UMEÅ UNIVERSITY
Department of Computing Science
Department of Physics

Examinator: Kenneth Bodin
Cell phone: 070-631 5520

Exam

Course:
Visual Interactive Simulation D, 5p
TDBD22

Date: Saturday June 4 2005 9.00 - 15.00

Social security number:

Name:
(print)

Signature

#	Done	Points	Max
1			8
2			7
3			7
4			6
5			6
6			7
7			7
Σ			48

UMEÅ UNIVERSITY
Department of Computing Science
Department of Physics

Examinator: Kenneth Bodin
Cell phone: 070-631 5520

Exam in Visual Interactive Simulation D (TDBD22) (5p, 7.5 ECTS credits)

Sal 2 Östra paviljongerna. Bokningsnummer: 81774

Time 9.00 - 15.00 Calculator and math formulas allowed.
Always start a new assignment on a new sheet of paper and put your name on all sheets handed in!
Maximum 48 credit points

1. **Knowledge check** (8p)
   - **a.** What do we mean by *plastic deformation*? (0.5p)
   - **b.** What is the difference between *brittle and ductile material*? (0.5p)
   - **c.** What does a *Poisson ratio of 0.5* imply? (0.5p)
   - **d.** Mention one possible effect of using the *linear approximation of Green-strain* in a simulation. (0.5p)
   - **e.** What are *Poisson's hypothesis* and *Stronge's hypothesis*? (0.5p)
   - **f.** What are *holonomic* and *non-holonomic* constraints? (0.5p)
   - **g.** Explain what a *tire slip* is, and why it occurs in vehicle dynamics. (0.5p)
   - **h.** In grid based methods, one uses the neighboring grid points to calculate derivatives, but there is no grid in SPH. So, how are derivatives calculated in SPH? (0.5p)
   - **i.** Draw a picture that shows the amplitude as a function of time for a damped spring, for the overdamped, underdamped and critically damped cases. What is the criterion for critical damping? (1p)
   - **j.** Explain the difference between explicit and implicit Euler for time integration, and why there is a difference in computational cost (1p)
   - **k.** For two interacting particles i and j, write down the force terms for pressure and viscosity in SPH for a viscous fluid and explain in words what they mean. (1p)
   - **l.** Write down the relation between pressure and density (i.e. the equation of state) that was used in the SPH lab project. How will a change in the value of the speed of sound affect the simulation? (1p)

2. **Intersection test/find** (7p)
   - **a.** Explain the separating axis method for intersection tests of convex polygons. Draw a picture of a 2D case! (2p)
   - **b.** For 3D obb-obb intersections, the naive approach results in 156 tests. What are these tests, i.e. explain where the number "156" comes from. What symmetries can be used to reduce the number of tests and what are these remaining 15 tests (explain in words)? (2p)
   - **c.** Describe the principles for generating a contact set for 3D obb-obb intersections, i.e. for finding contact points and contact normals. (2p)
   - **d.** A potential problem of the separating axis method is that a zero vector may incorrectly be interpreted as a separating axis. When can this occur? (1p)

3. **Collision impulses** (7p)
   You are to develop a particle simulation and you want to treat boundary conditions with an impulse law, reflecting the velocities in the normal.
   a. Derive the post-collision velocity $\boldsymbol{v}'$ in vector notation expressed in terms of the collision normal, the pre-collision velocity $\boldsymbol{v}$, and the coefficient of restitution ε. (2p)

   b. Assume that we also choose to reflect the position in order to avoid penetration, so that $\mathbf{x}' = \mathbf{x} + \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ can be chosen in different ways. We consider the two choices $\boldsymbol{\delta}_1 = (0, \Delta y)$ and $\boldsymbol{\delta}_2 = (\Delta x, 0)$. The total energy before collision is denoted $E(\mathbf{x},\mathbf{v})$ and after collision $E'(\mathbf{x}',\mathbf{v}')$. The total energy here includes kinetic and gravitational potential energy of the particle – gravity acts in the negative y-direction. Show that the use of δ_1 changes the total energy whereas the use of δ_2 does not! Estimate the size of the error in energy in a simulation with time step Δt and characteristic velocity $\mathbf{v}_{ch}$. (3p)
   c. Modify a) with a Newton-Coulomb impulse, i.e. also take dynamic friction into account. (2p)

