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Exam in Visual Interactive Simulation D (5DV058) (7.5 ECTS)

Sal 1 Östra paviljongerna. Bokningsnummer: 99020

Time 9.00 - 15.00. Calculator and math formulas allowed.

Always start a new assignment on a new sheet of paper. If solving a problem involving math and derivations, make sure you write down all important steps and not just the final result!

Maximum 48 credit points. Lycka till och ha en skön sommar! /Kenneth

1. **Knowledge check** (8p)
   - **a.** Briefly describe what the Rubin and Ungar theorem states when dealing with strong forces. (1p)
   - **b.** What are holonomic and non-holonomic constraints, respectively, and how do they affect the effective number of degrees of freedom in a rigid body system? (1p)
   - **c.** Write down Newton's second law (i.e. F=ma) for a 1D mass-spring system with damping. What form has the analytic solution in the case of weak damping? Assume that the system at time $t = 0$ is released from a state where the spring is stretched, and sketch the time evolution of the amplitude x(t) (1p)
   - **d.** Write down the relation between pressure and density (i.e. the equation of state) that was used in the SPH lab project. If the sound speed goes to infinity, what would be the resulting physical behavior of a real fluid, and what would be the resulting behavior in a simulation? (1p)
   - **e.** What is the so called gyroscopic term for a rigid body (write it down), and why is it often neglected in numerical integration? For what types of bodies is the gyroscopic term zero? (2p)
   - **f.** Start from the Leap-Frog integrator with timestep h,

     $$v_{t+h/2} = v_{t-h/2} + F_t\, h/m$$
     $$x_{t+h} = x_t + h\, v_{t+h/2}$$

     and rewrite it into a form without velocities so that the new position x_{t+h} can be expressed in terms of x_t , x_{t-h} and the force F_t, i.e. show that Leap-Frog and position Verlet have identical trajectories! (2p)

2. **Newton-Coulomb Impulse for a Particle** (8p)
   A particle of mass ***m*** is travelling at velocity ***v*** and collides with an infinite plane and bounces off with normal restitution e and a tangential friction coefficient μ.
   - **a.** Derive the post-collision velocity ***v'*** of the particle, as well as the entire impulse of the impact. Draw a picture that illustrates the situation and defines you notation. (2p)
   - **b.** What is the relative change in kinetic energy immediately before and after the impulsive change in velocity? (2p)
   - **c.** Assume the special case when the plane is horizontal and the particle falls vertically affected by gravity. Given Leap-Frog integration and a certain timestep Δt, the bounce can overcome gravity, so that the particle is travelling outwards at the end of the timestep, and ends up outside the plane. However, below this threshold the particle will start to "creep" into the plane. Derive this threshold.(2p)
   - **d.** Show that a velocity half step, just before the impact, balances the impulse integrator so that the threshold in b) becomes smaller. (2p)

3. Non-penetration constraint (8p)

Algebraic laws in mechanics can conveniently be described in terms of constraints, characterized by a Jacobian, and constraint forces. An inequality constraint, such as for non-penetration can be written as,

$$C(q,\dot{q};t) \geq 0,$$

where, $q,\dot{q}$ refer to any generalized coordinate and its velocity. The corresponding kinematic constraint is the time derivative of $C(q,\dot{q};t)$,

$$\dot{C}(q,\dot{q};t) \geq 0.$$

The Jacobian *G* maps the generalized velocities *V* onto the kinematic constraint,

$$GV = \dot{C}(q,\dot{q};t).$$

- **a.** For two rigid bodies i and j, with mass center coordinates $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$, with contact points $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ relative to the centers of mass, and contact normal $\boldsymbol{n}$ (relative to *i*), write down the interpenetration constraint *C*. Also derive the corresponding kinematic constraint $\dot{C}$. (2p)
- **b.** Explain when and why the time dependency of the normal, that is present in the kinematic constraint, can be neglected, and approximate it away. Derive the Jacobian for the interpenetration constraint. (2p)
- **c.** Show that for the non-penetration constraint the Shur complement $GM^{-1}G^T$ also can be written on the form $\mathrm{n}^{\mathrm{T}}\mathrm{Kn}$, where *K* is the collision matrix. (2p)
- **d.** Explain the role of each of the five terms in the SPOOK stepping equation for computing the lagrange multiplier λ for the constraint. (2p)

$$\left(GM^{-1}G^T + \varepsilon\right)\lambda = -aGq - bGV - hGf$$

4. 2D SPH Fluid (8p)

In this exercise we discuss several properties of the sph-approach. Since the different parts (a-d) are independent we want you to read through all questions before you start solving them.

Assume that we have the situation described in figure 1 below where a sph-fluid is close to a solid boundary corner. All particles have the mass $m = 2.9\,kg$ and interaction radius $h = 0.1m$ (indicated by dashed circle in figure 1). The density of particle 2 and 3 have been found to be 1000 kg/m^3 each.

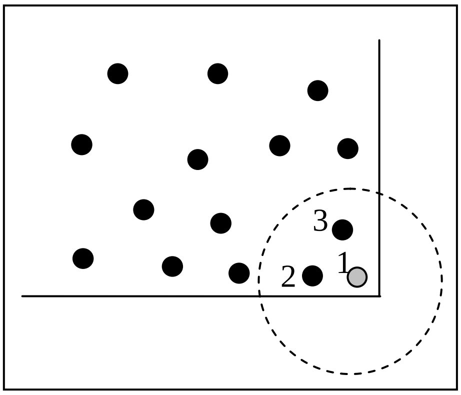

Figure 1.

