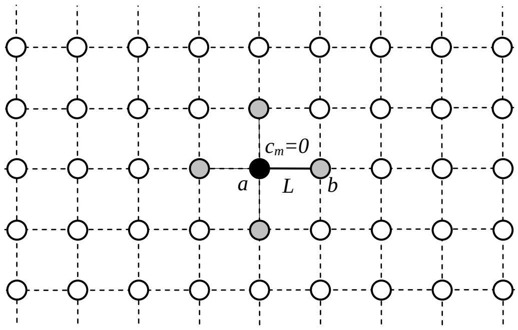


Figure A. Cloth modelled by particles and distance constraints. The nearest neighbours of particle *a* is shown in grey. The constraint $c_m=0$ on particle *a* and *b* is indivated with a solid line.

We collect the constraints in a vector $C = (c_1, c_2, ..., c_M)^T$ that has dimension *M*. We collect the particles position, velocity and force in vectors $X = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)^T$ (that has dimension *3N*) and similarly for *V* and *F*. From the lectures you recall that the constraints give rise to a force acting to maintain the constraint. This force is $F_{constr} = J^T \lambda$, where λ is the Lagrange multiplier (yet to be determined) and the *Jacobian* matrix is given by

$$J_{mn} = \frac{\partial C_m}{\partial X_n}$$

   b. What is the dimension of the Jacobian matrix and how many elements are nonzero? (2p)

From the lectures you also recall that in the *acceleration based method* for constraints the equations of motion for the particles are

$$\begin{aligned} \dot{X} &= V \\ \mathrm{M}\dot{V} &= F_{ext} + J^T \lambda \\ A\lambda &= b \end{aligned}$$

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where M is a $N \times N$ matrix with the particle mass on the diagonal (otherwise zero), b is a (known) vector and $A = JM^{-1}J^T$.

c. Of what dimension is the matrix A? (1p)

The time-stepping involves solving the matrix equation $A\lambda = b$. If A has dimension $n \times n$ a general method for solving the equation requires $O(n^3)$ operations. If on the other hand A is sparse and banded (band diagonal matrix), with bandwith m, there are solution methods that requires only $O(m \times n)$ operations.

d. Argue that A is a banded matrix (for a suitable ordering of the particles and constraints)! What is the bandwith of A with your ordering of particles? What is then the complexity for solving the matrix equation? Give the answer in terms of N (the number of particles in the cloth). Compare this to $O(n^3)$! (4p)