4. **Proximity tests for particles** (6p)
   a. Describe the spatial subdivision algorithm based on a regular grid and linked lists, for finding neighbors in a particle simulation with a particle interaction range given by h. The entire system is confined inside a cube of side $L = 100\ h$ and it contains $N = 10000$ particles. (2p)
   b. Write down an algorithm that shows how the real grid can be replaced by an infinite virtual grid and a hash index. (2p)
   c. What are the advantages and disadvantages of method a) and b) in comparison? Estimate relative performance of a) and b) in terms of *memory consumption* and *number of floating point operations*. (2p)

5. Cloth (6p)
Consider a filament made of N identical point particles with mass m. Each particle is attached to the nearest neighbor with a spring of stiffness k. This filament hangs vertically and the particle at the top is attached with a spring of stiffness k to a fixed object. Assuming uniform gravitational acceleration of magnitude, g, pointing downward, compute the elongation of a filament with N particles as a function of N, for the configuration where the filament is at equilibrium.

Hint: reduce the problem to one dimension. Draw a picture displaying the system and all the relevant forces.
Gauss' formula: $\sum_{n=1}^{M} n = M(M+1)/2$, will come in handy.

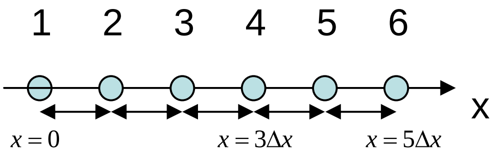
6. SPH and function approximation with kernels (7p)
In SPH, the value of a function $f(x)$, at particle i, can be approximated by

$$\langle f \rangle_i \approx \sum_j \frac{m_j}{\rho_j} f_j W(x_{ij})$$

where m_j, ρ_j, f_j is the mass, density and function value at particle j. The sum also includes i, and x_{ij} is the distance between particle i and j. In a 1D case we can use the following Kernel function:

$$W_{poly6}(x_{ij}) = N\left(h^2 - x_{ij}{}^2\right)^3$$

where N is a normalization constant, $N = 35/(32h^7)$, and h is the radius of the interaction region. Let us now consider the following uniform 1D particle distribution with particle spacing Δx:



In order to simplify the calculations we set $m_j / \rho_j = \Delta x = 1$. The SPH-average then becomes $\langle f \rangle_i = \sum_j f_j W(x_{ij})$. The interaction radius is chosen to be $h = 1.5 \cdot \Delta x = 1.5$.

- **a.** For the constant function $f(x) = 1$, calculate the SPH-value at particle 4 and 6 (calculate $\langle f \rangle_4, \langle f \rangle_6$). Plot the values together with $f(x) = 1$ in a diagram. (2 p)
- **b.** For the function $g(x) = x$, calculate the SPH-value at particle 4 and 6 (calculate $\langle g \rangle_4, \langle g \rangle_6$). Plot the values together with $g(x) = x$ in a diagram. (2 p)

c. Rescale $\langle f \rangle_6$ such that it becomes equal to one (in other words, change the normalizing constant N). Take this scaling and apply it on $\langle g \rangle_6$. Plot the new value together with $g(x) = x$ in a diagram. (2 p)

d. How can the above be used to improve your SPH lab project? (1p)

7. Constraints (7p)

Consider a pendulum in two dimensions which consists of a point particle of mass *m* at position *x(t)* and velocity *v(t)*. The constant pendulum length is unity and it is centered at the origin, *(0, 0)*. The force of gravity on the point mass is $F_g = -mg\hat{z}$ where $\hat{z}$ is a unit vector pointing upward, and $g>0$ is a constant.