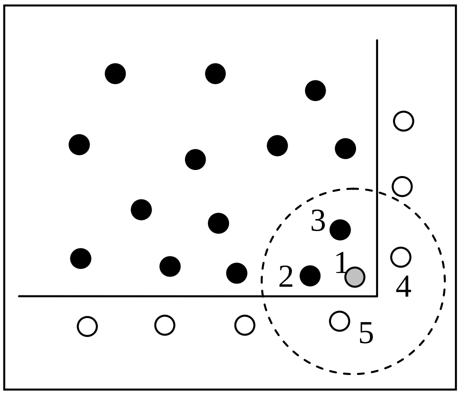

Figure 2.

a. The density difference between a particle inside the fluid and a particle near a boundary (particle 1) is in the SPH-community called "the particle deficiency problem" and it can cause severe difficulties in a simulation.
One simple solution to this problem is to glue particles into the boundary (see figure 2) and treat these as ordinary particles but without updating their positions. Such particles also prevent fluid particles to penetrate through the solid boundaries.
Calculate the density of particle 1 for the two situations described in figure 1 and figure 2.
Use the kernel $W(r_{ij}) = N\left(h^2 - r_{ij}^2\right)^3$ where $N = 4/\left(\pi h^8\right) \approx 1.2732 \cdot 10^9\ m^{-8}$ is a normalization constant and $r_{12} = r_{13} = r_{14} = r_{15} = 0.5h$ are the distances between particle 1 and the other.
By how many percent increases the density due to the glued particles? (3p)

b. Another way to handle the particle deficiency problem described above is to make use of a basic property of the kernel called *unit normalization condition* $\int_V W dV = 1$.
Calculate the density of particle 1, in the situation in figure1, using the unit condition corrected SPH. (2p)

c. Show that the unit condition $\int_V W dV = 1$ must be satisfied in order to reproduce a constant function $f(x) = a$ when we use the SPH method. Derive also the extra integral condition the kernel must satisfy in order to correctly reproduce the function $f(x) = a + bx$. (2p)

d. In some cases it may be useful to choose different kernel functions for different terms in a SPH-simulation. Two kernels that are commonly used in computer graphic community are the *poly6-kernel:*

$$W(r_{ij}) = N\left(h^2 - r^2\right)^3 \text{ with } N = 4/\left(\pi h^8\right) \text{ (the 2d-case)}$$

and the *spiky-kernel*

$$W_s(r_{ij}) = N_s\left(h - r\right)^3 \text{ with } N_s = 10/\left(\pi h^5\right) \text{ (the 2d-case).}$$

Describe a situation where it would be better to use ∇W_s instead of ∇W, explain also *why* it would be better. (1p)

5. Iterative solvers (8p)
Linear relaxational iterative solvers such as Gauss-Seidel and Jacobi solve linear systems of equations by solving for one unknown at a time, while using previous approximations for the other unknowns.
a. Solve the following simple system of equations approximately,

$$4x + 2y = 1$$
$$x + 3y = 4$$

using three iterations with Gauss-Seidel (make a table). Do the same thing with Jacobi iterations and compare the results. (2p)
b. The SPOOK stepper also results in a linear system of equations,

$$\left(GM^{-1}G^T + \varepsilon\right)\lambda = -aGq - bGV - hGf$$

If there are several Jacobian blocks and correspondingly also several λ, show how this equation can be solved using Gauss-Seidel. Also write down the complimentarity conditions if the G's all correspond to non-penetration constraints, as well as the clamping procedure for the solution vector. (2p)

c. Suggest at least two ways of measuring the error in **b**. and discuss what type of errors are important to reduce in a visual and interactive simulation (2p)

d. Gauss-Seidel typically has linear convergence, i.e. the error decreases linearly with the number of iterations. However, in practice convergence depends a lot on the direction of the iterations. Suggest and discuss a method for choosing the direction, and also explain the worst case scenario for example, when stacking N spheres on top of eachother. (2p)

6. **Penalty methods for e.g. wires and cloth (8p)**

Consider a filament made of *N* identical point particles with mass *m*. Each particle is attached to the nearest neighbor with a spring of stiffness *k*. This filament hangs vertically and the particle at the top is attached with a spring of stiffness *k* to a fixed object. Assume uniform gravitational acceleration of magnitude, *g*, pointing downward.

- **a.** Compute the elongation of a filament with *N* particles as a function of *N*, for the configuration where the filament is at equilibrium. (4p)

  Hint: reduce the problem to one dimension. Draw a picture displaying the system and all the relevant forces.

  Gauss' formula: $\sum_{n=1}^{M} n = M(M+1)/2$, will come in handy.

- **b.** Assume we're simulating a long wire with N = 100, m = 0.1, and the resting length of each spring being 0.5m. We want an elongation of max 5%. What should the value of the spring constant be? If you're using Leapfrog for the integration, estimate the maximum size of the timestep that will make this simulation stable. (4p)