a. Express the condition that the length of the pendulum is unity as an algebraic relation, $g(x) = 0$. (0.5p)

b. The Jacobian of a vector function $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))^T$ is the matrix (or vector if $m=1$):

$$G_{ij} = \frac{\partial g_i}{\partial x_j}.$$

In other words, $\dot{g}(x) = G\dot{x} = Gv$. Compute the matrix G for the constraint $g(x)$. (0.5p)

c. If the constraint is satisfied, we have $g(x(t)) = 0$ for any value $t > 0$. What is the numerical value of $\dot{g}(x(t))$ and $\ddot{g}(x(t))$ for any $t > 0$? (0.5p)

d. Compute the algebraic expression of $\ddot{g}(x(t))$ and show that it can be expressed as $\ddot{g}(x(t)) = G\ddot{x} + \alpha(t)$, i.e., give an analytic expression for the function $\alpha(t)$. (0.5p)

e. Compute the analytic solution of the unconstrained motion i.e. $m\dot{v} = F_g = mg\hat{y}$. Show that $g(x) \neq 0$ for this at $t>0$ if we start at the position $x = (1,0)^T$ and velocity $v = (0,0)^T$. This shows that the constraint must be producing a force on the particle. (0.5p)

f. Work is defined as the following integral over the trajectory,

$$W = \int_{t_0}^{t_1} F^T v dt.$$

Show that if the constraint force can be expressed as $F_c = G^T \lambda(t)$, where G is the Jacobian of some function $g(x)$, and λ is some real vector with the same dimension as $g(x)$, then $g(x) = 0$ is a sufficient condition for the work to be 0 along the trajectory, i.e. that the conditions g(x(t)) = 0 and $F_c = G^T \lambda(t)$ imply that $W_c = \int_{t_0}^{t_1} F^T v dt = 0.$ (0.5p)

g. Using the expression for the constraint force as $F_c = G^T \lambda$, the full set of equations of motion becomes:

$$\dot{x} = v$$
$$M\dot{v} = F_e + G^T \lambda$$
$$g(x) = 0.$$

This is 3 sets of equations in 3 sets of unknowns: $\dot{x}, \dot{v}$ and λ. Replacing $g(x) = 0$ by $\ddot{g}(x) = 0$ in the equations of motion, we have to solve the following 2 coupled

Examinator: Kenneth Bodin
Cell phone: 070-631 5520

equations in two unknowns $(\dot{v}, \lambda)$:

$$M\dot{v} = F_e + G^T \lambda$$

$$G\dot{v} = -\alpha(t)$$

where $\alpha(t)$ was computed previously. Show a derivation for the identiy: (0.5p)

$$GM^{-1}G^T \lambda = -\alpha + GM^{-1}F_e$$

h. Compute the *Schur complement matrix* $GM^{-1}G^T$ for the pendulum (0.5p).

i. Solve the equation for λ explicitly and derive the reduced equation of motion:

$$M\dot{v} = -M\left|v\right|^2 x + \left[I - xx^T\right]F_e$$

where *I* is the *2x2* identity matrix.(0.5p)

j. Show that the matrix $P = \left[I - xx^T\right]$ is a projection operator, i.e. that $P^2 = P$. (0.5p)

k. Give a geometric interpretation for this projection. (0.5p)

l. Let $S = GM^{-1}G^T$ be the Schur complement matrix and S^{-1} the inverse. Show that, in general, the the result of eliminating the constraints is: (0.5p)

$$M\dot{v} = -G^T S^{-1}\alpha + [I - G^T S^{-1} GM^{-1}]F_e$$

m. Show also that $P = [I - G^T S^{-1} GM^{-1}]$ is a projection operator in general.(0.5p)

n. Show also that $GM^{-1}P = 0$, i.e. that P projects into the null space of GM^{-1}.(0.5p)